Deformations of the Hill curves and isoperiodicity in the KdV and the sine-Gordon equations

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Dedicated to the memory of Sergei Petrovich Novikov (1938-2024).

Abstract

We consider a family of genus g hyperelliptic curves as double ramified coverings over the Riemann sphere with the set of branch points of the form $\{0, \infty, x_1, \dots, x_g, u_1, \dots, u_g\}$. The branch point at infinity P_∞ is selected to be a marked point on the Riemann surfaces. A meromorphic differential Ω with a unique pole being of order two at P_∞, is completely defined by the values of half of its periods, the a-periods. Fixing values of a-periods of Ω, we then find a continuous subfamily in the considered family of hyperelliptic curves along which all the periods of Ω are constant. This subfamily is defined by the functions $u_j(x_1, \dots, x_g)$, while $x_1, \dots, x_g$ are independent parameters. We derive a system of differential equations for the functions $u_j(x_1, \dots, x_g)$, which, remarkably, has rational coefficients. We call this subfamily the *isoperiodic deformations* of the hyperelliptic curves relative to the given differential of the second kind Ω. We deduce necessary and sufficient conditions for the existence and uniqueness of isoperiodic deformations. We discuss reality conditions as well. Using the obtained results, we solve the following problem for the Korteweg-de Vries and sine-Gordon equations: *starting from an algebro-geometric data which generate a real periodic solution of a period T, how to deform the data, so that the associated solutions remain periodic with the same period T.*

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1 Introduction

The integrals of Abelian differentials over closed contours on Riemann surfaces, called *periods* of the differentials, is an important concept in Riemann sufrace theory. Fixing a canonical homology basis of a surface, and fixing periods of a differential corresponding to half of the basis (normalization of the differential), the other half becomes an important tool for analysing the structure of the surface. Periods of holomorphic normalized differentials are combined in the *Riemann matrix*, the central piece of the theory. Remarkably, periods of normalized meromorphic differentials are an important component of algebro-geometric solutions to equations of mathematical physics. In particular, periods of normalized meromorphic differentials of the second kind with a unique pole at a branch point, play a significant role in solving the Korteweg-de Vries (KdV), sine-Gordon, Kadomtsev-Petviashvili equations, the associated integrable hierarchies, as well as in the expressions for finite gap potentials of the Schrödinger operator [3, 9, 10, 19, 20, 38, 39, 40]. Certain values of the periods, or more precisely, the Riemann surfaces they are realized on, correspond to periodic solutions. In this paper we focus on meromorphic differentials of the second kind on the hyperelliptic curves, with a unique pole being of order two at the branch point at infinity.

In 1895, Korteweg and de Vries derived cnoidal waves as solutions of the KdV equation expressed in terms of elliptic functions and periodic in the spatial variable x [40]. Eighty years later, very sophisticated algebro-geometric methods were developed for construction of periodic and quasi-periodic solutions of soliton equations, see e.g. [40, 10] and references therein. These methods are based mostly on the so-called isospectral deformations, Lax representations, and the Baker-Akhiezer functions. Along these lines, tremendous success has been achieved, including the solution of the Novikov conjecture, which resolved the classical Riemann-Schottke problem in algebraic geometry, see [41], [37]. Yet, for the KdV equation, as well as for many other soliton equations, the problem of singling out the periodic in x solutions of a given period among the quasi-periodic ones has not been effectively solved till nowadays, see e.g. [40], p. 266.

This motivates our present research, where we approach the problem of constructing periodic solutions in the following way: *starting from an algebro-geometric data which generate a periodic solution of a period T to a given soliton equation, how to deform the data, so that the deformed solutions remain periodic with the same period T.* This leads us to the study of *isoperiodic deformations*, as an antithesis to the isospectral defformations. Reality of solutions plays a significant role, in particular from the point of view of applications, for example, in physics. Thus, we pay a special attention to the reality conditions and apply them in the framework of the isoperiodic deformations of the KdV and sine-Gordon equations.

On the other hand, meromorphic differentials of the second kind with real periods were used in [16] to define and study foliations of moduli spaces of Riemann surfaces of a fixed genus with one marked point and a jet of local parameters at the marked point. A leaf of such a foliation is a set of triples $(\mathcal{L}, P, z)$ where $\mathcal{L}$ is a genus g Riemann surface, p is a marked point on $\mathcal{L}$ and z is a local coordinate at p and such that for each triple the periods of a chosen meromorphic differential with a unique second order pole at p are the same. These foliations were related in [16] to Calogero-Moser integrable system. A combinatorial approach to studying these foliations was proposed in [23].

The question we explore in this paper is closely related to such foliations of the moduli space of surfaces. We consider a family of genus g hyperelliptic coverings ramified over the set of the form $\{0, \infty, x_1, \dots, x_g, u_1, \dots, u_g\}$ in the complex sphere. This set of *branch points* defines the corresponding hyperelliptic covering completely. All coverings being ramified at the point at infinity P_∞, this point becomes the marked point on our Riemann surfaces. Our meromorphic differential Ω with its unique pole being the pole of order two at P_∞, is completely defined by the values of half of its periods, the a-periods. As the surface varies, we fix the a-periods to be some given constants. Then, the other half, the b-periods, are transcendental functions of the branch points $\{x_j, u_j\}_{j=1}^{g}$. Given a set of a-periods, we then find a continuous subfamily in the set of all considered hyperelliptic coverings for which all the periods of Ω are constant. This subfamily is defined by the functions $u_j(x_1, \dots, x_g)$, that is we have g independent parameters $x_1, \dots, x_g$. We derive a system of differential equations for the functions $u_j(x_1, \dots, x_g)$. Remarkably, the obtained system of differential equations has rational coefficients. We call this subfamily the *isoperiodic deformations* of the hyperelliptic curves relative to the given differential Ω. We deduce necessary and sufficient conditions for the existence and uniqueness of isoperiodic deformations.

As an important particular case of our deformations, we obtain deformations of the Hill curves, the spectral curves of the Schrödinger operator with periodic real potential. These deformations preserve the period of the corresponding potential while the spectrum of the Schrödinger operator is governed by the obtained differential equations. For classical results about Hill curves see e.g. [29, 33, 30, 31, 32].

A similar question was studied in [14]. However, our approach, the obtained equations, and derived consequences are completely different from [14]. Our main technical tools for doing analysis on Hurwitz spaces of families of hypereliptic curves are the fundamental Riemann bidifferential W on Riemann surfaces and the Rauch variational formulae. We present this in the next Section 2, see in particular Sections 2.2 and 2.3.

The paper is organized as follows. In Section 2.1, we give the necessary background on families of hyperelliptic curves, meromorphic differentials and their variations over such families; we also give a definition of Hill curves. Section 3 is devoted to definition of isoperiodic deformations for hyperelliptic surfaces relative to a differential of the second kind with a unique pole (of order two) at the ramification point at infinity of the curves. In Section 4 we consider isoperiodic deformations in genus one and obtain an ordinary differential equation for one of the two non-constant branch point as a function of another one which describes such deformations; we prove their existence and uniqueness. We then discuss the significance of isoperiodic deformation in the theory of one-gap potentials with applications to planar Neumann system and cnoidal waves. In Section 5, we derive a system of differential equations for g dependent branch points as functions of g independently varying branch points in order for the corresponding hyperelliptic curves to form an isoperiodic family relative to the chosen differential of the second kind. We prove the existence and unicity of isoperiodic families. Sections 6 - 9 are devoted to discussing applications of our isoperiodic deformations of hyperelliptic curves in the theory of integrable systems. In particular, in Section 6, we construct deformations of finite-gap periodic potentials preserving their period. In Section 7, we discuss periodic solutions to general Neumann problem and their isoperiodic deformations. In Section 8, we describe deformations preserving the wavevector of solutions to the KdV equation constructed from hyperelliptic curves as well as isoperiodic deformations of real periodic finite-gap solutions of the KdV equation. Section 9 is devoted to continuous families of real periodic solutions to the sine-Gordon equation for which the period is the same, that is constant along the family. In Section 10 we relate our isoperiodic deformations to the deformations of comb regions preserving the base of the comb while varying the lengths of the vertical slits.

2 Family of hyperelliptic surfaces

2.1 Hyperelliptic curves

Consider a general family of hyperelliptic algebraic curves defined by

$$\mathcal{C} = \{(\lambda, \mu) \in \mathbb{C}^2 \mid \mu^2 = \prod_{j=1}^{2g+1} (\lambda - \lambda_j)\}, \tag{2.1}$$

parametrized by complex variables $\lambda_1, \dots, \lambda_{2g+1}$ that may vary without coinciding. The compact projective curve $\hat{\mathcal{C}} \subset \mathbb{C}P^2$ obtained from a curve of this family by adding a point at infinity $[\lambda : \mu : z] = [0 : 1 : 0]$ is singular at the point at infinity. The associated compact hyperelliptic Riemann surface, which we denote by $\mathcal{C}_\Lambda$ with $\Lambda = (\lambda_1, \dots, \lambda_{2g+1})$, is of genus g and is such that there is a holomorphic map $\pi : \mathcal{C}_\Lambda \to \mathbb{C}P^2$ where $\pi(\mathcal{C}_\Lambda) = \hat{\mathcal{C}}$ and

$$\pi : \mathcal{C}_\Lambda \setminus \pi^{-1}([0 : 1 : 0]) \to \hat{\mathcal{C}} \setminus \{[0 : 1 : 0]\}. \tag{2.2}$$

is a biholomorphism. Using this application, and given that the algebraic curves (2.1) are non-singular, we identify the points of the Riemann surface with those of the projective curve $\hat{\mathcal{C}}$ or of algebraic curve

$\mathcal{C}$ as follows. We denote the point at infinity of $\mathcal{C}_\Lambda$ by P_∞ where $P_\infty = \pi^{-1}([0:1:0])$ and we say that $P = (\lambda, \mu)$ is a point of the surface $\mathcal{C}_\Lambda$ if $P = \pi^{-1}([\lambda : \mu : 1])$.
We now extend the function λ, defined naturally on the algebraic curve $\mathcal{C} \to \mathbb{C}$ as a projection to the first coordinate, to the Riemann surface $\lambda : \mathcal{C}_\Lambda \to \mathbb{C}P^1$ by setting $\lambda(P_\infty) = \infty$. This allows us to represent the hyperelliptic Riemann surface as the surface of the two-fold ramified covering $(\mathcal{C}_\Lambda, \lambda)$ ramified over the points $\lambda_1, \ldots, \lambda_{2g+1}, \infty$, called the branch points of the covering. We denote the corresponding ramification points by $P_{\lambda_j} = (\lambda_j, 0)$ for $j = 1, \ldots, 2g+1$ and P_∞.
The structure of this ramified covering induces the *standard local coordinates* on the surface $\mathcal{C}_\Lambda$ as follows:

$$
\begin{aligned}
&\zeta_{\lambda_k}(P) = \sqrt{\lambda(P) - \lambda_k} \quad \text{if} \quad P \sim P_{\lambda_k}, \quad \text{with} \quad k = 1, \ldots, 2g+1\,, \\
&\zeta_\infty(P) = \frac{1}{\sqrt{\lambda(P)}} \quad \text{if} \quad P \sim P_\infty, \\
&\zeta_Q(P) = \lambda(P) - \lambda(Q) \quad \text{if} \quad P \sim Q \quad \text{and } Q \text{ is a regular point.}
\end{aligned}
\tag{2.3}
$$

It is with these standard local charts that we work in this paper. The variation of λ_j induces a change of the curve in the family (2.1) and of the associated Riemann surface. We thus have a family of compact Riemann surfaces $\mathcal{C}_\Lambda$ associated with the family of algebraic curves (2.1).
The set of branch points $\Lambda = (\lambda_1, \ldots, \lambda_{2g+1})$ varies in $\mathbb{C}^{2g+1} \setminus \Delta$ where Δ is a union of all the diagonals, $\Delta = \{(\lambda_1, \ldots, \lambda_{2g+1}) \mid \lambda_i = \lambda_j \text{ for } i \neq j\}$. We work with a connected domain $\mathcal{S}$ in $\mathbb{C}^{2g+1} \setminus \Delta$ such that for the hyperelliptic compact Riemann surfaces from the family $\mathcal{C}_\Lambda$ with $\Lambda \in \mathcal{S}$, we may choose a canonical homology basis $a_1, \ldots, a_g; b_1, \ldots, b_g$ such that the images $\lambda(a_j)$ and $\lambda(b_j)$ of the basis cycles are fixed for all $\Lambda \in \mathcal{S}$, that is are the same for all surfaces from the family.

2.2 Abelian differentials

We are going to work with Abelian differentials, that is meromorphic differentials, on the compact Riemann surfaces $\mathcal{C}_\Lambda$ corresponding to hyperelliptic curves from the family (2.1). In this case, many Abelian differentials may be expressed in terms of the functions $\lambda(P)$ and $\mu(P)$ on the surface, after their extension to the point at infinity by $\lambda(P_\infty) = \infty$ and $\mu(P_\infty) = \infty$ as in Section 2.1. On such a compact hyperelliptic Riemann surface of genus g, we have a basis in the space of holomorphic differentials given by $\phi(P),\ \lambda(P)\phi(P), \ldots, \lambda(P)^{g-1}\phi(P)$ with

$$
\phi(P) = \frac{d\lambda(P)}{\mu(P)}. \tag{2.4}
$$

Another basis in this space is given by the holomorphic *normalized* differentials $\omega_1, \ldots, \omega_g$ satisfying the following conditions with respect to the chosen, as in Section 2.1, canonical homology basis $a_1, \ldots, a_g; b_1, \ldots, b_g$ of the surface:

$$
\oint_{a_j} \omega_k = \delta_{jk}, \qquad j, k = 1, \ldots, g. \tag{2.5}
$$

The *b-periods* of this basis define the *Riemann matrix* $\mathbb{B}$ of the surface:

$$
\oint_{b_j} \omega_k = \mathbb{B}_{jk}, \qquad j, k = 1, \ldots, g. \tag{2.6}
$$

Example 1 *If a hyperelliptic curve* (2.1) *is given by an equation with all* λ_j *real, then it admits an antiholomorphic involution*

$$\rho(\lambda,\mu) = (\bar\lambda,\bar\mu),$$

see e.g. [5]; the canonical basis of cycles can then be chosen so that $\rho a_j = a_j$, $\rho b_j = -b_j$, $j = 1,\dots,g$. *In this case, the holomorphic normalized differentials*

$$\omega_j = \frac{\sum_{k=0}^{g-1} c_{jk}\lambda^k}{\mu},\ j = 1,\dots,g$$

have all the coefficients c_{jk} *real. The Riemann matrix* $\mathbb{B}$ *is purely imaginary in this case, see e.g. [5]. More generally, a hyperelliptic curve of equation of the form* $y^2 = \mathcal{P}(\lambda)$ *where* $\mathcal{P}$ *is any real polynomial (having possibly pairs of complex roots) also admits the antiholomorphic involution. These are two classes of real hyperelliptic curves, where, by definition, real are those curves that admit an antiholomorphic involution. The former class is going to be used several times later, while the latter will be mentioned with respect to real solutions of the sine-Gordon equation.*
A real oval of a real curve is a connected component of the fixed points of the involution ρ. *If a curve is given by equation* (2.1) *with real* λ_j *such that* $\lambda_1 < \lambda_2 < \dots < \lambda_{2g+1}$, *then it has* $g+1$ *real ovals* $\mathcal{O}_j = \{(\lambda,\mu) | \lambda_{2j-1} \le \lambda \le \lambda_{2j}, \mu = \pm\sqrt{\prod_{k=1}^{2g+1}(\lambda-\lambda_k)}\}$, $j = 1,\dots,g+1$, *where* $\lambda_{2g+2} = \infty$.
If a real hyperelliptic curve is defined by a real polynomial, then at least one of λ_j *is real. The largest real* λ_j *belongs then to the real oval passing through the point at infinity.*

An important tool for working with Abelian differentials is the so-called Riemann fundamental bidifferential $W(P,Q)$ depending on the Riemann surface, a choice of a canonical homology basis, and two points P and Q of the surface. It can be expressed in terms of an odd theta-function or defined by its three characteristic properties as follows:

- Symmetry: $W(P,Q) = W(Q,P)$;
- A second order pole along the diagonal $P = Q$ with the following local expansion in terms of a local parameter ζ near $P = Q$ and no other singularities:

$$W(P,Q) \underset{P\sim Q}{=} \left(\frac{1}{(\zeta(P)-\zeta(Q))^2} + \mathcal{O}(1)\right) d\zeta(P)d\zeta(Q); \tag{2.7}$$

- Normalization by vanishing of the a-periods: $\oint_{a_j} W(P,Q) = 0$ for all $j = 1,\dots,g$. Due to the symmetry, these integrals can be computed with respect to either P or Q.

This definition can be shown to imply

$$\oint_{b_j} W(P,Q) = 2\pi\mathrm{i}\,\omega_j(P), \qquad j = 1,\dots,g. \tag{2.8}$$

We are also going to work with Abelian differentials *evaluated* at ramification points of the covering $(\mathcal{C}_\Lambda,\lambda)$. The evaluation is defined through the standard local parameters from Section 2.1 as follows. Let Υ be an Abelian differential on $\mathcal{C}_\Lambda$. We define its *value* at a point $\tilde{Q} \in \mathcal{C}_\Lambda$ which is not a pole of Υ as the constant term of the Taylor series expansion of the differential with respect to the standard local parameter ζ from the list (2.3) at $\tilde{Q}$, that is

$$\Upsilon(\tilde{Q}) = \frac{\Upsilon(P)}{d\zeta(P)}\Big|_{P=\tilde{Q}}. \tag{2.9}$$

As an example, for the value at a ramification point P_{λ_j} of the holomorphic differential (2.4), we use the local parameter $\zeta_{\lambda_j}(P) = \sqrt{\lambda(P) - \lambda_j}$ for which we have $d\lambda(P) = 2\zeta_{\lambda_j}(P) d\zeta_{\lambda_j}(P)$. Thus

$$\phi(P_{\lambda_j}) = \frac{\phi(P)}{d\zeta_{\lambda_j}(P)}\Big|_{\zeta_{\lambda_j}=0} = \frac{2}{\sqrt{\prod_{\substack{i=1\\ i\neq j}}^{2g+1}(\lambda_j - \lambda_i)}}. \tag{2.10}$$

This definition of evaluation is extended to the bidifferential W by considering one of the arguments of W fixed and thus obtaining a differential with respect to the other argument. For example,

$$W(P, P_\infty) = \frac{W(P,Q)}{d\zeta_\infty(Q)}\Big|_{\zeta_\infty=0}. \tag{2.11}$$

In this paper, the main role is played by the Abelian differential of the second kind defined by

$$\Omega(P) = W(P, P_\infty) + \alpha\omega(P), \tag{2.12}$$

where by ω we denote the column vector of holomorphic normalized differentials (2.5), $\omega = (\omega_1, \dots, \omega_g)^T$, and $\alpha = (\alpha_1, \dots, \alpha_g) \in \mathbb{C}^g$ is a constant row vector; its components define the a-periods of Ω. Differential (2.12) has a pole of the second order at the point P_∞ at infinity of $\mathcal{C}_\Lambda$, the compact hyperelliptic surface corresponding to a curve from the family (2.1) and no other singularities.

2.3 Rauch variation

Abelian differentials depend on the complex structure of the Riemann surface they are defined on. For our family of surfaces, the complex structure on each of them is given by the local charts (2.3). These complex structures are thus determined by the values of the finite branch points $\lambda_1, \dots, \lambda_{2g+1}$. In this section, we describe a way to study the dependence of the Abelian differentials on small variation of the λ_j.
We only consider variations of $\Lambda = (\lambda_1, \dots, \lambda_{2g+1})$ that leave Λ in $\mathcal{S}$ defined at the end of Section 2.1, so that the chosen canonical homology basis does not change under the variation. An Abelian differential $\Upsilon(P)$ depends on the Riemann surface $\mathcal{C}_\Lambda$ determined by Λ and on a point P on $\mathcal{C}_\Lambda$. The Rauch variational formulas [12] give the dependence of differentials $\Upsilon(P)$ on Λ provided the point P is fixed by the condition $\lambda(P) = \text{const}$. Let us call this derivation the *Rauch derivative*:

$$\frac{\partial^{\text{Rauch}}}{\partial\lambda_k}\Upsilon(P) := \frac{\partial}{\partial\lambda_k}\Big|_{\lambda(P)=const}\Upsilon(P)\,. \tag{2.13}$$

In the case of the Riemann bidifferential we need to require that both $\lambda(P)$ and $\lambda(Q)$ stay fixed. We have the following Rauch variational formula for the W, see [12, 22]:

$$\frac{\partial^{\text{Rauch}}}{\partial\lambda_k}W(P,Q) := \frac{\partial}{\partial\lambda_k}\Big|_{\substack{\lambda(P)=const\\ \lambda(Q)=const}} W(P,Q) = \frac{1}{2}W(P, P_{\lambda_k})W(P_{\lambda_k}, Q).$$

This variation of the Riemann bidifferential implies the following Rauch formulas for ω_j due to (2.8) and for the Riemann matrix $\mathbb{B}$ (2.6):

$$\frac{\partial^{\text{Rauch}}\omega_j(P)}{\partial\lambda_k} = \frac{1}{2}\omega_j(P_{\lambda_k})W(P, P_{\lambda_k}), \qquad \frac{\partial^{\text{Rauch}}\mathbb{B}_{ij}}{\partial\lambda_k} = \pi\mathrm{i}\,\omega_j(P_{\lambda_k})\omega_i(P_{\lambda_k})\,. \tag{2.14}$$

These Rauch variational formulas applied to definition (2.12) of our main Abelian differential of the second kind Ω, yield

$$\frac{\partial^{\mathrm{Rauch}}\Omega(P)}{\partial\lambda_k} = \frac{1}{2}\Omega(P_{\lambda_k})W(P, P_{\lambda_k})\,. \tag{2.15}$$

Note that in the above formulas we can evaluate both sides of the equation at $P = P_{\lambda_j}$ for some $j \neq k$, obtaining, for example

$$\frac{\partial^{\mathrm{Rauch}}\Omega(P_{\lambda_j})}{\partial\lambda_k} = \frac{1}{2}\Omega(P_{\lambda_k})W(P_{\lambda_j}, P_{\lambda_k})\,. \tag{2.16}$$

However, this method does not allow us to express the derivative of $\Omega(P_{\lambda_k})$ with respect to λ_k since, for example, that would require having in the right hand side of (2.16) the bidifferential W evaluated twice at the same ramification point. This cannot be done as W has a pole when both arguments coincide. In order to circumvent this difficulty, we prove the following result.

