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CoLoRFulNNLO for hadron collisions: regularizing initial-state double real emissions

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Abstract

We present the extension of the completely local subtraction scheme CoLoRFulNNLO to color-singlet production in hadron collisions. We provide explicit momentum mappings and the complete set of double-real counterterms required for this class of processes. The counterterms are systematically derived from the known infrared limit formulae of QCD matrix elements, and particular care has been taken to ensure their analytic integrability. The resulting construction involves a relatively small number of counter-events, preserving the locality and efficiency of the scheme. All formulae have been implemented within the publicly available `NNLOCAL` Monte Carlo program, and we explicitly validate all IR limits using arbitrary-precision computer algebra and present representative results. The counterterms presented here constitute a self-contained subset applicable to general hadronic processes within the CoLoRFulNNLO approach.

Contents

1 Introduction

Precision calculations in Quantum Chromodynamics (QCD) for hadron collider processes are indispensable for exploiting the full potential of the LHC and future experiments. The steadily improving experimental precision requires theoretical predictions at next-to-next-to-leading order (NNLO) in the perturbative expansion in order to reduce theory uncertainties and to provide reliable background estimates for searches of new physics. The explicit construction of NNLO corrections is, however, technically challenging because of the intricate pattern of overlapping soft and collinear infrared (IR) singularities that appear in the double real, real–virtual and double virtual contributions.

A productive strategy to tame these singularities is furnished by local subtraction methods, which introduce local counterterms that reproduce the singular limits of the QCD matrix elements point-by-point in phase space and thus permit stable numerical integration of the finite remainders. Representative local approaches include antenna subtraction [1], sector-improved residue subtraction [2], local analytic sector subtraction [3], nested soft-collinear subtractions [4], the projection-to-Born method [5], geometric IR subtraction [6] and the CoLoRFulNNLO framework [7]. Alternative strategies based on the definition of non-local counterterms and slicing parameters have also been successfully pursued, including q_T subtraction [8] and N-jettiness subtraction [9]. Although the local subtraction methods differ in technical implementation, the class of processes for which they are fully worked out, as well as in the specific treatment of phase space (e.g., mappings of momenta and whether or not phase space is partitioned), they share the common objective of achieving exact local cancellation of infrared divergences while enabling efficient Monte Carlo integration.

The CoLoRFulNNLO framework [7,10–15] is distinguished by its systematic construction of local counterterms that explicitly account for all spin and color degrees of freedom and by the partial availability of analytic results for the corresponding integrated subtraction terms. In particular, closed analytic expressions for the integrated counterterms have been obtained for classes of configurations relevant to final-state radiation off massive partons [13,15] and to the initial-state radiation discussed here. For final-state radiation from massless partons, integrated counterterms have been computed analytically up to and including the $\mathcal{O}(\varepsilon^{-1})$ poles in the dimensional-regularization expansion; at $\mathcal{O}(\varepsilon^0)$ all but some contributions are available in closed analytic form whereas the remaining pieces have been evaluated numerically and incorporated through interpolation [7, 14]. The availability of fully analytic integrated subtraction terms where possible is a significant practical advantage: such expressions simplify the combination of integrated counterterms with virtual contributions and can often be used directly in Monte Carlo programs. This in turn can improve numerical stability, since one need not worry about ensuring a sufficiently small interpolation error. Moreover, the availability of analytic integrated counterterms increases the portability of the subtraction framework.

An important structural feature of CoLoRFulNNLO is the freedom to redefine local counterterms by exploiting various degrees of freedom such as the choice of momentum mappings, constraining the subtractions to the neighborhood of singular regions and reorganizations of subtraction kernels. This flexibility can be used to optimize numerical behavior, to reduce the number of momentum mappings, and hence the number of counter-events, and to adapt the scheme to different classes of processes. In particular, these freedoms will be instrumental when extending the framework to accommodate more complex final states, for example by adding an observed jet to the color-singlet final state in hadronic collisions.

In this paper we extend the CoLoRFulNNLO methodology to treat double initial-state radiation in processes that produce a color-singlet final state at hadron colliders. Our construction provides a local subtraction scheme that cancels the infrared singularities associated with single and double emissions from the incoming partons, while preserving the desirable practical properties of the original framework:

- numerical stability, achieved by exact local cancellation of soft and collinear divergences at integrand level;
- computational efficiency, through a design that aims to keep the number of required momentum mappings, and thus counter-events, under control;
- analytic integrated counterterms, enabling a transparent treatment of pole cancellation with virtual contributions;
- modularity and flexibility, which facilitate portability to new processes and future extensions, for instance, to final states containing jets.

The results we present constitute both a concrete realization of double initial-state subtraction within CoLoRFulNNLO and a step towards a more general, practical local subtraction framework for NNLO QCD calculations at hadron colliders. The remainder of the paper is organized as follows.

