# THE INTERSECTION POLYNOMIALS OF A LONG VIRTUAL KNOT II: TWO SUPPORTING GENERA AND CHARACTERIZATIONS

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Abstract. We develop the study of the twelve intersection polynomials of long virtual knots, previously introduced in our preceding paper. We define two geometric invariants, the 1- and 2-supporting genera, using two distinct surface realizations. These genera yield a natural filtration of the set of long virtual knots, and we analyze the behavior of the intersection polynomials for long virtual knots with small supporting genera. Moreover, we investigate virtual 2-string tangles, analyzing how their sums with long virtual knots affect the intersection polynomials through right closures. As an application, we provide complete realizability criteria for all twelve intersection polynomials.

## 1. Introduction

Virtual knot theory, discovered by Kauffman [8], naturally leads to the study of long virtual knots. In the preceding paper [10], we introduced twelve polynomial invariants $F_{ab}(K;t)$, $G_{ab}(K;t)$, and $H_{ab}(K;t)$ $(a,b\in\{0,1\})$ of a long virtual knot $K$, collectively called the *intersection polynomials*. These invariants are defined via intersection numbers of cycles on an oriented closed surface, and were shown to be finite-type invariants of degree two with respect to crossing changes. In [10], we also established their fundamental properties, including their behavior under symmetries, crossing changes, and concatenation products.

This paper is a sequel to [10], focusing on the geometric structure and realizability of the intersection invariants. The first aim of this paper is to introduce two geometric invariants of a long virtual knot $K$, the 1- and 2-*supporting genera* $sg_1(K)$ and $sg_2(K)$, based on the minimal genus of surface realizations of $K$. These genera naturally yield a filtration of the set of long virtual knots

$$\mathcal{K}_1(0)\subset\mathcal{K}_2(0)\subset\mathcal{K}_1(1)\subset\mathcal{K}_2(1)\subset\mathcal{K}_1(2)\subset\cdots,$$

where $\mathcal{K}_i(g)=\{K\mid sg_i(K)\leq g\}$ for $i\in\{1,2\}$ and $g\geq 0$. We study several properties of the intersection polynomials of a long virtual knot with small supporting genera, which enable us to prove the strictness of the initial steps of the filtration: $\mathcal{K}_1(0)\neq\mathcal{K}_2(0)\neq\mathcal{K}_1(1)\neq\mathcal{K}_2(1)$.

The second aim of this paper is to establish a complete characterization for each of the twelve intersection polynomials. That is, we determine necessary and sufficient conditions for a given Laurent polynomial to be realized as each of the

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intersection polynomials of a long virtual knot. For example, we prove that a Laurent polynomial $f(t)$ is realized as $f(t) = F_{00}(K;t)$ for some long virtual knot $K$ if and only if $f(1) = 1$ and $f(t)$ is reciprocal. We further show that such a knot $K$ can always be chosen to satisfy $sg_1(K) \leq 1$.

This paper is organized as follows. In Section 2, we recall the definition and some fundamental properties of the intersection polynomials from [10]. Section 3 introduces the two supporting genera and proves the strictness of the small genus filtration (Theorem 3.8). Sections 4–6 establish essential tools and preliminary characterizations needed for the main result. Section 4 introduces six polynomial invariants of a virtual 2-string tangle $T$ via its right closure and analyzes how the sum $T+K$ affects the intersection polynomials (Theorem 4.3). Section 5 utilizes the right and left closures of $T$ to provide a relationship between $F$ and $H$ polynomials (Proposition 5.4). Section 6 is devoted to characterizing the writhe polynomial $W_a(K;t)$ under the condition $sg_2(K) = 0$ (Proposition 6.2). Finally, in Section 7, we synthesizes these preliminary results to establish complete characterizations of the intersection polynomials $F_{ab}(K;t)$, $G_{ab}(K;t)$, and $H_{ab}(K;t)$ (Theorems 7.1, 7.2, and 7.4).

## 2. Preliminaries

In this section, we review the definitions of the writhe and the intersection polynomials, together with several of their fundamental properties. Refer to [10] for more details.

A long virtual knot is presented by a diagram in $\mathbb{R}^2$ that coincides with the $x$-axis outside a 2-disk and is equipped with real and virtual crossings. It can also be described by a surface realization on an oriented closed surface with a basepoint corresponding to $\pm\infty$.

Let $D$ be a diagram of a long virtual knot $K$, $c_1, \dots, c_n$ the real crossings of $D$, and $(\Sigma_g, D)$ a surface realization of $D$. Here, $\Sigma_g$ denotes an oriented closed surface of genus $g$. For each $i$ with $1 \leq i \leq n$, traversing $D$ from the basepoint in the positive direction, if we pass $c_i$ first over and then under, we say that $c_i$ is *of type* 0; otherwise, it is *of type* 1. We define two sets by

$$I_a(D) = \{i \mid c_i \text{ is of type } a\} \ (a = 0, 1).$$

Smoothing $D$ at $c_i$ yields two cycles on $\Sigma_g$, one of which does not contain the basepoint, while the other does. We denote by $\alpha_i$ the cycle that does not contain the basepoint, and by $\beta_i$ the one that does.

Let $\varepsilon_i \in \{\pm 1\}$ be the sign of $c_i$ $(i = 1, \dots, n)$. For each $a \in \{0, 1\}$, the Laurent polynomial

$$W_a(D;t) = \sum_{i=1}^{n} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1)$$

is an invariant of $K$ [5, Lemma 4.1], where the dot between cycles denotes their intersection number. It is called the *a-writhe polynomial*, and is denoted by $W_a(K;t)$. Let $\gamma_D$ be the cycle on $\Sigma_g$ presented by $D$. Since $\alpha_i + \beta_i = \gamma_D$ holds for $1 \leq i \leq n$, we have

$$\alpha_i \cdot \beta_i = \alpha_i \cdot (\gamma_D - \alpha_i) = \alpha_i \cdot \gamma_D.$$

Therefore, in the definition of $W_a(K;t)$, we may use $\alpha_i \cdot \gamma_D$ instead of $\alpha_i \cdot \beta_i$.

For each $a, b \in \{0, 1\}$, the Laurent polynomials

$$F_{ab}(D;t) = \sum_{i\in I_a(D),\, j\in I_b(D)} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\alpha_j} - 1),$$
$$G_{ab}(D;t) = \sum_{i\in I_a(D),\, j\in I_b(D)} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\beta_j} - 1) - \omega_b(D)\cdot W_a(K;t), \text{ and}$$
$$H_{ab}(D;t) = \sum_{i\in I_a(D),\, j\in I_b(D)} \varepsilon_i\varepsilon_j(t^{\beta_i\cdot\beta_j} - 1) - \omega_a(D)\cdot W_b(K;t) - \omega_b(D)\cdot W_a(K;t^{-1})$$

are invariants of $K$ [10, Theorem 2.3], which are called the *intersection polynomials*, and denoted by $F_{ab}(K;t)$, $G_{ab}(K;t)$, and $H_{ab}(K;t)$, respectively. Here, $\omega_a(D) = \sum_{i\in I_a(D)} \varepsilon_i$ is the *a-writhe* of $D$.

Any long virtual knot can be presented by a diagram with $\omega_0(D) = \omega_1(D) = 0$, called an *untwisted diagram*. For such a diagram, the definitions of $G_{ab}(K;t)$ and $H_{ab}(K;t)$, like that of $F_{ab}(K;t)$, take much simple forms.

A Laurent polynomial $f(t)$ is called *reciprocal* if it holds that $f(t) = f(t^{-1})$.

**Lemma 2.1** ([10, Lemma 2.4 and Theorem 10.3(ii)]). *Any long virtual knot $K$ satisfies the following.*

(i) $F_{01}(K;t) = F_{10}(K;t^{-1})$ *and* $H_{01}(K;t) = H_{10}(K;t^{-1})$.
(ii) $F_{00}(K;t)$, $F_{11}(K;t)$, $H_{00}(K;t)$, *and* $H_{11}(K;t)$ *are reciprocal.*
(iii) $G'_{00}(K;1) = G'_{11}(K;1) = 0$. □

For a diagram $D$ of $K$, we define three diagrams $D^{\#}$, $-D$, and $D^*$ as follows:

- $D^{\#}$ is obtained from $D$ by switching the over/under information at each real crossing of $D$.
- $-D$ is obtained from $D$ by reversing the orientation of $D$.
- $D^*$ is obtained from $D$ by an orientation-reversing homeomorphism of $\mathbb{R}^2$ (or $\Sigma_g$).

Let $K^{\#}$, $-K$, and $K^*$ denote the long virtual knots presented by $D^{\#}$, $-D$, and $D^*$, respectively.

