THE INTERSECTION POLYNOMIALS OF A LONG VIRTUAL KNOT I: DEFINITIONS AND PROPERTIES

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Abstract. We introduce twelve polynomial invariants for long virtual knots, called intersection polynomials, extending and refining the three intersection polynomials for virtual knots. They are defined via intersection numbers of cycles on a closed surface, considering the order of over- and under-crossings. We study their fundamental properties including behavior under symmetries, crossing changes, and concatenation products. All are finite-type invariants of degree two under crossing changes, but not under virtualizations, and we examine their relation to the closure and the values at $t = 1$ of their derivatives.

1. Introduction

In 1999, Kauffman [12] introduced the notion of virtual knots as a generalization of classical knots, which naturally leads to the concept of long virtual knots. In 2000, Goussarov, Polyak, and Viro [5] studied long virtual knots from the point of view of finite-type invariants and showed that their theory differs substantially from that of virtual knots. Manturov [14] proved that the concatenation product of certain pair of long virtual knots is noncommutative, and Silver and Williams [17] constructed an infinite family of long virtual knots whose closures realize any prescribed virtual knot (see also [7]). Polynomial invariants for long virtual knots have also been studied extensively. Refer to [1, 2, 10, 11], for example.

In a series of papers [6]–[9], Higa with the first three authors introduced and studied three polynomial invariants of a virtual knot, called the first, second, and third intersection polynomials. Their construction is based on the intersection number of a pair of cycles on a closed surface. In this paper, we employ a similar idea to define twelve polynomial invariants of a long virtual knot K. These are also referred to as the intersection polynomials of K, and are denoted by

$$F_{ab}(K;t),\ G_{ab}(K;t),\ \text{and } H_{ab}(K;t) \text{ for } a,b \in \{0,1\}.$$

In the definition, we distinguish the type of each real crossing according to whether the overcrossing or undercrossing is encountered first along K.

This paper is the first in a series of two. Its aim is to introduce the twelve intersection polynomials and to study their fundamental properties. These invariants refine the three intersection polynomials previously defined for virtual knots. In the second paper [15] of this series, we introduce two supporting genera of a long

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virtual knot and establish their relationship with the intersection polynomials. We further provide a characterization of each intersection polynomial.

The rest of this paper is organized as follows. Section 2 introduces the twelve polynomial invariants. We prove their invariance in Section 3, and present examples of computations in Section 4. Section 5 investigates their behavior under symmetries such as orientation-reversal and mirror-reflection (Theorem 5.2). Section 6 provides formulae for the product of long virtual knots (Theorem 6.2). In Section 7, we construct invariants under crossing changes from the intersection polynomials (Theorems 7.2). In Sections 8 and 9, we prove that the intersection polynomials are finite-type invariants of degree two under crossing changes (Theorem 8.2), but not of any degree under virtualizations (Theorem 9.2). In Section 10, we study the relationship between the intersection polynomials of a long virtual knot and those of its closure (Proposition 10.1), and establish several results on the values at $t = 1$ of their first and second derivatives (Theorem 10.3).

2. Intersection polynomials

A *long virtual knot diagram* is an immersed line in a plane $\mathbb{R}^2$ such that

(i) it is identical to the x-axis of $\mathbb{R}^2$ outside some 2-disk, and
(ii) it has finitely many transversal double points called *real crossings* or *virtual crossings*.

Here, a real crossing has an over/under information, and a virtual crossing is encircled by a small circle. Figure 2.1 shows an example of a long virtual knot diagram with five real crossings $c_1, \dots, c_5$ and two virtual crossings. A *long virtual knot* is an equivalence class of such diagrams under generalized Reidemeister moves I–VII. Throughout this paper, all long virtual knots are oriented from $-\infty$ to ∞.

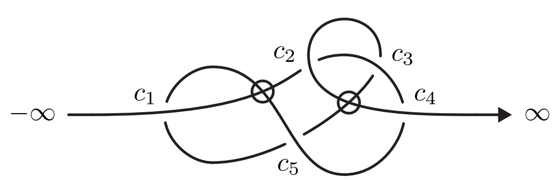

FIGURE 2.1. A long virtual knot diagram

Let $c_1, \dots, c_n$ be the real crossings of a long virtual knot diagram D, which are classified into two types as follows. We say that c_i is *of type* 0 if we pass c_i first going over and then under along D from $-\infty$ to ∞. Otherwise c_i is *of type* 1. Refer to [1, 10, 14]. The set of indices of real crossings of type $a \in \{0, 1\}$ is denoted by

$$I_a = I_a(D) = \{i \mid c_i \text{ is of type } a\}.$$

For example, the diagram in Figure 2.1 gives $I_0 = \{1, 3, 5\}$ and $I_1 = \{2, 4\}$.

Let Σ_g denote a closed, connected, and oriented surface of genus $g \geq 0$. For a long virtual knot diagram D, a *surface realization* of D is a pair of Σ_g and a knot diagram D' with a basepoint such that

(i) D' lies in Σ_g with no virtual crossings,
(ii) the set of real crossings of D coincides with that of D', and
(iii) the basepoint of D' corresponds to $\pm\infty$ of D.

Provided there is no ambiguity, we also use the notation D for D'. A long virtual knot is regarded as an equivalence class of such pairs (Σ_g, D) under Reidemeister moves I–III and (de)stabilizations (cf. [7, 12]). For example, the virtual knot diagram D in Figure 2.1 has a surface realization (Σ_2, D) as shown in Figure 2.2.

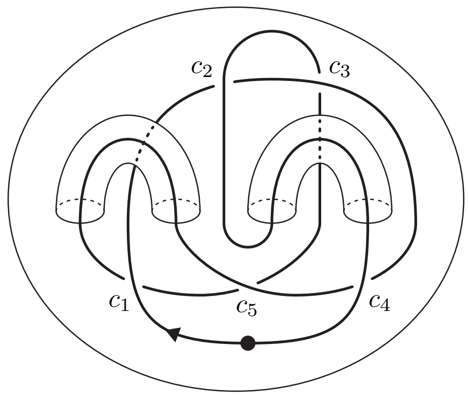

FIGURE 2.2. A surface realization (Σ_2, D)

We apply a smoothing operation to a surface realization (Σ_g, D) at a real crossing c_i to obtain a pair of oriented loops on Σ_g. We denote by α_i (resp. β_i) the loop that does not contain (resp. contains) the basepoint of D. These loops are regarded as homology cycles on Σ_g. As an example, consider a surface realization (Σ_1, D) with two real crossings c_1 and c_2 as shown on the left of Figure 2.3. The cycles α_1 and β_1 are as depicted in the center of the figure, while α_2 and β_2 are shown on the right.

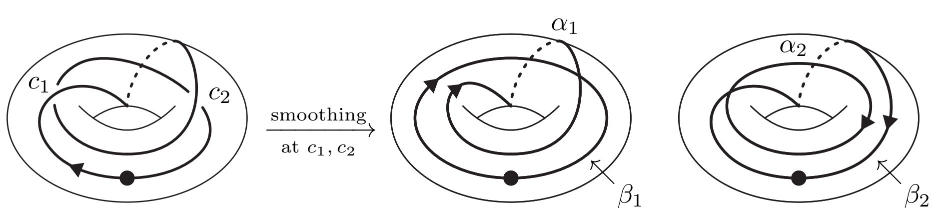

FIGURE 2.3. A surface realization (Σ_1, D) with four cycles

For two cycles $\gamma, \gamma' \in \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$, we denote by $\gamma \cdot \gamma' \in \mathbb{Z}$ the intersection number of γ and γ', which is invariant under (de)stabilization of Σ_g. We denote by $\varepsilon_i = \varepsilon(c_i)$ the sign of a real crossing c_i.

Theorem 2.1 ([7, Lemma 4.1])**.** *Let D be a diagram of a long virtual knot K, and $c_1, \ldots, c_n$ the real crossings of D. For any $a \in \{0, 1\}$, the Laurent polynomial*

$$W_a(D; t) = \sum_{i \in I_a} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) \in \mathbb{Z}[t, t^{-1}]$$

is an invariant of K. □

The Laurent polynomial in Theorem 2.1 is called the *a-writhe polynomial* of K, and is denoted by $W_a(K;t)$ for $a \in \{0,1\}$. For example, since (Σ_1, D) in Figure 2.3 satisfies $I_0 = \{1,2\}$, $I_1 = \emptyset$, $\varepsilon_1 = \varepsilon_2 = 1$, $\alpha_1 \cdot \beta_1 = -1$, and $\alpha_2 \cdot \beta_2 = 1$, we have

$$W_0(K;t) = t - 2 + t^{-1} \text{ and } W_1(K;t) = 0.$$

Remark 2.2. Let γ_D denote the cycle on Σ_g presented by D. Since $\alpha_i + \beta_i = \gamma_D$ holds, we have

$$\alpha_i \cdot \beta_i = \alpha_i \cdot (\gamma_D - \alpha_i) = \alpha_i \cdot \gamma_D.$$

In [7], we used $\alpha_i \cdot \gamma_D$ instead of $\alpha_i \cdot \beta_i$ to define the 0- and 1-writhe polynomials.

For each $a, b \in \{0,1\}$, we put

$$\begin{aligned} f_{ab}(D;t) &= \sum_{i \in I_a,\, j \in I_b} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \alpha_j} - 1), \\ g_{ab}(D;t) &= \sum_{i \in I_a,\, j \in I_b} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_j} - 1), \text{ and} \\ h_{ab}(D;t) &= \sum_{i \in I_a,\, j \in I_b} \varepsilon_i \varepsilon_j (t^{\beta_i \cdot \beta_j} - 1). \end{aligned}$$

We denote by $\omega_a = \omega_a(D)$ the sum of signs of real crossings of D of type $a \in \{0,1\}$; that is, $\omega_a = \sum_{i \in I_a} \varepsilon_i$. We call ω_a the *a-writhe* of D.

Theorem 2.3. *Let D be a diagram of a long virtual knot K. For any $a, b \in \{0,1\}$, the Laurent polynomials*

$$\begin{aligned} F_{ab}(D;t) &= f_{ab}(D;t), \\ G_{ab}(D;t) &= g_{ab}(D;t) - \omega_b W_a(K;t), \text{ and} \\ H_{ab}(D;t) &= h_{ab}(D;t) - \omega_a W_b(K;t) - \omega_b W_a(K;t^{-1}) \end{aligned}$$

are invariants of K.

The twelve Laurent polynomials in Theorem 2.3 are collectively called the *intersection polynomials* of a long virtual knot K, and are denoted by $F_{ab}(K;t)$, $G_{ab}(K;t)$, and $H_{ab}(K;t)$, respectively. We give the proof of Theorem 2.3 in Section 3.

A Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ is called *reciprocal* if $f(t^{-1}) = f(t)$ holds.

