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On the Weyl anomaly for chiral fermions.

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Abstract

We compute the parity-odd part of the Weyl anomaly for chiral fermions in a background gravitational field. We start from a manifestly real form of the Lagrangian (that is, not only real up to a total derivative), and we regularize it by means of Pauli-Villars fermions. All parity-odd terms in the anomaly cancel in the *integrand*, so that the result of the anomaly is necessarily parity-even.

Contents

1 Introduction

There has recently been some interest in the Weyl (conformal) anomaly of a chiral fermion on a background gravitational field with metric $g_{\mu\nu}$. This was motivated by a paper by Bonora [4] claiming that there is a parity-odd term in the trace anomaly proportional to the Pontryagin index. Symbollically

$$\langle g^{\mu\nu} T_{\mu\nu} \rangle \sim i \int d(vol)\, R_{\mu\nu\rho\sigma} \eta^{\rho\sigma\alpha\beta} R_{\alpha\beta}{}^{\mu\nu} \tag{1.1}$$

with an *imaginary* coefficient. Here $R_{\mu\nu\rho\sigma}$ are the components of Riemann's tensor and $d(vol)$ is the usual volume element

$$d(vol) \equiv \frac{1}{4!}\, \eta_{\mu\nu\rho\sigma} dx^{\mu\nu\rho\sigma} \equiv \frac{1}{4!} \sqrt{|g|} \eta_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{1.2}$$

This result would violate unitarity and point to a fatal inconsistency in the physics of such chiral fermions. The main reason is that the energy momentum tensor is the coefficient of the response of the action to an arbitrary (but real) variation of the metric

$$\delta S \equiv \int d(vol) T_{\mu\nu} \delta g^{\mu\nu} \tag{1.3}$$

If the trace of the energy momentum tensor is not real, then the action itself is also not real, which violates unitarity among other things.

This proposal however has been challenged by some authors [12, 1, 2, 7, 9, 11, 15], but in some other publications like [13] support it.

To be precise, in reference [4] it is argued that the total conformal anomaly of a chiral fermion is given by

$$T_\mu{}^\mu = \frac{1}{180 \times 16\pi^2} \left(\frac{11}{4} E_4 - \frac{9}{2} W^2 + i \frac{15}{4} P \right) \tag{1.4}$$

here E_4 is the integrand of the Euler characteristic, a topological invariant given by

$$\begin{aligned} \chi(M) \quad & \equiv \frac{1}{32\pi^2} \int \eta_{abcd} R^{ab} \wedge R^{cd} = \int d(vol) E_4 = \frac{1}{2} \frac{1}{(4\pi)^2} \int d(vol) R^* R^* = \\ & = \frac{1}{32\pi^2} \int d(vol) \left(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 \right) \end{aligned} \tag{1.5}$$

and W^2 is the square of Weyl's tensor, which can be characterized by being invariant under conformal transformations

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \tag{1.6}$$

namely

$$W^{\mu}{}_{\nu\rho\sigma} \to W^{\mu}{}_{\nu\rho\sigma} \tag{1.7}$$

It follows that $\sqrt{|g|}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ is a conformal singlet in $n=4$ dimensions.

$$W^2 \equiv W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2 \tag{1.8}$$

and $P_1(M)$ is the Pontryagin density (another topological invariant [6])

$$P_1(M) \equiv -\frac{1}{8\pi^2}\int \mathrm{tr}R\wedge R = \frac{1}{(4\pi)^2}\int d(vol)R^*_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{1.9}$$

The dual Riemann tensor is given by

$$R^*_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\eta_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} \tag{1.10}$$

Our aim in this paper is to compute this anomaly using a different, and in our opinion, safer procedure [16, 14, 10]. We will start from a real Lagrangian and regulate the theory using Pauli-Villars fermions in an attempt to avoid the ambiguities of dimensional regularization in the presence of γ_5.

Our main interest will be in the parity-odd part of the anomaly (the coefficient of the Pontryagin invariant) which is where there is disagreement in the literature.

Let us now be more precise about the definition of the problem.

Let $\omega_{\mu ab}(x)$ be the Lorentz (spin) connection defined in terms of the Levi-Civita one by

$$\omega_{\mu ab}(x) = -e^{\lambda}_{b}(x)\partial_\mu e_{\lambda a}(x) + e_{\lambda a}(x)\Gamma^{\lambda}_{\rho\mu}e^{\rho}_{b}(x).$$

Then, the classical action of the model reads

$$S = \frac{1}{2}\int d^4x\, e(x)\, e^{\mu}_{a}(x)\,[i\bar{\xi}_{\dot{\alpha}}(x)\bar{\sigma}^{\dot{\alpha}\alpha a}(\mathcal{D}_\mu\xi)_\alpha(x) - i(\mathcal{D}_\mu\bar{\xi})_{\dot{\alpha}}(x)\bar{\sigma}^{\dot{\alpha}\alpha a}\xi_\alpha(x)],$$

where

$$(\mathcal{D}_\mu\xi)_\alpha(x) = \partial_\mu\xi_\alpha(x) - \frac{i}{2}\omega_{\mu ab}(x)\,(\sigma^{ab}\xi)_\alpha(x), \quad (\mathcal{D}\bar{\xi})_{\dot{\alpha}}(x) = \partial_\mu\bar{\xi}_{\dot{\alpha}}(x) + \frac{i}{2}(\bar{\xi}(x)\bar{\sigma}^{ab})_{\dot{\alpha}}\,\omega_{\mu ab}(x).$$

We shall denote spinorial indices by the first letters of the Greek alphabet ($\alpha, \dot{\alpha}, \beta, \dot{\beta}\ldots. = 1,2$) and reserve the middle letters of the Greek alphabet for four-dimensional vector indices

($\mu, \nu, \ldots = 0, \ldots, 3$) Our convention [5] for the Pauli matrices is as follows

$$
\begin{aligned}
\sigma^0 &= \overline{\sigma}^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\sigma^1 &= -\overline{\sigma}^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma^2 &= -\overline{\sigma}^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\sigma^3 &= -\overline{\sigma}^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
\tag{1.11}
$$

There is another notation frequently used representing chiral spinors as left and right ones, ψ_L or ψ_R. The relationship with ours is

$$
\begin{aligned}
\psi_L &\leftrightarrow \xi_\alpha \\
\psi_R &\leftrightarrow \xi_{\dot{\alpha}}
\end{aligned}
\tag{1.12}
$$

In is important to remark that the action is real without total derivatives. This fact will be relevant later on.

