EXTREMES OF BROWNIAN DECISION TREES

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Abstract: We consider a Brownian motion with linear drift that splits at fixed time points into a fixed number of branches, which may depend on the branching point. For this process, which we shall refer to as the Brownian decision tree, we investigate the exact asymptotics of high exceedance probabilities in finite time horizon, including: the probability that at least one branch exceeds some high threshold, the probability that the largest distance between branches gets large and the probability that all branches simultaneously exceed some high barrier. Additionally, we find the asymptotics for the probability that all branches of at least one of M independent Brownian decision trees exceed a high threshold.
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1. Introduction

We investigate the exact asymptotics of high exceedance probabilities for the process, which we shall refer to as *the Brownian decision tree*. This process is a close relative of the standard *branching Brownian motion (BBm)* and it can be informally described as follows: at time $t = 0$ a Brownian motion $B(t)$ sets off from zero and runs freely until a non-random time $\tau_1 > 0$, at which it splits into $N_1 \geq 1$ conditionally on the common past independent Brownian motions

$$B(t) \xmapsto[\text{at } \tau_1]{\text{branching}} \boldsymbol{B}_1(t) = \begin{pmatrix} B(\tau_1) \\ \vdots \\ B(\tau_1) \end{pmatrix} + \boldsymbol{B}_1^*(t - \tau_1),$$

where $\boldsymbol{B}_1^*$ is an $\mathbb{R}^{N_1}$-valued Brownian motion. The resulting vector-valued process again runs freely up to some time point $\tau_2 > \tau_1$, where *each of its components* splits again into $N_2 \geq 1$ particles

$$\boldsymbol{B}_1(t) \xmapsto[\text{at } \tau_2]{\text{branching}} \boldsymbol{B}_2(t) = \begin{pmatrix} \boldsymbol{B}_1(\tau_2) \\ \vdots \\ \boldsymbol{B}_1(\tau_2) \end{pmatrix} + \boldsymbol{B}_2^*(t - \tau_2)$$

and the construction recursively repeats.

There are two differences between this and the classical BBm model as presented, for example, in the seminal paper by Bramson [1]. Firstly, the branching times are non-random, whereas in the standard model the distances between them are exponentially distributed. Secondly, all branches (that is, the components of the vector-valued process described above) undergo splitting into the same number of offsprings and at the same time (in the classical BBm model each branch has its own branching clock). This, along with the usual description of the classical BBm model as a process indexed by a tree, suggests the name *Brownian decision trees*, where the word "decision" refers to the specific type of trees branching at the same points and into the same amount of branches.

The main findings of the paper are collected in Section 3, which we begin with two preliminary results. In Section 3.1 we consider the exact asymptotics of the probability that at least one branch of the Brownian decision tree with drift exceeds u, as $u \to \infty$, that is,

$$\mathbb{P}\left\{\exists\, t \in [0, T],\ \exists\, \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\}, \tag{1}$$

where $\Gamma = \{0, \ldots, n-1\}$ is the set of indices of the decision tree branches and $c \in \mathbb{R}$ is a deterministic constant. In Theorem 3.1 we show that (1) is asymptotically equal to the product of the number of branches at time T and the probability that a single Brownian motion with drift c crosses level u in time interval $[0, T]$.

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A similar approach to the above can be applied for the exact asymptotics of the largest distance between the branches

$$\mathbb{P}\left\{\exists\, t \in [0,T], \exists\, \gamma_1, \gamma_2 \in \Gamma \colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\}, \tag{2}$$

as $u \to \infty$, which is derived in Theorem 3.2.
In Section 3.3 we focus on the probability

$$\mathbb{P}\left\{\exists\, t \in [0,T]\ \forall\, \gamma \in \Gamma \colon B_{\gamma}(t) - ct > u\right\} \tag{3}$$

that for some $t \in [0,T]$ all branches exceed threshold u. This problem is much more complicated and needs a more subtle approach than used in the analysis of (1) and (2). In order to get the exact asymptotics of (3) we develop the technique introduced in [2] for extremes of centered vector-valued Gaussian processes to the branching model considered in this contribution. The main result of this section is displayed in Theorem 3.4, which is supplemented by an extension of Korshunov-Wang inequality [3–5] (see Proposition 3.3) where a tight upper bound for (3), that is valid for all $u > 0$, is derived. Complementary to the above findings, we investigate the asymptotics of (3) for a version of Brownian decision tree with random numbers of offsprings (Corollary 3.6) and the limiting distribution of the corresponding conditional exceedance times (Corollary 3.7).
In Section 3.4 we present asymptotic results for the process which we call *Brownian decision forest*, that consists of a family of M independent Brownian decision trees $\boldsymbol{B}_{\Gamma_i}$, which grow from different points $x_i \in \mathbb{R}$. For this purpose we introduce a partial order on the set of tuples $(\boldsymbol{\tau}, \boldsymbol{N}, c, x)$, determining the trees of a forest, such that the trees which are high in this order are more likely to exceed the high barrier. In Theorem 3.10 we use this order to find the exact asymptotics of the probability that all branches of at least one tree in a forest exceed some high barrier simultaneously.
We note that in the context of the classical branching Brownian motion most of the asymptotical results focus either on the number of particles or the distribution of the highest branch, which in our setup would be $\max_{\gamma\in\Gamma} B_\gamma(t)$. In both cases the limiting parameter is t approaching infinity. We refer the reader to the classical papers [1, 6–9]. The two types of questions mentioned above have been investigated extensively in the last decades for various versions of the classical BBm model. Among these versions, the most popular is the Kesten's BBm with absorption (see the original paper [10]). Other models include BBm in random [11, 12] and periodic [13] environments, spatial selection such as in the Brownian bees model [14–16], spatially-inhomogenous branching rates [17], self-repulsive BBm [18] and many others.
As it turns out, the exact asymptotics of the counterpart of (3) for the classical BBm is relatively simpler to obtain than for the analyzed in this contribution Brownian decision trees. Namely, it suffices to observe that the most probable scenario in this case is that the BBm process does not manage to branch even once before hitting the high barrier. Hence, the probability (3) is asymptotically equivalent to the probability that the first branching event happens after T times the one-dimensional ruin probability of the Brownian motion. We present here the precise result, the proof of which the reader can find in the Appendix.

Proposition 1.1. *Let $\boldsymbol{B}(t)$, $t \ge 0$ be the classical Branching Brownian motion as described above, with $\tau \sim \mathrm{Exp}(1)$. Then, for any $c \in \mathbb{R}$ we have*

$$\mathbb{P}\left\{\exists\, t \in [0,T]\ \forall\, \gamma \in \Gamma \colon B_{\gamma}(t) - ct > u\right\} \sim e^{-T} \times \sqrt{\frac{2T}{\pi}} \exp\left(-\frac{(u+cT)^2}{2T}\right).$$

The paper is organized as follows. We begin with a construction of the Brownian decision tree process and establish some necessary notation and basic properties in Section 2. The main results are presented in Section 3. Most of the technical proofs are relegated to Sections 4 and 5.

2. Definitions and basic properties of Brownian decision trees

Let $0 < \tau_1 < \ldots < \tau_\eta < T$ be a finite collection of points, further referred to as *the branching points*, and a sequence of numbers $N_i \ge 1$ with $i = 1, \ldots, \eta$, interpreted as *the numbers of branches generated at τ_i*. Denote by P_i the total number of branches born up to time τ_i, that is,

$$P_i = \prod_{j=1}^{i} N_j,$$

and define the set Γ indexing the branches on $[0,T]$ by

$$\Gamma = \{0, 1, \dots, P_\eta - 1\}.$$

Let I_i be the $P_i \times P_i$ identity matrix, and $\mathbf{1}_i = (1, \dots, 1)^\top \in \mathbb{R}^{P_i}$. Denote by $i(t)$ the number of branching points before t

$$i(t) = \max\{i \in \{1, \dots, \eta\} \colon t > \tau_i\}.$$

Clearly, each t belongs to the corresponding interval of the form $(\tau_{i(t)}, \tau_{i(t)+1}]$.

Next, take a collection of mutually independent standard Brownian motions $\boldsymbol{B}_i^*(t)$, $t \ge 0$ in $\mathbb{R}^{P_i}$ indexed by $i = 1, \dots, \eta$ and a one dimensional Brownian motion $B_0^*(t)$, $t \ge 0$. Assume that all these processes are mutually independent. Using the ingredients described above, we construct a new process $\widetilde{\boldsymbol{B}}_\Gamma(t)$, $t \in [0,T]$, which we shall call *the Brownian decision tree*, as follows: for $t \in (\tau_i, \tau_{i+1}]$ set

$$\widetilde{\boldsymbol{B}}_\Gamma(t) \coloneqq \begin{pmatrix} \widetilde{\boldsymbol{B}}_\Gamma(\tau_i) \\ \vdots \\ \widetilde{\boldsymbol{B}}_\Gamma(\tau_i) \end{pmatrix} + \boldsymbol{B}_i^*(t - \tau_i) \quad \text{for} \quad t \in (\tau_i, \tau_{i+1}]$$

and

$$\widetilde{\boldsymbol{B}}_\Gamma(t) \coloneqq B_0^*(t) \quad \text{for} \quad t \in [0, \tau_1].$$

Note that $(\widetilde{\boldsymbol{B}}_\Gamma(t))_{t \in (\tau_i, \tau_{i+1}]}$ belongs to $C((\tau_i, \tau_{i+1}], \mathbb{R}^{P_i})$. We can extract the individual branch indexed by $\gamma \in \Gamma$ by taking

$$B_\gamma(t) \coloneqq \big(\widetilde{\boldsymbol{B}}_\Gamma(t)\big)_{(\gamma \bmod P_{i(t)})+1} \tag{4}$$

and denote for two fixed branches γ_1 and γ_2 their *separation moment* at which they diverge by

$$\kappa(\gamma_1, \gamma_2) \coloneqq \min\left\{n \in \{1, \dots, \eta\} \colon \gamma_1 \neq \gamma_2 \bmod P_n\right\}.$$

Clearly, $B_{\gamma_1}(t) = B_{\gamma_2}(t)$ for $t \le \tau_{\kappa(\gamma_1,\gamma_2)}$.

Next, we present some useful properties of Brownian decision trees.

Proposition 2.1. *For $\gamma \in \Gamma$ the process B_γ is a Brownian motion and for $t \in (\tau_1, T]$ it admits the following representation in terms of a collection of independent Brownian motions $\{\boldsymbol{B}_j^*\}_{j=0,\dots,\eta}$:*

$$B_\gamma(t) = B_0^*(\tau_1) + \sum_{j=1}^{i(t)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \big(\boldsymbol{B}_{i(t)}^*\big)_{(\gamma \bmod P_{i(t)})+1}(t - \tau_{i(t)}). \tag{5}$$

Moreover, for $\gamma_1 \neq \gamma_2$ and t_1, $t_2 \in [0,T]$ holds

$$\mathrm{Cov}(B_{\gamma_1}(t_1), B_{\gamma_2}(t_2)) = \min\{t_1, t_2, \tau_{\kappa(\gamma_1,\gamma_2)}\}. \tag{6}$$

The proof of Proposition 2.1 can be found in Appendix.

Sometimes it is more convenient to work with the process $\boldsymbol{B}_\Gamma$, which contains all the branches simultaneously

$$\boldsymbol{B}_\Gamma(t) = (B_\gamma(t))_{\gamma \in \Gamma} \in \mathbb{R}^{P_\eta}. \tag{7}$$

The difference between this process and $\widetilde{\boldsymbol{B}}_\Gamma$ is that its values belong to $\mathbb{R}^{P_\eta}$ for each t instead of $\mathbb{R}^{P_{i(t)}}$. The two processes are related as follows:

$$\widetilde{\boldsymbol{B}}_\Gamma(t) = (\boldsymbol{B}_\Gamma(t))_{\gamma \in \{0, \dots, P_{i(t)}-1\}}. \tag{8}$$

Unlike $\widetilde{\boldsymbol{B}}_\Gamma(t)$, the variance matrix of this process is degenerate for all $t \in [0, \tau_\eta]$:

$$\det\left(\mathrm{Var}\left(\boldsymbol{B}_\Gamma(t)\right)\right) = 0.$$

Proposition 2.2. *For $0 \le t_1 < t_2 \le T$, the random vector $\boldsymbol{B}_\Gamma(t_2) - \boldsymbol{B}_\Gamma(t_1)$ is independent of the process $\boldsymbol{B}_\Gamma(t)|_{t \in [0,t_1]}$.*

Let $\Sigma(t)$ denote the covariance matrix of the random vector $\widetilde{\boldsymbol{B}}_\Gamma(t)$:

$$\Sigma(t) \coloneqq \mathbb{E}\left\{\widetilde{\boldsymbol{B}}_\Gamma(t)\, \widetilde{\boldsymbol{B}}_\Gamma^\top(t)\right\}, \qquad t \in [0,T].$$

Proposition 2.3. *For $t \in (\tau_1, T]$,*

$$\Sigma(t) = \begin{pmatrix} \Sigma(\tau_{i(t)}) & \dots & \Sigma(\tau_{i(t)}) \\ \vdots & \ddots & \vdots \\ \Sigma(\tau_{i(t)}) & \dots & \Sigma(\tau_{i(t)}) \end{pmatrix} + (t - \tau_{i(t)}) I_{i(t)},$$

where the first term matrix has $N_{i(t)}^2$ blocks, all equal to $\Sigma(\tau_{i(t)})$.