Lemma 2.4. *For a long virtual knot K, we have the following.*

(i) $F_{01}(K;t) = F_{10}(K;t^{-1})$ *and* $H_{01}(K;t) = H_{10}(K;t^{-1})$.
(ii) $F_{00}(K;t)$, $F_{11}(K;t)$, $H_{00}(K;t)$, *and* $H_{11}(K;t)$ *are reciprocal.*

Proof. (i) Since $\alpha_i \cdot \alpha_j = -\alpha_j \cdot \alpha_i$ holds for any i and j, we have

$$F_{10}(K;t^{-1}) = \sum_{i \in I_1,\, j \in I_0} \varepsilon_i \varepsilon_j (t^{-\alpha_i \cdot \alpha_j} - 1) = \sum_{j \in I_0,\, i \in I_1} \varepsilon_j \varepsilon_i (t^{\alpha_j \cdot \alpha_i} - 1) = F_{01}(K;t).$$

The other equation can be proved similarly.

(ii) It holds that

$$F_{00}(K;t^{-1}) = \sum_{i,j \in I_0} \varepsilon_i \varepsilon_j (t^{-\alpha_i \cdot \alpha_j} - 1) = \sum_{i,j \in I_0} \varepsilon_j \varepsilon_i (t^{\alpha_j \cdot \alpha_i} - 1) = F_{00}(K;t).$$

The other equations can be proved similarly. □

A long virtual knot is called *classical* if it is presented by a diagram in $\mathbb{R}^2$ with no virtual crossings.

Lemma 2.5. *If K is a long classical knot, then we have the following.*

(i) $W_a(K;t) = 0$ *for any* $a \in \{0,1\}$.
(ii) $X_{ab}(K;t) = 0$ *for any* $X \in \{F, G, H\}$ *and* $a, b \in \{0,1\}$.

Proof. By definition, K is presented by some surface realization (Σ_0, D). The intersection number of any pair of cycles vanishes on a 2-sphere Σ_0. $\square$

3. Proof of Theorem 2.3

Let (Σ_g, D) be a surface realization of a long virtual knot K. To prove Theorem 2.3, we prepare Lemmas 3.1 and 3.2 stated below.

Lemma 3.1. *Let (Σ_g, D') be a surface realization obtained from (Σ_g, D) by a Reidemeister move* II *or* III. *For any $a, b \in \{0,1\}$, we have*

$$f_{ab}(D;t) = f_{ab}(D';t),\ \ g_{ab}(D;t) = g_{ab}(D';t),\ \ \text{and } h_{ab}(D;t) = h_{ab}(D';t).$$

Proof. The proof of the invariance is essentially the same as that of the intersection polynomials for a virtual knot, as presented in [6]. For example, assume that (Σ_g, D') is obtained from (Σ_g, D) by a Reidemeister move II canceling a pair of real crossings c_1 and c_2 of D. We remark that $\alpha_1 = \alpha_2$ and $\varepsilon_1 = -\varepsilon_2$ hold, and that c_1 and c_2 are of the same type. If both c_1 and c_2 are of type 0, then we have

$$\begin{aligned}
f_{00}(D) - f_{00}(D') = & \sum_{i,j\in\{1,2\}} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\alpha_j} - 1) \\
& + \sum_{j\in I_0(D)\setminus\{1,2\}} \varepsilon_1\varepsilon_j(t^{\alpha_1\cdot\alpha_j} - 1) + \sum_{j\in I_0(D)\setminus\{1,2\}} \varepsilon_2\varepsilon_j(t^{\alpha_2\cdot\alpha_j} - 1) \\
& + \sum_{i\in I_0(D)\setminus\{1,2\}} \varepsilon_i\varepsilon_1(t^{\alpha_i\cdot\alpha_1} - 1) + \sum_{i\in I_0(D)\setminus\{1,2\}} \varepsilon_i\varepsilon_2(t^{\alpha_i\cdot\alpha_2} - 1) \\
= & \ 0.
\end{aligned}$$

$\square$

Lemma 3.2. *Let (Σ_g, D') be a surface realization obtained from (Σ_g, D) by a Reidemeister move* I *removing a real crossing c_1 of D as shown in* Figure 3.1. *For any $a, b \in \{0,1\}$, we have the following.*

(i) $f_{ab}(D;t) = f_{ab}(D';t)$.

(ii) $g_{ab}(D;t) - g_{ab}(D';t) = \begin{cases} 0 & \text{for } 1 \notin I_b(D), \\ \varepsilon_1 W_a(K;t) & \text{for } 1 \in I_b(D). \end{cases}$

(iii) $h_{ab}(D;t) - h_{ab}(D';t)$

$$= \begin{cases} \varepsilon_1(W_b(K;t) + W_a(K;t^{-1})) & \text{for } a = b \text{ and } 1 \in I_a(D) = I_b(D), \\ 0 & \text{for } a = b \text{ and } 1 \notin I_a(D) = I_b(D), \\ \varepsilon_1 W_b(K;t) & \text{for } a \neq b \text{ and } 1 \in I_a(D), \\ \varepsilon_1 W_a(K;t^{-1}) & \text{for } a \neq b \text{ and } 1 \in I_b(D). \end{cases}$$

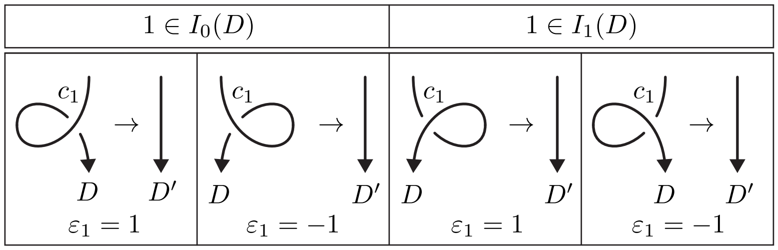

FIGURE 3.1. A Reidemeister move I

Proof. The proof is essentially the same as that of the intersection polynomials for a virtual knot, as presented in [6]. We prove (iii) only; (i) and (ii) can be proved similarly. Since we have $\alpha_1 = 0$ and $\beta_1 = \gamma_D$, it holds that

$$\beta_1 \cdot \beta_j = \gamma_D \cdot \beta_j = (\alpha_j + \beta_j) \cdot \beta_j = \alpha_j \cdot \beta_j.$$

First we consider the case $a = b$. If $1 \in I_a(D) = I_b(D)$, then we have

$$\begin{aligned}
&h_{aa}(D;t) - h_{aa}(D';t) \\
&= \varepsilon_1^2(t^{\beta_1\cdot\beta_1} - 1) + \sum_{j\in I_a(D)\setminus\{1\}} \varepsilon_1\varepsilon_j(t^{\beta_1\cdot\beta_j} - 1) + \sum_{i\in I_a(D)\setminus\{1\}} \varepsilon_i\varepsilon_1(t^{\beta_i\cdot\beta_1} - 1) \\
&= \varepsilon_1 \sum_{j\in I_a(D')} \varepsilon_j(t^{\alpha_j\cdot\beta_j} - 1) + \varepsilon_1 \sum_{i\in I_a(D')} \varepsilon_i(t^{-\alpha_i\cdot\beta_i} - 1) \\
&= \varepsilon_1 W_a(K;t) + \varepsilon_1 W_a(K;t^{-1}).
\end{aligned}$$

If $1 \notin I_a(D) = I_b(D)$, then we clearly have $h_{aa}(D;t) = h_{aa}(D';t)$.

Next we consider the case $a \neq b$. If $1 \in I_a(D)$, then we have

$$h_{ab}(D;t) - h_{ab}(D';t) = \sum_{j\in I_b(D)} \varepsilon_1\varepsilon_j(t^{\beta_1\cdot\beta_j} - 1) = \varepsilon_1 W_b(K;t).$$

If $1 \in I_b(D)$, then

$$h_{ab}(D;t) - h_{ab}(D';t) = \sum_{i\in I_a(D)} \varepsilon_i\varepsilon_1(t^{\beta_i\cdot\beta_1} - 1) = \varepsilon_1 W_a(K;t^{-1}).$$

□

Proof of Theorem 2.3. We prove only that $H_{ab}(D;t) = H_{ab}(D';t)$ holds for any two surface realizations (Σ_g, D) and $(\Sigma_{g'}, D')$ of K. The cases of F_{ab} and G_{ab} can be proved similarly. Since the intersection numbers $\beta_i \cdot \beta_j$ are preserved by a (de)stabilization of a surface, it suffices to consider the case $g = g'$.

Assume that (Σ_g, D') is obtained from (Σ_g, D) by a Reidemeister move II or III. Since we have $\omega_a(D) = \omega_a(D')$ and $\omega_b(D) = \omega_b(D')$, it follows from Theorem 2.1 and Lemma 3.1 that

$$\begin{aligned}
H_{ab}(D;t) &= h_{ab}(D;t) - \omega_a(D)W_b(K;t) - \omega_b(D)W_a(K;t^{-1}) \\
&= h_{ab}(D';t) - \omega_a(D')W_b(K;t) - \omega_b(D')W_a(K;t^{-1}) = H_{ab}(D';t).
\end{aligned}$$

Assume that (Σ_g, D') is obtained from (Σ_g, D) by a Reidemeister move I removing a real crossing c_1 of D as shown in Figure 3.1. If $a = b$ and $1 \in I_a(D) = I_b(D)$, then we have $\omega_a(D) = \omega_a(D') + \varepsilon_1$ and hence

$$\begin{aligned} H_{aa}(D;t) &= h_{aa}(D;t) - \omega_a(D)W_a(K;t) - \omega_a(D)W_a(K;t^{-1}) \\ &= h_{aa}(D';t) + \varepsilon_1(W_a(K;t) + W_a(K;t^{-1})) \\ &\quad - (\omega_a(D') + \varepsilon_1)W_a(K;t) - (\omega_a(D') + \varepsilon_1)W_a(K;t^{-1}) \\ &= h_{aa}(D';t) - \omega_a(D')W_a(K;t) - \omega_a(D')W_a(K;t^{-1}) \\ &= H_{aa}(D';t) \end{aligned}$$

by Theorem 2.1 and Lemma 3.2(iii). The other cases can be proved similarly. □

Definition 3.3. A long virtual knot diagram D is called *untwisted* if it satisfies $\omega_0(D) = \omega_1(D) = 0$.

Remark 3.4. By applying Reidemeister moves of type I appropriately, we see that any long virtual knot K has an untwisted diagram. For such a diagram D, we have

$$G_{ab}(K;t) = g_{ab}(D;t) \text{ and } H_{ab}(K;t) = h_{ab}(D;t) \quad (a, b \in \{0, 1\}).$$

4. Examples

For an integer $n \geq 2$, let $\Gamma(n)$ be a *flat* long virtual knot diagram with $n+1$ crossings $c_1, \dots, c_{n+1}$ as shown in Figure 4.1(a), where the real crossings have no crossing information.

