

Set theory, logic, and homeomorphism groups of manifolds

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ABSTRACT. We investigate the relationship between axiomatic set theory and the first-order theory of homeomorphism groups of manifolds in the language of group theory, concentrating on first-order rigidity and type versus conjugacy. We prove that under the axiom of constructibility (i.e. $V=L$), homeomorphism groups of arbitrary connected manifolds are first-order rigid, and that the conjugacy class of a homeomorphism of a manifold is determined by its type. In contradistinction, under projective determinacy (PD), we show that in all dimensions greater than one, there exist pairs of noncompact, connected manifolds whose homeomorphism groups are elementarily equivalent but which are not homeomorphic. We also show that under PD, every manifold of positive dimension admits pairs of homeomorphisms with the same type which are not conjugate to each other. Finally, we show that infinitary sentences do determine conjugacy classes of homeomorphisms and homeomorphism types of manifolds; specifically, the conjugacy class of a homeomorphism of an arbitrary manifold is determined by a single $L_{\omega_1\omega}$ sentence. Similarly, the homeomorphism type of an arbitrary connected manifold is determined by a single $L_{\omega_1\omega}$ sentence.

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1. INTRODUCTION

In this paper, we investigate the first-order theory of $\text{Homeo}(M)$, the homeomorphism group of an arbitrary connected manifold M , and its relation to underlying axioms in set theory. In [18], M. Rubin conjectured that under the set theoretic assumption of Gödel constructibility, if M and N are arbitrary, connected, boundaryless manifolds with elementarily equivalent homeomorphism groups, then M and N are homeomorphic. In this paper, we completely resolve Rubin's conjecture and furthermore show that set theoretic axioms beyond ZFC are necessary.

Throughout this paper, M will denote an arbitrary (not necessarily compact) connected manifold of positive dimension and with empty boundary; we will assume that M is second countable and Hausdorff. We write $\text{Homeo}(M)$ for the group of all homeomorphisms of M , which will be viewed as a first-order structure in the language $\mathcal{L}$ of group theory. We adopt the convention that this language has exactly two nonlogical symbols, namely the binary multiplication operation and the identity.

The *theory* of $\text{Homeo}(M)$, denoted $\text{Th}(\text{Homeo}(M))$, consists of all first-order $\mathcal{L}$ -sentences ψ such that $\text{Homeo}(M) \models \psi$. We will say that $\text{Homeo}(M)$ is *first-order rigid* if for all manifolds N , we have that M is homeomorphic to N if and only if $\text{Th}(\text{Homeo}(M)) = \text{Th}(\text{Homeo}(N))$. When the theories agree, we also write $\text{Homeo}(M) \equiv \text{Homeo}(N)$, and say that these two groups are *elementarily equivalent*. The first-order theory of homeomorphism groups of compact manifolds was investigated in [9, 10, 12, 11].

Similarly, if $g \in \text{Homeo}(M)$, the *type* of g , written $\text{tp}(g)$, consists of all first-order formulae $\psi(v)$ in one variable v such that $\text{Homeo}(M) \models \psi(g)$.

In this paper, we will consider two mutually inconsistent extensions of Zermelo–Frankel (ZF) set theory, namely Gödel constructibility and projective determinacy; see [6, 4] for background and discussion. Roughly, Gödel constructibility, often written $V=L$, is a restriction on the scope of the power set operation, and asserts that the only sets which exist at a particular level $V_{\alpha+1}$ of Zermelo's cumulative hierarchy are ones which are definable in the language of set theory, with parameters in the preceding level V_α . Under $V=L$, the Axiom of Choice and the Continuum Hypothesis are both theorems. Importantly for us, under $V=L$, there is a projectively definable well-ordering of $\mathbb{R}$, or more generally of an analytically presented Polish space.

Projective determinacy (PD) is another extension of ZFC, which adopts the axiom that for a 2-player infinite game of perfect information on $\mathbb{N}$ with a projective winning condition, one of the two players has a winning strategy. The specific consequence of PD which we will use is that every projective subset of Cantor space $2^{\mathbb{N}}$ is Lebesgue measurable; see [7].

Our first results relate first-order rigidity of homeomorphism groups of non-compact manifolds to extensions of ZFC:

Theorem 1.1. *Suppose $V=L$ holds. Then homeomorphism groups of manifolds are first-order rigid; that is, for all pairs M, N of connected manifolds without boundary, we have $\text{Homeo}(M) \equiv \text{Homeo}(N)$ if and only if M and N are homeomorphic.*

By contrast:

Theorem 1.2. *Suppose PD holds. Then for all $n \geq 2$, there exist manifolds M and N of dimension n that are not homeomorphic, but such that $\text{Homeo}(M) \equiv \text{Homeo}(N)$.*

In [9], the second and third authors together with Kim proved that first-order rigidity holds for compact manifolds within ZFC, and so the examples furnished by Theorem 1.2 are necessarily noncompact.

Theorem 1.2 does not require producing examples in all dimensions; we will show that it is a theorem of ZFC that failure of first-order rigidity in dimension d generally causes failure of first-order rigidity in all dimensions above d :

Theorem 1.3. *Suppose M and N are manifolds of some fixed dimension d and let S^k denote the k -dimensional sphere. Then*

$$\text{Homeo}(M) \equiv \text{Homeo}(N) \quad \Rightarrow \quad \text{Homeo}(M \times S^k) \equiv \text{Homeo}(N \times S^k).$$

In particular, if there are M and N such that $\text{Homeo}(M) \equiv \text{Homeo}(N)$ and $M \times S^k \not\equiv N \times S^k$, then first-order rigidity fails in dimension $d + k$.

In Theorem 1.3, the assumption that both M and N are not homeomorphic and $M \times S^k$ and $N \times S^k$ are also not homeomorphic is not due to an idle worry; two non-homeomorphic manifolds can become homeomorphic after taking a Cartesian product with a fixed manifold. However, the choice of S^k in the statement of Theorem 1.3 is not crucial; it suffices to consider any manifold X that is analytically presented as a Polish space (see Section 2.1) such that taking a Cartesian product with X is injective on homeomorphism classes.

As is well-known and of course also follows from results above, the assumptions $V=L$ and projective determinacy are two incompatible extensions of ZF. However, $V=L$ has the same consistency strength as ZF, meaning that if ZF is consistent then so is $\text{ZF} + V=L$. On the other hand, PD does have nontrivial

consistency strength over ZFC, in the sense that, to construct models of ZFC + PD, one needs to assume the existence of large cardinals. Although PD itself has nontrivial consistency strength, the failure of first-order rigidity does not. Indeed, the latter can be obtained by forcing:

Theorem 1.4. *Every model of ZFC has a forcing extension in which, for each $n \geq 2$, there is a pair of non-homeomorphic n -manifolds with elementarily equivalent homeomorphism groups.*

Corollary 1.5. *The first-order rigidity of homeomorphism groups of manifolds is independent of ZFC.*

The above results concern the question of whether one can distinguish different manifolds by the theory of their homeomorphism groups. Similarly, one may ask if one can distinguish different homeomorphisms of the same manifold via their type. We will call a homeomorphism $g \in \text{Homeo}(M)$ *type rigid* if for all $h \in \text{Homeo}(M)$, we have $\text{tp}(g) = \text{tp}(h)$ if and only if g and h are conjugate in $\text{Homeo}(M)$.

For specific manifolds, one can often find many homeomorphisms which are type rigid, by straightforward application of the machinery in [10]; for instance, a homeomorphism h of a sphere S^1 with north–south dynamics is type rigid. In fact, the conjugacy class of h is *isolated* by a single formula. That is, there is a formula $\phi(v)$ such that $\phi(h)$ holds, and any other homeomorphism of S^1 satisfying ϕ is conjugate to h in $\text{Homeo}(S^1)$. It follows then that the type of h is isolated by the single formula ϕ , i.e. for all other formulae $\psi(v) \in \text{tp}(h)$, we have $\text{Homeo}(S^1) \models \forall v (\phi(v) \rightarrow \psi(v))$.

From a model-theoretic point of view, type rigidity measures the homogeneity of the structure $\text{Homeo}(M)$; indeed, two conjugate elements of $\text{Homeo}(M)$ will certainly have the same type, and structures in which the type of an element determines the automorphism orbit of that element are *1-homogeneous*. For compact manifolds at least, a result of Whittaker [23] shows that the automorphism group of $\text{Homeo}(M)$ coincides with the inner automorphisms of $\text{Homeo}(M)$, i.e. conjugation. We show that for all manifolds, 1-homogeneity of $\text{Homeo}(M)$ depends on the set theory being used:

Theorem 1.6. *Suppose $V=L$ holds, and let M be an arbitrary connected manifold. Then every $g \in \text{Homeo}(M)$ is type rigid.*

By contrast:

Theorem 1.7. *Suppose PD holds. Then for all manifolds M , there exist pairs $g, h \in \text{Homeo}(M)$ such that $\text{tp}(g) = \text{tp}(h)$ but such that g and h are not conjugate in $\text{Homeo}(M)$.*

Finally, we consider rigidity and type rigidity within ZFC with infinitary logics. Specifically, we will be interested in $L_{\omega_1\omega}$ logic. Formulae in $L_{\omega_1\omega}$ have the same signature as in classical first-order logic, and terms and atomic formulae are defined identically. Negations of formulae are again formulae, and whenever $\phi(x)$ is a formula with a free variable x then $\exists x \phi(x)$ is also a formula. The difference with classical logic lies in the fact that countable disjunctions and countable conjunctions of formulae are again formulae. Formulae that are $L_{\omega_1\omega}$ play an important role in investigating the descriptive set theory of spaces of countable models of theories; see [15] for a detailed discussion.

Theorem 1.8. *Let M be an arbitrary connected manifold. There is a $L_{\omega_1\omega}$ sentence ψ_M such that for all manifolds N , we have $\text{Homeo}(N) \models \psi_M$ if and only if M and N are homeomorphic.*

Moreover:

Theorem 1.9. *Let M be an arbitrary connected manifold and let $g \in \text{Homeo}(M)$. Then there is an $L_{\omega_1\omega}$ formula $\psi_g(x)$ such that for all $h \in \text{Homeo}(M)$, we have $\text{Homeo}(M) \models \psi_g(h)$ if and only if h is conjugate to g in $\text{Homeo}(M)$.*

2. DEFINABILITY IN SECOND-ORDER ARITHMETIC

We recall a few notions from descriptive set theory, adapted to the various spaces that we will need; see [16, 7, 5], for instance. Recall that a *Polish space* is a separable topological space X whose topology can be induced by a complete metric on X . Thus, generally, the metric is not part of the given data. In this section, we discuss presentations of Polish spaces and relate them to parameter-free definability in second-order arithmetic.

2.1. Analytical presentation of Polish spaces and definability. In this section, we will gather some background on analytically presented Polish spaces; the fundamental reason for this discussion is that in the sequel, we wish for all of our constructions to be carried out *parameter-free in second-order arithmetic*. That is, we will be given the two-sorted structure $(\mathbb{N}, 2^{\mathbb{N}}, +, \times, <, \in)$, i.e. second-order arithmetic, and we will require all implicit formulae relating to Polish spaces to be parameter-free. The reader may find a discussion of the content of this subsection in [16, 20].

2.1.1. Integers and rational numbers. It is a standard fact that the structure $(\mathbb{Z}, +, \times, <)$ can be parameter-free interpreted in standard first-order arithmetic $(\mathbb{N}, +, \times, <)$ as a definable set of pairs of natural numbers up to a suitable parameter-free definable equivalence relation. A similar interpretation of the structure $(\mathbb{Q}, +, \times, <)$ can be carried out. Thus, via an effective pairing

$$\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

both $\mathbb{Q}$ and $\mathbb{Z}$ can be viewed as parameter-free definable subsets of $\mathbb{N}$ up to equivalence, and the operations/relations of addition, multiplication, and order are arithmetically definable. The Euclidean metric d on $\mathbb{Q}$ is a definable function.

2.1.2. Sequences and reals. Sequences of natural numbers are parameter-free definable in second-order arithmetic. Indeed, if σ is a sequence of natural numbers then σ can be coded as a subset of $\mathbb{N}$ via

$$\sigma \mapsto X_\sigma = \{\langle n, \sigma(n) \rangle \mid n \in \mathbb{N}\}.$$

By the definability of the predicates $(+, \times, <)$ over $\mathbb{Q}$, Cauchy sequences with respect to the metric d in $\mathbb{Q}$ are parameter-free definable. From this, we obtain a parameter-free definable sort $\mathbb{R}$ consisting of convergent Cauchy sequences, together with a definable extension of the metric d and the predicates $(+, \times, <)$ to $\mathbb{R}$. In the sequel, $\mathbb{R}$ will serve as a prototypical analytically presented Polish space.

2.1.3. Analytically presented Polish spaces. An *analytically presented Polish space* X is given by a countable set $D = \{x_n\}_{n \in \mathbb{N}}$ and a parameter-free definable function

$$d = d_X: D \times D \longrightarrow \mathbb{R}_{\geq 0},$$

where definability is understood by viewing d as a function from pairs of natural numbers to $\mathbb{R}$, satisfying the axioms for a metric. The points of X are identified with the set of (parameter-free definable set of) Cauchy sequences in D , up to the natural equivalence relation.

Observe that there are natural maps

$$n \mapsto x_n, \quad A \subseteq \mathbb{N} \mapsto X_A = \{x_n \mid n \in A\}.$$

The parameter-free definable subsets of the space (X, d) are, by definition, given by the parameter-free definable sets in the structure $(\mathbb{N}, 2^{\mathbb{N}}, +, \times, <, \in)$, via the maps above.

2.1.4. Coding open and closed sets. Let X be an analytically presented Polish space, with a dense subset $\{x_n\}_{n \in \mathbb{N}}$ and definable metric d . An open subset U of X is given as a union of countably many metric balls of rational radius contained in U , each of which is centered at a point in $\{x_n\}_{n \in \mathbb{N}}$. A sequence of such balls is simply given by a subset $A \subseteq \mathbb{N}$ and sequence σ of pairs of the form (n, q) encoding an open d -ball of radius $q \in \mathbb{Q}_{>0}$ about x_n , with $n \in A$. Thus, $x \in U$ if and only if there exists an $(n, q) \in \sigma$ such that $d(x, x_n) < q$. Subsets of $\mathbb{N}$ and sequences of positive rationals are parameter-free definable, so open subsets of X are parameter-free interpretable. Note that we are not actually defining open subsets of X , but rather codes for them. It is a straightforward

exercise to write down a definable equivalence relation for when two codes represent the same open set, and a predicate expressing when a point belongs to an open set.