In the next proposition we find the eigenvalues of $\Sigma(t)$.

Proposition 2.4. *For $t \in [0, T]$, the eigenvalues of matrix $\Sigma(t)$ are given by*

$$\mu_v(t) = (t - \tau_{i(t)}) + \sum_{l=v+1}^{i(t)} (\tau_l - \tau_{l-1}) \prod_{j=l}^{i(t)} N_j, \qquad v = 0, 1, \dots, i(t). \tag{9}$$

The multiplicity of $\mu_v(t)$ equals $P_v - P_{v-1}$, except for μ_0, the multiplicity of which is 1. Additionally, for any $t \in [0, T]$, the vector $\mathbf{1}_{i(t)}$ is an eigenvector of $\Sigma(t)$ corresponding to $\mu_0(t)$.

Using Proposition 2.4 we can obtain the following result.

Proposition 2.5. *For $c \in \mathbb{R}$ and $T > 0$*

$$\mathbb{P}\left\{\boldsymbol{B}_\Gamma(T) - cT\mathbf{1}_\eta > u\mathbf{1}_\eta\right\} \sim u^{-P_\eta} \frac{\mu_0^{P_\eta - 1/2}(T)}{(2\pi)^{P_\eta/2} \prod_{v=1}^{\eta} \mu_v^{(P_v - P_{v-1})/2}(T)} \exp\left(-\frac{(u+cT)^2 P_\eta}{2\mu_0(T)}\right)$$

as $u \to \infty$, where $\mu_v(T)$ are defined in (9).

Remark 2.6. *The result obtained in Proposition 2.5 still hold for $T = T_u$ with $T_u \to T$ as $u \to \infty$.*

The proofs of Proposition 2.4 and Proposition 2.5 are given in Appendix.

3. Main results

In this section we present the main findings of this contribution. We begin with the asymptotic analysis of the high exceedance probability of at least one branch of $\boldsymbol{B}_\Gamma$ with linear drift, then we proceed to the asymptotics of the largest distance between the branches. In Section 3.3 we investigate the probability that all branches of $\boldsymbol{B}_\Gamma$ with linear drift exceed high threshold and then extend it to the analysis of a finite collection of independent Brownian decision trees.

3.1. **High-exeedance of at least one branch.** Consider the probability that at least one branch of the branching Brownian decision tree with linear trend exceeds some large threshold u. Clearly, for each $u > 0$ and a standard Brownian motion $B(t), t \in [0, \infty)$

$$\mathbb{P}\left\{\exists\, t \in [0,T],\, \exists\, \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\} \le P_\eta \mathbb{P}\left\{\exists\, t \in [0,T] \colon B(t) - ct > u\right\},$$

where P_η is the total amount of the branches in the decision tree. It appears that, as $u \to \infty$, the above bound provides the exact asymptotics, as shown in the following theorem.

Theorem 3.1. *For $c \in \mathbb{R}$, as $u \to \infty$,*

$$\begin{aligned} \mathbb{P}\left\{\exists\, t \in [0,T],\, \exists\, \gamma \in \Gamma \colon B_\gamma(t) - ct > u\right\} &\sim P_\eta \mathbb{P}\left\{\exists\, t \in [0,T] \colon B(t) - ct > u\right\} \\ &\sim P_\eta \sqrt{\frac{2}{\pi}} \frac{\sqrt{T}}{u + cT} \exp\left(-\frac{(u+cT)^2}{2T}\right). \end{aligned}$$

3.2. **The largest distance between the branches.** Next, we investigate the probability that in time interval $[0, T]$ the largest distance between the branches of the Brownian branching tree with drift gets larger than $u > 0$

$$\mathbb{P}\left\{\exists\, t \in [0,T], \exists\, \gamma_1, \gamma_2 \in \Gamma \colon (B_{\gamma_1}(t) - ct) - (B_{\gamma_2}(t) - ct) > u\right\}.$$

We note that the drifts in the above formula cancel out. Thus in the following result we consider the driftless case.

Theorem 3.2. *As $u \to \infty$*

$$\mathbb{P}\left\{\exists\, t \in [0,T], \exists\, \gamma_1, \gamma_2 \in \Gamma \colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\} \sim P_\eta^2 \frac{N_1 - 1}{N_1} \frac{2\sqrt{T - \tau_1}}{u\sqrt{\pi}} \exp\left(-\frac{u^2}{4(T - \tau_1)}\right).$$

3.3. **Simultaneous high-exceedance of all branches.** In this section we analyze properties of the simultaneous all-branch high exceedance probability

$$\mathbb{P}\left\{\exists t\in[0,T]\ \forall\gamma\in\Gamma\colon B_\gamma(t)-ct>u\right\}. \tag{10}$$

We begin with a non-asymptotic result. Obviously, for each $u>0$

$$\mathbb{P}\left\{\exists t\in[0,T]\ \forall\gamma\in\Gamma\colon B_\gamma(t)-ct>u\right\}\geq\mathbb{P}\left\{\forall\gamma\in\Gamma\colon B_\gamma(T)-cT>u\right\}, \tag{11}$$

where the asymptotics, as $u\to\infty$, of the probability on the righthand side of the above inequality is given in Proposition 2.5.

In the following proposition we find an upper bound for (10) which differs from the bound (11) by some constant.

Proposition 3.3. *For $c\in\mathbb{R}$ and any $u>0$,*

$$\mathbb{P}\left\{\exists t\in[0,T]\ \forall\gamma\in\Gamma\colon B_\gamma(t)-ct>u\right\}\leq C\,\mathbb{P}\left\{\forall\gamma\in\Gamma\colon B_\gamma(T)-cT>u\right\},$$

where $C=\left(\mathbb{P}\left\{\xi>|c|\right\}\right)^{-P_\eta}$ with ξ a standard normal random variable.

In order to present the exact asymptotics of (10) as $u\to\infty$ we need to introduce the following constant

$$\mathcal{H}_{\mathcal{N},\lambda}:=\int_{\mathbb{R}^{\mathcal{N}}}\mathbb{P}\left\{\exists t\in(0,\infty),\ \forall i\in\{1,\ldots,\mathcal{N}\}\colon\lambda B_i^*(t)-\lambda^2t>x_i\right\}e^{\sum_{i=1}^{\mathcal{N}}x_i}\mathrm{d}\boldsymbol{x}, \tag{12}$$

where $B_i^*(t),\ t\in[0,\infty),\ i=1,...,\mathcal{N}\in\mathbb{N}$ are mutually independent standard Brownian motions and $\lambda>0$. The finiteness of this constant is shown in Lemma 4.4.

The following result constitutes the main finding of this section.

Theorem 3.4. *For $c\in\mathbb{R}$, as $u\to\infty$,*

$$\mathbb{P}\left\{\exists t\in[0,T]\ \forall\,\gamma\in\Gamma\colon B_\gamma(t)-ct>u\right\}\sim\mathcal{H}_{P_\eta,1/\mu_0(T)}\,\mathbb{P}\left\{\forall\,\gamma\in\Gamma\colon B_\gamma(T)-cT>u\right\}.$$

Remark 3.5. *The results obtained in Proposition 3.3 and Theorem 3.4 still hold for $T=T_u$ with $T_u\to T$ as $u\to\infty$.*

Interestingly, the above result can be extended to the version of Brownian decision tree with random numbers of offsprings N_i. For a random vector $\boldsymbol{N}$, let $\operatorname{essinf}(\boldsymbol{N})$ denote the componentwise essential infimum of $\boldsymbol{N}$.

Corollary 3.6. *Assume that $N_i\in\mathbb{N}$ are independent random variables. Then,*

$$\begin{aligned}\mathbb{P}\left\{\exists t\in[0,T]\ \forall\gamma\in\Gamma\colon B_\gamma(t)-ct>u\right\}\ &\sim\ \mathcal{H}_{P_\eta,1/\mu_0(T)}\prod_{i=1}^{\eta}\mathbb{P}\left\{N_i=\operatorname{essinf}(N_i)\right\}\\ &\quad\times\mathbb{P}\left\{\forall\,\gamma\in\Gamma\colon B_\gamma(T)-ct>u\ \mid\boldsymbol{N}=\operatorname{essinf}(\boldsymbol{N})\right\},\end{aligned}$$

as $u\to\infty$. The constant $\mathcal{H}_{P_\eta,1/\mu_0(T)}$ is calculated under the assumption that $\boldsymbol{N}=\operatorname{essinf}(\boldsymbol{N})$.

The results obtained in Theorem 3.4 and Proposition 2.5 allow us to derive the asymptotics for the conditional first time of simultaneous exceedance of all branches

$$\mathcal{T}(u):=\inf\{t\in[0,T]\colon\forall\,\gamma\in\Gamma\colon B_\gamma(t)-ct>u\}.$$

Corollary 3.7. *Then for any $x,y\in(0,+\infty)$ such that $x>y$, holds*

$$\lim_{u\to\infty}\mathbb{P}\left\{u^2(T-\mathcal{T}(u))\geq x\mid\mathcal{T}(u)\leq T-y/u^2\right\}=\exp\left(-\frac{(x-y)P_\eta}{2\mu_0^2(T)}\right).$$

Example 3.8 (Binary tree)**.** *Suppose that the difference between branching points equals one (i.e. $\tau_i-\tau_{i-1}=1$ for all $i\in\mathbb{N}$) and each process always splits into two (i.e. $N_i=2$ for all $i\in\mathbb{N}$). For the sake of simplicity, let T be integer. Then*

$$\mu_v(T)=2^{T-v}-1.$$

Combining Theorem 3.4 and Proposition 2.5 we obtain that as $u\to\infty$

$$\mathbb{P}\left\{\exists t\in[0,T]\ \forall\gamma\in\Gamma\colon B_\gamma(t)-ct>u\right\}\sim\frac{\mathcal{H}_{2^{T-1},1/(2^T-1)}(2^T-1)^{2^{T-1}-1/2}}{(2\pi)^{2^{T-2}}\prod_{v=1}^{T-1}(2^{T-v}-1)^{2^{v-2}}}u^{-2^{T-1}}\exp\left(-\frac{2^{T-2}}{2^T-1}(u+cT)^2\right).$$

3.4. **Brownian decision forests.** In this section we shall consider a set of $M \in \mathbb{N}$ independent Brownian decision trees with drift. That is, for $i = 1, \dots, M$ we take a triple $(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c)$, where $\boldsymbol{\tau}_i$ and $\boldsymbol{N}_i$ are two vectors and c_i is a constant, associate to each of them a Brownian decision tree $\boldsymbol{B}_{\Gamma_i}$ as described in Section 2 (such that all of them are independent), and set

$$\boldsymbol{W}_i(t) \coloneqq \boldsymbol{B}_{\Gamma_i}(t) + (x_i - c_i\, t)\, \mathbf{1}_{\eta_i},$$

where x_i's are interpreted as the starting points of these trees. Each tree $\boldsymbol{W}_i(t)$ is thus uniquely defined by its tuple $(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i)$. Let us define a partial order relation $\succcurlyeq$ on the set of trees as follows: we write $(\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \succcurlyeq (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2)$ if one of the following three conditions holds:

$$\text{(i)} \qquad \frac{\mu_{0,1}(T)}{P_{\eta_1}} > \frac{\mu_{0,2}(T)}{P_{\eta_2}}\,,$$

$$\text{(ii)} \qquad \frac{\mu_{0,1}(T)}{P_{\eta_1}} = \frac{\mu_{0,2}(T)}{P_{\eta_2}} \quad \text{and} \quad c_1 T - x_1 < c_2 T - x_2\,,$$

$$\text{(iii)} \qquad \frac{\mu_{0,1}(T)}{P_{\eta_1}} = \frac{\mu_{0,2}(T)}{P_{\eta_2}} \quad \text{and} \quad c_1 T - x_1 = c_2 T - x_2 \quad \text{and} \quad P_{\eta_1} \le P_{\eta_2}.$$

In other words, it is the lexicographic order on the tuples $(\mu_0(T)/P_\eta,\ x - cT,\ -P_\eta)$. One may notice that this order is full. Next, let

$$(\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \approx (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2) \iff \begin{cases} (\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \preccurlyeq (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2), \\ (\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \succcurlyeq (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2) \end{cases}$$

and

$$(\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \succ (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2) \iff \begin{cases} (\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \succcurlyeq (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2), \\ (\boldsymbol{\tau}_1, \boldsymbol{N}_1, c_1, x_1) \not\approx (\boldsymbol{\tau}_2, \boldsymbol{N}_2, c_2, x_2). \end{cases}$$

Combining Theorem 3.4 and Proposition 2.5 we straightforwardly obtain the following result.