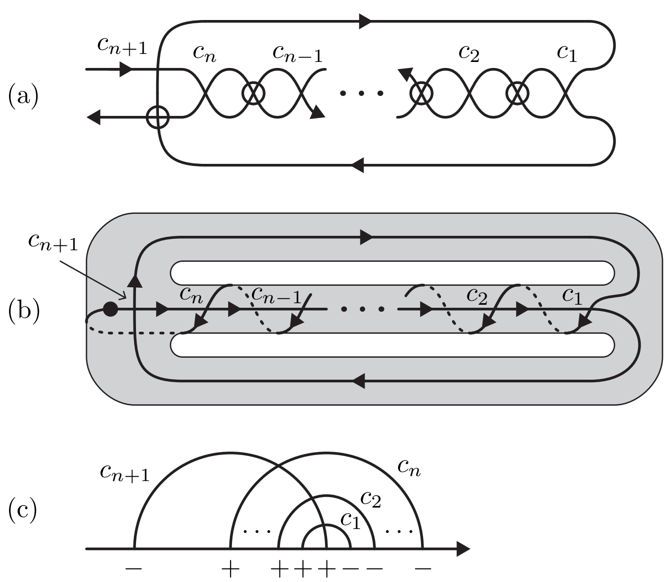

Figure 4.1. The flat long virtual knot diagram $\Gamma(n)$

Let D be a long virtual knot diagram whose underlying curve is $\Gamma(n)$. We remark that the intersection numbers are independent of crossing information at $c_1, \dots, c_{n+1}$. Since $\Gamma(n)$ is realized by an immersed circle on Σ_2 as shown in Figure 4.1(b), we see that

$$\alpha_i \cdot \gamma_D = \begin{cases} 1 & \text{for } 1 \leq i \leq n, \\ n & \text{for } i = n+1, \end{cases}$$

and

$$\alpha_i \cdot \alpha_j = \begin{cases} 0 & \text{for } 1 \leq i, j \leq n \text{ and } i = j = n+1, \\ j-1 & \text{for } i = n+1 \text{ and } 1 \leq j \leq n, \\ -(i-1) & \text{for } 1 \leq i \leq n \text{ and } j = n+1. \end{cases}$$

This is also obtained from the Gauss diagram of $\Gamma(n)$ as shown in Figure 4.1(c). Refer to [6, Section 3] for the calculation by using a Gauss diagram. By the equations

$$\alpha_i \cdot \beta_j = \alpha_i \cdot (\gamma_D - \alpha_j) = \alpha_i \cdot \gamma_D - \alpha_i \cdot \alpha_j \text{ and}$$
$$\beta_i \cdot \beta_j = (\gamma_D - \alpha_i) \cdot (\gamma_D - \alpha_j) = -\alpha_i \cdot \gamma_D + \alpha_j \cdot \gamma_D + \alpha_i \cdot \alpha_j,$$

we have

$$\alpha_i \cdot \beta_j = \begin{cases} 1 & \text{for } 1 \leq i, j \leq n, \\ (n+1)-j & \text{for } i = n+1 \text{ and } 1 \leq j \leq n, \\ i & \text{for } 1 \leq i \leq n \text{ and } j = n+1, \\ n & \text{for } i = j = n+1, \end{cases}$$

and

$$\beta_i \cdot \beta_j = \begin{cases} 0 & \text{for } 1 \leq i, j \leq n \text{ and } i = j = n+1, \\ -n+j & \text{for } i = n+1 \text{ and } 1 \leq j \leq n, \\ n-i & \text{for } 1 \leq i \leq n \text{ and } j = n+1. \end{cases}$$

These calculations are summarized in Table 4.1.

TABLE 4.1. The intersection numbers for $\Gamma(n)$

$\alpha_i \cdot \alpha_j$	1	$\cdots$	n	$n+1$
1	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
n	0	$\cdots$	0	$-n+1$
$n+1$	0	$\cdots$	$n-1$	0

$\alpha_i \cdot \beta_j$	1	$\cdots$	n	$n+1$
1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
n	1	$\cdots$	1	n
$n+1$	n	$\cdots$	1	n

$\beta_i \cdot \beta_j$	1	$\cdots$	n	$n+1$
1	0	$\cdots$	0	$n-1$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
n	0	$\cdots$	0	0
$n+1$	$-n+1$	$\cdots$	0	0

Example 4.1. Let $D(n)$ be the long virtual knot diagram obtained from $\Gamma(n)$ by specifying $\varepsilon_1, \dots, \varepsilon_n = -1$ and $\varepsilon_{n+1} = 1$. Equivalently, every real crossing c_i is of type 0. Let $K(n)$ be the long virtual knot presented by $D(n)$. Since it holds that

$$I_0 = \{1, \dots, n+1\},\ I_1 = \emptyset,\ \omega_0 = -n+1, \text{ and } \omega_1 = 0,$$

we have

$$W_0(K(n); t) = \sum_{i=1}^{n+1} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) = t^n - nt + n - 1,$$

$$\begin{aligned}
F_{00}(K(n);t) &= \sum_{1\le i,j\le n+1} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\alpha_j}-1)\\
&= -(t^{n-1}+t^{-n+1})-\cdots-(t+t^{-1})+2(n-1),\\
G_{00}(K(n);t) &= \sum_{1\le i,j\le n+1} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\beta_j}-1)-\omega_0 W_0(K(n);t)\\
&= (n-2)t^n-2(t^{n-1}+\cdots+t)+nt\\
&= \begin{cases} 0 & \text{for } n=2,\\ (n-2)t^n-2(t^{n-1}+\cdots+t^2)+(n-2)t & \text{for } n\ge 3,\end{cases}\\
H_{00}(K(n);t) &= \sum_{1\le i,j\le n+1} \varepsilon_i\varepsilon_j(t^{\beta_i\cdot\beta_j}-1)-\omega_0 W_0(K(n);t)-\omega_0 W_0(K(n);t^{-1})\\
&= (n-1)(t^n+t^{-n})-(t^{n-1}+t^{-n+1})-\cdots-(t+t^{-1})\\
&\quad -n(n-1)(t+t^{-1})+2n(n-1)\\
&= \begin{cases} (t^2+t^{-2})-3(t+t^{-1})+4 & \text{for } n=2,\\ (n-1)(t^n+t^{-n})-(t^{n-1}+t^{-n+1})-\cdots-(t^2+t^{-2}) & \\ -(n^2-n+1)(t+t^{-1})+2n(n-1) & \text{for } n\ge 3,\end{cases}
\end{aligned}$$

and the other polynomials are equal to zero.

Example 4.2. Let $D'(n)$ be the long virtual knot diagram obtained from $\Gamma(n)$ by specifying $\varepsilon_1,\dots,\varepsilon_{n+1}=1$. Equivalently, $c_1,\dots,c_n$ are of type 1 and c_{n+1} is of type 0. Let $K'(n)$ be the long virtual knot presented by $D'(n)$. Since it holds that

$$I_0=\{n+1\},\ I_1=\{1,\dots,n\},\ \omega_0=1,\ \text{and } \omega_1=n,$$

we have

$$\begin{aligned}
W_0(K'(n);t) &= \varepsilon_{n+1}(t^{\alpha_{n+1}\cdot\beta_{n+1}}-1)=t^n-1,\\
W_1(K'(n);t) &= \sum_{i=1}^{n}\varepsilon_i(t^{\alpha_i\cdot\beta_i}-1)=n(t-1),\\
F_{00}(K'(n);t) &= \varepsilon_{n+1}^2(t^{\alpha_{n+1}\cdot\alpha_{n+1}}-1)=0,\\
F_{01}(K'(n);t) &= \sum_{j=1}^{n}\varepsilon_{n+1}\varepsilon_j(t^{\alpha_{n+1}\cdot\alpha_j}-1)=t^{n-1}+\cdots+t-n+1,\\
F_{10}(K'(n);t) &= F_{01}(K'(n);t^{-1})=t^{-n+1}+\cdots+t^{-1}-n+1,\\
F_{11}(K'(n);t) &= \sum_{1\le i,j\le n}\varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\alpha_j}-1)=0,\\
G_{00}(K'(n);t) &= \varepsilon_{n+1}^2(t^{\alpha_{n+1}\cdot\beta_{n+1}}-1)-\omega_0W_0(K'(n);t)=0,\\
G_{01}(K'(n);t) &= \sum_{j=1}^{n}\varepsilon_{n+1}\varepsilon_j(t^{\alpha_{n+1}\cdot\beta_j}-1)-\omega_1W_0(K'(n);t)\\
&= -(n-1)t^n+t^{n-1}+\cdots+t,\\
G_{10}(K'(n);t) &= \sum_{i=1}^{n}\varepsilon_i\varepsilon_{n+1}(t^{\alpha_i\cdot\beta_{n+1}}-1)-\omega_0W_1(K'(n);t)=t^n+\cdots+t^2-(n-1)t,
\end{aligned}$$

$$
\begin{aligned}
G_{11}(K'(n);t) &= \sum_{1\leq i,j\leq n} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\beta_j}-1)-\omega_1W_1(K'(n);t)=0,\\
H_{00}(K'(n);t) &= \varepsilon_{n+1}^2(t^{\beta_{n+1}\cdot\beta_{n+1}}-1)-\omega_0W_0(K'(n);t)-\omega_0W_0(K'(n);t^{-1})\\
&= -t^n+2-t^{-n},\\
H_{01}(K'(n);t) &= \sum_{j=1}^{n}\varepsilon_{n+1}\varepsilon_j(t^{\beta_{n+1}\cdot\beta_j}-1)-\omega_0W_1(K'(n);t)-\omega_1W_0(K'(n);t^{-1})\\
&= -nt^{-n}+t^{-n+1}+\cdots+t^{-1}+(n+1)-nt,\\
H_{10}(K'(n);t) &= H_{01}(K'(n);t^{-1}) = -nt^n+t^{n-1}+\cdots+t+(n+1)-nt^{-1},\\
H_{11}(K'(n);t) &= \sum_{1\leq i,j\leq n}\varepsilon_i\varepsilon_j(t^{\beta_i\cdot\beta_j}-1)-\omega_1W_1(K(n);t)-\omega_1W_1(K(n);t^{-1})\\
&= -n^2(t-2+t^{-1}).
\end{aligned}
$$

5. Symmetries

In what follows, when no ambiguity arises, we identify a long virtual knot diagram D with its surface realization (Σ_g, D). We construct three diagrams $D^{\#}$, $-D$, and D^* from D as follows:

- $D^{\#}$ is obtained from D by switching over/under information at all real crossings of D.
- $-D$ is obtained from D by reversing the orientation of D.
- D^* is obtained from D by an orientation-reversing homeomorphism of $\mathbb{R}^2$ (or Σ_g).

Let K be the long virtual knot presented by D. We denote by $K^{\#}$, $-K$, and K^* the long virtual knots presented by $D^{\#}$, $-D$, and D^*, respectively.

In this section, we prove the following two theorems, where $a' = 1-a$ and $b' = 1-b$ for $a, b \in \{0,1\}$.

Theorem 5.1. *For any $a \in \{0,1\}$, we have the following.*

(i) $W_a(K^{\#};t) = -W_{a'}(K;t)$.
(ii) $W_a(-K;t) = W_{a'}(K;t)$.
(iii) $W_a(K^*;t) = -W_a(K;t^{-1})$.