Lemma 1 *Let $\mathcal{C}_\Lambda$ be the compact Riemann surfaces associated with the hyperelliptic curves of the family* (2.1)*. Let P_{λ_j}, $j = 1, \dots, 2g+1$, be the ramification points of the two-fold ramified covering $\lambda : \mathcal{C}_\Lambda \to \mathbb{C}P^1$. Let Ω be the meromorphic differential of the second kind on $\mathcal{C}_\Lambda$ defined by* (2.12) *for some fixed $\alpha \in \mathbb{C}$ and $\Omega(P_{\lambda_j})$ its values at the ramification points defined with respect to the standard local parameters $\zeta_{\lambda_j}(P) = \sqrt{\lambda(P) - \lambda_j}$ as in* (2.9)*. Then the Rauch derivatives of $\Omega(P_{\lambda_k})$ satisfy*

$$\sum_{\substack{j=1\\ j\neq k}}^{2g+1} \frac{\partial^{\mathrm{Rauch}}}{\partial\lambda_j}\Omega(P_{\lambda_k}) + \frac{\partial\Omega(P_{\lambda_k})}{\partial\lambda_k} = 0. \tag{2.17}$$

Proof. Let ε be a complex number sufficiently close to zero and let the ramified covering $\lambda_\varepsilon : \mathcal{C}_\Lambda \to \mathbb{C}P^1$ be defined by $\lambda_\varepsilon : P \mapsto \lambda(P) + \varepsilon$, where $P = (\lambda(P), \mu(P))$ is a point of the surface $\mathcal{C}_\Lambda$. For $\varepsilon = 0$ we obtain the covering λ from Section 2.1.
The covering λ_ε is ramified at the ramification points $P_{\lambda_j} = (\lambda_j, 0)$, $j = 1, \dots, 2g+1$, which are zeros of $d\lambda_\varepsilon$, and at $P_\infty = (\infty, \infty)$ for any ε; the branch points of λ_ε are at $\lambda_1 + \varepsilon, \dots, \lambda_{2g+1} + \varepsilon, \infty$.
For the meromorphic differential Ω defined by (2.12) on the surface $\mathcal{C}_\Lambda$, its value at P_{λ_k} may be obtained with respect to the standard local parameters induced either by the covering λ or by the covering λ_ε. This could give two quantities that we denote respectively by $\Omega(P_{\lambda_k})$ and $\Omega_\varepsilon(P_{\lambda_k})$. However, given that the local parameter $\zeta_{P_{\lambda_k}}$ (2.3) is unaffected by the shift $\lambda \mapsto \lambda + \varepsilon$, the two quantities coincide: $\Omega_\varepsilon(P_{\lambda_k}) = \Omega(P_{\lambda_k})$. This implies in particular $\frac{d}{d\varepsilon}\Omega_\varepsilon(P_{\lambda_k}) = 0$.
Note that the differential $W(P, P_\infty)$ from definition of Ω is unaffected by the shift $\lambda \mapsto \lambda + \varepsilon$ as it can be invariantly defined on the Riemann surface $\mathcal{C}_\Lambda$ as the unique meromorphic normalized ($\oint_{a_n} W(P, P_\infty) = 0$ for all $n = 1, \dots, g$) differential of the second kind with a pole of second order at P_∞ and no other singularities.
On the other hand, the quantity $\Omega(P_{\lambda_k})$ is a function of the branch points $\lambda_1, \dots, \lambda_{2g+1}$ of the covering $\lambda : \mathcal{C}_\Lambda \to \mathbb{C}P^1$. At the same time, $\Omega_\varepsilon(P_{\lambda_k})$ is a function of the branch points $\lambda_1 + \varepsilon, \dots, \lambda_{2g+1} + \varepsilon$ of the covering λ_ε and thus we have

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Omega_\varepsilon(P_{\lambda_k}) = \sum_{j=1}^{2g+1} \frac{\partial}{\partial(\lambda_j + \varepsilon)}\Omega_\varepsilon(P_{\lambda_k})\Big|_{\varepsilon=0} = \sum_{j=1}^{2g+1} \frac{\partial}{\partial\lambda_j}\Omega(P_{\lambda_k}),$$

which proves the lemma. □

This lemma allows us to obtain the derivative of $\Omega(P_{\lambda_k})$ with respect to λ_k through the derivatives of $\Omega(P_{\lambda_k})$ with respect to all the λ_j with $j \neq k$. By slight abuse of terminology, we call this derivative the Rauch derivative as well and obtain the following corollary.

Corollary 1 *In the situation of Lemma 1, the following holds*

$$\frac{\partial \Omega(P_{\lambda_k})}{\partial \lambda_k} = -\frac{1}{2} \sum_{\substack{j=1 \\ j \neq k}}^{2g+1} W(P_{\lambda_k}, P_{\lambda_j}) \Omega(P_{\lambda_j}) \,.$$

Proof. This is a direct consequence of Lemma 1 and the Rauch formulas (2.15). □
Analogously to the proof of Lemma 1, we can obtain that the sum of Rauch derivatives of $\omega_s(P_{\lambda_k})$ with respect to all finite branch points vanishes and thus

$$\frac{\partial \omega_s(P_{\lambda_k})}{\partial \lambda_k} = -\frac{1}{2} \sum_{\substack{j=1 \\ j \neq k}}^{2g+1} W(P_{\lambda_k}, P_{\lambda_j}) \omega_s(P_{\lambda_j}), \tag{2.18}$$

for every $k, s = 1, \dots, g$.

2.4 Hill curves

The Hill curves are particular curves in the hyperelliptic family (2.1); they are remarkable due to their relationship with the Korteweg-de Vries equation. One can give several equivalent definitions of a Hill curve [29], one of them being as follows. A curve of genus $g > 0$ from the family (2.1) is called Hill curve if $\lambda_j \in \mathbb{R}$ for all $j = 1, \dots, 2g+1$ and there exists a differential of the second kind on the associated compact Riemann surface having a pole of a second order at P_∞ with real leading coefficient $-T$ of the Laurent series at P_∞ and no other singularities and such that its a-periods vanish and the b-periods are integer multiples of $2\pi\mathrm{i}$.
In our notation, we say that a Hill curve is a curve from (2.1) with real λ_j carrying a differential $\Omega = \Omega_0$ defined by (2.12) with $\alpha = (0, \dots, 0)$ and such that

$$\oint_{b_j} \Omega_0 = 2\pi\mathrm{i}\,\frac{n_j}{T} \quad \text{with} \quad n_j \in \mathbb{Z} \quad \text{for some} \quad T > 0, \quad \text{for all} \quad j = 1, \dots, g. \tag{2.19}$$

Note that in the notation of [29], where $\mathrm{e}(P)$ is the *unit* on the surface (2.1), for the differential Ω_0 defined by (2.12) with $\alpha = 0$ and satisfying (2.19), we have $\Omega_0 = -\frac{1}{T}\mathrm{d}\log \mathrm{e}(P)$.

3 Deformations of Hill curves

In this section, we consider isoperiodic deformations of a differential of the second kind over the family of hyperelliptic curves. Deformations of Hill curves is a special case of such deformations.

3.1 Isoperiodic deformations of a differential of second kind

We want to consider continuous variations of the branch points of the curves (2.1) which leave all periods of the meromorphic differential Ω (2.12) invariant. More precisely, the a-periods of Ω are given by the vector $\alpha = (\alpha_1, \dots, \alpha_g) \in \mathbb{C}^g$ constant by construction. Let us denote by $\beta = (\beta_1, \dots, \beta_g) \in \mathbb{C}^g$ its b-periods:

$$\oint_{b_k} \Omega(P) = \beta_k. \tag{3.1}$$

Integrating (2.12), we have

$$\beta_k = 2\pi \mathrm{i}\omega_k(P_\infty) + \alpha\mathbb{B}_k, \tag{3.2}$$

with $\mathbb{B}_k$ being the kth column of the Riemann matrix $\mathbb{B}$. As we see from the Rauch formulas (2.14), a generic variation of λ_j does not guarantee the constancy of β_k. Therefore, in order to find deformations of the hyperelliptic curve (2.1) which allow for the periods of the differential (2.12) to stay constant, we need to reduce the number of degrees of freedom. To this end, we set $\lambda_1 = 0$ and split the remaining $2g$ finite branch points of our hyperelliptic curves into two sets: $u_1, \dots, u_g$ and $x_1, \dots, x_g$ and allow $u_1, \dots, u_g$ to depend on $x_1, \dots, x_g$. Thus, in the rest of the paper, our curves are given by

$$\mathcal{C}_{\mathbf{x}} = \{(\lambda, \mu) \in \mathbb{C}^2 \mid \mu^2 = \lambda \prod_{j=1}^{g}(\lambda - u_j) \prod_{j=1}^{g}(\lambda - x_j)\}, \tag{3.3}$$

where only g branch points $x_1, \dots, x_g$ are allowed to vary independently. Here $\mathbf{x}$ stands for $\mathbf{x} = (x_1, \dots, x_g)$ and we denote the compact hyperelliptic Riemann surfaces associated with these curves also by $\mathcal{C}_{\mathbf{x}}$.

The domain $\mathcal{S}$ from Section 2.1 is replaced now by a connected domain $\mathcal{S}_{\mathbf{x}} \subset \mathbb{C}^g \setminus \Delta_{\mathbf{x}}$ where $\Delta_{\mathbf{x}} = \{(x_1, \dots, x_g) \mid \exists\, i \neq j \text{ such that } x_i = x_j\} \cup \{(x_1, \dots, x_g) \mid x_i = u_j \text{ for some } 1 \leqslant i, j \leqslant g\}$. The domain $\mathcal{S}_{\mathbf{x}}$ should be chosen such that for the hyperelliptic compact Riemann surfaces from the family $\mathcal{C}_{\mathbf{x}}$ with $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$, we may choose a canonical homology basis $a_1, \dots, a_g; b_1, \dots, b_g$ such that the images $\lambda(a_j)$ and $\lambda(b_j)$ of the basis cycles are fixed for all $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$, that is are the same for all surfaces from the family.

Our objective is to obtain equations for $u_1, \dots, u_g$ as functions of $x_1, \dots, x_g$ that are satisfied when the surface $\mathcal{C}_{\mathbf{x}}$ deforms in a way to keep the quantities β_k (3.2) fixed. We refer to these deformations as *isoperiodic deformations* of the pair $(\mathcal{C}_{\mathbf{x}}, \Omega)$ with $\Omega = \Omega(\mathbf{x})$ given by (2.12) on $\mathcal{C}_{\mathbf{x}}$. We also say that the family of pairs $(\mathcal{C}_{\mathbf{x}}, \Omega(\mathbf{x}))$ parametrized by $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$ is an *isoperiodic family of hyperelliptic surfaces relative to differential* $\Omega(\mathbf{x})$. In Section 4, we derive such an equation for genus one surfaces while the general case is treated in Section 5.

4 Isoperiodic deformations in genus one

Here we consider the family of compact Riemann surfaces corresponding to the elliptic curves given by

$$\mathcal{C}_x = \{(\lambda, \mu) \in \mathbb{C}^2 \mid \mu^2 = \lambda(\lambda - u)(\lambda - x)\}, \tag{4.1}$$

with x being an independently varying branch point and u being a function of x. This is the family (3.3) in the case $g = 1$. The differential of the second kind Ω is defined on these surfaces by (2.12). The b-period of Ω is

$$\beta = 2\pi \mathrm{i}\omega(P_\infty) + \alpha\tau, \tag{4.2}$$

where ω is the holomorphic normalized differential on the Riemann surface and τ is its b-period. Differentiating this relation with respect to x assuming that u to be a function of x using the Rauch formula (2.14) with $\tau = \mathbb{B}_{11}$ and $\omega = \omega_1$, yields

$$0 = \pi \mathrm{i} W(P_\infty, P_x)\omega(P_x) + \pi \mathrm{i}\alpha\omega^2(P_x) + \left(\pi \mathrm{i} W(P_\infty, P_u) + \pi \mathrm{i}\alpha\omega^2(P_u)\right) u',$$

where u' stands for $\frac{du}{dx}$. From this, using the definition (2.12) of Ω and the evaluation (2.9) at P_x and P_u, we obtain an expression for the first derivative of u:

$$u' = -\frac{\omega(P_x)\Omega(P_x)}{\omega(P_u)\Omega(P_u)}. \tag{4.3}$$

4.1 The equation describing isoperiodic deformations in genus one

Let us now differentiate expression (4.3) for the first derivative of u the second time with respect to x. With the help of Corollary 1 and (2.18) for derivatives of differentials evaluated at P_x and P_u with respect to x and u, we have

$$\begin{aligned} u'' = & -\frac{\omega(P_x)\Omega(P_x)}{\omega(P_u)\Omega(P_u)} \left\{ \frac{\frac{d}{dx}\omega(P_x)}{\omega(P_x)} + \frac{\frac{d}{dx}\Omega(P_x)}{\Omega(P_x)} - \frac{\frac{d}{dx}\omega(P_u)}{\omega(P_u)} - \frac{\frac{d}{dx}\Omega(P_u)}{\Omega(P_u)} \right\} \\ = & -\frac{\omega(P_x)\Omega(P_x)}{2\omega(P_u)\Omega(P_u)} \left\{ -\frac{\omega(P_0)W(P_x,P_0)}{\omega(P_x)} - \frac{\omega(P_u)W(P_x,P_u)}{\omega(P_x)} + u'\frac{\omega(P_u)W(P_x,P_u)}{\omega(P_x)} \right. \\ & - \frac{\Omega(P_0)W(P_x,P_0)}{\Omega(P_x)} - \frac{\Omega(P_u)W(P_x,P_u)}{\Omega(P_x)} + u'\frac{\Omega(P_u)W(P_x,P_u)}{\Omega(P_x)} \\ & - \frac{\omega(P_x)W(P_x,P_u)}{\omega(P_u)} + u'\left(\frac{\omega(P_0)W(P_0,P_u)}{\omega(P_u)} + \frac{\omega(P_x)W(P_x,P_u)}{\omega(P_u)}\right) \\ & \left. -\frac{\Omega(P_x)W(P_x,P_u)}{\Omega(P_u)} + u'\left(\frac{\Omega(P_0)W(P_0,P_u)}{\Omega(P_u)} + \frac{\Omega(P_x)W(P_x,P_u)}{\Omega(P_u)}\right) \right\}. \end{aligned} \tag{4.4}$$

We now replace W evaluated at ramification points by the following expressions in terms of the function $\lambda(P)$ defined on the Riemann surface $\mathcal{C}_{\mathbf{x}}$ by a natural extension from the algebraic curve (4.1) to the point at infinity: $\lambda(P_\infty) = \infty$, see Section 2.4.

$$\begin{aligned} W(P,P_0) &= \left(\frac{1}{\lambda(P)\omega(P_0)} + I^0\right)\omega(P), \quad W(P,P_x) = \left(\frac{1}{(\lambda(P)-x)\omega(P_x)} + I^x\right)\omega(P), \\ W(P,P_u) &= \left(\frac{1}{(\lambda(P)-u)\omega(P_u)} + I^u\right)\omega(P), \end{aligned}$$

where I^0, I^x, I^u are normalization constants. From here, evaluating with respect to the second argument, we obtain

$$W(P_x,P_0) = \left(\frac{1}{x\omega(P_0)} + I^0\right)\omega(P_x) = \left(-\frac{1}{x\omega(P_x)} + I^x\right)\omega(P_0), \tag{4.5}$$

$$W(P_x,P_u) = \left(\frac{1}{(x-u)\omega(P_u)} + I^u\right)\omega(P_x) = \left(\frac{1}{(u-x)\omega(P_x)} + I^x\right)\omega(P_u). \tag{4.6}$$

$$W(P_0,P_u) = \left(-\frac{1}{u\omega(P_u)} + I^u\right)\omega(P_0) = \left(\frac{1}{u\omega(P_0)} + I^0\right)\omega(P_u). \tag{4.7}$$

From (4.5) and (4.6) we obtain the following relationship between the normalization constants:

$$I^0 = -\frac{1}{\omega(P_0)x} + \left(I^x - \frac{1}{x\omega(P_x)}\right)\frac{\omega(P_0)}{\omega(P_x)} \quad \text{and} \quad I^u = \left(\frac{1}{(u-x)\omega(P_x)} + I^x\right)\frac{\omega(P_u)}{\omega(P_x)} - \frac{1}{(x-u)\omega(P_u)}. \tag{4.8}$$

Note that for the holomorphic normalized differential ω, we can use the following expression in terms of the functions λ and μ on the elliptic curve (4.1):

$$\omega(P) = \frac{d\lambda}{I_0\mu} = \frac{d\lambda}{I_0\sqrt{\lambda(\lambda-u)(\lambda-x)}}, \tag{4.9}$$

where I_0 is the normalization constant depending on the curve. Evaluating this differential at the ramification points with respect to the standard local parameters according to (2.9), we have

$$\omega(P_0) = \frac{2}{I_0\sqrt{ux}}, \qquad \omega(P_u) = \frac{2}{I_0\sqrt{u(u-x)}}, \qquad \omega(P_x) = \frac{2}{I_0\sqrt{x(x-u)}}. \tag{4.10}$$

This implies the following identity:

$$\omega^2(P_0) + \omega^2(P_u) + \omega^2(P_x) = 0. \tag{4.11}$$

Now, we plug (4.5)-(4.7) as well as (4.3) in the expression (4.4) for the second derivative of u, choosing to use the normalization constant I^x when possible, and then simplify expressions with the help of (4.10) and (4.11). This gives

$$\begin{aligned} u'' = \frac{u'}{2}\left(\frac{1}{u} - \frac{1}{x} - \frac{x}{u(x-u)} + u'\left(\frac{x}{u(x-u)} + \frac{I^x\omega^2(P_u)}{\omega(P_x)}\right)\right) \\ + \frac{\omega(P_0)\Omega(P_0)}{2\omega(P_u)\Omega(P_u)}\left(-\frac{1}{x} + I^x\omega(P_x)\right) + \frac{1-u'}{2}\left(\frac{1}{u-x} + I^x\omega(P_x)\right) \\ + \frac{u'}{2}\left(-\frac{1}{u-x} + u'\left(\frac{1}{u} + I^0\omega(P_0) + \frac{1}{u-x} + I^x\omega(P_x)\right)\right) + \frac{(u')^2}{2}\left(\frac{x}{u(x-u)} + \frac{I^x\omega^2(P_u)}{\omega(P_x)}\right) \\ + \frac{(u')^2}{2}\frac{\omega(P_0)\Omega(P_0)}{\omega(P_u)\Omega(P_u)}\left(\frac{x}{u(u-x)} + \frac{I^0\omega^2(P_u)}{\omega(P_0)}\right) - \frac{(u')^3}{2}\left(\frac{x}{u(x-u)} + \frac{I^x\omega^2(P_u)}{\omega(P_x)}\right). \end{aligned} \tag{4.12}$$

Let us now consider the following identity on the elliptic curve

$$\omega(P_0)\Omega(P_0) + \omega(P_u)\Omega(P_u) + \omega(P_x)\Omega(P_x) = 0,$$

which can be obtained as the vanishing sum of residues of the differential $\frac{\omega(P)\Omega(P)}{d\lambda(P)}$ on the compact Riemann surface corresponding to the elliptic curve (4.1). Dividing this relation by $\omega(P_u)\Omega(P_u)$ and using (4.3) for u', we have

$$\frac{\omega(P_0)\Omega(P_0)}{\omega(P_u)\Omega(P_u)} = u' - 1. \tag{4.13}$$

Plugging (4.13) into (4.12) and expressing I^0 in terms of I^x as in (4.8), we see with the help of (4.11) that the terms with I^x cancel out and thus we obtain an ordinary differential equation (4.15) for $u(x)$ and prove the following theorem.

Theorem 1 *Let $\mathcal{S}_\mathbf{x}$ be a connected domain as discussed in Section 3.1 such that for $x \in \mathcal{S}_\mathbf{x}$ one can choose a canonical homology basis a, b on all Riemann surfaces corresponding to the curves from the family*

$$\mathcal{C}_x = \{(\lambda,\mu) \in \mathbb{C}^2 \mid \mu^2 = \lambda(\lambda - u(x))(\lambda - x)\} \tag{4.14}$$

in such a way that $\lambda(a)$ and $\lambda(b)$ are fixed for all $x \in \mathcal{S}_{\mathbf{x}}$. Let $u(x)$ for $x \in \mathcal{S}_{\mathbf{x}}$ be such that the compact Riemann surfaces associated with curves of subfamily (4.14) *of the family* (4.1) *are such that the differential of the second kind Ω defined on each of these surfaces by* (2.12) *has a- and b-periods that are constant for all $x \in \mathcal{S}$. Then $u(x)$ satisfies the following equation:*

$$u'' = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{u-x}\right) - \frac{u'}{2}\left(\frac{2}{x} + \frac{1}{u-x}\right) + \frac{(u')^2}{2}\left(\frac{2}{u} + \frac{1}{x-u}\right) - \frac{(u')^3}{2}\left(\frac{1}{u} + \frac{1}{x-u}\right). \tag{4.15}$$

In particular if the curves (4.14) *are Hill curves for every $x \in \mathcal{S}$, then $u(x)$ is a solution of* (4.15).

Conversely, some solutions of equation (4.15) correspond to isoperiodic deformations as explained in the next theorem.

