The effects of individual versus community-influenced isolation on SIS epidemic persistence on finite random graphs

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Abstract

The contact process, or SIS epidemic, is a continuous-time Markov process used to model the spread of infection on a graph. Each vertex is either healthy or infected, and each infected vertex independently infects each of its healthy neighbors at rate λ and recovers at rate 1. We study the contact process in the presence of additional intervention measures by introducing a third possible state for vertices, which we call isolated. Vertices may enter the isolated state either because of individual decisions or due to community-influenced decisions, which leads to two distinct models that we call the isolation model and the vigilance model, respectively. In the isolation model, infected vertices self-isolate at rate α. In the vigilance model, each healthy vertex causes each of its infected neighbors to isolate at rate α. Unlike the usual contact process, these models lack the key features of attractiveness and existence of a dual, which makes analyzing them more challenging. We study the persistence times of the infection on large, finite, degree-heterogeneous random graphs. We show that the infection in the isolation model persists for at least stretched exponential time in the size of the graph for all values of α and λ. By contrast, in the vigilance model, for every fixed α the persistence time of the infection exhibits a phase transition in λ: for small λ the infection persists for at most a linear time in the size of the graph, while for large λ the infection persists exponentially long. This contrast demonstrates that individual versus community-influenced isolation can substantially affect the persistence of an epidemic.

1 Introduction

The contact process, introduced by [14], has long been studied as stochastic epidemic model on graphs. Given a graph $G = (V, E)$ with vertices V and edges E, the *classical contact process* allows for two possible states on each vertex, healthy and infected, and has a single parameter λ that determines the rate at which infection spreads across edges from infected to healthy vertices. The contact process has been studied extensively on lattices, trees, and general random graphs .

Recently, there has been increased interest in variant contact process models that include additional vertex states or edge behaviors as a way to model more complex pathogen dynamics and human behavior. We consider two such models here. The first, the *contact process with isolation*, includes an additional isolated state in which vertices can neither transmit nor receive infection,

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and has infected vertices self-isolate at rate α. The second, the *contact process with vigilance*, also includes the isolated state but instead of self-isolation, each healthy vertex isolates each of its infected neighbors at rate α. These models are intended to model different ways in which individuals or communities might respond to the presence of infection.

An important difference between our models and the classical contact process is that the classical contact process has a type of monotonicity called *attractiveness* (or equivalently is an attractive particle system) while our models are not known to have this property. This property states that if we have initial infected sets $A \subseteq B$ and ξ_t^A and ξ_t^B are the infected sets at time t beginning from initial infected sets A and B respectively, then there exists a coupling between ξ_t^A and ξ_t^B such that $\xi_t^A \subseteq \xi_t^B$ for all $t \geq 0$. Attractiveness is an important ingredient in the proofs of many results about the classical contact process. Neither the contact process with isolation nor the contact process with vigilance is known to be an attractive particle system, and this presents technical challenges in deriving our results.

We study the contact process with isolation and contact process with vigilance on configuration model random graphs where the degree distribution follows a power law distribution with power law $\gamma > 2$. Our main result on the contact process with isolation shows that this model, like the classical contact process, exhibits at least stretched exponential survival for all choices of λ and α on these graphs. This is somewhat surprising because [4] and [17] previously showed that similar models exhibit polynomial survival on the star graph, and behavior on the star graph forms the basis of proofs showing the classical contact process exhibits long survival for all $\lambda > 0$ on high degree random graphs.

In contrast, we show that that the contact process with vigilance exhibits a phase transition in λ for every fixed α on graphs satisfying the isoperimetric inequality given in assumption 2, which include configuration model random graphs where the degree distribution follows a power law distribution with power $\gamma > 3$ and minimum degree 3. From this contrast, we observe that an individual versus a communal response to an epidemic can have a substantial impact on survival and extinction of an epidemic.

2 Main Results

We define the **contact process with isolation** as follows. Let $G = (V, E)$ be a graph with vertices V and edges E. Let $\xi_t(v)$ denote the state of vertex v at time t, where $\xi_t(v) \in \{0, 1, -1\}$ where state 0 is healthy, state 1 is infectious (or infected), and state -1 is isolating. Let $|\xi_t|$ Given an initial condition $\xi_0 \in \{0, 1, -1\}^V$ on the states of the vertices in G and parameters λ and α, the process evolves according to the following rules:

1. $\xi_t(v)$ goes from $0 \to 1$ at rate $\lambda \cdot |\{w \sim v : \xi_t(w) = 1\}|$.
2. $\xi_t(v)$ goes from $1 \to -1$ at rate α.
3. $\xi_t(v)$ goes from $\{1, -1\} \to 0$ at rate 1.

Here rate means that the time to event follows an Exponential distribution with the given rate parameter. The contact process with isolation is intended to model an individually-driven response to an epidemic.

We define the **contact process with vigilance** as follows. Let $G = (V, E)$ be a graph with vertices V and edges E. At any given time, each vertex v is in one of three states: 1 (infectious), 0

(healthy), or -1 (isolated). Let $\zeta_t(v)$ be the state of vertex v at time t. Given the initial states of all vertices $\zeta_0 \in \{0, 1, -1\}^V$, an infection rate parameter λ and a vigilance rate parameter α, the process evolves according to the following rules.

1. For v such that $\zeta_t(v) = 0$, $\zeta_t(v) \to 1$ at rate $\lambda|\{w \sim v : \zeta_t(w) = 1\}|$.
2. For v such that $\zeta_t(v) = 1$, $\zeta_t(v) \to 0$ at rate 1.
3. For v such that $\zeta_t(v) = 1$, $\zeta_t(v) \to -1$ at rate $\alpha|\{w \sim v : \zeta_t(w) = 0\}|$.
4. For v such that $\zeta_t(v) = -1$, $\zeta_t(v) \to 0$ at rate 1.

Again rate means that the time to event follows an Exponential distribution with the given rate parameter. The first rule states that each healthy vertex becomes infected at rate λ times the number of infected neighbors it has. The second rule states each infected vertex recovers and becomes healthy at rate 1. The third rule states that each infectious vertex is identified and isolated at rate α times the number of healthy neighbors is has. The fourth rule states that each isolated vertex becomes healthy and returns from isolation at rate 1. The contact process with vigilance is intended to model a community-driven response to an epidemic.

We now state our main results.

Assumption 1. *Suppose $G = (V, E)$ is a configuration model random graph with n vertices where the degree sequence $D_1, \ldots, D_n$ are sampled from a power law distribution with*

$$\mathbb{P}(D_i = k) \propto \frac{1}{k^\gamma}$$

for power $\gamma > 2$.

Assumption 2. *Suppose that $G = (V, E)$ is a graph such that for some $\epsilon' < \epsilon < 1/2$ there exists a $\delta > 0$ such that for any set of vertices B satisfying $\epsilon'|V| \leq |B| \leq \epsilon|V|$,*

$$|\partial B| \geq \delta |B|,$$

where $\partial B = \{v \in V \setminus B : \exists w \in B \text{ such that } \{w, v\} \in E\}$ denotes the external vertex boundary of B.

Theorem 1. *Let $G = (V, E)$ be a graph satisfying assumption 1 with $|V| = n$, and let $\tau_A = \inf\{t : |\{\xi_t = 1\}| = 0\}$ be the time to extinction of the contact process with isolation on G. Starting from initial condition $\xi_0(v) = 1$ for all $v \in V$, there exists $\epsilon = \epsilon(\alpha, \lambda) > 0$ such that $\mathbb{P}(\tau_A < e^{n^\epsilon}) \to 0$ as $n \to \infty$ for any $\alpha > 0$ and $\lambda > 0$.*

Theorem 2. *Let $G = (V, E)$ be any graph with $|V| = n$ and let $\tau_v = \inf\{t : |\{\zeta_t = 1\}| = 0\}$ be the time to extinction of the contact process with vigilance on G. Fix $\alpha > 0$ and any $\lambda < \alpha$. Starting from initial condition $\zeta_0(v) = 1$ for all $v \in V$, there exists $C > 0$ such that $\mathbb{P}(\tau_v > Cn) \to 0$ as $n \to \infty$.*

Theorem 3. *Let $G = (V, E)$ be a graph satisfying assumption 2 for some $\epsilon > 0$ and $\delta > 0$ with $|V| = n$ and let $\tau_v = \inf\{t : |\{\zeta_t = 1\}| = 0\}$ be the time to extinction of the contact process with vigilance on G. Fix $\alpha > 0$. Starting from initial condition $\zeta_0(v) = 1$ for all $v \in V$, there exists $\lambda_0 = \lambda_0(\alpha, \epsilon, \delta)$ and $C > 0$ such that $\mathbb{P}(\tau_v < e^{Cn}) \to 0$ as $n \to \infty$ for any $\lambda > \lambda_0$.*

In Theorem 4, we show that Assumption 2 is satisfied with high probability when G is a graph satisfying Assumption 1 where $\gamma > 3$ and the minimum degree is 3. This leads to the following corollary that gives a direct comparison to Theorem 1 for the contact process with vigilance.

Corollary 1. *Let $G = (V, E)$ be a graph satisfying assumption 1 with $\gamma > 3$ and minimum degree 3, and let $\tau_v = \inf\{t : |\{\zeta_t = 1\}| = 0\}$ be the time to extinction of the contact process with vigilance on G. Fix $\alpha > 0$. Starting from initial condition $\zeta_0(v) = 1$ for all $v \in V$, there exists $\lambda_0 = \lambda_0(\alpha, \gamma)$ and $C > 0$ such that $\mathbb{P}(\tau_v < e^{Cn}) \to 0$ as $n \to \infty$ for any $\lambda > \lambda_0$.*

3 Background and related results

One key question in the study of the contact process concerns the long-run survival of the infection. On infinite graphs it is possible for the infection to survive indefinitely, and it is well-known that on lattices and trees the contact process exhibits a phase transition in λ between almost sure extinction and positive probability of indefinite survival. While the contact process cannot survive indefinitely on a finite graph, it is known that on certain finite graphs such as $\mathbf{Z} \mod n$ that there is a phase transition in λ between survival time $O(\log n)$ and survival time $e^{\Theta(n)}$ asymptotically as the size of the graph $n \to \infty$. For a detailed introduction to and summary of known results about the contact process, we direct the reader to [18] and [7].

Of particular relevance to us are the results of [1], who show that the contact process has survival time $e^{\Theta(n)}$ on star graphs for all $\lambda > 0$, [19], who extend this result to random graphs with power law degree distribution, and [2], who extend this result to general random graphs whose degree distributions have subexponential tails. This contrasts with our main results showing that the survival time of the vigilance model has a phase transition between linear and exponential survival times on a class of graphs including power law random graphs.

Recently, there has been increased interest in variant contact process models that include additional vertex and edge behaviors intended to model more complex disease dynamics, behaviors, and population structures. An early example is the adaptive SIS model proposed by [12] in which SI edges randomly rewire. While this model has received considerable attention [11, 13, 20, 6], it has proven difficult to study rigorously, with the aforementioned work focusing on mean field approximations, moment closures, and computational results.

Some variants of the adaptive SIS model have proven more tractable. [4] consider a model in which directed edges temporarily deactivate rather than rewrite and show that this model exhibits a phase transition like that of the classical contact process on $\mathbb{Z}, \mathbb{Z} \mod n$ and exhibits asymptotic survival time polynomial in the size of the graph on the star graph. [22] study a similar model where bidirectional edges temporarily deactivate and derive an epidemic threshold on the complete graph using differential equation approximations. [15] study a model in which edges rewrite independently of vertex states and show that this model has a phase transition in λ for certain power law degree distribution graphs and sufficiently fast rewiring.

The evoSIR model, which considers the random rewiring of the adaptive SIS model in the context of an underlying SIR model, has also been studied. In general, SIR models tends to be easier to study than SIS models because in SIR models vertices are removed from the graph upon recovery and so each vertex can become infected at most once. Indeed, more is known about the evoSIR model, including the existence of phase transitions in λ and their continuity behavior on Erdos-Renyi graphs [16] and configuration model graphs [9].

Other variants of the contact process, including the isolation and vigilance models we study in this work, add additional vertex states rather than edge behaviors. [21] studies an ecologically-motivated model that we can translate into the language of our isolation model. In this model all vertices, rather than only infectious vertices, isolate at rate α and return from isolation at rate $\alpha\delta$. He shows this model is monotonic in λ, α, and δ individually, although changing multiple parameters can lead to incomparable processes. He also shows this model has phase transitions on $\mathbf{Z}$ in both λ and δ. For sufficiently small δ, the infection dies out almost surely for all λ, while for larger δ the infection has positive probability of persisting indefinitely only when λ is sufficiently large. We make a more detailed comparison of our isolation model with this model in section 5.1.

Another related model is the SIRS model, in which after recovering, individuals become temporarily immune to reinfection. Like the isolation model, the SIRS model is not monotonic in λ for reasons analogous to those demonstrated in figure 4. This has made rigorous study of the SIRS model difficult. However, some results have been derived. [8] show that for sufficiently large λ, the SIRS model has a nontrivial stationary distribution on $\mathbb{Z}^2$. However, their results rely on isoperimetric properties specific to $\mathbb{Z}^2$ and so do not generalize to other graphs.