Theorem 5.2. *For any $X \in \{F, G, H\}$ and $a, b \in \{0,1\}$, we have the following.*

(i) $X_{ab}(K^{\#};t) = X_{ab}(-K;t) = X_{a'b'}(K;t)$.
(ii) $X_{ab}(K^*;t) = X_{ab}(K;t^{-1})$.

For a real crossing c_i of D, we denote by $c_i^{\#}$, c_i^-, and c_i^* the corresponding real crossings of $D^{\#}$, $-D$, and D^*, respectively. The sign and the two cycles associated with each of these crossings are analogously indicated by appending the superscripts $\#$, $-$, and $*$, respectively.

Proof of Theorem 5.1. (i) Since it holds that

$$I_a(D^{\#}) = I_{a'}(D),\ \varepsilon_i^{\#} = -\varepsilon_i,\ \alpha_i^{\#} = \alpha_i, \text{ and } \beta_i^{\#} = \beta_i,$$

we have

$$W_a(K^{\#};t) = \sum_{i\in I_a(D^{\#})}\varepsilon_i^{\#}(t^{\alpha_i^{\#}\cdot\beta_i^{\#}}-1) = \sum_{i\in I_{a'}(D)}(-\varepsilon_i)(t^{\alpha_i\cdot\beta_i}-1) = -W_{a'}(K;t).$$

(ii) Since it holds that

$$I_a(-D) = I_{a'}(D),\ \varepsilon_i^- = \varepsilon_i,\ \alpha_i^- = -\alpha_i,\ \text{and}\ \beta_i^- = -\beta_i,$$

we have

$$\begin{aligned} W_a(-K;t) &= \sum_{i\in I_a(-D)} \varepsilon_i^-(t^{\alpha_i^-\cdot\beta_i^-}-1) = \sum_{i\in I_{a'}(D)} \varepsilon_i(t^{(-\alpha_i)\cdot(-\beta_i)}-1) \\ &= \sum_{i\in I_{a'}(D)} \varepsilon_i(t^{\alpha_i\cdot\beta_i}-1) = W_{a'}(K;t). \end{aligned}$$

(iii) Since it holds that

$$I_a(D^*) = I_a(D),\ \varepsilon_i^* = -\varepsilon_i,\ \text{and}\ \alpha_i^*\cdot\beta_i^* = -\alpha_i\cdot\beta_i,$$

we have

$$W_a(K^*;t) = \sum_{i\in I_a(D^*)} \varepsilon_i^*(t^{\alpha_i^*\cdot\beta_i^*}-1) = \sum_{i\in I_a(D)} (-\varepsilon_i)(t^{-\alpha_i\cdot\beta_i}-1) = -W_a(K;t^{-1}).$$

□

Proof of Theorem 5.2. We prove only the case $X = H$; the other cases can be proved similarly. We may assume that D is untwisted, and hence so are $D^{\#}$, $-D$, and D^*. By Remark 3.4, we have $H_{ab}(K;t) = h_{ab}(D;t)$.

(i) Similarly to the proofs of Theorem 5.1(i) and (ii), we have

$$\begin{aligned} H_{ab}(K^{\#};t) = h_{ab}(D^{\#};t) &= \sum_{\substack{i\in I_a(D^{\#})\\ j\in I_b(D^{\#})}} \varepsilon_i^{\#}\varepsilon_j^{\#}(t^{\beta_i^{\#}\cdot\beta_j^{\#}}-1) \\ &= \sum_{\substack{i\in I_{a'}(D)\\ j\in I_{b'}(D)}} (-\varepsilon_i)(-\varepsilon_j)(t^{\beta_i\cdot\beta_j}-1) = h_{a'b'}(D;t) = H_{a'b'}(K;t)\ \text{and} \\ H_{ab}(-K;t) = h_{ab}(-D;t) &= \sum_{\substack{i\in I_a(-D)\\ j\in I_b(-D)}} \varepsilon_i^-\varepsilon_j^-(t^{\beta_i^-\cdot\beta_j^-}-1) \\ &= \sum_{\substack{i\in I_{a'}(D)\\ j\in I_{b'}(D)}} \varepsilon_i\varepsilon_j(t^{(-\beta_i)\cdot(-\beta_j)}-1) = h_{a'b'}(D;t) = H_{a'b'}(K;t). \end{aligned}$$

(ii) Similarly to the proof of Theorem 5.1(iii), we have

$$\begin{aligned} H_{ab}(K^*;t) = h_{ab}(D^*;t) &= \sum_{\substack{i\in I_a(D^*)\\ j\in I_b(D^*)}} \varepsilon_i^*\varepsilon_j^*(t^{\beta_i^*\cdot\beta_j^*}-1) \\ &= \sum_{\substack{i\in I_a(D)\\ j\in I_b(D)}} (-\varepsilon_i)(-\varepsilon_j)(t^{-\beta_i\cdot\beta_j}-1) = h_{ab}(D;t^{-1}) = H_{ab}(K;t^{-1}), \end{aligned}$$

where we use the equation $\beta_i^*\cdot\beta_j^* = -\beta_i\cdot\beta_j$. □

Proposition 5.3. *There are infinitely many long virtual knots K such that*

$$\pm K,\ \pm K^{\#},\ \pm K^*,\ \textit{and}\ \pm K^{\#*}$$

are mutually distinct.

Proof. For an integer $n \geq 2$, we consider the long virtual knot $K'(n)$ given in Example 4.2. Since it holds that

$$W_0(K'(n);t) = t^n - 1 \text{ and } W_1(K'(n);t) = n(t-1),$$

the eight writhe polynomials

$$\pm W_0(K'(n);t),\ \pm W_1(K'(n);t),\ \pm W_0(K'(n);t^{-1}),\ \text{ and } \pm W_1(K'(n);t^{-1})$$

are mutually distinct. Therefore, the conclusion follows from Theorem 5.1. □

6. The product of long virtual knots

Let D and D' be diagrams of long virtual knots K and K', respectively. Concatenating D' after D yields a new diagram $D'' = D \circ D'$. The *product* of K and K' is defined as the long virtual knot presented by D'', and is denoted by $K \circ K'$. This operation is well-defined; that is, it does not depend on the choice of diagrams D and D'. The aim of this section is to prove the following two theorems.

Theorem 6.1. *For any $a \in \{0,1\}$, we have*

$$W_a(K \circ K';t) = W_a(K;t) + W_a(K';t).$$

Theorem 6.2. *For any $a, b \in \{0,1\}$, we have*
(i) $F_{ab}(K \circ K';t) = F_{ab}(K;t) + F_{ab}(K';t)$.
(ii) $G_{ab}(K \circ K';t) = G_{ab}(K;t) + G_{ab}(K';t)$.
(iii) $H_{ab}(K \circ K';t) = H_{ab}(K;t) + H_{ab}(K';t)$
$+W_a(K;t^{-1})W_b(K';t) + W_a(K';t^{-1})W_b(K;t)$.

Let $c_1, \dots, c_n$ and $c'_{n+1} \dots, c'_m$ be the real crossings of D and D', respectively. We attach primes and double primes to the cycles, signs, and index sets of D' and D'', respectively. We remark that $I''_a = I_a \cup I'_a$ holds for any $a \in \{0,1\}$.

Lemma 6.3. *The intersection numbers $\alpha''_i \cdot \alpha''_j$, $\alpha''_i \cdot \beta''_j$, and $\beta''_i \cdot \beta''_j$ among the cycles from D'' are given as shown in* Table 6.1.

Table 6.1. The intersection numbers for D''

$\alpha''_i \cdot \alpha''_j$	$1 \leq j \leq n$	$n+1 \leq j \leq m$
$1 \leq i \leq n$	$\alpha_i \cdot \alpha_j$	0
$n+1 \leq i \leq m$	0	$\alpha'_i \cdot \alpha'_j$

$\alpha''_i \cdot \beta''_j$	$1 \leq j \leq n$	$n+1 \leq j \leq m$
$1 \leq i \leq n$	$\alpha_i \cdot \beta_j$	$\alpha_i \cdot \beta_i$
$n+1 \leq i \leq m$	$\alpha'_i \cdot \beta'_i$	$\alpha'_i \cdot \beta'_j$

$\beta''_i \cdot \beta''_j$	$1 \leq j \leq n$	$n+1 \leq j \leq m$
$1 \leq i \leq n$	$\beta_i \cdot \beta_j$	$-\alpha_i \cdot \beta_i + \alpha'_j \cdot \beta'_j$
$n+1 \leq i \leq m$	$-\alpha'_i \cdot \beta'_i + \alpha_j \cdot \beta_j$	$\beta'_i \cdot \beta'_j$

Proof. Let (Σ_g, D) and $(\Sigma_{g'}, D')$ be surface realizations of D and D', respectively. The connected sum of Σ_g and $\Sigma_{g'}$ as shown in Figure 6.1 provides a surface realization $(\Sigma_{g+g'}, D'')$. Then it holds that

$$\alpha''_i = \begin{cases} \alpha_i & \text{for } 1 \leq i \leq n, \\ \alpha'_i & \text{for } n+1 \leq i \leq m, \end{cases}$$
$$\beta''_i = \begin{cases} \beta_i + \gamma_{D'} & \text{for } 1 \leq i \leq n, \\ \beta'_i + \gamma_D & \text{for } n+1 \leq i \leq m, \end{cases}$$
$$\gamma_{D''} = \gamma_D + \gamma_{D'}.$$

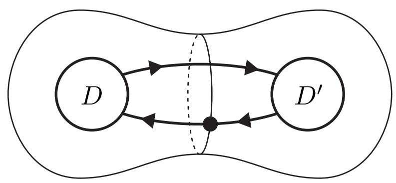

FIGURE 6.1. A surface realization $(\Sigma_{g+g'}, D'')$

Since the intersection number of a cycle on Σ_g and a cycle on $\Sigma_{g'}$ is equal to zero, the conclusion follows: For example, for $1 \leq i \leq n$ and $n+1 \leq j \leq m$, we have

$$\begin{aligned} \alpha''_i \cdot \alpha''_j &= \alpha_i \cdot \alpha'_j = 0, \\ \alpha''_i \cdot \beta''_j &= \alpha_i \cdot (\beta'_j + \gamma_D) = \alpha_i \cdot \gamma_D = \alpha_i \cdot (\alpha_i + \beta_i) = \alpha_i \cdot \beta_i, \text{ and} \\ \beta''_i \cdot \beta''_j &= (\beta_i + \gamma_{D'}) \cdot (\beta'_j + \gamma_D) = \beta_i \cdot \gamma_D + \gamma_{D'} \cdot \beta'_j \\ &= \beta_i \cdot (\alpha_i + \beta_i) + (\alpha'_j + \beta'_j) \cdot \beta'_j = -\alpha_i \cdot \beta_i + \alpha'_j \cdot \beta'_j. \end{aligned}$$