In constructing the quantum theory for the fermions in perturbation theory around Minkowski spacetime, we shall express the metric, $g_{\mu\nu}$, the vierbein e^a_μ, the inverse vierbein, e^μ_a, and the spin connection, $\omega_{a,bc}$ as follows (the perturbative parameter is related to Newton's constant by $\kappa^2 \equiv 8\pi G$)

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu}, \\
e^a_\mu &= \delta^a_\mu + \kappa \mathcal{E}^{(1)\,a}_\mu + \kappa^2 \mathcal{E}^{(2)\,a}_\mu + \ldots \\
e^\mu_a &= \delta^\mu_a + \kappa \mathcal{E}^{(1)\,\mu}_a + \kappa^2 \mathcal{E}^{(2)\,\mu}_a + \ldots \\
\omega_{a,bc} &= e^\mu_a \omega_{\mu bc} = \kappa \Omega^{(1)}_{a,bc} + \kappa^2 \Omega^{(2)}_{a,bc} + \ldots.,
\end{aligned}
$$

in the above the terms involving the external gravitons become

$$
\begin{aligned}
&\mathcal{E}^{(1)\,a}_\mu = \frac{1}{2} h^a_\mu, \quad \mathcal{E}^{(2)\,a}_\mu = -\tfrac{1}{8}\, h^{a\rho} h_{\rho\mu}, \quad \mathcal{E}^{(1)\,\mu}_a = \frac{1}{2} h^\mu_a, \quad \mathcal{E}^{(2)\,\mu}_a = \tfrac{3}{8}\, h^{\mu\rho} h_{\rho a} \\
&\Omega^{(1)}_{a,bc} = -\tfrac{1}{2}\left(\partial_b h_{ca} - \partial_c h_b a\right) \\
&\Omega^{(2)}_{a,bc} = \tfrac{1}{4} h^\mu_a (\partial_b h_{c\mu} - \partial_c h_{b\mu}) - \tfrac{1}{8}(h^\rho_b \partial_a h_{c\rho} - h^\rho_c \partial_a h_{b\rho}) + \tfrac{1}{4}(h^\rho_b \partial_\rho h_{ca} - h^\rho_c \partial_\rho h_{ba}) - \tfrac{1}{4}(h^\rho_b \partial_c h_{\rho a} - h^\rho_c \partial_b h_{\rho a}).
\end{aligned}
$$

The linearized conformal transformation reads

$$\delta_\theta h_{\mu\nu} = 2\left(\frac{\theta(x)}{\kappa}\eta_{\mu\nu} + h_{\mu\nu}\right) \tag{1.13}$$

The previous expansions in powers of κ, leads to the following expansion of the action S:

$$\begin{aligned}
&S = S_0 + S_{int}\\
&S_0 = \frac{1}{2}\int d^4x\; [i\bar{\xi}_{\dot\alpha}(x)\bar{\sigma}^{\dot\alpha\alpha\mu}(\partial_\mu\xi)_\alpha(x) - i(\partial_\mu\bar{\xi})_{\dot\alpha}(x)\bar{\sigma}^{\dot\alpha\alpha\mu}\xi_\alpha(x)],\\
&S_{int} = \sum_n \kappa^n S^{(n)}.
\end{aligned}$$

In both S_0 and S_{int} all the indices are flat. S_0 is the free action in Minkowski spacetime and S_{int} carry the interaction vertices between the graviton field $h_{\mu\nu}$ and the fermions. $S^{(n)}$ is quadratic on the fermion fields and involves n powers of $h_{\mu\nu}$ and/or its derivatives.

Let $\mathcal{Z}[h_{\mu\nu}]$ be partition function of the model is defined by perturbation theory around the Minkowski background

$$\begin{aligned}
&\mathcal{Z}[h_{\mu\nu}] = \frac{1}{\mathcal{N}}\int \mathcal{D}\xi_\alpha \mathcal{D}\bar{\xi}_{\dot\alpha}\; e^{iS} = \langle\, e^{iS_{int}}\,\rangle_0\\
&\mathcal{N} = \int \mathcal{D}\xi_\alpha \mathcal{D}\bar{\xi}_{\dot\alpha}\; e^{iS_0}.
\end{aligned}$$

$\langle\mathcal{O}\rangle_0$ denotes the average of $\mathcal{O}$ with regard to the free action S_0:

$$\langle\mathcal{O}\rangle_0 = \frac{1}{\mathcal{N}}\int \mathcal{D}\xi_\alpha \mathcal{D}\bar{\xi}_{\dot\alpha}\, \mathcal{O}\, e^{iS_0}$$

Then, the quantum action, $\mathcal{W}[h_{\mu\nu}]$, of the model is defined to be

$$\mathcal{W}[h_{\mu\nu}] = -i\,\mathrm{Ln}\mathcal{Z}[h_{\mu\nu}] = -i\sum_n \frac{i^n}{n!}\langle (S_{int})^n\rangle_0^{(connected)}, \tag{1.14}$$

The superscript "connected" signals that only connected contractions, as given by Wick's theorem, are to be kept.

1.1 Pauli-Villars fields and the regularized quantum action.

As it stands, $\mathcal{W}[h_{\mu\nu}]$, (1.14) is an ill-defined due to ultraviolet (UV) divergences. We shall associate to this ill-definite quantity a regularized –and, therefore, well-defined mathematically–

expression. This will be done by introducing several massive Pauli-Villars bispinor fields: $\xi^{(i)}_\alpha$, $\bar\xi^{(i)}_{\dot\alpha}$, $i=1...f$, with classical actions

$$
\begin{aligned}
&S^{(i)} = \\
&\tfrac{1}{2}\int d^4x\, e(x)\, e^\mu_a(x)\, [i\bar\xi^{(i)}_{\dot\alpha}(x)\bar\sigma^{\dot\alpha\alpha a}(\mathcal{D}_\mu\xi^{(i)})_\alpha(x) - i(\mathcal{D}\bar\xi^{(i)})_{\dot\alpha}(x)\bar\sigma^{\dot\alpha\alpha a}\xi^{(i)}_\alpha(x)] \\
&-\tfrac{m_i}{2}\int d^4x\, e(x)\ [\xi^{(i)\,\alpha}(x)\xi^{(i)}_\alpha(x) + \bar\xi^{(i)}_{\dot\alpha}(x)\bar\xi^{(i)\,\dot\alpha}(x)].
\end{aligned}
$$

By using these actions, we define

$$
\mathcal{W}^{(i)}[h_{\mu\nu}] = -i\,\mathrm{Ln}\mathcal{Z}[h_{\mu\nu}] = -i\sum_n \frac{i}{n!}\langle (S^{(i)}_{int})^n\rangle_0^{(connected)}, \quad i=1...f. \tag{1.15}
$$

Finally, we regularize $\mathcal{W}[h_{\mu\nu}]$ by replacing it with $\mathcal{W}^{(PV)}[h_{\mu\nu}]$, the latter defined as follows

$$
\mathcal{W}^{(PV)}[h_{\mu\nu}] = \sum_{i=0}^{f} c_i\,\mathcal{W}^{(i)}[h_{\mu\nu}], \tag{1.16}
$$

with the understanding that $i=0$ labels the physical field: $\xi^{(0)}_\alpha = \xi_\alpha$, $\bar\xi^{(0)}_{\dot\alpha} = \bar\xi_{\dot\alpha}$, $m_0=0$ and $c_0=1$.

The c_i 's above are the Pauli-Villars (PV) parameters and, along with Pauli-Villars masses, are to be chosen so that the loop integrals are UV finite by power-counting at non exceptional momenta. This regularization breaks fermion number owing to the PV mass terms, but respects diffeomorphism invariance. Any dependence on the PV masses will disappear in the physical theory.

It is well known that when dealing with anomalies it is of paramount importance not to carry out any kind of simplification in the integrands of the non-regularized UV divergent integrals. So, here, we shall be careful and choose the c_i 's and the Pauli-Villars masses so that all the loop integrals are regularized before carrying out any manipulations in the integrands of the loop integrals.