Since closed sets are complements of open sets, the closed subsets of X are also parameter-free interpretable. Compact subsets can be characterized by sequential compactness; we omit the details. A single open ball of rational radius q about a point x_n can be coded by a constant sequence (n, q) . This, together with an effective pairing $\mathbb{N} \times \mathbb{Q}_{>0} \rightarrow \mathbb{N}$ gives a parameter-free recursively enumerated sequence of codes for a basis for the topology on X ; we will reserve $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ for such a basis. The predicate $x \in \mathcal{U}_i$ is parameter-free definable whenever X is an analytically presented Polish space.

2.1.5. Some commonly used Polish spaces. We will need some explicit analytically presented Polish spaces. In each case, it is clear that the Polish space is in fact analytically presentable.

- For each $d \in \mathbb{N}$, the Euclidean space $\mathbb{R}^d$.
- For each $\mathbb{R}^d$ and each parameter-free definable $r \in \mathbb{R}_{\geq 0}$, the closed ball $B^d(r)$ of radius r centered at the origin; here, we implicitly use the standard Euclidean metric on $\mathbb{R}^d$.
- The intersection of $B^d(1)$ with coordinate planes in $\mathbb{R}^d$ (which are then homeomorphic to balls of smaller dimension).
- For each d , the $d - 1$ -dimensional sphere $S^{d-1} = \partial(B^d(1))$.
- For all $d, d' \geq 1$, the Banach space $C(B^d(1), \mathbb{R}^{d'})$ of continuous functions $B^d(1) \rightarrow \mathbb{R}^{d'}$ with the usual supremum metric and the space $C(B^d(1), \mathbb{R}^{d'})^{\mathbb{Z}}$ of bi-infinite sequences of such functions with an appropriate product metric. To lighten the notation, when the dimensions are obvious from the context, we shall often simply write B , E , $C(B, E)$ and $C(B, E)^{\mathbb{Z}}$ for $B^d(1)$, $\mathbb{R}^{d'}$, $C(B^d(1), \mathbb{R}^{d'})$, respectively $C(B^d(1), \mathbb{R}^{d'})^{\mathbb{Z}}$.

2.2. Analytical definability. If X is a Polish space, we let $K(X)$ denote the space of all compact subsets of X equipped with the *Victoris topology*, which is the topology generated by sets of the form

$$\{K \in K(X) \mid K \cap U \neq \emptyset\}, \quad \{K \in K(X) \mid K \subseteq U\},$$

where U ranges over open subsets of X . Similarly, $F(X)$ is the *Effros-Borel space* consisting of all closed subsets of X equipped with the σ -algebra generated by sets of the form

$$\{F \in F(X) \mid F \cap U \neq \emptyset\}$$

with U varying over open subsets of X . The space $K(X)$ is Polish, whereas $F(X)$ is standard Borel. Moreover, the canonical inclusion of $K(X)$ into $F(X)$ is a Borel embedding and $K(X)$ is a Borel subset of $F(X)$.

Suppose that X is an analytically presented Polish space. By the discussion in Subsection 2.1.4, the collections $F(X)$ and $K(X)$ are parameter-free interpretable (i.e. codeable in a way that depends only on the presentation of X), as are the open sets in X . From a fixed open set $U \subseteq X$, the collections of compact or closed sets meeting or contained in U is definable in the interpreted sorts $K(X)$ and $F(X)$ respectively, using only U as a parameter.

If X is an analytically presented Polish space, then the *projective* sets are the subsets of X which are definable with parameters. A subset which is definable without parameters, we will call an *analytical set*. It is straightforward that analytical sets are closed under complements, finite unions, and finite intersections. It is straightforward to show that these functions and sets, viewed as relations, are analytical whenever X is analytically presented.

- (1) $\{(x, F) \in X \times F(X) \mid x \in F\}$,
- (2) $\{(F_1, F_2) \in F(X) \times F(X) \mid F_1 \subseteq F_2\}$
- (3) $f \in C(B, E) \mapsto f[B] \in K(E)$,
- (4) $(F_i)_{i \in \mathbb{Z}} \in F(X)^{\mathbb{Z}} \mapsto \overline{\bigcup_{i \in \mathbb{Z}} F_i} \in F(X)$.

We have the following:

Lemma 2.1. *Suppose X is an analytically presented Polish space. The following properties and relations are analytical.*

- (1) $(x_n)_{n=1}^{\infty} \in X^{\mathbb{N}}$ is a sequence satisfying $x_1 = \lim_{n \rightarrow \infty} x_n$,
- (2) $F \in F(X)$ is the code for a connected set,
- (3) $f \in C(B, E)$ is an injective function,
- (4) $G \in F(X \times X)$ is the code for the graph of a homeomorphism between the two sets $F_1, F_2 \in F(X)$.
- (5) $G \in F(X \times X)$ is the code for the graph of a homeomorphism of pairs $(F_1, A_1) \longrightarrow (F_1, A_2)$, for $A_i \subseteq F_i$ and $A_1, A_2, F_1, F_2 \in F(X)$.
- (6) $G \in F(X \times X)$ is the code for the graph of a homeomorphism of triples

$$(F_1, A_1, x_1) \longrightarrow (F_1, A_2, x_2),$$

for $x_i \in A_i \subseteq F_i$ and $A_1, A_2, F_1, F_2 \in F(X)$.

Proof. Note that $F \in F(X)$ is connected if and only if

$$\forall F_1 \forall F_2 \in F(X) \ (F_1 \cup F_2 \neq X \text{ or } F \cap F_1 \cap F_2 \neq \emptyset \text{ or } F \cap F_1 = \emptyset \text{ or } F \cap F_2 = \emptyset),$$

which is clearly parameter-free definable in second-order arithmetic.

Similarly, $f \in C(B, E)$ is injective if and only if

$$\forall x \in X \ \forall y \in X \ (x = y \text{ or } f(x) \neq f(y)).$$

We have that $G \in F(X \times X)$ is the graph of a homeomorphism between the two sets $F_1, F_2 \in F(X)$ exactly when

G is the graph of a bijection between F_1 and F_2 and

$$\forall ((x_n, y_n))_{n \in \mathbb{N}} \left(\forall n (x_n, y_n) \in G \rightarrow (x_1 = \lim_{n \rightarrow \infty} x_n \leftrightarrow y_1 = \lim_{n \rightarrow \infty} y_n) \right)$$

which is clearly parameter-free definable in second-order arithmetic. Similarly for the last two properties. $\square$

Lemma 2.2. *Fix dimensions $1 \leq d \leq d'$. The collection*

$$\text{manif}_d = \{F \in F(\mathbb{R}^{d'}) \mid F \text{ is a connected } d\text{-dimensional manifold}\}$$

is analytical.

Proof. Since connectedness is already analytical, it suffices to show that being a d -dimensional submanifold of $\mathbb{R}^{d'}$ is analytical. Let $\mathbb{R}^d$ be embedded in $\mathbb{R}^{d'}$ on the first d coordinates, whereby $B^d(1) = \mathbb{R}^d \cap B^{d'}(1)$. Then a connected set $F_1 \in F(\mathbb{R}^{d'})$ is a d -dimensional submanifold if and only if, for all $x \in F_1$, there exists a $F_2 \in F(\mathbb{R}^{d'})$ such that we obtain a homeomorphism of triples

$$(F_2, F_1 \cap F_2, x) \cong (B^d(1), B^d(1), 0).$$

The conclusion now follows by Lemma 2.1. $\square$

The following corollaries are straightforward.

Corollary 2.3. *The homeomorphism relation on pairs of d -dimensional submanifolds of $\mathbb{R}^{d'}$ is analytical.*

Corollary 2.4. *Let $E = \mathbb{R}^{d'}$ be a fixed Euclidean space. The subset*

$$\text{HGrp}_d \subseteq F(E) \times F(E \times E)$$

consisting of pairs (M, G) , for which $M \in \text{manif}_d$ and G is the graph of a homeomorphism of M , is analytical.

Corollary 2.5. *The set*

$$\mathfrak{m}_d = \left\{ (f_n) \in C(B, E)^{\mathbb{Z}} \mid \overline{\bigcup_{n \in \mathbb{Z}} f_n[B]} \in \text{manif}_d \right\}$$

is analytical.

Recall that we fixed a recursively enumerated set $\{\mathcal{U}_i\}_{i \in \mathbb{N}}$ of codes for a countable basis for the topology on $C(B, E)^{\mathbb{Z}}$. For an element $(f_n) \in \mathfrak{m}_d$, we may think of (f_n) as encoding an atlas for an embedded submanifold of E , and the indices i for which (f_n) belongs to $\mathcal{U}_i$ can be viewed as an address for (f_n) .

Proposition 2.6. *Let HGrp_d be as in Corollary 2.4. The relation*

$$\text{conj}_d \subseteq \text{HGrp}_d \times \text{HGrp}_d$$

consisting of pairs (M_1, G_1) and (M_2, G_2) , for which there exists a homeomorphism $\phi: M_1 \rightarrow M_2$ conjugating the homeomorphism with graph G_1 to the homeomorphism with graph G_2 , is analytical.

Proof. Write $(x_1, y_1) \in G_1$ and $(x_2, y_2) \in G_2$ for typical points. The existence of the homeomorphism ϕ is equivalent to the existence of a graph G_3 of a homeomorphism between M_1 and M_2 such that

$$\forall (x_i, y_i) \left(((x_1, y_1) \in G_1 \wedge (y_1, y_2) \in G_3) \leftrightarrow ((x_1, x_2) \in G_3 \wedge (x_2, y_2) \in G_2) \right).$$

It is clear then that conj_d is analytical. $\square$

To investigate type rigidity of homeomorphisms of manifolds, we will require an analytical predicate encoding pairs of atlases for a manifold M , where one atlas is twisted by a fixed homeomorphism of M . We write mark_d for the set of pairs $((f_n), (g_n)) \in \mathbf{m}_d \times \mathbf{m}_d$ such that:

(1)

$$M := \overline{\bigcup_{n \in \mathbb{Z}} f_n[B]} = \overline{\bigcup_{n \in \mathbb{Z}} g_n[B]};$$

(2) There exists a G such that $(M, G) \in \text{HGrp}_d$ and such that for all $x \in B$ and all $n \in \mathbb{Z}$ we have $(f_n[x], g_n[x]) \in G$.

The conditions defining mark_d express that the atlas encoded by (g_n) is simply the atlas encoded by (f_n) composed with a fixed homeomorphism of M whose graph is encoded by G . For $(M_0, G_0) \in \text{HGrp}_d$ fixed, we let $\text{mark}_d(M_0, G_0)$ consists of pairs $((f_n), (g_n)) \in \mathbf{m}_d \times \mathbf{m}_d$ which each encode M_0 , and where for all $x \in B$ we require $(f_n[x], g_n[x]) \in G_0$. The following is straightforward:

Proposition 2.7. *The set mark_d is analytical.*

2.3. Uniform interpretations and first-order rigidity.

Definition 2.8. Let $\mathcal{F}$ be a family of structures in a countable signature $\mathcal{S}$ that is definable in second-order arithmetic. We write $\text{FOR}(\mathcal{F})$ for the set theoretical statement that expresses the fact that the class $\mathcal{F}$ is first-order rigid, that is that $\mathcal{A} \equiv \mathcal{B}$ implies $\mathcal{A} \cong \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in \mathcal{F}$. This is expressed by a formula on the same parameters used to define $\mathcal{F}$.

This work is can be understood as an inquiry into the set theoretical strength of statements of the form $\text{FOR}(\mathcal{F})$ where $\mathcal{F}$ is one of the following:

- the collection of all homeomorphism groups of manifolds of a fixed dimension m ,
- for a fixed compact manifold M , the collection of all expansions of the group structure $\text{Homeo}(M)$ by a constant that distinguishes an element of $\text{Homeo}(M)$, or some variation thereof.

Definition 2.9. Consider the a families of structures $\mathcal{F}$ and $\mathcal{F}'$, which have the countable signatures $\mathcal{S}$ from before, and

$$\mathcal{S}' = \{R^{(k_i)_i}, f_j^{(r_j)} \mid i \in I, j \in J\},$$

respectively. By a *uniform interpretation* ι of structures from $\mathcal{F}'$ in $\mathcal{F}$ we mean a collection of parameter-free $\mathcal{S}$ -formulae

$$\theta, \chi, \{\phi_i\}_{i \in I}, \{\psi_j\}_{j \in J}$$

such that for any $\mathcal{A} \in \mathcal{F}$, the formulae in question interpret a structure $\iota(\mathcal{A}) \in \mathcal{F}'$ in the usual sense; that is:

- χ defines an equivalence relation E on the subset $X \subseteq \mathcal{A}^k$ defined by θ ,
- for all $i \in I$ the predicate defined by ϕ_i factors through a k_i -ary predicate on X/E ,
- for all $j \in J$ the predicate defined by ψ_j factors through a r_j -ary function from X/E onto itself,
- $\iota(\mathcal{A})$ is the structure with universe X/E , where R_i is interpreted by ϕ_i and the predicate of f_j by ψ_j .

In this situation we may also write, more succinctly, that $\mathcal{A} \in \mathcal{F}$ uniformly interprets $\iota(\mathcal{A}) \in \mathcal{F}'$ without parameters. If $\iota(\mathcal{B}) \cong \iota(\mathcal{C})$ always implies $\mathcal{B} \cong \mathcal{C}$, then we say that ι is *faithful*, and we write $\mathcal{F} \leq^{ufi} \mathcal{F}'$.

It is a standard fact (i.e. consequence of ZFC) that in the situation above $\mathcal{M} \equiv \mathcal{M}'$. It immediately follows that the result below is a theorem of ZFC:

Lemma 2.10. *For $\mathcal{F}, \mathcal{F}'$ as in Definition 2.8 we have:*

$$\mathcal{F} \leq^{ufi} \mathcal{F}' \longrightarrow (\text{FOR}(\mathcal{F}') \rightarrow \text{FOR}(\mathcal{F})).$$

3. FIRST-ORDER PROPERTIES OF HOMEOMORPHISM GROUPS AND TOPOLOGICAL PRELIMINARIES

In this section, we gather some background material about first-order rigidity of homeomorphism groups of manifolds, as well as some topological tools adapted to the study of noncompact manifolds embedded in Euclidean space.

3.1. First-order theory of homeomorphism groups of manifolds. Recall that all manifolds M are assumed to be connected, second countable, and Hausdorff, though not necessarily compact. To avoid some technical difficulties, we always assume $\partial M = \emptyset$. The second countability of M is equivalent to σ -compactness of M , so that M admits a countable exhaustion by compact submanifolds.