**Lemma 2.2** ([10, Theorem 5.1]). *For any $a \in \{0, 1\}$, the following hold.*

(i) $W_a(K^{\#};t) = -W_{1-a}(K;t)$.
(ii) $W_a(-K;t) = W_{1-a}(K;t)$.
(iii) $W_a(K^*;t) = -W_a(K;t^{-1})$. □

**Lemma 2.3** ([10, Theorem 5.2]). *For any $X \in \{F, G, H\}$ and $a, b \in \{0, 1\}$, the following hold.*

(i) $X_{ab}(K^{\#};t) = X_{ab}(-K;t) = X_{1-a,1-b}(K;t)$.
(ii) $X_{ab}(K^*;t) = X_{ab}(K;t^{-1})$. □

Let $D$ and $D'$ be diagrams of long virtual knots $K$ and $K'$, respectively. We denote by $D \circ D'$ the diagram obtained by concatenating $D'$ after $D$. The *product* of $K$ and $K'$ is defined as the long virtual knot presented by $D \circ D'$, and is denoted by $K \circ K'$.

**Lemma 2.4** ([10, Theorems 6.1 and 6.2]). *For any $a, b \in \{0, 1\}$, the following hold.*

(i) $W_a(K \circ K';t) = W_a(K;t) + W_a(K';t)$.

(ii) $X_{ab}(K \circ K'; t) = X_{ab}(K; t) + X_{ab}(K'; t)$ *for any* $X \in \{F, G\}$.

(iii) $H_{ab}(K \circ K'; t) = H_{ab}(K; t) + H_{ab}(K'; t)$
$+W_a(K; t^{-1})W_b(K'; t) + W_a(K'; t^{-1})W_b(K; t)$. □

The writhe polynomial, defined independently in [1, 9, 11], and the three intersection polynomials introduced in [4] are polynomial invariants of (closed) virtual knots. For a long virtual knot $K$, let $\widehat{K}$ denote its closure. Then the writhe polynomial $W(\widehat{K}; t)$ and the first intersection polynomial $I(\widehat{K}; t)$ can be expressed in terms of those of $K$ as follows.

**Lemma 2.5** ([10, Proposition 10.1])**.** *Any long virtual knot $K$ satisfies the following.*

(i) $W(\widehat{K}; t) = W_0(K; t) + W_1(K; t^{-1})$.

(ii) $I(\widehat{K}; t) = F_{01}(K; t) + G_{00}(K; t) + G_{11}(K; t^{-1}) + H_{01}(K; t^{-1})$. □

## 3. Two supporting genera

The supporting genus of a virtual knot $\kappa$ is one of the important notions in virtual knot theory. It is defined as the minimal genus for all surface realizations of $\kappa$, and is denoted by $sg(\kappa)$. If $sg(\kappa) \leq 1$, that is, $\kappa$ is presented by a diagram on the torus $\Sigma_1$, then the following hold.

**Lemma 3.1** ([3, 7])**.** *If $sg(\kappa) \leq 1$ holds, then $W(\kappa; t)$ and $I(\kappa; t)$ are reciprocal.* □

In this section, we introduce two genera of a long virtual knot. The first one is defined via a surface realization as follows.

**Definition 3.2.** For a long virtual knot $K$,

$$sg_1(K) = \min\{g \mid (\Sigma_g, D) \text{ is a surface realization of } K\}$$

is called the 1-*supporting genus* of $K$.

By definition, $sg_1(\widehat{K}) \leq sg_1(K)$ holds for any long virtual knot $K$.

**Lemma 3.3.** *If $sg_1(K) = 0$ holds, then we have the following.*

(i) $W_a(K; t) = 0$ *for any* $a \in \{0, 1\}$.

(ii) $X_{ab}(K; t) = 0$ *for any* $X \in \{F, G, H\}$ *and* $a, b \in \{0, 1\}$.

*Proof.* Since $K$ is presented by a surface realization on the 2-sphere $\Sigma_0$, the intersection number of any pair of cycles vanishes. Therefore, the conclusion follows. □

We remark that $sg_1(K) = 0$ if and only if $K$ is a long classical knot.

**Lemma 3.4.** *If $sg_1(K) \leq 1$ holds, then we have the following.*

(i) $W_0(K; t) - W_1(K; t)$ *is reciprocal.*

(ii) $F_{01}(K; t) + G_{00}(K; t) - G_{11}(K; t) - H_{01}(K; t)$ *is reciprocal.*

(iii) $G_{00}(K; t) - G_{01}(K; t) - G_{10}(K; t) + G_{11}(K; t)$ *is reciprocal.*

*Proof.* (i) Since the closure of $K$ satisfies $sg(\widehat{K}) \leq 1$, the writhe polynomial $W(\widehat{K}; t)$ of $\widehat{K}$ is reciprocal by Lemma 3.1. Then it follows from Lemma 2.5(i) that

$$W_0(K; t) + W_1(K; t^{-1}) = W_0(K; t^{-1}) + W_1(K; t).$$

(ii) By Lemma 3.1, the first intersection polynomial $I(\widehat{K};t)$ is reciprocal. Therefore, it follows from Lemma 2.5(ii) that

$$\begin{aligned}&F_{01}(K;t)+G_{00}(K;t)+G_{11}(K;t^{-1})+H_{01}(K;t^{-1})\\&=F_{01}(K;t^{-1})+G_{00}(K;t^{-1})+G_{11}(K;t)+H_{01}(K;t).\end{aligned}$$

(iii) Let $K^d$ denote the descending long virtual knot associated with $K$ (cf. [2, Section 2.2]); that is, $K^d$ is presented a diagram $D$ with $I_1(D)=\emptyset$. Making a surface realization of $K$ on the torus $\Sigma_1$ descending, we obtain a surface realization of $K^d$ on $\Sigma_1$. Hence, we have $sg_1(K^d)\leq 1$, and

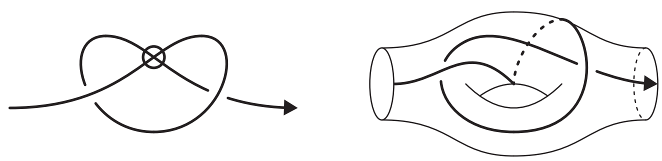
$$F_{01}(K^d;t)+G_{00}(K^d;t)-G_{11}(K^d;t)-H_{01}(K^d;t)$$

is reciprocal by (ii). On the other hand, it follows from [10, Corollary 7.4(ii)] that

$$\begin{aligned}&F_{01}(K^d;t)=G_{11}(K^d;t)=H_{01}(K^d;t)=0 \text{ and}\\&G_{00}(K^d;t)=G_{00}(K;t)-G_{01}(K;t)-G_{10}(K;t)+G_{11}(K;t).\end{aligned}$$

Therefore, the conclusion follows. □

The second genus of a long virtual knot is defined by using a 2-punctured surface as follows. Let $\Sigma_{g,2}$ be a connected, oriented, compact surface of genus $g$ with two boundary components. We regard a long virtual knot diagram $D$ as a tangle diagram with two boundary points. Then we can consider a surface realization of $D$ on $\Sigma_{g,2}$ such that the two boundary points of $D$ lie on distinct boundary components of $\Sigma_{g,2}$. See Figure 3.1.



$D$ $(\Sigma_{1,2},D)$

FIGURE 3.1. A surface realization $(\Sigma_{1,2},D)$

**Definition 3.5.** For a long virtual knot $K$,

$$sg_2(K)=\min\{g \mid (\Sigma_{g,2},D) \text{ is a surface realization of } K\}$$

is called the 2-*supporting genus* of $K$.

**Lemma 3.6.** *If $sg_2(K)=0$ holds, then we have the following.*

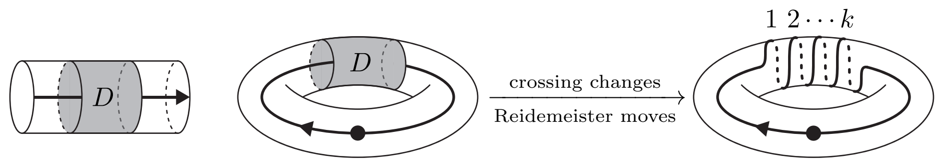
(i) $W_0(K;t)=W_1(K;t)$.

(ii) $F_{ab}(K;t)=G_{ab}(K;t)=0$ *for any* $a,b\in\{0,1\}$.

(iii) $H_{ab}(K;t)=W_0(K;t)W_0(K;t^{-1})$ *for any* $a,b\in\{0,1\}$.

*Proof.* (i) By identifying the boundary components of the annulus $\Sigma_{0,2}$, we obtain a surface realization of $K$ with a basepoint on the torus $\Sigma_1$. Then it can be deformed into a diagram with no crossings, which presents the trivial knot, by a finite sequence of crossing changes and Reidemeister moves. See Figure 3.2. Therefore, $W_0(K;t)-W_1(K;t)=0$ holds by [10, Theorem 7.1].