It is not difficult to come to the conclusion that the most UV divergent loop integrals which occur in $\mathcal{W}[h_{\mu\nu}]$ are

$$
\int\frac{d^4q}{(2\pi)^4}\ \frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2n}}}{(q+P_1)^2(q+P_2)^2\cdots(q+P_n)^2},
$$

where the P_i 's are (non-exceptional) linear combinations of the external momenta. The Pauli-

Villars regularization of the previous integral amounts to its replacement with

$$\int\frac{d^4q}{(2\pi)^4}\ \Big[\sum_{i=0}^{f} c_i\,\frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2n}}}{((q+P_1)^2-m_i^2)((q+P_2)^2-m_i^2)\cdots((q+P_n)^2-m_i^2)}\Big]. \tag{1.17}$$

Now, all these integrals will be UV finite by power-counting if the c_i's and m_i's satisfy the Pauli-Villars conditions, which, in the case at hand, become:

$$c_0=1,\qquad \sum_{i=0}^{f} c_i=0,\ \sum_{i=1}^{f} c_i\,m_i^2=0\qquad\text{and}\quad \sum_{i=1}^{f} c_i\,m_i^4=0.$$

The condition on m_i^4 has to do with the fact that the integrals in (1.17) diverge as the fourth power of the loop momentum.

But this is not all, for the mass terms of the Pauli-Villars fields also couple to gravity so that we shall have interaction vertices linear in the m_i's. These vertices do not occur in the massless physical theory. Further, unlike in the massless case, the free propagators

$$\langle\xi^{(i)}_\alpha(x)\xi^{(i)}_\beta(y)\rangle_0,\quad \langle\bar\xi^{(i)}_{\dot\alpha}(x)\bar\xi^{(i)}_{\dot\beta}(y)\rangle_0,\quad i=1...f,$$

have numerators linear in the $m_i's$. Hence, we will meet UV divergent integrals of the type

$$\int\frac{d^4q}{(2\pi)^4}\ \Big[\sum_{i=1}^{f} c_i\,m_i^{2n'}\frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2(n-n')}}}{((q+P_1)^2-m_i^2)((q+P_2)^2-m_i^2)\cdots((q+P_n)^2-m_i^2)}\Big],$$

with $n'=1,2$. Making these integrals UV finite does not require new Pauli-Villars conditions.

2 No parity-odd terms in the conformal anomaly.

A simple way of understanding the Weyl anomaly stems from a Weyl transformation in the path integral [8]. This will be enough to show that no parity-odd terms appear in the anomaly. Recall that

$$S[g,\xi,\bar\xi]=\frac{1}{2}\int d^4x\,e(x)\,e^\mu_a(x)\,[i\bar\xi_{\dot\alpha}(x)\bar\sigma^{\dot\alpha\alpha a}(\mathcal{D}_\mu\xi)_\alpha(x)-i(\mathcal{D}_\mu\bar\xi)_{\dot\alpha}(x)\bar\sigma^{\dot\alpha\alpha a}\xi_\alpha(x)],$$

$$S^{(i)}[g,\xi^{(i)},\bar\xi^{(i)}]=\frac{1}{2}\int d^4x\,e(x)\,e^\mu_a(x)\,[i\bar\xi^{(i)}_{\dot\alpha}(x)\bar\sigma^{\dot\alpha\alpha a}(\mathcal{D}_\mu\xi)^{(i)}_\alpha(x)-i(\mathcal{D}_\mu\bar\xi^{(i)})_{\dot\alpha}(x)\bar\sigma^{\dot\alpha\alpha a}\xi^{(i)}_\alpha(x)],$$

$$S^{(i)}_m[g,\xi^{(i)},\bar\xi^{(i)}]=-\tfrac{m_i}{2}\int d^4x\,e(x)\ [\xi^{(i)\,\alpha}(x)\xi^{(i)}_\alpha(x)+\bar\xi^{(i)}_{\dot\alpha}(x)\bar\xi^{(i)\,\dot\alpha}(x)],$$

and that in (1.16) we defined the regularized Pauli-Villars effective action $\mathcal{W}^{(PV)}[g_{\mu\nu}]$ by

$$\mathcal{W}^{(PV)}[g_{\mu\nu}] = -i \sum_{i=0}^{f} c_i \ln \mathcal{Z}_i[g_{\mu\nu}]$$

where

$$\mathcal{Z}_i[g_{\mu\nu}] = \frac{1}{\mathcal{N}_i} \int \mathcal{D}\xi_\alpha^{(i)} \mathcal{D}\bar{\xi}_{\dot{\alpha}}^{(i)} \; e^{iS^{(i)}[g,\xi^{(i)},\bar{\xi}^{(i)}]+iS_m^{(i)}[g,\xi^{(i)},\bar{\xi}^{(i)}]}$$

Then by performing the Weyl transformations

$$g_{\mu\nu} \to \Omega^2\, g_{\mu\nu}, \quad \xi \to \Omega^{-3/2}\xi, \quad \bar{\xi} \to \Omega^{-3/2}\bar{\xi}, \qquad \xi^{(i)} \to \Omega^{-3/2}\xi^{(i)}, \quad \bar{\xi}^{(i)} \to \Omega^{-3/2}\bar{\xi}^{(i)}, \tag{2.1}$$

we get

$$\mathcal{W}^{(PV)}[\Omega^2 g_{\mu\nu}] = -i \sum_{i=0}^{f} c_i \ln \mathcal{Z}_i[\Omega^2 g_{\mu\nu}] = -i \sum_{i=0}^{f} c_i \ln \frac{1}{\mathcal{N}_i} \int \mathcal{D}\xi_\alpha^{(i)} \mathcal{D}\bar{\xi}_{\dot{\alpha}}^{(i)} \; e^{iS^{(i)}[g,\xi^{(i)},\bar{\xi}^{(i)}]+iS_m^{(i)}[g,\xi^{(i)},\bar{\xi}^{(i)};\Omega]}, \tag{2.2}$$

where the transformed PV mass terms read

$$S_m^{(i)}[g,\xi^{(i)},\bar{\xi}^{(i)};\Omega] = -\frac{m_i}{2} \int d^4x\, \Omega(x) e(x) \; [\xi^{(i)\,\alpha}(x)\xi_\alpha^{(i)}(x) + \bar{\xi}_{\dot{\alpha}}^{(i)}(x)\bar{\xi}^{(i)\,\dot{\alpha}}(x)].$$

Let us linearize around the unit transformation

$$\Omega(x) = 1 + \theta(x), \quad 0 \leqslant \theta(x) << 1.$$

Now, the infinitesimal version of (2.2) reads

$$\delta_\theta \mathcal{W}^{(PV)}[g_{\mu\nu}] = \sum_{i=1}^{f} c_i \frac{1}{\mathcal{Z}_i[g]\mathcal{N}_i} \int \mathcal{D}\xi_\alpha^{(i)} \mathcal{D}\bar{\xi}_{\dot{\alpha}}^{(i)} \;\; (\delta_\theta S_m^{(i)}) \; e^{iS^{(i)}[g,\xi^{(i)},\bar{\xi}^{(i)}]+iS_m^{(i)}[g,\xi^{(i)},\bar{\xi}^{(i)}]}, \tag{2.3}$$

where

$$\delta_\theta S_m^{(i)} = -\frac{1}{2}\, m_i \int d^4x\, \theta(x)\, e(x) \; [\xi^{(i)\,\alpha}(x)\xi_\alpha^{(i)}(x) + \bar{\xi}_{\dot{\alpha}}^{(i)}(x)\bar{\xi}^{(i)\,\dot{\alpha}}(x)], \quad i \geqslant 1.$$