More recently, [17] showed that the survival time of the SIRS model on the star graph is asymptotically polynomial in the size of the graph. Because the result of [19] that the contact process has survival time $e^{\Theta(n)}$ on power law degree distribution random graphs for all $\lambda > 0$ relies heavily on the fact the contact process has $e^{\Theta(n)}$ on star graphs for all $\lambda > 0$, the results of [17] and [4] pose an interesting question of whether these variant models exhibit qualitatively different behavior from that of the classical contact process on power law degree distribution random graphs. Conversely, [10] show that the expected survival time of the SIRS process is exponential in the size of the graph for certain expander graphs, suggesting one might expect similar behavior on power law degree distribution random graphs. Theorem 1 answers this question for our isolation model rather than the SIRS model, but because these models are similar we suspect the same result holds in the case of the SIRS model and that this result could be proved using techniques similar to the ones we use here.

4 Graphical construction

A useful tool for studying the contact process and its variants is a graphical construction sometimes called the Harris construction. For the classical contact process, this construction is defined as follows. For a graph G, consider the spacetime region $G \times [0, \infty)$. At each vertex v and directed edge e of the graph, we will have a Poisson process of temporal marks with appropriate rates, defined as follows.

(C1) For each directed edge $e \in G$, define a Poisson process on $\{e\} \times [0, \infty)$ with intensity λ that generates infection arrows.

(C2) For each vertex $v \in G$, define a Poisson process on $\{v\} \times [0, \infty)$ with intensity 1 that generates recovery dots.

We can then realize the process on $G \times [0, \infty)$ given some initial state $\xi_0 \in \{0, 1\}^V$, where 1 indicates an infected vertex and 0 a healthy vertex, using these marks by doing the following.

1. Label infected vertices 1 (blue in Figure 4) vertically in time until a recovery dot is reached.

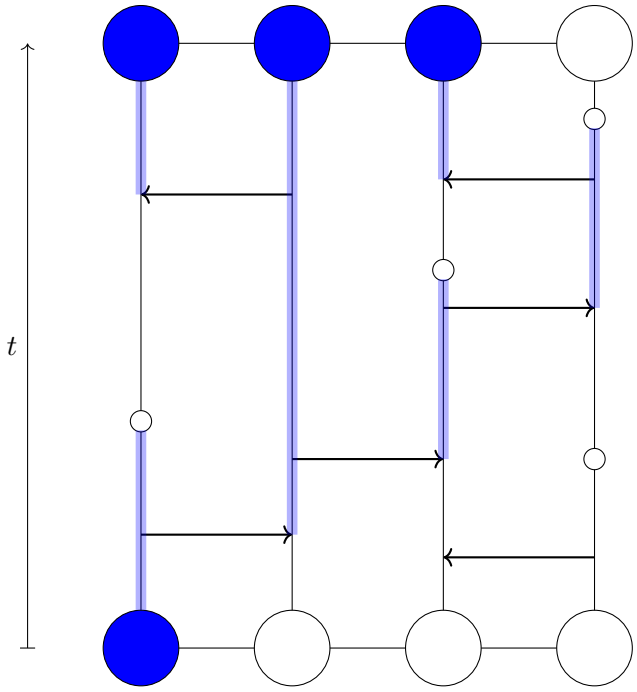

Figure 1: An example of the graphical construction for the classical contact process. Blue vertices are infected, and white vertices are susceptible. We follow the blue paths to determine which vertices are infected at time t.

2. When an infection arrow is observed, if the source vertex is infected (1, blue), infect the target vertex and repeat step 1. for this newly infected vertex.

The set of vertices in G infected at time t is exactly those vertices reached by a blue path at time t. See Figure 4 for an example. This graphical construction can be used to demonstrate that the classical contact process is attractive. Adding additional vertices to the initial infected set can only increase the number of vertices reached by a blue path at time t.

We can define a similar graphical construction for the contact process with isolation by adding Poisson processes to generate marks that govern isolation transitions.

(I3) For each vertex $v \in G$, define a Poisson process on $\{v\} \times [0, \infty)$ with intensity α that generates isolation crosses.

We can then realize the process on $G \times [0, \infty)$ given some initial state $\xi_0 \in \{-1, 0, 1\}^V$, where 1 indicates an infected vertex, 0 a healthy vertex, and -1 and isolated vertex using these marks by doing the following.

1. Label infected vertices 1 (blue in Figure 4) vertically in time until a recovery dot or an isolation cross is reached. If a recovery dot is reached first, return the vertex to the health (0) state and remove its color.

2. When an isolation cross is observed on a (1, blue in Figure 4) vertex, change its state to isolated (-1) and color the vertex red vertically in time until a recovery dot is reached. When a recovery dot is reached first, return the vertex to the health (0) state and remove its color.

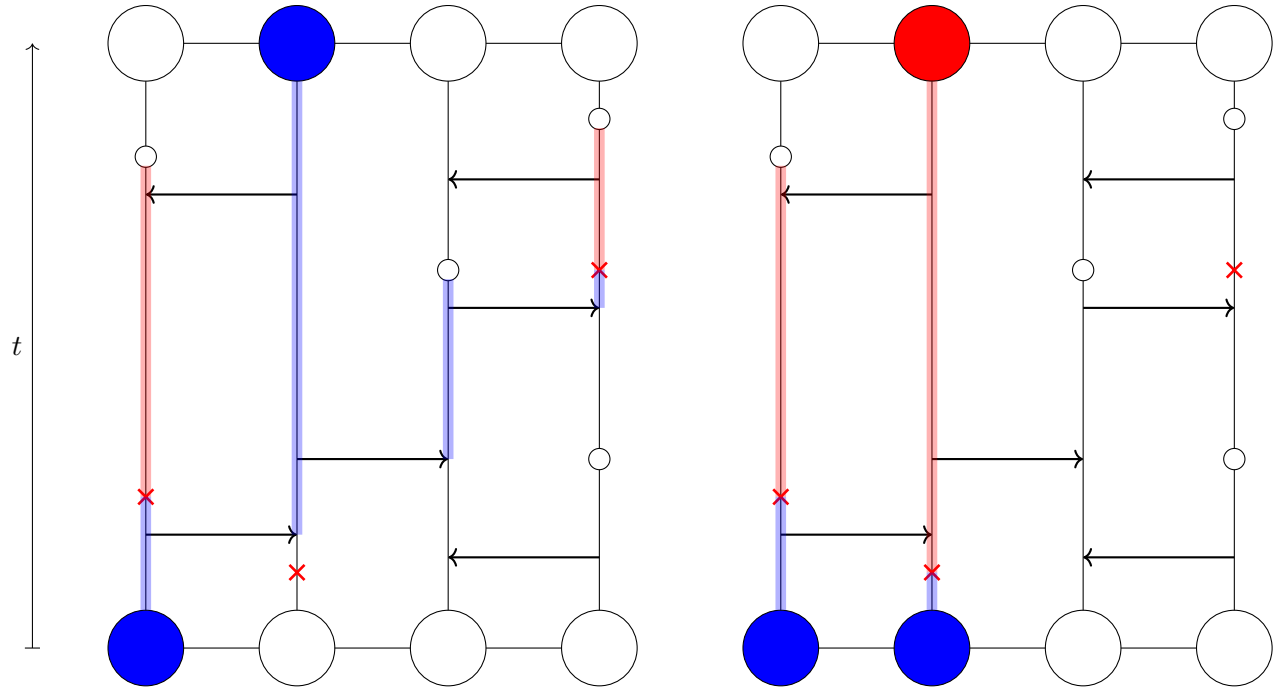

Figure 2: In the isolation model adding additional vertices to the initial infected set can lead to a smaller final infected set. Blue vertices are infected, white vertices are susceptible, and red vertices are isolated.

3. When an infection arrow is observed, if the source vertex is infected (1, blue), infect the target vertex and repeat step 1. and 2. for this newly infected vertex.

Again, the set of vertices in A infected at time t is exactly those vertices reached by a blue path at time t. We can also see from this construction why the argument used to show the classical contact process is attractive fails for the contact process with isolation. Adding additional vertices to the initial infected set may cause additional isolations and larger red regions. As in the example shown in figure 4, this can lead to a smaller infected set at time t.

5 Isolation model

5.1 Overview

Our goal in the first part of this paper is to prove Theorem 1. Rather than study the isolation model directly on random graphs, we begin by studying the process on what we call a **star of stars graph of order** m. We define this structure as follows.

Definition 1. *A graph $G = (V, E)$ is called a **star of stars graph of order** m if it is a rooted tree with the following properties.*

- *The root has m children. Call the root the **center** vertex.*
- *Call the m children of the center **hubs**. Each hub has m children, which we call **leaves**.*

We will call the subgraph induced by the center and hubs the **center star** and the subgraph induced by each hub and its leaves a **hub star**. We define the **process internal to a hub star** to mean the the realization of the process using only marks in the graphical construction internal to that hub star.

Our first goal is to show long survival on the star of stars structure for any $\alpha, \lambda > 0$ by proving the following lemma.

Lemma 1. *Let $G = (V, E)$ be a graph satisfying definition 1, and let $\tau_* = \inf\{t : |\xi_t(V)| = 0\}$ be the time to extinction of the contact process with isolation on G. Starting from initial condition $\xi_0(v) = 1$ for all $v \in V$, there exists $\epsilon = \epsilon(\alpha, \lambda)$ such that $\mathbb{P}(\tau < e^{m^\epsilon}) \to 0$ as $m \to \infty$.*

We begin by giving a high-level overview of our strategy to prove Lemma 1. If we consider the state of the center, we can classify the process as being in one of three phases at any given time:

- The one phase, when the center is infected.
- The zero phase, when the center is healthy.
- The minus one phase, when the center is isolating.

During the one phase, the center is spreading the infection to the hubs, potentially reintroducing it to any hub stars where it might have previously gone extinct. During the zero phase, any hub star can potentially reinfect the center, assuming its own infection has not gone extinct locally. During the minus one phase, the hub stars behave independently as stars of degree m, and can potentially see their infections go extinct locally.

From this, we can broadly see three reasons the infection might die out on our star of stars.

1. During a one phase, the center does not infect enough hubs.
2. During a zero phase, the non-extinct hub stars do not reinfect the center quickly enough.
3. During a minus one phase, too many hub stars go extinct locally.

Our goal will be to show that each of these is unlikely. However, 1. presents a problem. Namely, the one phase could be very short, and in this case we would not expect the center to infect many hubs. Thus, we will define long one phases to be those that are sufficiently long and change 1. to

1. During a long one phase, the center does not infect enough hubs.

Since we are interested in long one phases, we combine 2. and 3. into

2. Between long one phases, too many hub stars go extinct locally.

Before we examine 1.-2. more closely, we will establish lemma 2 to allow us to deal with the nonmonotonicity of the contact process with isolation. Because our goal is ultimately to consider the star of stars structure embedded within a random graph, we must consider the effect of infection from the outside. Unfortunately, unlike the classical contact process, the contact process with isolation is not an attractive particle system, and so it cannot be assumed that infection from the outside will only aid in survival. In particular, this is a problem when studying the one phase, as

infections of hubs from the outside shortly prior to the one phase could lead to those hubs isolating through the one phase when they would not have without the outside infection.

We overcome this difficulty by showing the contact process with isolation stochastically dominates a comparison process in which all vertices, not just infected vertices, isolate at rate α. This process is equivalent to the special case of the model studied by Remenik [21] where his parameter $\delta = 1/\alpha$. Unlike the isolation model, the comparison model is attractive in the same sense as the classical contact process, and Remenik [21] notes this is true of his model generally. We will first formally define the comparison process and establish stochastic domination before briefly discussing why the comparison process is attractive.

We define the comparison process as follows. Let $G = (V, E)$ be a graph and denote the state of a vertex v at time t by $\xi_t^R(v) \in \{-1, 0, 1\}$ where state 1 is infectious, state 0 is healthy (and susceptible), and state -1 is isolating. Given an initial state ξ_0^R, this comparison model evolves according to the following rules

1. If $\xi_t^R(v) = 0$, $\xi_t^R(v) \to 1$ at rate $\lambda \cdot |\{w \sim v : \xi_t(w) = 1\}|$.
2. If $\xi_t^R(v) \in \{0, 1\}$, $\xi_t^R(v) \to -1$ at rate α.
3. If $\xi_t^R(v) \in \{1, -1\}$, $\xi_t^R(v) \to 0$ at rate 1.

Lemma 2. *The infected set in the contact process with isolation stochastically dominates the infected set in the comparison process. That is, given initial conditions $\xi_0(V) = \xi_0^R(V)$ and shared parameters λ and α, there exists a coupling such that*

$$\xi_t^R(v) = 1 \implies \xi_t(v) = 1$$

for all $v \in V$ and all $t > 0$.

Proof. We can demonstrate an appropriate coupling using the graphical construction defined via the Poisson processes of marks (C1), (C2), and (I3) in section 4.

We can couple the isolation model with the comparison process by generating the same realization of symbols in the graphical constructions for both and considering an arbitrary initial configuration. However, the rules for realizing the process differ for the isolation model and the comparison process. In the isolation model:

1. Color infected (state 1) vertices blue vertically in time until either a recovery dot or isolation cross is reached. If a recovery dot is reached first, return the vertex to the healthy (0) state and remove its color.
2. When an isolation cross is observed on an infected (1, blue) vertex, change its state to isolated (-1) and color the vertex red vertically in time until a recovery dot is reached. When a recovery dot is reached, return the vertex to the healthy (0) state and remove its color.
3. When an infection arrow is observed, if the source vertex is infected (1, blue) and the target vertex is not isolated, infect the target vertex and repeat steps 1.-2. for this newly infected vertex.