(ii) We may assume that $D$ is untwisted; that is, $\omega_0(D)=\omega_1(D)=0$. Each cycle $\alpha_i$ satisfies $\alpha_i=k_i\mu$ for some $k_i\in\mathbb{Z}$, where $\mu$ is a meridian of $\Sigma_1$. By $\alpha_i\cdot\alpha_j=0$




$(\Sigma_{0,2}, D)$ $(\Sigma_1, D)$

FIGURE 3.2. Proof of Lemma 3.6(i)

for any $i$ and $j$, we have $F_{ab}(K;t) = 0$ for any $a, b \in \{0,1\}$. On the other hand, since it holds that

$$\alpha_i \cdot \beta_j = \alpha_i \cdot (\gamma_D - \alpha_j) = \alpha_i \cdot \gamma_D = \alpha_i \cdot (\alpha_i + \beta_i) = \alpha_i \cdot \beta_i,$$

we have

$$\begin{aligned} G_{ab}(K;t) &= \sum_{i \in I_a(D),\, j \in I_b(D)} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_j} - 1) \\ &= \sum_{i \in I_a(D),\, j \in I_b(D)} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_i} - 1) = \omega_b(D) \cdot W_a(K;t) = 0. \end{aligned}$$

(iii) It holds that

$$\beta_i \cdot \beta_j = (\gamma_D - \alpha_i) \cdot (\gamma_D - \alpha_j) = -\alpha_i \cdot \gamma_D + \alpha_j \cdot \gamma_D = -\alpha_i \cdot \beta_i + \alpha_j \cdot \beta_j.$$

By (i), we have

$$\begin{aligned} H_{ab}(K;t) = &\sum_{i \in I_a(D), j \in I_b(D)} \varepsilon_i \varepsilon_j (t^{-\alpha_i \cdot \beta_i + \alpha_j \cdot \beta_j} - 1) \\ = &\left( \sum_{i \in I_a(D)} \varepsilon_i (t^{-\alpha_i \cdot \beta_i} - 1) \right) \left( \sum_{j \in I_b(D)} \varepsilon_j (t^{\alpha_j \cdot \beta_j} - 1) \right) \\ &+ \sum_{i \in I_a(D), j \in I_b(D)} \varepsilon_i \varepsilon_j (t^{-\alpha_i \cdot \beta_i} - 1) + \sum_{i \in I_a(D), j \in I_b(D)} \varepsilon_i \varepsilon_j (t^{\alpha_j \cdot \beta_j} - 1) \\ = &\, W_a(K;t^{-1}) W_b(K;t) = W_0(K;t) W_0(K;t^{-1}). \end{aligned}$$

□

For an integer $g \geq 0$, we define two sets of long virtual knots by

$$\mathcal{K}_1(g) = \{K \mid sg_1(K) \leq g\} \text{ and}$$
$$\mathcal{K}_2(g) = \{K \mid sg_2(K) \leq g\}.$$

**Lemma 3.7.** *We have $sg_2(K) \leq sg_1(K) \leq sg_2(K) + 1$, and hence*

$$\mathcal{K}_1(0) \subset \mathcal{K}_2(0) \subset \mathcal{K}_1(1) \subset \mathcal{K}_2(1) \subset \mathcal{K}_1(2) \subset \cdots.$$

*Proof.* If $K$ has a surface realization $(\Sigma_g, D)$, then removing a pair of disks near the basepoint from $\Sigma_g$ provides a surface realization $(\Sigma_{g,2}, D)$. See the left and middle of Figure 3.3. Therefore, we have $sg_2(K) \leq sg_1(K)$.

On the other hand, if $K$ has a surface realization $(\Sigma_{g,2}, D)$, then identifying the boundary components of $\Sigma_{g,2}$ provides a surface realization $(\Sigma_{g+1}, D)$. Therefore, we have $sg_1(K) \leq sg_2(K) + 1$. See the middle and right of the figure. □

**Theorem 3.8.** *We have the following.*

(i) $\mathcal{K}_1(0) \neq \mathcal{K}_2(0)$.

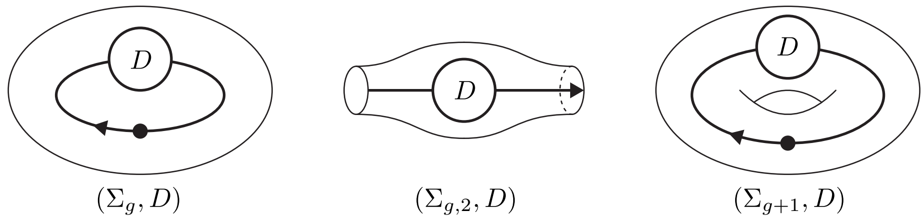



FIGURE 3.3. Proof of Lemma 3.7

(ii) $\mathcal{K}_2(0) \neq \mathcal{K}_1(1)$.
(iii) $\mathcal{K}_1(1) \neq \mathcal{K}_2(1)$.

*Moreover, there are infinitely many long virtual knots which realize the difference between the two sets in each of* (i)–(iii)*.*

*Proof.* (i) For an integer $n \geq 1$, we consider the long virtual knot $K_n$ presented by a surface realization on the annulus $\Sigma_{0,2}$ as shown in Figure 3.4. We have $sg_2(K_n) = 0$, and hence $sg_1(K_n) \leq 1$ by Lemma 3.7. Since it can be verified that

$$W_0(K_n; t) = n(t - 1) \neq 0,$$

we have $sg_1(K_n) \neq 0$ by Lemma 3.3(i), and hence $sg_1(K_n) = 1$. Therefore, $K_n \in \mathcal{K}_2(0) \setminus \mathcal{K}_1(0)$ holds. Moreover, $K_n$'s are all distinct.

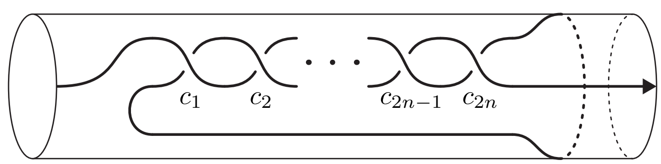



FIGURE 3.4. A surface realization of $K_n$

(ii) For an integer $n \geq 1$, we consider the long virtual knot $K'_n$ presented by a surface realization on the torus $\Sigma_1$ as shown in Figure 3.5. We have $sg_1(K'_n) \leq 1$, and hence $sg_2(K'_n) \leq 1$ by Lemma 3.7. Since it can be verified that

$$F_{00}(K'_n; t) = n(t - 2 + t^{-1}) \neq 0,$$

we have $sg_2(K'_n) = 1$ by Lemma 3.6(ii) and $sg_1(K'_n) = 1$ by Lemma 3.7. Therefore, $K'_n \in \mathcal{K}_1(1) \setminus \mathcal{K}_2(0)$ holds. Moreover, $K'_n$'s are all distinct.

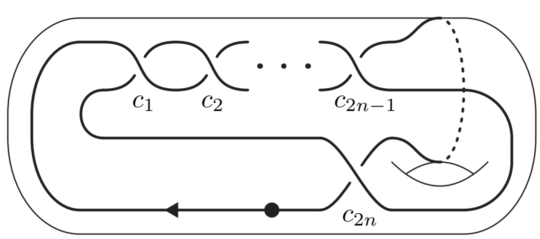



FIGURE 3.5. A surface realization of $K'_n$

(iii) For an integer $n \geq 1$, we consider the long virtual knot $K''_n$ presented by a surface realization on the 2-punctured torus $\Sigma_{1,2}$ as shown in Figure 3.6. We have $sg_2(K''_n) \leq 1$, and hence, $sg_1(K''_n) \leq 2$ by Lemma 3.7. Since

$$W_0(K''_n;t) - W_1(K''_n;t) = \big((n-1)t^2 + 2t - (n+1)\big) - n(t^2-1) = -t^2 + 2t - 1$$

is not reciprocal, we have $sg_1(K''_n) = 2$ by Lemma 3.4(i) and $sg_2(K''_n) = 1$ by Lemma 3.7. Therefore, $K''_n \in \mathcal{K}_2(1) \setminus \mathcal{K}_1(1)$ holds. Moreover, $K''_n$'s are all distinct. $\square$

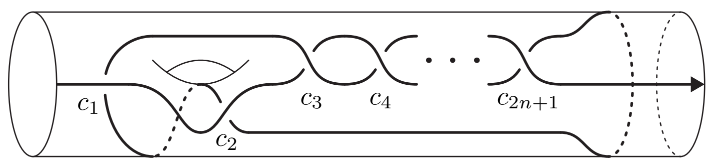



FIGURE 3.6. A surface realization of $K''_n$

**Remark 3.9.** By definition, the 1- and 2-supporting genera of the product of long virtual knots $K_1$ and $K_2$ satisfy

(i) $sg_1(K_1 \circ K_2) \leq sg_1(K_1) + sg_1(K_2)$ and

(ii) $sg_2(K_1 \circ K_2) \leq sg_2(K_1) + sg_2(K_2)$.