The right hand side of (2.3) yields the anomaly, and at order two in the number of graviton fields $h_{\mu\nu}$ it is given by

$$\begin{aligned}\mathcal{A}_2 = -\tfrac{1}{2} \sum_{i=1}^{f} c_i \langle \delta_\theta^{(0)} S_m^{(i)} S_{kin}^{(i),1} S_{kin}^{(i),1} \rangle_0^{(c)} + i \sum_{i=1}^{f} c_i \langle \delta_\theta^{(0)} S_m^{(i)} S_{kin}^{(i),2} \rangle_0^{(c)} + i \sum_{i=1}^{f} c_i \langle \delta_\theta^{(0)} S_m^{(i)} S_{spin}^{(i),2} \rangle_0^{(c)} + \\ i \sum_{i=1}^{f} c_i \langle \delta_\theta^{(1)} S_m^{(i)} S_{kin}^{(i),1} \rangle_0^{(c)} + \sum_{i=1}^{f} c_i \langle \delta_\theta^{(2)} S_m^{(i)} \rangle_0^{(c)},\end{aligned} \tag{2.4}$$

Let us recall that the superscript (c) stands for the connected contributions, the subscript 0 denotes vaccum expectation vvalue when $h_{\mu\nu}$ is set to zero, and the Weyl variation of the PV mass terms become

$$
\begin{aligned}
\delta_\theta^{(0)} S_m^{(i)} &= -\tfrac{1}{2}\, m_i \int d^4x\, \theta(x)\, [\xi^{(i)\alpha}(x)\xi_\alpha^{(i)}(x) + \bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\xi}^{(i)\,\dot\alpha}(x)],\\
\delta_\theta^{(1)} S_m^{(i)} &= -\tfrac{1}{4}\, m_i \int d^4x\, \theta(x)\, h_\mu^\mu(x)\, [\xi^{(i)\alpha}(x)\xi_\alpha^{(i)}(x) + \bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\xi}^{(i)\,\dot\alpha}(x)],\\
\delta_\theta^{(2)} S_m^{(i)} &= -\tfrac{1}{4}\, m_i \int d^4x\, \theta(x)\, H_2(x)\, [\xi^{(i)\alpha}(x)\xi_\alpha^{(i)}(x) + \bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\xi}^{(i)\,\dot\alpha}(x)],
\end{aligned}
$$

and the kinetic and spin parts are defined as

$$
\begin{aligned}
S_{kin}^{(i),1} &= \int d^4x\, \Big\{ \tfrac{i}{2}\, H^\mu_{1,\,\nu}(x)\, [\bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\sigma}^{\dot\alpha\alpha\nu}\partial_\mu\xi_\alpha^{(i)}(x) - \partial_\mu\bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\sigma}^{\dot\alpha\alpha\nu}\xi_\alpha^{(i)}(x)]\\
&\qquad\qquad -\tfrac{m_i}{4} h_\mu^\mu(x)\, [\xi^{(i)\alpha}(x)\xi_\alpha^{(i)}(x) + \bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\xi}^{(i)\,\dot\alpha}(x)]\Big\},\\
S_{kin}^{(i),2} &= \int d^4x\, \Big\{ \tfrac{i}{2}\, H^\mu_{2,\,\nu}(x)\, [\bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\sigma}^{\dot\alpha\alpha\nu}\partial_\mu\xi_\alpha^{(i)}(x) - \partial_\mu\bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\sigma}^{\dot\alpha\alpha\nu}\xi_\alpha^{(i)}(x)]\\
&\qquad\qquad -\tfrac{m_i}{4} H_2(x)\, [\xi^{(i)\alpha}(x)\xi_\alpha^{(i)}(x) + \bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\xi}^{(i)\,\dot\alpha}(x)]\Big\},\\
S_{spin}^{(i),2} &= \int d^4x\, \tfrac{1}{16}\, \epsilon^{\mu\nu\rho\sigma}\, h_\mu^\lambda(x)\partial_\nu h_{\rho\lambda}(x)\bar{\xi}_{\dot\alpha}^{(i)}(x)\bar{\sigma}_\sigma^{\dot\alpha\alpha}\xi_\alpha^{(i)}(x),
\end{aligned}
$$

$$
\begin{aligned}
H^\mu_{1,\,\nu}(x) &= \tfrac{1}{2}\,(h_\rho^\rho(x)\delta_\nu^\mu - h_\nu^\mu(x)),\\
H^\mu_{2,\,\nu}(x) &= -\tfrac{1}{4}h_\rho^\rho(x)h_\nu^\mu(x) + \tfrac{1}{8}(h_\rho^\rho(x))^2\delta_\nu^\mu - \tfrac{1}{4}h^{\rho\sigma}(x)h_{\rho\sigma}(x) + \tfrac{3}{8}h^{\mu\rho}(x)h_{\rho\nu}(x),\\
H_2(x) &= \tfrac{1}{4}(h_\rho^\rho(x))^2 - \tfrac{1}{2}h^{\rho\sigma}(x)h_{\rho\sigma}(x).
\end{aligned}
$$

2.1 No parity-odd anomaly from Triangles.

There are terms including a variation of a PV mass and two terms from the kinetic terms that would give rise to Feynman diagrams type *triangle* upon Wick contractions. Let us examine them

First of all,

$$
-\frac{1}{2}\sum_{i=1}^{f} c_i\, \langle \delta_\theta^{(0)} S_m^{(i)} S_{kin}^{(i),1} S_{kin}^{(i),1}\rangle_0^{(c)} = -\frac{1}{2}(\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3)
$$

This (2.4) yields no parity-odd anomalous contribution. In the previous equation, $\mathcal{T}_m$, $m = 1,2,3$, is given by

$$
\begin{aligned}
&\mathcal{T}_1 = \tfrac{1}{8}\int dx_1\, dx_2\, dx_3\, \theta(x_1)\, H^{\mu_2}_{1,\nu_2}(x_2) H^{\mu_3}_{1,\nu_3}(x_3)\Big\{\\
&\sum_{i=1}^{f} c_i\, m_i\, [\xi^{(i)\alpha}\xi_\alpha^{(i)}(x_1) + \bar{\xi}_{\dot\alpha}^{(i)}\bar{\xi}^{(i)\,\dot\alpha}(x_1)][j^{(i)\,\nu_2}_{R\,\mu_2}(x_2) - j^{(i)\,\nu_2}_{L\,\mu_2}(x_2)][j^{(i)\,\nu_3}_{R\,\mu_3}(x_3) - j^{(i)\,\nu_3}_{L\,\mu_3}(x_3)]\rangle_0^{(c)}\Big]\Big\},
\end{aligned}
\tag{2.5}
$$