In sec. 2 we present the basic layout of our subtraction scheme, while sec. 3 reviews the elements of the CoLoRFulNNLO construction relevant to our extension. Sec. 4 introduces the approximate cross sections that regularize the double real contribution. The new counterterms and momentum mappings for double initial-state radiation are then presented in full detail in secs. 5–7, where we also define the integrated counterterms. Sec. 8 discusses the numerical validation of our construction. Finally, sec. 9 summarizes our findings and outlines directions for future work, including possible refinements of the counterterm definitions in view of adding jets to the final state.

2 The subtraction scheme

We consider the production of m jets plus any number of colorless particles collectively denoted by X in the collision of hadrons A and B of momenta p_A and p_B, $A + B \to m$ jets $+ X$. The cross section for this process is given by

$$\sigma(p_A, p_B) = \sum_{a,b} \int_0^1 \mathrm{d}x_a\, f_{a/A}(x_a, \mu_F^2) \int_0^1 \mathrm{d}x_b\, f_{b/B}(x_b, \mu_F^2) \sigma_{ab}(p_a, p_b; \mu_F^2)\,, \tag{2.1}$$

i.e., it is a convolution of the partonic cross section with parton density functions (PDFs). Here $p_a = x_a p_A$ and $p_b = x_b p_B$, while the summations are over parton flavors. The partonic cross section can be computed in perturbation theory and its perturbative expansion up to NNLO reads

$$\sigma_{ab}(p_a, p_b; \mu_F^2) = \sigma_{ab}^{\mathrm{LO}}(p_a, p_b; \mu_R^2, \mu_F^2) + \sigma_{ab}^{\mathrm{NLO}}(p_a, p_b; \mu_R^2, \mu_F^2) + \sigma_{ab}^{\mathrm{NNLO}}(p_a, p_b; \mu_R^2, \mu_F^2) + \dots\,. \tag{2.2}$$

In the following, when no confusion can arise, the dependence of the cross sections on partonic momenta as well as the renormalization and the factorization scales will be suppressed.

The leading order (LO) partonic cross section is just the integral of the fully differential Born cross section over the phase space of the produced final state,

$$\sigma_{ab}^{\mathrm{LO}} = \int_{m+X} \mathrm{d}\sigma_{ab}^{\mathrm{B}}\, J_{m+X}\,. \tag{2.3}$$

Above, J_{m+X} is the value of some infrared and collinear-safe physical observable J evaluated on the phase space of m partons plus the colorless particles in X. By virtue of the infrared and collinear safety of J, the phase space integral in eq. (2.3) is finite and can be computed by standard numerical methods directly in four spacetime dimensions.

In contrast, higher-order corrections are given by sums of real-emission and/or virtual contributions which are separately IR divergent and require regularization.[1] In particular, at N^kLO there are precisely k extra emissions which may become unresolved and each of these may be either real or virtual. E.g., at NLO only a single extra emission is allowed, hence the NLO correction is the just the sum of the (single) real-emission contribution $\mathrm{d}\sigma_{ab}^{\mathrm{R}}$ and the virtual contribution $\mathrm{d}\sigma_{ab}^{\mathrm{V}}$,

$$\sigma_{ab}^{\mathrm{NLO}} = \int_{m+X+1} \mathrm{d}\sigma_{ab}^{\mathrm{R}}\, J_{m+X+1} + \int_{m+X} \left(\mathrm{d}\sigma_{ab}^{\mathrm{V}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1}\right) J_{m+X}\,. \tag{2.4}$$

Note that due to the renormalization of the PDFs, the collinear remnant $\mathrm{d}\sigma_{ab}^{\mathrm{C}_1}$ must also be included to obtain a finite NLO cross section. The IR singularities appearing in eq. (2.4) can be handled by any number of well-established methods [17, 18].

At NNLO, two extra emissions are allowed which may both be real or virtual. Thus the NNLO correction is a sum of the double real contribution $\mathrm{d}\sigma_{ab}^{\mathrm{RR}}$, the real-virtual contribution $\mathrm{d}\sigma_{ab}^{\mathrm{RV}}$ and the double virtual contribution $\mathrm{d}\sigma_{ab}^{\mathrm{VV}}$,

$$\sigma_{ab}^{\mathrm{NNLO}} = \int_{m+X+2} \mathrm{d}\sigma_{ab}^{\mathrm{RR}}\, J_{m+X+2} + \int_{m+X+1} \left(\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1}\right) J_{m+X+1} + \int_{m+X} \left(\mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_2}\right) J_{m+X}\,, \tag{2.5}$$