Moreover, we can construct a pair of long virtual knots which does not attain the equality in (i) as follows. Suppose that $K_1$ and $K_2$ are nonclassical long virtual knots with $sg_2(K_1) = sg_2(K_2) = 0$, and that $K_1 \circ K_2$ is also nonclassical. Since each of $K_1$, $K_2$, and $K_1 \circ K_2$ has a surface realization on the torus $\Sigma_1$, we have

$$sg_1(K_1) = sg_1(K_2) = sg_1(K_1 \circ K_2) = 1.$$

## 4. Invariants of a virtual tangle

We consider an oriented virtual tangle diagram $E$ consisting of two strings $A$ and $B$, each connecting two of the four endpoints as shown on the left of Figure 4.1. A *virtual 2-string tangle* $T$ is defined as an equivalence class of such diagrams $E$ under Reidemeister moves keeping the endpoints fixed. Let $R(E)$ denote the long virtual knot diagram obtained from $E$ by closing the pair of the right endpoints, as shown on the right of the figure. The *right closure* of $T$ is the long virtual knot presented by $R(E)$, and is denoted by $R(T)$.

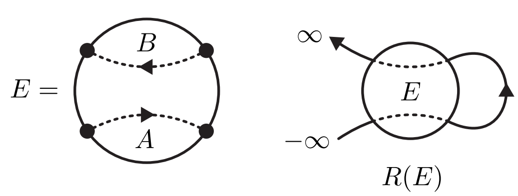



FIGURE 4.1. A tangle diagram and its right closure

Any invariant of a long virtual knot induces an invariant of a virtual 2-string tangle by using the right closure. In particular, the writhe polynomials $W_0(R(T);t)$ and $W_1(R(T);t)$ are regarded as invariants of $T$. These invariants can be elaborated as follows. Let $c_1, \cdots, c_m$ be the real crossings of $E$. For each $i$ with $1 \leq i \leq m$, we classify $c_i$ as follows;

- $c_i$ is *of type* $(A, A; 0)$ if both intersecting paths at $c_i$ belong to the string $A$, and we pass $c_i$ first over and then under along $A$,
- $c_i$ is *of type* $(A, A; 1)$ if both paths belong to $A$, and we pass $c_i$ first under and then over along $A$,
- $c_i$ is *of type* $(B, B; 0)$ if both paths belong to $B$, and we pass $c_i$ first over and then under along $B$,
- $c_i$ is *of type* $(B, B; 1)$ if the paths at $c_i$ both belong to $B$, and we pass $c_i$ first going under and then over along $B$,
- $c_i$ is *of type* $(A, B)$ if the over-path at $c_i$ belongs to $A$ and the under-path to $B$, and
- $c_i$ is *of type* $(B, A)$ if the over-path belongs to $B$ and the under-path to $A$.

We define six sets by

$J_0^A(E) = \{i \mid c_i \text{ is of type } (A, A; 0)\}$, $J_1^A(E) = \{i \mid c_i \text{ is of type } (A, A; 1)\}$,
$J_0^B(E) = \{i \mid c_i \text{ is of type } (B, B; 0)\}$, $J_1^B(E) = \{i \mid c_i \text{ is of type } (B, B; 1)\}$,
$J_0'(E) = \{i \mid c_i \text{ is of type } (A, B)\}$, $J_1'(E) = \{i \mid c_i \text{ is of type } (B, A)\}$.

Moreover, we put

$$J_0(E) = J_0^A(E) \cup J_0^B(E) \text{ and } J_1(E) = J_1^A(E) \cup J_1^B(E).$$

For any $a \in \{0, 1\}$, it follows by definition that

$$I_a(R(E)) = J_a(E) \cup J_a'(E).$$

We take a surface realization $(\Sigma_g, R(E))$ of the right closure $R(E)$. Let $\alpha_i$ and $\beta_i$ be the cycles defined at $c_i$ associated with $(\Sigma_g, R(E))$. For $a \in \{0, 1\}$ and $X \in \{A, B\}$, we define six Laurent polynomials by

$$U_a^X(E; t) = \sum_{i \in J_a^X(E)} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) \text{ and } V_a(E; t) = \sum_{i \in J_a'(E)} \varepsilon_i t^{\alpha_i \cdot \beta_i}.$$

**Lemma 4.1.** *The Laurent polynomials*

$$U_0^A(E; t),\ U_0^B(E; t),\ U_1^A(E; t),\ U_1^B(E; t),\ V_0(E; t),\ \textit{and } V_1(E; t)$$

*are invariants of* $T$.

*Proof.* For each $a \in \{0, 1\}$ and $X \in \{A, B\}$, the invariance of the Laurent polynomials $U_a^X(E; t)$ and $V_a(E; t)$ can be proved in a similar manner to the proof of [5, Lemma 4.1]. □

We may replace the letter $E$ with $T$ in the invariants of Lemma 4.1. For $a \in \{0, 1\}$, we put

$$\lambda_a(T) = V_a(T; 1) = \sum_{i \in J_a'(E)} \varepsilon_i.$$

The integers $\lambda_0(T)$ and $\lambda_1(T)$ are called the *linking numbers* of $T$, and $U_a^X(T; t)$ and $V_a(T; t)$ are the *writhe polynomials* of $T$. We also put

$$U_a(T; t) = U_a^A(T; t) + U_a^B(T; t) = \sum_{i \in J_a(E)} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1).$$

**Lemma 4.2.** *For any* $a \in \{0, 1\}$*, we have*

$$W_a(R(T); t) = U_a(T; t) + V_a(T; t) - \lambda_a(T).$$

*Proof.* Since $I_a(R(E)) = J_a(E) \cup J'_a(E)$, we have

$$\begin{aligned} W_a(R(T);t) &= \sum_{i\in J_a(E)} \varepsilon_i(t^{\alpha_i\cdot\beta_i}-1) + \sum_{i\in J'_a(E)} \varepsilon_i(t^{\alpha_i\cdot\beta_i}-1) \\ &= U_a(T;t) + V_a(T;t) - \lambda_a(T). \end{aligned}$$

□

Inspired by Conway's tangle sum, we define a sum of a virtual 2-string tangle $T$ and a long virtual knot $K$ as follows. Let $E$ and $D$ be diagrams of $T$ and $K$, respectively. We form a long virtual knot diagram, denoted by $E+D$, by connecting the endpoints of $E$ and $D$ as shown in Figure 4.2. The *sum* of $T$ and $K$, written $T+K$, is the long virtual knot presented by $E+D$.

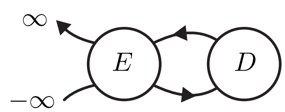



FIGURE 4.2. A long virtual knot digram $E+D$

**Theorem 4.3.** *For any $a, b \in \{0,1\}$, we have the following.*

(i) $W_a(T+K;t) = W_a(R(T);t) + W_a(K;t)$.

(ii) $F_{ab}(T+K;t) = F_{ab}(R(T);t) + F_{ab}(K;t)$
$+\lambda_b(T)\cdot W_a(K;t) + \lambda_a(T)\cdot W_b(K;t^{-1})$.

(iii) $G_{ab}(T+K;t) = G_{ab}(R(T);t) + G_{ab}(K;t)$
$-\lambda_b(T)\cdot W_a(K;t) + V_a(T;t)\cdot W_b(K;t)$.

(iv) $H_{ab}(T+K;t) = H_{ab}(R(T);t) + H_{ab}(K;t)$
$+(U_b(T;t)-\lambda_b(T))\cdot W_a(K;t^{-1})$
$+(U_a(T;t^{-1})-\lambda_a(T))\cdot W_b(K;t)$.

We take a surface realization $(\Sigma_g, R(E))$ of $R(E)$. Let $c_1,\dots,c_m$ be the real crossings of $R(E)$, and $\alpha_i$ and $\beta_i$ $(1\le i\le m)$ the cycles at $c_i$ on $\Sigma_g$. We may assume that

$$J_0(E)\cup J_1(E) = \{1,\dots,\ell\} \text{ and } J'_0(E)\cup J'_1(E) = \{\ell+1,\dots,m\}$$

for some $\ell$. We also take a surface realization $(\Sigma_h, D)$ of $D$. Let $c_{m+1},\dots,c_n$ be the real crossings of $D$, and $\alpha_i$ and $\beta_i$ $(m+1\le i\le n)$ the cycles at $c_i$ on $\Sigma_h$.

Taking a connected sum of $(\Sigma_g, R(E))$ and $(\Sigma_h, D)$ yields a surface realization $(\Sigma_{g+h}, E+D)$ of $E+D$ as shown in Figure 4.3. Let $\alpha'_i$ and $\beta'_i$ $(1\le i\le n)$ be the cycles at $c_i$ on $\Sigma_{g+h}$. We also regard the cycles $\alpha_i$ and $\beta_i$ as lying on $\Sigma_{g+h}$ $(1\le i\le n)$. To prove Theorem 4.3, we prepare the following lemma.

**Lemma 4.4.** *The intersection numbers $\alpha'_i\cdot\alpha'_j$, $\alpha'_i\cdot\beta'_j$, and $\beta'_i\cdot\beta'_j$ are given as shown in* Tables 4.1–4.3.