Theorem 2 *Let $x_0 \in \mathbb{C} \setminus \{0\}$, and let $\Omega = \Omega(x_0)$ be the meromorphic differential of the second kind defined by* (2.12) *on a compact Riemann surface $\mathcal{C}_{x_0}$ of the elliptic curve* (4.1) *with $x = x_0$. Assume that x_0 is such that $\Omega(x_0; P_u) \neq 0$. Let $\{a, b\}$ be a canonical homology basis on $\mathcal{C}_{x_0}$ such that the projections $u(a)$ and $u(b)$ do not intersect a certain neighbourhood $\hat{\mathcal{X}}$ of x_0. Then there exists a unique continuous family $(\mathcal{C}_x, \Omega(x))$ providing an isoperiodic deformation of the pair $(\mathcal{C}_{x_0}, \Omega(x_0))$ for x varying in some neighbourhood $\mathcal{X} \subset \hat{\mathcal{X}}$ of x_0.*

Proof. Isoperiodic deformations of $(\mathcal{C}_{x_0}, \Omega(x_0))$ are defined by the condition $\oint_b \Omega(x) = \text{const}$, as the a-period of $\Omega(x)$ is constant by definition (2.12). By the implicit function theorem, the relation $\oint_b \Omega(x) = \text{const}$ defines a function $u(x)$ in a neighbourhood of $x = x_0$ if the derivative $\frac{\partial}{\partial u}\oint_b \Omega(x)$ is non-zero at $x = x_0$. The b-period of Ω is given by (4.2). Thus, the derivative in question can be calculated by Rauch variational formulas (2.14):

$$\frac{\partial}{\partial u}\oint_b \Omega = \frac{\partial^{\text{Rauch}}}{\partial u}(2\pi \mathrm{i}\omega(P_\infty) + \alpha\tau) = \pi \mathrm{i} W(P_u, P_\infty)\omega(P_u) + \pi \mathrm{i}\alpha\omega^2(P_u) = \pi \mathrm{i}\Omega(P_u)\omega(P_u).$$

The factor of $\omega(P_u)$ is non-zero since a holomorphic differential on a genus one surface does not vanish, and $\Omega(P_u)$ is non-zero for the curve $\mathcal{C}_{x_0}$ given the choice of x_0. This proves the existence of an isoperiodic deformation in some neighbourhood $\mathcal{X} \subset \hat{\mathcal{X}}$ of x_0.
On the other hand, Theorem 1 states that for every continuous isoperiodic deformation, the function $u(x)$ satisfies equation (4.15). In addition, for an isoperiodic deformation, $u'(x_0)$ is given by (4.3). Thus the initial values $u(x_0)$ and $u'(x_0)$ determine a unique solution to (4.15). This proves the unicity of an isoperiodic deformation of the pair $(\mathcal{C}_{x_0}, \Omega(x_0))$. We assume $\mathcal{X}$ small enough so that $u(x)$ does not belong to $u(a)$ or $u(b)$ for $x \in \mathcal{X}$. □

4.2 Isoperiodic deformations of one-gap potentials

In this section, we discuss an application of Theorems 1 and 2 to the theory of one-gap potentials. Let $\wp(z)$ be the standard Weierstrass function associated with the elliptic curve

$$\Gamma_{e_2,e_3} = \{(\lambda, w) \in \mathbb{C}^2 \,|\, w^2 = 4\lambda^3 - g_2\lambda - g_3\}. \tag{4.16}$$

Denoting the ramification points of the covering $\lambda : \Gamma_{e_2,e_3} \to \mathbb{C}P^1$ by $P_{e_j} = (\lambda = e_j, w = 0)$ and P_∞, the standard choice of the canonical homology basis on Γ_{e_2,e_3} is as follows: the cycle a encircles the points P_∞ and P_{e_1} while the cycle b encircles the points P_∞ and P_{e_2}. The elliptic curve can be parameterized by the points of the torus $\mathbb{T}^2 = \mathbb{C}/\{2m\omega + 2n\omega'\}$ with $\lambda = \wp(z)$ and $w = \wp'(z)$ for

$z \in \mathbb{T}^2$. The Weierstrass functions ζ and σ are defined in the standard manner (see e.g. [2]) by $\zeta'(z) = -\wp(z)$ and $\sigma'(z)/\sigma(z) = \zeta(z)$.
In 1940, Ince [18] showed that $v_1(X) = 2\wp(X),\ X \in \mathbb{C}$ is a one-gap potential for the Lamé operator

$$\mathbb{L} = -\frac{\partial^2}{\partial X^2} + v_1(X).$$

In [17], it was shown that there are no other one-gap potentials apart from those constructed by Ince. Following e.g. [10] and [2], let us introduce the Baker-Akhiezer function

$$\psi(X,P) = \frac{\sigma(z - z_1 - X)\exp(X\zeta(z))}{\sigma(z - z_1)\sigma(z_1 + X)},$$

where $P = (\wp(z), \wp'(z))$. Then, $\psi(X,P)$ is an eigenfunction of the Lamé operator $\mathbb{L} = -\frac{\partial^2}{\partial X^2} + 2\wp(X)$:

$$\mathbb{L}\psi(X,P) = \lambda\psi(X,P), \quad \lambda = \wp(z).$$

The Lamé potential $v_1(X)$ is a doubly periodic function with periods $2w_1$ and $2w_2$, where

$$2w_1 = \oint_a \phi, \qquad 2w_2 = \oint_b \phi.$$

Here $\phi = \frac{d\lambda}{w}$ is a non-normalized holomorphic differential on Γ, and a and b form a canonical basis of homology of Γ, where we denote the compact Riemann surface associated with the elliptic curve by Γ as well.
It could thus be interesting to study deformations of the elliptic curve Γ which preserve one of the periods of the potential $v_1(X) = 2\wp(X)$.
In the standard way, let us denote the zeros of the polynomial $4z^3 - g_2 z - g_3$ by $e_1,\ e_2,\ e_3$. Then $e_1 = -e_2 - e_3$.

Theorem 3 *There exists a unique continuous family of elliptic curves* (4.16) *parameterized by interdependent values of $e_2,\ e_3 \in \mathbb{C}$ such that the period $2w_1$ is constant over all curves of the family. Moreover, for this family $u = 2e_2 + e_3$ satisfies equation* (4.15) *as a function of $x = e_2 + 2e_3$.*

Proof. Let us perform a shift by $-e_1$ in the λ-sphere which maps the curves of the family Γ_{e_2,e_3} into the curves $\hat{\Gamma}_{x,u}$ with branch points at $0,\ x = 2e_3 + e_2,\ u = 2e_2 + e_3$ and the point at infinity. The canonical homology basis a and b chosen above is shifted accordingly. The periods $2w_1,\ 2w_2$ of the holomorphic differential $\phi = \frac{d\lambda}{w}$ are unchanged under the change of variables.
Let Ω_0 be the differential of the second kind (2.12) with $\alpha = 0$ defined on the curves from the family $\hat{\Gamma}_{x,u}$. Applying (4.2) for $\alpha = 0$ we see that the b-period of Ω_0 is given by

$$\oint_b \Omega_0 = 2\pi \mathrm{i}\omega(P_\infty). \tag{4.17}$$

Here ω is the holomorphic normalized differential on $\hat{\Gamma}_{x,u}$, that is $\omega = \frac{\phi}{2w_1}$, and the evaluation at the point at infinity of $\hat{\Gamma}_{x,u}$ is done as in (2.9) with respect to the standard local parameter $\zeta(P) = 1/\sqrt{\lambda(P)}$:

$$\omega(P_\infty) = \frac{\omega(P)}{d\zeta(P)}\Big|_{\zeta=0} = \frac{d\left(\frac{1}{\zeta^2}\right)}{2w_1\sqrt{\frac{4}{\zeta^6} - \frac{g_2}{\zeta^2} - g_3}}\Big|_{\zeta=0} = -\frac{1}{2w_1}.$$

Thus we obtain that $2w_1$ is constant for a family $\hat{\Gamma}_{x,u}$ if and only if the b-period of Ω_0 is constant for the same family. Thus the required family of curves is an isoperiodic family $(\hat{\Gamma}_{x,u}, \Omega_0)$; its existence and uniqueness is given by Theorem 2. The fact that $u(x)$ satisfies equation (4.15) follows from Theorem 1. □

Corollary 2 *The function $u(x)$ from Theorem 3 satisfies equation* (4.15) *with the initial condition given by* (4.3) *with $\Omega = \Omega_0$ given by* (2.12) *with $\alpha = 0$ and $\omega = \frac{d\lambda}{2w_1 w}$ being the holomorphic normalized differential on the curve $\hat{\Gamma}_{x,u}$ for some initial value of x.*

Corollary 3 *There exists a unique continuous deformation of the curve $\Gamma_{e_2,3_3}$* (4.16) *which preserves the period $2w_1$ of the potential $v_1(X) = 2\wp(X)$. This deformation is such that $u = 2e_2 + e_3$ satisfies equation* (4.15) *as a function of $x = 2e_3 + e_2$ with the initial condition expressed by* (4.3) *with respect to the following differentials on the curve $\hat{\Gamma}_{x,u}$ obtained from Γ_{e_2,e_3} by a shift by $-e_1$ in the λ-sphere: $\Omega = \Omega_0$ given by* (2.12) *with $\alpha = 0$ and $\omega = \frac{d\lambda}{2w_1 w}$.*

We now give two examples providing an application of the theory of this section to isoperiodic deformations of solutions to the planar version of the Neumann system and of the cnoidal waves expressed as periodic solutions to the Korteweg-de Vries equation.

Example 2 *[Period-preserving deformations of solutions of the planar Neumann system] Let us consider a system of two uncoupled oscillators with frequencies $\sqrt{A_1}$, $\sqrt{A_2}$ with $\sqrt{A_1} < \sqrt{A_2}$, under the assumption that the external force F constrains the particles to remain on a unit circle. The equations of motion are*

$$\ddot{q}_1 + A_1 q_1 = -F q_1, \tag{4.18}$$
$$\ddot{q}_2 + A_2 q_2 = -F q_2, \tag{4.19}$$

where $F = \dot{q}_1^2 + \dot{q}_2^2 - A_1 q_1^2 - A_2 q_2^2$ and $q_1^2 + q_2^2 = 1$ with q_1, q_2 being the positions of the two particles. We can use the Jacobi elliptic coordinates which map a point (q_1, q_2) from the unit circle to the conical Jacobi coordinate μ_1, such that

$$\frac{q_1^2}{A_1 - \mu} + \frac{q_2^2}{A_2 - \mu} = 0.$$

Thus, $\mu_1 = A_2 q_1 + A_1 q_2$ satisfies $A_1 \leq \mu \leq A_2$ and

$$q_1 = \sqrt{\frac{\mu - A_1}{A_2 - A_1}}, \quad q_2 = \sqrt{\frac{A_2 - \mu}{A_2 - A_1}}.$$

After differentiation with respect to time, we get a Weierstrass-type differential equation for μ:

$$(\dot{\mu})^2 = -4(\mu - A_1)(\mu - A_2)(\mu - A_1 - A_2 - 2h),$$

where

$$h = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + A_1 q_1^2 + A_2 q_2^2)$$

is the first integral of the system equal to the energy of the uncoupled pair of oscillators. Thus,

$$\mu(t) = \wp(\mathrm{i}t + t_0) - z_0, \quad z_0 = -\frac{2}{3}(h + A_1 + A_2),$$

with some constant t_0, which corresponds to the initial conditions for the planar Neumann system, where $\wp$ is the Weierstrass function of the elliptic curve Γ (4.16) with $\lambda = \mu + z_0$ and the standard choice of the canonical homology basis as above. According to Corollary 3, the solution $\mu(t)$ is deformed by varying the underlying elliptic curve, in a way that the period $2w_1/\mathrm{i}$ of $\mu(t)$ is preserved if and only if $u = 2e_2 + e_3$ satisfies equation (4.15) as a function of $x = e_2 + 2e_3$ with the initial condition (4.3), where $\Omega = \Omega_0$ from the proof of Theorem 3, see (4.17).
The three-dimensional version of the Neumann problem was studied by C. Neumann in 1859. Higher-dimensional generalizations of the Neumann system were studied by Moser [35], [36] and many others. We discuss isoperiodic deformations of solutions to the Neumann system in dimensions three and higher in Section 7.

Example 3 *[Period-preserving deformations of cnoidal waves] The function*

$$v(X,t) \equiv v(X - ct) = 2\wp(X - ct - X_0) - \frac{c}{6}$$

provides a periodic solution to the Korteweg-de Vries equation

$$v_t - 6vv_X + v_{XXX} = 0,$$

of the form of cnoidal wave. According to Corollary 3, the solution $v(X,t)$ can be continuously deformed by varying the underlying elliptic curve in a way that the period $2w_1$ is preserved. This deformation is such that $u = 2e_2 + e_3$ satisfies equation (4.15) as a function of $x = e_2 + 2e_3$ with the initial condition (4.3), where $\Omega = \Omega_0$ from (4.17).

5 Isoperiodic deformations in genus greater than one

5.1 Variation of branch points

Consider now the family of curves (3.3) for any $g > 0$ assuming that the branch points $u_1, \dots, u_g$ are functions of the independently varying branch points $x_1, \dots, x_g$. With this assumption in mind, we differentiate equation (3.2) with respect to x_j and obtain, in Theorem 5 equations for the functions $\{u_k\}$ with respect to $\{x_j\}$.
Differentiating (3.2) with respect to an independently varying branch point x_j with $j = 1, \dots, g$, using Rauch formulas (2.14), we have

$$0 = \frac{\partial \beta_k}{\partial x_j} = \pi\mathrm{i}\omega_k(P_{x_j})(W(P_{x_j}, P_\infty) + \alpha\omega(P_{x_j})) + \pi\mathrm{i}\sum_{m=1}^{g}\omega_k(P_{u_m})(W(P_{u_m}, P_\infty) + \alpha\omega(P_{u_m}))\frac{\partial u_m}{\partial x_j},$$

which can be rewritten in terms of differential Ω (2.12) in the form

$$\sum_{m=1}^{g}\omega_k(P_{u_m})\Omega(P_{u_m})\frac{\partial u_m}{\partial x_j} = -\omega_k(P_{x_j})\Omega(P_{x_j}). \tag{5.1}$$

The latter equation can be seen as a system of g linear equations for g variables $\frac{\partial u_m}{\partial x_j}$ for every fixed j. In the equations of this system, the differential ω_k can be replaced by any combination of holomorphic normalized differentials $\omega_1, \dots, \omega_g$. To obtain a solution to this linear system, it is convenient to introduce a new basis $v_1, \dots, v_g$ in the space of holomorphic differentials defined by the condition

$$v_j(P_{u_i}) = \delta_{ij} \quad \text{for } i, j = 1, \dots, g, \tag{5.2}$$

see (5.6) below for a more explicit definition.
Differentials v_j are linear combinations of $\omega_1, \dots, \omega_g$ and thus we can replace ω_k by v_k in (5.1). Due to relation $v_k(P_{u_m}) = \delta_{km}$ this gives the following theorem generalizing (4.3).

Theorem 4 *Let $\mathcal{S}_{\mathbf{x}}$ be such as discussed in Section 3.1. Let Ω be the differential defined by* (2.12) *on the curves of the family $\mathcal{C}_{\mathbf{x}}$* (3.3) *with $u_j = u_j(x_1, \dots, x_g)$ and the basis of a- and b-cycles chosen consistently for all curves $\mathcal{C}_{\mathbf{x}}$ with $\mathbf{x} = (x_1, \dots, x_g) \in \mathcal{S}_{\mathbf{x}}$. If the periods of Ω stay constant for all the curves of the family for $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$, then derivatives of $u_1, \dots, u_g$ with respect to $x_1, \dots, x_g$ are given by*

$$\frac{\partial u_k}{\partial x_j} = -\frac{v_k(P_{x_j})\Omega(P_{x_j})}{\Omega(P_{u_k})}\,, \tag{5.3}$$

where P_{x_j} and P_{u_k} are the ramification points corresponding to the branch points x_j and u_k, respectively, and v_j are holomorphic differentials defined by conditions (5.2) *on the curves from the family $\mathcal{C}_{\mathbf{x}}$.*

Let us note the following useful relation. The sum of residues of the meromorphic differential $\frac{v_m(P)\Omega(P)}{d\lambda(P)}$ on a curve from the family $\mathcal{C}_{\mathbf{x}}$ (2.1) vanishes and therefore we have

$$\Omega(P_0)v_m(P_0) + \sum_{i=1}^{g} \Omega(P_{x_i})v_m(P_{x_i}) + \Omega(P_{u_m}) = 0, \tag{5.4}$$

which, dividing out $\Omega(P_{u_m})$ and using (5.3) becomes

$$\frac{\Omega(P_0)v_m(P_0)}{\Omega(P_{u_m})} = \sum_{i=1}^{g} \frac{\partial u_m}{\partial x_i} - 1. \tag{5.5}$$

5.2 Coordinate representation of Abelian differentials

The holomorphic differentials v_j determined by conditions (5.2) admit the following expressions in terms of the functions λ and μ on a surface of the family $\mathcal{C}_{\mathbf{x}}$ ((2.1)):

$$v_j(P) = \frac{\phi(P)\prod_{i\neq j}(\lambda - u_i)}{\phi(P_{u_j})\prod_{i\neq j}(u_j - u_i)}, \quad j = 1, \dots, g, \tag{5.6}$$

where ϕ is the holomorphic non-normalized differential (2.4) and the index range is $i = 1, \dots, g$ in the products.
Analogously to formulas (4.5) - (4.7) in genus one, we can write expressions for W with one of the two arguments evaluated at a branch point in terms of the function $\lambda(P)$ on the compact surface $\mathcal{C}_{\mathbf{x}}$ at the expense of introducing normalization constants denoted by $I_i^{x_k}$ and $I_i^{u_m}$ below.

$$W(P, P_{x_k}) = \frac{\phi(P)}{\phi(P_{x_k})(\lambda(P) - x_k)} + \sum_{i=1}^{g} I_i^{x_k} v_i(P), \quad k = 1, \dots, g, \tag{5.7}$$

$$W(P, P_{u_m}) = \frac{\phi(P)}{\phi(P_{u_m})(\lambda(P) - u_m)} + \sum_{i=1}^{g} I_i^{u_m} v_i(P), \quad m = 1, \dots, g. \tag{5.8}$$

Evaluating further, as in (2.9), the second argument at another ramification point, and using definition (5.2) or (5.6) of the differentials v_j, we obtain

$$W(P_{u_m}, P_{x_k}) = \frac{\phi(P_{u_m})}{\phi(P_{x_k})(u_m - x_k)} + I_m^{x_k} = \frac{\phi(P_{x_k})}{\phi(P_{u_m})(x_k - u_m)} + \sum_{i=1}^{g} I_i^{u_m} v_i(P_{x_k}), \tag{5.9}$$

$$W(P_{x_n}, P_{x_k}) = \frac{\phi(P_{x_n})}{\phi(P_{x_k})(x_n - x_k)} + \sum_{i=1}^{g} I_i^{x_k} v_i(P_{x_n}), \tag{5.10}$$

$$W(P_{u_j}, P_{u_m}) = \frac{\phi(P_{u_j})}{\phi(P_{u_m})(u_j - u_m)} + \ I_j^{u_m}. \tag{5.11}$$

Let us now prove five useful technical lemmas.

Lemma 2 *Let W be the Riemann fundamental bidifferential* (2.7) *on a surface $\mathcal{C}_{\mathbf{x}}$ from the family* (3.3)*, let ϕ and v_j be the holomorphic differentials* (2.4) *and* (5.6)*, respectively. Let the differentials be evaluated at ramification points according to* (2.9) *with respect to the standard local parameters* (2.3)*. For any two integers $1 \leqslant k, n \leqslant g$, the following identity holds:*

$$\sum_{j=1}^{g} W(P_{u_j}, P_{x_k}) v_j(P_{x_n}) = W(P_{x_n}, P_{x_k}) + \frac{\phi(P_{x_n})}{\phi(P_{x_k})} \frac{1}{x_k - x_n} \frac{\prod_{i=1}^{g}(x_n - u_i)}{\prod_{i=1}^{g}(x_k - u_i)}\,, \tag{5.12}$$

where P_{x_j} and P_{u_j} are the ramification points of $\lambda : \mathcal{C}_{\mathbf{x}} \to \mathbb{C}P^1$ corresponding to the branch points x_j and u_j, respectively.