For compact manifolds M (possibly with boundary), the first-order theory of the homeomorphism group $\text{Homeo}(M)$ in the language $\mathcal{L}$ was developed extensively in [9]. Most importantly for our purposes, the content of [9] is that homeomorphism groups of compact manifolds uniformly interpret many auxiliary sorts together with definable predicates relating them to each other and to the home sort, which are useful for establishing rigidity results.

Suppose M is a manifold of some dimension $d \geq 1$. A homeomorphic embedding $\phi: B^d(1) \rightarrow M$ that extends to an embedding $B^d(2) \rightarrow M$ is called a *parametrized collared ball* in M ; we will simply call the image of ϕ a *collared ball*.

A subset of a Polish space is definable (with parameters, generally) in second-order arithmetic if and only if it is projective, in the sense of descriptive set theory; see [20].

We will assume all abstract topological spaces to be at least Hausdorff. In a topological space, an open set is called *regular* if it is the interior of its closure. The regular open sets of a topological space X are written $\text{RO}(X)$, and form a Boolean algebra. A *regular compact set* is a compact set whose complement is regular open. A compact set is regular if it is the closure of its interior. The regular compact subsets of X are written $\text{RC}(X)$.

The following result summarizes much of the technical content of [9]. For each (imaginary) sort listed in the result, we say that the sort is *uniformly interpretable* in $\text{Homeo}(M)$ if there is a formula ϕ in the language $\mathcal{L}$ that interprets this sort in $\text{Homeo}(M)$, and ϕ does not depend on M ; that is, the manifold M may be treated as a variable, and the same formula ϕ defines the corresponding sort for each manifold. All definitions and interpretations are parameter-free, unless stated otherwise. The term *uniform(ly)*, present in isolation, is always to be understood as “independent(ly) of the manifold M ”.

Theorem 3.1 (See [9]). *Let M be an arbitrary compact, connected manifold, possibly with boundary and $\text{Homeo}_0 \leq G \leq \text{Homeo}(G)$. Several collections of objects derived from M can be shown to admit a canonical bijection with a series of imaginary sorts in G , as a structure in the language of groups. We also claim that certain predicates defined over said derived data are definable, i.e., definable modulo the canonical identification in question.*

- (1) *There is a uniform imaginary sort in canonical bijection with the collection $\text{RO}(M)$ of regular open subsets of M . Moreover, there are uniformly definable predicates encoding:*
 - (a) *The Boolean algebra structure on $\text{RO}(M)$;*
 - (b) *The action structure on $\text{RO}(M)$; that is, the graph of the map $\text{act} : G \times \text{RO}(M) \longrightarrow \text{RO}(M)$ mapping (g, U) to $g \cdot U$.*
- (2) *There is a uniform imaginary sort admitting a canonical bijection with the natural numbers $\mathbb{N}$, and another one admitting a canonical bijection with the collection $2^{\mathbb{N}}$ of subsets of $\mathbb{N}$, so that the following predicates in the sorts above are uniformly definable:*
 - (a) *the graph of the arithmetic operations $+, \times$ on $\mathbb{N}$;*
 - (b) *the order on $\mathbb{N}$;*
 - (c) *the membership predicate $\in \subseteq \mathbb{N} \times 2^{\mathbb{N}}$;*
 - (d) *the collection of $(n, U) \in \mathbb{N} \times \text{RO}(M)$ for which U has n connected components.*
- (3) *There is a uniform imaginary sort in canonical bijection with the set of points of M with respect to which the membership predicate $[\in'] \subseteq M \times \text{RO}(M)$ and the predicate of the action of G on M are uniformly definable.*
- (4) *There is a uniform imaginary sort whose elements are in canonical bijection with the collection $\mathcal{P}^{<\omega}(M)$ of finite sequences of points in M . Moreover, the predicate $\in (\pi, k, U)$, that holds precisely when the k^{th} term of $\pi \in \mathcal{P}^{<\omega}(M)$ lies in $U \in \text{RO}(M)$ is uniformly definable.*
- (5) *The collection of tuples $(g, n, U, V) \in G \times \mathbb{N} \times \text{RO}(M)^2$ for which $g^n(U) = V$ is uniformly definable. The same holds if U and V are replaced by points in M .*
- (6) *The set of pairs $(p, U) \in M \times \text{RO}(M)$ for which $p \in M$ lies in the closure of $U \in \text{RO}(M)$ is uniformly definable.*
- (7) *For each $d \in \mathbb{N}$, there is a sentence dim_d such that $G \models \text{dim}_d$ if and only if M has dimension d .*
- (8) *For each fixed dimension d , there is an imaginary sort $\text{RC}(M)$, uniform on all d -dimensional manifolds M , in canonical bijection with the regular compact subsets of M . Moreover, via this bijection, the action structure on $\text{RC}(M)$ is given by uniformly definable predicates.*
- (9) *For each fixed dimension d , there is an imaginary sort $\text{ball}_s(M)$, uniform on all d -dimensional manifolds M , which is in canonical bijection with parametrized collared balls $\beta : B^d(1) \rightarrow M$. Moreover, the following predicates are uniformly definable for fixed d :*
 - (a) *The collection $\text{open} - \text{ball} \subseteq \text{RO}(M)$ of all U which are the interior of some collared ball $\beta(B)$ in M .*

- (b) The collection $\text{closed} - \text{ball}(C) \subseteq \text{RC}(M)$ of sets of the form $\beta(B)$ for some collared ball in M .
- (c) The collection param of all $(\beta, C, q, p) \in \text{ball}_s \times \text{RC}(M) \times \mathcal{P}(M)^2$ such that:
 - (i) $C = \beta(B)$;
 - (ii) $q \in B \cap \mathbb{Q}^d$, or more generally q is projectively definable in B ;
 - (iii) $p \in C$;
 - (iv) $\beta(q) = p$.

Some remarks are in order. First, in the proofs of the items in Theorem 3.1 as given in [10], many of the interpretations rely essentially on the compactness of the underlying manifold M , though many do not. For the regular open sets $\text{RO}(M)$ and the action of $\text{Homeo}(M)$ on $\text{RO}(M)$, the interpretation can be carried out in a much more general context of locally moving actions (see [2, 18, 19]), and so there is no difficulty replacing M by a noncompact manifold for interpreting regular open sets. It follows that regular closed sets are immediately interpretable in $\text{Homeo}(M)$. The interpretation of $\mathbb{N}$ and the attendant predicates can be carried out in any compact set with compact closure in M , even when M is compact. Similar considerations apply to the interpretations of points. For dimension, since M is connected, we have that the dimension of M coincides with the dimension of any nonempty open subset of M . In particular, one can express the dimension of a manifold from a nonempty open set U contained in a collared open ball; a similar consideration applies to parametrized collared balls in M .

For interpreting regular compact sets when M is not necessarily compact, some further steps are necessary. The following proposition is easy, and follows from straightforward dimension theory of manifolds and transitivity of the action of the homeomorphism group of a manifold on collared balls; see [9, 12, 11, 14]:

Proposition 3.2. *Let M be an arbitrary connected manifold of dimension d , let $\emptyset \neq U \in \text{RO}(M)$, and let $K \subseteq M$ be arbitrary.*

- (1) *The closure of U lies inside a collared ball in M if and only if, for all nonempty $W \in \text{RO}(M)$, there is a $g \in \text{Homeo}(M)$ such that $g(U) \subseteq W$.*
- (2) *The subset K lies in a compact submanifold of M if and only if there exists a finite collection $\{U_1, \dots, U_m\} \subseteq \text{RO}(M)$ such that each point $p \in K$ lies in some U_i , and such that each U_i has closure contained in a collared ball in M .*

- (3) *Conversely, if $N \subseteq M$ is a compact, connected submanifold, then there is an $m \in \mathbb{N}$ depending only on the dimension d of M such that N is contained in the union of regular open sets $\{U_1, \dots, U_m\} \subseteq \text{RO}(M)$, each of which has closure contained in a collared ball in M .*

It follows from Proposition 3.2 that a uniform imaginary sort in canonical bijection with $\text{RC}(M)$ so that the relevant predicates are all uniformly definable in $\text{Homeo}(M)$ by $\mathcal{L}$ -formulae, provided that the dimension of M is a fixed value d .

The uniform interpretation of $\mathbb{N}$ and the related predicates means that analytically presented Polish spaces, the Borel hierarchy, and the projective hierarchy, are all uniformly interpretable across homeomorphism groups of manifolds. Thus, all homeomorphism groups of manifolds “agree” on what B is, what Euclidean spaces are, what continuous functions are, etc.

Corollary 3.3. *The conclusions of Theorem 3.1 hold for arbitrary connected manifolds.*

A further sort which is uniformly interpretable for manifolds of fixed dimension d is that of sequences of points; this interpretation was carried out in [12]:

Proposition 3.4. *Let d be fixed. Given a group*

$$\text{Homeo}_0 \leq G \leq \text{Homeo}(M),$$

where here M is a compact manifold of dimension d . There is an uniform imaginary sort $\text{seq}(M)$, which admits a canonical bijection with the collection of countable sequences of points in M . Moreover, for fixed d the set of pairs

$$(p, k, \sigma) \in M \times \mathbb{N} \times \text{seq}(M)$$

for which p is the k^{th} term of σ is uniformly definable.

Proposition 3.4 is generalized to noncompact manifolds below in Corollary 3.8.

Let X be a fixed separable Hausdorff topological space that is parameter-free interpretable in second-order arithmetic; that is to say, the points $\mathcal{P}(X)$ and open sets $\mathcal{U}$ can be canonically identified with imaginary sorts in second-order arithmetic, so that the inclusion relation $p \in U$ of a point in an open set is uniformly definable; for instance, X may be a analytically presented Polish space. Then, homeomorphisms of X are determined by their values on a dense subset of X , and countable sequences of points in X are a definable sort in second-order arithmetic. Combining with Proposition 3.4, there is a uniformly definable sort $\text{seq}(M \times X)$ in canonical bijection with countably sequences in

$M \times X$, together with a uniformly definable predicate defining the k^{th} term of a sequence, for all $k \in \mathbb{N}$.

It is not difficult, from direct access to regular open sets in M and the topology of X , to construct definable predicates which define sequences in $M \times X$ encoding graphs of homeomorphisms of $M \times X$. This was done explicitly for compact manifolds in [12], and we spell out some details in Corollary 3.8 below in the case of general connected boundaryless manifolds.

3.2. Dynamics of homeomorphisms of noncompact manifolds. Crucial to our investigation of noncompact manifolds and encoding such manifolds with finite data is the notion of a *topologically transitive* homeomorphism. A homeomorphism g of a topological space X is topologically transitive if for all nonempty open $U, V \subseteq X$, there exists an $n \in \mathbb{Z}$ such that $g^n(U) \cap V \neq \emptyset$. Clearly there are no topologically transitive homeomorphisms of $\mathbb{R}$, though in dimension two or more, topologically transitive homeomorphisms always exist (see [1]):

Theorem 3.5. *Let M be a connected σ -compact manifold of dimension at least two. Then M admits a topologically transitive homeomorphism.*

For us, topologically transitive homeomorphisms are a parameter-free definable subset of $\text{Homeo}(M)$.

Proposition 3.6. *The set $\mathcal{T} \subseteq \text{Homeo}(M)$ consisting of topologically transitive homeomorphisms is uniformly definable in the language $\mathcal{L}$ without parameters.*

Proof. We simply define $\mathcal{T}$ to consist of all $g \in \text{Homeo}(M)$ such that for all nonempty $U, V \in \text{RO}(M)$, there exists a $p \in U$ and an $n \in \mathbb{Z}$ such that $g^n(p) \in V$. That these conditions are first-order expressible follows from Theorem 3.1 and Corollary 3.3. $\square$

For us, topologically transitive homeomorphisms of a manifold M are merely a means to an end. For an open set $U \subseteq M$ and a homeomorphism $g \in \text{Homeo}(M)$, we write

$$M_{g,U} = \bigcup_{i \in \mathbb{Z}} g^i(U).$$

If $M_{g,U}$ is a dense subset of M then we say that $M_{g,U}$ is a *g -dense* submanifold of M (or just a *dense* submanifold if g and U are not relevant for the discussion), and we say that g is *sufficiently transitive*. We have the following, which can be proved the same way as Proposition 3.6:

Proposition 3.7. *The set $\mathcal{G}(M)$ consisting of all $(g, U) \in \text{Homeo}(M) \times \text{RO}(M)$ such that*

$$U \text{ is the interior of a collared ball \quad \& \quad } M_{g,U} \text{ is } g\text{-dense}$$

is uniformly definable without parameters. Furthermore, for all connected, boundaryless manifolds M , we have $\mathcal{G}(M) \neq \emptyset$.

Proposition 3.7 allows us to generalize Proposition 3.4 to connected, boundaryless manifolds that are not necessarily compact.

Corollary 3.8. *Let M be a connected, boundaryless manifold of dimension d . There exists a uniformly imaginary sort in $\text{Homeo}(M)$ admitting a canonical bijection with the collection $\text{seq}^{\mathcal{G}}(M)$ of countable sequences σ of points in M , and a uniformly definable predicate on triples (p, k, σ) expressing that p is the k^{th} entry of σ .*

Proof. We first claim that there exists a uniformly definable sort $\text{seq}^{\mathcal{G}}(M)$, in canonical bijection with countable sequences σ of points in M for which there is a $(g, U) \in \mathcal{G}(M)$ such that

$$\sigma \subseteq M_{g,U} \subseteq M,$$

together with the corresponding uniformly definable entry extraction predicate.

Fix a pair $(g, U) \in \mathcal{G}(M)$, with B denoting the closure of U . An identical argument as for establishing Proposition 3.4 in [12] shows that the imaginary sort of countable sequences in B is uniformly $\mathcal{L}$ -interpretable, with (g, U) as a parameter. If σ is a sequence of points in B and $\tau \in \mathbb{Z}^{\mathbb{N}}$ then we obtain a sequence $\xi(\sigma, \tau)$ in $M_{g,U}$ by specifying its n^{th} term for all n : $\xi(n) = g^{\tau(n)}(\sigma(n))$. It is clear that every sequence of points in $M_{g,U}$ arises as $\xi(\sigma, \tau)$ for some σ and τ , and so $\text{seq}^{\mathcal{G}}(M)$ can be defined by

$$\left\{ \xi \mid \exists g \exists U \exists \sigma \exists \tau \left((g, U) \in \mathcal{G}(M) \wedge \forall n \xi(n) = g^{\tau(n)}(\sigma(n)) \right) \right\}$$

That the entry extraction predicate is uniformly definable is trivial.