Lemma 3.9. *The following equivalences hold:*

$$(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i) \succ (\boldsymbol{\tau}_j, \boldsymbol{N}_j, c_j, x_j) \iff \lim_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T] :\ \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\}}{\mathbb{P}\left\{\exists t \in [0,T] :\ \boldsymbol{W}_j(t) > u\mathbf{1}_{\eta_j}\right\}} = \infty,$$

$$(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i) \prec (\boldsymbol{\tau}_j, \boldsymbol{N}_j, c_j, x_j) \iff \lim_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T] :\ \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\}}{\mathbb{P}\left\{\exists t \in [0,T] :\ \boldsymbol{W}_j(t) > u\mathbf{1}_{\eta_j}\right\}} = 0,$$

$$(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i) \approx (\boldsymbol{\tau}_j, \boldsymbol{N}_j, c_j, x_j) \iff \lim_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T] :\ \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\}}{\mathbb{P}\left\{\exists t \in [0,T] :\ \boldsymbol{W}_j(t) > u\mathbf{1}_{\eta_j}\right\}} \in (0,\infty).$$

Lemma 3.9 allows us to find the asymptotics for the probability that all branches of at least one of our M trees exceed the threshold u.

Theorem 3.10. *Let us define by A the set of indices $i \in \{1, \dots, M\}$ for which the corresponding set $(\boldsymbol{\tau}_i, \boldsymbol{N}_i, c_i, x_i)$ is the maximal among all given. Then, as $u \to \infty$,*

$$\mathbb{P}\left\{\exists\, i \in \{1, \dots, M\}, t \in [0,T] \colon \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\} \sim \sum_{i\in A} \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{W}_i(t) > u\mathbf{1}_{\eta_i}\right\},$$

where the asymptotics of each term is given in Theorem 3.4.

4. Proofs

4.1. **Proof of Proposition 1.1.** Let $\tau \sim \mathrm{Exp}(1)$ be the first branching moment. Define by $\mathfrak{d}$ the dimension of vector $\boldsymbol{B}(t)$ (i.e. the amount of branches at the point t), and $\mathbf{1}(t) = (1, \dots, 1) \in \mathbb{R}^{\mathfrak{d}}$. Then,

$$\begin{aligned} &\mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t),\ \tau \ge T\right\} \\ &\qquad \le \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t)\right\} \\ &\qquad \le \mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t),\ \tau \le T - 2u^{-1}\right\} \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}\left\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t),\ \tau \in [T-2u^{-1}, T]\right\}\\
&+\mathbb{P}\left\{\exists t\ \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t),\ \tau \geq T\right\}\\
=:&\ S_1 + S_2 + S_3
\end{aligned}
$$

Consider each term separately. For S_3, using that $(\boldsymbol{B} \mid \tau \geq T)$ for $t \in [0,T]$ is a Brownian motion independent of τ, we have

$$
\begin{aligned}
S_3 &= \mathbb{P}\{\tau \geq T\}\ \mathbb{P}\{\exists\, t \in [0,T] \colon \boldsymbol{B}(t) - ct\mathbf{1}(t) > u\mathbf{1}(t) \mid \tau \geq T\}\\
&\sim e^{-T}\sqrt{\frac{2T}{\pi}}\, u^{-1} \exp\left(-\frac{(u+cT)^2}{2T}\right)
\end{aligned}
$$

as $u \to \infty$. Considering S_2, again using that B_1 is Brownian motion independent of τ, as $u \to \infty$

$$
\begin{aligned}
S_2 &\leq \mathbb{P}\left\{\tau \in [T-2u^{-1}, T]\right\}\ \mathbb{P}\{\exists\, t \in [0,T] \colon B_1(t) - ct > u\}\\
&\sim (e^{2u^{-1}} - 1)\, e^{-T}\sqrt{\frac{2T}{\pi}}\, u^{-1} \exp\left(-\frac{(u+cT)^2}{2T}\right),
\end{aligned}
$$

implies that

$$
\lim_{u\to\infty} \frac{S_2}{S_3} \to 0.
$$

Then, for S_1

$$
\begin{aligned}
S_1 \ \leq\ & \mathbb{P}\left\{\exists\, t \in [0, T-u^{-1}] \colon B_1(t) - ct > u,\ \tau \leq T - 2u^{-1}\right\}\\
&+\mathbb{P}\left\{\exists\, t \in [T-u^{-1}, T] \colon \frac{B_1(t)+B_2(t)}{2} - ct > u,\ \tau \leq T-2u^{-1}\right\} =: Z_1 + Z_2.
\end{aligned}
$$

For Z_1, using that

$$
Z_1 \sim (1-e^{-T})\sqrt{\frac{2T}{\pi}}\, u^{-1}\exp\left(-\frac{(u+c(T-u^{-1}))^2}{2(T-u^{-1})}\right),
$$

we obtain

$$
\begin{aligned}
\lim_{u\to\infty}\frac{Z_1}{S_3} &= \lim_{u\to\infty}\frac{1-e^{-T}}{e^{-T}}\exp\left(\frac{(u+cT)^2}{2T} - \frac{(u+c(T-u^{-1}))^2}{2(T-u^{-1})}\right)\\
&= \lim_{u\to\infty}\frac{1-e^{-T}}{e^{-T}}\exp\left(\frac{-u^{-1}(u+cT)^2 - 4cT^2 + 2Tc^2u^{-2}}{2T(T-u^{-1})}\right) = 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
Z_2 &= \mathbb{E}\left\{\mathbb{I}_{\{\tau < T-2u^{-1}\}}\,\mathbb{P}\left\{\exists\, t \in [T-u^{-1}, T] \colon \frac{B_1(t)+B_2(t)}{2} - ct > u \,\middle|\, \tau\right\}\right\}\\
&\leq \sup_{\mathfrak{t}\in[0,T-2u^{-1}]}\mathbb{P}\left\{\exists\, t \in [T-u^{-1}, T] \colon \frac{B_1(t)+B_2(t)}{2} > u - |c|\, T \,\middle|\, \tau = \mathfrak{t}\right\}.
\end{aligned}
$$

Since $(B_1(t)+B_2(t) \mid \tau = \mathfrak{t})$ is a Gaussian process and for any $\mathfrak{t} \in [0, T-u^{-1}]$

$$
\mathrm{Var}\left(\frac{B_1(t)+B_2(t)}{2} \,\middle|\, \tau = \mathfrak{t}\right) = \begin{cases} \frac{t+\mathfrak{t}}{2}, & t > \mathfrak{t},\\ t, & t \leq \mathfrak{t}, \end{cases}
$$

we can apply Piterbarg inequality (see, e.g., [19, Theorem 8.1]) obtaining that for $u > |c|\, T$

$$
\mathbb{P}\left\{\exists\, t \in [T-u^{-1}, T] \colon \frac{B_1(t)+B_2(t)}{2} > u - |c|\, T \,\middle|\, \tau = \mathfrak{t}\right\} \leq C(u - |c|\, T)^{\alpha}\exp\left(-\frac{(u-|c|\, T)^2}{2\sigma_{\mathfrak{t}}}\right),
$$

where

$$
\sigma_{\mathfrak{t}} = \max_{t\in[0,T]}\mathrm{Var}\left(\frac{B_1(t)+B_2(t)}{2}\right) = \frac{T+\mathfrak{t}}{2}
$$

and $C > 0$, $\alpha \in \mathbb{R}$ are some constants. Using that C, α does not depend ot $\mathfrak{t}$, we get

$$Z_2 \leq C u^{\alpha} \sup_{\mathfrak{t} \in [0, T - 2u^{-1}]} \exp\left(-\frac{(u - |c| T)^2}{2\sigma_{\mathfrak{t}}} \right) = C u^{\alpha} \exp\left(-\frac{(u - |c| T)^2}{2(T - u^{-1})} \right).$$

It remains to note that $Z_2/S_3 \to 0$ as $u \to \infty$ by the same reason as Z_1/S_3. □

4.2. **Proof ot Theorem 3.1.** We begin with the observation that

$$\begin{aligned} \mathbb{P}\left\{ \exists\, t \in [0,T], \gamma \in \Gamma \colon B_\gamma(t) - ct > u \right\} \;\;\geq\;\; & \sum_{\gamma \in \Gamma} \mathbb{P}\left\{ \exists\, t \in [0,T] \colon B_\gamma(t) - ct > u \right\} \\ & - \sum_{\substack{\gamma_1, \gamma_2 \in \Gamma \\ \gamma_1 \neq \gamma_2}} \mathbb{P}\left\{ \exists\, t \in [0,T] \colon \begin{matrix} B_{\gamma_1}(t) - ct > u, \\ B_{\gamma_2}(t) - ct > u \end{matrix} \right\} \end{aligned}$$

and

$$\mathbb{P}\left\{ \exists\, t \in [0,T], \gamma \in \Gamma \colon B_\gamma(t) - ct > u \right\} \leq \sum_{\gamma \in \Gamma} \mathbb{P}\left\{ \exists\, t \in [0,T] \colon B_\gamma(t) - ct > u \right\},$$

where branch B_γ is a Brownian motion. By [20, Formula 1.1.4 and Appendix 2, Section 8], we have

$$\mathbb{P}\left\{ \exists\, t \in [0,T] \colon B_\gamma(t) - ct > u \right\} \sim \frac{2\sqrt{T}}{\sqrt{2\pi}(u + cT)} \exp\left(-\frac{(u + cT)^2}{2T} \right) \tag{13}$$

as $u \to \infty$. For the double sum on the left hand side, we have the following upper bound

$$\mathbb{P}\left\{ \exists\, t \in [0,T] \colon \begin{matrix} B_{\gamma_1}(t) - ct > u, \\ B_{\gamma_2}(t) - ct > u \end{matrix} \right\} \leq C\, \mathbb{P}\left\{ \begin{matrix} B_{\gamma_1}(T) - cT > u, \\ B_{\gamma_2}(T) - cT > u \end{matrix} \right\}.$$

Here C is some positive constant independent of u; see [21, Theorem 3.1]. Since $(B_{\gamma_1}(T), B_{\gamma_2}(T))$ is a Gaussian vector with covariance matrix

$$\Sigma = \begin{pmatrix} T & \tau_{\kappa(\gamma_1,\gamma_2)} \\ \tau_{\kappa(\gamma_1,\gamma_2)} & T \end{pmatrix}$$

we have for some positive constant $C_2 > 0$ as $u \to \infty$

$$\mathbb{P}\left\{ B_{\gamma_1}(T) + cT > u,\, B_{\gamma_2}(T) - cT > u \right\} \sim C_2\, u^{-2} \exp\left(-\frac{(u + cT)^2}{2\left(T + \tau_{\kappa(\gamma_1,\gamma_2)} \right)} \right).$$

Using that $\tau_{\kappa(\gamma_1,\gamma_2)} < T$ for $\gamma_1 \neq \gamma_2$, we obtain by (13)

$$\mathbb{P}\left\{ \exists\, t \in [0,T] \colon \begin{matrix} B_{\gamma_1}(t) - ct > u, \\ B_{\gamma_2}(t) - ct > u \end{matrix} \right\} = o\left(\mathbb{P}\left\{ \exists\, t \in [0,T] \colon B_\gamma(t) - ct > u \right\} \right)$$

for any $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, establishing the claim. □

4.3. **Proof of Theorem 3.2.** Note that

$$\begin{aligned} & \sum_{\gamma_1, \gamma_2 \in \Gamma} \mathbb{P}\left\{ \exists\, t \in [0,T] \colon\; B_{\gamma_1}(t) - B_{\gamma_2}(t) > u \right\} \\ & \quad \geq \mathbb{P}\left\{ \exists\, t \in [0,T], \exists\, \gamma_1, \gamma_2 \in \Gamma \colon\; B_{\gamma_1}(t) - B_{\gamma_2}(t) > u \right\} \\ & \quad \geq \sum_{\gamma_1, \gamma_2 \in \Gamma} \mathbb{P}\left\{ \exists\, t \in [0,T] \colon\; B_{\gamma_1}(t) - B_{\gamma_2}(t) > u \right\} \\ & \qquad - \sum_{\substack{\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma \\ \{\gamma_1, \gamma_2\} \neq \{\gamma_3, \gamma_4\}}} \mathbb{P}\left\{ \exists\, t, s \in [0,T] \colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u, B_{\gamma_3}(s) - B_{\gamma_4}(s) > u \right\}. \end{aligned}$$