Proof. For some fixed k and n, consider the following differential on the compact surface $\mathcal{C}_{\mathbf{x}}$ having simple poles at P_{x_n}, P_{x_k} and P_{u_i} with $i = 1, \dots, g$, and no other singularities:

$$\Upsilon_1(P) = \frac{W(P, P_{x_k}) d\lambda(P)}{\phi(P)(\lambda(P) - x_n) \prod_{i=1}^{g}(\lambda(P) - u_i)}.$$

The vanishing of the sum of all residues of this differential is equivalent to the statement of the lemma. The residues at P_{x_n} and P_{u_i} are

$$\operatorname*{res}_{P=P_{x_n}} \Upsilon_1(P) = \frac{2W(P_{x_n}, P_{x_k})}{\phi(P_{x_n}) \prod_{i=1}^{g}(x_n - u_i)}, \qquad \operatorname*{res}_{P=P_{u_j}} \Upsilon_1(P) = \frac{2W(P_{u_j}, P_{x_k})}{\phi(P_{u_j})(u_j - x_n) \prod_{\substack{i=1 \\ i \neq j}}^{g}(u_j - u_i)}.$$

Let us compute the residue at P_{x_k}, making use of expression (5.7) for $W(P, P_{x_k})$ and the standard local parameter $\zeta_{x_k}(P) = \sqrt{\lambda(P) - x_k}$ at P_{x_k}:

$$\operatorname*{res}_{P=P_{x_k}} \Upsilon_1(P) = \operatorname*{res}_{P=P_{x_k}} \frac{2\zeta_{x_k}(P) d\zeta_{x_k}(P) \left(\frac{\phi(P)}{\phi(P_{x_k})\zeta_{x_k}^2(P)} + \sum_{i=1}^{g} I_i^{x_k} v_i(P) \right)}{\phi(P)(\lambda(P) - x_n) \prod_{i=1}^{g}(\lambda - u_i)} = \frac{2}{\phi(P_{x_k})(x_k - x_n) \prod_{i=1}^{g}(x_k - u_i)}.$$

The statement of the lemma is now obtained by equating to zero the sum of residues using expression (5.6) for v_j. $\square$

Lemma 3 *Assuming notation of Lemma 2, let m be fixed, $1 \leqslant m \leqslant g$. Let $I_m^{u_m}$ be the normalization constant from* (5.8)*. Then the following identity holds*

$$\sum_{\substack{j=1, \\ j \neq m}}^{g} W(P_{u_j}, P_{u_m}) v_j(P_{x_n}) = W(P_{x_n}, P_{u_m}) - \frac{v_m(P_{x_n})}{x_n - u_m} + v_m(P_{x_n}) \sum_{\substack{i=1, \\ i \neq m}}^{g} \frac{1}{u_m - u_i} - v_m(P_{x_n}) I_m^{u_m}\,. \tag{5.13}$$

Proof. The following differential on the compact surface $\mathcal{C}_{\mathbf{x}}$ has simple poles at P_{x_n}, P_{u_i} for $i \neq m$, a pole of order three at P_{u_m} and no other singularities:

$$\Upsilon_2(P) = \frac{W(P, P_{u_m}) d\lambda(P)}{\phi(P)(\lambda(P) - x_n) \prod_{i=1}^{g}(\lambda(P) - u_i)}.$$

The statement of the lemma is obtained by equating to zero the sum of the residues of Υ_2. Let us compute the residue at P_{u_m}. Using (5.8) for $W(P, P_{u_m})$ and the local parameter $\zeta_{u_m}(P) = \sqrt{\lambda(P) - u_m}$, we have

$$\begin{aligned}\operatorname*{res}_{P=P_{u_m}} \Upsilon_2(P) &= \operatorname*{res}_{P=P_{u_m}} \frac{2d\zeta_{u_m}(P)\left(\frac{\phi(P)}{\phi(P_{u_m})\zeta_{u_m}^2(P)} + \sum_{i=1}^{g} I_i^{u_m} v_i(P)\right)}{\zeta_{u_m}(P)\phi(P)(\lambda(P) - x_n)\prod_{\substack{i=1\\ i\neq m}}^{g}(\lambda(P) - u_i)} \\ &= -\frac{2}{\phi(P_{u_m})(u_m - x_n)\prod_{\substack{i=1\\ i\neq m}}^{g}(u_m - u_i)}\left(\frac{1}{u_m - x_n} + \sum_{\substack{i=1\\ i\neq m}}^{g}\frac{1}{u_m - u_i} - I_m^{u_m}\right).\end{aligned}$$

It remains to compute residues at the simple poles and write the sum of residues using expression (5.6) for $v_j(P_{x_n})$. □

Lemma 4 *Assuming notation of Lemma 2, let $I_j^{x_k}$ for $j = 1, \ldots, g$ be the normalization constants from* (5.7). *For any integer k, $1 \leqslant k \leqslant g$, the following identity holds:*

$$\sum_{j=1}^{g} W(P_{u_j}, P_{x_k}) v_j(P_{x_k}) = \sum_{j=1}^{g} I_j^{x_k} v_j(P_{x_k}) - \sum_{j=1}^{g} \frac{1}{x_k - u_j}. \tag{5.14}$$

Proof. This lemma is proved looking at the sum of residues of the following differential on the compact surface $\mathcal{C}_{\mathbf{x}}$:

$$\Upsilon_3(P) = \frac{W(P, P_{x_k}) d\lambda(P)}{\phi(P)(\lambda(P) - x_k) \prod_{i=1}^{g}(\lambda(P) - u_i)}.$$

This differential has simple poles at P_{u_i} for $i = 1, \ldots, g$, a pole of order three at P_{x_k} and no other singularities. Its residue at P_{x_k} is obtained writing expression (5.7) for $W(P, P_{x_k})$ and using $\zeta_{x_k} = \sqrt{\lambda - x_k}$ as a local parameter:

$$\operatorname*{res}_{P=P_{x_k}} \Upsilon_3(P) = -\frac{2}{\phi(P_{x_k})\prod_{i=1}^{g}(x_k - u_i)} \sum_{i=1}^{g} \frac{1}{x_k - u_i} + \frac{2\sum_{j=1}^{g} I_j^{x_k} v_j(P_{x_k})}{\phi(P_{x_k})\prod_{i=1}^{g}(x_k - u_i)}.$$

Using again expression (5.6) for v_j, we see that the vanishing of the sum of residues of Υ_3 is equivalent to the equality claimed in the lemma. □

Lemma 5 *Let $1 \leqslant m \leqslant g$. Let $I_m^{u_m}$ be the normalization constant from* (5.8). *Assuming notation of Lemma 2, with P_0 being the ramification point of $\lambda : \mathcal{C}_{\mathbf{x}} \to \mathbb{C}P^1$ such that $\lambda(P_0) = 0$, and Ω being the* *differential* (2.12) *evaluated at ramification points according to* (2.9)*, we have the following identity:*

$$\frac{W(P_{u_m},P_0)\Omega(P_0)}{\Omega(P_{u_m})}+\sum_{i=1}^{g}\frac{W(P_{u_m},P_{x_i})\Omega(P_{x_i})}{\Omega(P_{u_m})}+\sum_{\substack{j=1\\ j\neq m}}^{g}\frac{W(P_{u_m},P_{u_j})\Omega(P_{u_j})}{\Omega(P_{u_m})}$$
$$=\left(\sum_{i=1}^{g}\frac{\partial u_m}{\partial x_i}-1\right)\left(\frac{\prod_{\substack{i=1\\ i\neq m}}^{g}(u_m-u_i)}{\prod_{i=1}^{g}(-u_i)}+\sum_{\substack{j=1\\ j\neq m}}^{g}\frac{u_m\prod_{i\neq m,j}(u_m-u_i)}{u_j\prod_{i\neq j}(u_j-u_i)}\right)$$
$$-\sum_{j=1}^{g}\frac{\prod_{\substack{i=1\\ i\neq m}}(u_m-u_i)}{\prod_{i=1}^{g}(x_j-u_i)}\frac{\partial u_m}{\partial x_j}-\sum_{\substack{j=1\\ j\neq m}}^{g}\sum_{i=1}^{g}\frac{(x_i-u_m)\prod_{s\neq m,j}(u_m-u_s)}{(x_i-u_j)\prod_{s\neq j}(u_j-u_s)}\frac{\partial u_m}{\partial x_i}-I_m^{u_m}. \quad (5.15)$$

Proof. Let us denote by S the quantity from the lemma which we seek to express:

$$S:=\frac{W(P_{u_m},P_0)\Omega(P_0)}{\Omega(P_{u_m})}+\sum_{i=1}^{g}\frac{W(P_{u_m},P_{x_i})\Omega(P_{x_i})}{\Omega(P_{u_m})}+\sum_{\substack{j=1\\ j\neq m}}^{g}\frac{W(P_{u_m},P_{u_j})\Omega(P_{u_j})}{\Omega(P_{u_m})}$$

and note that $S\Omega(P_{u_m})$ can be obtained through the sum of residues of the differential $\frac{\Omega(P)W(P,P_{u_m})}{d\lambda(P)}$ on a compact Riemann surface from the family (2.1):

$$0=\sum_{P\in\mathcal{C}_{\mathbf{x}}}\operatorname*{res}_{P}\frac{\Omega(P)W(P,P_{u_m})}{d\lambda(P)}=\frac{1}{2}S\Omega(P_{u_m})+\operatorname*{res}_{P=P_{u_m}}\frac{\Omega(P)W(P,P_{u_m})}{d\lambda(P)}.$$

Now rewriting $W(P,P_{u_m})$ as in (5.8), we have for the last residue:

$$\operatorname*{res}_{P=P_{u_m}}\frac{\Omega(P)W(P,P_{u_m})}{d\lambda(P)}=\operatorname*{res}_{P=P_{u_m}}\frac{\Omega(P)\phi(P)}{\phi(P_{u_m})(\lambda(P)-u_m)d\lambda(P)}+\frac{1}{2}\Omega(P_{u_m})I_m^{u_m}.$$

The sum of residues of the differential $\frac{\Omega(P)\phi(P)}{(\lambda(P)-u_m)d\lambda(P)}$ vanishes and thus we have

$$0=\sum_{P\in\mathcal{C}_{\mathbf{x}}}\operatorname*{res}_{P}\frac{\Omega(P)\phi(P)}{(\lambda(P)-u_m)d\lambda(P)}=\operatorname*{res}_{P=P_{u_m}}\frac{\Omega(P)\phi(P)}{(\lambda(P)-u_m)d\lambda(P)}$$
$$+\frac{1}{2}\left(\frac{\Omega(P_0)\phi(P_0)}{-u_m}+\sum_{j=1}^{g}\frac{\Omega(P_{x_j})\phi(P_{x_j})}{x_j-u_m}+\sum_{\substack{j=1\\ j\neq m}}^{g}\frac{\Omega(P_{u_j})\phi(P_{u_j})}{u_j-u_m}\right).$$

Putting these results together, we obtain

$$S=\frac{1}{\Omega(P_{u_m})\phi(P_{u_m})}\left(\frac{\Omega(P_0)\phi(P_0)}{-u_m}+\sum_{j=1}^{g}\frac{\Omega(P_{x_j})\phi(P_{x_j})}{x_j-u_m}+\sum_{\substack{j=1\\ j\neq m}}^{g}\frac{\Omega(P_{u_j})\phi(P_{u_j})}{u_j-u_m}\right)-I_m^{u_m}.$$

Let us rewrite the terms with $\Omega(P_{x_i})$ and $\Omega(P_0)$ with the help of differentials v_j making use of their expression from (5.6). In the last term, we express $\Omega(P_{u_j})$ as in (5.4) using j instead of m in (5.4). This, together with expression (5.3) for derivatives $\frac{\partial u_m}{\partial x_j}$, yields

$$S = \frac{\Omega(P_0)v_m(P_0)}{\Omega(P_{u_m})}\frac{\prod_{\substack{i=1\\i\neq m}}^g(u_m-u_i)}{\prod_{i=1}^g(-u_i)} - \prod_{\substack{i=1\\i\neq m}}(u_m-u_i)\sum_{j=1}^g\frac{1}{\prod_{i=1}^g(x_j-u_i)}\frac{\partial u_m}{\partial x_j}$$
$$-\frac{1}{\phi(P_{u_m})\Omega(P_{u_m})}\sum_{\substack{j=1\\j\neq m}}^g\frac{\phi(P_{u_j})(\Omega(P_0)v_j(P_0)+\sum_{i=1}^g\Omega(P_{x_i})v_j(P_{x_i}))}{u_j-u_m} - I_m^{u_m}.$$

Rewriting the expressions in terms of differentials v_j (5.6) once again and expressing $\frac{\Omega(P_0)v_m(P_0)}{\Omega(P_{u_m})}$ via derivatives of u_m due to (5.5), we prove the lemma. □

Lemma 6 *Let m and k be fixed integers, $1\leqslant m\leqslant g$ and let $I_j^{x_k}$ for $j=1,\dots,g$ be the normalization constants from (5.7). Assuming notation of Lemma 5, we have the following identity:*

$$\frac{\partial u_m}{\partial x_k}\left(\frac{W(P_{x_k},P_0)\Omega(P_0)}{\Omega(P_{x_k})}+\sum_{\substack{j=1\\j\neq k}}^g\frac{W(P_{x_k},P_{x_j})\Omega(P_{x_j})}{\Omega(P_{x_k})}+\sum_{j=1}^g\frac{W(P_{x_k},P_{u_j})\Omega(P_{u_j})}{\Omega(P_{x_k})}\right)=$$
$$\left(\sum_{i=1}^g\frac{\partial u_m}{\partial x_i}-1\right)\left(\frac{\prod_{i\neq m}(x_k-u_i)}{x_k\prod_{i\neq m}(-u_i)}-\sum_{j=1}^g\frac{u_m}{u_j}\frac{\prod_{i\neq m,j}(x_k-u_i)}{\prod_{i\neq j}(u_j-u_i)}\right)-\frac{\partial u_m}{\partial x_k}\sum_{j=1}^g I_j^{x_k}v_j(P_{x_k})$$
$$+\sum_{\substack{j=1\\j\neq k}}^g\frac{\prod_{i\neq m}(x_k-u_i)}{\prod_{i\neq m}(x_j-u_i)}\frac{1}{(x_j-x_k)}\frac{\partial u_m}{\partial x_j}-\sum_{i,j=1}^g\frac{\partial u_m}{\partial x_i}\frac{x_i-u_m}{(u_j-x_k)(x_i-u_j)}\frac{\prod_{s\neq m}(x_k-u_s)}{\prod_{s\neq j}(u_j-u_s)}. \quad (5.16)$$

Proof. Let us denote by T the quantity in the parentheses in the left hand side:

$$T:=\frac{W(P_{x_k},P_0)\Omega(P_0)}{\Omega(P_{x_k})}+\sum_{\substack{j=1\\j\neq k}}^g\frac{W(P_{x_k},P_{x_j})\Omega(P_{x_j})}{\Omega(P_{x_k})}+\sum_{j=1}^g\frac{W(P_{x_k},P_{u_j})\Omega(P_{u_j})}{\Omega(P_{x_k})}.$$

Then note that $T\Omega(P_{x_k})$ can be expressed through the sum of residues of the differential $\frac{\Omega(P)W(P,P_{x_k})}{d\lambda(P)}$ on a compact Riemann surface corresponding to a curve from the family (2.1):

$$0=\sum_{P\in\mathcal{C}_{\mathbf{x}}}\operatorname*{res}_P\frac{\Omega(P)W(P,P_{x_k})}{d\lambda(P)}=\frac{1}{2}T\Omega(P_{x_k})+\operatorname*{res}_{P=P_{x_k}}\frac{\Omega(P)W(P,P_{x_k})}{d\lambda(P)}.$$

Representing $W(P,P_{x_k})$ in coordinates as in (5.7), we have

$$\operatorname*{res}_{P=P_{x_k}}\frac{\Omega(P)W(P,P_{x_k})}{d\lambda(P)}=\frac{1}{\phi(P_{x_k})}\operatorname*{res}_{P=P_{x_k}}\frac{\Omega(P)\phi(P)}{(\lambda(P)-x_k)d\lambda(P)}+\frac{1}{2}\Omega(P_{x_k})\sum_{j=1}^g I_j^{x_k}v_j(P_{x_k}).$$

The last residue can be obtained through the sum of residues of the differential $\frac{\Omega(P)\phi(P)}{(\lambda(P)-x_k)d\lambda(P)}$ vanishing on a compact Riemann surface. This differential has simple poles at all finite ramification points except P_{x_k} where it has a pole of order three. Thus we have

$$0 = \sum_{P\in\mathcal{C}_{\mathbf{x}}} \operatorname*{res}_{P} \frac{\Omega(P)\phi(P)}{(\lambda(P)-x_k)d\lambda(P)} = \operatorname*{res}_{P=P_{x_k}} \frac{\Omega(P)\phi(P)}{(\lambda(P)-x_k)d\lambda(P)} + \frac{1}{2}\frac{\Omega(P_0)\phi(P_0)}{-x_k}$$
$$+ \frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{g} \frac{\Omega(P_{x_j})\phi(P_{x_j})}{x_j - x_k} + \frac{1}{2}\sum_{j=1}^{g} \frac{\Omega(P_{u_j})\phi(P_{u_j})}{u_j - x_k}.$$

Putting these results together, we obtain

$$T = -\frac{\Omega(P_0)\phi(P_0)}{x_k\Omega(P_{x_k})\phi(P_{x_k})} + \sum_{\substack{j=1\\ j\neq k}}^{g} \frac{\Omega(P_{x_j})\phi(P_{x_j})}{\Omega(P_{x_k})\phi(P_{x_k})(x_j-x_k)} + \sum_{j=1}^{g} \frac{\Omega(P_{u_j})\phi(P_{u_j})}{\Omega(P_{x_k})\phi(P_{x_k})(u_j-x_k)} - \sum_{j=1}^{g} I_j^{x_k} v_j(P_{x_k}).$$

Let us now multiply T by $\frac{\partial u_m}{\partial x_k} = -\frac{v_m(P_{x_k})\Omega(P_{x_k})}{\Omega(P_{u_m})}$ to express the left hand side of the equation of the lemma. This leads to derivatives $\frac{\partial u_m}{\partial x_j}$ appear in one of the sums in the above expression for T. At the same time, we replace $\Omega(P_{u_j})$ from (5.4) with $m = j$. This yields

$$T\frac{\partial u_m}{\partial x_k} = \frac{v_m(P_{x_k})}{x_k\phi(P_{x_k})}\frac{\Omega(P_0)\phi(P_0)}{\Omega(P_{u_m})} + \sum_{\substack{j=1\\ j\neq k}}^{g} \frac{v_m(P_{x_k})}{\phi(P_{x_k})(x_j-x_k)}\frac{\phi(P_{x_j})}{v_m(P_{x_j})}\frac{\partial u_m}{\partial x_j}$$
$$+ \sum_{j=1}^{g} \frac{v_m(P_{x_k})\phi(P_{u_j})}{\phi(P_{x_k})} \frac{(\Omega(P_0)v_j(P_0) + \sum_{i=1}^{g}\Omega(P_{x_i})v_j(P_{x_i}))}{\Omega(P_{u_m})(u_j-x_k)} - \frac{\partial u_m}{\partial x_k}\sum_{j=1}^{g} I_j^{x_k} v_j(P_{x_k}).$$

Using explicit expressions (5.6) for differentials v_j, grouping the terms with $\frac{\Omega(P_0)v_m(P_0)}{\Omega(P_{u_m})}$ and replacing this quantity by $\sum_{i=1}^{g}\frac{\partial u_m}{\partial x_i} - 1$ due to (5.5), we prove the lemma. $\square$

5.3 System of equations for the branch points under isoperiodic deformations

Now we are in position to derive the system of equations for the functions $u_1, \ldots, u_g$ depending on the variables $x_1, \ldots, x_g$ which needs to be satisfied to provide a deformation for a Hill curve.

Theorem 5 *Let $\mathcal{S}_{\mathbf{x}}$ be a connected domain as discussed in Section 3.1, that is such that for $\mathbf{x} = (x_1, \ldots, x_g) \in \mathcal{S}_{\mathbf{x}}$ one can choose a canonical homology basis $\{a_j, b_j\}_{j=1}^{g}$ on all Riemann surfaces corresponding to the curves from the family (3.3) in a consistent way meaning that $\lambda(a_j)$ and $\lambda(b_j)$ are constant for all j and all $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$. Let $u_1(\mathbf{x}), \ldots, u_g(\mathbf{x})$ with $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$ be such that the compact Riemann surfaces associated with curves of subfamily*

$$\mathcal{C}_{\mathbf{x}} = \{(\lambda,\mu) \in \mathbb{C}^2 \mid \mu^2 = \lambda\prod_{j=1}^{g}(\lambda - u_j(\mathbf{x}))\prod_{j=1}^{g}(\lambda - x_j)\} \tag{5.17}$$

of the family (2.1) are such that the differential of the second kind Ω defined on each of these surfaces by (2.12) has a- and b-periods that are constant for all $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$. Then the functions $u_1(\mathbf{x}), \ldots, u_g(\mathbf{x})$ satisfy the following system of equations. For $1 \leqslant k, m, n \leqslant g$ and $n \neq k$

$$
\begin{aligned}
\frac{\partial^2 u_m}{\partial x_k \partial x_n} &= \frac{1}{2}\frac{\partial u_m}{\partial x_k}\left(\frac{1}{x_k - x_n} + \frac{1}{x_n - u_m}\right) + \frac{1}{2}\frac{\partial u_m}{\partial x_n}\left(\frac{1}{x_n - x_k} + \frac{1}{x_k - u_m}\right) \\
&\quad + \frac{1}{2}\frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_n}\left(\frac{1}{u_m} + \sum_{\substack{i=1\\ i\neq k,n}}^{g}\frac{1}{u_m - x_i} - \sum_{\substack{i=1\\ i\neq m}}^{g}\frac{2}{u_m - u_i}\right) \\
&\quad + \frac{1}{4}\frac{\partial u_m}{\partial x_k}\sum_{\substack{j=1\\ j\neq m}}^{g}\left(\frac{1}{u_m - u_j} - \frac{1}{x_k - u_j}\right)\frac{\partial u_j}{\partial x_n} + \frac{1}{4}\frac{\partial u_m}{\partial x_n}\sum_{\substack{j=1\\ j\neq m}}^{g}\left(\frac{1}{u_m - u_j} - \frac{1}{x_n - u_j}\right)\frac{\partial u_j}{\partial x_k} \\
&\quad - \frac{1}{2}\frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_n}\left(\sum_{i=1}^{g}\frac{\partial u_m}{\partial x_i} - 1\right)\prod_{\substack{i=1\\ i\neq m}}^{g}\frac{u_i - u_m}{u_i}\left(\frac{1}{u_m} - \sum_{\substack{j=1\\ j\neq m}}^{g}\frac{1}{u_m - u_j}\prod_{\substack{s=1\\ s\neq j}}^{g}\frac{u_s}{u_s - u_j}\right) \\
&\quad - \frac{1}{2}\frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_n}\left(\sum_{j=1}^{g}\frac{1}{x_j - u_m}\prod_{\substack{i=1\\ i\neq m}}^{g}\frac{u_m - u_i}{x_j - u_i}\frac{\partial u_m}{\partial x_j} + \sum_{j=1}^{g}\sum_{\substack{i=1\\ i\neq m}}^{g}\frac{x_j - u_m}{(x_j - u_i)(u_m - u_i)}\frac{\prod_{s\neq m}(u_m - u_s)}{\prod_{s\neq i}(u_i - u_s)}\frac{\partial u_m}{\partial x_j}\right)
\end{aligned} \tag{5.18}
$$

and

$$
\begin{aligned}
\frac{\partial^2 u_m}{\partial x_k^2} &= \frac{1}{2}\frac{\partial u_m}{\partial x_k}\left(-\frac{1}{x_k} - \sum_{\substack{j=1\\ j\neq k}}^{g}\frac{1}{x_k - x_j} + \sum_{\substack{j=1\\ j\neq m}}^{g}\frac{2}{x_k - u_j} + \frac{1}{x_k - u_m}\right) \\
&\quad + \frac{1}{2}\left(\frac{\partial u_m}{\partial x_k}\right)^2\left(\frac{1}{u_m} + \sum_{\substack{j=1\\ j\neq k}}^{g}\frac{1}{u_m - x_j} - \sum_{\substack{j=1\\ j\neq m}}^{g}\frac{2}{u_m - u_j} + \frac{1}{x_k - u_m}\right) \\
&\quad + \frac{1}{2}\frac{\partial u_m}{\partial x_k}\sum_{\substack{j=1\\ j\neq m}}^{g}\left(\frac{1}{u_m - u_j} - \frac{1}{x_k - u_j}\right)\frac{\partial u_j}{\partial x_k} \\
&\quad - \frac{1}{2}\left(\sum_{i=1}^{g}\frac{\partial u_m}{\partial x_i} - 1\right)\prod_{\substack{i=1\\ i\neq m}}^{g}\frac{u_i - x_k}{u_i}\left(\frac{1}{x_k} - \sum_{j=1}^{g}\frac{1}{x_k - u_j}\prod_{\substack{s=1\\ s\neq j}}^{g}\frac{u_s}{u_s - u_j}\right) \\
&\quad - \frac{1}{2}\left(\frac{\partial u_m}{\partial x_k}\right)^2\left(\sum_{i=1}^{g}\frac{\partial u_m}{\partial x_i} - 1\right)\prod_{\substack{i=1\\ i\neq m}}^{g}\frac{u_i - u_m}{u_i}\left(\frac{1}{u_m} - \sum_{\substack{j=1\\ j\neq m}}^{g}\frac{1}{u_m - u_j}\prod_{\substack{s=1\\ s\neq j}}^{g}\frac{u_s}{u_s - u_j}\right) \\
&\quad - \frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{g}\frac{1}{x_j - x_k}\prod_{\substack{i=1\\ i\neq m}}^{g}\frac{x_k - u_i}{x_j - u_i}\frac{\partial u_m}{\partial x_j} - \frac{1}{2}\sum_{i,j=1}^{g}\frac{x_j - u_m}{(x_j - u_i)(x_k - u_m)}\prod_{\substack{s=1\\ s\neq i}}^{g}\frac{x_k - u_s}{u_i - u_s}\frac{\partial u_m}{\partial x_j}
\end{aligned}
$$

$$-\frac{1}{2}\left(\frac{\partial u_m}{\partial x_k}\right)^2\left(\sum_{j=1}^{g}\frac{1}{x_j-u_m}\prod_{\substack{i=1\\i\neq m}}^{g}\frac{u_m-u_i}{x_j-u_i}\frac{\partial u_m}{\partial x_j}+\sum_{j=1}^{g}\sum_{\substack{i=1\\i\neq m}}^{g}\frac{x_j-u_m}{(x_j-u_i)(u_m-u_i)}\frac{\prod_{s\neq m}(u_m-u_s)}{\prod_{s\neq i}(u_i-u_s)}\frac{\partial u_m}{\partial x_j}\right). \tag{5.19}$$

Proof. We subdivide the proof into two parts, proving the two equations claimed in the theorem in the two parts, respectively.