To conclude the proof of the Lemma, recall the standard fact that there is a bijection $(m, n) \mapsto \langle m, n \rangle$ between $\mathbb{N}^2$ and $\mathbb{N}$ definable without parameters in arithmetic. Now, consider the collection $\text{seq}^{\mathcal{G}^*}(M)$ of sequences $\sigma \in \text{seq}^{\mathcal{G}}(M)$ such that for each $m \in \mathbb{N}$ the sequence $\sigma(\langle m, n \rangle)$ converges to a point $p_m \in M$. It is trivial from all the definability results presented so far that this collection is uniformly definable and that so is the graph of the map

$$\mathbb{N} \times \text{seq}^{\mathcal{G}^*}(M) \longrightarrow M, \quad (k, \sigma) \mapsto \lim_n \sigma(m, n).$$

Clearly, for all $\tau \in \text{seq}^{\mathcal{G}}(M)$ there is $\sigma \in \text{seq}^{\mathcal{G}^*}(M)$ such that $\tau(m) = \lim_n \sigma(\langle m, n \rangle)$ for all n , so it can be easily concluded that $\text{seq}^{\mathcal{G}}(M)$ can be identified with a suitable quotient of $\text{seq}^{\mathcal{G}^*}(M)$ by a uniformly definable equivalence relation. $\square$

From this point the same arguments as in [12] one can conclude the following:

Corollary 3.9. *Fix a dimension $d \geq 1$. For all imaginary sorts S in the structure $\text{Homeo}(M)$, there exists another sort, defined by formulae that are uniform in all boundaryless manifolds M of dimension d , which encodes the collection $\text{seq}^S(M)$ of sequences of elements of S . Moreover, the collection of triples $(e, k, \sigma) \in S \times \mathbb{N} \times \text{seq}^S(M)$ such that $\sigma(k) = e$ is uniformly definable for fixed d .*

For X a separable Hausdorff topological space that is parameter-free definable in second-order arithmetic, we write $\text{seq}^{\mathcal{G}}(M \times X)$ for sequences in $M \times X$ where the projection to the first coordinate is an element of $\text{seq}^{\mathcal{G}}(M)$.

Corollary 3.10. *Let M be a connected, boundaryless manifold of dimension d and let X be a separable Hausdorff topological space that is parameter-free definable in second-order arithmetic. There is a uniformly definable imaginary sort in $\text{seq}^{\mathcal{G}}(M \times X)$ which is in canonical bijection with elements of $\text{Homeo}(M \times X)$. Moreover, the group operation on $\text{Homeo}(M \times X)$ is given by a definable predicate.*

3.3. Language expansion and dimension jumping. Now, consider the class of connected manifolds up to homeomorphism, and let M be a manifold of fixed dimension d . By Theorem 3.1, there is an $\mathcal{L}$ -interpretation in $\text{Homeo}(M)$ of points p in M and regular open sets U of M , together with the predicate $p \in U$; these interpretations are parameter-free and uniform in M . We thus uniformly interpret an uncountable structure which consists of two sorts (points and regular open sets) and which has an underlying signature consisting of the non-logical symbol $\in$. We (conservatively) expand this structure to a larger structure $[M]$ which includes a uniform, parameter-free interpretation of full second-order arithmetic, i.e. the sort $(\mathbb{N}, 2^{\mathbb{N}}, \in, +, \times, <)$. It is clear that $[M]$ contains all the data of the topology of M , and the uniformity of the interpretation implies that if $\text{Homeo}(M) \equiv \text{Homeo}(M')$ then $[M] \equiv [M']$.

By Corollary 3.8, the sort $\text{seq}^{\mathcal{G}}(M)$ of sequences of points lying in dense submanifolds is uniformly $\mathcal{L}$ -interpretable. We write $[M]^+$ for the extended structure consisting of $[M]$ together with the sort $\text{seq}^{\mathcal{G}}(M)$. Again, if $\text{Homeo}(M) \equiv \text{Homeo}(N)$ then $[M]^+ \equiv [N]^+$.

Now, for a fixed k , we fix a description of the k -sphere S^k as an analytically presented Polish space. By Corollary 3.10, we see the following:

Proposition 3.11. *The structure $[M]^+$ parameter-free interprets the structure $\text{Homeo}(M \times S^k)$, uniformly for manifolds M of fixed dimension d , together with a predicate expressing the group operation.*

Because the interpretation of $\text{Homeo}(M \times S^k)$ is parameter-free and uniform, we have $[M]^+ \equiv [N]^+$ implies that $\text{Homeo}(M \times S^k) \equiv \text{Homeo}(N \times S^k)$. We thus obtain Theorem 1.3 from the introduction as an immediate corollary of Proposition 3.11.

3.4. Noncompact surfaces and ordinals. From Corollary 3.9 above, we have the following fact relating the topology of subspaces of the Cantor space $2^{\mathbb{N}}$ to first-order rigidity:

Corollary 3.12. *Let $\alpha < \omega_1$ be an ordinal. Consider the surface S_α obtained as the complement in the 2-sphere S^2 of a set of points homeomorphic to α with the order topology. Let COrd be a uniformly definable sort in $\text{Homeo}(S)$ that interprets the collection of all countable ordinals. Then there is a formula $\phi(x)$, independent of the underlying surface, such that $\text{Homeo}(S_\alpha) \models \phi(\beta)$ for a countable ordinal β if and only if $\beta = \alpha$.*

Proof. By applying Corollary 3.9 one can successively encode the following data as imaginary classes, uniformly in all manifolds of dimension d :

- open subsets of M , seen sequences of interior of embedded balls quotiented by the equivalence relation of having the same union;
- the binary relation which holds between open sets when one is a connected component of the other;
- compact subsets of M ;
- infinite sequences $((K_n, U_n))_{n \in \mathbb{N}}$, where $K_1 \subseteq K_2 \dots$ is an exhaustion of M by compact sets and U_n is a connected component of $M \setminus K_n$;
- the equivalence relation between the sequences

$$((K_n, U_n))_{n \in \mathbb{N}} \quad \text{and} \quad ((K'_n, U'_n))_{n \in \mathbb{N}}$$

given by $(U_n)_{n \in \mathbb{N}}$ and $(U'_n)_{n \in \mathbb{N}}$ are mutually cointial for the inclusion, and whose equivalence classes are in bijection with the space of ends;

- the relation between open sets $U \subseteq M$ and ends E of M , given by $E \in \bar{U}$.

Countable ordinals exist as an imaginary sort S in second-order arithmetic, and the order topology on any such ordinal is definable in a way that does not depend on the ordinal. It is thus trivial to write down a formula without

parameters $\phi(\alpha)$ which expresses the existence of a homeomorphism between α and the space of ends. $\square$

Corollary 3.13. *Suppose that all elements of the group $\text{Homeo}_0([0, 1])$ are type rigid and that $\alpha \neq \alpha'$ are countable ordinals. Then $\text{Homeo}(S_\alpha) \not\cong \text{Homeo}(S_{\alpha'})$.*

Proof. It is not hard to check that in second-order arithmetic, there is map that is definable without parameters, which to each countable ordinal α assigns an element $g(\alpha) \in \text{Homeo}_0([0, 1])$ in such a way that $g(\alpha)$ and $g(\alpha')$ are conjugate in $\text{Homeo}([0, 1])$ only if $\alpha = \alpha'$. Such a map can be obtained, for instance, by blowing up some canonical (up to an orientation preserving homeomorphism) embedding of the ordinal into $[0, 1]$. Corollary 3.12 and Lemma 2.10 give the desired conclusion. $\square$

3.5. A topological fact. In this section, we record a fact from point-set topology, which will be useful in the sequel.

Proposition 3.14. *Let X and Y be separable, locally compact, complete metric spaces. Assume that $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ and $\{y_n\}_{n \in \mathbb{N}} \subseteq Y$ are dense subsets of X , respectively of Y such that:*

- (1) *for each regular compact $C \subseteq X$ there exists a regular compact $K \subseteq Y$ such that*

$$x_n \in C \Leftrightarrow y_n \in K;$$

- (2) *for each regular compact $K \subseteq Y$ there exists a regular compact $C \subseteq X$ such that*

$$x_n \in C \Leftrightarrow y_n \in K.$$

Then the map $x_n \mapsto y_n$ extends to a homeomorphism

$$\theta: X \longrightarrow Y.$$

Proof. Because the set $\{x_n\}_{n \in \mathbb{N}}$ is dense in X , we see that a regular closed set $C \subseteq X$ is completely determined determined by the intersection $C \cap \{x_n\}_{n \in \mathbb{N}}$. Similarly for regular closed subsets of Y .

Let $x \in X$ be any given point and find, by local compactness, a sequence $(C_m)_{m \in \mathbb{N}}$ of regular compact subsets of X such that $\{x\} = \bigcap_m C_m$. We let K_m be the (unique) regular open set in Y corresponding via (1) to C_m and we consider $\bigcap_m K_m$. Because the K_m are nested, compact, and nonempty, we have that this intersection contains at least one point y . Suppose for a contradiction that $y' \neq y$ is another point in this intersection and choose disjoint regular compact neighbourhoods $K \ni y$ and $K' \ni y'$. Set

$$L_m = \overline{\text{int}(K_m \cap K)} \quad \& \quad L'_m = \overline{\text{int}(K_m \cap K')}$$

and note that $(L_m)_{m \in \mathbb{N}}$ and $(L'_m)_{m \in \mathbb{N}}$ form disjoint nested sequences of regular compact subsets with

$$y \in \bigcap_m L_m \quad \& \quad y' \in \bigcap_m L'_m.$$

The L_m and L'_m correspond via (2) to regular compact sets $N_m \subseteq C_m$ and $N'_m \subseteq C_n$ such that $N_m \cap N'_m = \emptyset$. Moreover, by compactness again, we have

$$\emptyset \neq \bigcap_m N_m \subseteq \bigcap_m C_m = \{x\}$$

and

$$\emptyset \neq \bigcap_m N'_m \subseteq \bigcap_m C_m = \{x\},$$

which is absurd.

Thus, $\bigcap_n K_n$ consists of exactly the one point y . We define $\theta(x) = y$. It can be verified then that a sequence in $\{x_n\}_{n \in \mathbb{N}}$ converges to a point $x \in X$ if and only if the sequence in $\{y_n\}_{n \in \mathbb{N}}$ with the same indices converges to a point $y \in Y$, and $\theta(x) = y$. This implies that θ is a continuous bijection with a continuous inverse. $\square$

4. CONSTRUCTIBILITY AND RIGIDITY

Let M be a connected, abstract manifold. To keep a succinct notation, in this section, we set $G_M = \text{Homeo}(M)$ and let $g \in G_M$. We wish to show that under $V=L$, the group G_M is first-order rigid and the homeomorphism g is type rigid. All interpretations of structures in G_M are over the language $\mathcal{L}$.

We will use the following general fact that relates constructibility to descriptive set theory:

Proposition 4.1 (See [16]). *Assume $V=L$, and let X be an analytically presented Polish space. Then there is an analytical well-ordering on X .*

4.1. first-order rigidity. For this section, let M be fixed of dimension d , and let N be an arbitrary connected manifold with $G_N = \text{Homeo}(N)$.

Theorem 4.2. *Assume $V=L$. Then $G_M \equiv G_N$ if and only if M and N are homeomorphic manifolds.*

Obviously we need only prove that if $G_M \equiv G_N$ then $M \cong N$. The first reduction we can make is to assume that N also has dimension d , as follows from Theorem 3.1.

In general, we will need to build a nexus between the first-order theory of G_M and the descriptive set theory of $C(B, E)^{\mathbb{Z}}$; this will be provided almost entirely by Theorem 3.1 and Theorem 3.5. We begin by fixing a uniform

interpretation of second-order arithmetic, which allows for direct access to all analytically presented Polish spaces, including $B = B^d(1)$ and $C(B, E)^{\mathbb{Z}}$. We fix a uniform interpretation of $\text{ball}(M)$, and we let $\mathcal{T}(M)$ denote the (uniformly definable) set of sufficiently transitive homeomorphisms of M . We fix a $\beta \in \text{ball}(M)$ and $\tau \in \mathcal{T}(M)$ such that

$$M(\tau, \beta) = \bigcup_{n \in \mathbb{Z}} \tau^n \beta[B] \subseteq M$$

is dense. We let $B_{\mathbb{Q}} \subseteq B$ denote the set of rational points (which may be viewed as the countable subset giving the effective presentation of B), and we set

$$D = D(\beta, \tau) = \bigcup_{n \in \mathbb{Z}} \tau^n \beta[B_{\mathbb{Q}}] \subseteq M.$$

By fixing a definable enumeration of $\mathbb{Q}^d \times \mathbb{Z}$, we obtain a fixed enumeration of $D = \{d_n\}_{n \in \mathbb{N}}$.

From the data of τ and β , we obtain a (parameter-free definable) family of elements $\mathcal{D}(\tau, \beta) \subseteq C(B, E)^{\mathbb{Z}}$ consisting of sequences (f_n) such that:

- (1) We have $(f_n) \in \mathfrak{m}_d$;
- (2) For all $x, y \in B$ and all $i, j \in \mathbb{Z}$, we have $f_i(x) = f_j(y)$ if and only if $\tau^i \beta[x] = \tau^j \beta[y]$.

In particular, the set $\mathcal{D}(\tau, \beta) \subseteq C(B, E)^{\mathbb{Z}}$ consists of sequences (f_n) for which $\bigcup_{n \in \mathbb{Z}} f_n[B]$ is homeomorphic to $M(\tau, \beta)$.

For $(f_n) \in C(B, E)^{\mathbb{Z}}$, we let

$$D_{(f_n)} = \bigcup_{n \in \mathbb{Z}} f_n(B_{\mathbb{Q}}).$$

From the fixed enumeration of $\mathbb{Q}^d \times \mathbb{Z}$, we obtain an enumeration $D_{(f_n)} = \{e_n\}_{n \in \mathbb{N}}$.