□

Proof of Theorem 6.1. By Lemma 6.3, we have

$$\begin{aligned} W_a(K \circ K'; t) &= \sum_{i \in I''_a} \varepsilon''_i (t^{\alpha''_i \cdot \beta''_i} - 1) \\ &= \sum_{i \in I_a} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) + \sum_{i \in I'_a} \varepsilon'_i (t^{\alpha'_i \cdot \beta'_i} - 1) \\ &= W_a(K; t) + W_a(K'; t). \end{aligned}$$

□

Proof of Theorem 6.2. We may assume that D and D' are untwisted, and hence, so is D''. Then it follows from Lemma 6.3 that

$$\begin{aligned} F_{ab}(K \circ K'; t) &= \sum_{i \in I_a,\, j \in I_b} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \alpha_j} - 1) + \sum_{i \in I'_a,\, j \in I'_b} \varepsilon'_i \varepsilon'_j (t^{\alpha'_i \cdot \alpha'_j} - 1) \\ &= F_{ab}(K; t) + F_{ab}(K'; t) \text{ and} \\ G_{ab}(K \circ K'; t) &= \sum_{i \in I_a,\, j \in I_b} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_j} - 1) + \sum_{i \in I_a,\, j \in I'_b} \varepsilon_i \varepsilon'_j (t^{\alpha_i \cdot \beta_i} - 1) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i\in I'_a,\, j\in I_b}\varepsilon'_i\varepsilon_j(t^{\alpha'_i\cdot\beta'_i}-1)+\sum_{i\in I'_a,\, j\in I'_b}\varepsilon'_i\varepsilon'_j(t^{\alpha'_i\cdot\beta'_j}-1)\\
&=G_{ab}(K;t)+\omega_b(D')W_a(K;t)+\omega_b(D)W_a(K';t)+G_{ab}(K';t)\\
&=G_{ab}(K;t)+G_{ab}(K';t).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
H_{ab}(K\circ K';t)=&\sum_{i\in I_a,\, j\in I_b}\varepsilon_i\varepsilon_j(t^{\beta_i\cdot\beta_j}-1)+\sum_{i\in I_a,\, j\in I'_b}\varepsilon_i\varepsilon'_j(t^{-\alpha_i\cdot\beta_i+\alpha'_j\cdot\beta'_j}-1)\\
&+\sum_{i\in I'_a,\, j\in I_b}\varepsilon'_i\varepsilon_j(t^{-\alpha'_i\cdot\beta'_i+\alpha_j\cdot\beta_j}-1)+\sum_{i\in I'_a,\, j\in I'_b}\varepsilon'_i\varepsilon'_j(t^{\beta'_i\cdot\beta'_j}-1).
\end{aligned}
$$

The second sum in the right hand side is equal to

$$
\begin{aligned}
&\sum_{i\in I_a,\, I'_b}\varepsilon_i\varepsilon'_j(t^{-\alpha_i\cdot\beta_i}-1)(t^{\alpha'_j\cdot\beta'_j}-1)\\
&+\sum_{i\in I_a,\, j\in I'_b}\varepsilon_i\varepsilon'_j(t^{-\alpha_i\cdot\beta_i}-1)+\sum_{i\in I_a,\, j\in I'_b}\varepsilon_i\varepsilon'_j(t^{\alpha'_j\cdot\beta'_j}-1)\\
=&W_a(K;t^{-1})W_b(K';t)+\omega_b(D')W_a(K;t^{-1})+\omega_a(D)W_b(K';t)\\
=&W_a(K;t^{-1})W_b(K';t),
\end{aligned}
$$

and the third sum is equal to $W_a(K';t^{-1})W_b(K;t)$. Therefore, we have

$$
\begin{aligned}
H_{ab}(K\circ K';t)=&H_{ab}(K;t)+W_a(K;t^{-1})W_b(K';t)\\
&+W_a(K';t^{-1})W_b(K;t)+H_{ab}(K';t).
\end{aligned}
$$

□

7. Invariants under crossing changes

A *crossing change* is a local move on a long virtual knot diagram that switches the over/under information at a real crossing. The aim of section is to prove the following two theorems.

Theorem 7.1. *The Laurent polynomial*

$$\widetilde{W}(K;t)=W_0(K;t)-W_1(K;t)$$

is invariant under crossing changes.

Theorem 7.2. *For any $X\in\{F,G,H\}$, the Laurent polynomial*

$$\widetilde{X}(K;t)=X_{00}(K;t)-X_{01}(K;t)-X_{10}(K;t)+X_{11}(K;t)$$

is invariant under crossing changes.

Let D be a long virtual knot diagram of K with n real crossings $c_1,\dots,c_n$ such that

(i) $c_1,\dots,c_k$ are of type 0,
(ii) $c_{k+1},\dots,c_n$ are of type 1, and
(iii) the sign of c_i is ε_i ($1\le i\le n$).

Let D' be the diagram obtained from D by a crossing change at c_k, and K' the long virtual knot presented by D'. We remark that c_k in D' is of type 1 and has the sign $-\varepsilon_k$.

Proof of Theorem 7.1. It follows by assumption that

$$I_0(D) = \{1, \dots, k-1, k\},\ I_1(D) = \{k+1, \dots, n\},$$
$$I_0(D') = \{1, \dots, k-1\},\ \text{and } I_1(D') = \{k, k+1, \dots, n\}.$$

Since it holds that

$$\widetilde{W}(D;t) = \left(\sum_{i=1}^{k-1} \varepsilon_i(t^{\alpha_i \cdot \beta_i} - 1) + \varepsilon_k(t^{\alpha_k \cdot \beta_k} - 1)\right) - \sum_{i=k+1}^{n} \varepsilon_i(t^{\alpha_i \cdot \beta_i} - 1) \text{ and}$$
$$\widetilde{W}(D';t) = \sum_{i=1}^{k-1} \varepsilon_i(t^{\alpha_i \cdot \beta_i} - 1) - \left((-\varepsilon_k)(t^{\alpha_k \cdot \beta_k} - 1) + \sum_{i=k+1}^{n} \varepsilon_i(t^{\alpha_i \cdot \beta_i} - 1)\right),$$

we have $\widetilde{W}(D;t) = \widetilde{W}(D';t)$. □

Lemma 7.3. *For any $x \in \{f, g, h\}$, the Laurent polynomial*

$$\widetilde{x}(D;t) = x_{00}(D;t) - x_{01}(D;t) - x_{10}(D;t) + x_{11}(D;t)$$

is invariant under crossing changes.

Proof. We prove only the case $x = g$; the other cases can be proved similarly.
Put $p_{ij} = \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \beta_j} - 1)$ and

$$P_1 = \sum_{\substack{1 \le i \le k-1 \\ 1 \le j \le k-1}} p_{ij}, \quad P_2 = \sum_{1 \le i \le k-1} p_{ik}, \quad P_3 = \sum_{\substack{1 \le i \le k-1 \\ k+1 \le j \le n}} p_{ij},$$
$$P_4 = \sum_{1 \le j \le k-1} p_{kj}, \quad P_5 = p_{kk}, \quad P_6 = \sum_{k+1 \le j \le n} p_{kj},$$
$$P_7 = \sum_{\substack{k+1 \le i \le n \\ 1 \le j \le k-1}} p_{ij}, \quad P_8 = \sum_{k+1 \le i \le n} p_{ik}, \quad P_9 = \sum_{\substack{k+1 \le i \le n \\ k+1 \le j \le n}} p_{ij}.$$

Since c_k in D is of type 0, it holds that

$$g_{00}(D;t) = P_1 + P_2 + P_4 + P_5, \quad g_{01}(D;t) = P_3 + P_6,$$
$$g_{10}(D;t) = P_7 + P_8, \quad g_{11}(D;t) = P_9.$$

On the other hand, since c_k in D' is of type 1 with the sign $-\varepsilon_k$, it holds that

$$g_{00}(D';t) = P_1, \quad g_{01}(D';t) = -P_2 + P_3,$$
$$g_{10}(D';t) = -P_4 + P_7, \quad g_{11}(D';t) = P_5 - P_6 - P_8 + P_9.$$

Therefore, we have $\widetilde{g}(D;t) = \widetilde{g}(D';t)$. □

Proof of Theorem 7.2. We prove only the case $X = G$; the other cases can be proved similarly. It follows by definition that

$$\widetilde{G}(K;t) = \widetilde{g}(D;t) - (\omega_0(D) - \omega_1(D))\widetilde{W}(K;t).$$

Since it holds that

$$\omega_0(D) - \omega_1(D) = \left(\sum_{i=1}^{k-1} \varepsilon_i + \varepsilon_k\right) - \sum_{i=k+1}^{n} \varepsilon_i$$

$$= \sum_{i=1}^{k-1} \varepsilon_i - \left((-\varepsilon_k) + \sum_{i=k+1}^{n} \varepsilon_i \right) = \omega_0(D') - \omega_1(D'),$$

we have $\widetilde{G}(K;t) = \widetilde{G}(K';t)$ by Theorem 7.1 and Lemma 7.3. □

Theorems 7.1 and 7.2 show that $\widetilde{W}$, $\widetilde{F}$, $\widetilde{G}$, and $\widetilde{H}$ are invariants of *flat* long virtual knots. Since the long virtual knots $K(n)$ and $K'(n)$ given in Section 4 have the same underlying curve $\Gamma(n)$, it holds that

$$\begin{aligned}
&W_0(K(n);t) = W_0(K'(n);t) - W_1(K'(n);t),\\
&F_{00}(K(n);t) = -F_{01}(K'(n);t) - F_{10}(K'(n);t),\\
&G_{00}(K(n);t) = -G_{01}(K'(n);t) - G_{10}(K'(n);t), \text{ and}\\
&H_{00}(K(n);t) = H_{00}(K'(n);t) - H_{01}(K'(n);t) - H_{10}(K'(n);t) + H_{11}(K'(n);t).
\end{aligned}$$

A long virtual knot is called *descending* if it is presented by a descending diagram; that is, its all real crossings are of type 0. By performing crossing changes to make a diagram descending, we obtain a natural map from the set of long virtual knots to that of descending long virtual knots (cf. [4, Section 2.2]). Let K^d denote the descending long virtual knot associated with a (possibly non-descending) long virtual knot K. Then we have the following.