where once more, $\mathrm{d}\sigma_{ab}^{\mathrm{C}_1}$ and $\mathrm{d}\sigma_{ab}^{\mathrm{C}_2}$ are collinear remnants coming from PDF renormalization. In the CoLoRFulNNLO scheme, the IR singularities in eq. (2.5) associated with one or both of the extra emissions becoming unresolved are regularized by subtracting and adding back various approximate cross sections. In the double real part, up to two partons can become unresolved. This is reflected in the structure of subtractions,

$$\sigma_{ab,\mathrm{reg.}}^{\mathrm{RR}} = \int_{m+X+2} \left\{\mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+X+2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_1} J_{m+X+1} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_2} J_{m+X} + \mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_{12}} J_{m+X}\right\}, \tag{2.6}$$

where $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_1}$ and $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_2}$ are the single and double unresolved approximate cross sections that match the singular behavior of $\mathrm{d}\sigma_{ab}^{\mathrm{RR}}$ point-wise in IR limits where one and two partons become unresolved. The last term, $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_{12}}$, is introduced in order to avoid double subtraction in regions of phase space where single and double limits overlap, and hence plays a dual role. Conceptually, it regularizes $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_1}$ in double unresolved limits and $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_2}$ in single unresolved ones. However, the true structure of cancellations is somewhat more involved. In fact, in single unresolved limits, in addition to regularizing $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_2}$, this approximate cross section must also cancel all spurious singularities (i.e., singularities of subtraction terms unrelated to

[1] In this work, we employ dimensional regularization in $d = 4 - 2\varepsilon$ dimensions in the CDR scheme [16] to regulate IR singularities.

the specific single unresolved limit being considered) that develop in $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}$. We will refer to $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}}$ as the iterated single unresolved approximate cross section. Turning to the real-virtual contribution, we recall that in this case at most one parton can become unresolved, hence the subtractions take the form

$$\begin{aligned}\sigma_{ab,\mathrm{reg.}}^{\mathrm{RV}} = \int_{m+X+1} \Bigg\{ \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1}} + \int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right] J_{m+X+1} \\ - \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV,A_1}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1,A_1}} + \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] J_{m+X} \Bigg\}\,.\end{aligned} \tag{2.7}$$

Notice in particular that the integrated version of $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}$, which is subtracted in eq. (2.6), is added back here and together with $\mathrm{d}\sigma_{ab}^{\mathrm{C_1}}$ it cancels the explicit ε-poles of $\mathrm{d}\sigma_{ab}^{\mathrm{RV}}$ coming from the extra loop. However, all three terms on the first line also have kinematic singularities as a parton becomes unresolved, which are regularized by the three single unresolved approximate cross sections on the second line. Note that the latter must be constructed in such a way that their sum is also free of explicit ε-poles. Thus, ε-poles cancel within the brackets, while IR singularities cancel between them. Finally, the double virtual contribution is free of kinematic singularities since J is infrared and collinear safe, but due to the two extra loops, it contains explicit IR ε-poles. These poles are then canceled by the integrated forms of the various subtraction terms that we have not yet added back,

$$\begin{aligned}\sigma_{ab,\mathrm{reg.}}^{\mathrm{VV}} = \int_{m+X} \Bigg\{ \mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_2}} + \int_2 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} \right] \\ + \int_1 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV,A_1}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1,A_1}} + \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] \Bigg\} J_{m+X}\,.\end{aligned} \tag{2.8}$$

As a result of these manipulations, the five terms in eq. (2.5) are rearranged into the three contributions in eqs. (2.6)–(2.8), each separately finite in four dimensions.

Of course, the formal approximate cross sections must be precisely defined before eqs. (2.6)–(2.8) can be applied in practical calculations. In the following, we focus on double real emission and present explicit formulae for the approximate cross sections $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}$, $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}}$ and $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}}$ for the case of color-singlet production in hadronic collisions (i.e., $m = 0$). We emphasize that all subtraction terms remain valid for the more general case when final-state radiation is present already at Born level. For such processes, the subtraction terms presented here will need to be supplemented by additional ones regulating those unresolved configurations that do not occur for color-singlet production.

3 Building subtraction terms

In the CoLoRFulNNLO framework, approximate cross sections are built from subtraction terms that are in turn constructed starting from QCD IR factorization formulae. These formulae capture the universal behavior of QCD squared matrix elements as some number of partons becomes unresolved (i.e., soft and/or collinear) and are completely known up to NNLO accuracy [19–31] and in some cases beyond [32–49]. Their general structure can be given as follows. Let us denote the k-fold real emission squared matrix element for some specific partonic subprocess as[2] $\left|\mathcal{M}_{ab,m+X+k}^{(0)}(p_a, p_b; \{p\}_{m+X+k})\right|^2$. Here the subscripts a and b denote the *flavors* of the incoming partons, while the subscript on the set of final-state momenta in the argument gives the *multiplicity.* For later convenience, the dependence of the matrix element on initial-state momenta is shown explicitly. Furthermore, we denote by $\boldsymbol{U}_j$ the formal operator that takes some