We can then follow the blue paths to determine which vertices in G are infected at time t. Note that the path of the infection cannot travel through the red regions indicating isolated vertices.

To construct the comparison process on the same realization of marks in the graphical construction, step 2. above is replaced by $2'$ below.

$2'$. When an isolation cross is observed on any vertex, color the vertex red vertically in time until a recovery dot is reached, at which time its state reverts to healthy (0) and its color removed.

Thus, any red region that exists in the isolation model will exist in the comparison model, as the isolation cross that initiated the red region in the isolation model will also initiate a red region in the comparison model, and the red regions in both models will continue until they encounter the next recovery dot, which occurs at the same time for both processes. The comparison model may have additional red regions not present in the isolation model.

We now consider that if we were to ignore the red regions, the path of infection in both models will be the same, as they share the same infection arrows and recovery dots. However, we truncate any infection paths that attempt to cross a red region. Thus, if the red regions in the isolation model are contained in the red regions in the comparison model, any infection path in the comparison model that is not truncated by a red region is also not truncated by a red region in the isolation model. This demonstrates a coupling in which under any shared realization of marks in the graphical construction and any initial configuration on $G \times [0, \infty)$, for every vertex v and time t,

$$\xi_t^R(v) = 1 \implies \xi_t(v) = 1.$$

That is, the set of infected vertices in the comparison model at time t is contained in the set of infected vertices in the isolation model at time t. □

We briefly pause to observe why the comparison model is attractive. The isolation model is not attractive because additional infections can create additional isolations (red regions) that might block infection paths that would otherwise successfully persist from time 0 to time t in our graphical construction. However, in the comparison model, the isolated vertices and thus the red regions are determined completely by the isolation crosses and recovery dots. The configuration of infection arrows has no effect on when isolations start and end. Thus, adding more infected vertices to the initially infected set or adding more infection arrows to a realization of the graphical construction can only ever increase the number of infection paths that persist from time 0 to time t.

Having set forth our strategy and established lemma 2, we now move to examine the process during the different center phases. We will prove lemma 1 and later theorem 1 for the comparison process, and because the contact process with isolation dominates the comparison process, these results will also hold for the contact process with isolation.

5.2 The one phase

Recall that a one phase occurs when the center becomes infected and ends when the center stops being infected. Let S_i be the time of the ith reinfection of the center, and let T_i be the next time after S_i that the center is not infected. Note that S_i and T_i are stopping times, and on the event $S_i < \infty$, $T_i - S_i$ is exponentially distributed and independent of the σ-field generated by all marks in the graphical construction outside the center during $[S_i, T_i]$. We will call the ith one phase **long** if $T_i - S_i > 1/(1 + \alpha)$.

For the process to survive, we need the center to spread infection to the hub stars sufficiently so that enough hub stars can survive independently of the center until the next long one phase. To this end, we will call a hub star **lit** whenever its hub is infected and it has at least δm infected leaves. The value of $\delta \in (0,1)$ will be chosen later.

Our goal in studying the one phase is to show that there is some $\eta > 0$ such that at the end of a long one phase, there are at least ηm lit hub stars with high probability irrespective of the configuration at the start of the long one phase. Since S_i is a stopping time, the strong Markov property allows us to consider the process starting from time 0 with the center infected and show that, conditioned on this initial one phase being long, we are likely to light enough hub stars. We formalize this in the following lemma:

Lemma 3. *Fix $\alpha, \lambda > 0$ and suppose we start the process at time 0 from an initial configuration ξ_0 such that the center is infected. Let T_0 be the first time that the center is not infected, let G_0 be the event that $T_0 > 1/(1+\alpha)$, and let L_0 be the number of lit hub stars at time T_0. Then there exist $\delta, \eta > 0$ depending on α and λ but not on m such that*

$$\mathbb{P}^{G_0}(L_0 < \eta m) \leq \exp\left(\frac{-\eta m}{4}\right)$$

where $\mathbb{P}^{G_0}$ means the probability conditional on the event G_0.

Proof. Consider a hub and let $t = 1/(1+\alpha)$. Then on the event G_0, the following sequence of events ensures the hub is infected at time T_i.

(a) Hub has a recovery dot in $(T_i - t, T_i - 2t/3)$

(b) Hub has no recovery dots in $(T_i - 2t/3, T_i)$

(c) Hub has no isolation crosses in $(T_i - t, T_i)$

(d) Center has an infection arrow to the hub in $(T_i - 2t/3, T_i - t/3)$

Call the intersection of events (a)–(d) above G_1. We can use the independence of these events to compute the probability of G_1

$$\mathbb{P}^{G_0}(G_1) = (1 - e^{-t/3})(e^{-2t/3})(e^{-\alpha t})(1 - e^{-\lambda t/3}) =: \gamma_1. \tag{1}$$

Since $t = \frac{1}{1+\alpha}$, we have $\gamma_1 > 0$ depends on α and λ but not on m. However, it is not enough to infect the hub; we want the hub star to be lit at time T. Thus, we now consider a sequence of events that will cause the hub to be lit at time T.

G2 At least δm leaves of the hub star have

(e) an infection arrow from the hub in $(T_i - t/3, T_i)$, and

(f) a recovery dot in $(T_i - t, T_i - t/3)$, and

(g) no isolation crosses in $(T_i - t, T_i)$.

For a fixed hub star at leaf v, let $G_2(v)$ denote the intersection of events in (e)–(g) above. Observe that if $G_1 \cap G_2(v)$ occurs, then v will be infected at time T_i. Thus, if we let X be the number of leaves such that $G_2(v)$ occurs, we have

$$(X|T_i) \sim \text{Binomial}\Big(m, (1-e^{-\lambda t/3})(1-e^{-2t/3})(e^{-\alpha t})\Big)$$

Given our $t = \frac{1}{1+\alpha}$, we can choose $\delta > 0$ small depending on λ and α so that

$$E(X) \geq 2\delta m,$$

and then applying a Chernoff bound gives

$$\mathbb{P}^{G_0}(X < \delta m) \leq \exp\left(-\frac{(1/4)(2\delta m)}{2}\right) = \exp\left(-\frac{\delta m}{4}\right). \tag{2}$$

Finally, let G_2 be the event that $G_2(v)$ occurs for at least δm leaves of the hub star, so

$$\mathbb{P}^{G_0}(G_1 \cap G_2) \geq \mathbb{P}^{G_0}(G_1) \cdot \left(1 - \exp\left(-\frac{\delta m}{2}\right)\right) \geq \gamma_1/2$$

for sufficiently large m.

On the event G_0, the number of hubs X_L^i that will be lit at time T_i stochastically dominates Y where

$$Y \sim \text{Binomial}(m, \gamma_1/2).$$

Choose $\eta > 0$ depending on λ and α such that $E(Y) \geq 2\eta m$. By a Chernoff bound,

$$\mathbb{P}^{G_0}(X_L < \eta m) \leq \mathbb{P}^{G_0}(Y < \eta m) \leq \exp\left(\frac{-\eta m}{4}\right),$$

which completes the proof. □

5.3 Survival of lit hub stars between consecutive long one phases

We next consider the survival of the infection in hub stars between consecutive long one phases. We show an initially lit hub star can survive independently of its surroundings for polynomial time with probability at least 1/2. Because the end of the long one phase is a stopping time, the strong Markov property allows us to consider this starting from time 0.

Lemma 4. *Consider the process internal to a hub star, and let N_t be the number of infected leaves at time t. Suppose $N_0 \geq \delta m$ and that the hub is infected at time 0, and let $\tau_H = \inf\{t > 0 : N_t = 0\}$. Then there exists $1 > \gamma > 0$ such that*

$$\mathbb{P}(\tau_H > (2 + 2/\alpha)m^\gamma) \geq 1/2.$$

Proof. Consider the process internal to a hub star with the hub and at least δm leaves initially infected, and let N_{t^*} be the number of infected leaves at time t^*. We want to show there exists $1 > \gamma > 0$ such that when $t^* = (2 + 2/\alpha)m^\gamma$, then $\mathbb{P}(N_{t^*} > 0) \geq 1/2$.

For our fixed hub vertex, let S_i^H be the time of the ith reinfection of the hub vertex, and let T_i^H be the next time after S_i^H at which the hub is no longer infected. Call the ith time interval $[S_i^H, T_i^H)$ the ith **infectious phase**, provided $S_i^H < \infty$. We say the ith infectious phase is **long** if $T_i^H - S_i^H > 1/(1+\alpha)$. Broadly, the process can survive between long one phases if

1. the hub infects a sufficient number of leaves during a long infectious phase, and

2. a sufficient number of leaves stay infected until the start of the next long infectious phase.

We consider 1. first. Suppose T_i^H is the end of a long infectious phase, let $t = 1/(1+\alpha)$, and consider the time interval $[T_i^H - t, T_i^H]$. For each leaf, we can ensure it is infected at time T_i^H if the leaf has

(L1) A recovery dot in $[T_i^H - t, T_i^H - 2t/3]$,

(L2) An infection arrow from the hub in time $[T_i^H - 2t/3, T_i^H]$,

(L3) No recovery dots in $[T_i^H - 2t/3, T_i^H]$, and

(L4) No isolation crosses in $[T_i^H - t, T_i^H]$,

The probability of satisfying (L1)-(L4) is

$$p^* = (e^{-t/3})(1 - e^{-2\lambda t/3})(e^{-2t/3})(e^{-\alpha t}).$$

The number of infected leaves at the end of a long infectious phase stochastically dominates X^*, where

$$X^* \sim \text{Binomial}(m, p^*).$$

Let D be the event that there are fewer than $mp^*/2$ infected leaves at time T_i^H. Then using a Chernoff bound,

$$\mathbb{P}(D) \leq \mathbb{P}(X^* < mp^*/2) \leq \exp\left(\frac{-p^* m}{4}\right). \tag{3}$$

We now consider the survival of these infections until the next long one phase. Let T_ℓ be the time from T_i^H until the next long one phase. For $C > 0$ to be chosen later, define the following bad events.

(D1) Consider the first $2C \log m$ time intervals after T_i^H during which the hub is not in state 0. Let T_ℓ^* be the total amount of this time, and let D_1 be the event $T_\ell^* > 4C \log m$. Since $T_\ell^* \sim \text{Gamma}(2C \log m, 1)$, we have

$$\mathbb{P}(D_1) \leq \left(\frac{2}{e}\right)^{C \log m} = m^{-C(1-\log 2)}$$

by a Chernoff bound.

(D2) Consider the first $2C \log m$ time intervals after T_i^H during which the hub is in state 0. Let T_ℓ^{**} be the total amount of this time, and let D_2 be the event that $T_\ell^{**} > \frac{4C \log m}{\alpha}$. Since T_ℓ^{**} is dominated by a $\text{Gamma}(2C \log m, \alpha)$ random variable, we have

$$\mathbb{P}(D_2) \leq \left(\frac{2}{e}\right)^{C \log m} = m^{-C(1-\log 2)}$$

by a Chernoff bound.

(D3) There is no leaf that is state 1 at time T_i^H and stays in state 1 until time $T_i^H + 4C\log m + \frac{4C\log m}{\alpha}$. Call this event D_3.

(D4) More than $\frac{2\lambda}{\lambda+\alpha}C\log m$ of the next $2C\log m$ time intervals after T_i^H during which the hub is in state 0 are ended by transitions of the hub to state -1. Call this event D_4.

(D5) None of the next $\frac{2\lambda}{\lambda+\alpha}C\log m$ infectious phases after time T_i^H are long or there are fewer than $C\log m$ infectious phases after time T_i^H. Call this event D_5.

We observe on the event $D^c \cap D_1^c \cap D_2^c \cap D_3^c \cap D_4^c \cap D_5^c$ that $T_\ell \le 4C\log m + \frac{4C\log m}{\alpha}$. We also observe

$$\begin{aligned}(D^c \cap D_1^c \cap D_2^c \cap D_3^c \cap D_4^c \cap D_5^c)^c &= D \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \\ &\subseteq D \cup D_1 \cup D_2 \cup (D_3 \cap D^c) \cup (D_4 \cap D_3^c \cap D_1^c \cap D_2^c) \cup (D_5 \cap D_4^c)\end{aligned}$$

First, $(D_3 \cap D^c)$ requires that there are at least $mp^*/2$ infected leaves at time T_i^H. The probability that a leaf that is in state 1 at time T_i^H does not leave state 1 before time $T_i^H + 4C\log m + \frac{4C\log m}{\alpha}$ is $e^{-(4C\log m + \frac{4C\log m}{\alpha})(1+\alpha)}$, and this occurs independently for each leaf that is in state 1 at time T_i^H. Thus,

$$\mathbb{P}(D_3 \cap D^c) \le \exp\left(-(mp^*/4)(e^{-(4C\log m + \frac{4C\log m}{\alpha})(1+\alpha)}\right).$$

If we choose C depending on λ and α such that $e^{-(4C\log m + \frac{4C\log m}{\alpha})(1+\alpha)} = m^{-1/2}$, then,

$$\mathbb{P}(D_3 \cap D^c) \le e^{-p^*\sqrt{m}/4}$$

Next, consider $(D_4 \cap D_3^c \cap D_1^c \cap D_2^c)$. On the event $D_1^c \cap D_2^c$, there are at least $2C\log m$ time intervals during which the hub is in state 0 that occur before time $T_i^H + 4C\log m + \frac{4C\log m}{\alpha}$, and on the event D_3^c there exists some leaf that is in state 1 from time T_i^H until time $T_i^H + 4C\log m + \frac{4C\log m}{\alpha}$. On the event $D_3^c \cap D_1^c \cap D_2^c$, there exists some leaf that is infected continuously through the next $2C\log m$ time intervals during which the hub is in state 0, so during each of these intervals there is probability at least $\frac{\lambda}{\alpha+\lambda}$ that the 0 phase is ended by a transition to state 1. Therefore, the number of 0 phases ended by a transition of the hub to state 1 dominates $Y^* \sim \text{Binomial}(2C\log m, \frac{\lambda}{\alpha+\lambda})$. Thus,

$$\mathbb{P}(D_4 \cap D_3^c \cap D_1^c \cap D_2^c) \le P(Y^* \le C\log m) = P(Y^* \le E(Y^*)/2) \le \exp\left(-\frac{\lambda C\log m}{2(\alpha+\lambda)}\right) = m^{-\frac{\lambda C}{2(\alpha+\lambda)}}.$$

by a Chernoff bound.