*Proof.* It follows by definition that

(i) $\alpha'_i = \alpha_i$ and $\beta'_i = \beta_i + \gamma_D$ for $1\le i\le \ell$,
(ii) $\alpha'_i = \alpha_i + \gamma_D$ and $\beta'_i = \beta_i$ for $\ell+1\le i\le m$, and
(iii) $\alpha'_i = \alpha_i$ and $\beta'_i = \beta_i + \gamma_{R(E)}$ for $m+1\le i\le n$.

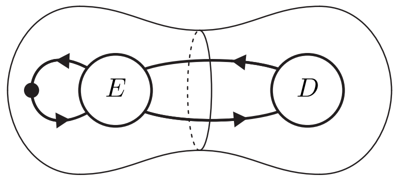



FIGURE 4.3. The surface realization $(\Sigma_{g+h}, E+D)$

TABLE 4.1. The intersection numbers $\alpha'_i \cdot \alpha'_j$

<table>
<tr><th></th><th>$1 \le j \le \ell$</th><th>$\ell+1 \le j \le m$</th><th>$m+1 \le j \le n$</th></tr>
<tr><td>$1 \le i \le \ell$</td><td colspan="2" rowspan="2">$\alpha_i \cdot \alpha_j$</td><td>$0$</td></tr>
<tr><td>$\ell+1 \le i \le m$</td><td>$-\alpha_j \cdot \beta_j$</td></tr>
<tr><td>$m+1 \le i \le n$</td><td>$0$</td><td>$\alpha_i \cdot \beta_i$</td><td>$\alpha_i \cdot \alpha_j$</td></tr>
</table>

TABLE 4.2. The intersection numbers $\alpha'_i \cdot \beta'_j$

<table>
<tr><th></th><th>$1 \le j \le \ell$</th><th>$\ell+1 \le j \le m$</th><th>$m+1 \le j \le n$</th></tr>
<tr><td>$1 \le i \le \ell$</td><td colspan="2" rowspan="2">$\alpha_i \cdot \beta_j$</td><td>$\alpha_i \cdot \beta_i$</td></tr>
<tr><td>$\ell+1 \le i \le m$</td><td>$\alpha_i \cdot \beta_i + \alpha_j \cdot \beta_j$</td></tr>
<tr><td>$m+1 \le i \le n$</td><td>$\alpha_i \cdot \beta_i$</td><td>$0$</td><td>$\alpha_i \cdot \beta_j$</td></tr>
</table>

TABLE 4.3. The intersection numbers $\beta'_i \cdot \beta'_j$

<table>
<tr><th></th><th>$1 \le j \le \ell$</th><th>$\ell+1 \le j \le m$</th><th>$m+1 \le j \le n$</th></tr>
<tr><td>$1 \le i \le \ell$</td><td colspan="2" rowspan="2">$\beta_i \cdot \beta_j$</td><td>$-\alpha_i \cdot \beta_i + \alpha_j \cdot \beta_j$</td></tr>
<tr><td>$\ell+1 \le i \le m$</td><td>$-\alpha_i \cdot \beta_i$</td></tr>
<tr><td>$m+1 \le i \le n$</td><td>$-\alpha_i \cdot \beta_i + \alpha_j \cdot \beta_j$</td><td>$\alpha_j \cdot \beta_j$</td><td>$\beta_i \cdot \beta_j$</td></tr>
</table>

We consider only the case of the intersection numbers $\alpha'_i \cdot \beta'_j$; the other cases are treated analogously.

$\underline{1 \le i, j \le m.}$ Since $\alpha_i \cdot \gamma_D = \beta_i \cdot \gamma_D = 0$, it follows from (i) and (ii) that

$$\alpha'_i \cdot \beta'_j = (\alpha_i \text{ or } \alpha_i + \gamma_D) \cdot (\beta_j + \gamma_D \text{ or } \beta_j) = \alpha_i \cdot \beta_j.$$

$\underline{1 \le i \le \ell \text{ and } m+1 \le j \le n.}$ Since $\alpha_i \cdot \beta_j = 0$, it follows from (i) and (iii) that

$$\alpha'_i \cdot \beta'_j = \alpha_i \cdot (\beta_j + \gamma_{R(E)}) = \alpha_i \cdot \gamma_{R(E)} = \alpha_i \cdot (\alpha_i + \beta_i) = \alpha_i \cdot \beta_i.$$

$\underline{\ell+1 \le i \le m \text{ and } m+1 \le j \le n.}$ Since $\alpha_i \cdot \beta_j = \gamma_D \cdot \gamma_{R(E)} = 0$, it follows from (ii) and (iii) that

$$\alpha'_i \cdot \beta'_j = (\alpha_i + \gamma_D) \cdot (\beta_j + \gamma_{R(E)}) = \alpha_i \cdot \gamma_{R(E)} + \gamma_D \cdot \beta_j = \alpha_i \cdot \beta_i + \alpha_j \cdot \beta_j.$$

$\underline{m+1 \le i \le n \text{ and } 1 \le j \le \ell.}$ Since $\alpha_i \cdot \beta_j = 0$, it follows from (i) and (iii) that

$$\alpha'_i \cdot \beta'_j = \alpha_i \cdot (\beta_j + \gamma_D) = \alpha_i \cdot \gamma_D = \alpha_i \cdot \beta_i.$$

$\underline{m+1 \leq i \leq n \text{ and } \ell+1 \leq j \leq m.}$ It follows from (ii) and (iii) that

$$\alpha'_i \cdot \beta'_j = \alpha_i \cdot \beta_j = 0.$$

$\underline{m+1 \leq i, j \leq n.}$ Since $\alpha_i \cdot \gamma_{R(E)} = 0$, it follows from (iii) that

$$\alpha'_i \cdot \beta'_j = \alpha_i \cdot (\beta_j + \gamma_{R(E)}) = \alpha_i \cdot \beta_j.$$

□

*Proof of* Theorem 4.3. (i) Since $I_a(E+D) = I_a(R(E)) \cup I_a(D)$, it follows from Table 4.2 that

$$\begin{aligned}
W_a(T+K;t) &= \sum_{i \in I_a(E+D)} \varepsilon_i (t^{\alpha'_i \cdot \beta'_i} - 1) \\
&= \sum_{i \in I_a(R(E))} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) + \sum_{i \in I_a(D)} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) \\
&= W_a(R(T);t) + W_a(K;t).
\end{aligned}$$

(ii) By Table 4.1, we have

$$\begin{aligned}
&F_{ab}(T+K;t) - F_{ab}(R(T);t) - F_{ab}(K;t) \\
&= \sum_{\substack{i \in J'_a(E) \\ j \in I_b(D)}} \varepsilon_i \varepsilon_j (t^{-\alpha_j \cdot \beta_j} - 1) + \sum_{\substack{i \in I_a(D) \\ j \in J'_b(E)}} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_i} - 1) \\
&= \left( \sum_{i \in J'_a(E)} \varepsilon_i \right) \left( \sum_{j \in I_b(D)} \varepsilon_j (t^{-\alpha_j \cdot \beta_j} - 1) \right) + \left( \sum_{j \in J'_b(E)} \varepsilon_j \right) \left( \sum_{i \in I_a(D)} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) \right) \\
&= \lambda_a(T) \cdot W_b(K;t^{-1}) + \lambda_b(T) \cdot W_a(K;t).
\end{aligned}$$

(iii) By applying Reidemeister moves I to $E$ and $D$ if necessary, we may assume that $R(E)$ and $D$ are untwisted. By Table 4.2, we have

$$\begin{aligned}
&G_{ab}(T+K;t) - G_{ab}(R(T);t) - G_{ab}(K;t) \\
&= g_{ab}(E+D;t) - g_{ab}(R(E);t) - g_{ab}(D;t) \\
&= \sum_{\substack{i \in J_a(E) \\ j \in I_b(D)}} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_i} - 1) + \sum_{\substack{i \in J'_a(E) \\ j \in I_b(D)}} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_i + \alpha_j \cdot \beta_j} - 1) \\
&\quad + \sum_{\substack{i \in I_a(D) \\ j \in J_b(E)}} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_i} - 1).
\end{aligned}$$

Since $D$ is untwisted, we have $\omega_b(D) = 0$. Therefore, the first sum is equal to

$$\left( \sum_{j \in I_b(D)} \varepsilon_j \right) \left( \sum_{i \in J_a(E)} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) \right) = \omega_b(D) \cdot U_a(T;t) = 0.$$

By the same reason, the second sum is equal to

$$\begin{aligned}
&\left( \sum_{i \in J'_a(E)} \varepsilon_i t^{\alpha_i \cdot \beta_i} \right) \left( \sum_{j \in I_b(D)} \varepsilon_j (t^{\alpha_j \cdot \beta_j} - 1) \right) + \left( \sum_{j \in I_b(D)} \varepsilon_j \right) \left( \sum_{i \in J'_a(E)} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) \right) \\
&= V_a(T;t) \cdot W_b(K;t) + \omega_b(D) \cdot (V_a(T;t) - \lambda_a(T)) \\
&= V_a(T;t) \cdot W_b(K;t).
\end{aligned}$$

Moreover, since $R(E)$ is untwisted, we have $\sum_{j\in J_b(E)}\varepsilon_j+\lambda_b(T)=\omega_b(R(E))=0$. Therefore, the third sum is equal to

$$\left(\sum_{j\in J_b(E)}\varepsilon_j\right)\left(\sum_{i\in I_a(D)}\varepsilon_i(t^{\alpha_i\cdot\beta_i}-1)\right)=-\lambda_b(T)\cdot W_a(K;t).$$

Therefore, the required equation follows.