Corollary 7.4. *For a long virtual knot K, we have the following.*

(i) $W_a(K^d;t) = \begin{cases} W_0(K;t) - W_1(K;t) & \textit{for } a = 0, \\ 0 & \textit{for } a = 1. \end{cases}$

(ii) *For any $X \in \{F, G, H\}$,*

$$X_{ab}(K^d;t) = \begin{cases} X_{00}(K;t) - X_{01}(K;t) - X_{10}(K;t) + X_{11}(K;t) & \textit{for } a = b = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since a descending diagram of K^d has no real crossing of type 1, $W_1(K^d;t) = X_{ab}(K^d;t) = 0$ holds for any $a, b \in \{0,1\}$ except $a = b = 0$. Moreover, since K and K^d are related by a finite sequence of crossing changes, it follows from Theorems 7.1 and 7.2 that

$$W_0(K^d;t) = W_0(K^d;t) - W_1(K^d;t) = W_0(K;t) - W_1(K;t)$$

and

$$\begin{aligned}
X_{00}(K^d;t) &= X_{00}(K^d;t) - X_{01}(K^d;t) - X_{10}(K^d;t) + X_{11}(K^d;t)\\
&= X_{00}(K;t) - X_{01}(K;t) - X_{10}(K;t) + X_{11}(K;t).
\end{aligned}$$

□

8. Finite-type invariants under crossing changes

There are two definitions of finite-type invariants for long virtual knots; one due to Vassiliev [18] (see also [12]) under crossing changes, and the other due to Goussarov, Polyak, and Viro [5] under virtualizations. In this section, we study finite-type invariants under crossing changes, and prove that the 0- and 1-writhe polynomials are finite-type invariants of degree one, and the intersection polynomials are of degree two. It is known that the same property also holds for the writhe polynomial and for the intersection polynomials of (closed) virtual knots [9].

We recall the definition of finite-type invariants under crossing changes. A k-*marked* long virtual knot diagram (D, C_k) is a pair of a diagram D and a set of k specified real crossings $C_k = \{c_1, \dots, c_k\}$ of D. For k integers $\delta_1, \dots, \delta_k \in \{0, 1\}$, we denote by $D_{\delta_1,\dots,\delta_k}$ the long virtual knot diagram obtained from (D, C_k) by modifying each $c_i \in C_k$ according to $\delta_i \in \{0, 1\}$ with the rule shown in Figure 8.1, where we mark the real crossings belonging to C_k with an astarisk $*$. Let K be the long virtual knot presented by D. We denote by $K_{\delta_1,\dots,\delta_k}$ the long virtual knot presented by $D_{\delta_1,\dots,\delta_k}$.

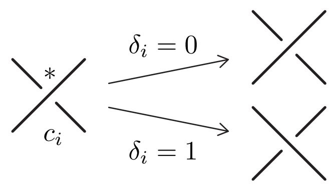

FIGURE 8.1. Performing a crossing change for $\delta_i = 1$

Let v be an invariant of long virtual knots taking values in an abelian group. Such an invariant is a finite-type invariant of degree k under crossing changes if and only if

$$v(D, C_{k+1}) = \sum_{\delta_1,\dots,\delta_{k+1} \in \{0,1\}} (-1)^{\delta_1+\cdots+\delta_{k+1}} v(K_{\delta_1,\dots,\delta_{k+1}}) = 0$$

holds for any (D, C_{k+1}), and there is (D', C_k) with $v(D', C_k) \neq 0$ (cf. [12, 18]).

Theorem 8.1. *For any $a \in \{0, 1\}$, the a-writhe polynomial $W_a(K; t)$ is a finite-type invariant of degree one under crossing changes.*

Proof. Let (D, C_2) be a 2-marked long virtual knot diagram with $C_2 = \{c_1, c_2\}$, and $c_3, \dots, c_n$ the non-marked real crossings of D. For $1 \leq i \leq n$ and $\delta_1, \delta_2 \in \{0, 1\}$, we put

$$\Phi_i^{\delta_1,\delta_2}(t) = \begin{cases} \varepsilon_i(t^{\alpha_i\cdot\beta_i} - 1) & \text{for } i \in I_a(D_{\delta_1,\delta_2}), \\ 0 & \text{for } i \notin I_a(D_{\delta_1,\delta_2}). \end{cases}$$

Here, the sign ε_i of c_i and the intersection number $\alpha_i \cdot \beta_i$ are taken in D_{δ_1,δ_2}. Then it holds that

$$\sum_{\delta_1,\delta_2\in\{0,1\}} (-1)^{\delta_1+\delta_2} \left(\sum_{i\in I_a(D_{\delta_1,\delta_2})} \varepsilon_i(t^{\alpha_i\cdot\beta_i} - 1) \right) = \sum_{i=1}^{n} \left(\sum_{\delta_1,\delta_2\in\{0,1\}} (-1)^{\delta_1+\delta_2} \Phi_i^{\delta_1,\delta_2}(t) \right).$$

We prove that the second sum in the right hand side of the above equation is equal to zero for each i with $1 \leq i \leq n$.

For $i = 1$, it follows by definition that the type and sign of c_1 in D_{δ_1,δ_2} and the intersection number $\alpha_1 \cdot \beta_1$ are independent of the choice of $\delta_2 \in \{0, 1\}$. Hence, $\Phi_1^{\delta_1,\delta_2}(t)$ is also independent of δ_2; that is, $\Phi_1^{\delta_1,0}(t) = \Phi_1^{\delta_1,1}(t)$. Therefore, we have

$$\sum_{\delta_1,\delta_2\in\{0,1\}} (-1)^{\delta_1+\delta_2} \Phi_1^{\delta_1,\delta_2}(t) = \sum_{\delta_1\in\{0,1\}} \Big((-1)^{\delta_1} \Phi_1^{\delta_1,0}(t) + (-1)^{\delta_1+1} \Phi_1^{\delta_1,1}(t) \Big) = 0.$$

Similarly, the second sum for $i = 2$ is also equal to zero.

For each i with $3 \leq i \leq n$, the type and sign of c_i and the intersection number $\alpha_i \cdot \beta_i$ in D_{δ_1,δ_2} are independent of both δ_1 and δ_2. Hence, $\Phi_i^{\delta_1,\delta_2}(t)$ is also independent of δ_1 and δ_2, and the second sum for such i is again equal to zero.

On the other hand, we consider a 1-marked long virtual knot diagram (D, C_1) with $C_1 = \{c_1\}$ as shown on the left of Figure 8.2. Then $W_a(D_0;t) = t-1$ holds for $a \in \{0,1\}$. Furthermore, since D_1 presents the trivial long virtual knot, $W_a(D_1;t) = 0$ holds for $a \in \{0,1\}$. See the middle and right of the figure. Therefore, we have

$$W_a(D_0;t) - W_a(D_1;t) = t - 1 \neq 0,$$

which shows that $W_a(K;t)$ is a finite-type invariant of degree one under crossing changes. $\square$

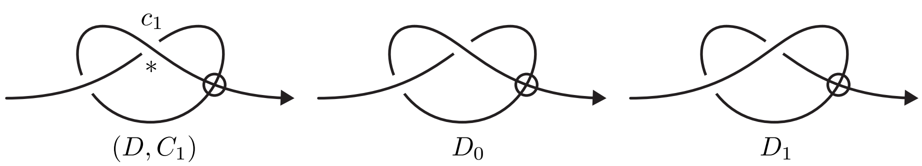

FIGURE 8.2. (D, C_1) with D_0 and D_1

Theorem 8.2. *For any $X \in \{F, G, H\}$ and $a, b \in \{0,1\}$, the intersection polynomial $X_{ab}(K;t)$ is a finite-type invariant of degree two under crossing changes.*

To prove Theorem 8.2, we prepare Lemmas 8.3 and 8.4 stated below.

Lemma 8.3. *Let (D, C_3) be a 3-marked long virtual knot diagram with $C_3 = \{c_1, c_2, c_3\}$. For any $x \in \{f, g, h\}$ and $a, b \in \{0,1\}$, we have*

$$\sum_{\delta_1,\delta_2,\delta_3 \in \{0,1\}} (-1)^{\delta_1+\delta_2+\delta_3} x_{ab}(D_{\delta_1,\delta_2,\delta_3};t) = 0.$$

Proof. We prove only the case $x = f$; the other cases can be proved similarly. Let $c_4, \dots, c_n$ be the non-marked real crossings of D. For $1 \leq i,j \leq n$ and $\delta_1, \delta_2, \delta_3 \in \{0,1\}$, we put

$$\Psi_{i,j}^{\delta_1,\delta_2,\delta_3}(t) = \begin{cases} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \alpha_j} - 1) & \text{for } i \in I_a(D_{\delta_1,\delta_2,\delta_3}) \text{ and } j \in I_b(D_{\delta_1,\delta_2,\delta_3}), \\ 0 & \text{otherwise.} \end{cases}$$

Then it holds that

$$\begin{aligned} &\sum_{\delta_1,\delta_2,\delta_3 \in \{0,1\}} (-1)^{\delta_1+\delta_2+\delta_3} \left(\sum_{\substack{i \in I_a(D_{\delta_1,\delta_2,\delta_3}) \\ j \in I_b(D_{\delta_1,\delta_2,\delta_3})}} \varepsilon_i \varepsilon_j (t^{\alpha_i \cdot \alpha_j} - 1) \right) \\ &= \sum_{1 \leq i,j \leq n} \left(\sum_{\delta_1,\delta_2,\delta_3 \in \{0,1\}} (-1)^{\delta_1+\delta_2+\delta_3} \Psi_{i,j}^{\delta_1,\delta_2,\delta_3}(t) \right). \end{aligned}$$

It suffices to prove that the second sum in the right hand side of the above equation is equal to zero for each (i,j) with $1 \leq i,j \leq n$. The proof is similar to that of Theorem 8.1. In fact, we have the following.

- For (i,j) with $1 \leq i,j \leq 3$, $\Psi_{i,j}^{\delta_1,\delta_2,\delta_3}(t)$ is independent of δ_ℓ, where $\ell \in \{1,2,3\} \setminus \{i,j\}$.
- For (i,j) with $1 \leq i \leq 3$ and $4 \leq j \leq n$, $\Psi_{i,j}^{\delta_1,\delta_2,\delta_3}(t)$ is independent of δ_ℓ, where $\ell \in \{1,2,3\} \setminus \{i\}$.
- For (i,j) with $4 \leq i \leq n$ and $1 \leq j \leq 3$, $\Psi_{i,j}^{\delta_1,\delta_2,\delta_3}(t)$ is independent of δ_ℓ, where $\ell \in \{1,2,3\} \setminus \{j\}$.
- For each (i,j) with $4 \leq i,j \leq n$, $\Psi_{i,j}^{\delta_1,\delta_2,\delta_3}(t)$ is independent of $\delta_1, \delta_2, \delta_3$.

Therefore, the second sum for each (i,j) is equal to zero. □

Lemma 8.4. *Let (D, C_3) be a 3-marked long virtual knot diagram with $C_3 = \{c_1, c_2, c_3\}$. For any $a, b \in \{0,1\}$, we have*

$$\sum_{\delta_1,\delta_2,\delta_3 \in \{0,1\}} (-1)^{\delta_1+\delta_2+\delta_3} \omega_a(D_{\delta_1,\delta_2,\delta_3}) W_b(K_{\delta_1,\delta_2,\delta_3}; t) = 0.$$

Proof. Let $c_4, \dots, c_n$ be the non-marked real crossings of D. For i with $1 \leq i \leq n$, we put

$$\varepsilon_i^{\delta_1,\delta_2,\delta_3} = \begin{cases} \varepsilon_i & \text{for } i \in I_a(D_{\delta_1,\delta_2,\delta_3}), \\ 0 & \text{for } i \notin I_a(D_{\delta_1,\delta_2,\delta_3}). \end{cases}$$

By $\omega_a(D_{\delta_1,\delta_2,\delta_3}) = \sum_{i=1}^n \varepsilon_i^{\delta_1,\delta_2,\delta_3}$, it holds that

$$\begin{aligned} &\sum_{\delta_1,\delta_2,\delta_3 \in \{0,1\}} (-1)^{\delta_1+\delta_2+\delta_3} \omega_a(D_{\delta_1,\delta_2,\delta_3}) W_b(K_{\delta_1,\delta_2,\delta_3}; t) \\ &= \sum_{i=1}^n \left(\sum_{\delta_1,\delta_2,\delta_3 \in \{0,1\}} (-1)^{\delta_1+\delta_2+\delta_3} \varepsilon_i^{\delta_1,\delta_2,\delta_3} W_b(K_{\delta_1,\delta_2,\delta_3}; t) \right). \end{aligned}$$

We prove that the second sum in the right hand side of the above equation is equal to zero for each i with $1 \leq i \leq n$.