Here the left and right currents are defined as

$$j_{R\ \mu}^{(i)\ \nu}(x) \equiv \bar{\xi}_{\dot{\alpha}}^{(i)}(x)\bar{\sigma}^{\dot{\alpha}\alpha\nu}\partial_\mu\xi_\alpha^{(i)}(x)$$
$$j_{L\ \mu}^{(i)\ \nu}(x) \equiv \partial_\mu\bar{\xi}_{\dot{\alpha}}^{(i)}(x)\bar{\sigma}^{\dot{\alpha}\alpha\nu}\xi_\alpha^{(i)}(x) \tag{2.6}$$

$$\mathcal{T}_2 = \tfrac{1}{8}\, i \int dx_1\, dx_2\, dx_3\, \theta(x_1)\, h_{\mu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i^2\, [\xi^{(i)\alpha}\xi_\alpha^{(i)}(x_1) + \bar{\xi}_{\dot{\alpha}}^{(i)}\bar{\xi}^{(i)\,\dot{\alpha}}(x_1)][\xi^{(i)\alpha}\xi_\alpha^{(i)}(x_2) + \bar{\xi}_{\dot{\alpha}}^{(i)}\bar{\xi}^{(i)\,\dot{\alpha}}(x_2)][j_{R\,\mu_3}^{(i)\,\nu_3}(x_3) - j_{L\,\mu_3}^{(i)\,\nu_3}(x_3)]\rangle_0^{(c)} \Big] \Big\} \tag{2.7}$$

and

$$\mathcal{T}_3 = -\tfrac{1}{32} \int dx_1\, dx_2\, dx_3\, \theta(x_1)\, h_{\mu_2}^{\mu_2}(x_2) h_{\mu_3}^{\mu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i^3\, [\xi^{(i)\alpha}\xi_\alpha^{(i)}(x_1) + \bar{\xi}_{\dot{\alpha}}^{(i)}\bar{\xi}^{(i)\,\dot{\alpha}}(x_1)][\xi^{(i)\alpha}\xi_\alpha^{(i)}(x_2) + \bar{\xi}_{\dot{\alpha}}^{(i)}\bar{\xi}^{(i)\,\dot{\alpha}}(x_2)][\xi^{(i)\alpha}\xi_\alpha^{(i)}(x_3) + \bar{\xi}_{\dot{\alpha}}^{(i)}\bar{\xi}^{(i)\,\dot{\alpha}}(x_3)] \Big\}. \tag{2.8}$$

2.1.1 No Parity-odd Weyl anomaly from $\mathcal{T}_1$.

Let us express $\mathcal{T}_1$ in (2.5) in terms of the three different products of the left and right currents and their symmetrizations

$$\mathcal{T}_1 = \frac{1}{8}\sum_{m=1}^{6}\mathcal{T}_{1m},$$

where

$$\mathcal{T}_{11} = \int dx_1\, dx_2\, dx_3\, \theta(x_1)\, H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i \langle \xi^{(i)\alpha}\xi_\alpha^{(i)}(x_1) j_{R\,\mu_2}^{(i)\,\nu_2}(x_2) j_{R\,\mu_3}^{(i)\,\nu_3}(x_3)\rangle_0^{(c)} \Big] \Big\},$$

$$\mathcal{T}_{12} = \int dx_1\, dx_2\, dx_3\, \theta(x_1)\, H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i\, \langle \bar{\xi}_{\dot{\alpha}}^{(i)}\bar{\xi}^{(i)\,\dot{\alpha}}(x_1) j_{R\,\mu_2}^{(i)\,\nu_2}(x_2) j_{R\,\mu_3}^{(i)\,\nu_3}(x_3)\rangle_0^{(c)} \Big] \Big\},$$

$$\mathcal{T}_{13} = \int dx_1\, dx_2\, dx_3\, \theta(x_1)\, H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i \langle \xi^{(i)\alpha}\xi_\alpha^{(i)}(x_1) j_{L\,\mu_2}^{(i)\,\nu_2}(x_2) j_{L\,\mu_3}^{(i)\,\nu_3}(x_3)\rangle_0^{(c)} \Big] \Big\},$$

$$\mathcal{T}_{14} = \int dx_1\, dx_2\, dx_3\, \theta(x_1)\, H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i\, \langle \bar{\xi}_{\dot{\alpha}}^{(i)}\bar{\xi}^{(i)\,\dot{\alpha}}(x_1) j_{L\,\mu_2}^{(i)\,\nu_2}(x_2) j_{L\,\mu_3}^{(i)\,\nu_3}(x_3)\rangle_0^{(c)} \Big] \Big\},$$

$$\mathcal{T}_{15} = -2\int dx_1\, dx_2\, dx_3\, \theta(x_1)\, H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i \langle \xi^{(i)\alpha}\xi_\alpha^{(i)}(x_1) j_{R\,\mu_2}^{(i)\,\nu_2}(x_2) j_{L\,\mu_3}^{(i)\,\nu_3}(x_3)\rangle_0^{(c)} \Big] \Big\},$$

$$\mathcal{T}_{16} = -2\int dx_1\, dx_2\, dx_3\, \theta(x_1)\, H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i\, \langle \bar{\xi}_{\dot{\alpha}}^{(i)}\bar{\xi}^{(i)\,\dot{\alpha}}(x_1) j_{L\,\mu_2}^{(i)\,\nu_2}(x_2) j_{R\,\mu_3}^{(i)\,\nu_3}(x_3)\rangle_0^{(c)} \Big] \Big\}.$$

Recall that $\mathcal{T}_1$ is symmetric under the exchange of (μ_2, ν_2, x_2) and (μ_3, ν_3, x_3).

The application of Wick's theorem readily leads to the following results for the parity-odd contributions of $\mathcal{T}_{1m}$:

$$
\begin{aligned}
&\mathcal{T}_{11}[odd] = 4\int \prod_{i=1}^{3} \frac{d^4p_j}{(2\pi)^4}(2\pi)^4\, \delta(p_1+p_2+p_3)\,\theta(p_1)\, H^{\mu_2}_{1,\nu_2}(p_2) H^{\mu_3}_{1,\nu_3}(p_3)\Big\{ \\
&\int\frac{d^4q}{(2\pi)^4}\,\Big[\sum_{i=1}^{f} c_i m_i^2 \frac{\epsilon^{\nu_4\nu_5\nu_2\nu_3}\big[(q+p_3)_{\mu_2}(q+p_3)_{\mu_3}(p_2+p_3)_{\nu_4}q_{\nu_5} - 2(q+p_2+p_3)_{\mu_2}(q+p_3)_{\mu_3}p_{3\nu_4}q_{\nu_5}\big]}{(q^2-m_i^2)((q+p_3)^2-m_i^2)((q+p_2+p_3)^2-m_i^2)}\Big]\Big\} \\
&\mathcal{T}_{12}[odd] = 4\int \prod_{i=1}^{3} \frac{d^4p_j}{(2\pi)^4}(2\pi)^4\, \delta(p_1+p_2+p_3)\,\theta(p_1)\, H^{\mu_2}_{1,\nu_2}(p_2) H^{\mu_3}_{1,\nu_3}(p_3)\Big\{ \\
&\int\frac{d^4q}{(2\pi)^4}\,\Big[\sum_{i=1}^{f} c_i m_i^2 \frac{\epsilon^{\nu_4\nu_5\nu_2\nu_3}\big[2(q+p_2+p_3)_{\mu_2}(q+p_3)_{\mu_3}p_{2\nu_4}q_{\nu_5} - (q+p_2+p_3)_{\mu_2}q_{\mu_3}(p_2+p_3)_{\nu_4}q_{\nu_5}\big]}{(q^2-m_i^2)((q+p_3)^2-m_i^2)((q+p_2+p_3)^2-m_i^2)}\Big]\Big\} \\
&\mathcal{T}_{13}[odd] = -\mathcal{T}_{12}[odd], \\
&\mathcal{T}_{14}[odd] = -\mathcal{T}_{11}[odd], \\
&\mathcal{T}_{15}[odd] = -8\int \prod_{i=1}^{3} \frac{d^4p_j}{(2\pi)^4}(2\pi)^4\, \delta(p_1+p_2+p_3)\,\theta(p_1)\, H^{\mu_2}_{1,\nu_2}(p_2) H^{\mu_3}_{1,\nu_3}(p_3)\Big\{ \\
&\int\frac{d^4q}{(2\pi)^4}\,\Big[\sum_{i=1}^{f} c_i m_i^2 \frac{\epsilon^{\nu_4\nu_5\nu_2\nu_3}\big[p_{2\nu_4}p_{3\nu_5}(q+p_3)_{\mu_2}(q+p_3)_{\mu_3} + p_{2\nu_4}q_{\nu_5}(q+p_3)_{\mu_2}p_{3\mu_3} + p_{3\nu_4}q_{\nu_5}p_{2\mu_2}q_{\mu_3}\big]}{(q^2-m_i^2)((q+p_3)^2-m_i^2)((q+p_2+p_3)^2-m_i^2)}\Big]\Big\}, \\
&\mathcal{T}_{16}[odd] = -\mathcal{T}_{15}[odd].
\end{aligned}
$$

It follows that,

$$
\mathcal{T}_{11}[odd] + \mathcal{T}_{14}[odd] = 0, \quad \mathcal{T}_{12}[odd] + \mathcal{T}_{13}[odd] = 0, \quad \mathcal{T}_{15}[odd] + \mathcal{T}_{16}[odd] = 0,
$$

so that no parity-odd contribution to the Weyl anomaly comes from $\mathcal{T}_1$.

2.1.2 No Parity-odd Weyl anomaly from $\mathcal{T}_2$.

The only contributions to $\mathcal{T}_2$ in (2.7) which may give rise to a trace over four Pauli matrices, and thus an odd parity contribution are:

$$\mathcal{T}_{21} = \left(\tfrac{1}{8}\, i\right) \int dx_1\, dx_2\, dx_3 \,[\theta(x_1)\, h^{\mu_2}_{\mu_2}(x_2) + \theta(x_2)\, h^{\mu_1}_{\mu_1}(x_1)] H^{\mu_3}_{1,\nu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i^2 \Big[\langle \xi^{(i)\alpha_1}(x_1) \xi^{(i)}_{\alpha_1}(x_1) \bar{\xi}^{(i)}_{\dot{\alpha}_2}(x_2) \bar{\xi}^{(i)\,\dot{\alpha}_2}(x_2)\, j^{(i)\,\nu_3}_{R\,\mu_3}(x_3) \rangle_0^{(c)} \Big] \Big\},$$

$$\mathcal{T}_{22} = \left(-\tfrac{1}{8}\, i\right) \int dx_1\, dx_2\, dx_3 \,[\theta(x_1)\, h^{\mu_2}_{\mu_2}(x_2) + \theta(x_2)\, h^{\mu_1}_{\mu_1}(x_1)] H^{\mu_3}_{1,\nu_3}(x_3) \Big\{ \sum_{i=1}^{f} c_i\, m_i^2 \Big[\langle \bar{\xi}^{(i)}_{\dot{\alpha}_1}(x_1) \bar{\xi}^{(i)\,\dot{\alpha}_1}(x_1) \xi^{(i)\alpha_2}(x_2) \xi^{(i)}_{\alpha_2}(x_2) j^{(i)\,\nu_3}_{L\,\mu_3}(x_3) \rangle_0^{(c)} \Big] \Big\},$$

A tedious computation shows that the odd parity contribution to $\mathcal{T}_{21}$ reads

$$\mathcal{T}_{21}[odd] = -\int \prod_{i=1}^{3} \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4\, \delta(p_1+p_2+p_3) [\theta(p_1)\, h^{\mu_2}_{\mu_2}(p_2) + \theta(p_2)\, h^{\mu_1}_{\mu_1}(p_1)] H^{\mu_3}_{1,\nu_3}(p_3) \Big\{ \int \frac{d^4 q}{(2\pi)^4}\, \Big[\sum_{i=1}^{f} c_i m_i^2 \frac{\epsilon^{\nu_1\nu_2\nu_3\nu_4}\, p_{2\nu_1} p_{3\nu_2} (q+p_3)_{\mu_3} q_{\nu_4}}{(q^2-m_i^2)((q+p_3)^2-m_i^2)((q+p_2+p_3)^2-m_i^2)} \Big] \Big\},$$

whereas the odd parity contribution to $\mathcal{T}_{21}[odd]$ is

$$\mathcal{T}_{21}[odd] = -\mathcal{T}_{22}[odd].$$

Putting it all together we conclude that $\mathcal{T}_2$ carries no parity-odd contributions.

2.1.3 No parity-odd anomaly from $\mathcal{T}_3$.

$\mathcal{T}_3$ in (2.8) never yields parity-odd contributions since the maximum number of Pauli Matrices to trace over is two.

2.2 No parity-odd anomaly from bubble-type contributions.

There are also terms that would give rise under contractions to *bubble* type Feynman diagrams. These terms carry one variation of the PV mass term and a kinetic term to second order. Here, we shall show that no parity-odd contribution to the Weyl anomaly will arise from either

$$\mathcal{B}_1 = \sum_{i=1}^{f} c_i \,\langle \delta^{(0)}_{\theta} S^{(i)}_m S^{(i),2}_{kin} \rangle_0^{(c)} \tag{2.9}$$

or else

$$\mathcal{B}_2 = \sum_{i=1}^{f} c_i \, \langle \delta_\theta^{(0)} S_m^{(i)} S_{spin}^{(i),2} \rangle_0^{(c)} \tag{2.10}$$

or again,

$$\mathcal{B}_3 = \sum_{i=1}^{f} c_i \, \langle \delta_\theta^{(1)} S_m^{(i)} S_{kin}^{(i),1} \rangle_0^{(c)} \tag{2.11}$$

in (2.4).