Part 1: proof of equation (5.18)**.** Let us differentiate expression (5.3) for $\frac{\partial u_m}{\partial x_k}$ with respect to x_n with $n\neq k$:

$$\frac{\partial^2 u_m}{\partial x_k\partial x_n}=\frac{\partial}{\partial x_n}\left\{-\frac{v_m(P_{x_k})\Omega(P_{x_k})}{\Omega(P_{u_m})}\right\}=\frac{\partial u_m}{\partial x_k}\left(\frac{\frac{\partial v_m(P_{x_k})}{\partial x_n}}{v_m(P_{x_k})}+\frac{\frac{\partial\Omega(P_{x_k})}{\partial x_n}}{\Omega(P_{x_k})}-\frac{\frac{\partial\Omega(P_{u_m})}{\partial x_n}}{\Omega(P_{u_m})}\right). \tag{5.20}$$

Let us write these three terms one by one. Note that derivatives with respect to x_j contain derivatives with respect to the dependent branch points u_m. Differentiation of $v_m(P_{x_k})$ can be done using explicit expressions (5.6) and (2.4), (2.10):

$$\frac{1}{v_m(P_{x_k})}\frac{\partial v_m(P_{x_k})}{\partial x_n}=\frac{1}{2}\left(\frac{1}{x_k-x_n}-\frac{1}{u_m-x_n}-\sum_{\substack{j=1\\j\neq m}}^{g}\frac{1}{x_k-u_j}\frac{\partial u_j}{\partial x_n}\right.$$
$$\left.+\left(\frac{1}{u_m}+\sum_{\substack{i=1\\i\neq k}}^{g}\frac{1}{u_m-x_i}\right)\frac{\partial u_m}{\partial x_n}-\sum_{\substack{j=1\\j\neq m}}^{g}\frac{1}{u_m-u_j}\left(\frac{\partial u_m}{\partial x_n}-\frac{\partial u_j}{\partial x_n}\right)\right). \tag{5.21}$$

The Rauch formulas (2.16) and the chain rule, imply the following expression for the derivative of $\Omega(P_{x_k})$:

$$\frac{1}{\Omega(P_{x_k})}\frac{\partial\Omega(P_{x_k})}{\partial x_n}=\frac{W(P_{x_k},P_{x_n})\Omega(P_{x_n})}{2\Omega(P_{x_k})}+\sum_{j=1}^{g}\frac{W(P_{x_k},P_{u_j})\Omega(P_{u_j})}{2\Omega(P_{x_k})}\frac{\partial u_j}{\partial x_n}$$
$$=\frac{\Omega(P_{x_n})}{2\Omega(P_{x_k})}\left(W(P_{x_k},P_{x_n})-\sum_{j=1}^{g}W(P_{x_k},P_{u_j})v_j(P_{x_n})\right),$$

where in the second equality we rewrote the derivative $\frac{\partial u_j}{\partial x_n}$ according to (5.3). Applying now Lemma 3, we obtain

$$\frac{1}{\Omega(P_{x_k})}\frac{\partial\Omega(P_{x_k})}{\partial x_n}=\frac{\Omega(P_{x_n})}{2\Omega(P_{x_k})}\frac{\phi(P_{x_n})}{\phi(P_{x_k})}\frac{1}{x_n-x_k}\frac{\prod_{i=1}^{g}(x_n-u_i)}{\prod_{i=1}^{g}(x_k-u_i)},$$

and thus, using again (5.3) and (5.6), we have

$$\frac{\partial u_m}{\partial x_k}\frac{\frac{\partial\Omega(P_{x_k})}{\partial x_n}}{\Omega(P_{x_k})}=-\frac{\Omega(P_{x_n})v_m(P_{x_n})}{2\Omega(P_{u_m})}\frac{x_n-u_m}{(x_n-x_k)(x_k-u_m)}=\frac{1}{2}\frac{\partial u_m}{\partial x_n}\left(\frac{1}{x_n-x_k}+\frac{1}{x_k-u_m}\right). \tag{5.22}$$

To differentiate $\Omega(P_{u_m})$ we Corollary 1 with $\{\lambda_1,\dots,\lambda_{2g+1}\}=\{0,x_1,\dots,x_g,u_1,\dots,u_g\}$ to obtain the derivative of $\Omega(P_{u_m})$ with respect to u_m :

$$\begin{aligned}\frac{1}{\Omega(P_{u_m})}\frac{\partial\Omega(P_{u_m})}{\partial x_n}&=\frac{1}{\Omega(P_{u_m})}\left(\frac{\partial^{\mathrm{Rauch}}\Omega(P_{u_m})}{\partial x_n}+\sum_{\substack{j=1\\j\neq m}}^{g}\frac{\partial^{\mathrm{Rauch}}\Omega(P_{u_m})}{\partial u_j}\frac{\partial u_j}{\partial x_n}+\frac{\partial^{\mathrm{Rauch}}\Omega(P_{u_m})}{\partial u_m}\frac{\partial u_m}{\partial x_n}\right)\\&=\frac{W(P_{u_m},P_{x_n})\Omega(P_{x_n})}{2\Omega(P_{u_m})}+\frac{1}{2}\sum_{\substack{j=1\\j\neq m}}^{g}\frac{W(P_{u_m},P_{u_j})\Omega(P_{u_j})}{\Omega(P_{u_m})}\frac{\partial u_j}{\partial x_n}\\&-\frac{1}{2}\left(\frac{W(P_{u_m},P_0)\Omega(P_0)}{\Omega(P_{u_m})}+\sum_{i=1}^{g}\frac{W(P_{u_m},P_{x_i})\Omega(P_{x_i})}{\Omega(P_{u_m})}+\sum_{\substack{j=1\\j\neq m}}^{g}\frac{W(P_{u_m},P_{u_j})\Omega(P_{u_j})}{\Omega(P_{u_m})}\right)\frac{\partial u_m}{\partial x_n}.\end{aligned}$$

After rewriting derivative $\frac{\partial u_j}{\partial x_n}$ in terms of Ω and v_j as in (5.3), we can apply Lemma 3 in the first line. In the second line, Lemma 5 can be applied. This yields

$$\begin{aligned}\frac{1}{\Omega(P_{u_m})}\frac{\partial\Omega(P_{u_m})}{\partial x_n}&=\frac{\Omega(P_{x_n})v_m(P_{x_n})}{2\Omega(P_{u_m})}\left(\frac{1}{x_n-u_m}-\sum_{\substack{i=1\\i\neq m}}^{g}\frac{1}{u_m-u_i}+I_m^{u_m}\right)\\&-\frac{1}{2}\left(\sum_{i=1}^{g}\frac{\partial u_m}{\partial x_i}-1\right)\left(\frac{\prod_{\substack{i=1\\i\neq m}}^{g}(u_m-u_i)}{\prod_{i=1}^{g}(-u_i)}+\sum_{\substack{j=1\\j\neq m}}^{g}\frac{u_m\prod_{i\neq m,j}(u_m-u_i)}{u_j\prod_{i\neq j}(u_j-u_i)}\right)\frac{\partial u_m}{\partial x_n}\\&+\frac{1}{2}\left(\sum_{j=1}^{g}\frac{\prod_{\substack{i=1\\i\neq m}}(u_m-u_i)}{\prod_{i=1}^{g}(x_j-u_i)}\frac{\partial u_m}{\partial x_j}+\sum_{\substack{j=1\\j\neq m}}^{g}\sum_{i=1}^{g}\frac{(x_i-u_m)\prod_{s\neq m,j}(u_m-u_s)}{(x_i-u_j)\prod_{s\neq j}(u_j-u_s)}\frac{\partial u_m}{\partial x_i}+I_m^{u_m}\right)\frac{\partial u_m}{\partial x_n}.\end{aligned}\tag{5.23}$$

Note that the terms containing $I_m^{u_m}$ cancel out due to (5.3), and the remaining terms are rational in u_j and their derivatives, also due to (5.3) as for the very first factor in the right hand side we have $\frac{\Omega(P_{x_n})v_m(P_{x_n})}{\Omega(P_{u_m})}=-\frac{\partial u_m}{\partial x_n}$.

Let us now use (5.21), (5.22), (5.23) in expression (5.20) for the derivative $\frac{\partial^2 u_m}{\partial x_k\partial x_n}$. This yields a right hand side rational in u_j and their derivatives. To obtain equation (5.18) claimed in the theorem, it remains to symmetrize the following term. We show that this term is symmetric with respect to exchanging k and n by rewriting the derivatives in terms of Ω and v_m as in (5.3):

$$\begin{aligned}\frac{\partial u_m}{\partial x_k}\sum_{\substack{j=1\\j\neq m}}^{g}\left(\frac{1}{u_m-u_j}-\frac{1}{x_k-u_j}\right)\frac{\partial u_j}{\partial x_n}&=\frac{\Omega(P_{x_k})\Omega(P_{x_n})}{\Omega(P_{u_m})}\sum_{\substack{j=1\\j\neq m}}^{g}\left(\frac{1}{u_m-u_j}-\frac{1}{x_k-u_j}\right)\frac{v_j(P_{x_n})v_m(P_{x_k})}{\Omega(P_{u_j})}\\=\frac{\Omega(P_{x_k})\Omega(P_{x_n})}{\Omega(P_{u_m})\phi(P_{u_m})}\sum_{\substack{j=1\\j\neq m}}^{g}&\frac{\phi(P_{x_k})\phi(P_{x_n})\prod_{s=1}^{g}(x_k-u_s)\prod_{i=1}^{g}(x_n-u_i)}{\Omega(P_{u_j})\phi(P_{u_j})(u_m-u_j)(x_n-u_j)(x_k-u_j)\prod_{i\neq j}(u_j-u_i)\prod_{s\neq m}(u_m-u_s)}.\end{aligned}$$

The last expression being symmetric in n and k, we may symmetrize the left hand side and write

$$\frac{1}{2}\frac{\partial u_m}{\partial x_k}\sum_{\substack{j=1\\ j\neq m}}^{g}\left(\frac{1}{u_m-u_j}-\frac{1}{x_k-u_j}\right)\frac{\partial u_j}{\partial x_n}$$
$$=\frac{1}{4}\frac{\partial u_m}{\partial x_k}\sum_{\substack{j=1\\ j\neq m}}^{g}\left(\frac{1}{u_m-u_j}-\frac{1}{x_k-u_j}\right)\frac{\partial u_j}{\partial x_n}+\frac{1}{4}\frac{\partial u_m}{\partial x_n}\sum_{\substack{j=1\\ j\neq m}}^{g}\left(\frac{1}{u_m-u_j}-\frac{1}{x_n-u_j}\right)\frac{\partial u_j}{\partial x_k}.$$

This explains the third line in (5.18) and finishes the proof of the first part of the theorem.

Part 2: proof of equation (5.19)**.** Now we proceed to differentiation of expression (5.3) for $\frac{\partial u_m}{\partial x_k}$ second time with respect to the same variable:

$$\frac{\partial^2 u_m}{\partial x_k^2}=\frac{\partial}{\partial x_k}\left\{-\frac{v_m(P_{x_k})\Omega(P_{x_k})}{\Omega(P_{u_m})}\right\}=\frac{\partial u_m}{\partial x_k}\left(\frac{\frac{\partial v_m(P_{x_k})}{\partial x_k}}{v_m(P_{x_k})}+\frac{\frac{\partial\Omega(P_{x_k})}{\partial x_k}}{\Omega(P_{x_k})}-\frac{\frac{\partial\Omega(P_{u_m})}{\partial x_k}}{\Omega(P_{u_m})}\right). \tag{5.24}$$

As in Part 1, the derivative of $v_m(P_{x_k})$ is obtained by differentiating the evaluation at P_{x_k} of explicit expression (5.6) for v_m using also expressions (2.10) for ϕ evaluated at ramification points:

$$\frac{1}{v_m(P_{x_k})}\frac{\partial v_m(P_{x_k})}{\partial x_k}=\frac{1}{v_m(P_{x_k})}\frac{\partial}{\partial x_k}\left\{\frac{\phi(P_{x_k})\prod_{i\neq m}(x_k-u_i)}{\phi(P_{u_m})\prod_{i\neq m}(u_m-u_i)}\right\}=\frac{1}{2}\left(\frac{1}{u_m}\frac{\partial u_m}{\partial x_k}+\sum_{\substack{j=1\\ j\neq k}}^{g}\frac{1}{u_m-x_j}\frac{\partial u_m}{\partial x_k}\right.$$
$$\left.-\frac{1}{x_k}-\sum_{\substack{j=1\\ j\neq k}}^{g}\frac{1}{x_k-x_j}+\sum_{\substack{j=1\\ j\neq m}}^{g}\frac{1}{x_k-u_j}\left(1-\frac{\partial u_j}{\partial x_k}\right)-\sum_{\substack{j=1\\ j\neq m}}^{g}\frac{1}{u_m-u_j}\left(\frac{\partial u_m}{\partial x_k}-\frac{\partial u_j}{\partial x_k}\right)\right). \tag{5.25}$$

To obtain the second term in (5.24) with the derivative of $\Omega(P_{x_k})$, we need to use the Rauch formulas (2.16) and Corollary 1 with $\{\lambda_1,\dots,\lambda_{2g+1}\}=\{0,x_1,\dots,x_g,u_1,\dots,u_g\}$ as in (2.1):

$$\frac{1}{\Omega(P_{x_k})}\frac{\partial\Omega(P_{x_k})}{\partial x_k}=\frac{1}{\Omega(P_{x_k})}\left(\frac{\partial^{\mathrm{Rauch}}\Omega(P_{x_k})}{\partial x_k}+\sum_{j=1}^{g}\frac{\partial^{\mathrm{Rauch}}\Omega(P_{x_k})}{\partial u_j}\frac{\partial u_j}{\partial x_k}\right)=-\frac{W(P_{x_k},P_0)\Omega(P_0)}{2\Omega(P_{x_k})}$$
$$-\sum_{\substack{j=1\\ j\neq k}}^{g}\frac{W(P_{x_k},P_{x_j})\Omega(P_{x_j})}{2\Omega(P_{x_k})}-\sum_{j=1}^{g}\frac{W(P_{x_k},P_{u_j})\Omega(P_{u_j})}{2\Omega(P_{x_k})}+\sum_{j=1}^{g}\frac{W(P_{x_k},P_{u_j})\Omega(P_{u_j})}{2\Omega(P_{x_k})}\frac{\partial u_j}{\partial x_k}. \tag{5.26}$$

Let us replace in the last sum the derivative $\frac{\partial u_j}{\partial x_k}$ by its expression (5.3). This allows us to write the last sum as $-\frac{1}{2}\sum_{j=1}^{g}W(P_{x_k},P_{u_j})v_j(P_{x_k})$ and to replace it by the right hand side of (5.14) from Lemma 4. Let us at the same time multiply the whole expression by $\frac{\partial u_m}{\partial x_k}$ as in (5.24) in order to use Lemma 6 for the remaining terms of (5.26). After cancelling the terms containing the normalization constants

$I_j^{x_k}$, this gives

$$\frac{\partial u_m}{\partial x_k}\frac{\frac{\partial \Omega(P_{x_k})}{\partial x_k}}{\Omega(P_{x_k})} = -\frac{1}{2}\left(\sum_{i=1}^{g}\frac{\partial u_m}{\partial x_i}-1\right)\left(\frac{\prod_{i\neq m}(x_k-u_i)}{x_k\prod_{i\neq m}(-u_i)}-\sum_{j=1}^{g}\frac{u_m}{u_j}\frac{\prod_{i\neq m,j}(x_k-u_i)}{\prod_{i\neq j}(u_j-u_i)}\right)$$
$$-\frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{g}\frac{\prod_{i\neq m}(x_k-u_i)}{\prod_{i\neq m}(x_j-u_i)}\frac{1}{(x_j-x_k)}\frac{\partial u_m}{\partial x_j}$$
$$-\frac{1}{2}\sum_{i,j=1}^{g}\frac{x_i-u_m}{x_i-u_j}\frac{\prod_{s\neq m,j}(x_k-u_s)}{\prod_{s\neq j}(u_j-u_s)}\frac{\partial u_m}{\partial x_i}+\frac{1}{2}\frac{\partial u_m}{\partial x_k}\sum_{j=1}^{g}\frac{1}{x_k-u_j}. \quad (5.27)$$

The third term in (5.24), the quantity $\frac{1}{\Omega(P_{u_m})}\frac{\partial\Omega(P_{u_m})}{\partial x_k}$, is already obtained in Part 1: we need to use the result in (5.23) upon replacing the index n by k. It remains now to plug the results (5.25), (5.27) and the appropriately modified (5.23) into (5.24). □

Example 4 *If $g=2$, system* (5.18), (5.19) *from Theorem 5 has the form:*

$$\frac{\partial^2 u_1}{\partial x_1\partial x_2} = \frac{1}{2}\frac{\partial u_1}{\partial x_1}\left(\frac{1}{x_1-x_2}+\frac{1}{x_2-u_1}\right)+\frac{1}{2}\frac{\partial u_1}{\partial x_2}\left(\frac{1}{x_2-x_1}+\frac{1}{x_1-u_1}\right)+\frac{1}{2}\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}\left(\frac{2}{u_1}-\frac{3}{u_2-u_1}\right)$$
$$+\frac{1}{4}\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\left(\frac{1}{u_1-u_2}-\frac{1}{x_1-u_2}\right)+\frac{1}{4}\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\left(\frac{1}{u_1-u_2}-\frac{1}{x_2-u_2}\right)$$
$$-\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2\frac{\partial u_1}{\partial x_2}\left(\frac{1}{u_1}+\frac{1}{x_1-u_1}\right)-\frac{1}{2}\frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2}\right)^2\left(\frac{1}{u_1}+\frac{1}{x_2-u_1}\right) \quad (5.28)$$

and

$$\frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{2}\left(\frac{1}{x_1}-\frac{1}{x_1-u_1}\right)+\frac{1}{2}\frac{\partial u_1}{\partial x_1}\left(-\frac{2}{x_1}-\frac{1}{x_1-x_2}+\frac{1}{x_1-u_2}+\frac{1}{x_1-u_1}\right)-\frac{1}{2}\frac{\partial u_1}{\partial x_2}\left(\frac{1}{x_1}+\frac{1}{x_2-x_1}\right)$$
$$+\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2\left(\frac{2}{u_1}+\frac{1}{u_1-x_2}-\frac{1}{u_1-u_2}+\frac{1}{x_1-u_1}\right)+\frac{1}{2}\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1}\left(\frac{1}{u_1-u_2}-\frac{1}{x_1-u_2}\right)$$
$$-\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^3\left(\frac{1}{u_1}+\frac{1}{x_1-u_1}\right)-\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2\frac{\partial u_1}{\partial x_2}\left(\frac{1}{u_1}+\frac{1}{x_2-u_1}\right). \quad (5.29)$$