Finally, consider the event $(D_5 \cap D_4^c)$. On D_4^c there are at least $C\log m$ infectious phases after time T_i^H. Since each infectious phase is long with probability e^{-1},

$$\mathbb{P}(D_5 \cap D_4^c) \le \exp\left(-\frac{e^{-1}C\log m}{4}\right) = m^{-\frac{C}{4e}}$$

We now compute

$$\begin{aligned}\mathbb{P}(D^c \cap D_1^c \cap D_2^c \cap D_3^c \cap D_4^c \cap D_5^c)^c &\leq \mathbb{P}(D) + \mathbb{P}(D_1) + \mathbb{P}(D_2) + \\ &+ \mathbb{P}(D_3 \cap D^c) + \mathbb{P}(D_4 \cap D_3^c \cap D_1^c \cap D_2^c) + \mathbb{P}(D_5 \cap D_4^c) \\ &\leq 5 \max\left(\exp\left(\frac{-p^* m}{4}\right), m^{-C(1-\log 2)}, e^{-p^*\sqrt{m}/4}, m^{-\frac{\lambda C}{2(\alpha+\lambda)}}, , m^{-\frac{C}{4e}} \right) \\ &= 5 \max\left(m^{-C(1-\log 2)}, m^{-\frac{\lambda C}{2(\alpha+\lambda)}}, , m^{-\frac{C}{4e}} \right)\end{aligned} \tag{4}$$

for large enough m.

If we consider cycles of the process internal to the hub moving between long infectious phases, then equation (4) gives an upper bound on probability that following a long infectious phase the hub fails to survive long enough to have a subsequent long infectious phase. Now choose $1 > \gamma > 0$ small enough such that

$$(2 + 2/\alpha)m^{\gamma}(1+\alpha) \leq (1/2)\left(5 \max\left(m^{-C(1-\log 2)}, m^{-C\frac{\lambda}{\alpha+\lambda}}, m^{-\frac{5C}{108}} \right)\right)^{-1}.$$

Then the probability of seeing fewer than $(2 + 2/\alpha)m^{\gamma}(1+\alpha)$ successful cycles is at most $1/2$ for large enough m. Each cycle must last at least $1/(1+\alpha)$ time, since the initial long infectious phase lasts at least this much time, and so this guarantees

$$\mathbb{P}(N_{t^*} > 0) \geq 1/2$$

for $t^* = (2 + 2/\alpha)m^{\gamma}$ as desired.

Since the comparison process is attractive, hub stars are positively correlated. Thus, if we begin with $2\eta m$ lit hub stars, the number of non-extinct hub stars X_h after time $(3 + 2/(\lambda+\alpha))m^{\gamma \log(C^*)}$ stochastically dominates a

$$\text{Binomial}(2\eta m, 1/2)$$

random variable, and so if we choose $\eta^* = 1/2\eta$, then

$$\mathbb{P}^{G_0}(X_h < \eta^* m) \leq \exp\left(\frac{-\eta m}{2}\right)$$

□

Since the infected set in the contact process with isolation stochastically dominates the infected set in the comparison model, this also holds for the contact process with isolation.

5.4 Time between long one phases

We now study the time between consecutive long one phases. We begin with the following lemma.

Lemma 5. *Suppose at time* 0 *the process is in the zero phase and there are* h *non-extinct hub stars. Then the probability the center is infected at time* 1 *with no other state changes to the center in the interim is at least* $q\mathbf{1}_{h>0}$ *where*

$$q = (e^{-3\alpha})(e^{-8/3})(1 - e^{-1/3})(e^{-2\lambda/3})$$

Proof. We first observe that if $h = 0$, the process is extinct and so $q\mathbf{1}_{h>0} = 0$ as expected. If $h > 0$, then there exists some hub star where either the hub is infected at time 0 or some leaf is infected at time 0. In this case, the following sequence of marks in the graphical construction ensures the center is infected at time 1.

(E1) None of the center, our chosen hub, or our chosen leaf have an avoidance cross in time $(0, 1)$. This event has probability $e^{-3\alpha}$.

(E2) Neither the center nor the leaf have a recovery dot in time $(0, 1)$. This event has probability e^{-2}.

(E3) The hub has a recovery dot in time $(0, 1/3)$. This event has probability $1 - e^{-1/3}$.

(E4) The hub does not have a recovery dot in $(2/3, 1)$. This event has probability $e^{-2/3}$.

(E5) The leaf has an infection arrow to the hub in time $(1/3, 2/3)$. This event has probability $1 - e^{\lambda/3}$.

(E6) The hub has an infection arrow to the center in time $(2/3, 1)$. This event has probability $1 - e^{-\lambda/3}$

Since events (E1)-(E6) are independent in our graphical construction, we can multiply the probabilities and obtain $q\mathbf{1}_{h>0} = (e^{-3\alpha})(e^{-8/3})(1 - e^{-1/3})(e^{-2\lambda/3})$ as desired. □

Note that the strong Markov property implies that lemma 5 can also be applied starting from any stopping time T^*.

Lemma 6. *Recall that T_i is the next time after the ith reinfection of the center that the center is no longer infected, and let W^{long} be the amount of time from T_i until the start of the next long one phase. Then given the configuration ξ_{T_i} is such that there are at least ηm lit hub stars at time T_i, there exists $\gamma > 0$ such that*

$$\mathbb{P}^{\xi_{T_i}}\left(W^{long} > (2 + 2/\alpha)m^\gamma\right) \leq 5e^{-Cm^\gamma}$$

where C is a constant depending on α and λ but not on m and $\mathbb{P}^{\xi_{T_i}}$ means probability conditioned on ξ_{T_i}.

Proof. Let T_i^{long} be the amount of time from T_i until the next long one phase. For any $\gamma > 0$, define the following bad events:

(B1) Consider the first intervals m^γ time intervals after time T_i during which the center is in states $\{-1, 1\}$. Let T_g^* the total amount of this time, and let B_1 be the event that $T_g^* > 2m^\gamma$. Call this event B_1. Since $T_g^* \sim \text{Gamma}(m^\gamma, 1)$ for every possible ξ_{T_i}, we can observe

$$\mathbb{P}^{\xi_{T_i}}(B_1) \leq \left(\frac{2}{e}\right)^{m^\gamma} = e^{-m^{-\gamma}(1-\log(2))}$$

by the Chernoff bound.

(B2) Consider the first m^γ time intervals after time T_i during which the center is in state 0. Let T_g^{**}, and let B_2 be the event that $T_g^{**} > \frac{2m^\gamma}{\alpha}$. Since T_g^{**} is dominated by a Gamma(m^γ, α) random variable for every possible ξ_{T_i}, we can observe

$$\mathbb{P}^{\xi_{T_i}}(B_2) \leq \left(\frac{2}{e}\right)^{m^\gamma} = e^{-m^{-\gamma}(1-\log(2))}$$

by the Chernoff bound.

(B3) Let B_3 be the event that $h_t = 0$ for some $T_i < t \leq T_i + (2 + 2/\alpha)m^\gamma$. To compute the probability of B_3, we can observe that by lemma 4, the number of non-extinct hub stars at time $T_i + (2 + 2/\alpha)m^\gamma$ dominates $X \sim$ Binomial$(L_i, 1/2)$, where L_i is the number of lit hub stars at time T_i and $L_i \geq \eta m$ by our assumption on ξ_{T_i}. Then

$$\mathbb{P}^{\xi_{T_i}}(B_3) \leq (1/2)^{\eta m}$$

by a Chernoff bound on X.

(B4) Fewer than $(q/2)m^\gamma$ of the next m^γ times after T_i the center changes from state 0 to either state 1 or state -1 result in a one phase. Call this event B_4. Consider that each time the process returns to state 0, it is state 1 unit of time later with no other changes of the state of the center in the interim with probability at least $q \min\{T_i < t \leq T_i + (2 + 2/\alpha)m^\gamma : \mathbf{1}_{h_t>0}\}$ by lemma 5. Thus, the number of the next m^γ time intervals after T_i during which the center is in state $\{-1, 1\}$ that are state 1 is dominated by $X \sim$ Binomial$(m^\gamma, q \min\{T_i < t \leq T_i + (2 + 2/\alpha)m^\gamma : \mathbf{1}_{h_t>0}\})$.

(B5) There are fewer than $(q^*/2)m^\gamma$ one phases after time T_i or none of the first $(q^*/2)m^\gamma$ one phases after time T_i is long. Call this event B_5

We ccan observe on the event $B_1^c \cap B_2^c \cap B_3^c \cap B_4^c \cap B_5^c$ that $W^{long} \leq (2 + 2/\alpha)m^\gamma$. We can also observe

$$\begin{aligned}(B_1^c \cap B_2^c \cap B_3^c \cap B_4^c \cap B_5^c)^c &= B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \\ &\subseteq B_1 \cup B_2 \cup B_3 \cup (B_4 \cap B_1^c \cap B_2^c \cap B_3^c) \cup (B_5 \cap B_4^c)\end{aligned}$$

First, $(B_4 \cap B_1^c \cap B_2^c \cap B_3^c)$ requires that $\min\{T_i < t \leq T_i + (2 + 2/\alpha)m^\gamma : \mathbf{1}_{h_t>0}\} = 1$ and that the next m^γ times the center changes from state 0 to either state 1 or state -1 occur before time $T_i + (2 + 2/\alpha)m^\gamma$. Thus

$$\mathbb{P}^{\xi_{T_i}}(B_4 \cap B_1^c \cap B_2^c \cap B_3^c) \leq \exp\left(-\frac{qm^\gamma}{8}\right).$$

We obtain this bound by observing that $\mathbb{P}^{\xi_{T_i}}(B_4 \cap B_1^c \cap B_2^c \cap B_3^c) \leq P(X^* \leq (q/2)m^\gamma)$ where $X^* \sim$ Binomial(m^γ, q) and applying a Chernoff bound.

We can now compute

$$\begin{aligned}\mathbb{P}^{\xi_{T_i}}((B_1^c \cap B_2^c \cap B_3^c \cap B_4^c \cap B_5^c)^c) &\leq \mathbb{P}^{\xi_{T_i}}(B_1) + \mathbb{P}^{\xi_{T_i}}(B_2) + \mathbb{P}^{\xi_{T_i}}(B_3) \\ &\quad + \mathbb{P}^{\xi_{T_i}}(B_4 \cap B_1^c \cap B_2^c \cap B_3^c) + \mathbb{P}^{\xi_{T_i}}(B_5 \cap B_4^c) \\ &\leq 5 \max\left(e^{-m^\gamma(1-\log(2))}, (1/2)^{\eta m}, \exp\left(-\frac{qm^\gamma}{8}\right)\right) \\ &\leq 5e^{-Cm^\gamma}\end{aligned}$$

for some constant C, completing the proof. □

5.5 Main result for star of stars

We can now prove lemma 1.

Proof. Suppose we start the comparison process from time 0 with initial configuration ξ_0^R such that the center is initially healthy (state 0) and there are at least ηn lit hub stars. Because the comparison process is attractive, this provides a lower bound for the initial configuration with all vertices infected. Then since we have at least ηm lit hub stars, we can apply Lemma 6 to conclude that if W^{long} is the time until the first long one phase, then

$$\mathbb{P}^{\xi_0^R}\Big(W^{\text{long}} > (2+2/\alpha)m^\gamma\Big) \leq 5e^{-Cm^\gamma}.$$

Supposing that $W^{\text{long}} < \infty$, we want to apply Lemma 3. However, W^{long} is not a stopping time because it requires that there are no recovery dots or isolation crosses on the center in the time interval $[W^{\text{long}}, W^{\text{long}} + 1/(1+\alpha)]$. However, the event $\{W^{\text{long}} \leq t\}$ is measurable with respect to the Poisson processes on all edges and vertices apart from the center through time t and on the Poisson processes at the center vertex through time $t + 1/(1+\alpha)$. It follows that W^{long} is a stopping time with respect to the filtration where we look ahead by time $1/(1+\alpha)$ just at the center vertex, and the distribution of the process after time W^{long} is the same as the distribution of ξ_t^R conditioned not to have any marks at the center vertex through time $1/(1+\alpha)$ and with initial configuration $\xi^R_{W^{\text{long}}}$.