Lemma 4.3. *There is a nonempty $\mathcal{L}$ -definable subset $\mathcal{M}(\tau, \beta) \subseteq \mathcal{D}(\tau, \beta)$ consisting of sequences (f_n) such that*

$$\overline{\bigcup_{n \in \mathbb{Z}} f_n[B]} \cong M.$$

Proof. By the Whitney Embedding Theorem, the manifold M embeds in E , and therefore such a sequence (f_n) exists. Because the class of regular compact subsets of E is analytical and because the sort of regular compact subsets of M is uniformly $\mathcal{L}$ -interpretable, we may require that for every regular compact subset

$$C \subseteq \overline{\bigcup_{n \in \mathbb{Z}} f_n[B]} \cong M,$$

there exists a regular compact $K \subseteq M$ such that for all n , we have $e_n \in C$ if and only if $d_n \in K$, and conversely switching the roles of C and K . Then, the hypotheses of Proposition 3.14 are satisfied, and the bijection $\theta: e_n \mapsto d_n$ extends to a homeomorphism

$$\Theta: \overline{\bigcup_{n \in \mathbb{Z}} f_n[B]} \longrightarrow M,$$

as desired. $\square$

We set

$$\mathcal{M}(M) = \bigcup_{\tau, \beta} \mathcal{M}(\tau, \beta),$$

which is clearly definable, since $(f_n) \in \mathcal{M}(M)$ if and only if

$$\exists \tau \exists \beta (f_n) \in \mathcal{M}(\tau, \beta).$$

We think of $\mathcal{M}(M)$ as the set of codes for M . We can now establish first-order rigidity.

Proof of Theorem 4.2. Suppose $G_M \equiv G_N$, and let $(f_n)^M$ and $(f_n)^N$ be the minimal elements of $\mathcal{M}(M)$ and $\mathcal{M}(N)$, respectively. We have $M \cong N$ if and only if $(f_n)^M = (f_n)^N$.

Let χ_i be the first-order $\mathcal{L}$ -sentence which expresses that the minimal code $(f_n)^M \in C(B, E)^{\mathbb{Z}}$ for M lies in the basis element $\mathcal{U}_i$ for the topology of $C(B, E)^{\mathbb{Z}}$. We have $G_M \equiv G_N$ if and only if for all i we have

$$(G_M \models \chi_i) \Leftrightarrow (G_N \models \chi_i),$$

if and only if $(f_n)^M = (f_n)^N$, as desired. $\square$

4.2. Type rigidity. Now, let (M, g) be a fixed pair, where M is an abstract connected manifold of dimension d and where $g \in \text{Homeo}(M)$.

Theorem 4.4. *Suppose $V=L$ and let $h \in \text{Homeo}(M)$ be arbitrary. Then $\text{tp}(g) = \text{tp}(h)$ if and only if g and h are conjugate in $\text{Homeo}(M)$.*

As in the case of first-order rigidity, it suffices to prove that if $\text{tp}(g) = \text{tp}(h)$ then g and h are conjugate. We retain the setup of Subsection 4.1 above. Thus, for each suitable pair (τ, β) , we have an $\mathcal{L}$ -definable collection $\mathcal{M}(\tau, \beta) \subseteq C(B, E)^{\mathbb{Z}}$ of codes for M adapted to τ and β . We can augment $\mathcal{M}(\tau, \beta)$ to $\mathcal{M}(\tau, \beta, g) \subseteq C(B, E)^{\mathbb{Z}} \times C(B, E)^{\mathbb{Z}}$ by adjoining an atlas for M that is twisted by g . Precisely, we set $\mathcal{M}(\tau, \beta, g) \subseteq C(B, E)^{\mathbb{Z}} \times C(B, E)^{\mathbb{Z}}$ to consist of pairs of sequences $((f_n), (h_n))$ such that:

$$(1) \ (f_n), (h_n) \in \mathcal{M}(\tau, \beta);$$

(2)

$$\overline{\bigcup_{n \in \mathbb{Z}} f_n[B]} = \overline{\bigcup_{n \in \mathbb{Z}} h_n[B]}.$$

(3) For all $x \in B$ and $n \in \mathbb{Z}$, we have $h_n(x) = f_n(x)$ if and only if $g\tau^n\beta[x] = \tau^n\beta[x]$.

We then set $\mathcal{M}(g) = \bigcup_{\tau, \beta} \mathcal{M}(\tau, \beta, g)$, which is clearly $\mathcal{L}$ -definable with g as a parameter. The elements of $\mathcal{M}(g)$ are the codes of g .

Proposition 4.5. *Let $g, h \in G_M$ and suppose $\mathcal{M}(g) \cap \mathcal{M}(h) \neq \emptyset$. Then $\mathcal{M}(g) = \mathcal{M}(h)$ and g and h are conjugate in G_M . Conversely, if g and h are conjugate in G_M then $\mathcal{M}(g) = \mathcal{M}(h)$.*

Proof. Suppose first that

$$((\alpha_n), (\beta_n)) \in \mathcal{M}(g) \cap \mathcal{M}(h).$$

Then, there exist sufficiently transitive homeomorphisms $\{\tau_g, \tau_h\}$ and collared balls $\{\beta_g, \beta_h\}$ such that $((\alpha_n), (\beta_n)) \in \mathcal{M}(\tau_g, \beta_g, g)$ and $((\alpha_n), (\beta_n)) \in \mathcal{M}(\tau_h, \beta_h, h)$. Since $(\alpha_n) \in \mathcal{M}(\tau_g, \beta_g) \cap \mathcal{M}(\tau_h, \beta_h)$, there is a homeomorphism ϕ of M conjugating (τ_g, β_g) to (τ_h, β_h) . Specifically, for all $x \in B$ and $n \in \mathbb{Z}$, we have $\phi(\tau_g^n \beta_g[x]) = \tau_h^n \beta_h[x]$. That ϕ extends to all of M follows from the definition of (α_n) and Proposition 3.14. Similarly, ϕ conjugates g to h as well. Thus, if $\mathcal{M}(g) \cap \mathcal{M}(h) \neq \emptyset$ then g and h are conjugate.

Conversely, it is clear that if g and h are conjugate then $\mathcal{M}(g) = \mathcal{M}(h)$, and so the proposition follows. $\square$

Proof of Theorem 4.4. Let $h \in \text{Homeo}(M)$ be arbitrary. We have that g and h are conjugate if and only if $\mathcal{M}(g) = \mathcal{M}(h)$. These two sets are analytically projective. We fix an analytical well-ordering on $C(B, E)^{\mathbb{Z}} \times C(B, E)^{\mathbb{Z}}$ and a recursive basis $\{\mathcal{U}_i\}_{i \in \mathbb{N}}$ for the topology, and let σ_g and σ_h be the minimal codes in $\mathcal{M}(g)$ and $\mathcal{M}(h)$ respectively. As in the case of first-order rigidity, we set $\chi_i(x)$ to be the $\mathcal{L}$ -formula in one variable such that $G_M \models \chi(g)$ if and only if the minimal code for g lies in $\mathcal{U}_i$. Thus, if $\text{tp}(g) = \text{tp}(h)$ then for all i , we have

$$G_M \models [\chi_i(g) \leftrightarrow \chi_i(h)],$$

which in turn implies that g and h are conjugate. $\square$

5. INFINITARY LOGIC AND RIGIDITY

From now on, we will no longer assume that $V=L$. However, the arguments needed to establish Theorem 1.8 and Theorem 1.9 are very similar to those used to establish rigidity under $V=L$. Constructibility allows us to pick out a canonical code for a manifold or for a manifold marked with a homeomorphism,

whereas infinitary logics allow us to completely describe arbitrary codes with a single sentence. For the remainder of this section, we retain the setup from Subsections 4.1 and 4.2.

Proof of Theorem 1.8. Suppose M has dimension d . Fix a code

$$(f_n) \in \mathcal{M}(\tau, \beta) \subseteq \mathcal{M}(M),$$

and let $\chi_i(x)$ be the $\mathcal{L}$ -formula expressing that an element $(h_n) \in C(B, E)^{\mathbb{Z}}$ lies in the basis element $\mathcal{U}_i$. Let $I = \{i \mid (f_n) \in \mathcal{U}_i\}$. We set $\phi(x)$ to be the conjunction

$$\phi: \bigwedge_{i \in I} \chi_i(x) \wedge \bigwedge_{j \notin I} \neg \chi_j(x).$$

Then, if N is an arbitrary connected manifold, we have

$$G_N \models \dim_d \wedge \exists \tau \exists \beta \exists (g_n) \left((g_n) \in \mathcal{M}(\tau, \beta) \wedge \phi((g_n)) \right)$$

if and only if N has dimension d and N admits (f_n) as a code, in which case $M \cong N$. $\square$

Proof of Theorem 1.9. Let $g \in G_M$, and let $((f_n), (h_n)) \in \mathcal{M}(\tau, \beta, g) \subseteq \mathcal{M}(g)$ be a code for g . Let $\chi_i(x, y)$ be the $\mathcal{L}$ -formula expressing that a pair

$$((\alpha_n), (\beta_n)) \in C(B, E)^{\mathbb{Z}} \times C(B, E)^{\mathbb{Z}}$$

lies in the basis element $\mathcal{U}_i$. Let $I = \{i \mid ((f_n), (h_n)) \in \mathcal{U}_i\}$. We set $\phi(x, y)$ to be the conjunction

$$\phi: \bigwedge_{i \in I} \chi_i(x, y) \wedge \bigwedge_{j \notin I} \neg \chi_j(x, y).$$

Now, for $h \in G_M$ arbitrary, we see that

$$G_M \models \exists \tau \exists \beta \exists ((\gamma_n), (\delta_n)) \left(((\gamma_n), (\delta_n)) \in \mathcal{M}(\tau, \beta, h) \wedge \phi((\gamma_n), (\delta_n)) \right)$$

if and only if $((f_n), (h_n))$ is a code for g , in which case g and h are conjugate. $\square$

6. PROJECTIVE DETERMINACY AND NON-CLASSIFICATION

In this section, we will show that under projective determinacy, there are pairs of connected, noncompact manifolds with empty boundary whose homeomorphism groups are elementarily equivalent but which are not homeomorphic to each other. Similarly, we will show that two homeomorphisms of the same closed manifold can have the same type and not be conjugate; this will establish Theorems 1.2 and 1.7.

6.1. Non-homeomorphic surfaces with elementarily equivalent homeomorphism groups. Recall that, if X is a compact metrizable space, $K(X)$ denotes the hyperspace of all non-empty closed subsets of X equipped with the Vietoris topology.

In the following, let S^2 denote the 2-sphere and fix a homeomorphic embedding

$$2^{\mathbb{N}} \xrightarrow{\iota} S^2$$

of the Cantor space $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ into S^2 . Let also $\mathcal{C} = \iota[2^{\mathbb{N}}]$ denote its image, whereby $S^2 \setminus \mathcal{C}$ is the sphere with a Cantor set of punctures, also known as the *Cantor tree surface*. We will decorate $S^2 \setminus \mathcal{C}$ by appropriately attaching handles in such a way that under projective determinacy two such decorated surfaces will end up having elementarily equivalent homeomorphism groups, but nevertheless will be non-homeomorphic. We now proceed to describe this decoration.

For all finite binary strings $t \in 2^{<\mathbb{N}}$, fix a closed disk $D_t \subseteq S^2 \setminus \mathcal{C}$ with boundary curve C_t so that

- (1) $D_t \cap \overline{\bigcup_{s \neq t} D_s} = \emptyset$,
- (2) $\overline{\bigcup_{t \in 2^{<\mathbb{N}}} D_t} = \mathcal{C} \cup \bigcup_{t \in 2^{<\mathbb{N}}} D_t$,
- (3) for all $\alpha \in 2^{\mathbb{N}}$, the sequence of closed sets $(D_{\alpha|n})_{n=1}^{\infty}$ converges to $\{\iota(\alpha)\}$ in the Vietoris topology on $K(S^2)$.

We let M be the compact metrizable space with $S^2 \subseteq M$ obtained by attaching a handle H_t to S^2 along each boundary curve C_t in such a way that

- (1) $H_t \cap \overline{\bigcup_{s \neq t} H_s} = \emptyset$,
- (2) $\overline{\bigcup_{t \in 2^{<\mathbb{N}}} H_t} = \mathcal{C} \cup \bigcup_{t \in 2^{<\mathbb{N}}} H_t$,
- (3) for all $\alpha \in 2^{\mathbb{N}}$, the sequence of closed sets $(H_{\alpha|n})_{n=1}^{\infty}$ converges to $\{\iota(\alpha)\}$ in the Vietoris topology on $K(M)$.

Thus $D_t \cap H_t = C_t$ for all $t \in 2^{<\mathbb{N}}$.

We let

$$P = S^2 \setminus \overline{\bigcup_{t \in 2^{<\mathbb{N}}} D_t} = M \setminus \overline{\bigcup_{t \in 2^{<\mathbb{N}}} (D_t \cup H_t)}$$

and note that P is an open subset of S^2 and M .

For $t \in 2^{<\mathbb{N}}$, let

$$N_t = \{\alpha \in 2^{\mathbb{N}} : t \text{ is an initial segment of } \alpha\},$$

which is a clopen subset of $2^{\mathbb{N}}$. Thus, for every $t \in 2^{<\mathbb{N}}$, the set

$$\{F \in K(2^{\mathbb{N}}) : F \cap N_t \neq \emptyset\}$$

is clopen. Also, for $F \in K(2^{\mathbb{N}})$, we let

$$S_F = P \cup \bigcup_{F \cap N_t \neq \emptyset} H_t \cup \bigcup_{F \cap N_t = \emptyset} D_t$$

and note that S_F is a decorated Cantor tree surface where an end $\iota(\alpha)$ with $\alpha \in 2^{\mathbb{N}}$ is accumulated by genus if and only if $\alpha \in F$. By the homeomorphic classification of surfaces [8, 17], it follows that two such surfaces S_F and $S_{F'}$ are homeomorphic if and only if there exists a homeomorphism h of the full Cantor space $2^{\mathbb{N}}$ so that $h[F] = F'$.

Note also that, for all closed subsets $F \subseteq 2^{\mathbb{N}}$, $S_F \subseteq M$ and that $\overline{S_F}$ is the disjoint union of S_F and $\mathcal{C}$. We thus see that, for any two closed sets $F, F' \subseteq 2^{\mathbb{N}}$, the following two properties are equivalent.

- (1) There is a homeomorphism h of Cantor space $2^{\mathbb{N}}$ so that $h[F] = F'$,
- (2) there is a homeomorphism h of $\overline{S_F}$ with $\overline{S_{F'}}$ so that $h[\mathcal{C}] = \mathcal{C}$.

Let $Q = [0, 1]^{\mathbb{N}}$ denote the Hilbert cube and recall the notion of Z -sets in Q . First of all, the collection of Z -sets forms an ideal of closed subsets of Q with the property that any homeomorphism $L \xrightarrow{h} L'$ between two Z -sets extends to a homeomorphism $Q \xrightarrow{\tilde{h}} Q$. Furthermore, Q may be homeomorphically embedded into itself as a Z -set. As these are the only facts needed for our construction, we need not worry about the exact definition of Z -sets and instead refer the reader to [21, Chapter 5] for further details.