We can now consider separately terms independent of the PV masses and those which are not.

$$\mathcal{B}_1 = \mathcal{B}_{11} + \mathcal{B}_{12},$$

with

$$\begin{aligned}
\mathcal{B}_{11} &= -\tfrac{1}{4}\, i \int dx_1 dx_2 \; \theta(x_1) H_{2\nu_2}^{\mu_2}(x_2) \big\{ \\
&\qquad \sum_{i=0}^{f} c_i \, m_i \langle [\xi^{(i)\alpha_1}(x_1)\xi_{\alpha_1}^{(i)}(x_1) + \bar{\xi}_{\dot{\alpha}_1}^{(i)}(x_1)\bar{\xi}^{(i)\dot{\alpha}_1}(x_1)][j_{R\mu_2}^{(i)\nu_2}(x_2) - j_{L\mu_2}^{(i)\nu_2}(x_2)] \rangle_0^{(c)} \big\}, \\
\mathcal{B}_{12} &= \tfrac{1}{8} \int dx_1 dx_2 \; \theta(x_1) H_2(x_2) \big\{ \\
&\sum_{i=0}^{f} c_i \, m_i^2 \langle [\xi^{(i)\alpha_1}(x_1)\xi_{\alpha_1}^{(i)}(x_1) + \bar{\xi}_{\dot{\alpha}_1}^{(i)}(x_1)\bar{\xi}^{(i)\dot{\alpha}_1}(x_1)][\xi^{(i)\alpha_2}(x_2)\xi_{\alpha_2}^{(i)}(x_2) + \bar{\xi}_{\dot{\alpha}_2}^{(i)}(x_2)\bar{\xi}^{(i)\dot{\alpha}_2}(x_2)] \rangle_0^{(c)} \big\}.
\end{aligned}$$

Now, $\mathcal{B}_{11}$ and $\mathcal{B}_{12}$ cannot give rise to parity-odd contributions since, at most, the trace of two Pauli matrices occurs. Hence, $\mathcal{B}_1$ in (2.9) yields no parity-odd contribution.

Let us move on and analyze $\mathcal{B}_2$ in (2.10). Let us express $\mathcal{B}_2$ as follows

$$\mathcal{B}_2 = \mathcal{B}_{21} + \mathcal{B}_{22},$$

where

$$\begin{aligned}
\mathcal{B}_{21} &= -\tfrac{1}{2} \int dx_1 dx_2 \; \theta(x_1) H_{\mu\nu\rho}(x_2) \big\{ \\
&\qquad \epsilon^{\mu\nu\rho\nu_2} \sum_{i=0}^{f} c_i \, m_i \langle \xi^{(i)\alpha_1}(x_1)\xi_{\alpha_1}^{(i)}(x_1) \, \bar{\xi}_{\dot{\alpha}_2}^{(i)}(x_2)\bar{\sigma}_{\nu_2}^{\dot{\alpha}_2\alpha_2}\xi_{\alpha_2}^{(i)}(x_2) \rangle_0^{(c)} \big\}, \\
\mathcal{B}_{22} &= -\tfrac{1}{2} \int dx_1 dx_2 \; \theta(x_1) H_{\mu\nu\rho}(x_2) \big\{ \\
&\qquad \epsilon^{\mu\nu\rho\nu_2} \sum_{i=0}^{f} c_i \, m_i \langle \bar{\xi}_{\dot{\alpha}_1}^{(i)}(x_1)\bar{\xi}^{(i)\dot{\alpha}_1}(x_1)\bar{\xi}_{\dot{\alpha}_2}^{(i)}(x_2) \, \bar{\sigma}_{\nu_2}^{\dot{\alpha}_2\alpha_2}\xi_{\alpha_2}^{(i)}(x_2) \rangle_0^{(c)} \big\}, \\
H_{\mu\nu\rho}(x) &= \tfrac{1}{16}\, h_\mu^\lambda(x)\partial_\nu h_{\rho\lambda}(x).
\end{aligned} \tag{2.12}$$

A little computation yields

$$\mathcal{B}_{21} = -2\int\prod_{i=1}^{2}\frac{d^4p_j}{(2\pi)^4}(2\pi)^4\,\delta(p_1+p_2)\theta(p_1)H_{\mu\nu\rho}(p_2)\Big\{\int\frac{d^4q}{(2\pi)^4}\;\Big[\sum_{i=1}^{f}c_i\,m_i^2\frac{\epsilon^{\mu\nu\rho\lambda}(q+p_1)_\lambda}{(q^2-m_i^2)((q+p_1)^2-m_i^2)}\Big]\Big\}$$

$$\mathcal{B}_{22} = -\mathcal{B}_{21},$$

so that

$$\mathcal{B}_{21}+\mathcal{B}_{22}=0.$$

Hence, we conclude that $\mathcal{B}_2$ in (2.10) gives rise to no parity-odd anomalous term. Finally,

$$\mathcal{B}_3=\mathcal{B}_{31}+\mathcal{B}_{32},$$

with

$$\mathcal{B}_{31} = -\tfrac{1}{8}\,i\int dx_1dx_2\;\theta(x_1)h^{\mu_1}_{\mu_1}(x_1)\,H^{\mu_2}_{1\nu_2}(x_2)\Big\{ \sum_{i=0}^{f}c_i\,m_i\,\langle[\xi^{(i)\alpha_1}(x_1)\xi^{(i)}_{\alpha_1}(x_1)+\bar{\xi}^{(i)}_{\dot{\alpha}_1}(x_1)\bar{\xi}^{(i)\dot{\alpha}_1}(x_1)][j^{(i)\nu_2}_{R\mu_2}(x_2)-j^{(i)\nu_2}_{L\mu_2}(x_2)]\rangle_0^{(c)}\Big\},$$

$$\mathcal{B}_{32} = \tfrac{1}{16}\int dx_1dx_2\;\theta(x_1)h^{\mu_1}_{\mu_1}(x_1)\,h^{\mu_2}_{\mu_2}(x_2)\Big\{ \sum_{i=0}^{f}c_i\,m_i^2\,\langle[\xi^{(i)\alpha_1}(x_1)\xi^{(i)}_{\alpha_1}(x_1)+\bar{\xi}^{(i)}_{\dot{\alpha}_1}(x_1)\bar{\xi}^{(i)\dot{\alpha}_1}(x_1)][\xi^{(i)\alpha_2}(x_2)\xi^{(i)}_{\alpha_2}(x_2)+\bar{\xi}^{(i)}_{\dot{\alpha}_2}(x_2)\bar{\xi}^{(i)\dot{\alpha}_2}(x_2)]\rangle_0^{(c)}\Big\}.$$

But $\mathcal{B}_{31}$ and $\mathcal{B}_{32}$ cannot give rise to parity-odd contributions, since the maximum number of Pauli matrices involved is two. Hence, $\mathcal{B}_3$ in (2.11) produces no parity-odd contribution.

2.3 No parity-odd anomaly from tadpole-type contributions.

The terms in the second order variation of the PV mass terms would give rise to tadpoles upon Wicks contractions. The tadpole-type contributions to the anomaly, if any, comes from

$$\sum_{i=1}^{f}c_i\,\langle\delta_\theta^{(2)}S_m^{(i)}\rangle_0^{(c)}.$$

Since this contribution involves no Pauli matrices, there is no parity-odd term coming from it.