Thus, Lemma 3 and the strong Markov property lets us conclude that if T is the end of the long phase starting at time W^{long} and L_T is the number of lit hub stars at time T,

$$\begin{aligned}
\mathbb{P}^{\xi_0^R}(L_T < \eta m) &\leq \mathbb{P}^{\xi_0^R}(L_T < \eta m, W^{\text{long}} < \infty) + \mathbb{P}^{\xi_0^R}(W^{\text{long}} > (2+2/\alpha)m^\gamma) \\
&\leq \mathbb{P}^{G_0}(L_0 < \eta m) + 5e^{-Cm^\gamma} \\
&\leq e^{-\eta m/4} + 5e^{-Cm^\gamma}.
\end{aligned}$$

If $W^{\text{long}} < \infty$ and $T < \infty$ and $L_T \geq \eta m$, then by the strong Markov property we can apply Lemma 6 again starting from time T and repeat the cycle. Thus, the number of long one phases we will see before the infection dies out dominates the random variable

$$X \sim \text{Geometric}\left(\exp\left(\frac{-\eta m}{4}\right) + 5e^{-Cm^\gamma}\right)$$

and there exists $\epsilon > 0$ such that $P(X \geq (1+\alpha)e^{m^\epsilon}) \to 1$. Since each long one phase lasts at least $1/(1+\alpha)$, we conclude that

$$P(\tau < e^{m^\epsilon}) \to 0 \text{ as } n \to \infty.$$

□

5.6 Main result for power law random graphs

To prove theorem 1, we will first show that our configuration model power law degree distribution random graph of size n contains a star of stars of order $m = n^\delta$ for some $\delta > 0$ with high probability as $n \to \infty$.

Lemma 7. *Let $G = (V, E)$ be a graph with n vertices generated according to a configuration model with power law degree distribution with power law $\gamma > 2$ and minimum degree 3. Let A be the event that there exists some $\delta > 0$ such that G contains a subgraph G' satisfying definition 1 with order $m = n^\delta$. Then $P(A) \to 1$ as $n \to \infty$.*

Proof. Generate the degree sequence of half edges $\{D_1, \ldots, D_n\}$ iid according to the power law $\gamma > 2$ distribution where $\mu = E(D_i) < \infty$ by our choice of $\gamma > 2$. Now partition V into $V_1 = \{v_1, \ldots, v_{n/3}\}$, $V_2 = \{v_{n/3+1}, \ldots, v_{2n/3}\}$, and $V_3 = \{v_{2n/3+1}, \ldots, v_n\}$. Define $S_i = \sum_{v \in V_i} D_v$ for $i = 1, 2, 3$ and $S = S_1 + S_2 + S_3$. By the Weak Law of Large Numbers, there exists a constant $b_0 > 1$ such that

$$\mathbb{P}(n < S_i < b_0 n) = 1 - o(1) \text{ for } i = 1, 2, 3 \text{ and}$$
$$\mathbb{P}(3n < S < 3b_0 n) = 1 - o(1).$$

Let A_1 be the event that $n < S_i < b_0 n$ for $i = 1, 2, 3$ and $3n < S < 3b_0 n$ so that $\mathbb{P}(A_1) = 1 - o(1)$.

Choose $k_\gamma = 2\gamma - 1$ and $\alpha = \frac{1}{(k_\gamma + 1)\gamma}$. First, we look for some vertex in V_1 with at least $n^{k_\gamma \alpha}$ half edges. Let A_2 be the event that there is some vertex $v^* \in V_1$ such that $D_{v^*} \geq n^{k_\gamma \alpha}$. We can observe that given $v_i \in V_1$

$$\mathbb{P}(D_i \geq n^{k_\gamma \alpha}) \geq (C_1/3) n^{-k_\gamma \alpha \gamma},$$

for some constant C_1. Thus, the expected number of vertices in V_1 that have at least $n^{k_\gamma \alpha}$ half edges is at least

$$(C_1/3) n^{1 - k_\gamma \alpha \gamma} \to \infty \text{ as } n \to \infty \text{ since } \alpha < \frac{1}{k_\gamma \gamma}.$$

in which case $\mathbb{P}(A_2) = 1 - o(1)$.

Now we consider the number of vertices in V_2 with at least $n^{2\alpha}$ half edges. Given a vertex $v_i \in V_2$

$$\mathbb{P}(n^{2\alpha} \geq D_i) \geq C_2 n^{-2\alpha\gamma}$$

so if B_2 is the set of vertices in V_2 with at least $n^{2\alpha}$ half edges

$$|B_2| \succeq \text{Binomial}(n/3, C_2 n^{-2\alpha\gamma}).$$

Let A_3 be the event that $|B_2| > (C_2/6) n^{1 - 2\alpha\gamma}$. Then $\mathbb{P}(A_3) = 1 - o(1)$.

Next we consider matching the first $n^{K_\gamma \alpha}$ half edges edges of v^*, which we will label $e_1, \ldots, e_{n^{K_\gamma \alpha}}$ and call this set E_1. Take the first $(C_2/6) n^{1 - 2\alpha\gamma}$ vertices in B_2, call this set B_2', let E_{v_i} be the set of the first $n^{2\alpha}$ half edges for each vertex $v_i \in B_2'$, and let $E_2' = \bigcup_{i=1}^{|B_2'|} E_{v_i}$. Let $K = |E_2'| = n^{2\alpha}(C_2/6) n^{1 - 2\alpha\gamma}$. Then if X_{v^*} is the number of half edges of v^* in E_1 that match to half edges in E_2'

$$X_{v^*} \sim \text{Hypergeometric}(K, S, |E_1|).$$

Let $\mu_X = \mathbb{E}(X_{v^*})$ and observe

$$\mu_X = |E_1|\frac{K}{S} \geq \frac{n^{(k_\gamma+2)\alpha}(C_2/6)n^{1-2\alpha\gamma}}{3b_0 n} = \frac{C_2 n^{(k_\gamma+2-2\gamma)\alpha}}{18b_0}$$
$$\mu_X = |E_1|\frac{K}{S} \leq \frac{n^{(k_\gamma+2)\alpha}(C_2/6)n^{1-2\alpha\gamma}}{2b_0 n} = \frac{C_2 n^{(k_\gamma+2-2\gamma)\alpha}}{12b_0}$$

From our choice of $k_\gamma = 2\gamma - 1$ and $\alpha = \frac{1}{(k_\gamma+1)\gamma}$, we see that $(k_\gamma + 2 - 2\gamma)\alpha = \alpha$. Thus, using a Chernoff bound

$$\mathbb{P}\bigg(\frac{C_2 n^\alpha}{36b_0} \leq X_{v^*} \leq \frac{C_2 n^\alpha}{6b_0}\bigg) \geq 1 - 2\exp\bigg(-\frac{C_2 n^\alpha}{54b_0}\bigg)$$

Let A_4 be the event that $\frac{C_2 n^\alpha}{36b_0} \leq X_{v^*} \leq \frac{C_2 n^\alpha}{6b_0}$ so that $P(A_4) \geq 1 - 2\exp\bigg(-\frac{C_2 n^\alpha}{54b_0}\bigg)$.

Now we consider how many of these X_{v^*} matches go to distinct vertices in B_2'. Suppose we match each half edge $e_1, \ldots, e_{X_{v^*}}$ sequentially to a half edge in B_2. Let Y_i be an indicator that is 1 if e_i attaches to a vertex in B_2' that none of $e_1, \ldots, e_{i-1}$ attach to and 0 otherwise. Then we can observe that

$$\mathbb{P}\bigg(Y_i = 1 \bigg| \sum_{j=1}^{i-1} Y_i < |B_2'|/2\bigg) \geq 1/2.$$

From this, we can observe that if we let $u = \min\bigg(\frac{C_2 n^\alpha}{36b_0}, (C_2/12)n^{1-2\alpha\gamma n}\bigg)$ then $\sum_{i=1}^{X_{v^*}} Y_i$ stochastically dominates the random variable

$$U \sim \text{Binomial}(u, 1/2)$$

Let A_5 be the event that $U \geq (1/4)u$. Then by a Chernoff bound

$$\mathbb{P}(A_5) \geq 1 - \exp\bigg(-\frac{u}{16}\bigg) = 1 - o(1) \text{ as } n \to \infty,$$

and on the event $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$ there exists a vertex of degree at least $n^{k_\gamma \alpha}$ with at least $\frac{C_2 n^\alpha}{144b_0}$ neighbors of degree at least $n^{2\alpha}$.

Let $B_3 = \{v_i \in B_2' : v_i \sim v^*\}$ and let $E_3 = \bigcup_{v_i \in B_3} E_{v_i}$. We now want to match the half edges in E_3 to half edges of the vertices in V_3. However, we must consider that some of the half edges of vertices in V_3 may have been exhausted when we explored the half edges of v^* in E_1. Let X_3 count the number of vertices in V_3 that have at least unmatched one half edge remaining after matching the half edges of v^*. Since each vertex has degree at least 3, there must be at least $n/3 - n^{k_\gamma\alpha}/3$ such vertices, so $X_3 \geq n/3 - n^{k_\gamma\alpha}/3$

We must also consider that some of the half-edges in E_3 may have been exhausted when we explored the half edges of v^* in E_1. However, on the event A_4, the total number of half edges in E_3 matched to half edges in E_1 is at most $\frac{C_2 n^\alpha}{6b_0}$ and so each vertex in B_3 has at least $n^{2\alpha} - \frac{C_2 n^\alpha}{6b_0} > n^{2\alpha}/2$ unmatched half edges for large enough n. Let E_3' denote this subset of unmatched half edges in E_3.

Now take each vertex $v_i \in B_3$ and match the half edges in $E_3' \cap E_{v_i}$, $e'_{v_i,1}, \ldots, e'_{v_i,|E_3' \cap E_{v_i}|}$ sequentially to unmatched half edges in the entire graph. Let $Y'_{i,\ell}$ be an indicator that is 1 if $e'_{,v_i\ell}$ attaches to a vertex in V_3 that no other $e'_{v_j,h} \in E_3'$ attach to and 0 otherwise. Then we can observe

$$\mathbb{P}\left(Y'_{i,\ell}=1\middle|\sum_{j=1}^{i-1}\sum_{h=1}^{|E'_3\cap E_{v_j}|}Y'_{j,h}\le K\right)\ge\frac{n/3-n^{k_\gamma\alpha}/3-K}{3b_0n}\ge\frac{1}{10b_0}$$

for large enough n by our choice of $\alpha=\frac{1}{(k_\gamma+1)\gamma}$ and because $\sum_{j=1}^{i-1}\sum_{h=1}^{|E'_3\cap E_{v_j}|}Y'_{j,h}\le|E'_3|\le K$. Thus if W_{v_i} is the number of unique neighbors $v_i\in B_3$ has relative to other vertices in B_3, then $W_{v_i}\succeq W$ where

$$W\sim\text{Binomial}\left(n^{2\alpha}/2,\frac{1}{10b_0}\right)$$

Applying a Chernoff bound,

$$\mathbb{P}\left(W_{v_i}\le\frac{n^{2\alpha}}{40b_0}\right)\le\exp\left(-\frac{n^{2\alpha}}{60b_0}\right)$$

and then applying a union bound over all $v_i\in B_3$,

$$\mathbb{P}\left(W_{v_i}\ge\frac{n^{2\alpha}}{40b_0}\text{ for all }v_i\in B_3\right)\ge1-n\exp\left(-\frac{n^{2\alpha}}{60b_0}\right)$$

Let A_6 be the event that $W_{v_i}\ge\frac{n^{2\alpha}}{40b_0}$ for all $v_i\in B_3$ and so $\mathbb{P}(A_6)=1-o(1)$.

We finish by observing that on the event $A_1\cap A_2\cap A_3\cap A_4\cap A_5\cap A_6$ the graph $G=(V,E)$ contains a subgraph G' satisfying definition 1 for m where

$$m=\min\left(\frac{C_2n^\alpha}{144b_0},\frac{n^{2\alpha}}{40b_0}\right)$$

Since we have shown $\mathbb{P}(A_i)=1-o(1)$ for $i=1,\dots,6$, it follows by a union bound that

$$\mathbb{P}(A_1\cap A_2\cap A_3\cap A_4\cap A_5\cap A_6)=1-o(1),$$

completing the proof. □

We are now ready to prove theorem 1.

Proof. By lemma 7 there exists a subgraph G' such that G' is a star of stars of order n^δ for some $\delta>0$, and initial condition $\xi_0(v)=1$ for all $v\in G$ implies $\xi_0(v)=1$ for all $v\in G'$. Fix $\alpha>0$ and $\lambda<0$. By lemma 1, there exists some $\epsilon^*>0$ depending on α and λ such that the comparison process survives on G' for at least time $\frac{(1/6)e^{n^{\delta\epsilon^*}}}{1+\alpha}$ with probability going to 1 as $n\to\infty$. Since G contains G', the comparison process also survives on G for at least time $\frac{(1/6)e^{n^{\delta\epsilon^*}}}{1+\alpha}$ with probability going to 1 as $n\to\infty$. Finally, by lemma 2, this implies the contact process with isolation survives for at least time $\frac{(1/6)e^{n^{\delta\epsilon^*}}}{1+\alpha}$ with probability going to 1 as $n\to\infty$. Let $\epsilon=\delta\epsilon^*/2$. Then

$$\mathbb{P}(\tau<e^{n^\epsilon})\to0\text{ as }n\to\infty.$$

□

6 Vigilance model

6.1 Overview

Our goal for the vigialnce model is to show that for graphs satisfying assumption 2, for every fixed $\alpha > 0$ there is a phase transition in λ between linear and exponential asymptotic survival regimes. We will study the size of the infected set over time, and so introduce the following notation. Let S_t, I_t, and A_t denotes the sets of healthy, infected, and isolated vertices at time t respectively, and let $|S_t|$, $|I_t|$, and $|A_t|$ denote the sizes of these sets.