Because every compact metric space embeds into Q , we can, by composing with the embedding of Q into Q as a Z -set, suppose that M is itself a Z -subset of Q . In particular, this means that, for any two closed subsets

$$L, L' \subseteq M \subseteq Q$$

and every homeomorphism $L \xrightarrow{h} L'$, there is a homeomorphism $\tilde{h}$ of Q extending h . It follows that (1) and (2) above are equivalent with

- (3) there is a homeomorphism h of Q with $h[\overline{S_F}] = \overline{S_{F'}}$ and $h[\mathcal{C}] = \mathcal{C}$.

Observe that the set

$$\mathbb{X} = \{(L, h) \in K(Q) \times \text{Homeo}(Q) \mid h[L] = L \text{ \& } h[\mathcal{C}] = \mathcal{C}\}$$

is closed and the map

$$F \in K(2^{\mathbb{N}}) \mapsto \overline{S_F} \in K(Q)$$

is continuous. It follows from this that the set

$$\mathbb{H} = \{(F, h) \in K(2^{\mathbb{N}}) \times \text{Homeo}(Q) \mid h[\overline{S_F}] = \overline{S_F} \text{ \& } h[\mathcal{C}] = \mathcal{C}\}$$

is closed.

Proposition 6.1. *Suppose that ϕ is a first-order sentence in the language of group theory. Then*

$$\{F \in K(2^{\mathbb{N}}) \mid \text{Homeo}(S_F) \models \phi\}$$

is a projective set.

Proof. By rewriting ϕ in prenex form with matrix in conjunctive normal form, we see that ϕ is equivalent to a sentence of the form

$$\exists x_1 \forall x_2 \cdots \exists x_{n-1} \forall x_n \bigwedge_{i=1}^p \bigvee_{j=1}^q (w_{ij} = u_{ij})^{\epsilon_{ij}},$$

where $w_{ij}(\bar{x})$ and $u_{ij}(\bar{x})$ are two words in the language of group theory and $\epsilon_{ij} \in \{-1, 1\}$. Here $(w = u)^1$ designates the formula $w = u$ itself, whereas $(w = u)^{-1}$ designates its negation $w \neq u$.

Observe that, for every $F \in K(2^{\mathbb{N}})$, the vertical section

$$\mathbb{H}_F = \{h \in \text{Homeo}(Q) \mid h[\overline{S_F}] = \overline{S_F} \ \& \ h[\mathcal{C}] = \mathcal{C}\}$$

is a closed subgroup of $\text{Homeo}(Q)$. Also, because every homeomorphism h of S_F extends (in multiple ways) to a homeomorphism of the ambient space Q preserving $\overline{S_F}$ and $\mathcal{C}$, the restriction to S_F defines a continuous epimorphism

$$\mathbb{H}_F \xrightarrow{\pi_F} \text{Homeo}(S_F).$$

Therefore, the statement ϕ holds in $\text{Homeo}(S_F)$ if and only if

$$\begin{aligned} &\exists f_1 \in \mathbb{H}_F \forall f_2 \in \mathbb{H}_F \cdots \exists f_{n-1} \in \mathbb{H}_F \forall f_n \in \mathbb{H}_F \\ &\bigwedge_{i=1}^p \bigvee_{j=1}^q \left(w_{ij}(\pi_F(f_1), \dots, \pi_F(f_n)) = u_{ij}(\pi_F(f_1), \dots, \pi_F(f_n)) \right)^{\epsilon_{ij}}. \end{aligned}$$

Suppose now that $w(x_1, \dots, x_n)$ and $u(x_1, \dots, x_n)$ are two words in the language of group theory. We let $[w = u]$ denote the set of all tuples

$$(f_1, \dots, f_n, F) \in K(2^{\mathbb{N}}) \times \text{Homeo}(Q)^n$$

satisfying

$$w(\pi_F(f_1), \dots, \pi_F(f_n)) = u(\pi_F(f_1), \dots, \pi_F(f_n)),$$

that is, so that

$$\forall z \in Q \left(z \in S_F \rightarrow w(f_1, \dots, f_n)(z) = u(f_1, \dots, f_n)(z) \right).$$

By the latter expression, we find that $[w = u]$ is a coanalytic subset of $K(2^{\mathbb{N}}) \times \text{Homeo}(Q)^n$. Expanding the quantifiers $\exists f \in \mathbb{H}_F$ and $\forall f \in \mathbb{H}_F$ as

$$\exists f \left((F, f) \in \mathbb{H} \ \& \ \dots \right) \quad \text{and} \quad \forall f \left((F, f) \in \mathbb{H} \rightarrow \dots \right),$$

we finally see that

$$\exists f_1 \in \mathbb{H}_F \forall f_2 \in \mathbb{H}_F \cdots \exists f_{n-1} \in \mathbb{H}_F \forall f_n \in \mathbb{H}_F \\ \bigwedge_{i=1}^p \bigvee_{j=1}^q \left(w_{ij}(\pi_F(f_1), \dots, \pi_F(f_n)) = u_{ij}(\pi_F(f_1), \dots, \pi_F(f_n)) \right)^{\epsilon_{ij}}$$

defines a projective condition on F . In other words, the set

$$\{F \in K(2^{\mathbb{N}}) : \text{Homeo}(S_F) \models \phi\}$$

is projective. $\square$

Theorem 6.2 (Assume projective determinacy). *There are non-homeomorphic orientable surfaces with elementarily equivalent homeomorphism groups.*

Proof. Let E_0 denote the equivalence relation of eventual agreement of infinite binary sequences, that is, for $\alpha, \beta \in 2^{\mathbb{N}}$, we set

$$\alpha E_0 \beta \Leftrightarrow \exists m \forall n \geq m \ \alpha_n = \beta_n.$$

By [3, Theorem 3], there is Borel measurable map

$$2^{\mathbb{N}} \xrightarrow{\kappa} K(2^{\mathbb{N}})$$

so that

$$\alpha E_0 \beta \Leftrightarrow \exists h \in \text{Homeo}(2^{\mathbb{N}}) \ h[\kappa(\alpha)] = \kappa(\beta).$$

By Proposition 6.1, for every first-order sentence of the language of group theory ϕ , the set

$$\{F \in K(2^{\mathbb{N}}) \mid \text{Homeo}(S_F) \models \phi\}$$

is projective. As the inverse image of a projective set by a Borel function is also projective, it follows that

$$A_\phi = \{\alpha \in 2^{\mathbb{N}} \mid \text{Homeo}(S_{\kappa(\alpha)}) \models \phi\}$$

is projective too. Moreover, for all $\alpha, \beta \in 2^{\mathbb{N}}$,

$$\begin{aligned} \alpha E_0 \beta &\Rightarrow \exists h \in \text{Homeo}(2^{\mathbb{N}}) \ h[\kappa(\alpha)] = \kappa(\beta) \\ &\Rightarrow S_{\kappa(\alpha)} \cong S_{\kappa(\beta)} \\ &\Rightarrow \text{Homeo}(S_{\kappa(\alpha)}) \cong \text{Homeo}(S_{\kappa(\beta)}) \\ &\Rightarrow (\alpha \in A_\phi \leftrightarrow \beta \in A_\phi). \end{aligned}$$

In other words, A_ϕ is an E_0 -invariant projective set.

Now, under the assumption of projective determinacy, every projective subset of $2^{\mathbb{N}}$ is Lebesgue measurable. It thus follows from Kolmogorov's zero-one

law that A_ϕ is either null or conull. If A_ϕ is conull, let $C_\phi = A_\phi$ and otherwise let $C_\phi = 2^\mathbb{N} \setminus A_\phi$. It thus follows that

$$C = \bigcap_{\phi} C_\phi,$$

where the intersection runs over the countable collection of all first-order sentences ϕ of the language of group theory, is conull in $2^\mathbb{N}$. Furthermore, for all $\alpha, \beta \in C$, we have

$$\text{Homeo}(S_{\kappa(\alpha)}) \equiv \text{Homeo}(S_{\kappa(\beta)}).$$

It thus suffices to pick $\alpha, \beta \in C$ that are E_0 -inequivalent and hence so that $S_{\kappa(\alpha)} \not\equiv S_{\kappa(\beta)}$, but nevertheless $\text{Homeo}(S_{\kappa(\alpha)}) \equiv \text{Homeo}(S_{\kappa(\beta)})$. $\square$

6.2. Nonhomeomorphic manifolds with elementarily equivalent homeomorphism groups. We now wish to show that the failure of first-order rigidity for homeomorphism groups of noncompact surfaces implies failure of first-order rigidity for noncompact manifolds in all higher dimensions. For an arbitrary connected surface S , we will write $P_k(S)$ for $S \times S^k$, the Cartesian product of S with the k -sphere S^k . we will suppress the notation S and k when they are clear from context or irrelevant.

The goal of this subsection is to show the following.

Theorem 6.3. *For every fixed $k \geq 1$ the manifold $P_k(S)$ and its homeomorphism group $\text{Homeo}(P_k(S))$ are uniformly interpretable in $\text{Homeo}(S)$.*

Observe that there is a natural map $\pi: P_k(S) \rightarrow S$ by projecting onto the first factor. Moreover, if $K \subseteq S$ is a compact subspace then $\pi^{-1}(K)$ is a compact subspace of $P_k(S)$. It is straightforward then that the end-space $\mathcal{E}(S)$ is homeomorphic to the end-space $\mathcal{E}(P_k(S))$.

Let

$$K_0 \subseteq K_1 \subseteq \dots$$

be a cofinal sequence of increasing compact subspaces of S and let $U_i = S \setminus K_i$. We let $(\hat{U}_i)$ denote a coherent choice of connected component of each U_i , so that $\hat{U}_i$ is a connected component of U_i for each i and $\hat{U}_{i+1} \subseteq \hat{U}_i$ for each i . Thus, $(\hat{U}_i)$ defines an end of the surface S . It follows then that $(\hat{V}_i) = (\hat{U}_i \times S^k)$, given by taking the Cartesian product of each $\hat{U}_i$ with S^k , defines an end of $P_k(S)$. We will say that $(\hat{V}_i)$ is *accumulated by genus* if $(\hat{U}_i)$ is. The set of ends accumulated by genus is written $\mathcal{E}^g(P_k(S))$.

Lemma 6.4. *The inclusion $\mathcal{E}^g(P_k(S)) \subseteq \mathcal{E}(P_k(S))$ is $\text{Homeo}(P_k(S))$ -invariant.*

Proof. It suffices to distinguish topologically between the ends of $P_k(S)$ which are accumulated by genus and those which are not, since a topological distinction will be preserved by all homeomorphisms. We observe that an end $(\hat{V}_i)$ is accumulated by genus if and only if there exists a disjoint collection of compact submanifolds $\{W_i\}_{i \in \mathbb{N}}$ of $P_k(S)$ with $W_i \subseteq \hat{V}_i$ for all i and such that:

- (1) For all i , the manifold W_i admits a locally separating submanifold

$$M_i = S^1 \times S^k \times \{0\} \subseteq W_i$$

such that $W_i \cong M_i \times [-1, 1]$;

- (2) For all i , the complement $N_i = P_k(S) \setminus W_i$ is connected.

We claim that such a sequence $\{W_i\}_{i \in \mathbb{N}}$ exists if and only if for all i , the component $\hat{V}_i$ contains a simple closed curve which is nonseparating in S ; this latter property is possible if and only if the end $(\hat{V}_i)$ is accumulated by genus.

Note that if $(\hat{V}_i)$ is accumulated by genus then such a collection $\{W_i\}_{i \in \mathbb{N}}$ clearly exists. Indeed, in each $\hat{U}_i$ there is a nonseparating essential closed curve γ_i , and we may simply take W_i to be the closure of a small tubular neighborhood of γ_i crossed with S^k . Conversely, if $(\hat{V}_i)$ is not accumulated by genus then no such construction is possible.

In general, we may assume M_i locally separates $P_k(S)$ but does not globally separate $P_k(S)$. Choose a loop δ in $P_k(S)$ which intersects M_i exactly once. There is a compact essential subsurface $\Sigma \subseteq S$ such that $\delta \cup M_i \subseteq \Sigma \times S^k$, and we claim that Σ must have positive genus, and suppose the contrary for a contradiction.

We view M_i as an element of $H_{k+1}(\Sigma \times S^k, \mathbb{Z}/2\mathbb{Z})$. Writing ∂ for the boundary of $\Sigma \times S^k$, Poincaré–Lefschetz duality implies that intersection number with elements of $H_{k+1}(\Sigma \times S^k, \mathbb{Z}/2\mathbb{Z})$ furnishes an isomorphism with $H^1(\Sigma \times S^k, \partial, \mathbb{Z}/2\mathbb{Z})$. If Σ has genus 0 then the long exact sequence on relative cohomology shows that the inclusion of the boundary ∂ of $\Sigma \times S^k$ induces a surjection

$$H^0(\partial, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(\Sigma \times S^k, \partial, \mathbb{Z}/2\mathbb{Z}).$$

On the other hand, M_i furnishes a nontrivial $\mathbb{Z}/2\mathbb{Z}$ -valued first cohomology class of $\Sigma \times S^k$ relative to the boundary, which is not in the image of $H^0(\partial, \mathbb{Z}/2\mathbb{Z})$. This is a contradiction. $\square$

Corollary 6.5. *Assume that PD holds and let $d \geq 2$. There exist pairs of non-homeomorphic d -manifolds whose homeomorphism groups are elementarily equivalent.*

Proof. By Theorem 1.3 and Lemma 6.4, it suffices to find two surfaces which are not homeomorphic but have elementarily equivalent homeomorphism groups,

and which remain non-homeomorphic after taking a Cartesian product with spheres of positive dimension. This follows immediately from the construction of the surfaces S_F in Section 6.1 above and Lemma 6.4. $\square$

6.3. Failure of type rigidity under projective determinacy. In the present section, we let M be a fixed manifold of dimension $d \geq 1$. For this section, we will not assume that M has no boundary nor that it be connected.

Let D_∞ denote the infinite dihedral group viewed as the group of order-preserving and order-reversing automorphisms of the linear order $(\mathbb{Z}, <)$. We define an action $D_\infty \curvearrowright \{-1, 1\}^\mathbb{Z}$ by letting

$$(\sigma \cdot \alpha)_{\sigma(i)} = \begin{cases} \alpha_i & \text{if } \sigma \text{ is order-preserving,} \\ -\alpha_i & \text{if } \sigma \text{ is order-reversing,} \end{cases}$$

for all $\sigma \in D_\infty$, $\alpha \in \{-1, 1\}^\mathbb{Z}$ and $i \in \mathbb{Z}$. Observe also that by restricting to the subgroup $\mathbb{Z} \leq D_\infty$ of translations, we obtain the usual shift-action $\mathbb{Z} \curvearrowright \{-1, 1\}^\mathbb{Z}$. Note that the shift-action $\mathbb{Z} \curvearrowright \{-1, 1\}^\mathbb{Z}$ is mixing for the $\{\frac{1}{2}, \frac{1}{2}\}$ -product measure μ on $\{-1, 1\}^\mathbb{Z}$ and so every $\mathbb{Z}$ -invariant μ -measurable set $B \subseteq \{-1, 1\}^\mathbb{Z}$ is either null or conull.