For any $\gamma_1 \neq \gamma_2$, the process $B_{\gamma_1} - B_{\gamma_2}$ admits the following representation

$$B_{\gamma_1}(t) - B_{\gamma_2}(t) = \begin{cases} 0, & t \leq \tau_{\kappa(\gamma_1,\gamma_2)}, \\ B^*(2(t - \tau_{\kappa(\gamma_1,\gamma_2)}), & t > \tau_{\kappa(\gamma_1,\gamma_2)}, \end{cases}$$

where $B^*(t)$ is Brownian motion. Hence, as $u \to \infty$,

$$\begin{aligned}\mathbb{P}\left\{\exists t \in [0,T]\colon B_{\gamma_1}(t) - B_{\gamma_2} > u\right\} &= \mathbb{P}\left\{\exists t \in [\tau_{\kappa(\gamma_1,\gamma_2)}, T]\colon B^*(2(t - \tau_{\kappa(\gamma_1-\gamma_2)})) > u\right\}\\ &= \mathbb{P}\left\{\exists t \in [0, 2(T - \tau_{\kappa(\gamma_1,\gamma_2)})]\colon B^*(t) > u\right\}\\ &\sim \frac{2\sqrt{2(T - \tau_{\kappa(\gamma_1,\gamma_2)})}}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{4(T - \tau_{\kappa(\gamma_1,\gamma_2)})}\right).\end{aligned}$$

For $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$, if $\kappa(\gamma_1, \gamma_2) < \kappa(\gamma_3, \gamma_4)$, then as $u \to \infty$

$$\frac{\mathbb{P}\left\{\exists t \in [0,T]\colon B_{\gamma_3}(t) - B_{\gamma_4} > u\right\}}{\mathbb{P}\left\{\exists t \in [0,T]\colon B_{\gamma_1}(t) - B_{\gamma_2} > u\right\}} \to 0.$$

The latter implies that

$$\begin{aligned}\sum_{\gamma_1,\gamma_2\in\Gamma} \mathbb{P}\left\{\exists t \in [0,T]\colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\} &\sim \sum_{\substack{\gamma_1,\gamma_2\in\Gamma\\ \kappa(\gamma_1,\gamma_2)=1}} \frac{2\sqrt{2(T-\tau_1)}}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{4(T-\tau_1)}\right)\\ &= N_1(N_1 - 1)\left(\frac{P_\eta}{N_1}\right)^2 \frac{2\sqrt{2(T-\tau_1)}}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{4(T-\tau_1)}\right).\end{aligned}$$

Next, consider the double events' probabilities. For $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$,

$$\begin{aligned}&\mathbb{P}\left\{\exists t, s \in [0,T]\colon B_{\gamma_1}(t) - B_{\gamma_2}(t) > u, B_{\gamma_3}(s) - B_{\gamma_4}(s) > u\right\}\\ &\quad\le \mathbb{P}\left\{\exists t, s \in [0,T]\colon \frac{B_{\gamma_1}(t) - B_{\gamma_2}(t) + B_{\gamma_3}(s) - B_{\gamma_4}(s)}{2} > u\right\}.\end{aligned}$$

Using that $(B_{\gamma_1}(t) - B_{\gamma_2}(t) + B_{\gamma_3}(s) - B_{\gamma_4}(s))/2$ is a Gaussian random field and

$$\mathrm{Var}\left(\frac{B_{\gamma_1}(t) - B_{\gamma_2}(t) + B_{\gamma_3}(s) - B_{\gamma_4}(s)}{2}\right) \le \frac{3T + \tau_\eta - 4\tau_1}{2},$$

by Piterbarg inequality (see, e.g., [19, Theorem 8.1]) there exist some constants $C > 0$, $\alpha \in \mathbb{R}$

$$\mathbb{P}\left\{\exists t, s \in [0,T]\colon B_{\gamma_1}(t) - B_{\gamma_2}(t) + B_{\gamma_3}(s) - B_{\gamma_4}(s) > 2u\right\} \le C u^\alpha \exp\left(-\frac{u^2}{3T + \tau_\eta - 4\tau_1}\right).$$

The above inequality implies that

$$\frac{\displaystyle\sum_{\substack{\gamma_1,\gamma_2,\gamma_3,\gamma_4\in\Gamma\\ \{\gamma_1,\gamma_2\}\neq\{\gamma_3,\gamma_4\}}} \mathbb{P}\left\{\exists t, s \in [0,T]\colon \begin{array}{l} B_{\gamma_1}(t) - B_{\gamma_2}(t) > u,\\ B_{\gamma_3}(s) - B_{\gamma_4}(s) > u \end{array}\right\}}{\displaystyle\sum_{\gamma_1,\gamma_2\in\Gamma} \mathbb{P}\left\{\exists t \in [0,T] : B_{\gamma_1}(t) - B_{\gamma_2}(t) > u\right\}} \to 0$$

as $u \to \infty$, establishing the proof. □

4.4. **Proofs of results on simultaneous high-exceedance probability of all branches.** Until the end of this secion, let us define

$$\boldsymbol{W}(t) = \boldsymbol{B}_\Gamma(t) - \boldsymbol{c}t, \qquad \boldsymbol{c} = c\,\mathbf{1}_\eta, \tag{14}$$

where $c \in \mathbb{R}$ is fixed. Then the the all-branch high exceedance probability can be stated as

$$\mathbb{P}\left\{\exists t \in [0,T],\, \forall \gamma \in \Gamma\colon\ B_\gamma(t) - ct > u\right\} = \mathbb{P}\left\{\exists t \in [0,T]\colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}.$$

Proof of Proposition 3.3. As in the proof of [3, Theorem 1.1] define a stopping time by

$$\mathcal{T} = \inf\{t \in [0,T]\colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\} = \inf\{t \in [0,T]\colon \forall\gamma \in \Gamma\ B_\gamma(t) - ct > u\},$$

using the convention $\inf \varnothing = T + 1$ and write $F_{\mathcal{T}}$ for its distribution function. Since the paths of $\boldsymbol{W}$ are continuous, $\mathcal{T}$ is well-defined. By Proposition 2.2,

$$\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\} = \int_0^T dF_{\mathcal{T}}(t)\, \mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta \mid \mathcal{T} = t\right\}$$

$$= \int_0^T dF_{\mathcal{T}}(t)\, \mathbb{P}\left\{\forall \gamma \in \Gamma\colon B_\gamma(T) - cT > u \mid \mathcal{T} = t\right\}$$

$$\geq \int_0^T dF_{\mathcal{T}}(t)\, \mathbb{P}\left\{\forall \gamma \in \Gamma\colon B_\gamma(T) - B_\gamma(t) > c(T-t)\right\}.$$

Consider the last probability. Define for $t \in [0, T_1]$ and $\gamma \in \Gamma$ a Gaussian random variable $\overline{B}_\gamma(t)$ by

$$\overline{B}_\gamma(t) = \left(B_\gamma(T) - B_\gamma(t)\right).$$

Its variance is

$$\mathrm{Var}\left(\overline{B}_\gamma(t)\right) = \mathrm{Var}\left(B_\gamma(T) - B_\gamma(t)\right) = T_1 - t,$$

and the covariance between them

$$\mathrm{Cov}\left(\overline{B}_{\gamma_1}(t), \overline{B}_{\gamma_2}(t)\right) = \min\{T, \tau_{\kappa(\gamma_1,\gamma_2)}\} - \min\{t, \tau_{\kappa(\gamma_1,\gamma_2)}\} \geq 0.$$

Hence, using Slepian inequality (see e.g., Theorem 2.2.1 in [22]) we arrive at

$$\mathbb{P}\left\{\forall\, \gamma \in \Gamma\colon \overline{B}_\gamma(t) > c(T_1 - t)\right\} \geq \prod_{i=1}^{P_\eta} \mathbb{P}\left\{\overline{B}_i(t) > |c|\,(T_1 - t)\right\} = \left(\mathbb{P}\left\{\xi > |c|\,\sqrt{T_1 - t}\right\}\right)^{P_\eta} \geq \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{P_\eta},$$

where ξ is a standard normal random variable. Hence, we can continue our initial inequality as follows

$$\begin{aligned}
\mathbb{P}\left\{\boldsymbol{W}(T_1) > u\mathbf{1}_\eta\right\} &\geq \int_0^{T_1} dF_{\mathcal{T}}(t)\, \mathbb{P}\left\{\forall \gamma \in \Gamma\colon B_\gamma(T_1) - B_\gamma(t) > c(T_1 - t)\right\} \\
&\geq \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{P_\eta} \int_0^{T_1} dF_{\mathcal{T}}(t) \\
&= \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{P_\eta} \mathbb{P}\left\{\mathcal{T} \leq T_1\right\} \\
&= \left(\mathbb{P}\left\{\xi > |c|\right\}\right)^{P_\eta} \mathbb{P}\left\{\exists\, t \in [0, T_1] : \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}
\end{aligned}$$

establishing the claim. □

Before proving Theorem 3.4 we need to introduce some useful notation and present some lemmas that are used in the proof.
In the remainder we shall need some results concerning the so-called quadratic programming problem: for a positive-definite $d \times d$ matrix Σ and real-valued vector $\boldsymbol{a} \in \mathbb{R}^d$

$$\Pi_\Sigma(\boldsymbol{a})\colon \quad \text{minimise} \quad \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} \quad \text{under the linear constraint} \quad \boldsymbol{x} \geq \boldsymbol{a}. \tag{15}$$

As shown in [23], (15) has a unique solution $\tilde{\boldsymbol{a}}$, see Appendix, Lemma 5.1. We would also use further the sets I, J defined in Lemma 5.1. In particular, $\tilde{\boldsymbol{a}}_I = \boldsymbol{a}_I$ with some index set I. The components of $\tilde{\boldsymbol{a}}$ with indices in $J = \{1, \ldots, d\} \setminus I$ can be expressed in terms of $\boldsymbol{a}_I$. The next result shows the solution of $\Pi_{\Sigma(T)}(\mathbf{1}_\eta)$ for our model.

Lemma 4.1. *The unique solution of* $\Pi_{\Sigma(T)}(\mathbf{1}_\eta)$ *satisfies*

$$\tilde{\boldsymbol{a}} = \mathbf{1}_\eta, \qquad I = \{1, \ldots, P_\eta\}, \qquad J = \varnothing.$$

The proof of Lemma 4.1 is given in Appendix.
Let $\delta_u(L) = Lu^{-2}$ for $L > 0$ and denote into two

$$\begin{aligned}
M(u, L) &:= \mathbb{P}\left\{\exists\, t \in [T - \delta_u(L), T]\colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}, \\
m(u, L) &:= \mathbb{P}\left\{\exists\, t \in [0, T - \delta_u(L)]\colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}.
\end{aligned}$$

Clearly,

$$M(u, L) \leq \mathbb{P}\left\{\exists\, t \in [0, T]\colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\} \leq M(u, L) + m(u, L).$$

We will prove that $m(u, L)$ is negligible with respect to $M(u, L)$.

Lemma 4.2. *For fixed $L > 0$, as $u \to \infty$*

$$M(u, L) \sim H(L)\, \mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\},$$

where $\boldsymbol{\lambda} = \Sigma^{-1}(T)\mathbf{1}_\eta$, and the constant $H(L)$ is given by

$$H(L) = e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}} \int_{\mathbb{R}^{P_\eta}} \mathbb{P}\left\{\exists\, t \in [0, L] \colon \boldsymbol{\lambda}\boldsymbol{B}^*(t) > \boldsymbol{x}\right\} e^{\sum_{i=1}^{P_\eta} x_i} \mathrm{d}\boldsymbol{x}, \tag{16}$$

where $\boldsymbol{B}^*(t) \in \mathbb{R}^{P_\eta}$ is a Brownian motion with independent components.