3 Conclusions.

When computing the Weyl anomaly in perturbation theory one has to deal with the follow type of vacuum expectation values

$$\langle 0|\delta S_{int}S_{int}\cdots S_{int}|0\rangle = \int dx\,dx_1\cdots dx_n\,\langle 0|\delta\mathcal{L}_{int}(x)\mathcal{L}_{int}(x_1)\cdots\mathcal{L}_{int}(x_n)|0\rangle \qquad (3.1)$$

where $|0\rangle$ is the Fock vacuum of the theory and S_{int} the part of the action which contains the interaction and δS_{int} is its symmetry variation. Obviously, we assume that S_{int} is hermitian and so is δS_{int}. Formally, the vacuum expectation value in question is real; but this is not a mathematically rigorous statement, for, in general, the correlation functions on the right hand side of (3.1) contain UV divergences, which must be regularized. If there is a regularization method that explicitly preserves the formal reality of correlation functions under scrutiny, it is clear that no parity-odd contribution to the Weyl anomaly should show up. In this paper we have put forward such a regularization method –a Pauli-Villars regularization– based on the use of a real Lagrangian (not just a real Lagrangian modulo a total derivative) for each Pauli-Villars field. Notice, however, that the regularized correlation functions are well-defined once they are represented in momentum space, and once the loop contributions coming from the physical fields and the Pauli-Villars fields have been collected as a linear combination inside the loop integrations. Hence, to show that the correlation functions which may give rise to a parity-odd contribution to the anomaly do cancel, we have checked that for each contribution of this type there is always a contribution with opposite sign. Finally, we believe that the formal reality of the correlation functions should be preserved in the regularized theory since it comes from the reality of the classical action and should be taken as a fundamental property of the quantum theory.

Let us end with a comment. There is no known topological interpretation of the Weyl anomaly; in particular no known relationship with any index theorem. In this respect it is interesting to highlight the work [3] where the exact form of the one-loop beta function of QCD is recovered from an index theorem on twistor space .

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References

[1] S. Abdallah, S. A. Franchino-Viñas and M. B. Fröb,
"Trace anomaly for Weyl fermions using the Breitenlohner-Maison scheme for γ_*," JHEP **03** (2021), 271 doi:10.1007/JHEP03(2021)271 [arXiv:2101.11382 [hep-th]].
"Trace anomalies for Weyl fermions: too odd to be true?," J. Phys. Conf. Ser. **2531** (2023) no.1, 012004 doi:10.1088/1742-6596/2531/1/012004 [arXiv:2304.08939 [hep-th]].

[2] F. Bastianelli and R. Martelli, "On the trace anomaly of a Weyl fermion," JHEP **11** (2016), 178 doi:10.1007/JHEP11(2016)178 [arXiv:1610.02304 [hep-th]].
F. Bastianelli and M. Broccoli, "Axial gravity and anomalies of fermions," Eur. Phys. J. C **80** (2020) no.3, 276 doi:10.1140/epjc/s10052-020-7782-4 [arXiv:1911.02271 [hep-th]].
F. Bastianelli and M. Broccoli, "On the trace anomaly of a Weyl fermion in a gauge background," Eur. Phys. J. C **79** (2019) no.4, 292 doi:10.1140/epjc/s10052-019-6799-z [arXiv:1808.03489 [hep-th]].
F. Bastianelli and M. Broccoli, "Weyl fermions in a non-abelian gauge background and trace anomalies," JHEP **10** (2019), 241 doi:10.1007/JHEP10(2019)241 [arXiv:1908.03750 [hep-th]].
F. Bastianelli and F. Comberiati, "Path integral calculation of heat kernel traces with first order operator insertions," Nucl. Phys. B **960** (2020), 115183 doi:10.1016/j.nuclphysb.2020.115183 [arXiv:2005.08737 [hep-th]].

[3] R. Bittleston and K. Costello, "The One-Loop QCD β-Function as an Index," [arXiv:2510.26764 [hep-th]].

[4] L. Bonora, S. Giaccari and B. Lima de Souza, "Trace anomalies in chiral theories revisited," JHEP **07** (2014), 117 doi:10.1007/JHEP07(2014)117 [arXiv:1403.2606 [hep-th]].
L. Bonora and R. Soldati, "On the trace anomaly for Weyl fermions," [arXiv:1909.11991 [hep-th]].
"Anomaly footprints in SM+Gravity," [arXiv:2510.25217 [hep-th]].

[5] H. K. Dreiner, H. E. Haber and S. P. Martin, "Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry," Phys. Rept. **494** (2010), 1-196 doi:10.1016/j.physrep.2010.05.002 [arXiv:0812.1594 [hep-ph]].

[6] T. Eguchi, P. B. Gilkey and A. J. Hanson, "Gravitation, Gauge Theories and Differential Geometry," Phys. Rept. **66** (1980), 213 doi:10.1016/0370-1573(80)90130-1

[7] M. B. Fröb and J. Zahn, "Trace anomaly for chiral fermions via Hadamard subtraction," JHEP **10** (2019), 223 doi:10.1007/JHEP10(2019)223 [arXiv:1904.10982 [hep-th]].
"The Weyl anomaly in interacting quantum field theory on curved spacetimes," doi:10.1007/s00023-025-01635-2 [arXiv:2504.17854 [hep-th]].

[8] K. Fujikawa, 'Comment on Chiral and Conformal Anomalies," Phys. Rev. Lett. **44** (1980), 1733 doi:10.1103/PhysRevLett.44.1733

[9] J. M. Gipson, "Path Integral Derivation of the Chiral Anomalies of Dirac Fermions Coupled to Gauge and Gravitational Fields in Even Space-time Dimensions," Phys. Rev. D **29** (1984), 2989 doi:10.1103/PhysRevD.29.2989

[10] H. Godazgar and H. Nicolai, "A rederivation of the conformal anomaly for spin-$\frac{1}{2}$," Class. Quant. Grav. **35** (2018) no.10, 105013 doi:10.1088/1361-6382/aaba97 [arXiv:1801.01728 [hep-th]].

[11] R. Larue, J. Quevillon and R. Zwicky, "Gravity-gauge anomaly constraints on the energy-momentum tensor," JHEP **05** (2024), 307 doi:10.1007/JHEP05(2024)307 [arXiv:2312.13222 [hep-th]].
R. Larue, J. Quevillon and R. Zwicky, "Trace anomaly of weyl fermions via the path integral," JHEP **12** (2023), 064 doi:10.1007/JHEP12(2023)064 [arXiv:2309.08670 [hep-th]].

[12] H. Leutwyler and S. Mallik, "GRAVITATIONAL ANOMALIES," Z. Phys. C **33** (1986), 205 doi:10.1007/BF01411138

[13] C. Y. Liu, "Investigation of Pontryagin trace anomaly using Pauli-Villars regularization," Nucl. Phys. B **980** (2022), 115840 doi:10.1016/j.nuclphysb.2022.115840 [arXiv:2202.13893 [hep-th]].

[14] K. Okuyama and H. Suzuki, "Gauge invariant Pauli-Villars regularization for chiral fermion," Prog. Theor. Phys. **98** (1997), 463-484 doi:10.1143/PTP.98.463 [arXiv:hep-th/9603062 [hep-th]].

[15] Y. S. Stanev, "Correlation Functions of Conserved Currents in Four Dimensional Conformal Field Theory," Nucl. Phys. B **865** (2012), 200-215 doi:10.1016/j.nuclphysb.2012.07.027 [arXiv:1206.5639 [hep-th]].

[16] A. Vilenkin, "Pauli-Villars Regularization and Trace Anomalies." Nuovo Cim. L. 44A, N. 3 1 April 1978