Proposition 6.6. *There is a continuous map $\{-1, 1\}^\mathbb{Z} \xrightarrow{f} \text{Homeo}_0([0, 1])$ such that*

$$\exists \sigma \in \mathbb{Z} \ (\sigma \cdot \alpha = \beta) \quad \Leftrightarrow \quad \exists h \in \text{Homeo}_0(B^d) \ (hf_\alpha h^{-1} = f_\beta)$$

and

$$\exists \sigma \in D_\infty \ (\sigma \cdot \alpha = \beta) \quad \Leftrightarrow \quad \exists h \in \text{Homeo}(B^d) \ (hf_\alpha h^{-1} = f_\beta).$$

Proof. Let $i \in \mathbb{Z} \mapsto p_i \in (0, 1)$ be an order-embedding of $\mathbb{Z}$ into the open interval $(0, 1)$ so that $\lim_{i \rightarrow \infty} p_{-i} = 0$ and $\lim_{i \rightarrow \infty} p_i = 1$. Fix also homeomorphism ζ_i of $J_i = (p_i, p_{i+1})$ such that $x < \zeta_i(x)$ for all $x \in J_i$ and therefore also $\zeta_i^{-1}(x) < x$ for all $x \in J_i$.

For each $\alpha \in \{-1, 1\}^\mathbb{Z}$, we construct

$$f_\alpha \in \text{Homeo}_0([0, 1])$$

by letting $f_\alpha(0) = 0$, $f_\alpha(1) = 1$ and

$$f_\alpha \upharpoonright_{J_i} = \zeta_i^{\alpha_i} \quad \& \quad f_\alpha(p_i) = p_i$$

for all $i \in \mathbb{Z}$. The map $\alpha \mapsto f_\alpha$ is evidently continuous and $\{0, 1\} \cup \{p_i \mid i \in \mathbb{Z}\}$ is exactly the set of fixed points of f_α .

Suppose $f_\beta h = h f_\alpha$ for some $h \in \text{Homeo}([0, 1])$ and $\alpha, \beta \in \{-1, 1\}^\mathbb{Z}$. Then h maps the fixed points of f_α to those of f_β and so there is some $\sigma \in D_\infty$ such that $h[J_i] = J_{\sigma(i)}$ for all $i \in \mathbb{Z}$. In particular, σ is order-preserving if and only

h is orientation-preserving. Note that, if h is orientation-preserving, then for all $i \in \mathbb{Z}$

$$\begin{aligned} \alpha_i = 1 &\Leftrightarrow \forall x \in J_i \quad x < f_\alpha(x) \\ &\Leftrightarrow \forall x \in J_i \quad h(x) < hf_\alpha(x) = f_\beta h(x) \\ &\Leftrightarrow \forall y \in J_{\sigma(i)} \quad y < f_\beta(y) \\ &\Leftrightarrow \beta_{\sigma(i)} = 1, \end{aligned}$$

whereas, if h is orientation-reversing,

$$\begin{aligned} \alpha_i = 1 &\Leftrightarrow \forall x \in J_i \quad x < f_\alpha(x) \\ &\Leftrightarrow \forall x \in J_i \quad f_\beta h(x) = hf_\alpha(x) < h(x) \\ &\Leftrightarrow \forall y \in J_{\sigma(i)} \quad f_\beta(y) < y \\ &\Leftrightarrow \beta_{\sigma(i)} = -1. \end{aligned}$$

Thus, in either case, we find that $\beta = \sigma \cdot \alpha$.

For the converse, note that any two ζ_i and ζ_j are conjugate by an orientation-preserving homeomorphism $J_i \rightarrow J_j$, whereas ζ_i and ζ_j^{-1} are conjugate by an orientation-reversing homeomorphism $J_i \rightarrow J_j$. Using this, one easily sees that, when $\beta = \sigma \cdot \alpha$ for some $\sigma \in D_\infty$, then also $f_\beta h = hf_\alpha$ for some $h \in \text{Homeo}([0, 1])$.

This shows that

$$\exists \sigma \in D_\infty \quad (\sigma \cdot \alpha = \beta) \quad \Leftrightarrow \quad \exists h \in \text{Homeo}(M) \quad (hf_\alpha h^{-1} = f_\beta).$$

To get the first equivalence, we simply note that σ is order-preserving if and only if $\sigma \in \mathbb{Z}$, whereas h is order-preserving if and only if $h \in \text{Homeo}_0([0, 1])$. $\square$

Proposition 6.7. *For every $d \geq 2$ there is a continuous map $\{-1, 1\}^{\mathbb{Z}} \xrightarrow{g} \text{Homeo}_\partial(B^d(1))$ such that*

$$\begin{aligned} \exists \sigma \in D_\infty \quad (\sigma \cdot \alpha = \beta) &\quad \Leftrightarrow \quad \exists h \in \text{Homeo}(B^d(1)) \quad (hg_\alpha h^{-1} = g_\beta) \\ &\quad \Leftrightarrow \quad \exists h \in \text{Homeo}_0(B^d(1)) \quad (hg_\alpha h^{-1} = g_\beta). \end{aligned}$$

Proof. Recall that, for a topological space X , ΣX denotes the suspension over X , that is, the product $X \times [0, 1]$ with $X \times \{1\}$ and $X \times \{0\}$ collapsed to single points p^+ and p^- respectively. Observe that every homeomorphism of X extends canonically to a homeomorphism of the suspension ΣX .

We let $\Sigma^{d-1}[0, 1]$ denote the $d - 1$ -fold suspension

$$\Sigma^{d-1}[0, 1] = \underbrace{\Sigma(\cdots \Sigma(\Sigma[0, 1]) \cdots)}_{d-1 \text{ times}} \cong B^d(1).$$

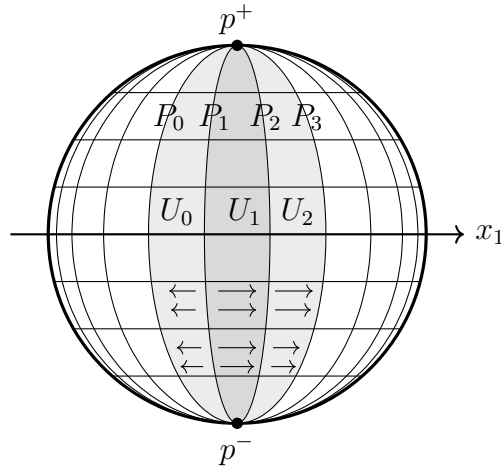
So, by induction, the $f_\alpha \in \text{Homeo}_\partial([0, 1])$ constructed in the proof of Proposition 6.6 extend to homeomorphisms $g_\alpha \in \text{Homeo}_\partial(\Sigma^{d-1}[0, 1])$ so that

$$\overline{\{x \in \Sigma^{d-1}[0, 1] \mid g_\alpha(x) \neq x\}} = \Sigma^{d-1}[0, 1].$$

We also note that there is an orientation-reversing $k \in \text{Homeo}(\Sigma^{d-1}[0, 1])$ such that $g_\alpha k = k g_\alpha$ for all α . Indeed, as $d \geq 2$, we may let κ be the orientation-reversing homeomorphism of $\Sigma[0, 1]$ defined by $\kappa(r, t) = (r, 1 - t)$ and then simply let k be the induced homeomorphism of $\Sigma^{d-1}[0, 1]$. It thus follows that if $h \in \text{Homeo}(\Sigma^{d-1}[0, 1])$ is orientation-reversing and $g_\beta h = h g_\alpha$, then also $g_\beta k h = k h g_\alpha$ and $k h \in \text{Homeo}_0(\Sigma^{d-1}[0, 1])$. This verifies the last equivalence of the proposition.

Suppose now that $\sigma \cdot \alpha = \beta$ for some $\sigma \in D_\infty$ and $\alpha, \beta \in \{-1, 1\}^\mathbb{Z}$. Then $f_\beta h = h f_\alpha$ for some $h \in \text{Homeo}([0, 1])$ and, if $\tilde{h}$ denotes the induced homeomorphism $\tilde{h} \in \text{Homeo}(\Sigma^{d-1}[0, 1])$, then $g_\beta \tilde{h} = \tilde{h} g_\alpha$.

For the converse, note first that as $\Sigma^{d-1}[0, 1] \cong B^d(1)$ we may talk about its interior, which can be seen to be a union of $\mathbb{Z}$ -indexed regions $\{U_i\}_{i \in \mathbb{Z}}$, each of which is invariant under all g_α . The interior of each U_i is itself homeomorphic to $\mathbb{R}^d$, whereas two regions U_{i-1} and U_i meet along a copy P_i of $\mathbb{R}^{d-1}$ that is pointwise fixed by all g_α . Furthermore, by appropriately choosing the homeomorphisms $\text{int}(U_i) \cong \mathbb{R}^d$, up to conjugacy, all g_α act on each such copy of $\mathbb{R}^d$ by translation either to the left or to the right with respect to the coordinate axis x_1 according to whether $\alpha(i) = -1$ or $\alpha(i) = 1$. In particular, $\bigcup_{i \in \mathbb{Z}} P_i$ is exactly the collection of points in the interior of $\Sigma^{d-1}[0, 1]$ fixed by the g_α .



Observe that, for all $\alpha \in \{-1, 1\}^{\mathbb{Z}}$, $i \in \mathbb{Z}$, and $x \in \text{int}(U_i)$,

$$\begin{aligned}\alpha_i = -1 &\Leftrightarrow \lim_{n \rightarrow \infty} g_\alpha^n(x) \in P_i \\ \alpha_i = 1 &\Leftrightarrow \lim_{n \rightarrow \infty} g_\alpha^n(x) \in P_{i+1}.\end{aligned}$$

Suppose now that $g_\beta h = h g_\alpha$ for some $h \in \text{Homeo}(\Sigma^{d-1}[0, 1])$. Then h preserves the interior of $\Sigma^{d-1}[0, 1]$ and must therefore map $\bigcup_{i \in \mathbb{Z}} P_i$ to itself. It follows that there is some $\sigma \in D_\infty$ so that $h[U_i] = U_{\sigma(i)}$ for all $i \in \mathbb{Z}$. On the other hand, since $P_i = U_{i-1} \cap U_i$, we have

$$h[P_i] = h[U_{i-1}] \cap h[U_i] = U_{\sigma(i-1)} \cap U_{\sigma(i)} = \begin{cases} P_{\sigma(i)} & \text{if } \sigma \text{ is order-preserving,} \\ P_{\sigma(i)+1} & \text{if } \sigma \text{ is order-reversing.} \end{cases}$$

Suppose σ is order-preserving. Then, for all i

$$\begin{aligned}\alpha_i = -1 &\Leftrightarrow \forall x \in \text{int}(U_i) \lim_{n \rightarrow \infty} g_\alpha^n(x) \in P_i \\ &\Leftrightarrow \forall x \in \text{int}(U_i) \lim_{n \rightarrow \infty} g_\beta^n h(x) = \lim_{n \rightarrow \infty} h g_\alpha^n(x) = h\left(\lim_{n \rightarrow \infty} g_\alpha^n(x)\right) \\ &\hspace{15em} \in h[P_i] = P_{\sigma(i)} \\ &\Leftrightarrow \forall y \in \text{int}(U_{\sigma(i)}) \lim_{n \rightarrow \infty} g_\beta^n(y) \in P_{\sigma(i)} \\ &\Leftrightarrow \beta_{\sigma(i)} = -1.\end{aligned}$$

On the other had, if σ is order-reversing

$$\begin{aligned}\alpha_i = -1 &\Leftrightarrow \forall x \in \text{int}(U_i) \lim_{n \rightarrow \infty} g_\alpha^n(x) \in P_i \\ &\Leftrightarrow \forall x \in \text{int}(U_i) \lim_{n \rightarrow \infty} g_\beta^n h(x) = \lim_{n \rightarrow \infty} h g_\alpha^n(x) = h\left(\lim_{n \rightarrow \infty} g_\alpha^n(x)\right) \\ &\hspace{15em} \in h[P_i] = P_{\sigma(i)+1} \\ &\Leftrightarrow \forall y \in \text{int}(U_{\sigma(i)}) \lim_{n \rightarrow \infty} g_\beta^n(y) \in P_{\sigma(i)+1} \\ &\Leftrightarrow \beta_{\sigma(i)} = 1.\end{aligned}$$

Since this holds for all $i \in \mathbb{Z}$ we find that $\beta = \sigma \cdot \alpha$. $\square$

We can now prove the failure of type rigidity under projective determinacy.

Theorem 6.8. *Assume PD. For every manifold M of dimension $d \geq 1$, there are $g_1, g_2 \in \text{Homeo}_0(M)$ that are non-conjugate in $\text{Homeo}(M)$, whereas*

$$\text{tp}(g_1) = \text{tp}(g_2).$$

Proof. Recall our notation $S^{d-1} = \partial(B^d(1))$. Suppose $h \in \text{Homeo}_0(B^d(1))$, whereby

$$h \upharpoonright_{S^{d-1}} \in \text{Homeo}_0(S^{d-1}).$$

Since $\text{Homeo}_0(S^{d-1})$ is path connected, we may find a continuous path

$$(h_t)_{t \in [1,2]} \in \text{Homeo}_0(S^{d-1})$$

beginning at $h_1 = h \upharpoonright_{S^{d-1}}$ and so that $h_2 = \text{id}_{S^{d-1}}$. We may therefore define a homeomorphism $\tilde{h} \in \text{Homeo}_\partial(B^d(2))$ that extends h by letting

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } \|x\| \leq 1, \\ rh_r(\frac{x}{r}) & \text{if } 1 < \|x\| = r \leq 2. \end{cases}$$

Fix now a homeomorphic embedding of $B^d(2)$ into M and identify $B^d(2)$ with its image. Then every element of $\text{Homeo}_\partial(B^d(2))$ extends to an element of $\text{Homeo}(M)$ by setting it to be the identity on $M \setminus B^d(2)$. By the argument above, we thus see that every $h \in \text{Homeo}_0(B^d(1))$ canonically extends to a full homeomorphism of M , which is the identity on $M \setminus B^d(2)$. Similarly, every $g \in \text{Homeo}_\partial(B^d(1))$ extends to all of M by setting it to be the identity on $M \setminus B^d(1)$.