(iv) We may assume that $R(E)$ and $D$ are untwisted. By Table 4.3, we have

$$\begin{aligned}
&H_{ab}(T+K;t)-H_{ab}(R(T);t)-H_{ab}(K;t)\\
&=h_{ab}(E+D;t)-h_{ab}(R(E);t)-h_{ab}(D;t)\\
&=\sum_{\substack{i\in J_a(E)\\ j\in I_b(D)}}\varepsilon_i\varepsilon_j(t^{-\alpha_i\cdot\beta_i+\alpha_j\cdot\beta_j}-1)+\sum_{\substack{i\in J'_a(E)\\ j\in I_b(D)}}\varepsilon_i\varepsilon_j(t^{-\alpha_i\cdot\beta_i}-1)\\
&\quad+\sum_{\substack{i\in I_a(D)\\ j\in J_b(E)}}\varepsilon_i\varepsilon_j(t^{-\alpha_i\cdot\beta_i+\alpha_j\cdot\beta_j}-1)+\sum_{\substack{i\in I_a(D)\\ j\in J'_b(E)}}\varepsilon_i\varepsilon_j(t^{\alpha_j\cdot\beta_j}-1).
\end{aligned}$$

Since $R(E)$ and $D$ are untwisted, the first sum is equal to

$$\begin{aligned}
&\left(\sum_{i\in J_a(E)}\varepsilon_i(t^{-\alpha_i\cdot\beta_i}-1)\right)\left(\sum_{j\in I_b(D)}\varepsilon_j(t^{\alpha_j\cdot\beta_j}-1)\right)\\
&\quad+\left(\sum_{j\in I_b(D)}\varepsilon_j\right)\left(\sum_{i\in J_a(E)}\varepsilon_i(t^{-\alpha_i\cdot\beta_i}-1)\right)+\left(\sum_{i\in J_a(E)}\varepsilon_i\right)\left(\sum_{j\in I_b(D)}\varepsilon_j(t^{\alpha_j\cdot\beta_j}-1)\right)\\
&=U_a(T;t^{-1})\cdot W_b(K;t)+\omega_b(D)\cdot U_a(T;t^{-1})-\lambda_a(T)\cdot W_b(K;t)\\
&=(U_a(T;t^{-1})-\lambda_a(T))\cdot W_b(K;t).
\end{aligned}$$

Since the third sum is obtained from the first sum by replacing $t$ with $t^{-1}$ and by exchanging $a$ and $b$, it is equal to $(U_b(T;t)-\lambda_b(T))\cdot W_a(K;t^{-1})$.

On the other hand, since $D$ is untwisted, the second sum is equal to

$$\left(\sum_{j\in I_b(D)}\varepsilon_j\right)\left(\sum_{i\in J'_a(E)}\varepsilon_i(t^{-\alpha_i\cdot\beta_i}-1)\right)=\omega_b(D)\cdot(V_a(T;t^{-1})-\lambda_a(T))=0.$$

Similarly, the fourth sum is equal to zero. Therefore, the conclusion follows. □

## 5. The left closure of a tangle

For a diagram $E$ of a virtual 2-string tangle $T$, let $L(E)$ denote the long virtual knot diagram obtained from $E$ by closing the pair of the left endpoints, as shown in Figure 5.1. The *left closure* of $T$ is the long virtual knot presented by $L(E)$, and is denoted by $L(T)$.

**Definition 5.1.** A virtual 2-string tangle $T$ is called *simply linked* if it is presented by a diagram $E$ with

$$J_0(E)=J_1(E)=\emptyset,$$

which means that any real crossing of $E$ occurs between $A$ and $B$.

**Lemma 5.2.** *For any long virtual knot $K$, there is a virtual 2-string tangle $T$ such that*

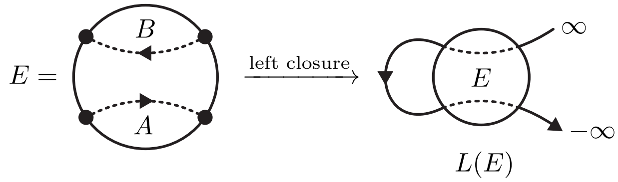



FIGURE 5.1. The left closure

(i) *$T$ is simply linked,*
(ii) $\lambda_0(T) = \lambda_1(T) = 0$,
(iii) $R(T) = K$,
(iv) $\widehat{L(T)} = \widehat{K}$, *and*
(v) $sg_1(L(T)) \leq sg_1(K)$.

*Proof.* Let $(\Sigma_g, D)$ be a surface realization of $K$ with $sg_1(K) = g$, and $p$ the base-point on $D$. By applying Reidemeister moves I if necessary, we may assume that $D$ is untwisted; that is, $\omega_0(D) = \omega_1(D) = 0$.

First, we choose a point $q$ on $D$ ahead of $p$ with respect to the orientation of $D$. Let $A$ and $B$ be the subarcs of $D$ such that $A$ runs from $p$ to $q$ and $B$ runs from $q$ to $p$. Note that there is no real crossing between two paths of $A$. While traveling $B$, let $c$ be the crossing between two paths of $B$ closest to $q$. Next, we move $c$ to a position behind $q$ by a finite sequence of Reidemeister moves as shown in Figure 5.2, where $B$ is indicated by a thick line. This procedure reduces the number of crossings between two paths of $B$, introduces no new real crossing between two paths of $A$, and preserves both the 0- and 1-writhes of $D$. By repeating this process, we finally obtain an untwisted diagram $D$ on $\Sigma_g$ with two points $p$ and $q$ such that any crossing occurs between $A$ and $B$.

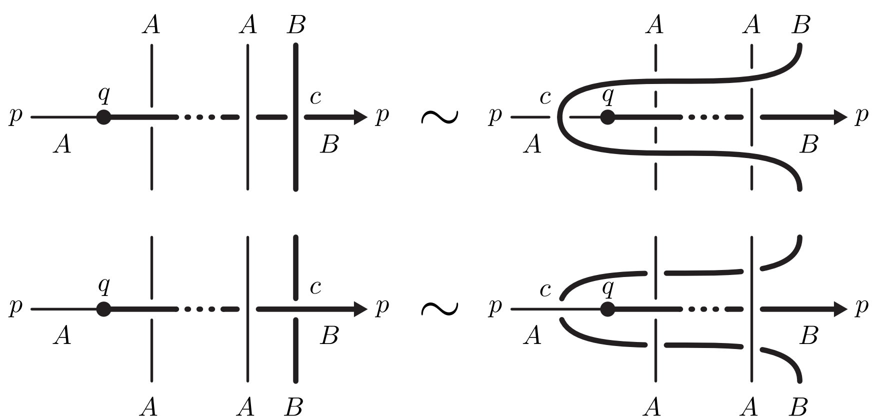



FIGURE 5.2. Moving $c$ to a position behind $q$

Let $E$ be a virtual 2-string tangle diagram obtained from the resulting diagram $D$ by cutting it at $p$ and $q$, and $T$ the tangle presented by $E$. Since there is no crossing between two paths of $A$ or two paths of $B$, we have $J_0(E) = J_1(E) = \emptyset$, which means that $T$ is simply linked. Moreover, we obviously have $R(T) = K$, and

$$\lambda_a(T) = \lambda_a(E) = \omega_a(D) = 0$$

for any $a \in \{0,1\}$.

Since the closure of $L(T)$ is presented by $D$ with the basepoint $q$, we have $\widehat{L(T)} = \widehat{K}$ and $sg_1(L(T)) \leq g = sg_1(K)$. □

**Lemma 5.3.** *If a simply linked tangle $T$ stasifies $\lambda_0(T) = \lambda_1(T) = 0$, then*

$$F_{ab}(L(T);t) = H_{a'b'}(R(T);t) \text{ and } H_{ab}(L(T);t) = F_{a'b'}(R(T);t)$$

*hold for any $a, b \in \{0,1\}$, where $a' = 1-a$ and $b' = 1-b$.*

*Proof.* Let $E$ be a diagram of $T$ with $J_0(E) = J_1(E) = \emptyset$. We regard the real crossings $c_1, \dots, c_m$ of $E$ as those of the right closure $R(E)$. Note that $R(E)$ is untwisted by $\lambda_0(E) = \lambda_1(E) = 0$.