For $i = 1$, since $\varepsilon_1^{\delta_1,\delta_2,\delta_3}$ is independent of both δ_2 and δ_3, Theorem 8.1 induces that the second sum is equal to

$$\sum_{\delta_1 \in \{0,1\}} (-1)^{\delta_1} \varepsilon_1^{\delta_1,0,0} \left(\sum_{\delta_2,\delta_3 \in \{0,1\}} (-1)^{\delta_2+\delta_3} W_b(K_{\delta_1,\delta_2,\delta_3}; t) \right) = 0.$$

Similarly, the second sums for $i = 2$ and 3 are also equal to zero.

For each i with $4 \leq i \leq n$, $\varepsilon_i^{\delta_1,\delta_2,\delta_3}$ is independent of δ_1, δ_2, and δ_3. Hence, the second sum for such i is again equal to zero. □

Proof of Theorem 8.2. By Lemmas 8.3 and 8.4, we have

$$\sum_{\delta_1,\delta_2,\delta_3 \in \{\pm 1\}} (-1)^{\delta_1+\delta_2+\delta_3} X_{ab}(K_{\delta_1,\delta_2,\delta_3}; t) = 0$$

for any $X \in \{F, G, H\}$ and $a, b \in \{0,1\}$.

On the other hand, we consider a 2-marked diagram (D, C_2) with $C_2 = \{c_1, c_2\}$ as shown on the top left of Figure 8.3. Then it holds that

$$\sum_{\delta_1,\delta_2 \in \{0,1\}} (-1)^{\delta_1+\delta_2} F_{ab}(D_{\delta_1,\delta_2}; t) = t - 2 + t^{-1} \neq 0$$

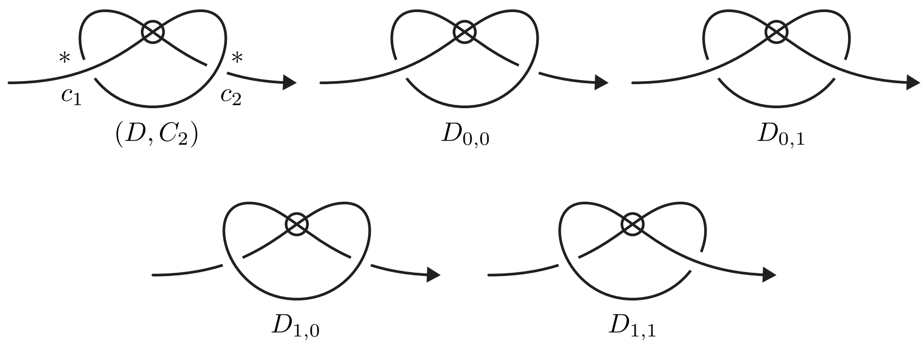

FIGURE 8.3. (D, C_2) with $D_{0,0}$, $D_{0,1}$, $D_{1,0}$, and $D_{1,1}$

and

$$\sum_{\delta_1,\delta_2\in\{0,1\}} (-1)^{\delta_1+\delta_2} G_{ab}(D_{\delta_1,\delta_2};t) = \sum_{\delta_1,\delta_2\in\{0,1\}} (-1)^{\delta_1+\delta_2} H_{ab}(D_{\delta_1,\delta_2};t)$$
$$= -t + 2 - t^{-1} \neq 0,$$

which shows that $X_{ab}(K;t)$ is a finite-type invariant of degree two under crossing changes. □

9. Finite-type invariants under virtualizations

In this section, we study finite-type invariants under virtualizations. The definition is quite similar to that under crossing changes, and the only difference lies in whether we use virtualizations or crossing changes (cf. [5]).

Let (D, C_k) be a k-marked long virtual knot diagram with $C_k = \{c_1, \dots, c_k\}$. For k integers $\delta_1, \dots, \delta_k \in \{0,1\}$, we also, by abuse of notation, denote by $D_{\delta_1,\dots,\delta_k}$ the long virtual knot diagram obtained from (D, C_k) by modifying $c_i \in C_k$ with the rule shown in Figure 9.1. Let $K_{\delta_1,\dots,\delta_k}$ be the long virtual knot presented by $D_{\delta_1,\dots,\delta_k}$.

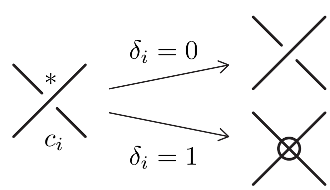

FIGURE 9.1. Performing a virtualization for $\delta_i = 1$

An invariant v of long virtual knots taking values in an abelian group is *not* a finite-type invariant under virtualizations if and only if there exist infinitely many integers $n \geq 0$ such that there is an n-marked diagram $(D(n), C_n)$ satisfying

$$v(D(n), C_n) = \sum_{\delta_1,\dots,\delta_n\in\{0,1\}} (-1)^{\delta_1+\cdots+\delta_n} v(K(n)_{\delta_1,\dots,\delta_n}) \neq 0.$$

Theorem 9.1. *For any $a \in \{0,1\}$, the a-writhe polynomial $W_a(K;t)$ is not a finite-type invariant under virtualizations.*

Proof. By Theorem 5.1(ii), it suffices to consider the 0-writhe polynomial W_0. Let $D(n)$ $(n \geq 2)$ be the long virtual knot diagram given in Example 4.1. We consider the $(n-2)$-marked diagram $(D(n), C_{n-2})$ with $C_{n-2} = \{c_1, \ldots, c_{n-2}\}$. Let s denote the number of 0's among the integers $\delta_1, \ldots, \delta_{n-2} \in \{0, 1\}$. Since $D(n)_{\delta_1,\ldots,\delta_{n-2}}$ represents the long virtual knot $K(s+2)$, we have

$$\begin{aligned}&\sum_{\delta_1,\ldots,\delta_{n-2}\in\{0,1\}} (-1)^{\delta_1+\cdots+\delta_{n-2}} W_0(D(n)_{\delta_1,\ldots,\delta_{n-2}}; t)\\ &= \sum_{s=0}^{n-2} (-1)^{n-2-s} \binom{n-2}{s} W_0(K(s+2); t).\end{aligned}$$

By the equation $W_0(K(s+2); t) = t^{s+2} - (s+2)t + s + 1$ given in Example 4.1, the maximal degree of the sum is equal to n, which completes the proof. □

Theorem 9.2. *For any $a, b \in \{0, 1\}$ and $X \in \{F, G, H\}$, the intersection polynomial $X_{ab}(K; t)$ is not a finite-type invariant under virtualizations.*

Proof. For the polynomials F_{00}, G_{00}, and H_{00}, the proofs are similar to that of W_0 in Theorem 9.1. In fact, the maximal degrees of $F_{00}(K(n); t)$, $G_{00}(K(n); t)$, and $H_{00}(K(n); t)$ $(n \geq 3)$ are equal to $n - 1$, n, and n, respectively.

On the other hand, for the polynomials F_{01}, G_{01}, and H_{10}, we may use the $(n-2)$-marked diagram $(D'(n), C'_{n-2})$ with $C'_{n-2} = \{c_1, \ldots, c_{n-2}\}$, where $D'(n)$ is the long virtual knot diagram given in Example 4.2. In this case, the maximal degrees of $F_{01}(K'(n); t)$, $G_{01}(K'(n); t)$, and $H_{10}(K'(n); t)$ are equal to $n-1$, n, and n, respectively. Therefore, F_{01}, G_{01}, and H_{10} are not finite-type invariants.

By Theorem 5.2(i), the remaining intersection polynomials are also not finite-type invariants. □

10. The intersection polynomials of a virtual knot

For a diagram D of a long virtual knot K, we denote by $\widehat{D}$ the diagram obtained from D by identifying $-\infty$ with ∞. The *closure* of K is the virtual knot presented by $\widehat{D}$, and is denoted by $\widehat{K}$.

We recall the definitions of the writhe and intersection polynomials of a virtual knot κ. Let Δ be a diagram of a virtual knot κ, and $c_1, \ldots, c_n$ the real crossings of Δ. Consider a surface realization (Σ_g, Δ) of Δ. For each real crossing c_i, let γ_i (resp. $\overline{\gamma}_i$) be the cycle on Σ_g presented by the part of Δ running from the overcrossing to the undercrossing (resp. from the undercrossing to the overcrossing) at c_i. See Figure 10.1. We denote by $\omega(\Delta)$ the sum of signs of real crossings of Δ. Then the following Laurent polynomials are invariants of κ;

$$\begin{aligned}W(\kappa; t) &= \sum_{i=1}^{n} \varepsilon_i (t^{\gamma_i \cdot \overline{\gamma}_i} - 1),\\ I(\kappa; t) &= \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j (t^{\gamma_i \cdot \overline{\gamma}_j} - 1) - \omega(\Delta) W(\kappa; t), \text{ and}\\ II(\kappa; t) &= \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j (t^{\gamma_i \cdot \gamma_j} - 1) + \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j (t^{\overline{\gamma}_i \cdot \overline{\gamma}_j} - 1) - \omega(\Delta)\big(W(\kappa; t) + W(\kappa; t^{-1})\big).\end{aligned}$$

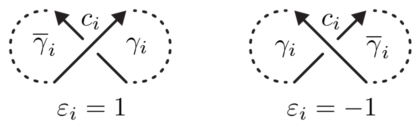

FIGURE 10.1. The two cycles γ_i and $\overline{\gamma}_i$ at a real crossing c_i of Δ

These are called the *writhe polynomial* [16], *first intersection polynomial*, and *second intersection polynomial* [6] of κ, respectively. We remark that the writhe polynomial is also defined in [3, 13] independently.

For a long virtual knot K, the polynomials W, I, and II of the closure $\widehat{K}$ is expressed by using W_a, F_{ab}, G_{ab} and H_{ab} of K as follows.

Proposition 10.1. *For a long virtual knot K, we have the following.*

(i) $W(\widehat{K};t) = W_0(K;t) + W_1(K;t^{-1})$.

(ii) $I(\widehat{K};t) = F_{01}(K;t) + G_{00}(K;t) + G_{11}(K;t^{-1}) + H_{01}(K;t^{-1})$.