It is straightforward to show that the infection dies out quickly when $\alpha > \lambda$. If we consider a single SI pair, then the I will become isolated before infecting the S with probability $\frac{\alpha}{\alpha+\lambda} > 1/2$. This suggests that when $\alpha > \lambda$, $|I_t|$ will have negative drift, and we can use a standard random walk comparison to show the process dies out in linear time.

When λ is much large than $\max(\alpha, 1)$, a single SI pair is more likely to result in an infection of the S vertex rather than an isolation or recovery of the I vertex. This suggests it is possible that $|I_t|$ has positive drift. However, there are two complications. First, we must consider that $|A_t|$ could be large. Second, $|\partial I_t|$ could be small, and combined with large $|A_t|$, lead to a configuration in which the number of SI pairs is smaller than $\lambda|I_t|$. If this is the case, then $|I_t|$ will have negative drift, as the next recovery occurs at rate $|I_t|$ while the next infection occurs at rate λ times the number of SI pairs at time t.

To show such configurations are not typical, we prove lemma 10 and use assumption 2. Assumption 2 guarantees that when $|I_t|$ is not too large compared to the size of the graph $|V|$, $|\partial I_t|$ is not too small compared to $|I_t|$. Lemma 10 shows that if we choose λ sufficiently large compared to α, then we can obtain a uniform bound on $|A_t|$ in terms of $|I_t|$ that holds for all $t \leq e^{Cn}$ with high probability, where the constant C depends on α and λ as well as the constants ϵ and δ in assumption 2.

We can then construct a renewal argument for the survival of the process. We start from all vertices infected and show in lemma 11 that there is some small $\epsilon^* > 0$ such that whenever $|I_t|$ drops below $(2/3)\epsilon^* n$, so long as our uniform bound on $|A_t|$ holds, the probability that $|I_t|$ hits $(1/3)\epsilon^* n$ before it hits $\epsilon^* n$ is e^{-Cn} for some constant C that depends on α and λ as well as the constants ϵ and δ in assumption 2. Thus, we expect to observe $O(e^{-Cn})$ successful cycles of this renewal, which ensures the infection survives so long as no cycle has failed.

6.2 Isoperimetric assumption

We study the vigilance model on graphs satisfying isoperimetric condition in assumption 2. Before studying the vigilance model such graphs, we show that a configuration model power law random graph with minimum degree 3 satisfies this assumption with high probability.

Theorem 4. *Let $G = (V, E)$ be a configuration model graph with n vertices and degree sequence $D_1, \ldots, D_n$ sampled iid such that*

$$\mathbb{P}(D_i = k) \propto \frac{1}{k^\gamma}, k = 3, 4, \ldots$$

where $\gamma > 3$. Then G satisfies assumption 2 with probability $1 - o(1)$ as $n \to \infty$.

Proof. Observe that $\mathbb{E}(D_i) < \infty$ and $\mathrm{Var}(D_i) < \infty$ by our choice of $\gamma > 3$. Define $\bar{D} = \frac{1}{n}\sum_{i=1}^{n} D_i$. By the Weak Law of Large Numbers, there exists a constant b_0 such that

$$\mathbb{P}(3 < \bar{D} < b_0) \to 1 \qquad \text{as } n \to \infty$$

and call this event A_0.

Suppose $1/2 > \epsilon > 0$ and suppose we arbitrarily choose a set U of ϵn vertices. Then if $S(U)$ is the sum of the number of half edges of these vertices,

$$3\epsilon n \le S(U) \le \sum_{i=n-\epsilon n}^{n} D_{(i)}$$

where $D_{(i)}$ is the ith order statistic of $\{D_1, \ldots, D_n\}$. To bound $\sum_{i=n-\epsilon n}^{n} D_{(i)}$, we use the results in [5], which provide a central limit theorem for trimmed means that can be applied to the special case of the upper tail sum. Following their notation, we let F be the cdf of the degree distribution, $q_\epsilon = \sup\{x : F(x) \le 1-\epsilon\}$, and define

$$F_{\text{trimmed}}(k) = \begin{cases} 0, & \text{if } k \le q_\epsilon \\ \frac{F(k)-(1-\epsilon)}{\epsilon}, & \text{if } k > q_\epsilon. \end{cases}$$

When the variance of F_{trimmed} is finite, which our choice of $\gamma > 3$ guarantees, the results of [5] imply that $S(U)/n \xrightarrow{P} \mu_t$ where μ_t is the mean of F_{trimmed}. Observe $\mu_t = E(D|D \ge q_\epsilon)$. We see

$$\mathbb{P}(D > k) = \sum_{\ell=k+1}^{\infty} \frac{C_1}{\ell^\gamma} \sim \int_k^\infty \frac{C_1}{x^\gamma} dx = \frac{C_1 k^{1-\gamma}}{(\gamma-1)},$$

and setting $\mathbb{P}(D > k) = \epsilon$, we get that $q_\epsilon \propto \epsilon^{-1/(\gamma-1)}$. Thus

$$\mathbb{E}(D|D \ge q_\epsilon) = \frac{\sum_{k=q_\epsilon}^{\infty} k^{-\gamma+1}}{\sum_{k=q_\epsilon}^{\infty} k^{-\gamma}} \sim \frac{\int_{q_\epsilon}^{\infty} k^{-\gamma+1} dk}{\int_{q_\epsilon}^{\infty} k^{-\gamma} dk} = \frac{\gamma-1}{\gamma-2} q_\epsilon.$$

Thus we have

$$\epsilon n \mathbb{E}(D|D \ge q_\epsilon) = C_2 n \epsilon^{(\gamma-2)/(\gamma-1)}(1+o(1))$$

and so we can choose $k_\epsilon = C\epsilon^{(\gamma-2)/(\gamma-1)}$ for some constant $C > C_2$ such that

$$\mathbb{P}(S(U) \le k_\epsilon n) \to 1 \text{ as } n \to \infty.$$

Let A_1 be the event that $S(U) \le k_\epsilon n$. Then $\mathbb{P}(A_1) = 1 - o(1)$.

We now count the external vertex boundary of U. Order the half edges in U, $u_1, \ldots, u_{S(U)}$. Beginning from u_1, we pair each half edge of U to another half edge of V. If a half edge u_j has already been matched with another half edge u_i for $i < j$, then we skip the half edge u_j. We can skip over at most $S(U)/2$ half edges, and so we are guaranteed at least $S(U)/2$ matchings. When we match each non-skipped half edge u_i, there are three possible outcomes:

1. u_i matches to u_j for some $j > i$,
2. u_i matches to the half edge of some vertex $v \notin U$ that has a half edge which is already matched to another half edge u_ℓ with $\ell < i$, or

3. u_i matches to the half edge at some vertex $v \notin U$ such that no half edge u_ℓ with $\ell < i$ has previously matched with a half edge at v.

Whenever we observe outcome 3., we increase our count of the external vertex boundary of U by 1. The probability of observing outcomes 1. or 2. is at most $2S(U)/(\bar{D}n - 2S(U))$. If we choose ϵ sufficiently small, then on the event $A_0 \cap A_1$ we have $\bar{D} - 2S(U) \geq 3 - 2k_\epsilon > 2$. Thus, if we let X count the number of non-skipped half edges of vertices in U that result in outcomes 1. or 2., then given $A_0 \cap A_1$, if $S(U) = sn$ with $s \leq k_\epsilon$, we have $X \preceq Y + sn/2$ where

$$Y \sim \text{Binomial}(sn/2, s),$$

and $\mu = \mathbb{E}(Y) = \frac{s^2 n}{\bar{D} - 2s}$.

Let $k > 1$ (to be chosen later) and let A_2 be the event $Y \leq \frac{sn}{2k}$. Using a Chernoff bound [3]

$$\mathbb{P}\bigg(Y \leq \frac{sn}{2k} \bigg| A_0 \cap A_1\bigg) \geq 1 - \exp\bigg(-\frac{sn}{2k}\bigg(\log\frac{.9\bar{D}}{ks} - 1\bigg) + \frac{s^2 n}{.9\bar{D}}\bigg).$$

Let $k' > k$, to be chosen later. Then for sufficiently small $\epsilon > 0$ and $s \leq k_\epsilon$,

$$1 - \exp\bigg(-\frac{sn}{2k}\bigg(\log\frac{.9\bar{D}}{k} + \log(1/s) - 1\bigg) + \frac{s^2 n}{.9\bar{D}}\bigg) \geq 1 - \exp\bigg(-\frac{sn}{2k'}\log(1/s)\bigg). \tag{5}$$

Since we chose $k > 1$, on the event $Y \leq \frac{sn}{2k}$, the external vertex boundary of U is at least size

$$sn - \bigg(sn/2 + \frac{sn}{2k}\bigg) = \frac{(k-1)sn}{2k}.$$

Thus, on the event $A_0 \cap A_1 \cap A_2$, the set of vertices U has external vertex boundary at least $\frac{(k-1)sn}{2k} \geq \frac{3(k-1)}{2k}|U|$, since the minimum degree is 3 so that $S(U) \geq 3|U|$.

Now consider that the number of possible sets of vertices of size ϵn is $\binom{n}{\epsilon n}$. Using Stirling's approximation

$$\binom{n}{\epsilon n} \leq \exp\bigg(1.1\epsilon\log(1/\epsilon)n\bigg) \tag{6}$$

for all sufficiently large n when ϵ is sufficiently small. On the event A_1, $S(U) = s$ for some $3\epsilon n \leq s \leq k_\epsilon n$, and the preceding estimates hold uniformly for all sets U and numbers of half edges $S(U)$ in this range. Let A_2^i be the event that A_2 holds for set U^i where $i = 1, \ldots \binom{n}{\epsilon n}$ enumerates all possible sets of size ϵn. Using a union bound and the fact that $s\log(1/s)$ is increasing for $s \in (0, 1/e)$, by choosing ϵ so that $k_\epsilon < 1/e$ we have

$$\mathbb{P}(A_0 \cap A_1 \cap A_2^1 \cap \ldots \cap A_2^{\binom{n}{\epsilon n}}) \geq 1 - o(1) - \exp\bigg(-\frac{k_\epsilon n}{2k'}\log(1/k_\epsilon) + 1.1\epsilon\log(1/\epsilon)\bigg).$$

Noting that $k_\epsilon = C\epsilon^{(\gamma-2)/(\gamma-1)} \gg \epsilon$, we can choose ϵ small enough such that $\frac{k_\epsilon}{2k'}\log(1/k_\epsilon) - 1.1\epsilon\log(1/\epsilon) > \epsilon' > 0$, so that

$$1 - \exp\bigg(-\frac{k_\epsilon n}{2k'}\log(1/k_\epsilon) + 1.1\epsilon\log(1/\epsilon)n\bigg) \geq 1 - e^{-\epsilon' n}.$$

□

6.3 The subcritical regime

The following lemma establishes the existence of a subcritical regime when $\alpha < \lambda$.

Lemma 8. *Suppose we have any graph $G = (V, E)$ and rates $\lambda, \alpha > 0$ such that $\alpha > \lambda$. Then the epidemic dies out quickly.*

Proof. Recall $|I_t|$ is the count of infected vertices at time t. We will show that $|I_t|$ is stochastically dominated by a random walk that is biased toward 0.

Starting at any time t there are four possibilities for the next event: infection, recovery, isolation, and return from isolation. Infection increases $|I_t|$ by 1, recovery and isolation decrease $|I_t|$ by 1, and return from isolation does not change $|I_t|$ but increases the number of SI edges in the system. We can observe

1. The rate of the next infection is $\lambda(\#SI \text{ edges})$
2. The rate of the next recovery event is $|I_t|$
3. The rate of the next isolation event is $\alpha(\#SI \text{ edges})$

The the ratio of the rate of events that increase $|I_t|$ by 1 to the rate of all events that change $|I_t|$ is

$$\frac{\lambda(\#SI \text{ edges})}{\lambda(\#SI \text{ edges}) + |I_t| + \alpha(\#SI \text{ edges})} \leq \frac{\lambda}{\lambda + \alpha}$$

where the right-hand side does not depend on the number of SI edges. Thus, the probability that the next event that changes $|I_t|$ increases $|I_t|$ by 1 is at most $\frac{\lambda}{\lambda+\alpha}$ and the probability that the next event that changes $|I_t|$ decreases $|I_t|$ by 1 is at least $\frac{\alpha}{\lambda+\alpha}$. Thus, we can stochastically dominate $|I_t|$ via coupling with a continuous time process W_t such that $W_0 = |I(0)|$ and W_0 evolves according to the following rules:

1. For all t such that $|I_t| > 0$, W_t transitions when $|I_t|$ transitions.
2. For all t such that $|I_t| = 0$, W_t transitions at rate 1.
3. When W_t transitions, it transitions to $W_t + 1$ with probability $\frac{\lambda}{\lambda+\alpha}$ and transitions to $W_t - 1$ with probability $\frac{\alpha}{\lambda+\alpha}$.