For each $i$ with $1 \leq i \leq m$, let $c_i'$ denote the crossing of the left closure $L(E)$ corresponding to $c_i$, and $\alpha_i'$ and $\beta_i'$ the cycles at $c_i'$ of $L(E)$. Then it holds that $\alpha_i' = \beta_i$ and $\beta_i' = \alpha_i$ for $1 \leq i \leq m$, and

$$I_a(L(E)) = J_{a'}(E) = I_{a'}(R(E)).$$

Therefore, we have

$$\begin{aligned} F_{ab}(L(T);t) = f_{ab}(L(E);t) &= \sum_{i \in I_a(L(E)), j \in I_b(L(E))} \varepsilon_i \varepsilon_j (t^{\alpha_i' \cdot \alpha_j'} - 1) \\ &= \sum_{i \in I_{a'}(R(E)), j \in I_{b'}(R(E))} \varepsilon_i \varepsilon_j (t^{\beta_i \cdot \beta_j} - 1) \\ &= h_{a'b'}(R(E);t) = H_{a'b'}(R(T);t). \end{aligned}$$

Similarly, we have $H_{ab}(L(T);t) = F_{a'b'}(R(T);t)$. □

**Proposition 5.4.** *For any long virtual knot $K$, there exists a long virtual knot $K'$ such that*

(i) $F_{ab}(K';t) = H_{a'b'}(K;t)$,
(ii) $H_{ab}(K';t) = F_{a'b'}(K;t)$, *and*
(iii) $sg_1(K') \leq sg_1(K)$

*for any $a, b \in \{0,1\}$, where $a' = 1-a$ and $b' = 1-b$.*

*Proof.* By Lemma 5.2, we may choose a simply linked tangle $T$ with $R(T) = K$ and $\lambda_0(T) = \lambda_1(T) = 0$. Let $K' = L(T)$. The conclusion then follows from Lemmas 5.2 and 5.3. □

## 6. Characterization of the writhe polynomials

In [5], the writhe polynomials $W_0(K;t)$ and $W_1(K;t)$ are characterized as follows.

**Lemma 6.1** ([5, Proposition 4.4])**.** *Let $f(t) \in \mathbb{Z}[t, t^{-1}]$ be a Laurent polynomial. Then the following are equivalent.*

(i) *There exists a long virtual knot $K$ with $W_0(K;t) = f(t)$.*
(ii) $f(1) = 0$.

*The same equivalence holds for $W_1$.* □

In this section, we prove the following, which is slightly stronger than Lemma 6.1.

**Proposition 6.2.** *Let $f(t) \in \mathbb{Z}[t, t^{-1}]$ be a Laurent polynomial. Then the following are equivalent.*

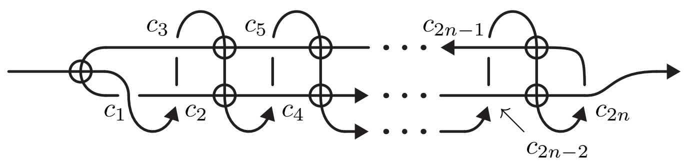
(i) *There eixsts a long virtual knot $K$ with $W_0(K;t) = f(t)$ and $sg_2(K) = 0$.*

(ii) $f(1) = 0$.

*The same equivalence holds for* $W_1$.

We remark that, if $f(t) \neq 0$, then any long virtual knot $K$ in Proposition 6.2(i) satisfies $sg_1(K) = 1$; in fact, $K$ is nonclassical.

For $n \geq 1$, let $J_n$ denote the long virtual knot presented by a diagram $D_n$ with $2n$ real crossings $c_1, \ldots, c_{2n}$, as shown at the top of Figure 6.1.



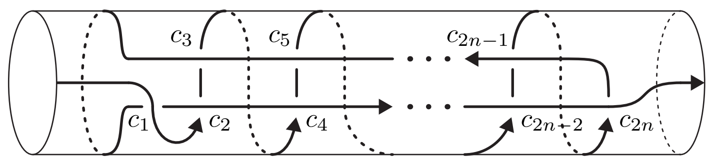




FIGURE 6.1. The long virtual knot $J_n$

**Lemma 6.3.** *The long virtual knot* $J_n$ $(n \geq 1)$ *satisfies the following.*

(i) $W_0(J_n;t) = W_1(J_n;t) = t^n - 1$.
(ii) $F_{ab}(J_n;t) = G_{ab}(J_n;t) = 0$ *for any* $a, b \in \{0,1\}$.
(iii) $H_{ab}(J_n;t) = -t^n + 2 - t^{-n}$ *for any* $a, b \in \{0,1\}$.
(iv) $sg_1(J_n) = 1$ *and* $sg_2(J_n) = 0$.

*Proof.* Since the diagram $D_n$ has a surface realization on the annulus $\Sigma_{0,2}$ as shown at the bottom of Figure 6.1, we have $sg_2(J_n) = 0$. Moreover, since it hols that

$$I_0(D_n) = \{1\}, \ \varepsilon_1 = 1, \text{ and } \alpha_1 \cdot \beta_1 = n,$$

we have $W_0(J_n;t) = t^n - 1$, and hence $sg_1(J_n) = 1$. The remaining assertions follow immediately from Lemma 3.6. □

We define a set of Laurent polynomials by

$$\mathcal{P} = \{W_0(K;t) \mid K\text{: a long virtual knot with } sg_2(K) = 0\}.$$

**Lemma 6.4.** *For any Laurent polynomials* $f(t)$ *and* $g(t) \in \mathcal{P}$, *we have*

(i) $-f(t) \in \mathcal{P}$,
(ii) $f(t^{-1}) \in \mathcal{P}$, *and*
(iii) $f(t) + g(t) \in \mathcal{P}$.

*Proof.* Let $K$ and $K'$ be long virtual knots such that

$$W_0(K;t) = f(t), \ W_0(K';t) = g(t), \text{ and } sg_2(K) = sg_2(K') = 0.$$

(i) By Lemmas 2.2(i) and 3.6(i), it holds that

$$W_0(K^{\#};t) = -W_1(K;t) = -W_0(K;t) = -f(t).$$

Since $sg_2(K^{\#}) = sg_2(K) = 0$, we have $-f(t) \in \mathcal{P}$.

(ii) By Lemmas 2.2(i), (iii), and 3.6(i), it holds that

$$W_0(K^{\#*};t) = W_1(K;t^{-1}) = W_0(K;t^{-1}) = f(t^{-1}).$$

Since $sg_2(K^{\#*}) = sg_2(K) = 0$, we have $f(t^{-1}) \in \mathcal{P}$.

(iii) By Lemma 2.4(i), it holds that

$$W_0(K \circ K';t) = W_0(K;t) + W_0(K';t) = f(t) + g(t).$$

Since $sg_2(K \circ K') \leq sg_2(K) + sg_2(K') = 0$, we have $f(t) + g(t) \in \mathcal{P}$. □

*Proof of* Proposition 6.2. The implication (i)⇒(ii) follows from $W_0(K;1) = 0$. We prove (ii)⇒(i). Since $f(1) = 0$, we can write

$$f(t) = \sum_{k \neq 0} c_k(t^k - 1)$$

for some integers $c_k$. Since $t^k - 1 \in \mathcal{P}$ $(k > 0)$ by Lemma 6.3(i), we have $f(t) \in \mathcal{P}$ by Lemma 6.4.

Since $W_0(K;t) = W_1(K;t)$ by Lemma 3.6(i), the same characterization holds for $W_1$ as well. □

## 7. Characterization of the intersection polynomials

In this section, we address the realizability problem for the intersection polynomials of long virtual knots, and establish a complete characterization for each of the twelve polynomials. We first focus on the polynomials $F_{00}$, $F_{11}$, $H_{00}$, and $H_{11}$.

**Theorem 7.1.** *Let $f(t) \in \mathbb{Z}[t, t^{-1}]$ be a Laurent polynomial. Then the following are equivalent.*

(i) *There eixsts a long virtual knot $K$ with $F_{00}(K;t) = f(t)$.*
(ii) *There eixsts a long virtual knot $K'$ with $F_{00}(K';t) = f(t)$ and $sg_1(K') \leq 1$.*
(iii) *$f(1) = 0$ and $f(t) = f(t^{-1})$.*

*The same equivalence holds for $F_{11}$, $H_{00}$, and $H_{11}$.*

*Proof.* The implication (ii)⇒(i) is trivial, and (i)⇒(iii) follows from Lemma 2.1(ii).

We prove (iii)⇒(ii). Since $f(1) = 0$ and $f(t) = f(t^{-1})$, we can write

$$f(t) = g(t) + g(t^{-1})$$

for some $g(t) \in \mathbb{Z}[t, t^{-1}]$ with $g(1) = 0$; in fact, there are integers $c_k$ such that $f(t) = \sum_{k>0} c_k(t^k - 2 + t^{-k})$, and we may take $g(t) = \sum_{k>0} c_k(t^k - 1)$. Since $g(1) = 0$, there exists a long virtual knot $K$ such that $W_0(K;t) = g(t)$ and $sg_2(K) = 0$ by Proposition 6.2. Moreover, $K$ satisfies $F_{00}(K;t) = 0$ by Lemma 3.6(ii).