Assume first $d \geq 2$. Then, by combining the above discussion with Proposition 6.7, we obtain a continuous map

$$\{-1, 1\}^{\mathbb{Z}} \xrightarrow{g} \text{Homeo}_0(M)$$

such that

$$\exists \sigma \in D_\infty (\sigma \cdot \alpha = \beta) \quad \Leftrightarrow \quad \exists h \in \text{Homeo}(M) (hg_\alpha h^{-1} = g_\beta).$$

Observe that, for every first-order formula $\phi(x)$ of the language of group theory, the set

$$\{f \in \text{Homeo}_0(M) \mid \text{Homeo}(M) \models \phi(f)\}$$

is projective and invariant under conjugacy by $\text{Homeo}(M)$. It follows that

$$A_\phi = \{\alpha \in \{-1, 1\}^{\mathbb{Z}} \mid \text{Homeo}(M) \models \phi(g_\alpha)\}$$

is an D_∞ -invariant projective set, which is thus either null or conull. Let again $C_\phi = A_\phi$ if A_ϕ is conull and $C_\phi = \{-1, 1\}^{\mathbb{Z}} \setminus A_\phi$ otherwise, we find that

$$C = \bigcap_{\phi} C_\phi,$$

where the intersection runs over the countable collection of all first-order formulae $\phi(x)$ of the language of group theory, is conull in $\{-1, 1\}^{\mathbb{Z}}$. Furthermore, for all $\alpha, \beta \in C$, we have

$$\text{tp}(g_\alpha) = \text{tp}(g_\beta).$$

It thus suffices to pick $\alpha, \beta \in C$ that are D_∞ -orbit inequivalent and hence such that g_α and g_β are non-conjugate in $\text{Homeo}(M)$.

The argument for the case $d = 1$ is similar and uses Proposition 6.6 in place of 6.7. $\square$

7. CONSISTENCY OF NON-CLASSIFICATION OVER ZFC ALONE

Projective determinacy has nontrivial consistency strength over ZFC. As such, it is natural to wonder whether the conclusion of Theorem 6.2 in itself has any consistency strength beyond that of ZFC. In this short section, we will show that it does not. In particular, we will show that the consistency of ZFC implies the consistency of ZFC + “for every $d \geq 2$, there are non-homeomorphic orientable d -dimensional manifolds with elementarily equivalent homeomorphism groups” by showing that this statement holds after forcing to collapse $\mathfrak{c}^+$ to be ω (where here $\mathfrak{c} = |\mathbb{R}|$ is the cardinality of the continuum). In this section, we will use standard arguments from forcing; see [6, 13, 22] for relevant background.

Fix a countable transitive model V of ZFC and let $\kappa = \mathfrak{c}^+$ as computed in V . Let also $\text{Col}(\omega, \kappa)$ be the partial order consisting of all partial functions

$$p: \omega \rightarrow \kappa$$

with finite domain and in which $p \leq q$ when p extends q as a function. Thus $p \leq \emptyset$ where $\emptyset$ denotes the function with empty domain. Elements of $\text{Col}(\omega, \kappa)$ are called *forcing conditions*. For all $m < \omega$ and $\alpha < \kappa$, let

$$D_{m,\alpha} = \{p \in \text{Col}(\omega, \kappa) \mid m \in \text{dom}(p) \text{ \& \; } \alpha \in \text{im}(p)\}$$

and note that $D_{m,\alpha}$ is *dense* in $\text{Col}(\omega, \kappa)$, meaning that each q has a minorant $p \leq q$ in $D_{m,\alpha}$. A *generic extension* $V[G]$ of V is the smallest transitive model of ZFC containing the submodel V such that $G \in V[G]$, where G is some $\text{Col}(\omega, \kappa)$ -*generic filter*. The latter means that $G \subseteq \text{Col}(\omega, \kappa)$ is an upwards closed subset such that any two $p, q \in G$ has a common minorant $r \in G$ and furthermore such that G meets every dense subset of $\text{Col}(\omega, \kappa)$. In particular, in $V[G]$, we obtain a surjection

$$g = \bigcup G: \omega \longrightarrow \kappa$$

and therefore $|\kappa| = |\omega| = \aleph_0$ in $V[G]$. It follows that every ordinal $\alpha < \kappa$ is countable in $V[G]$.

We briefly recall a few facts about the forcing relation $\Vdash_{\text{Col}(\omega, \kappa)}$. For a parameter-free formula $\chi(x_1, \dots, x_m)$, elements $a_1, \dots, a_m \in V$ and a forcing condition $p \in \text{Col}(\omega, \kappa)$, we have

$$p \Vdash_{\text{Col}(\omega, \kappa)} \chi(\check{a}_1, \dots, \check{a}_m) \quad \Leftrightarrow \quad \forall G \ni p \quad V[G] \models \chi(a_1, \dots, a_m).$$

Similarly, for a fixed generic filter G , we have

$$V[G] \models \chi(a_1, \dots, a_m) \iff \exists q \in G \quad q \Vdash_{\text{Col}(\omega, \kappa)} \chi(\check{a}_1, \dots, \check{a}_m).$$

The partial order $\text{Col}(\omega, \kappa)$ is easily seen to be *homogeneous*, meaning that, for every two elements $p, q \in \text{Col}(\omega, \kappa)$, there is an automorphism θ of $\text{Col}(\omega, \kappa)$ so that $\theta(p)$ and q have a common minorant. We also note that, if $\chi(x_1, \dots, x_m)$ is parameter-free and $a_1, \dots, a_m \in V$, then

$$p \Vdash_{\text{Col}(\omega, \kappa)} \chi(\check{a}_1, \dots, \check{a}_m) \iff \theta(p) \Vdash_{\text{Col}(\omega, \kappa)} \chi(\check{a}_1, \dots, \check{a}_m)$$

[13, pp. 155-157]. It thus follows from homogeneity that the empty condition forces every formula or its negation, i.e. either

$$\emptyset \Vdash_{\text{Col}(\omega, \kappa)} \chi(\check{a}_1, \dots, \check{a}_m)$$

or

$$\emptyset \Vdash_{\text{Col}(\omega, \kappa)} \neg \chi(\check{a}_1, \dots, \check{a}_m).$$

Lemma 7.1. *Suppose that $\phi(x, y)$ is a parameter-free formula that defines a function from ω_1 to 2^ω . Then there is a formula $\psi(x, y)$ with unique parameter κ that defines a function from κ to 2^ω such that, for any $\alpha < \kappa$,*

$$\emptyset \Vdash_{\text{Col}(\omega, \kappa)} \phi(\check{\alpha}, \check{r}) \quad \text{if and only if} \quad \psi(\alpha, r).$$

Proof. The relation $\emptyset \Vdash_{\text{Col}(\omega, \kappa)} \phi(\check{\alpha}, \check{r})$ is definable in the parameter κ by the uniform definability of $\text{Col}(\omega, \kappa)$ and the definability of forcing, so let $\psi(x, y)$ be a formula with unique parameter κ defining this relation. It is immediate that for any $\alpha < \kappa$ there is at most one $r \in 2^\omega$ such that $\psi(\alpha, r)$, so all we need to verify is that such an r exists for any such $\alpha < \kappa$.

So fix $\alpha < \kappa$. Since $\text{Col}(\omega, \kappa)$ is homogeneous and ϕ has no parameters, we have that for each $n \in \omega$, either

$$\emptyset \Vdash_{\text{Col}(\omega, \kappa)} \forall y (\phi(\check{\alpha}, y) \rightarrow \check{n} \in y),$$

or

$$\emptyset \Vdash_{\text{Col}(\omega, \kappa)} \exists y (\phi(\check{\alpha}, y) \wedge \check{n} \notin y).$$

Let

$$r = \{n \in \omega \mid \emptyset \Vdash_{\text{Col}(\omega, \kappa)} \forall y (\phi(\check{\alpha}, y) \rightarrow \check{n} \in y)\}$$

Since ZFC proves that ϕ defines a function, it is immediate that

$$\emptyset \Vdash_{\text{Col}(\omega, \kappa)} \phi(\check{\alpha}, \check{r}).$$

Therefore $\psi(\alpha, r)$ holds. □

Observe that if $\phi(x, y)$ is a formula as in Lemma 7.1, then, as $\kappa > \mathfrak{c}$, the function defined by ψ cannot be injective and therefore there are $\alpha < \beta < \kappa$ and $r \in 2^\omega$ so that

$$\emptyset \Vdash_{\text{Col}(\omega, \kappa)} \phi(\check{\alpha}, \check{r}) \wedge \phi(\check{\beta}, \check{r}),$$

whereby

$$V[G] \models \phi(\alpha, r) \wedge \phi(\beta, r)$$

for any choice of generic filter G . Recall however that, in $V[G]$, both α and β are countable ordinals.

The connection between ordinals and first-order theories is captured by the following result of Mazurkiewicz and Sierpiński. For this, if $d \geq 2$ and α is a countable ordinal, we let

$$M_\alpha^d = S^d \setminus F$$

where F is some closed subset of S^d homeomorphic with the countable compact space $\omega^\alpha + 1$. Note that, up to homeomorphism, M_α^d is independent of the specific choice of F .

Theorem 7.2 (Mazurkiewicz–Sierpiński). *For all $d \geq 2$ and all countable ordinals $\alpha \neq \beta$, the spaces M_α^d and M_β^d are not homeomorphic.*

Fix a canonical enumeration $\{\sigma_n\}_{n < \omega}$ of all $\mathcal{L}$ -sentences. We let $\vartheta(\alpha, n)$ be a formula expressing that α is a countable ordinal such that

$$\text{Homeo}(S^{(n)_1} \setminus F) \models \sigma_{(n)_2}$$

for any $F \subseteq S^{(n)_1}$ that is homeomorphic to $\omega^\alpha + 1$. Here $(n)_1$ and $(n)_2$ refer to the first and second coordinates of the image of n under some recursive bijection $\omega \rightarrow \omega \times \omega$.

Using ϑ , we may define a function $\Phi: \omega_1 \rightarrow 2^\omega$ by setting

$$\Phi(\alpha) = \{n < \omega \mid \vartheta(\alpha, n)\}.$$

Observe that Φ is defined by a formula $\phi(x, y)$ without parameters. Therefore, by our previous discussion, there are ordinals $\alpha \neq \beta$ that are countable in $V[G]$ and some $r \in 2^\omega$ such that

$$V[G] \models \phi(\alpha, r) \wedge \phi(\beta, r).$$

It thus follows that in $V[G]$, the following equivalence holds for all n

$$\text{Homeo}(M_\alpha^{(n)_1}) \models \sigma_{(n)_2} \Leftrightarrow \text{Homeo}(M_\beta^{(n)_1}) \models \sigma_{(n)_2}$$

and so

$$\text{Homeo}(M_\alpha^d) \equiv \text{Homeo}(M_\beta^d)$$

for all $d \geq 2$.

Summing up, by forcing with $\text{Col}(\omega, \mathfrak{c}^+)$, in the generic extension $V[G]$ there is a pair of non-homeomorphic orientable d -dimensional manifolds with elementarily equivalent homeomorphism groups. This establishes Theorem 1.4.

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REFERENCES

1. Steve Alpern and V. S. Prasad, *Typical dynamics of volume preserving homeomorphisms*, Cambridge Tracts in Mathematics, vol. 139, Cambridge University Press, Cambridge, 2000. MR1826331
2. James Belk, Luna Elliott, and Francesco Matucci, *A short proof of Rubin’s theorem*, Israel J. Math. **267** (2025), no. 1, 157–169. MR4930683
3. Riccardo Camerlo and Su Gao, *The completeness of the isomorphism relation for countable Boolean algebras*, Trans. Amer. Math. Soc. **353** (2001), no. 2, 491–518. MR1804507
4. Keith J. Devlin, *Constructibility*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1984. MR750828
5. Su Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics (Boca Raton), vol. 293, CRC Press, Boca Raton, FL, 2009. MR2455198
6. Peter G. Hinman, *Fundamentals of mathematical logic*, A K Peters, Ltd., Wellesley, MA, 2005. MR2179857
7. Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597
8. Béla Kerékjártó, *Vorlesungen über Topologie: I, Flächentopologie*, Verlag von Julius Springer, Berlin, 1923.
9. Sang-hyun Kim, Thomas Koberda, and J. de la Nuez González, *First order rigidity of homeomorphism groups of manifolds*, Commun. Am. Math. Soc. **5** (2025), 144–194. MR4904300
10. Sang-hyun Kim, Thomas Koberda, and J. de la Nuez González, *Elementary equivalence and diffeomorphism groups of smooth manifolds*, 2025, arXiv:2507.07427.
11. Thomas Koberda and J. de la Nuez González, *Locally approximating groups of homeomorphisms of manifolds*, 2024, arXiv:2410.16108.
12. ———, *Uniform first order interpretation of the second order theory of countable groups of homeomorphisms*, To appear in Pacific Journal of Mathematics (2025), arXiv:2312.16334.
13. Jean-Louis Krivine, *Théorie des ensembles*, Nouvelle Bibliothèque Mathématique, Cassini, Paris, 1998.
14. Frédéric Le Roux, *On closed subgroups of the group of homeomorphisms of a manifold*, J. Éc. polytech. Math. **1** (2014), 147–159. MR3322786

15. David Marker, *Lectures on infinitary model theory*, Lecture Notes in Logic, vol. 46, Association for Symbolic Logic, Chicago, IL; Cambridge University Press, Cambridge, 2016. MR3558585
16. Yiannis N. Moschovakis, *Descriptive set theory*, second ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009. MR2526093
17. Ian Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. **106** (1963), 259–269. MR143186
18. Matatyahu Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, Trans. Amer. Math. Soc. **312** (1989), no. 2, 487–538. MR988881
19. ———, *Locally moving groups and reconstruction problems*, Ordered groups and infinite permutation groups, Math. Appl., vol. 354, Kluwer Acad. Publ., Dordrecht, 1996, pp. 121–157. MR1486199
20. Stephen G. Simpson, *Subsystems of second order arithmetic*, second ed., Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, 2009. MR2517689
21. Jan van Mill, *The infinite-dimensional topology of function spaces*, North-Holland Mathematical Library, vol. 64, North-Holland Publishing Co., Amsterdam, 2001. MR1851014
22. Nik Weaver, *Forcing for mathematicians*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014. MR3184751
23. James V. Whittaker, *On isomorphic groups and homeomorphic spaces*, Ann. of Math. (2) **78** (1963), 74–91. MR150750

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