Remark 4.3. *In view of Lemma 4.1 and Lemma 5.1 (see Appendix) it follows that $\boldsymbol{\lambda} > \mathbf{0}$.*

Proof of Lemma 4.2. Let φ_t denote the pdf of $\boldsymbol{W}(t)$. We can assume that u is large enough to ensure that $i(T - \delta_u(L)) = i(T)$. Then,

$$M(u, L) = u^{-P_\eta} \int_{\mathbb{R}^{P_\eta}} J(u, T, L, \boldsymbol{x})\, \varphi_{T-\delta_u(L)} \left(u\, \mathbf{1}_\eta - \frac{\boldsymbol{x}}{u}\right) d\boldsymbol{x}, \tag{17}$$

where

$$J(u, T, L, \boldsymbol{x}) \coloneqq \mathbb{P}\left\{\exists\, t \in [T - \delta_u(L), T] \colon \boldsymbol{W}(t) > u\, \mathbf{1}_\eta \mid \boldsymbol{W}(T - \delta_u(L)) = u\, \mathbf{1}_\eta - \frac{\boldsymbol{x}}{u}\right\}.$$

Next, using Proposition 2.2 we can rewrite the probability $J(u, T, L, \boldsymbol{x})$ as follows

$$\begin{aligned}
J(u, T, L, \boldsymbol{x}) &= \mathbb{P}\left\{\exists\, t \in [T - \delta_u(L), T] \colon \boldsymbol{W}(t) - \boldsymbol{W}(T - \delta_u(L)) > \frac{\boldsymbol{x}}{u}\right\} \\
&= \mathbb{P}\left\{\exists\, t \in [T - \delta_u(L), T] \colon \boldsymbol{B}^*(t) - \boldsymbol{B}^*(T - \delta_u(L)) > \frac{\boldsymbol{x}}{u} + \boldsymbol{c}(t - T + \delta_u(L))\right\} \\
&= \mathbb{P}\left\{\exists\, t \in [0, \delta_u(L)] \colon \boldsymbol{B}^*(t) > \frac{\boldsymbol{x}}{u} + \boldsymbol{c}\, t\right\} \\
&= \mathbb{P}\left\{\exists\, t \in [0, L] \colon \boldsymbol{B}^*(t) > \boldsymbol{x} + \frac{\boldsymbol{c}\, t}{u}\right\},
\end{aligned}$$

where $\boldsymbol{B}^*(t) \in \mathbb{R}^{P_\eta}$ is a Brownian motion with independent components. We have

$$\mathbb{P}\left\{\exists\, t \in [0, L] \colon \boldsymbol{B}^*(t) > \boldsymbol{x} + \boldsymbol{c}\, t u^{-1}\right\} \xrightarrow[u\to\infty]{} \mathbb{P}\left\{\exists\, t \in [0, L] \colon \boldsymbol{B}^*(t) > \boldsymbol{x}\right\}$$

for almost all $\boldsymbol{x} \in \mathbb{R}^{P_\eta}$.
Next, let us consider the factor $\varphi_{T-\delta_u(L)}$. We have

$$\varphi_{T-\delta_u(L)} \left(u\, \mathbf{1}_\eta - \frac{\boldsymbol{x}}{u}\right) = (2\pi)^{-P_\eta/2}\, |\Sigma(T - \delta_u(L))|^{-1/2} \exp(-G(u, T, L, \boldsymbol{x})),$$

where

$$G(u, T, L, \boldsymbol{x}) \coloneqq -\frac{1}{2}\left(u\, \mathbf{1}_\eta - \frac{\boldsymbol{x}}{u} + \boldsymbol{c}(T - \delta_u(L))\right)^\top \times \Sigma^{-1}(T - \delta_u(L)) \left(u\, \mathbf{1}_\eta - \frac{\boldsymbol{x}}{u} + \boldsymbol{c}(T - \delta_u(L))\right).$$

Using

$$\Sigma^{-1}(T - \delta_u(L)) = \Sigma^{-1}(T) - \delta_u(L)\, \Sigma^{-2}(T) + o\left(\delta_u^2(L)\right),$$

we find that the prefactor converges to $(2\pi)^{-P_\eta/2}\, |\Sigma(T)|^{-1/2}$, and we can focus on the asymptotics of the exponent. We have

$$B(u, T, L, \boldsymbol{x}) = \frac{1}{2}\left(u\, \mathbf{1}_\eta - \boldsymbol{c}T\right)^\top \Sigma^{-1}(T)\left(u\, \mathbf{1}_\eta + \boldsymbol{c}\, T\right) + \boldsymbol{x}^\top\Sigma^{-1}(T)\, \mathbf{1}_\eta - \frac{L}{2}\, \mathbf{1}_\eta^\top\Sigma^{-2}(T)\mathbf{1}_\eta + O(L/u),$$

where we used Proposition 2.3. Hence, as $u \to \infty$,

$$\varphi_{T-\delta_u(L)}(u\mathbf{1}_\eta - \boldsymbol{x}/u) \sim \varphi_T(u\mathbf{1}_\eta)\, e^{\boldsymbol{x}^\top\boldsymbol{\lambda}}\, e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}.$$

Finally, we can bound the pdf under the integral (17), for all large enough u, as follows:

$$\frac{\varphi_{T-\delta_u(L)}(u\mathbf{1}_\eta - \boldsymbol{x}/u)}{\varphi_T(u\mathbf{1}_\eta)} \le A \exp\left(\boldsymbol{x}^\top\boldsymbol{\lambda}_{\boldsymbol{x}}(\varepsilon)\right) \eqqcolon \bar{\varphi}(\boldsymbol{x}), \tag{18}$$

where $\boldsymbol{\lambda}_{\boldsymbol{x}}(\varepsilon) := \boldsymbol{\lambda} + \operatorname{sign}(\boldsymbol{x})\, \varepsilon > \mathbf{0}$ for all small enough ε and

$$A := \max_{t \in [(\tau_\eta+T)/2, T]} \sqrt{\frac{|\Sigma(T)|}{|\Sigma(t)|}} \times \max_{\substack{t \in [(\tau_\eta+T)/2, T] \\ \boldsymbol{y}, \boldsymbol{z} \in [-\mathbf{1}, \mathbf{1}] \subset \mathbb{R}^{P_\eta}}} \exp\Big(\boldsymbol{y}^\top \Sigma^{-1}(t) (\mathbf{1}_\eta + \boldsymbol{z}) \Big) < \infty$$

In proving (18) we used the fact that both $\boldsymbol{c} L u^{-1}$ and $\boldsymbol{c} T u^{-1}$ belong to $\in [-\mathbf{1}, \mathbf{1}]$ for large enough u. The constant A is clearly finite since $\Sigma(t)$ is continuous.
To find the asymptotics of $M(u, L)$, we apply dominated convergence theorem. The probability under the integral $J(u, T, L, \boldsymbol{x})$ can bounded as follows. For $\boldsymbol{x} \in \mathbb{R}^{P_\eta}$, define

$$F_+(\boldsymbol{x}) := \{ i \colon x_i > 0 \},$$

and two associated sets

$$\mathcal{S}_F := \big\{ \boldsymbol{x} \in \mathbb{R}^{P_\eta} \colon F_+(\boldsymbol{x}) = F \big\}, \qquad \mathcal{C} := \left\{ \boldsymbol{x} \in \mathbb{R}^d \colon \sum_{i \in F_+(\boldsymbol{x})} (x_i + \varepsilon) < 0 \right\}.$$

By Piterbarg inequality (see, e.g., [19, Theorem 8.1]) for each $\boldsymbol{x} \in \mathcal{C}^c$ we have

$$\begin{aligned} J(u, T, L, \boldsymbol{x}) &\le \mathbb{P}\left\{ \exists\, t \in [0, L]\ \forall i \in F \colon B_i^*(t) > x_i + \varepsilon \right\} \\ &\le \mathbb{P}\left\{ \exists\, t \in [0, L] \colon \sum_{i \in F_+(\boldsymbol{x})} B_i^*(t) > \sum_{i \in F_+(\boldsymbol{x})} (x_i + \varepsilon) \right\} \\ &\le C \left(\sum_{i \in F_+(\boldsymbol{x})} (x_i + \varepsilon) \right)^{\gamma} \exp\left(-\delta \sum_{i \in F_+(\boldsymbol{x})} (x_i + \varepsilon)^2 \right) =: \bar{R}(\boldsymbol{x}) \end{aligned}$$

for some positive constants C, γ, δ and ε independent of $\boldsymbol{x}$. Thus, it is enough to show that the integral $\int_{\mathbb{R}^{P_\eta}} \bar{R}(\boldsymbol{x})\, \bar{\varphi}(\boldsymbol{x}) dx$ is finite. Setting

$$\mathcal{C}_F := \left\{ \boldsymbol{x} \in \mathbb{R}^{P_\eta} \colon \boldsymbol{x} > \mathbf{0},\ \sum_{i \in F} (x_i + c) < 0 \right\},$$

we obtain

$$\begin{aligned} \int_{\mathbb{R}^{P_\eta}} \bar{R}(\boldsymbol{x})\, \bar{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} \ &= \ CA \sum_{F \subset \{1, \dots, P_\eta\}} \left[\int_{\mathcal{S}_F \setminus \mathcal{C}} \left(\sum_{i \in F} (x_i + \varepsilon) \right)^{\gamma} \exp\left(\boldsymbol{x}^\top \boldsymbol{\lambda}_{\boldsymbol{x}}(\varepsilon) - \delta \sum_{i \in F} (x_i + \varepsilon)^2 \right) d\boldsymbol{x} \right. \\ &\qquad \left. + \int_{\mathcal{S}_F \cap \mathcal{C}} \exp\Big(\boldsymbol{x}^\top \boldsymbol{\lambda}_{\boldsymbol{x}}(\varepsilon) \Big)\, d\boldsymbol{x} \right] =: CA \sum_{F \subset \{1, \dots, P_\eta\}} \Big[A_1 + A_2 \Big]. \end{aligned}$$

The integral A_1 may be estimated as follows:

$$\begin{aligned} A_1 \le \exp&\left(-\frac{1}{4\delta}\, \|\boldsymbol{\lambda}_F + \varepsilon\, \mathbf{1}_F\|_2^2 - \varepsilon\, \mathbf{1}_F^\top (\boldsymbol{\lambda}_F + \varepsilon\, \mathbf{1}_F) \right) \\ &\times \int_{\{\boldsymbol{x}_F > \mathbf{0}\} \setminus \mathcal{C}_F} \exp\left(-\delta \left\| \boldsymbol{x}_F + \varepsilon\, \mathbf{1}_F - \frac{1}{2\delta}\, \boldsymbol{\lambda}_{\boldsymbol{x}, F}(\varepsilon) \right\|_2^2 \right) \|\boldsymbol{x}_F + \varepsilon\, \mathbf{1}_F\|_1^{\gamma}\, d\boldsymbol{x}_F \\ &\times \int_{\boldsymbol{x}_{F^c} \le \mathbf{0}_{F^c}} \exp\Big(\boldsymbol{x}_{F^c}^\top \boldsymbol{\lambda}_{\boldsymbol{x}, F^c}(\varepsilon) \Big)\, d\boldsymbol{x}_{F^c}, \end{aligned}$$

where $\|\cdot\|_p$ denotes the ℓ_p norm. Next, we bound A_2 as follows:

$$A_2 \le \int_{\boldsymbol{x}_F^c \le \mathbf{0}_F^c} \exp\Big(\boldsymbol{x}_{F^c}^\top\, \boldsymbol{\lambda}_{\boldsymbol{x}, F^c}(\varepsilon) \Big)\, d\boldsymbol{x} \times \int_{\mathcal{C}_F} d\boldsymbol{x}_F\, .$$

Combining the two bounds together, we obtain

$$\int_{\mathbb{R}^{P_\eta}} \bar{R}(\boldsymbol{x})\,\bar{\varphi}(\boldsymbol{x})\,d\boldsymbol{x} \le CA \sum_{S\subset\{1,\dots,P_\eta\}} \prod_{i\in F^c} (\lambda_i+\varepsilon)^{-1} \exp\left(-\frac{1}{4\delta}\,\|\boldsymbol{\lambda}_F+\varepsilon\,\mathbf{1}_F\|_2^2 - \varepsilon\,\mathbf{1}_F^\top(\boldsymbol{\lambda}_F+\varepsilon\,\mathbf{1}_F)\right) A_3(F),$$

where

$$A_3(F) := \int_{\mathbb{R}^{|F|}} \exp\left(-\delta\left\|\boldsymbol{x}_F+\varepsilon\,\mathbf{1}_F-\frac{1}{2\delta}\,\boldsymbol{\lambda}_{\boldsymbol{x},F}(\varepsilon)\right\|_2^2\right)\|\boldsymbol{x}_F+\varepsilon\,\mathbf{1}_F\|_1^\gamma\,d\boldsymbol{x}_F+\frac{\varepsilon^{|S|}}{|S|!}<\infty$$

for all $\delta>0$.
Hence, by the dominated convergence theorem, it follows that as $u\to\infty$

$$\begin{aligned} M(u,L) &\sim u^{-P_\eta}\,\varphi_T(u\mathbf{1}_\eta)\,e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}\int_{\mathbb{R}^{P_\eta}} \mathbb{P}\left\{\exists\, t\in[0,L]:\ \boldsymbol{B}^*(t)>\boldsymbol{x}\right\}\,e^{-\boldsymbol{x}^\top\boldsymbol{\lambda}}\mathrm{d}\boldsymbol{x} \\ &= \frac{\varphi_T(u\mathbf{1}_\eta)\,e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}}{\prod_{i=1}^{P_\eta}\lambda_i}\,u^{-P_\eta}\int_{\mathbb{R}^{P_\eta}}\mathbb{P}\left\{\exists\, t\in[0,L]\colon \boldsymbol{\lambda}\boldsymbol{B}^*(t)>\boldsymbol{x}\right\}e^{-\sum_{i=1}^{P_\eta}x_i}\,\mathrm{d}\boldsymbol{x}, \end{aligned}\tag{19}$$

which, together with the following asymptotic formula (see [21, Lemma 4.4])

$$\mathbb{P}\left\{\boldsymbol{W}(T)>u\mathbf{1}_\eta\right\}\sim\frac{u^{-P_\eta}}{\prod_{i=1}^{P_\eta}\lambda_i}\,\varphi_T(u\mathbf{1}_\eta),\qquad u\to\infty,\tag{20}$$

provides us the claimed assertion.