Let $T_1$ be the virtual tangle shown on the left of Figure 7.1. Then we have

$$F_{00}(R(T_1);t) = 0 \text{ and } \lambda_0(T_1) = 1.$$

Therefore, it follows from Theorem 4.3(ii) that

$$F_{00}(T_1 + K) = F_{00}(K;t) + W_0(K;t) + W_0(K;t^{-1}) = 0 + g(t) + g(t^{-1}) = f(t).$$

Moreover, since $sg_2(K) = 0$, the right of the figure shows that $sg_1(T_1 + K) \leq 1$.

For $F_{11}$, since it follows from Lemma 2.3(i) that

$$F_{11}((T_1 + K)^{\#};t) = F_{00}(T_1 + K;t) = f(t) \text{ and}$$
$$sg_1((T_1 + K)^{\#}) = sg_1(T_1 + K) \leq 1,$$

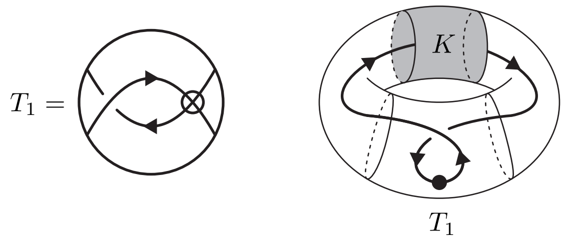



FIGURE 7.1. The tangle $T_1$ and a surface realization of $T_1 + K$

the same characterization holds for $F_{11}$ as well. For $H_{00}$ and $H_{11}$, by Lemma 2.3(i) and Proposition 5.4(ii) and (iii), there is a long virtual knot $K'$ such that

$$H_{00}(K'^{\#}; t) = H_{11}(K'; t) = F_{00}(T_1 + K; t) = f(t) \text{ and}$$
$$sg_1(K'^{\#}) = sg_1(K') \leq sg_1(T_1 + K) \leq 1.$$

Therefore, the same characterization holds for $H_{00}$ and $H_{11}$. □

We remark that, if $f(t) \neq 0$, then any long virtual knot $K'$ in Theorem 7.1(ii) satisfies $sg_1(K') = 1$; in fact, $K'$ is nonclassical.

The next theorem gives a characterization for $G_{00}$ and $G_{11}$.

**Theorem 7.2.** *Let $f(t) \in \mathbb{Z}[t, t^{-1}]$ be a Laurent polynomial. Then the following are equivalent.*

(i) *There eixsts a long virtual knot $K$ with $G_{00}(K; t) = f(t)$.*
(ii) *There eixsts a long virtual knot $K'$ with $G_{00}(K'; t) = f(t)$ and $sg_1(K') \leq 1$.*
(iii) *$f(1) = f'(1) = 0$.*

*The same equivalence holds for $G_{11}$.*

*Proof.* The implication (ii)⇒(i) is trivial, and (i)⇒(iii) follows from Lemma 2.1(iii).

We prove (iii)⇒(ii). Since $f(1) = f'(1) = 0$, we can write

$$f(t) = (1 - t^{-1})g(t)$$

for some $g(t) \in \mathbb{Z}[t, t^{-1}]$ with $g(1) = 0$. Then there exists a long virtual knot $K$ such that $W_0(K; t) = g(t)$ and $sg_2(K) = 0$ by Proposition 6.2. Moreover, $K$ satisfies $G_{00}(K; t) = 0$ by Lemma 3.6(ii).

Let $T_2$ be the virtual tangle shown on the left of Figure 7.2. Then we have

$$G_{00}(R(T_2); t) = 0, \ \lambda_0(T_2) = -1, \text{ and } V_0(T_2; t) = -t^{-1}.$$

Therefore, it follows from Theorem 4.3(iii) that

$$G_{00}(T_2 + K) = G_{00}(K; t) + (1 - t^{-1})W_0(K; t) = 0 + (1 - t^{-1})g(t) = f(t).$$

Moreover, since $sg_2(K) = 0$, the right of the figure shows that $sg_1(T_2 + K) \leq 1$.

For $G_{11}$, since it follows from Lemma 2.3(i) that

$$G_{11}((T_2 + K)^{\#}; t) = G_{00}(T_2 + K; t) = f(t) \text{ and}$$
$$sg_1((T_2 + K)^{\#}) = sg_1(T_2 + K) \leq 1,$$

the same characterization holds for $G_{11}$ as well. □

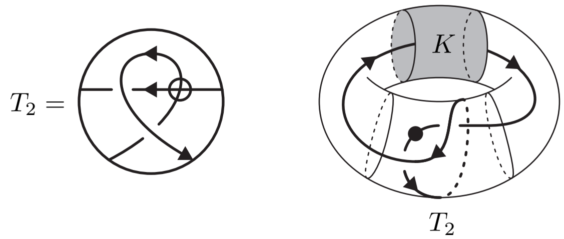



FIGURE 7.2. The tangle $T_2$ and a surface realization of $T_2 + K$

**Remark 7.3.** (i) In the proof of Theorem 7.2, we have $G_{00}(R(T_2); t) = 0$, which can be generalized as follows. Suppose that a long virtual knot $K$ has a diagram where a real crossing $c_1$ is of type 0, and the others are of type 1. Then we have

$$F_{00}(K;t) = \varepsilon_1^2(t^{\alpha_1\cdot\alpha_1} - 1) = 0 \text{ and}$$
$$G_{00}(K;t) = \varepsilon_1^2(t^{\alpha_1\cdot\beta_1} - 1) - \varepsilon_1 \cdot \varepsilon_1(t^{\alpha_1\cdot\beta_1} - 1) = 0.$$

(ii) If $f(t) \neq 0$, then any knot $K'$ in Theorem 7.2(ii) satisfies $sg_1(K') = 1$.
(iii) For (closed) virtual knots, the same characterization as in Theorem 7.2 also holds for the writhe polynomial [11] and for the first intersection polynomial [6].

We conclude this paper by giving a characterization of the remaining intersection polynomials as follows.

**Theorem 7.4.** *Let $f(t) \in \mathbb{Z}[t, t^{-1}]$ be a Laurent polynomial. Then the following are equivalent.*

(i) *There eixsts a long virtual knot $K$ with $F_{01}(K;t) = f(t)$.*
(ii) *There eixsts a long virtual knot $K'$ with $F_{01}(K';t) = f(t)$ and $sg_1(K') \leq 1$.*
(iii) $f(1) = 0$.

*The same equivalence holds for $F_{10}$, $G_{01}$, $G_{10}$, $H_{01}$ and $H_{10}$.*

*Proof.* The implications (ii)⇒(i) and (i)⇒(iii) are trivial.

We first prove (iii)⇒(ii) for $F_{01}$ and $G_{01}$. Since $f(1) = 0$, there exists a long virtual knot $K$ such that $W_0(K;t) = f(t)$ and $sg_2(K) = 0$ by Proposition 6.2. Moreover, $K$ satisfies $F_{01}(K;t) = G_{01}(K;t) = 0$ by Lemma 3.6(ii).

Let $T_3$ and $T_4$ be the virtual tangles as shown in Figure 7.3. Then we have

$$F_{01}(R(T_3);t) = 0,\ \lambda_0(T_3) = 0,\ \lambda_1(T_3) = 1,$$
$$G_{01}(R(T_4);t) = 0,\ \lambda_1(T_4) = -1, \text{ and } V_0(T_4;t) = 0.$$

Therefore, it follows from Theorem 4.3(ii) and (iii) that

$$F_{01}(T_3 + K;t) = F_{01}(K;t) + W_0(K;t) = 0 + f(t) = f(t) \text{ and}$$
$$G_{01}(T_4 + K;t) = G_{01}(K;t) + W_0(K;t) = 0 + f(t) = f(t).$$

Moreover, since $sg_2(K) = 0$, we have $sg_1(T_i + K) \leq 1$ $(i = 3, 4)$.

Similarly to the proof of Theorem 7.1, the same characterization holds for the remaining polynomials $F_{10}$, $G_{10}$, $H_{01}$, and $H_{10}$. □

We remark that, if $f(t) \neq 0$, then any long virtual knot $K'$ in Theorem 7.4(ii) satisfies $sg_1(K') = 1$.

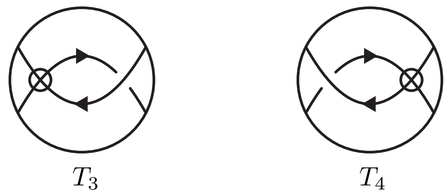



FIGURE 7.3. The tangles $T_3$ and $T_4$

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