Replacing u_1 by u_2 and vice versa in (5.28) *and in* (5.29) *we obtain equations for $\frac{\partial^2 u_2}{\partial x_1\partial x_2}$ and for $\frac{\partial^2 u_2}{\partial x_1^2}$, respectively. Replacing x_1 by x_2 and vice versa in* (5.29) *we obtain the equation for $\frac{\partial^2 u_1}{\partial x_2^2}$.*

Theorem 6 *Let $\mathbf{x}_0\in\mathbb{C}^g\setminus\{(x_1,\dots,x_g)\mid \exists\, i\neq j \text{ with } x_i=x_j\}$ be such that $x_{0j}\neq 0$ for $j=1,\dots,g$. Let $\{a_1,\dots,a_g,b_1,\dots,b_g\}$ be a canonical homology basis on $\mathcal{C}_{\mathbf{x}_0}$ such that the projections $u(a_j)$ and $u(b_j)$ with $j=1,\dots,g$ do not intersect a certain neighbourhood $\hat{\mathcal{X}}$ of $\mathbf{x}_0$. Let $\Omega=\Omega(\mathbf{x}_0)$ be the meromorphic differential of the second kind defined by* (2.12) *on a compact Riemann surface $\mathcal{C}_{\mathbf{x}_0}$ of the hyperelliptic curve* (3.3) *of genus g with $\mathbf{x}=\mathbf{x}_0$. Assume that $\mathbf{x}_0$ is such that $\Omega(\mathbf{x}_0;P_{u_j})\neq 0$ for $j=1,\dots,g$. Then there exists a unique continuous family $(\mathcal{C}_{\mathbf{x}},\Omega(\mathbf{x}))$ providing an isoperiodic deformation of the pair $(\mathcal{C}_{\mathbf{x}_0},\Omega(\mathbf{x}_0))$ for $\mathbf{x}$ varying in some neighbourhood $\mathcal{X}\subset\hat{\mathcal{X}}$ of $\mathbf{x}_0$.*

Proof. The defining condition for isoperiodic deformations of $(\mathcal{C}_{\mathbf{x}_0}, \Omega(\mathbf{x}_0))$ are given by $\beta_k = \oint_{b_k} \Omega(x) =$ const, $k = 1, \dots, g$. By the implicit function theorem, these relations define functions $u_1(\mathbf{x}), \dots, u_g(\mathbf{x})$ in a neighbourhood of $\mathbf{x} = \mathbf{x}_0$ if the the Jacobian det$J$ for a $g \times g$ matrix J such that $J_{jk} = \frac{\partial \beta_k}{\partial u_j}$ is non-zero at $\mathbf{x} = \mathbf{x}_0$. From (3.2) and (2.12) we obtain these derivatives using the Rauch formulas (2.14):

$$J_{jk} = \pi \mathrm{i}\, W(P_\infty, P_{u_j}) \omega(P_{u_j}) + \pi \mathrm{i}\, \alpha \omega(P_{u_j}) \omega_k(P_{u_j}) = \pi \mathrm{i}\, \Omega(P_{u_j}) \omega_k(P_{u_j}).$$

Thus $\det J = \pi \mathrm{i} \det \hat{J} \prod_{m=1}^{g} \Omega(P_{u_m})$ where by $\hat{J}$ we denote the matrix $\hat{J}_{jk} = \omega_k(P_{u_j})$. The product of $\Omega(P_{u_m})$ being non-zero at $\mathbf{x} = \mathbf{x}_0$, we only need to show that $\det \hat{J} \neq 0$. Note that replacing the basis $\omega_1, \dots, \omega_g$ by the basis $\phi(P),\ \lambda(P)\phi(P), \dots, \lambda(P)^{g-1}\phi(P)$ (2.4) in the space of holomorphic differentials does not affect the non-vanishing of $\det \hat{J}$. On the other hand, this change of basis transforms $\det \hat{J}$ into the Vandermonde determinant $V(u_1, \dots, u_g)$, which is non-zero for distinct $u_1, \dots, u_g$. This is fulfilled in a neighbourhood of $\mathbf{x}_0$. This proves the existence of local continuous isoperiodic deformation of $(\mathcal{C}_{\mathbf{x}_0}, \Omega(\mathbf{x}_0))$.
Now, Theorem 5 shows that for every continuous isoperiodic deformation, the functions $u_1(\mathbf{x}), \dots, u_g(\mathbf{x})$ satisfy equations (5.18) and (5.19) for all $1 \leqslant m, n, k \leqslant g$. In addition, due to Theorem 4, the condition $\oint_{b_k} \Omega(x) =$ const implies that the first derivatives of $\partial_{x_k} u_j(\mathbf{x})$ are given by (5.3) for any $1 \leqslant k, j \leqslant g$ and $\mathbf{x} = (x_1, \dots, x_g)$. Thus, for $\mathbf{x}_0 = (x_1^0, \dots, x_g^0)$, the initial values $u_j(\mathbf{x}_0)$ and derivatives $\partial_{x_k} u_j(\mathbf{x})$ (5.3) for any $\mathbf{x}$ specify a unique solution of system (5.18), (5.19) as follows. On the set of points $(x_1, x_2^0 \dots, x_g^0)$ for x_1 varying in some neighbourhood of x_1^0, the system reduces to a system of ordinary differential equations of second order with respect to the variable x_1, and thus the initial values $u_j(\mathbf{x}_0)$ and the derivatives $\partial_{x_1} u_j(\mathbf{x}_0)$ single out a unique solution $u_j(x_1, x_2^0 \dots, x_g^0)$ for $j = 1, \dots, g$ and $x_1 \in S_1$ for some open neighbourhood S_1 of x_1^0. Now, the obtained functions $u_j(x_1, x_2^0 \dots, x_g^0)$ give us initial values for the system of ordinary differential equations of second order with respect to the variable x_2 at any point of the set $\{(x_1, x_2^0 \dots, x_g^0) |\ x_1 \in S_1\}$. Together with the values of derivatives $\partial_{x_2} u_j((x_1, x_2^0 \dots, x_g^0))$, they single out a unique solution $u_j(x_1, x_2, x_3^0 \dots, x_g^0)$ for $j = 1, \dots, g$ and $(x_1, x_2) \in S_2$ for some open neighbourhood S_2 of (x_1^0, x_2^0). Proceeding in this way, we obtain uniqueness of an isoperiodic deformation of $(\mathcal{C}_{\mathbf{x}_0}, \Omega(\mathbf{x}_0))$ locally in an open neighbourhood of $\mathbf{x}_0$.
□

Corollary 4 *Let the situation and notation be those of Theorem 6. Let the initial hyperelliptic curve* $\mathcal{C}_{\mathbf{x}_0}$ (3.3) *of genus* g *with* $\mathbf{x} = \mathbf{x}_0$ *have all branch points* x_j *and* u_j *real and satisfy the Hill condition* (2.19). *Then, there exists a canonical homology basis on* $\mathcal{C}_{\mathbf{x}_0}$ *such that there exists a unique continuous isoperiodic deformation of the pair* $(\mathcal{C}_{\mathbf{x}_0}, \Omega(\mathbf{x}_0))$ *for* $\mathbf{x}$ *varying in some real neighbourhood* $\mathcal{X} \cap \mathbb{R}^g \subset \hat{\mathcal{X}}$ *of* $\mathbf{x}_0$ *for which all the functions* $u_j = u_j(x_1, \dots, x_g)$ *are real-valued, and the obtained hyperelliptic curves satisfy the Hill condition* (2.19) *with* T *being the same for all curves.*

Proof. The reality condition follows from [34] and the uniqueness of the isoperiodic deformation proven in Theorem 6. Theorem 3 of [34] shows that Hill curves can alternatively be characterized through the spectrum of Schrödinger operators $-\frac{\partial^2}{\partial X^2} + v(X)$ with periodic potentials v, see also [29]. Using the notation of [34] and [40], the set of branch points of the curves (3.3) is denoted by $\{E_0, E_1, \dots, E_{2g}\} \subset \mathbb{R}$ with $E_{2j-2} < E_{2j-1} < E_{2j}$ for $j = 1, \dots, g$, where the intervals of the form (E_{2j}, E_{2j+1}) form the Bloch spectrum of the Schrödinger operator while the intervals of the form $[E_{2j-1}, E_{2j}]$ are called *gaps*. Let us now choose the canonical homology basis on $\mathcal{C}_{\mathbf{x}_0}$ in such a way that the projections on the λ-sphere of the a_j cycle goes once around the gap interval $[E_{2j-1}, E_{2j}]$ for all $j = 1, \dots g$. The projection of the cycle b_j may be chosen to go once around the set of intervals

$[E_0, E_1], \dots [E_{2j-2}, E_{2j-1}]$. For this choice, the cycles a_j and b_j satisfy relations $\rho a_j = a_j$, $\rho b_j = -b_j$ with ρ being the antiholomorphic involution from Example 1.
Theorem 3 of [34] states that the set of branch points of the hyperelliptic curve 3.3 corresponds to a periodic Schrödinger operator if and only if equations (1.23), (1.24) of [34] are satisfied by the differential $dp(\lambda)$ of the *quasi-momentum* function $p(\lambda)$ as well as the asymptotic at the point at infinity given by $p(\lambda - E_0) = \sqrt{\lambda - E_0}$. This asymptotic implies that $dp(\lambda) = -\frac{d\zeta_\infty}{\zeta_\infty^2}$ in terms of the local coordinate (2.3). With the above choice of a- and b-cycles, equations (1.24), (1.23) from [34] imply that $\oint_{a_j} dp = 0$ and $\oint_{b_j} dp = \frac{2\pi i n_j}{T}$, respectively. These conditions imply that $dp = -\Omega_0$ where Ω_0 is the differential on $\mathcal{C}_{\mathbf{x}_0}$ given by (2.12) with $\alpha = 0$, see also formula (29), p. 145 in [40]. These conditions also imply that the corresponding surface is a Hill curve.
As explained in [34], Hill curves may also be characterized as solutions to the "g-band extremal problem", which is a minimum deviation problem for holomorphic functions on the set of intervals $[E_0, E_1], \dots, [E_{2g-2}, E_{2g-1}]$, $[E_{2g}, \infty]$, where the endpoints of the intervals are also to be found. In Section 4 of [34], it is shown that one may fix $g+1$ real parameters consisting of E_0 (we set $E_0 = 0$ in this paper), E_{2g} and one endpoint of each of the intervals $[E_{2j}, E_{2j+1}]$, for $j = 1, \dots, g-1$ and find the other g endpoints, that is E_1 and the second endpoint of each of the intervals $[E_{2j}, E_{2j+1}]$, for $j = 1, \dots, g-1$, as real functions of the $g+1$ parameters (one of which is here set equal to zero) in order for the set of intervals to carry a holomorphic function with minimal deviation. For this discussion, see [34] p. 843, paragraphs around formula (4.17). In our case, we know that T (denoted by a in [34]) is "large enough", because we start here from a Hill curve $\mathcal{C}_{\mathbf{x}_0}$ that determines the value of T. This result implies the existence of a family of Hill curves parametrized by g real parameters (as we set here $E_0 = 0$) for which we may take g branch points of the curves. By uniqueness proven in Theorem 6, these deformations coincide with isoperiodic deformations of the pair $(\mathcal{C}_{\mathbf{x}_0}, \Omega_0)$ restricted to the real line. $\square$

6 Isoperiodic deformations of two-gap potentials

Let us now show how Theorems 5 and 6 as well as Corollary 4 imply the existence of isoperiodic deformations of finite-gap potentials. According to the theory of finite-gap potentials, developed starting from mid 1970's by Novikov, Dubrovin, Lax, Its, Matveev, Mumford, McKean, Krichever, van Moerbeke, and others, see e.g. [9], all such potentials can be obtained as stationary solutions of the hierarchy of Korteweg - de Vries equations. For the case of two-gap potentials, following [9], [10], [40], let us consider a pair of linear differential operators $\mathbb{L}, \mathbb{A}$ satisfying the Novikov equation

$$[\mathbb{L}, \mathbb{A}] = 0.$$

We consider the operators

$$\mathbb{L} = \frac{d^2}{dX^2} + v(X) \tag{6.1}$$

and

$$\mathbb{A} = 16\frac{d^5}{dX^5} + 20\left(v\frac{d^3}{dX^3} + \frac{d^3}{dX^3}v\right) + 30v\frac{d}{dX}v - 5\left(v''\frac{d}{dX} + \frac{d}{dX}v''\right) + c_1\left(4\frac{d^3}{dX^3} + 3\Big(v\frac{d}{dX} + \frac{d}{dX}v\Big)\right) + c_2\frac{d}{dX}.$$

The resulting Novikov equation for $v(x)$ has a well-known Lagrangian form with the Lagrangian

$$\mathcal{L}(v, v', v'') = \frac{v''}{2} - \frac{5}{2}v''v^2 + \frac{5}{2}v^4 + c_1\left(\frac{v'^2}{2} + v^3\right) + c_2v^2 + c_3v.$$

It is equivalent to a Hamiltonian system (see [40]) with the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = p_1 p_2 + V(q_1, q_2), \tag{6.2}$$

where the potential is $V(q_1, q_2) = \frac{1}{8}(-5q_1^4 - 20q_1^2 q_2 + 4c_2 q_1^2 - 4q_2^2 + c_3 q_1)$. Here we assume that $c_1 = 0$, which can be accomplished by a shift of v by a constant and we set

$$q_1 = v, \quad q_2 = v'' - 5v^2, \quad p_1 = q_1', \quad p_2 = q_1'.$$

This Hamiltonian system has two first integrals, the energy H and

$$J = p_1^2 + p_2^2(2q_2 - c_2) + 2q_1 p_1 p_2 + q_1^5 + c_2 q_1^3 - 4q_1 q_2^2 + 2c_2 q_1 q_2 + 2c_2 q_2.$$

Thus, the system is completely integrable. Consider the polynomial

$$\mathbb{P}_5(\lambda) = \lambda^5 + \frac{c_2\lambda^3}{8} + \frac{c_3\lambda^2}{16} + \frac{1}{32}\left(H + 2c_2^2\right)\lambda + \frac{J - c_2 c_3}{2^8},$$

and the associated genus two hyperelliptic curve

$$\Gamma = \{(\lambda, w) \in \mathbb{C}^2 \mid w^2 = \mathbb{P}_5(\lambda)\}. \tag{6.3}$$

Denote the zeros of $\mathbb{P}_5$ by λ_j, $j = 1, \dots, 5$. The compactification of the ramified covering $\lambda : \Gamma \to \mathbb{C}$, $\lambda(\lambda, w) = \lambda$ is ramified at the point P_∞ over $\lambda = \infty$. Denote the other ramification points by $P_j = (\lambda = \lambda_j, w = \mathbb{P}_5(z_j))$ and denote the compact Riemann surface associated with the curve (6.3) by Γ as well. Let us fix a canonical homology basis $\{a_1, a_2; b_1, b_2\}$ on Γ as follows. Denote by a_1 the contour going around the points P_3 and P_2, by a_2 the contour around P_5 and P_4, by b_1 the contour going around P_2 and P_1, and by b_2 the contour going around the points P_1, P_2, P_3 and P_4. We may think of the contours b_1, b_2 as staying on one of the sheets of the ramified covering of the λ-sphere. The directions on the contours are chosen in a way to have the intersection indices $a_k \circ b_j = \delta_{kj}$. Denote by ω_1 and ω_2 the Abelian differentials of the first kind normalized with respect to the chosen homology basis.

The integration [40] of the Novikov equation and the Hamiltonian system with Hamiltonian H (6.2) results in the following solution

$$v(X) = 2\frac{\partial^2}{\partial X^2}\log\theta(XU + z_0) + \text{const}. \tag{6.4}$$

Here U is, up to a constant factor, the vector of b-periods of a differential of the second kind having a pole of order 2 at P_∞ with principal part $d\sqrt{\lambda}$, no other poles, and zero a-periods. In our notation, this differential is $-\Omega_0$ where Ω_0 is defined by (2.12) with $\alpha = 0$. More precisely,

$$U = -\frac{1}{2\pi \mathrm{i}}\left(\oint_{b_1}\Omega_0, \oint_{b_2}\Omega_0\right)^T. \tag{6.5}$$

Every solution $v(X)$ is codified by a two-dimensional vector

$$z_0 = -\mathcal{A}(Q_1 + Q_2) - K, \tag{6.6}$$

where Q_1, Q_2 is a non-special divisor on Γ and K is the vector of Riemann constants and $\mathcal{A}(P)$ is the Abel map, based at some point $P_0 \in \Gamma$,

$$\mathcal{A}(P) = \Big(\int_{P_0}^P \omega_1, \int_{P_0}^P \omega_2\Big)^T.$$

In the real case we assume $\lambda_5 < \lambda_4 < \cdots < \lambda_1$. The spectrum of $\mathbb{L}$ as an operator on $\mathcal{L}_2(-\infty,\infty)$ is the ray $(-\infty,\lambda_1]$ without two *gap intervals* (λ_5,λ_4) and (λ_3,λ_2). This is why $v(X)$ is called a two-gap potential. The Riemann surface Γ is called the spectral curve of the operator $\mathbb{L}$. It is natural to consider the eigenfunction of $\mathbb{L}$, denoted by ψ, such that

$$\mathbb{L}\psi = \lambda\psi.$$

This is a Baker-Akhiezer function, meromorphic on $\Gamma \setminus P_\infty$, with poles at Q_1 and Q_2, of the form

$$\psi(X,P) = \exp\Big(-X\int_{P_0}^{P}\Omega_0\Big)\frac{\theta(\mathcal{A}(P)+XU+z_0)\theta(z_0)}{\theta(\mathcal{A}(P)+z_0)\theta(XU+z_0)}. \tag{6.7}$$

The group of periods with respect to X of the derivative of $\log\psi$ coincides with the group of periods of the potential $v(X)$ (5.6). An important part of modern theory of integrable systems is a study of isospectral deformations, that is deformations of v which preserve Γ and thus the spectrum of deformed operators $\mathbb{L}$. Here we study *isoperiodic* deformations, that is those that deform v and Γ, while preserving U, the period of the deformed differential Ω_0. Here U can be seen as the wavevector of solutions ψ. In particular, in periodic cases, preserving the wavevector may lead to preserving the period (or one of the periods) of ψ. Thus, our isoperiodic deformations are *wavevector-preserving* and in the periodic cases, they can be *period-preserving*.
The analogous construction for hyperelliptic curves of genus $g \geqslant 1$,

$$\Gamma_g = \{(\lambda,w)\in\mathbb{C}^2 \mid w^2 = \mathbb{P}_{2g+1}(\lambda)\},$$

where $\lambda_{2g+1} < \cdots < \lambda_1$ are the zeros of the polynomial $\mathbb{P}_{2g+1}$, generates g-gap potentials $v(X)$, see [40], [10]. An obtained g-gap potential $v(X)$ is periodic with period T if and only if the vector U (6.5) satisfies

$$TU = (n_1e_1+\cdots+n_ge_g) + \mathbb{B}(m_1e_1+\cdots+m_ge_g), \tag{6.8}$$

with some integers $n_j,\ m_j,\ j=1,\dots,g$, where e_j is the j-th vector of the canonical basis of $\mathbb{C}^g$ and $\mathbb{B}$ is the Riemann matrix. In particular, for $m_1=\ldots=m_g=0$, one recognizes the Hill curves condition (2.19).

Proposition 1 *Let $v(X)$ be a real periodic g-gap potential corresponding to the spectral curve*

$$\Gamma = \{(\lambda,w)\in\mathbb{C}^2 \mid w^2 = \prod_{j=1}^{2g+1}(\lambda-\lambda_j)\} \qquad \textit{with} \quad \lambda_{2g+1} < \lambda_{2g} < \ldots < \lambda_1\,. \tag{6.9}$$

Assume that the a- and b-cycles of the spectral curve are selected as in Example 1. Denote by P_∞ the unique point at infinity of the compact Riemann surface Γ of this hyperelliptic curve and let Ω_0 be the differential on Γ defined by (2.12) with $\alpha=0$. Introduce $x_j=\lambda_{2j-1}-\lambda_{2g+1}$, $u_j=\lambda_{2j}-\lambda_{2g+1}$, $j=1,\dots,g$. The pair (Γ,Ω_0) is subject to an isoperiodic deformation if and only if $u_1,\dots,u_g$ as functions of $x_1,\dots,x_g$ satisfy equations (5.18) and (5.19) of Theorem 5 and the initial conditions (5.3), where $\Omega=\Omega_0$ and v_j are defined by (5.6). These deformations preserve reality of x_j, u_j and also preserve the wavevector of $\psi(X)$ (6.7), of the eigenfunction of the operator $\mathbb{L}$ (6.1). If the initial potential is periodic with the period $T>0$, then the potentials corresponding to the deformed curves are also periodic with the period T.

Proof. The first part of the Proposition follows directly from Theorems 4, 5 and 6. The wavevector U is related to the vector of b-periods of the differential Ω_0 via equation (6.5) and it is further related to the normalized holomorphic differentials via the vector form of (4.17):

$$U = \frac{1}{2\pi \mathrm{i}} \oint_b \Omega_0 = \omega(P_\infty), \tag{6.10}$$

where $b = (b_1, \ldots, b_g)$ and $\omega(P_\infty)$ is the column consisting of the normalized differentials of the basis of holomorphic differentials, evaluated at P_∞ according to (2.9). Due to the reality of the branch points, Γ is a real curve, see Example 1, and thus $\omega_j(P_\infty)$ is real for all $j = 1, \ldots, g$. Therefore, U is a real vector, and it is preserved under the isoperiodic deformations of (Γ, Ω_0). These deformations preserve reality of x_j, u_j according to Corollary 4. Thus, if the initial potential (6.4) is periodic with the period T, then TU satisfies (6.8). Since for a real curve with the given choice of canonical homology basis the Riemann matrix $\mathbb{B}$ is purely imaginary, then TU satisfies (6.8) with zero m_j's. Therefore, T remains to be a period under the isoperiodic deformations, since U is real and remains unchanged under the deformations. □

Example 5 *[Deformations of two-gap Lamé potentials] It was shown in [18] that Lamé potentials $v_g(x) = g(g+1)\wp(x)$ have g gaps. In particular $v_2(x) = 6\wp(x)$ is a two-gap potential. In the real case, denoting by $e_1 < e_2 < e_3$ the zeros of the polynomial $4\lambda^3 - g_2\lambda - g_3$, following [9], the endpoints of the gaps are given by:*

$$\lambda_5 = -3(e_2 + e_3), \quad \lambda_4 = -\sqrt{3g_2}, \quad \lambda_3 = 3e_2, \quad \lambda_2 = -\lambda_4, \quad \lambda_1 = 3e_3.$$

Here $g_2 = 4(e_2^2 + e_3^2 + e_2 e_3)$. The shift by λ_5 brings the genus two hyperelliptic curve with branch points $\lambda_1, \ldots, \lambda_5$ to the curve of the form (3.3) *with branch points at* 0, $x_1 = 3e_2 + 6e_3$, $x_2 = 6e_2 + 3e_3$, $u_1 = \sqrt{12(e_2^2 + e_3^2 + e_2 e_3)} + 3(e_2 + e_3)$ *and* $u_2 = -\sqrt{12(e_2^2 + e_3^2 + e_2 e_3)} + 3(e_2 + e_3)$*. It is immediate that*

$$e_3 = \frac{2x_1 - x_2}{9}, \qquad e_2 = \frac{2x_2 - x_1}{9}.$$

This example is setting stage for the following Proposition.