□

Lemma 4.4. *For $H(L)$ defined in (16) we have*

$$\mathcal{H}_{P_\eta,1/\mu_0(T)}=\lim_{L\to\infty}H(L)\in(0,\infty).$$

Proof of Lemma 4.4. Applying the Fubini-Tonelli theorem yields

$$\begin{aligned} H(L) &= e^{-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}\int_{\mathbb{R}^{P_\eta}}\mathbb{P}\left\{\exists\ t\in[0,L]:\ \boldsymbol{\lambda}\boldsymbol{B}_\eta^*(t)>\boldsymbol{x}\right\}e^{\sum_{i=1}^{P_\eta}x_i}\mathrm{d}\boldsymbol{x} \\ &= \mathbb{E}\left\{e^{-\boldsymbol{\lambda}^\top\boldsymbol{B}_\eta^*(L)-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}+\boldsymbol{\lambda}^\top\boldsymbol{B}_\eta^*(L)}\int_{\mathbb{R}^{P_\eta}}\mathbb{I}(\exists\, t\in[0,L]\colon \boldsymbol{\lambda}\boldsymbol{B}_\eta^*(t)>\boldsymbol{x})e^{\sum_{i=1}^{P_\eta}x_i}\mathrm{d}\boldsymbol{x}\right\} \\ &= \mathbb{E}\left\{e^{-\boldsymbol{\lambda}^\top\boldsymbol{B}_\eta^*(L)-\frac{L}{2}\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}\int_{\mathbb{R}^{P_\eta}}\mathbb{I}(\exists\, t\in[0,L]\colon \boldsymbol{\lambda}\left(\boldsymbol{B}_\eta^*(t)-\boldsymbol{B}_\eta^*(L)\right)>\boldsymbol{x})e^{\sum_{i=1}^{P_\eta}x_i}\mathrm{d}\boldsymbol{x}\right\} \\ &= \mathbb{E}\left\{\int_{\mathbb{R}^{P_\eta}}\mathbb{I}(\exists\, t\in[0,L]\colon \boldsymbol{\lambda}\left(\boldsymbol{B}_\eta^*(t)-\boldsymbol{B}_\eta^*(L)\right)-\boldsymbol{\lambda}^2(t-L)>\boldsymbol{x})e^{\sum_{i=1}^{P_\eta}\boldsymbol{x}}\mathrm{d}\boldsymbol{x}\right\} \\ &= \int_{\mathbb{R}^{P_\eta}}\mathbb{P}\left\{\exists\, t\in[0,L]\colon \boldsymbol{\lambda}\boldsymbol{B}_\eta^*(t)-\boldsymbol{\lambda}^2 t>\boldsymbol{x}\right\}e^{\sum_{i=1}^{P_\eta}x_i}\mathrm{d}\boldsymbol{x}, \end{aligned}$$

where the second to last step follows from [24, Lem B.6] and the last is a direct consequence of the Brownian motion increments' properties. Hence, $H(L)$ is an increasing function. Using the following inequality, which follows from Proposition 3.3,

$$(H(L)-\varepsilon)\mathbb{P}\left\{\boldsymbol{B}_\Gamma(T_1)>u\mathbf{1}_{i(T_1)}\right\}\le M(u,L)\le\mathbb{P}\left\{\exists\, t\in[0,T]:\ \boldsymbol{B}_\Gamma(t)>u\mathbf{1}_{i(t)}\right\}\le 2^{P_\eta}\,\mathbb{P}\left\{\boldsymbol{B}_\Gamma(T_1)>u\mathbf{1}_{i(T_1)}\right\}$$

for any small positive ε and large enough u, we obtain that

$$H(L)\le 2^{P_\eta}+\varepsilon<\infty$$

implies that $H(L)$ is bounded. In order to complete the proof it remains to apply the monotone convergence theorem and notice that according to Proposition 2.4

$$\boldsymbol{\lambda}=\frac{1}{\mu_0(t)}\mathbf{1}_\eta.$$

□

Lemma 4.5. *For $L > 0$ and large enough u,*

$$\frac{m(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \le C_1 e^{-C_2 L}$$

for some positive constants C_1, C_2, which do not depend on u or L.

Proof of Lemma 4.5. Applying Proposition 3.3

$$\frac{m(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\,\mathbf{1}_\eta\right\}} \le C\,\frac{\mathbb{P}\left\{\boldsymbol{W}(T-\delta_u(L)) > u\,\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\,\mathbf{1}_\eta\right\}} = C\,\frac{\mathbb{P}\left\{\boldsymbol{W}(T) > \boldsymbol{a}(u,L,T)\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\,\mathbf{1}_\eta\right\}},$$

where C defined in Proposition 3.3 and

$$\boldsymbol{a}(u,L,T) \coloneqq \sqrt{\frac{T}{T-\delta_u(L)}}\Big(u\,\mathbf{1}_\eta + \boldsymbol{c}(T-\delta_u(L))\Big) - \boldsymbol{c}T.$$

Hence, regarding (20), for any $\varepsilon > 0$ and large enough u

$$\begin{aligned}\frac{m(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\,\mathbf{1}_\eta\right\}} &\le C\,(1+\varepsilon)\left(\frac{T}{T-\delta_u(L)}\right)^{P_\eta/2}\frac{\varphi_T\left(\boldsymbol{a}(u,L,T)\right)}{\varphi_T\left(u\mathbf{1}_\eta\right)} \\ &\le C\,(1+\varepsilon)^{1+P_\eta/2}\,\frac{\varphi_T\left(\boldsymbol{a}(u,L,T)\right)}{\varphi_T\left(u\mathbf{1}_\eta\right)}.\end{aligned} \tag{21}$$

It remains to bound the quotient $\varphi_T(\boldsymbol{a}(u,L,T))/\varphi_T(u\mathbf{1}_\eta)$. To this end, let us first find the asymptotics of $\boldsymbol{a}(u,L,T)$, as $u\to\infty$

$$\begin{aligned}\boldsymbol{a}(u,L,T) &= u\mathbf{1}_\eta\left(1-\frac{\delta_u(L)}{T}\right)^{-1/2} - \boldsymbol{c}T\left(1-\left(1-\frac{\delta_u(L)}{T}\right)^{1/2}\right) \\ &\sim u\mathbf{1}_\eta\left(1+\frac{L}{2Tu}\right) + O(Lu^{-2}).\end{aligned}$$

In this computation we used that $\delta_u(L) = Lu^{-2}$. Therefore,

$$\begin{aligned}\Big(\boldsymbol{a}(u,L,T)+\boldsymbol{c}T\Big)^\top\Sigma^{-1}(T)\Big(\boldsymbol{a}(u,L,T)+\boldsymbol{c}T\Big) \;&=\; \left(u\mathbf{1}_\eta+\boldsymbol{c}T\right)^\top\Sigma^{-1}(T)\left(u\mathbf{1}_\eta+\boldsymbol{c}T\right) \\ &\quad +\frac{L}{T}\,\mathbf{1}_\eta^\top\Sigma^{-1}(T)\mathbf{1}_\eta + O(Lu^{-1}),\end{aligned}$$

which yields

$$\frac{\varphi_T(\boldsymbol{a}(u,L,T))}{\varphi_T(u\mathbf{1}_\eta)} \le C\,\exp\left(\frac{-L}{2T}\,\mathbf{1}_\eta^\top\Sigma^{-1}\mathbf{1}_\eta\right).$$

Combining this with (21), we obtain the desired result. ☐

We can now proceed to the proof of Theorem 3.4.

Proof of Theorem 3.4. Using Lemma 4.5, we obtain that for any positive L and large enough u

$$\begin{aligned}\frac{M(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} &\le \frac{\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \\ &\le \frac{M(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} + \frac{m(u,L)}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}},\end{aligned}$$

where the last term on the right is at most $C_1\exp(-C_2L)$ with some positive constants C_1 and C_2. Hence, letting $u\to\infty$ and using Lemma 4.2 we obtain that

$$\begin{aligned}H(L) &\le \liminf_{u\to\infty}\frac{\mathbb{P}\left\{\exists t\in[0,T]:\ \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \\ &\le \limsup_{u\to\infty}\frac{\mathbb{P}\left\{\exists t\in[0,T]:\ \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \le H(L) + C_1e^{-C_2L}.\end{aligned}$$

Pushing further $L \to \infty$ and applying Lemma 4.4 we obtain

$$\mathcal{H}_{P_\eta,1/\mu_0(T)} \le \liminf_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \le \limsup_{u\to\infty} \frac{\mathbb{P}\left\{\exists t \in [0,T]:\ \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta\right\}} \le \mathcal{H}_{P_\eta,1/\mu_0(T)}.$$

Hence, the claim follows. □

Proof of Corollary 3.6. Let us define $N_i^* = \operatorname{essinf}(N_i)$ and take $\tilde{\boldsymbol{N}}_1, \tilde{\boldsymbol{N}}_2 \in \mathbb{N}^\eta$ such that $\tilde{\boldsymbol{N}}_1 \ge \tilde{\boldsymbol{N}}_2$ and $\tilde{\boldsymbol{N}}_1 \ne \tilde{\boldsymbol{N}}_2$. Then we can show the following inequality

$$\mathbb{P}\left\{\exists t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \;\middle|\; \boldsymbol{N} = \tilde{\boldsymbol{N}}_1\right\} \le \mathbb{P}\left\{\exists t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \;\middle|\; \boldsymbol{N} = \tilde{\boldsymbol{N}}_2\right\}. \tag{22}$$

To show (22) it is enough to consider $\tilde{\boldsymbol{N}}_1, \tilde{\boldsymbol{N}}_2$ such that

$$\tilde{\boldsymbol{N}}_1 - \tilde{\boldsymbol{N}}_2 = \boldsymbol{e}_j$$

for some $j \in \{1, \dots, \eta\}$, where $\boldsymbol{e}_j$ is the j-th basis vector in $\mathbb{N}^\eta$. It is easy to see that the process $\boldsymbol{W}(t) \mid \boldsymbol{N} = \tilde{\boldsymbol{N}}_2$ may be obtained from $\boldsymbol{W}(t) \mid \boldsymbol{N} = \tilde{\boldsymbol{N}}_1$ by deleting some branches. Thus, the inequality (22) follows.