(iii) $II(\widehat{K};t) = F_{00}(K;t) + F_{11}(K;t) + G_{01}(K;t) + G_{01}(K;t^{-1})$
$+G_{10}(K;t) + G_{10}(K;t^{-1}) + H_{00}(K;t) + H_{11}(K;t)$.

To prove Proposition 10.1, we prepare the following lemma. Let (Σ_g, D) be a surface realization of K, and $c_1, \dots, c_n$ the real crossings of D. Recall that $I_a(D)$ is the set of indices of real crossings of type $a \in \{0,1\}$. The surface realization (Σ_g, D) can also be regarded as a surface realization $(\Sigma_g, \widehat{D})$ of the closure $\widehat{K}$ by ignoring the basepoint of D. By abuse of notation, we also use c_i to denote the corresponding crossing of $\widehat{D}$.

Lemma 10.2. *For any integer $i \in \{1, \dots, n\}$, we have*

$$(\gamma_i, \overline{\gamma}_i) = \begin{cases} (\alpha_i, \beta_i) & \text{for } i \in I_0(D), \\ (\beta_i, \alpha_i) & \text{for } i \in I_1(D). \end{cases}$$

Proof. See Figure 10.2, where "•" on a dotted line denotes the basepoint of D. □

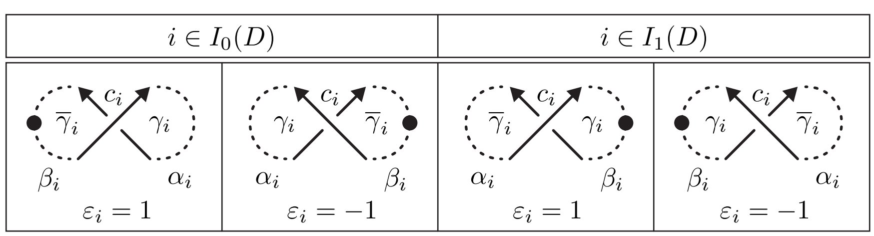

FIGURE 10.2. Proof of Lemma 10.2

Proof of Proposition 10.1. We may assume that D is untwisted.

(i) By Lemma 10.2, we have

$$\begin{aligned} W(\widehat{K};t) &= \sum_{i=1}^{n} \varepsilon_i (t^{\gamma_i \cdot \overline{\gamma}_i} - 1) = \sum_{i \in I_0} \varepsilon_i (t^{\alpha_i \cdot \beta_i} - 1) + \sum_{i \in I_1} \varepsilon_i (t^{\beta_i \cdot \alpha_i} - 1) \\ &= W_0(K;t) + W_1(K;t^{-1}). \end{aligned}$$

(ii) Since we have $\omega(\widehat{D})=0$, it follows from Lemma 10.2 that

$$\begin{aligned}
I(\widehat{K};t) &= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j(t^{\gamma_i\cdot\overline{\gamma}_j}-1)\\
&= \sum_{i,j\in I_0} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\beta_j}-1) + \sum_{i\in I_0,\, j\in I_1} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\alpha_j}-1)\\
&\quad + \sum_{i\in I_1,\, j\in I_0} \varepsilon_i\varepsilon_j(t^{\beta_i\cdot\beta_j}-1) + \sum_{i,j\in I_1} \varepsilon_i\varepsilon_j(t^{\beta_i\cdot\alpha_j}-1)\\
&= g_{00}(D;t)+f_{01}(D;t)+h_{10}(D;t)+g_{11}(D;t^{-1})\\
&= G_{00}(K;t)+F_{01}(K;t)+H_{10}(K;t)+G_{11}(K;t^{-1}).
\end{aligned}$$

Since $H_{10}(K;t)=H_{01}(K;t^{-1})$ holds by Lemma 2.4(i), we have the conclusion.

(iii) Similarly to the proof of (ii), we have

$$\begin{aligned}
I\!I(\widehat{K};t) &= \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j(t^{\gamma_i\cdot\gamma_j}-1) + \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j(t^{\overline{\gamma}_i\cdot\overline{\gamma}_j}-1)\\
&= \sum_{i,j\in I_0} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\alpha_j}-1) + \sum_{i\in I_0,\, j\in I_1} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\beta_j}-1)\\
&\quad + \sum_{i\in I_1,\, j\in I_0} \varepsilon_i\varepsilon_j(t^{\beta_i\cdot\alpha_j}-1) + \sum_{i,j\in I_1} \varepsilon_i\varepsilon_j(t^{\beta_i\cdot\beta_j}-1)\\
&\quad + \sum_{i,j\in I_0} \varepsilon_i\varepsilon_j(t^{\beta_i\cdot\beta_j}-1) + \sum_{i\in I_0,\, j\in I_1} \varepsilon_i\varepsilon_j(t^{\beta_i\cdot\alpha_j}-1)\\
&\quad + \sum_{i\in I_1,\, j\in I_0} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\beta_j}-1) + \sum_{i,j\in I_1} \varepsilon_i\varepsilon_j(t^{\alpha_i\cdot\alpha_j}-1)\\
&= f_{00}(D;t)+g_{01}(D;t)+g_{01}(D;t^{-1})+h_{11}(D;t)\\
&\quad + h_{00}(D;t)+g_{10}(D;t^{-1})+g_{10}(D;t)+f_{11}(D;t)\\
&= F_{00}(K;t)+G_{01}(K;t)+G_{01}(K;t^{-1})+H_{11}(K;t)\\
&\quad + H_{00}(K;t)+G_{10}(K;t^{-1})+G_{10}(K;t)+F_{11}(K;t).
\end{aligned}$$

□

From some known properties of the writhe and intersection polynomials for virtual knots, we obtain the following relations among the values at $t=1$ of the first and second derivatives of the polynomials for a long virtual knot.

Theorem 10.3. *For a long virtual knot K, we have the following.*

(i) $W_0'(K;1)=W_1'(K;1)$.
(ii) $G_{00}'(K;1)=G_{11}'(K;1)=0$.
(iii) $G_{01}'(K;1)+G_{10}'(K;1)=0$.
(iv) $F_{01}'(K;1)=H_{01}'(K;1)$ *and* $F_{10}'(K;1)=H_{10}'(K;1)$.
(v) $F_{00}''(K;1)+F_{11}''(K;1)+H_{00}''(K;1)+H_{11}''(K;1)\equiv 0 \pmod 4$.

Proof. We may assume that D is untwisted; that is, $\sum_{i\in I_0}\varepsilon_i=\sum_{j\in I_1}\varepsilon_j=0$.

(i) It was proved in [16, Proposition 4.2] that $W'(\kappa;1)=0$ holds for any virtual knot κ. Therefore, the equation follows from Proposition 10.1(i).

(ii) It holds that

$$
\begin{aligned}
G'_{00}(K;1) = g'_{00}(D;1) &= \sum_{i,j\in I_0} \varepsilon_i\varepsilon_j(\alpha_i\cdot\beta_j) \\
&= \Big(\sum_{i\in I_0}\varepsilon_i\alpha_i\Big)\cdot\Big(\sum_{j\in I_0}\varepsilon_j\beta_j\Big) = \Big(\sum_{i\in I_0}\varepsilon_i\alpha_i\Big)\cdot\Big(\sum_{j\in I_0}\varepsilon_j(\gamma_D-\alpha_j)\Big) \\
&= \Big(\sum_{i\in I_0}\varepsilon_i\alpha_i\Big)\cdot\Big(\Big(\sum_{j\in I_0}\varepsilon_j\Big)\gamma_D - \sum_{j\in I_0}\varepsilon_j\alpha_j\Big) \\
&= -\Big(\sum_{i\in I_0}\varepsilon_i\alpha_i\Big)\cdot\Big(\sum_{j\in I_0}\varepsilon_j\alpha_j\Big) = 0.
\end{aligned}
$$

By Theorem 5.2(i), we have $G_{11}(K;t) = G_{00}(K^{\#};t)$, and hence $G'_{11}(K;1) = 0$.

(iii) Since it holds that

$$\alpha_i\cdot\beta_j + \alpha_j\cdot\beta_i = \alpha_i\cdot(\gamma_D-\alpha_j) + \alpha_j\cdot(\gamma_D-\alpha_i) = \alpha_i\cdot\gamma_D + \alpha_j\cdot\gamma_D$$

for any i, j, we have

$$
\begin{aligned}
&G'_{01}(K;1) + G'_{10}(K;1) = g'_{01}(D;1) + g'_{10}(D;1) \\
&= \sum_{i\in I_0,\, j\in I_1}\varepsilon_i\varepsilon_j(\alpha_i\cdot\beta_j) + \sum_{i\in I_1,\, j\in I_0}\varepsilon_i\varepsilon_j(\alpha_i\cdot\beta_j) \\
&= \sum_{i\in I_0,\, j\in I_1}\varepsilon_i\varepsilon_j(\alpha_i\cdot\beta_j + \alpha_j\cdot\beta_i) \\
&= \sum_{i\in I_0,\, j\in I_1}\varepsilon_i\varepsilon_j(\alpha_i\cdot\gamma_D + \alpha_j\cdot\gamma_D) \\
&= \Big(\sum_{j\in I_1}\varepsilon_j\Big)\Big(\sum_{i\in I_0}\varepsilon_i(\alpha_i\cdot\gamma_D)\Big) + \Big(\sum_{i\in I_0}\varepsilon_i\Big)\Big(\sum_{j\in I_1}\varepsilon_j(\alpha_j\cdot\gamma_D)\Big) = 0.
\end{aligned}
$$

(iv) It was proved in [8, Theorem 3.2] that $I'(\kappa;1) = 0$ holds for any virtual knot κ. By Proposition 10.1(ii), we have

$$F'_{01}(K;1) + G'_{00}(K;1) - G'_{11}(K;1) - H'_{01}(K;1) = 0.$$

Therefore, it follows from (ii) that $F'_{01}(K;1) = H'_{01}(K;1)$. Furthermore, we have $F'_{10}(K;1) = H'_{10}(K;1)$ by Theorem 5.2(i).

(v) We put

$$P(K;t) = G_{01}(K;t) + G_{10}(K;t) \text{ and } Q(K;t) = P(K;t) + P(K;t^{-1}).$$

Then we have $P(K;1) = 0$ by definition and $P'(K;1) = 0$ by (iii). Therefore, it follows from [8, Lemma 3.5] that $Q''(K;1) \equiv 0 \pmod 4$. Since $I\!I''(\kappa;1) \equiv 0 \pmod 4$ holds for any virtual knot κ [8, Theorem 3.3], we have

$$
\begin{aligned}
&F''_{00}(K;1) + F''_{11}(K;1) + H''_{00}(K;1) + H''_{11}(K;1) + Q''(K;1) \\
&\equiv F''_{00}(K;1) + F''_{11}(K;1) + H''_{00}(K;1) + H''_{11}(K;1) \equiv 0 \pmod 4
\end{aligned}
$$

by Proposition 10.1(iii). □

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