Proposition 2 *Given the curve Γ_{e_2,e_3}* (4.16) *with $e_1 < e_2 < e_3$, denote $x = e_2 + 2e_3$ and $u = 2e_2 + e_3$. There exists a unique continuous deformation of the curve Γ_{e_2,e_3}* (4.16) *which preserves the real period $2w_1$ of the potential v_2, such that $u = u(x)$ is the real-valued function of x. Under this deformation, the genus two hyperelliptic curve $\mathcal{C}_{x_1,x_2}$ of the form* (3.3) *with the branch points at* 0, $x_1 = 3e_2 + 6e_3$, $x_2 = 6e_2 + 3e_3$, $u_1 = \sqrt{12(e_2^2 + e_3^2 + e_2 e_3)} + 3(e_2 + e_3)$ *and* $u_2 = -\sqrt{12(e_2^2 + e_3^2 + e_2 e_3)} + 3(e_2 + e_3)$ *deforms in a way that $x_1 = 3x$, $x_2 = 3u(x_1/3)$ and $u_1 = u_1(x_1, x_2)$ and $u_2 = u_2(x_1, x_2)$ are real-valued functions of real variables that satisfy equations from Example 4 and* (5.3) *with Ω and v_k given by* (2.12) *and* (5.6) *on the surface C_{x_1,x_2}.*

Proof. According to Corollary 3, there exists a unique continuous deformation of the curve Γ_{e_2,e_3} (4.16) which preserves the period $2w_1$ of the potential $v_1(X) = 2\wp(X)$, such that $u = 2e_2 + e_3$ satisfies equation (4.15) as a function of $x = 2e_3 + e_2$ with the initial condition (4.3), where $\Omega = \Omega_0$ on Γ_{e_2,e_3} is from the proof of Theorem 3, see (4.17). Due to Corollary 4, this deformation has a unique restriction to a real neighborhood $\mathcal{X}$ of x, for which $u = u(x)$ is a real-valued function. Using Example 5, this generates a deformation of the genus two hyperelliptic curve $\mathcal{C}_{x_1,x_2}$ (3.3) with x_1, x_2, u_1, u_2 as in the statement of the proposition, where $x_1 = 3x$ and $x_2 = 3u(x_1/3)$ are both real and where the associated periodic potential is v_2. Note now that, due to the fact that e_1, e_2, e_3 are real, the $\wp$-function has a real and a purely imaginary periods, denoted by w_1 and w_2 respectively (see e.g. [43], Example 1, Ch.20-32, p. 444). Thus, the $\wp$-function is real-valued along the boundary of the fundamental rectangle with the vertices 0, w_1, $w_1 + w_2$, w_2 (see e.g. [43], Example 2, Ch.20-32, p. 444). Thus, given that the curve $\mathcal{C}_{x_1,x_2}$ for $x \in \mathcal{X}$ is associated to a real-periodic potential, it is a real Hill curve [34, 29], see also the proof of Corollary 4 and thus the deformation in question is an isoperiodic deformation of the pair $(\mathcal{C}_{x_1,x_2}, \Omega_0)$, where Ω is defined in (2.12) for $\alpha = 0$ on $\mathcal{C}_{x_1,x_2}$ and satisfies the Hill condition (2.19) with $T = 2w_1$. By Theorem 5, the functions $u_1 = u_1(x_1, x_2)$ and $u_2 = u_2(x_1, x_2)$ satisfy equations from Example 4 and (5.3), with $\Omega = \Omega_0$. The reality of u_1 and u_2 through the deformation follows from reality of the endpoints of the gaps for a real-periodic potential. $\square$

7 Isoperiodic deformations of solutions to the Neumann problem

In this section we apply Theorems 5, 6 and Corollary 4 to obtain isoperiodic deformations of periodic solutions to the general Neumann system on S^n. The Neumann system from classical mechanics describes motion of a particle on an n-dimensional sphere

$$S^n : \quad \sum_{j=1}^{n+1} q_j^2 = 1$$

subject to the potential force with the quadratic potential

$$V(q) = \frac{1}{2} \sum_{j=1}^{n+1} A_j q_j^2, \quad A_j = \text{const.}$$

The equations of motion are

$$\ddot{q}_j = -A_j q_j + F(t) q_j, \quad j = 1, \dots, n+1.$$

Here $F(t) = \sum_{j=1}^{n+1} (A_j q_j^2 - \dot{q}_j^2)$ is the Lagrange multiplier corresponding to the reaction of the constraint which forces the particle to stay on the sphere S^n. This system is Hamiltonian restricted to the sphere S^n with the Hamiltonian

$$H = \frac{1}{2} \Big(\sum_{j=1}^{n+1} A_j q_j^2 + q^2 p^2 - (qp)^2 \Big),$$

where $q = (q_1, \dots, q_{n+1})$, $p = (p_1, \dots, p_{n+1})$ and qp is the euclidean scalar product. The independent involutive first integrals of motion are

$$F_k(q, p) = q_k^2 + \sum_{j \neq k} \frac{(q_k p_j - q_j p_k)^2}{A_j - A_k}, \quad k = 1, \dots, n+1.$$

Thus, the Neumann system is a completely integrable system in the Liouville sense and $2H = \sum_{j=1}^{n+1} A_j F_j$. The case $n = 1$ was considered in Section 4.2, see Example 2. For positive A_j and $n \geqslant 2$, the Neumann system is related to another famous integrable system, the Jacobi problem of geodesics on the ellipsoid in $\mathbb{R}^{n+1}$

$$\sum_{j=1}^{n+1} \frac{\tilde{q}_j^2}{A_j} = 1.$$

The correspondence between the Neumann and the Jacobi problems is as follows

$$\tilde{q} = p, \quad \tilde{p} = -q, \quad \tilde{H} = \sum_{j=1}^{n} A_j^{-1} F_j.$$

There is a remarkable connection between the Neumann system on S^n and n-gap potentials, according to [35], [36], [42], [37]. Consider an n-gap differential operator

$$\mathbb{L} = -\frac{d^2}{dx^2} + v(x), \tag{7.1}$$

where the potential $v(x)$ is periodic. Let the curve Γ_n given by the equation $w^2 = P_{2n+1}(z)$ be the spectral curve of $\mathbb{L}$, that is, denoting $z_{2n+1} < \cdots < z_1$ the zeros of the polynomial P_{2n+1}, we have that the Bloch spectrum of $\mathbb{L}$ is composed of the intervals $[z_{2j+1}, z_{2j}]$ for $j = 1, \ldots, n$ and $[z_1, \infty]$. Denote by P_{z_k} the point of Γ_n for which $z = z_k$. Assume that the a- and b-cycles of the spectral curve Γ_n are selected as in Example 1.
Denote also $-A_j = z_{2j+1}$ for $j = 1, \ldots, n$. The eigenvectors of $\mathbb{L}$ corresponding to eigenvalues $-A_j$ are constructed by considering the Baker-Akhiezer function $\psi(x, P)$ with $P \in \Gamma_n$ and $x \in \mathbb{R}$ corresponding to the operator $\mathbb{L}$. Setting

$$\psi_j(x) = \kappa_j \psi(x, P_{z_{2j+1}}),$$

with

$$\kappa_j = \Big(\prod_{k \neq j} (A_j - A_k)\Big)^{-\frac{1}{2}}, \quad j = 1, \ldots, n+1,$$

we have

$$\mathbb{L}\psi_j = -A_j \psi_j \,.$$

This last equation is equivalent to the Neumann system

$$\ddot{\psi}_j = -A_j \psi_j + v(x)\psi_j, \quad j = 1, \ldots, n+1,$$

with the following correspondence $x \to t$, $\psi_j \to q_j$, $v(x) \to F(t)$, taking into account that, see [42],

$$\sum_{j=1}^{n+1} \psi_j^2 = 1.$$

The original classical Neumann system is for $n = 2$ and corresponds to two-gap potentials. The solutions of the classical Neumann system are given by the formulae:

$$\begin{aligned} q_1(t) &= \kappa_1 \frac{\theta[(0,1/2),(0,0)](tU+z_0)\theta(z_0)}{\theta[(0,1/2),(0,0)](z_0)\theta(tU+z_0)}, \\ q_2(t) &= \kappa_2 \frac{\theta[(1/2,0),(0,1/2)](tU+z_0)\theta(z_0)}{\theta[(1/2,0),(0,1/2)](z_0)\theta(tU+z_0)}, \\ q_3(t) &= \kappa_3 \frac{\theta[(0,0),(1/2,1/2)](tU+z_0)\theta(z_0)}{\theta[(0,0),(1/2,1/2)](z_0)\theta(tU+z_0)}. \end{aligned} \tag{7.2}$$

Here θ is the theta function associated with the real curve Γ_2, and κ_j are defined above and U (6.5) is the same as in Section 6 for the curve Γ_2. The solutions of a general Neumann system on S^n for

arbitrary n have practically the same form in terms of θ-functions related to genus n hyperelliptic real curves Γ_n, see [37]. Solutions of the Neumann system are periodic with period $T \in \mathbb{R}$ if and only if the corresponding n-gap potential $v(x)$ is periodic, thus if and only if the period vector U of Ω_0 satisfies (6.8) with some $T \in \mathbb{R}$. This is true if and only if the Hill condition (2.19) is satisfied. Note that all m_j are zero in (6.8), since U is a real vector. The reality of U follows from the fact that the relevant hyperelliptic curves are real, see Example 1, the reality of the coefficients of the normalized holomorphic differentials and formula (6.10) expressing U in terms of those differentials.
In the case $n = 2$, by a rescaling of time t we may assume that $A_3 = 0$. (Similarly, we may assume that $A_{n+1} = 0$ for general n.) Proposition 1, implies the following statement.

Proposition 3 *Let $q = (q_1, q_2, q_3)$ be a periodic solution (7.2) to the classical Neumann system for $n = 2$ with the potential $V(q) = \frac{1}{2}(A_1 q_1^2 + A_2 q_2^2)$ and A_1, $A_2 \in \mathbb{R}$. Denote the period of the solution by $T \in \mathbb{R}$. Let Γ be the hyperelliptic curve of genus 2 represented as a two-fold ramified covering of z-sphere with a branch point at $z = \infty$ and finite branch points $z_5 = -A_3 = 0 < z_4 < z_3 = -A_2 < z_2 < z_1 = -A_1$, the spectral curve of the system. Assume now that $\{\Gamma_A, q_A\}$ is a continuous family of hyperelliptic curves with finite branch points $z_5 = -A_3 = 0 < z_4 < z_3 = -A_2 < z_2 < z_1 = -A_1$ and denote the corresponding solutions by q_A (7.2). Denote $x_1 = -A_1,\ x_2 = -A_2,\ u_1 = z_2$, and $u_2 = z_4$. Let x_1 and x_2 (that is A_1 and A_2) vary in some real neighborhood. The obtained solutions q_A are periodic with the same period T if and only if u_1, u_2 as real-valued functions of x_1, x_2 satisfy equations of Example 4 and the initial conditions (5.3). Here Ω and v_j in (5.3) are differentials on Γ_A defined by (2.12) and (5.6) with $\lambda = z$, respectively. More generally, assume that $\{\Gamma_A, q_A\}$ is a continuous family of hyperelliptic curves Γ_A of genus n represented as a two-fold ramified coverings of the z-sphere with finite branch points $z_{2n+1} < \cdots < z_1$, with $z_{2j-1} = -A_j$, $j = 1, \ldots, n+1$ where $A_{n+1} = 0$. Denote by q_A the solution to the general Neumann system on S^n with the potential $V(q) = \frac{1}{2}\sum_{j=1}^{n} A_j q_j^2$ constructed as in [37] from the corresponding spectral curve Γ_A, which is periodic with a period $T \in \mathbb{R}$. Denote $x_j = z_{2j-1} = -A_j$, $u_j = z_{2j}$, $j = 1, \ldots, n$. By varying $A_1, \ldots, A_n$ in some real neighborhoods, the obtained solutions q_A are periodic with the same period T if and only if $u_1, \ldots, u_n$ as real-valued functions of $x_1, \ldots, x_n$ satisfy equations (5.18) and (5.19) of Theorem 5 and the initial conditions (5.3). As before, Ω and v_j in (5.3) are differentials on Γ_A defined by (2.12) and (5.6) with $\lambda = z$, respectively.*

Proof. A periodic solution to the Neumann system corresponds to a periodic finite-gap potential [35, 36, 42], and thus the corresponding spectral curve has real branch points and satisfies the Hill condition (2.19) [29]. Therefore, the curves Γ_A in the family are curves with real branch points, satisfying the Hill condition (2.19). The statement thus follows from Theorem 5, Theorem 6 and Corollary 4. □

8 Isoperiodic deformations of the finite-gap periodic solutions to the KdV equation

In this section, we obtain isoperiodic deformation of periodic solutions to the KdV equation as a consequence of Theorems 5 and 6 and Corollary 4 from Section 5.3.
We consider a family $\mathcal{C}_{\mathbf{x}}$ of compact genus g hyperelliptic Riemann surfaces corresponding to the curves of equation (3.3). The hyperelliptic coverings of the λ-sphere have a ramification point P_∞ over $\lambda = \infty$. Denoting, as before, the other ramification points of the hyperelliptic covering by P_0, P_{u_j} and P_{x_j}, let $\{a_1, \ldots, a_g; b_1, \ldots, b_g\}$ be a fixed canonical homology basis on the surfaces $\mathcal{C}_{\mathbf{x}}$ for

$\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$, such that a_j goes around the points P_{x_j} and P_{u_j} while b_j goes around the points P_0, P_{x_k}, P_{u_k} and P_{x_j} for $k = 1, \ldots, j-1$. The cycles are oriented in a way so that the intersection indices be $a_k \circ a_j = b_k \circ b_j = 0$ and $a_k \circ b_j = \delta_{kj}$. The domain $\mathcal{S}_{\mathbf{x}}$ is chosen as in Section 3.1.
Consider differentials of the second kind $\Omega_{P_\infty}^{(n+1)}(P)$ with $P \in \mathcal{C}_{\mathbf{x}}$ normalized by vanishing of their a-periods and having a single pole at P_∞ of order $n+2$ with the local behaviour in terms of the local parameter ζ_∞ (2.3)

$$\Omega_{P_\infty}^{(n+1)}(P) = \left(\frac{1}{\zeta_\infty^{n+2}(P)} + \mathcal{O}(1) \right), \quad P \sim P_\infty, \tag{8.1}$$

This way, for the differential Ω_0 defined by (2.12) with $\alpha = 0$, we have $\Omega_0 = \Omega_{P_\infty}^{(1)}$. Denote further

$$U = -\frac{1}{2\pi \mathrm{i}} \left(\oint_{b_1} \Omega_{P_\infty}^{(1)}, \ldots, \oint_{b_g} \Omega_{P_\infty}^{(1)} \right)^T \quad \text{and} \quad W = -\frac{3}{2\pi \mathrm{i}} \left(\oint_{b_1} \Omega_{P_\infty}^{(3)}, \ldots, \oint_{b_g} \Omega_{P_\infty}^{(3)} \right)^T. \tag{8.2}$$

We follow [10] closely in this section while adjusting some constants to continue using our normalization (2.5), (2.8). For a fixed $\mathbf{x}$, the meromorphic function $\lambda : \mathcal{C}_{\mathbf{x}} \to \mathbb{C}P^1$ has a unique pole which is of the second order at P_∞ on $\mathcal{C}_{\mathbf{x}}$. With the help of this function, one obtains a reduction of the Kadomtsev-Petviashvili equation (KP) related to the Baker-Akhiezer function ϕ with an essential singularity at $\mathbb{P}_\infty$ defined by the polynomial

$$q(k) = kX + k^3 t,$$

where k is the reciprocal of the local parameter ζ_∞ around P_∞ on $\mathcal{C}_{\mathbf{x}}$. The Baker-Akhiezer function at $P \sim P_\infty$ behaves as

$$\phi(X, t; P) = e^{kX + k^3 t} \big(1 + \frac{\xi_1}{k} + \frac{\xi_2}{k^2} + \ldots \big),$$

where ξ_i, $i = 1, 2$ are functions of X and t to be determined below. Consider the operators

$$\mathbb{L} = \partial_X^2 + v, \quad \mathbb{A} = \partial_X^3 + \frac{3}{2} v \partial_X + w,$$

where

$$v = -2 \frac{\partial \xi_1}{\partial X}, \quad w = 3\xi_1 \frac{\partial \xi_1}{\partial X} + 3 \frac{\partial^2 \xi_1}{\partial X^2} - 3 \frac{\partial \xi_2}{\partial X}.$$

Then the Baker-Akhiezer function ϕ satisfies the system

$$\mathbb{L}\phi = \lambda \phi, \qquad \frac{\partial \phi}{\partial t} = \mathbb{A}\phi.$$

The compatibility condition for this system leads to the L-A equation

$$\frac{\partial}{\partial t} \mathbb{L} = \big[\mathbb{L}, \mathbb{A}\big],$$

which is equivalent to the system of nonlinear PDEs for v and w:

$$\frac{3}{4} v_{XX} - w_X = 0, \qquad -v_t + v_{XXX} + \frac{3}{2} v v_X - w_{XX} = 0.$$

By excluding w from the last two equations, one gets the following form of the Korteweg-de Vries (KdV) equation [10]:

$$v_t = \frac{1}{4}(6 v v_X + v_{XXX}). \tag{8.3}$$

As indicated in e.g. [10], the function

$$v(X,t) = 2\frac{\partial^2}{\partial X^2}\ln\theta(XU + tW + z_0) + c, \tag{8.4}$$

solves the KdV equation (8.3), where $\theta(z)$ is the theta-function associated with $\mathcal{C}_{\mathbf{x}}$ and U and W are the b-periods of the differentials of the second kind (8.2) (8.1) with a unique pole at P_∞. Considering now a variation of $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$, as an application of Theorems 5 and 6, we have the following statement.

Proposition 4 *Let $\mathcal{C}_{\mathbf{x}_0}$ be a compact Riemann surface of the curve* (3.3) *with $\mathbf{x} = \mathbf{x}_0$ and let $v = v_{\mathbf{x}_0}$ be a solution* (8.4) *of the KdV equation* (8.3)*, constructed from $\mathcal{C}_{\mathbf{x}_0}$ and the point $P_\infty \in \mathcal{C}_{\mathbf{x}_0}$. Then there exists a unique continuous family of surfaces $\mathcal{C}_{\mathbf{x}}$ parametrized by $\mathbf{x} \in \mathcal{X}$ with $\mathbf{x}_0 \in \mathcal{X}$ such that all solutions from the corresponding continuous family of solutions $v_{\mathbf{x}}$ of the KdV equation obtained from $\mathcal{C}_{\mathbf{x}}$ and $P_\infty \in \mathcal{C}_{\mathbf{x}}$ by* (8.4) *have the same vector U (the wavevector). Moreover, for the family $\mathcal{C}_{\mathbf{x}}$, the functions $u_1(\mathbf{x}), \dots, u_g(\mathbf{x})$ from* (3.3) *satisfy equations* (5.18), (5.19) *of Theorem 5 and* (5.3)*, where Ω in* (5.3) *is $\Omega_0 = \Omega^{(1)}_{P_\infty}$ and v_j are defined by* (5.6).

Proof. The family $\mathcal{C}_{\mathbf{x}}$ is an isoperiodic family $(\mathcal{C}_{\mathbf{x}}, \Omega^{(1)}_{P_\infty})$ relative to the normalized differential $\Omega^{(1)}_{P_\infty}$ (8.1). The existence and uniqueness of such a family is given by Theorem 6. The conditions (5.3) come from Theorem 4. The statement concerning equations of Theorem 5 holds due to the fact that these equations are derived starting from the condition that the b-periods of the differential Ω (2.12) are constant for all surfaces from the family $\mathcal{C}_{\mathbf{x}}$. The statement of Theorem 5 applies to the normalized differentials $\Omega^{(1)}_{P_\infty}$ and their b-periods equal to $-2\pi\mathrm{i}U$ (6.5), by setting $\alpha = 0$ in (2.12). □

If there exists a real nonzero number T, such that TU belongs to the lattice of the Jacobian of $\mathcal{C}_{\mathbf{x}}$ that is if (6.8) is satisfied for some integers n_j and m_j, then v (8.4) is periodic in X with a period T. In this case, we say that v is a *spatial T-periodic function.*

It follows from [9], that all real smooth finite-gap solutions of the KdV equation are obtained by (8.4) from a hyperelliptic curve Γ (6.9) of genus g which has all branch points real $\lambda_{2g+1} < \lambda_{2g} < \cdots < \lambda_1$, with the canonical basis of cycles chosen as in Exemple 1, and with an additional condition as follows. It is required that the non-special degree g divisor $\mathcal{D} = \sum_{j=1}^{g}(\gamma_j, w_j)$ of poles of the Baker-Akhiezer function ϕ be such that $\lambda_{2j} \le \gamma_j \le \lambda_{2j-1}$, for $j = 1, \dots g$. The divisor $\mathcal{D}$ determines the constant z_0 in (8.4) up to the vector of Riemann constants K of Γ (note that $\mathcal{D} = Q_1 + Q_2$ in (6.6) for $g = 2$). In this case, U is real and thus such a solution v (8.4) is spatially periodic with the period T if and only if the curve Γ satisfies the Hill condition (2.19).