Hence,

$$\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{N} = \boldsymbol{N}^*\right\} \mathbb{P}\left\{\exists t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^*\right\} \\
&\qquad \le \mathbb{P}\left\{\exists t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta\right\} \\
&\qquad \le \mathbb{P}\left\{\boldsymbol{N} = \boldsymbol{N}^*\right\} \mathbb{P}\left\{\exists t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^*\right\} \\
&\qquad\quad + \sum_{i=1}^{\eta} \mathbb{P}\left\{\boldsymbol{N} \ge \boldsymbol{N}^* + \boldsymbol{e}_i\right\} \mathbb{P}\left\{\exists t \in [0,T] \colon \boldsymbol{W}(t) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i\right\},
\end{aligned}$$

and the claim follows from Theorem 3.4 and the fact that

$$\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i\right\} = o\left(\mathbb{P}\left\{\boldsymbol{W}(T) > u\mathbf{1}_\eta \mid \boldsymbol{N} = \boldsymbol{N}^*\right\}\right).$$

To show the latter, we use Proposition 2.5 as follows. Combining

$$(P_\eta \mid \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i) = \left(1 + \frac{1}{N_i^*}\right)(P_\eta \mid \boldsymbol{N} = \boldsymbol{N}^*),$$

with

$$\begin{aligned}
(\mu_0(T) \mid \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i) &= (\mu_0(T) \mid \boldsymbol{N} = \boldsymbol{N}^*) + \sum_{l=1}^{i} (\tau_l - \tau_{l-1}) \prod_{\substack{j=l \\ j\ne i}}^{\eta} N_j \\
&= (\mu_0(T) \mid \boldsymbol{N} = \boldsymbol{N}^*) + \frac{1}{N_i} \sum_{l=1}^{i} (\tau_l - \tau_{l-1}) \prod_{j=l}^{\eta} N_j \\
&\le \left(1 + \frac{1}{N_i}\right)(\mu_0(T) \mid \boldsymbol{N} = \boldsymbol{N}^*),
\end{aligned}$$

we obtain

$$\left(\frac{P_\eta}{\mu_0(T)} \;\middle|\; \boldsymbol{N} = \boldsymbol{N}^* + \boldsymbol{e}_i\right) \le \left(\frac{P_\eta}{\mu_0(T)} \;\middle|\; \boldsymbol{N} = \boldsymbol{N}^*\right),$$

and invoke Proposition 2.5. □

Proof of Corollary 3.7. Using Theorem 3.4, we obtain

$$\begin{aligned}
\mathbb{P}\left\{u^2(T - \mathcal{T}(u)) \ge x \;\middle|\; \mathcal{T}(u) \le T - \frac{y}{u^2}\right\} &= \frac{\mathbb{P}\left\{\mathcal{T}(u) \le T - xu^{-2}\right\}}{\mathbb{P}\left\{\mathcal{T}(u) \le T - yu^{-2}\right\}} \\
&\sim \frac{\mathcal{H}_{P_\eta,1/\mu_0(T-yu^{-2})}\, \mathbb{P}\left\{\boldsymbol{W}(T - xu^{-2}) > u\,\mathbf{1}_\eta\right\}}{\mathcal{H}_{P_\eta,1/\mu_0(T-xu^{-2})}\, \mathbb{P}\left\{\boldsymbol{W}(T - yu^{-2}) > u\,\mathbf{1}_\eta\right\}} \sim \frac{\mathbb{P}\left\{\boldsymbol{W}(T - xu^{-2}) > u\,\mathbf{1}_\eta\right\}}{\mathbb{P}\left\{\boldsymbol{W}(T - yu^{-2}) > u\,\mathbf{1}_\eta\right\}}.
\end{aligned}$$

Applying Proposition 2.5, we find that the latter is asymptotically equivalent to

$$\frac{\mu_0^{P_\eta-1/2}(T-yu^{-2})\prod_{v=1}^{\eta}\mu_v^{(P_v-P_{v-1})/2}(T-xu^{-2})}{\mu_0^{P_\eta-1/2}(T-yu^{-2})\prod_{v=1}^{\eta}\mu_v^{(P_v-P_{v-1})/2}(T-yu^{-2})}\exp\left(-\frac{(u+c(T-xu^{-2}))^2P_\eta}{2\mu_0(T-xu^{-2})}+\frac{(u+c(T-yu^{-2}))^2P_\eta}{2\mu_0(T-yu^{-2})}\right).$$

Since the pre-exponential factor clearly tends to 1, it remains to compute the asymptotics of the exponent. We have

$$\begin{aligned}
&-\frac{(u+c(T-xu^{-2}))^2P_\eta}{2\mu_0(T-xu^{-2})}+\frac{(u+c(T-yu^{-2}))^2P_\eta}{2\mu_0(T-yu^{-2})}\\
&\qquad\qquad\sim-\frac{(u+cT)^2P_\eta}{2\mu_0(T-xu^{-2})\mu_0(T-yu^{-2})}\Big[\mu_0(T-yu^{-2})-\mu_0(T-xu^{-2})\Big].
\end{aligned}$$

It thus remains to note that

$$\mu_0(T-yu^{-2})-\mu_0(T-xu^{-2})\sim u^{-2}(x-y)\quad\text{and}\quad \mu_0(T-xu^{-2}),\ \mu_0(T-yn^{-2})\sim\mu_0(T)$$

to conclude the proof. $\square$

4.5. **Proof of Theorem 3.10.** We note that

$$\begin{aligned}
\mathbb{P}\left\{\exists\, i\in\{1,\dots,M\},\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\} \ \geq\ &\sum_{i=1}^{M}\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}\\
&-\sum_{\substack{i,j=1\\ i\neq j}}^{M}\mathbb{P}\begin{Bmatrix}\exists\, t_1\in[0,T]\colon \boldsymbol{W}_i(t_1)>u\mathbf{1}_{\eta_i}\\ \exists\, t_2\in[0,T]\colon \boldsymbol{W}_j(t_2)>u\mathbf{1}_{\eta_j}\end{Bmatrix}
\end{aligned}$$

and

$$\mathbb{P}\left\{\exists\, i\in\{1,\dots,M\},\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}\leq\sum_{i=1}^{M}\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}.$$

Using that $\boldsymbol{W}_i$ are independent,

$$\begin{aligned}
&\mathbb{P}\left\{\exists\, t_1,\, t_2\in[0,T]\colon \boldsymbol{W}_i(t_1)>u\mathbf{1}_{\eta_i},\ \boldsymbol{W}_j(t_2)>u\mathbf{1}_{\eta_j}\right\}\\
&\quad=\ \mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_j(t)>u\mathbf{1}_{\eta_j}\right\}\\
&\qquad+o\left(\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}+\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_j(t)>u\mathbf{1}_{\eta_j}\right\}\right).
\end{aligned}$$

We find

$$\mathbb{P}\left\{\exists\, i\in\{1,\dots,M\},\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}\sim\sum_{i=1}^{M}\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}.$$

Finally, Lemma 3.9 implies that

$$\sum_{i\notin A}\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}=o\left(\sum_{i\in A}\mathbb{P}\left\{\exists\, t\in[0,T]\colon \boldsymbol{W}_i(t)>u\mathbf{1}_{\eta_i}\right\}\right),$$

establishing the proof. $\square$

5. Appendix

Proof of Proposition 2.1. Using mathematical induction, we shall show that for all $i\in\{1,\dots,N\}$ holds

$$B_\gamma(\tau_i)=B_0^*(\tau_1)+\sum_{j=1}^{i-1}(\boldsymbol{B}_j^*)_{(\gamma \bmod P_j)+1}(\tau_{j+1}-\tau_j). \tag{23}$$

Since for $i=1$ the claim is clear, assuming it holds for $i=k$, we have for $i=k+1$

$$B_\gamma(\tau_{k+1})=\big(\widetilde{\boldsymbol{B}}_\Gamma\big)_{(\gamma \bmod P_k)+1}(\tau_{k+1}) \tag{24}$$

$$= \begin{pmatrix} \widetilde{\boldsymbol{B}}_\Gamma(\tau_k) \\ \vdots \\ \widetilde{\boldsymbol{B}}_\Gamma(\tau_k) \end{pmatrix}_{(\gamma \bmod P_k)+1} + \left(\boldsymbol{B}_k^*\right)_{(\gamma \bmod P_k)+1}(\tau_{k+1} - \tau_k). \tag{25}$$

For the first term we have

$$\begin{aligned} \begin{pmatrix} \widetilde{\boldsymbol{B}}_\Gamma(\tau_k) \\ \vdots \\ \widetilde{\boldsymbol{B}}_\Gamma(\tau_k) \end{pmatrix}_{(\gamma \bmod P_k)+1} &= \left(\widetilde{\boldsymbol{B}}_\Gamma(\tau_k)\right)_{\left((\gamma_k \bmod P_k) \bmod P_{k-1}\right)+1} \\ &= \left(\widetilde{\boldsymbol{B}}_\Gamma(\tau_k)\right)_{(\gamma_k \bmod P_{k-1})+1} = B_\gamma(\tau_k). \end{aligned}$$

By the induction hypothesis,

$$B_\gamma(\tau_k) = B_0^*(\tau_1) + \sum_{j=1}^{k-1} \left(\boldsymbol{B}_j^*\right)_{(\gamma \bmod P_j)+1}(\tau_{j+1} - \tau_j). \tag{26}$$

Subsume the second term of (24) into the sum (26) and note that this concludes the proof of this assertion for $t = \tau_{k+1}$. Similarly, if $t \in (\tau_1, T]$, then

$$B_\gamma(t) = \begin{pmatrix} \widetilde{\boldsymbol{B}}_\Gamma(\tau_{i(t)}) \\ \vdots \\ \widetilde{\boldsymbol{B}}_\Gamma(\tau_{i(t)}) \end{pmatrix}_{(\gamma \bmod P_{i(t)})+1} + \left(\boldsymbol{B}_k^*\right)_{(\gamma \bmod P_{i(t)})+1}(t - \tau_{i(t)}),$$

and therefore

$$B_\gamma(t) = B_0^*(\tau_1) + \sum_{j=1}^{i(t)-1} \left(\boldsymbol{B}_j^*\right)_{(\gamma \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \left(\boldsymbol{B}_{i(t)}^*\right)_{(\gamma \bmod P_{i(t)})+1}(t - \tau_{i(t)}).$$

Thus, the claim (5) follows, justifying that B_γ is a Brownian motion.

Consider now the assertion (6). Let $t_1 \le \tau_{\kappa(\gamma_1,\gamma_2)}$. Using that $B_{\gamma_1}(t) = B_{\gamma_2}(t)$ for any $t \le \tau_{\kappa(\gamma_1,\gamma_2)}$, we have

$$B_{\gamma_1}(t_1) = B_{\gamma_2}(t_1).$$

Since B_{γ_2} is a Brownian motion, we obtain

$$\mathrm{Cov}(B_{\gamma_1}(t_1), B_{\gamma_2}(t_2)) = \mathrm{Cov}(B_{\gamma_2}(t_1), B_{\gamma_2}(t_2)) = \min\{t_1, t_2\} = \min\{t_1, t_2, \tau_{\kappa(\gamma_1,\gamma_2)}\}.$$

The same holds in the case $t_2 \le \tau_{\kappa(\gamma_1,\gamma_2)}$. In the case $\tau_{\kappa(\gamma_1,\gamma_2)} < t_1, t_2$, using (5), we can see that

$$B_{\gamma_1}(t_1) = B_0^*(\tau_1) + \sum_{j=1}^{i(t_1)-1} \left(\boldsymbol{B}_j^*\right)_{(\gamma_1 \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \left(\boldsymbol{B}_{i(t_1)}^*\right)_{(\gamma_1 \bmod P_{i(t_1)})+1}(t_1 - \tau_{i(t_1)}).$$

Next, we split the middle sum at the branches' separation point $\kappa(\gamma_1, \gamma_2)$ and use the fact that

$$B_0^*(\tau_1) + \sum_{j=1}^{\kappa(\gamma_1,\gamma_2)-1} \left(\boldsymbol{B}_j^*\right)_{(\gamma_1 \bmod P_j)+1}(\tau_{j+1} - \tau_j) = B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)})$$

to obtain

$$B_{\gamma_1}(t_1) B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)}) + \sum_{j=\kappa(\gamma_1,\gamma_2)}^{i(t_1)-1} \left(\boldsymbol{B}_j^*\right)_{(\gamma_1 \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \left(\boldsymbol{B}_{i(t_1)}^*\right)_{(\gamma_1 \bmod P_{i(t_1)})+1}(t_1 - \tau_{i(t_1)}).$$

By the same reason,

$$B_{\gamma_2}(t_2) = B_{\gamma_2}(\tau_{\kappa(\gamma_1,\gamma_2)}) + \sum_{j=\kappa(\gamma_1,\gamma_2)}^{i(t_2)-1} \left(\boldsymbol{B}_j^*\right)_{(\gamma_2 \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \left(\boldsymbol{B}_{i(t_2)}^*\right)_{(\gamma_2 \bmod P_{i(t_2)})+1}(t_2 - \tau_{i(t_2)}).$$

We have used the fact that for all indices $j \geq \kappa(\gamma_1, \gamma_2)$ holds $(\gamma_1 \bmod P_j) \neq (\gamma_2 \bmod P_j)$, so the sums

$$\sum_{j=\kappa(\gamma_1,\gamma_2)}^{i(t_1)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma_1 \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \big(\boldsymbol{B}_{i(t_1)}^*\big)_{(\gamma_1 \bmod P_{i(t_1)})+1}(t_1 - \tau_{i(t_1)})$$

and

$$\sum_{j=\kappa(\gamma_1,\gamma_2)}^{i(t_2)-1} \big(\boldsymbol{B}_j^*\big)_{(\gamma_2 \bmod P_j)+1}(\tau_{j+1} - \tau_j) + \big(\boldsymbol{B}_{i(t_2)}^*\big)_{(\gamma_2 \bmod P_{i(t_2)})+1}(t_2 - \tau_{i(t_2)})$$

are independent. Additionally, each of them is independent of $B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)})$, which is equal to $B_{\gamma_2}(\tau_{\kappa(\gamma_1,\gamma_2)})$. Hence,

$$\operatorname{Cov}(B_{\gamma_1}(t_1), B_{\gamma_2}(t_2)) = \operatorname{Cov}(B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)}), B_{\gamma_2}(\tau_{\kappa(\gamma_1,\gamma_2)})) = \operatorname{Var}(B_{\gamma_1}(\tau_{\kappa(\gamma_1,\gamma_2)})) = \tau_{\kappa(\gamma_1,\gamma_2)}.$$