Let $\tau_0 = \inf\{t : W_t = 0\}$ and observe that $|I_t| \leq W_t$ for all $t \leq \tau_0$. In addition, observe that W_t is a random walk biased toward 0, and so if $\tau_0 = \inf\{t : W_t = 0\}$, there exists $C > 0$ such that $P(\tau_0 > Cn) \to 0$ as $n \to \infty$. □

6.4 The supercritical regime

Consider the vigilance model with infection rate λ and vigilance rate α. Define the continuous time process $V_t = |A_t| - \eta|I_t|$. We study the behavior of this process in lemma 9 and use the results to prove lemma 10.

Lemma 9. *Choose $\eta < \epsilon\delta/24$ where ϵ and δ satisfy Assumption 2 for G and $\delta \leq 12$. Choose $\lambda > \frac{2\alpha}{\eta}$ and let $V_t = |A_t| - \eta|I_t|$. Suppose V_0 is in the interval $(2\eta n, 2.1\eta n)$. Then $|V_t|$ hits $[0, \eta n]$ before it hits $(3\eta n, \infty)$ with probability at least $1 - \exp\left(\frac{-\eta^2 n}{43}\right)$.*

Proof. Define the jump process $\{(\hat{S}_k, \hat{I}_k, \hat{A}_k) : k \in \mathbb{Z}_{\geq 0}\}$ to be the embedded discrete time Markov chain of (S_t, I_t, A_t), which increments whenever an event (infection, isolation or recovery) occurs. Let p_k be the probability that the next event in this process is either an infection or an isolation. Then

$$p_k = \frac{(\alpha + \lambda)(\#\text{SI edges in } (\hat{S}_k, \hat{I}_k, \hat{A}_k))}{(\alpha + \lambda)(\#\text{SI edges in } (\hat{S}_k, \hat{I}_k, \hat{A}_k)) + |\hat{A}_k| + |\hat{I}_k|}.$$

Define

$$U_k = |\hat{A}_k| - \eta|\hat{I}_k|,$$

and observe that since $\eta < 1/4$ and $\lambda > \frac{2\alpha}{\eta} > 8\alpha$, while $U_k > \eta n$

$$\begin{aligned}
\mathbb{E}(U_{k+1} - U_k | (\hat{S}_k, \hat{I}_k, \hat{A}_k)) &= p_k \left(\frac{\alpha}{\alpha + \lambda}(1 + \eta) - \eta \frac{\lambda}{\lambda + \alpha} \right) + (1 - p_k) \left(\frac{-(|\hat{A}_k| - \eta|\hat{I}_k|)}{|\hat{A}_k| + |\hat{I}_k|} \right) \\
&\leq p_k \left(\frac{\alpha}{\lambda}(1 + \eta) - \eta \frac{8\alpha}{\alpha + 8\alpha} \right) + (1 - p_k) \left(\frac{-U_k}{n} \right) \\
&\leq p_k \Big((1/2)\eta(1 + \eta) - (8/9)\eta \Big) + (1 - p_k) \Big(- \eta \Big) \\
&\leq -\frac{1}{4}\eta.
\end{aligned}$$

Let $\mathcal{K} = \inf\{k : U_k \leq \eta n\}$ be the first time that U_k hits $[0, \eta n]$. Now define the discrete time process W_k by

$$W_k = U_{k \wedge \mathcal{K}} + \frac{1}{4}\eta(k \wedge \mathcal{K})$$

and note that W_k is a supermartingale. Let time $t = 0$ correspond to increment $k = 0$ in the embedded discrete process $(\hat{S}_k, \hat{I}_k, \hat{A}_k)$, so that $W_0 = U_0 = V_0$. Note that for all k, $|W_{k+1} - W_k| \leq 2$, so we can apply Azuma's inequality to see that

$$\mathbb{P}(W_k - W_0 > .9\eta n) \leq \exp\left(\frac{-(.81)\eta^2 n^2}{8k} \right).$$

Observe that when $W_k - W_0 \leq .9\eta n$ holds, we have

$$\begin{aligned}
U_k + \frac{1}{4}\eta k - U_0 &\leq .9\eta n \\
U_k &\leq U_0 + .9\eta n - \frac{1}{4}\eta k
\end{aligned}$$

Suppose $W_k - W_0 \leq .9\eta n$ holds for $k = 1, \ldots, \mathcal{K}$. This ensures that U_k hits the interval $[0, \eta n]$ before it hits the interval $(3\eta n, \infty)$. It also ensures we must have $\mathcal{K} - 1 \leq 8n$, otherwise

$$U_{\mathcal{K}-1} \leq U_0 - 2\eta n + .9\eta n < \eta n$$

which is a contradiction since $\mathcal{K} = \inf\{k : U_k \leq \eta n\}$. Thus we can observe

$$\begin{aligned}
\mathbb{P}(\mathcal{K} > 8n+1) &\leq P(W_k - W_0 > .9\eta n \text{ for some } k = 1, \ldots, m) \\
&\leq \sum_{k=1}^{\mathcal{K}} \exp\left(\frac{-(.81)\eta^2 n^2}{8k}\right) \\
&\leq \sum_{k=1}^{8n+1} \exp\left(\frac{-(.81)\eta^2 n^2}{8k}\right) \\
&\leq (8n+1)\exp\left(\frac{-\eta^2 n}{40}\right) \\
&\leq \exp\left(\frac{-\eta^2 n}{41}\right) \text{ for large enough } n.
\end{aligned}$$

So long as U_k does not hit the interval $(3\eta n, \infty)$, V_t does not hit the interval $(3\eta n, \infty)$, and when U_κ hits the interval $(0, \eta n]$, V_τ hits the interval $(0, \eta n]$. Thus for large enough n, with probability at least $1 - \exp\left(\frac{-\eta^2 n}{41}\right)$, after V_t hits the interval $(2\eta n, 2.1\eta n)$, it enters the interval $(0, \eta n]$ and does so before visiting $(3\eta n, \infty)$.

□

Lemma 10. *Chose $\eta < \epsilon\delta/24$ where ϵ and δ satisfy Assumption 2 for G and $\delta \leq 12$. Choose $\lambda > \frac{2\alpha}{\eta}$ and let $V_t = |A_t| - \eta|I_t|$. Suppose V_0 is in the interval $(2\eta n, 2.1\eta n)$. Let $\tau = \inf\{t > 0 : V_t > 3\eta n\}$. Then there is $c > 0$ (depending on λ, α, η) such that $\mathbb{P}(\tau > e^{cn}) \geq 1 - e^{-cn}$.*

Proof. Recall that V_0 is in the interval $(2\eta n, 2.1\eta n)$ and let τ_1 be the first time V_t leaves $(\eta n, 3\eta n)$. We can observe τ_1 is bounded below by the time it takes for $(.9\eta n)/(1+\eta) \geq \eta n/2$ transitions that change V_t to occur, since any transition can change the value of V_t by at most $1 + \eta$. The possible transitions are infections, isolations, recoveries of infected vertices, and recoveries of isolated vertices, and these happen at combined rate

$$\begin{aligned}
&(\lambda+\alpha)(\#\text{number SI edges}_t) + |A_t| + |I_t| \\
&\leq (\lambda+\alpha)\binom{n}{2} + 2n \qquad (7) \\
&\leq (\lambda+\alpha)n^2 \text{ for large enough } n
\end{aligned}$$

Therefore the amount of time for the continuous time process V_t to hit $(0, \eta n]$ starting from $(2\eta n, 2.1\eta n)$ stochastically dominates the random variable $X \sim \text{Gamma}(\eta n/2, (\lambda+\alpha)n^2)$. Thus using a Chernoff bound

$$\mathbb{P}\left(\tau_1 < \frac{\eta/2}{(\lambda+\alpha)n^2}\right) \leq \mathbb{P}\left(X < \frac{\eta/2}{(\lambda+\alpha)n^2}\right) \leq \exp(-n).$$

Now consider cycles of V_t hitting the interval $(2\eta n, 2.1\eta n)$ and then hitting either $(0, \eta n]$ or $(3\eta n, \infty)$. Say that a cycle fails if either after hitting the interval $(2\eta n, 2.1\eta n)$ V_t hits $(3\eta n, \infty)$

before it hits $(0, \eta n]$ or V_t leaves $(\eta n, 3\eta n)$ in less time than $\eta n/2$. Using a union bound, the probability of failure on a cycle is at most

$$\exp\left(\frac{-\eta^2 n}{41}\right) + \exp(-n)$$

and thus the probability of any failures in $\exp\left(\frac{\eta^2 n}{84}\right)$ cycles is at most

$$\begin{aligned} &\exp\left(\frac{\eta^2 n}{84}\right)\left(\exp\left(\frac{-\eta^2 n}{41}\right) + \exp(-n)\right) \\ &\leq \exp\left(\frac{-\eta^2 n}{84}\right) \text{ for large enough } n \end{aligned} \tag{8}$$

Therefore, if we observe no failures in $\exp\left(\frac{\eta^2 n}{84}\right)$ cycles we will have that if V_t begins below $2.1\eta n$, it will remain below $3\eta n$ for at least time

$$\begin{aligned} &\frac{\eta/2}{(\lambda+\alpha)n^2}\exp\left(\frac{\eta^2 n}{84}\right) \\ &\geq \exp\left(\frac{\eta^2 n}{85}\right) \text{ for large enough } n \end{aligned} \tag{9}$$

and this happens with probability at least $1 - \exp\left(\frac{-\eta^2 n}{84}\right)$. □

We now prove our main result.

Lemma 11. *Suppose we consider the vigilance model on a graph $G = (V, E)$ where $|V| = n$ satisfying Assumption 2. Fix $\alpha > 0$ and let $T_k = \inf\{t : |I_t| = k\}$. Then there exist ϵ and η such that if $|I_0| = (2/3)\epsilon n$, $|A_0| < 2\eta n$, and $\lambda > \lambda_0$ where λ_0 depends on α, ϵ, and η, then $\mathbb{P}(T_{\epsilon n} < T_{(1/3)\epsilon n}) \geq 1 - \exp\left(\frac{-\epsilon\eta^2 n}{85}\right)$*

Proof. Start by fixing $\alpha > 0$, choosing $\eta < \epsilon\delta/24$ where ϵ and δ satisfy Assumption 2 for G, $\delta \leq 12$, and $\lambda_0 = \max\{\frac{2\alpha}{\eta}, \alpha + 9/\delta\}$.

From Lemma 10 we have that with probability at least $1 - \exp\left(\frac{-\eta^2 n}{84}\right)$, $V_t = |A_t| - \eta|I_t| \leq 3\eta n$ for all $t \leq \exp\left(\frac{\eta^2 n}{85}\right)$. Thus for $t < \min\{\exp\left(\frac{\eta^2 n}{85}\right), T_{(1/3)\epsilon n}, T_{\epsilon n}\}$, we have $|I_t| \in ((1/3)\epsilon n, \epsilon n)$ and $V_t \leq 3\eta n$, which together imply

$$\begin{aligned} |A_t| &\leq (3+\epsilon)\eta n \\ &\leq 4\eta n. \end{aligned} \tag{10}$$

From our initial condition on $|I_0|$, the definitions of $T_{(1/3)\epsilon n}$ and $T_{\epsilon n}$, and Assumption 2, we can observe that $|\partial I_t| \geq \delta(\epsilon/3)n$ for all $t < \min\{\exp\left(\frac{\eta^2 n}{85}\right), T_{(1/3)\epsilon n}, T_{\epsilon n}\}$. We can further observe that since $|A_t| \leq 4\eta n$,

$$\begin{aligned} |\partial I_t \cap S_t| &\geq \delta(\epsilon/3)n - 4\eta n \\ &\geq \delta\epsilon n/6 \end{aligned} \tag{11}$$

by our choice of η.

Now consider events that change $|I_t|$. $|I_t|$ decreases by 1 when either an infected vertex recovers or becomes isolated, and I_t increases by 1 when a susceptible vertex becomes infected. Thus, for $t < \min\{\exp\left(\frac{\eta^2 n}{85}\right), T_{(1/3)\epsilon n}, T_{\epsilon n}\}$ the probability that the next event that changes $|I_t|$ increases it by 1 rather than decreases it by 1 is

$$\begin{aligned} &\frac{(\lambda-\alpha)|\partial I_t \cap S_t|}{(\lambda-\alpha)|\partial I_t \cap S_t| + |I_t|} \\ &\geq \frac{(\lambda-\alpha)\epsilon\delta n/6}{(\lambda-\alpha)\epsilon\delta n/6 + \epsilon n} \\ &\geq 3/5 \end{aligned} \tag{12}$$

by our choice of $\lambda > \lambda_0 \geq \alpha + 9/\delta$.

Recall the embedded discrete time process $(\hat{S}_k, \hat{I}_k, \hat{A}_k)$. For all increments k that occur before time $\min\{\exp\left(\frac{\eta^2 n}{85}\right), T_{(1/3)\epsilon n}, T_{\epsilon n}\}$, $|\hat{I}_k|$ stochastically dominates a random walk R_k that starts at $R_0 = (2/3)\epsilon n$, increases by 1 with probability 3/5 and decreases by 1 with probability 2/5 each increment. We continue this coupling until one of the following occurs:

1. $|\hat{I}_k|$ hits ϵn
2. $|\hat{I}_k|$ hits $(1/3)\epsilon n$
3. k hits K such that increment K occurs after time $\frac{18n}{(\alpha+\lambda)\delta}$

after which we let R_k continue independently of $|\hat{I}_k|$. We can observe that if all of the following hold

1. $R_{2\epsilon n} \geq \epsilon n$
2. R_k hits ϵn before it hits $(1/3)\epsilon n$
3. $K > 2\epsilon n$

then our coupling ensures that $|\hat{I}_k|$ hits ϵn before it hits $(1/3)\epsilon$ and this happens for some increment $k < \exp\left(\frac{\eta^2 n}{85}\right)$. This in turn implies that $T_{\epsilon n} \leq \min\{\exp\left(\frac{\eta^2 n}{85}\right), T_{(1/3)}\epsilon n\} \leq T_{(1/3)\epsilon n}$. We will bound each of failure of each of these events.