Proposition 5 *Given $v_{\mathbf{x}_0} = v_{\mathbf{x}_0}(X,t)$, an arbitrary smooth real finite gap solution* (8.4) *of the KdV equation* (8.3) *which is spatially periodic with the period T, denote by $\mathcal{C}_{\mathbf{x}_0}$ a curve* (3.3) *with $\mathbf{x} = \mathbf{x}_0$ and the point $P_\infty \in \mathcal{C}_{\mathbf{x}_0}$, from which $v_{\mathbf{x}_0}$ is constructed. There exists a continuous family of solutions $v_{\mathbf{x}}(X,t)$ for $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$ with $\mathcal{X}$ being some open neighbourhood of $\mathbf{x}_0$ for which all solutions $v_{\mathbf{x}}(X,t)$ of the KdV equation are spatially periodic with period T. Moreover, there exists a family of curves $\mathcal{C}_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$, given by* (3.3)*, which satisfy the Hill condition* (2.19) *relative to a canonical homology basis chosen as in Example 1. The family of curves $\mathcal{C}_{\mathbf{x}}$ is such that $v_{\mathbf{x}}(X,t)$ is constructed by* (8.4) *from $\mathcal{C}_{\mathbf{x}}$ and $P_\infty \in \mathcal{C}_{\mathbf{x}}$ and the real-valued functions $u_1(\mathbf{x}), \dots, u_g(\mathbf{x})$ from* (3.3) *satisfy equations* (5.18), (5.19) *of Theorem 5 as well as* (5.3)*. The differentials in* (5.3) *are $\Omega_0 = \Omega^{(1)}_{P_\infty}$* (8.1) *and v_j* (5.6) *defined on the curves $\mathcal{C}_{\mathbf{x}}$; Ω_0 is normalized by vanishing of its periods with respect to the a-cycles invariant under the antiholomorphic involution.*

Proof. From the fact that $v_{\mathbf{x}_0}$ is a real smooth solution of KdV equation, it follows that the covering $\lambda : \mathcal{C}_{\mathbf{x}_0} \to \mathbb{C}P^1$ has all branch points real. Therefore, we may choose a canonical homology basis on $\mathcal{C}_{\mathbf{x}_0}$ to satisfy relations $\rho a_j = a_j$, $\rho b_j = -b_j$ with respect to the antiholomorphic involution ρ as in Example 1. From the periodicity of $v_{\mathbf{x}_0}$ and from the properties of quasi-periodicity of the theta-function, it also follows that the curve $\mathcal{C}_{\mathbf{x}_0}$ satisfies the periodicity condition (6.8) with T and $m_j = 0$, $j = 1, \dots, g$. The integers m_j in (6.8) vanish due to the definition (8.2) of the vector U and its reality; recall that the Riemann matrix is purely imaginary for this choice of homology basis. Theorem 6 gives the existence and the uniqueness of an isoperiodic family $(\mathcal{C}_{\mathbf{x}}, \Omega^{(1)}_{P_\infty}(\mathbf{x}))$ where $\Omega^{(1)}_{P_\infty}(\mathbf{x})$ is the normalized differential of the second kind (8.1) on $\mathcal{C}_{\mathbf{x}}$ and where $\mathbf{x} \in \mathcal{X}$ for some open neighbourhood $\mathcal{X}$ of $\mathbf{x}_0$. Given that this family is isoperiodic, after restricting it to $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$, we obtain that the branch points of the obtained coverings $\lambda : \mathcal{C}_{\mathbf{x}} \to \mathbb{C}P^1$ are all real (the functions $u_j(\mathbf{x})$ are real due to Corollary 4). For this restriction, the homology basis chosen on $\mathcal{C}_{\mathbf{x}_0}$ extends to a basis on $\mathcal{C}_{\mathbf{x}}$ satisfying the same conditions with respect to the anitholomorphic involution on $\mathcal{C}_{\mathbf{x}}$ with $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$, that is $\rho a_j = a_j$, $\rho b_j = -b_j$. The differential $\Omega_0 = \Omega^{(1)}_{P_\infty}(\mathbf{x})$ is thus normalized using this basis and the Hill condition (2.19) holds for $\Omega_0 = \Omega^{(1)}_{P_\infty}(\mathbf{x})$ with the same T for all $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$ and the above homology basis. Thus (6.8) is satisfied with the same T and $m_j = 0$, $j = 1, \dots, g$, for all the curves $\mathcal{C}_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$. As a consequence, solutions $v_{\mathbf{x}}(X, t)$ constructed from $\mathcal{C}_{\mathbf{x}}$ are spatial periodic with period T. This proves the existence of a continuous family of solutions to the KdV sharing the same spatial period T. Equations (5.18), (5.19) of Theorem 5 are satisfied since the family $(\mathcal{C}_{\mathbf{x}}, \Omega^{(1)}_{P_\infty}(\mathbf{x}))$ is isoperiodic, that is since the b-period vector U is preserved over the family. The derivatives are expressed by (5.3), due to Theorem 4 . □

9 Isoperiodic deformations and the sine-Gordon equation

In this section we discuss one more application of Theorems 5 and 6 and Corollary 4 from Section 5.3, an application to the theory of periodic solutions of the sine-Gordon equation. Here, the family of curves $\mathcal{C}_{\mathbf{x}}$ for $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$, their canonical homology bases and ramification points of the coverings $\lambda : \mathcal{C}_{\mathbf{x}} \to \mathbb{C}P^1$ are as in Section 8.

Following [5], let us pick two ramification points of the covering λ and denote them by $\hat{P}$ and $\hat{Q}$. Let p and q be local parameters at $\hat{P}$ and $\hat{Q}$, respectively, proportional to the respective standard local parameters (2.3) ζ_{λ_k}, where the proportionality coefficients satisfy a precise condition from [5] unimportant for the present discussion. Denote by $U = (U_1, \dots, U_g)$ and $V = (V_1, \dots, V_g)$ the vectors defined by

$$U_j = \omega_j(\hat{P}), \quad V_j = \omega_j(\hat{Q}). \tag{9.1}$$

Here $\omega_1, \dots, \omega_g$ is a basis of normalized holomorphic differentials, as before, and the evaluation at the points $\hat{P}$ and $\hat{Q}$ is done with respect to the local parameters p and q, respectively, as in (2.9). Thus, $2\pi\mathrm{i}\,U$ and $2\pi\mathrm{i}\,V$ are the vectors of b-periods of differentials $\Omega_{\hat{P}}$ and $\Omega_{\hat{Q}}$ of the second kind with poles of the second order only at $\hat{P}$ and only at $\hat{Q}$, respectively, and with the leading coefficients of the Laurent series in p and q being equal to one. Define

$$\Delta = \Big(\int_{\hat{Q}}^{\hat{P}} \omega_1, \dots, \int_{\hat{Q}}^{\hat{P}} \omega_g \Big)^T.$$

Then, since $\hat{P}$ and $\hat{Q}$ are ramification points, Δ is a vector of half-periods:

$$\Delta = \frac{1}{2} M + \frac{1}{2} \mathbb{B} N, \quad M, N \in \mathbb{Z}^g,$$

where $\mathbb{B}$ is the Riemann matrix. According to [5], the function

$$u(X,Y) = \frac{1}{\mathrm{i}}\Big[\ln\frac{\theta(XU+YV+\Delta+z_0)\theta(XU+YV-\Delta+z_0)}{\theta^2(XU+YV+z_0)} + \pi\mathrm{i}\langle N,\Delta\rangle\Big], \tag{9.2}$$

is a solution of the sine-Gordon equation

$$u_{XY} = -4\kappa\sin u, \tag{9.3}$$

where

$$\kappa = \exp(-\pi\mathrm{i}\langle N,\Delta\rangle),$$

and z_0 is an arbitrary vector in the Jacobian of $\mathcal{C}_{\mathbf{x}}$. According to [5], real solutions to the sine-Gordon equation correspond to real hyperelliptic curves, that is curves defined by real polynomials and the ramification points $\hat{P},\hat{Q}$ belong to the same real oval, as defined in Example 1. In this case Δ is real,

$$\Delta = \frac{1}{2}M, \quad M\in\mathbb{Z}^g,$$

and solution (9.2) simplifies to

$$u(X,Y) = -\mathrm{i}\ln\left[\frac{\theta(XU+YV-\frac{M}{2}+z_0)}{\theta(XU+YV+z_0)}\right]^2, \tag{9.4}$$

while the sine-Gordon equation simplifies to

$$u_{XY} = -4\sin u. \tag{9.5}$$

Solution $u(X,Y)$ given by (9.4) of equation (9.5) is real [5] under some condition on the vector z_0 (see two paragraphs below).
Our isoperiodicity results in this paper (Theorems 5, 6, Corollary 4) are formulated for a differential Ω (2.12) of the second kind having a pole of the second order at P_∞, the ramification point at infinity of $\mathcal{C}_{\mathbf{x}}$. These results can be reformulated in a straightforward, though tedious way for a pole at any other ramification point. For simplicity and without loss of generality, we will assume from now on, that $\hat{P} = P_\infty$.
Let us assume now that we have a real periodic solution $\hat{u}_{\mathbf{x}_0}$ constructed from a real hyperelliptic curve (3.3), where *all the branch points* x_j^0, u_j^0, $j = 1,\dots g$ *are real* (note that not all real finite-gap solutions of the sine-Gordon equation satisfy that). The homology basis chosen at the beginning of the section, see Section 8, satisfy relations from Example 1 with respect to the antiholomorphic involution on this curve. Here we have that $\hat{Q}$ corresponds to the largest real finite branch point, so that it belongs to the same real oval as P_∞, see Example 1. In this case, for the reality of solutions, it is also required that $\bar{z}_0 \equiv -\Delta - z_0$. (Here the sign $\equiv$ means modulo the Jacobian lattice.)
Assume that x_g^0 is strictly larger than all other finite branch points $u_1^0,\dots u_g^0, x_1^0,\dots,x_{g-1}^0, 0$, and that $\hat{Q} = P_{x_g^0}$ is the ramification point corresponding to x_g^0. Since the canonical basis of cycles satisfies conditions of Example 1, the corresponding holomorphic normalized differentials $\omega_1,\dots,\omega_g$ are all real. The vector U from (9.1) is also real, and the solution $\hat{u}_{\mathbf{x}_0}$ given by (9.4) to the sine-Gordon equation (9.5) is periodic in X if and only if there exists a positive T such that TU is an integer-valued vector, $TU\in\mathbb{Z}^g$.

Proposition 6 *Let $\mathcal{C}_{\mathbf{x}_0}$ be a curve of equation* (3.3) *with $\mathbf{x} = \mathbf{x}_0$ with all branch points of the covering λ being real. Let $\hat{u}_{\mathbf{x}_0} = \hat{u}_{\mathbf{x}_0}(X, Y)$ be the real solution* (9.4) *to the sine-Gordon equation* (9.5) *constructed from the surface $\mathcal{C}_{\mathbf{x}_0}$ and from the ramification points $\hat{P} = P_\infty \in \mathcal{C}_{\mathbf{x}_0}$ and $\hat{Q}$ such that $\lambda(\hat{Q}) = \max\{0, x_j^0, u_j^0|, j = 1, \dots, g\}$. Assume that $\hat{u}_{\mathbf{x}_0}$ is periodic in X with the period T. There exists a continuous family of real solutions $\hat{u}_{\mathbf{x}}(X, Y)$ for $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$ with $\mathcal{X}$ being some open neighbourhood of $\mathbf{x}_0$ for which all solutions $\hat{u}_{\mathbf{x}}(X, Y)$ of the sine-Gordon equation* (9.5) *are periodic with the period T. Moreover, there exists a family of curves $\mathcal{C}_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$, given by* (3.3) *such that $\hat{u}_{\mathbf{x}}(X, Y)$ is constructed by* (9.4) *from $\mathcal{C}_{\mathbf{x}}$, $\hat{P} = P_\infty \in \mathcal{C}_{\mathbf{x}}$ and $\hat{Q}$ with $\lambda(\hat{Q}) = \max\{0, x_j, u_j|, j = 1, \dots, g\}$. The real-valued functions $u_1(\mathbf{x}), \dots, u_g(\mathbf{x})$ from* (3.3) *corresponding to the family $\mathcal{C}_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$, satisfy equations* (5.18), (5.19) *of Theorem 5.*

Proof. We have that the curve $\mathcal{C}_{\mathbf{x}_0}$ produces a real periodic solution $\hat{u}_{\mathbf{x}_0} = \hat{u}_{\mathbf{x}_0}(X, Y)$ (9.4) of the sine-Gordon equation and that all branch points of the covering $\lambda : \mathcal{C}_{\mathbf{x}_0} \to \mathbb{C}P^1$ are real. In particular, the canonical homology basis satisfies $\rho a_j = a_j$, $\rho b_j = -b_j$ where ρ is the antiholomorphic involution on $\mathcal{C}_{\mathbf{x}_0}$. Thus the vector U (9.1) is real and the Riemann matrix $\mathbb{B}$ is imaginary. From this and the periodicity of the solution $\hat{u}_{\mathbf{x}_0}$ (9.4) and from the properties of quasi-periodicity of the theta-function, we have that the vector U satisfies (6.8) with $m_j = 0$ and $T > 0$ being the period of $\hat{u}_{\mathbf{x}_0}$. This implies that the curve $\mathcal{C}_{\mathbf{x}_0}$ satisfies the Hill condition (2.19), where $\Omega_0 = \Omega^{(1)}_{P_\infty}$ is the differential (8.1) normalized with respect to the above homology basis. Theorem 6 and Corollary 4 give the existence of an isoperiodic real family $(\mathcal{C}_{\mathbf{x}}, \Omega^{(1)}_{P_\infty})$ for $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$ for some open neighbourhood $\mathcal{X}$ of $\mathbf{x}_0$ providing isoperiodic deformations of the pair $(\mathcal{C}_{\mathbf{x}_0}, \Omega_0)$. The curves $\mathcal{C}_{\mathbf{x}}$ are of the form (3.3) where $u_j(\mathbf{x})$ are real by Corollary 4. Therefore, the deformed curves are real and the homology basis of $\mathcal{C}_{\mathbf{x}_0}$ induces naturally a canonical homolgy basis on $\mathcal{C}_{\mathbf{x}}$ with $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$ satisfying relations $\rho a_j = a_j$, $\rho b_j = -b_j$ with respect to the antiholomorphic involution. The differentials $\Omega^{(1)}_{P_\infty}$ on $\mathcal{C}_{\mathbf{x}}$ are normalized with respect to this homology basis. Given that this family is isoperiodic, the Hill curve condition (2.19) holds for $\Omega_0 = \Omega^{(1)}_{P_\infty}$ on all the curves $\mathcal{C}_{\mathbf{x}}$ for $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$. The point $\hat{P}$ remains fixed at infinity, while the branch point $\lambda(\hat{Q})$ remains real and in the same real oval as $\hat{P}$. Thus, the half-integer valued vector Δ remains fixed. This means that the deformed curves generate real solutions (9.4) of the sine-Gordon equation. Since (6.8) is satisfied with the same $T > 0$ and $m_j = 0$, $j = 1, \dots, g$, for all the curves $\mathcal{C}_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{X} \cap \mathbb{R}^g$, as a consequence, solutions $\hat{u}_{\mathbf{x}}(X, Y)$ constructed by (9.4) from $\mathcal{C}_{\mathbf{x}}$ are periodic in X with period $T > 0$. This proves existence of a continuous family of real periodic in X solutions to the sine-Gordon equation sharing the same period. Equations (5.18), (5.19) of Theorem 5 are satisfied since the family $(\mathcal{C}_{\mathbf{x}}, \Omega^{(1)}_{P_\infty})$ is isoperiodic, that is since the b-period vector U is preserved over the family. □

10 Isoperiodic deformations and comb regions

As before, we consider a family $\mathcal{C}_{\mathbf{x}}$ of genus g hyperelliptic curves given by equation (3.3). The hyperelliptic coverings of the λ-sphere have ramification points P_∞ over $\lambda = \infty$ along with P_0, P_{u_j} and P_{x_j}, $j = 1, \dots, g$. We assume $0 < u_1 < x_1 < u_2 < \dots < x_{j-1} < u_j < \dots < x_g < \infty$. Here again $\{a_1, \dots, a_g; b_1, \dots, b_g\}$ is a fixed canonical homology basis on the surfaces $\mathcal{C}_{\mathbf{x}}$ for $\mathbf{x} \in \mathcal{S}_{\mathbf{x}}$, such that a_j goes around the points P_{x_j} and P_{u_j} for $j = 1, \dots, g$, while b_j goes around the points $P_0, \dots, P_{u_j}$ for $j = 1, \dots, g$. The domain $\mathcal{S}_{\mathbf{x}}$ is chosen as in Section 3.1. The normalized differential $\Omega_0 = \Omega^{(1)}_{P_\infty}$ is defined by (8.1).

Define a comb region $\mathcal{CL}$ in the complex upper half plane $\mathbb{H}$, as follows:

$$\mathcal{CL} = \{w \in \mathbb{C} | \Im w > 0,\ 0 \le \Re w \le \infty\} \setminus \bigcup_{j=1}^{g} \{w | \Re w = q_j,\ 0 \le \Im w \le h_j\}$$

with some given positive real values $q_1, \dots, q_g$ and $h_1, \dots, h_g$.
Differential $\Omega_0 = \Omega^{(1)}_{P_\infty}$ having a second order pole at P_∞ on the curve (3.3) as its only singularity, can be defined as a polynomial of degree g times the holomorphic differential (2.4). The coefficients of the polynomial are functions of $\mathbf{x}$ and $u_j(\mathbf{x})$ ensuring the correct normalization. Denote the zeros of the polynomial by ξ_j, $j = 1, \dots, g$. We construct a Schwarz–Christoffel map Θ, following [28], as a conformal map $\Theta : \mathbb{H} \to \mathcal{CL}$:

$$\Theta(z) = \frac{\mathrm{i}}{2} \int_0^z \frac{\prod_j^g (z - \xi_j)\, dz}{\sqrt{z \prod_{j=1}^g (z - x_j)(z - u_j)}}, \tag{10.1}$$

mapping the point at infinity to the point at infinity.
The comb region is a vertical semi-strip with a finite number of vertical slits where the *basis* of the comb region is the semi-infinite ray $[0, \infty) \subset \mathbb{R} = \{z | \Im z = 0\}$ together with the marked points q_j on it. The values q_j correspond to the b-periods of the differential Ω_0 (up to a constant factor), see [28]. The point $q_j + \mathrm{i}h_j$ is the Θ-image of the unique zero ξ_j of the polynomial defining Ω_0 belonging to the interval with the endpoints $\{x_j, u_j\}$, $j = 1, \dots, g$, [28].

Proposition 7 *Consider the differential Ω_0 defined by* (2.12) *with $\alpha = 0$ on a genus g real hyperelliptic curve $\mathcal{C}_{\mathbf{x}}$ with the branch points $\{0, \infty, x_j, u_j | j = 1, \dots, g\}$ of the covering λ satisfying the ordering conditions stated in this section. Consider the corresponding map Θ* (10.1) *from the upper half-plane to the comb region defined by positive values q_j, h_j, $j = 1, \dots, g$, with q_j being the Θ images of $\{u_j | j = 1, \dots, g\}$. Apply an isoperiodic deformation on the pairs $(\mathcal{C}_{\mathbf{x}}, \Omega_0)$ with $\mathbf{x}$ varying in some small neighbourhood $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^g$. Let $\Theta(\mathbf{x})$ be the conformal map* (10.1) *for every $\mathbf{x} \in \mathcal{X}$. Then, in the image of the $\Theta(\mathbf{x})$ for $\mathbf{x} \in \mathcal{X}$, the points q_j remain independent of $\mathbf{x}$, i.e. the basis of the comb is invariant under the isoperiodic deformations. The deformations apply only vertically by varying h_j, the imaginary parts of the tips of the vertical slits of the comb, that are the imaginary parts of the Θ-images of the zeros of the differential Ω_0, along the vertical rays, based at q_j, for each $j = 1, \dots, g$. Such a family of conformal maps of the comb regions with a finite number of slits and with a semi-infinite basis of the comb, that are obtained by varying h_j vertically are governed by equations* (5.18), (5.19) *as well as* (5.3), *as equations for the real-valued functions $u_1(\mathbf{x}), \dots, u_g(\mathbf{x})$ from* (3.3).

Proof. The values q_j correspond to the b-periods of the differential Ω_0, by construction and properties of the Schwarz–Christoffel map Θ, as explained in [28]. The isoperiodic deformations $(\mathcal{C}_{\mathbf{x}}, \Omega_0)$ preserve the periods of the deformed differential, thus they preserve the basis of the comb. Therefore, during the deformation, the tips of the vertical slits, defined by h_j deform vertically, along the slit. The isoperiodic deformations are governed by equations (5.18), (5.19) for the real-valued functions $u_1(\mathbf{x}), \dots, u_g(\mathbf{x})$ from (3.3), according to Theorem 5 and Corollary 4, and satisfy (5.3). □

Thus, Proposition 7 provides an explicit rectification of isoperiodic deformations, seen as vertical deformations of the teeth of the comb, with an unchanged base of the comb, together with the exact equations which govern deformations of the Θ-preimages of the base of the obtained comb regions.

Remark 1 *In the case $g = 1$, the comb regions have one vertical slit. The vertical deformations of that slit are governed by differential equation* (4.15) *from Theorem 1. Thus, equation* (4.15) *is related to a version of the Loewner differential equation, which governs deformations of conformal maps induced by deformations of regions with one slit. Therefore, we can see equations* (5.18), (5.19) *of Theorem 5 as a sort of PDE generalizations related to multi-slit regions. For more about Loewner differential equation, see [27, 24, 25, 11, 13].*

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