This establishes (6). □

Proof of Proposition 2.4. We are going to prove the claim of the theorem by induction in $t \in \{\tau_0, \dots, \tau_\eta\}$. The case $t = \tau_0$ is clear. Assume that the claim is true for $t = \tau_k$. We claim that the eigenvalues of

$$\begin{pmatrix} \Sigma(\tau_k) & \dots & \Sigma(\tau_k) \\ \vdots & & \vdots \\ \Sigma(\tau_k) & \dots & \Sigma(\tau_k) \end{pmatrix}$$

are the eigenvalues of $\Sigma(\tau_k)$ multiplied by N_k. Moreover, they are of at least the same multiplicity. Indeed, for any eigenvalue μ and a corresponding eigenvector $\boldsymbol{v} \in \mathbb{R}^{P_{k-1}}$ of $\Sigma(\tau_k)$ we can construct a vector $\boldsymbol{v}^* = (\boldsymbol{v}^\top, \dots, \boldsymbol{v}^\top)^\top \in \mathbb{R}^{P_k}$, which satisfies

$$\begin{pmatrix} \Sigma(\tau_k) & \dots & \Sigma(\tau_k) \\ \vdots & & \vdots \\ \Sigma(\tau_k) & \dots & \Sigma(\tau_k) \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \vdots \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} N_k \Sigma(\tau_k)\boldsymbol{v} \\ \vdots \\ N_k \Sigma(\tau_k)\boldsymbol{v} \end{pmatrix} = \begin{pmatrix} N_k \mu \boldsymbol{v} \\ \vdots \\ N_k \mu \boldsymbol{v} \end{pmatrix} = N_k\, \mu\, \boldsymbol{v}^*.$$

In particular, it means that as $\mathbf{1}_{k-1}$ is an eigenvector of $\Sigma(\tau_k)$ with the eigenvalue $\mu_0(\tau_k)$, then $\mathbf{1}_k$ is an eigenvector of the matrix mentioned above with the corresponding eigenvalue $N_k\, \mu_0(\tau_k)$. Since

$$\operatorname{rank} \begin{pmatrix} \Sigma(\tau_k) & \dots & \Sigma(\tau_k) \\ \vdots & & \vdots \\ \Sigma(\tau_k) & \dots & \Sigma(\tau_k) \end{pmatrix} = \operatorname{rank} \Sigma(\tau_k) = P_{k-1},$$

0 is an eigenvalue of the matrix mentioned above of multiplicity $P_k - P_{k-1}$. Using Proposition 2.3 for any eigenvalue $\mu_v(\tau_k)$ of $\Sigma(\tau_k)$ we can find an eigenvalue $\mu_v(\tau_{k+1})$ of $\Sigma(\tau_{k+1})$ as follows

$$\begin{aligned} \mu_v(\tau_{k+1}) &= N_k \mu_v(\tau_k) + (\tau_{k+1} - \tau_k) \\ &= (\tau_{k+1} - \tau_k) + N_k \left((\tau_k - \tau_{k-1}) + \sum_{l=v+1}^{k-1} (\tau_l - \tau_{l-1}) \prod_{j=l}^{k-1} N_j \right) \\ &= (\tau_{k+1} - \tau_k) + N_k(\tau_k - \tau_{k-1}) + \sum_{l=v+1}^{k-1} (\tau_l - \tau_{l-1}) \prod_{j=l}^{k} N_j \\ &= (\tau_{k+1} - \tau_k) + \sum_{l=v+1}^{k} (\tau_l - \tau_{l-1}) \prod_{j=l}^{k} N_j. \end{aligned}$$

The multiplicity of $\mu_v(\tau_{k+1})$ equals to the multiplicity of $\mu_v(\tau_k)$, i.e. $P_v - P_{v-1}$. In particular, for $i = 0$ we have that $\mu_0(\tau_{k+1})$ has exactly one eigenvector $\mathbf{1}_k$. Additionally, $\Sigma(\tau_{k+1})$ has the eigenvalue $\tau_{k+1} - \tau_k$ with multiplicity $P_k - P_{k-1}$. Hence, the claim follows for the matrix $\Sigma(\tau_{k+1})$. Finally, using Proposition 2.3 we can find eigenvalues and eigenvectors of $\Sigma(t)$ if we know them for $\Sigma(\tau_{i(t)})$. □

Proof of Proposition 2.5. Using that $\boldsymbol{\lambda} = \Sigma^{-1}\mathbf{1}_\eta$ and Proposition 2.4 we obtain

$$\boldsymbol{\lambda} = \frac{1}{\mu_0(T)}\mathbf{1}_\eta, \qquad \mathbf{1}_\eta^\top \Sigma^{-1}(T)\mathbf{1}_\eta = \frac{P_\eta}{\mu_0(T)}.$$

Additionally, using Proposition 2.4 we know that

$$|\Sigma(T)| = \mu_0(T)\prod_{v=1}^{\eta}\mu_v^{P_v - P_{v-1}}(T).$$

Hence, the claim follows from (20):

$$\begin{aligned}\mathbb{P}\left\{\boldsymbol{B}_\Gamma(T) - cT\mathbf{1}_\eta > u\mathbf{1}_\eta\right\} &\sim \frac{1}{\prod_{i=1}^{P_\eta}\lambda_i}u^{-P_\eta}\varphi_T(u\mathbf{1}_\eta)\\ &= \frac{u^{-P_\eta}}{\prod_{i=1}^{P_\eta}\lambda_i}\frac{1}{(2\pi)^{P_\eta/2}\left|\Sigma(T)\right|^{1/2}}\exp\left(-\frac{1}{2}(u+cT)^2\mathbf{1}_\eta^\top\Sigma^{-1}(T)\mathbf{1}_\eta\right).\end{aligned}$$

Recall that $\varphi(T)$ stands for the pdf of $\boldsymbol{B}_\Gamma(T) - cT\mathbf{1}_\eta$. □

Proof of Lemma 4.1. Note that any element $\gamma \in \Gamma$ has unique representation of the following form

$$\gamma = \sum_{i=1}^{\eta} a_i P_{i-1},$$

where $a_i \in \{0, 1, \ldots, N_i - 1\}$. Hence, the map

$$\gamma \mapsto (a_1(\gamma), a_2(\gamma), \ldots, a_\eta(\gamma))$$

is a bijection. Let us introduce the following class of permutations on Γ:

$$\pi_{j,b,c}(\gamma) = \gamma', \qquad i \in \{1, 2, \ldots, \eta\}, \quad b,\, c \in \{0, 1, \ldots, N_i - 1\},$$

where for any $i \in \{1, 2, \ldots, \eta\}$

$$a_i(\gamma') = \begin{cases} a_i(\gamma), & \text{if } i \neq j,\\ a_j(\gamma), & \text{if } i = j,\ a_j(\gamma) \neq c \text{ and } a_j(\gamma) \neq b,\\ b, & \text{if } i = j \text{ and } a_j(\gamma) = c,\\ c, & \text{if } i = j \text{ and } a_j(\gamma) = b.\end{cases}$$

Next, define a new Gaussian random vector $\tilde{\boldsymbol{X}}(j,b,c) = (\tilde{X}_0, \ldots, \tilde{X}_{P_\eta - 1}) \in \mathbb{R}^{P_\eta}$

$$\tilde{X}_i = B_{\pi_{j,b,c}(i)}(T)$$

and denote its covariance function by $\Sigma(j,b,c)$. Consider a new quadratic programming problem (see Lemma 5.1)

$$\Pi_{\Sigma(j,b,c)}(\mathbf{1}_\eta).$$

Define its corresponding sets by $I(j,b,c)$ and $J(j,b,c)$. Using that our vector $\tilde{\boldsymbol{X}}(j,b,c)$ is a permutation of $\boldsymbol{B}_\Gamma(T)$,

$$I(j,b,c) = \pi_{j,b,c}(I).$$

On the other hand, from the definition of $\pi_{j,b,c}$ we have that for any $i \in \{1, \ldots, \eta\}$

$$a_i(\gamma_1) = a_i(\gamma_2) \iff a_i(\pi_{j,b,c}(\gamma_1)) = a_i(\pi_{j,b,c}(\gamma_2)).$$

Using the formula for γ in terms of $a_i(\gamma)$ we obtain that

$$\gamma_1 = \gamma_2 \bmod P_i \iff \forall\, l \in \{1, 2, \ldots, i\}\ a_l(\gamma_1) = a_l(\gamma_2).$$

Combining these two facts, we find

$$\gamma_1 = \gamma_2 \bmod P_i \iff \pi_{j,b,c}(\gamma_1) = \pi_{j,b,c}(\gamma_2) \bmod P_i,$$

which implies that

$$\kappa(\gamma_1, \gamma_2) = \kappa(\pi_{j,b,c}(\gamma_1), \pi_{j,b,c}(\gamma_2)).$$

Consequently, we have

$$\begin{aligned}\operatorname{Cov}(\tilde{X}_{\gamma_1}, \tilde{X}_{\gamma_2}) &= \operatorname{Cov}(B_{\pi_{j,b,c}(\gamma_1)}(T), B_{\pi_{j,b,c}(\gamma_2)}(T)) \\ &= \min\{T, \tau_{\kappa(\pi_{j,b,c}(\gamma_1), \pi_{j,b,c}(\gamma_2))}\} \\ &= \min\{T, \tau_{\kappa(\gamma_1,\gamma_2)}\} = \operatorname{Cov}(B_{\gamma_1}(T), B_{\gamma_2}(T)).\end{aligned}$$

Hence $\Sigma(j,b,c) = \Sigma(T)$, and

$$I(j,b,c) = I.$$

It means that for any $j \in \{1,\dots,\eta\}$, $b, c \in \{0,\dots,N_j\}$

$$\pi_{j,b,c}(I) = I.$$

Finally, let us notice that for any γ_1, $\gamma_2 \in \Gamma$ we can find a sequence of the permutations of the class mentioned above which send γ_1 into γ_2:

$$\gamma_2 = \pi_{1,a_1(\gamma_1),a_1(\gamma_2)}(\pi_{2,a_2(\gamma_1),a_2(\gamma_2)}(\cdots \pi_{\eta,a_\eta(\gamma_1),a_\eta(\gamma_2)}(\gamma_1)\cdots)).$$

This implies that set I can either be empty, or contain all elements of Γ. Since by Lemma 5.1 it cannot be empty, the claim follows. □

Below, we cite the result from [25, Lemma 4].

Lemma 5.1. *Let $d \geq 2$ and Σ a $d\times d$ symmetric positive definite matrix with inverse Σ^{-1}. If $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$, then the quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$ defined in (15) has a unique solution $\tilde{\boldsymbol{a}}$ and there exists a unique non-empty index set $I \subset \{1,\dots,d\}$ with $|I| \leq d$ elements such that*

(i) $\tilde{\boldsymbol{a}}_I = \boldsymbol{a}_I \neq \boldsymbol{0}_I$;
(ii) $\tilde{\boldsymbol{a}}_J = \Sigma_{IJ}^{-1}\Sigma_{II}^{-1}\boldsymbol{a}_I \geq \boldsymbol{a}_J$, *and* $\Sigma_{II}^{-1}\boldsymbol{a}_I > \boldsymbol{0}_I$;
(iii) $\min_{\boldsymbol{x}\geq\boldsymbol{a}} \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x} = \tilde{\boldsymbol{a}}^\top\Sigma^{-1}\tilde{\boldsymbol{a}} = \boldsymbol{a}^\top\Sigma^{-1}\tilde{\boldsymbol{a}} = \boldsymbol{a}_I^\top\Sigma_{II}^{-1}\boldsymbol{a}_I > 0$,

with $\boldsymbol{\lambda} = \Sigma^{-1}\tilde{\boldsymbol{a}}$ satisfying $\boldsymbol{\lambda}_I = \Sigma_{II}^{-1}\boldsymbol{a}_I > \boldsymbol{0}_I$, and $\boldsymbol{\lambda}_J = \boldsymbol{0}_J$.

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