1. Applying a Binomial Chernoff bound, we observe that

$$\mathbb{P}(R_{2\epsilon n} < \epsilon n) < \exp\left(\frac{-\epsilon^2 n^2}{225}\right).$$

2. The probability that R_k hits $(1/3)\epsilon n$ before it hits ϵn is given by a standard Gambler's ruin analysis:

$$\mathbb{P}(R_k \text{ hits } (1/3)\epsilon n \text{ before it hits } \epsilon n) = \frac{1-(3/2)^{(1/3)\epsilon n}}{1-(3/2)^{(2/3)\epsilon n}} \leq \exp\left(\frac{-\epsilon n}{9}\right).$$

3. Observe that $|\hat{I}_k|$ changes when either an infection occurs, an isolation occurs, or an infected vertext recovers. This happens at combined rate

$$\begin{aligned}&(\alpha+\lambda)(\#\text{ SI edges}) + |\hat{I}_k| \\ &\geq (\alpha+\lambda)(\delta\epsilon n/6).\end{aligned} \tag{13}$$

Thus the number of increments k that occur in time $\frac{18n}{(\alpha+\lambda)\delta}$ stochastically dominates a random variable $Y \sim \text{Poisson}(3\epsilon n)$. Applying a Chernoff bound for Y

$$\mathbb{P}(K < 2\epsilon n) \leq \mathbb{P}(Y < 2\epsilon n) \leq \exp\left(\frac{-\epsilon n}{4}\right) \tag{14}$$

Therefore we can bound the probability of failure by

$$\begin{aligned}\mathbb{P}(T_{(1/3)\epsilon n} < T_{\epsilon n}) \leq \mathbb{P}(R_{2\epsilon n} < \epsilon n) + \mathbb{P}(R_k \text{ hits } (1/3)\epsilon n \text{ before it hits } \epsilon n) \\ + \mathbb{P}(K < 2\epsilon n) + \mathbb{P}(\text{Lemma 10 fails})\end{aligned} \tag{15}$$

We have already established $P(R_{2\epsilon n} < \epsilon n) < \exp\left(\frac{-\epsilon^2 n^2}{225}\right)$, $\mathbb{P}(R_k \text{ hits } (1/3)\epsilon n \text{ before it hits } \epsilon n) \leq \exp\left(\frac{-\epsilon n}{9}\right)$, $\mathbb{P}(K < 2\epsilon n) < \frac{-\epsilon n}{4}$ and $P(\text{Lemma 10 fails}) < \exp\left(\frac{-\eta^2 n}{84}\right)$.

Substituting these into equation 15, we have

$$\begin{aligned}\mathbb{P}(T_{\epsilon n} < T_{(1/3)\epsilon n}) &\geq 1 - \left(\exp\left(\frac{-\epsilon^2 n^2}{225}\right) + \exp\left(\frac{-\epsilon n}{9}\right) + \exp\left(\frac{-\epsilon n}{4}\right) + \exp\left(\frac{-\eta^2 n}{84}\right)\right) \\ &\geq 1 - \exp\left(\frac{-\epsilon\eta^2 n}{85}\right)\end{aligned} \tag{16}$$

for large enough n. □

We are now ready to prove theorem 3.

Proof. Choose some $\epsilon > 0$ and $\delta \in (0, 12)$ for which G satisfies assumption 2, and fix $\alpha > 0$. Start from all vertices infected so $|I_0| = n$, let $\tau_1 = \inf\{t : |I_t| = (2/3)\epsilon n\}$, and let $T_k^1 = \inf\{t \geq \tau_1 : |I_t| = k\}$. For $i = 2, 3, \ldots$, let $\tau_i = \inf\{t > \tau_{i-1} : |I_t| = (2/3)\epsilon n\}$ and let $T_k^i = \inf\{t \geq \tau_i : |I_t| = k\}$. By lemma 11, if we choose $\eta < \epsilon\delta/24$ and $\lambda > \lambda_0 = \max(\frac{2\alpha}{\eta}, \alpha + 9/\delta)$ then

$$\mathbb{P}(T_{\epsilon n}^1 < T_{(1/3)\epsilon n}^1) \geq 1 - \exp\left(\frac{-\epsilon\eta^2 n}{86}\right).$$

We can also observe for each $i = 1, 2, \ldots$ that if $T_{\epsilon n}^i > T_{(1/3)\epsilon n}^i$ this guarantees $\tau_{i+1} < \infty$ because $\lim_{t\to\infty} |I_t| = 0$ and to reach 0 from ϵn, $|I_t|$ must pass through $(2/3)\epsilon n$. We can thus conclude

$$\{T^j_{\epsilon n} < \infty\} \supseteq \{T^j_{\epsilon n} < T^j_{(1/3)\epsilon n}\} \supseteq \{T^j_{\epsilon n} < T^j_{(1/3)\epsilon n} | T^{j-1}_{\epsilon n} < T^{j-1}_{(1/3)\epsilon n}\} \cap \ldots \cap \{T^1_{\epsilon n} < T^1_{(1/3)\epsilon n}\}$$

and use the strong Markov property to apply lemma 11 to see that for each $i = 1, 2, \ldots$

$$\mathbb{P}(T^i_{\epsilon n} > T^i_{(1/3)\epsilon n} | T^{i-1}_{\epsilon n} > T^{i-1}_{(1/3)\epsilon n}) \geq 1 - \exp\left(\frac{-\epsilon\eta^2 n}{86}\right).$$

Taking a union bound on the probability of failure of lemma 11 for $i = 1, \ldots, j$, we see that

$$\mathbb{P}(T^j_{\epsilon n} < \infty) \geq \mathbb{P}(T^j_{\epsilon n} > T^j_{(1/3)\epsilon n}) \geq 1 - j\exp\left(\frac{-\epsilon\eta^2 n}{86}\right).$$

If we choose $j = \exp\left(\frac{\epsilon\eta^2 n}{172}\right)$ then this probability is $1 - \exp\left(\frac{-\epsilon\eta^2 n}{172}\right)$, so with high probability we will have $T^j_{\epsilon n} < \infty$ for $j = \exp\left(\frac{\epsilon\eta^2 n}{172}\right)$.

To obtain a lower bound on $T^j_{\epsilon n}$, consider that for each $i = 1, \ldots, j$ there must be at least one event that involves a vertex changing states between $T^{i-1}_{\epsilon n}$ and $T^i_{\epsilon n}$. Given our graph $G = (V, E)$, the total rate at which any such event occurs is bounded above by $(\lambda + \alpha)|E| + |V|$. Since $|V| = n$ and $|E| < n^2$, the total rate is bounded above by $(\lambda + \alpha + 1)n^2$. Thus, if for notational convenience we denote $T^0_{\epsilon n} = 0$ and consider the sum

$$S_j = \sum_{i=1}^{j} T^i_{\epsilon n} - T^{i-1}_{\epsilon n},$$

S_j dominates the random variable $X \sim \text{Gamma}(j, (\lambda + \alpha + 1)n^2)$. We can apply Chebyshev's inequality to see

$$\begin{aligned}
\mathbb{P}\left(S_j < \frac{j}{2(\lambda+\alpha+1)n^2}\right) &\leq \mathbb{P}\left(X < \frac{j}{2(\lambda+\alpha+1)n^2}\right) \\
&\leq \mathbb{P}\left(|X - \frac{j}{(\lambda+\alpha+1)n^2}| > \frac{j}{2(\lambda+\alpha+1)n^2}\right) \\
&\leq \frac{j4(\lambda+\alpha+1)^2n^4)}{j^2(\lambda+\alpha+1)^2n^4} \\
&= \frac{4}{j}.
\end{aligned} \tag{17}$$

On the event $\{T^j_{\epsilon n} < \infty\} \cap \{S_j \geq \frac{j}{2(\lambda+\alpha+1)n^2}\}$ we have that $\tau_v \geq \frac{j}{2(\lambda+\alpha+1)n^2}$. Settting $j = \exp\left(\frac{\epsilon\eta^2 n}{172}\right)$ and taking a union bound on the probability of failure for large n we observe

$$
\begin{aligned}
\mathbb{P}\left(\tau_v < \exp\left(\frac{\epsilon\eta^2 n}{344}\right)\right) &\leq \mathbb{P}\left(\tau_v < \frac{\exp\left(\frac{\epsilon\eta^2 n}{172}\right)}{2(\lambda+\alpha+1)n^2}\right) \\
&\leq \mathbb{P}(T^j_{\epsilon n} = \infty) + \mathbb{P}\left(S_j < \frac{\exp\left(\frac{\epsilon\eta^2 n}{172}\right)}{2(\lambda+\alpha+1)n^2}\right) \\
&\leq \mathbb{P}(T^j_{\epsilon n} = \infty) + \mathbb{P}\left(X < \frac{\exp\left(\frac{\epsilon\eta^2 n}{172}\right)}{2(\lambda+\alpha+1)n^2}\right) \\
&\leq \exp\left(\frac{-\epsilon\eta^2 n}{172}\right) + 4\exp\left(\frac{-\epsilon\eta^2 n}{172}\right) \\
&\to 0 \text{ as } n \to \infty,
\end{aligned} \tag{18}
$$

and this completes the proof. □

References

[1] N. Berger, C. Borgs, J. T. Chayes, and A. Saberi. On the spread of viruses on the internet. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '05, page 301–310, USA, 2005. Society for Industrial and Applied Mathematics.

[2] S. Bhamidi, D. Nam, O. Nguyen, and A. Sly. Survival and extinction of epidemics on random graphs with general degree. *The Annals of Probability*, 49(1):244 – 286, 2021.

[3] S. Boucheron, G. Lugosi, and P. Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 02 2013.

[4] S. Chatterjee, D. Sivakoff, and M. Wascher. The effect of avoiding known infected neighbors on the persistence of a recurring infection process. *Electronic Journal of Probability*, 27:1 – 40, 2022.

[5] H. David and H. Nagaraja. *Order Statistics*. John Wiley & Sons, 2003.

[6] G. Demirel, E. Barter, and T. Gross. Dynamics of epidemic diseases on a growing adaptive network. *Scientific Reports*, 10(7):42352, 2017.

[7] R. Durrett. *Dynamics on Graphs*. 2024.

[8] R. Durrett and C. Neuhauser. Epidemics with Recovery in $D = 2$. *The Annals of Applied Probability*, 1(2):189 – 206, 1991.

[9] R. Durrett and D. Yao. Susceptible–infected epidemics on evolving graphs. *Electronic Journal of Probability*, 27(none):1 – 66, 2022.

[10] T. Friedrich, A. Göbel, N. Klodt, M. Krejca, and M. Pappik. Analysis of the survival time of the sirs process via expansion. *Electronic Journal of Probability*, 29, 01 2024.

[11] T. Gross and B. Blasius. Adaptive coevolutionary networks: a review. *Journal of the Royal Society, Interface*, 5(20):259–271, 2007.

[12] T. Gross, C. J. D. D'Lima, and B. Blasius. Epidemic dynamics on an adaptive network. *Phys. Rev. Lett.*, 96:208701, May 2006.

[13] D. Guo, S. Trajanovski, R. van de Bovenkamp, H. Wang, and P. Van Mieghem. Epidemic threshold and topological structure of susceptible-infectious-susceptible epidemics in adaptive networks. *Phys. Rev. E*, 88:042802, Oct 2013.

[14] T. E. Harris. Contact Interactions on a Lattice. *The Annals of Probability*, 2(6):969 – 988, 1974.

[15] E. Jacob and P. Mö"rters. The contact process on scale-free networks evolving by vertex updating. *Royal Society open science*, 4(5):170081, 2017.

[16] Y. Jiang, R. Kassem, G. York, M. Junge, and R. Durrett. Sir epidemics on evolving graphs, 2019.

[17] P. Lam, O. Nguyen, and I. Yang. Optimal bound for survival time of the sirs process on star graphs, 2024.

[18] T. M. Liggett. *Stochastic Interacting Systems*. Springer-Verlag Berlin Heidelberg, 1999.

[19] T. Mountford, J.-C. Mourrat, D. Valesin, and Q. Yao. Exponential extinction time of the contact process on finite graphs. *Stochastic Processes and their Applications*, 126(7):1974–2013, 2016.

[20] N. Pipatsart, W. Triampo, and C. Modchang. Stochastic models of emerging infectious disease transmission on adaptive random networks. *Comput Math Methods Med*, 2017(2403851), 2017.

[21] D. Remenik. The contact process in a dynamic random environment. *The Annals of Applied Probability*, 18(6):2392–2420, 2008.

[22] I. Tunc, M. S. Shkarayev, and L. B. Shaw. Epidemics in adaptive social networks with temporary link deactivation. *Journal of statistical physics*, 151(1–2), 2013.