[2]If the Born process is loop-induced, the k-fold real emission squared matrix element will not literally correspond to the square of zero-loop matrix elements. In this case, the notation and in particular the (0) superscript should be understood to refer to the squared matrix element with zero-loop corrections compared to the Born process.

specific j-fold unresolved limit ($j \leq k$). Then, the IR factorization formula can be written in the following symbolic form

$$\boldsymbol{U}_j \big|\mathcal{M}^{(0)}_{ab,m+X+k}(p_a, p_b; \{p\}_{m+X+k})\big|^2 = \left(\frac{\alpha_s}{2\pi}\right)^j \mathrm{Sing}^{(0)}_j \big|\mathcal{M}^{(0)}_{\bar{a}\bar{b},m+X+k-j}(\bar{p}_a, \bar{p}_b; \{\bar{p}\}_{m+X+k-j})\big|^2 , \qquad (3.1)$$

i.e., it is a product of a tree-level, j-fold unresolved singular structure $\mathrm{Sing}^{(0)}_j$ times a reduced matrix element with j partons removed. The singular structures involve Altarelli-Parisi splitting functions, eikonal factors and their multi-emission generalizations and are generally matrices in color and/or spin space. Thus, the product in eq. (3.1) is to be understood accordingly, see appendix A for details. Moreover, the parton flavors in the reduced matrix element may be different from the original ones as indicated by the $\bar{a}$ and $\bar{b}$ subscripts. We note in passing that up to NNLO accuracy, the singular structures $\mathrm{Sing}^{(0)}_j$ are known to be universal, i.e., they do not depend on the specific process being considered. This opens the door to the construction of general NNLO subtraction schemes. However, strict process-independent factorization as implied by eq. (3.1) may be violated beyond NNLO for initial-state radiation [50, 51]. This issue is not relevant for our immediate purposes, but of course must be addressed by any calculational setup for computing perturbative corrections beyond NNLO accuracy.

As is well-known, the IR limit formulae of eq. (3.1) cannot be used directly as subtraction terms because of the following reasons. First, at any given order beyond LO, the regions of phase space corresponding to the various IR limits generally overlap. Indeed, already at NLO, the momentum of a parton can be soft and collinear to another momentum, hence the soft and collinear singular regions are not distinct. Clearly, the structure of overlaps becomes more elaborate at higher orders. This then implies that care must be taken to avoid multiple subtraction in regions of phase space where singular limits overlap. Second, IR factorization formulae are only well-defined in the strict limit which they describe. To appreciate this, it is enough to examine the prescription for the set of momenta $\{\bar{p}\}$ entering the factorized matrix element on the right hand side of eq. (3.1). For example in soft limits (single or multiple), the relevant IR factorization theorem instructs us to simply drop all soft momenta from the original set of momenta $\{p\}$ in order to obtain $\{\bar{p}\}$. However, this prescription clearly violates momentum conservation unless all soft momenta are strictly zero, i.e., the right hand side is only well-defined in the strict soft limit. In collinear limits, according to the IR factorization theorem, $\{\bar{p}\}$ is obtained simply by replacing the collinear momenta with their sum. This prescription however does not respect the mass-shell condition for the parent momentum, unless the combined momenta are strictly collinear. Moreover, the momentum fractions entering the Altarelli-Parisi splitting functions are also only well-defined if the collinear momenta are strictly proportional to the parent momentum, i.e., in the strict collinear limit. Thus, proper extensions of the IR limit formulae must be defined away from the limits.

3.1 Dealing with overlapping limits

Let us begin by formalizing the concept of overlapping limit, i.e., a limit in which several unresolved regions are approached simultaneously, as follows. An overlapping limit refers simply to the successive application of limits and we say that some j-fold unresolved limits overlap if their subsequent application leads to a configuration that is also j-fold unresolved. It should be noted that taking two j-fold unresolved limits one after the other can lead to a configuration that is more than j-fold unresolved.[3] In these cases, we do not consider the limits as overlapping. It can be shown [10] that the basic structure of IR limit formulae for overlapping limits is the same as in eq. (3.1). Hence, in the following, the formal limit operators $\boldsymbol{U}_j$ will refer to any generic j-fold unresolved limit, including overlapping limits.

The issue of over-counting of overlapping singular regions can then be dealt with easily by using the

[3]A simple example is the successive application of two single unresolved ($j = 1$) soft limits $p^\mu_r \to 0$ and $p^\mu_s \to 0$, which obviously produces a double soft configuration in which $j = 2$ momenta are unresolved.

inclusion-exclusion principle: we start by summing all basic limits, then subtract the pairwise overlaps of limits, add the triple overlaps and so on.[4] E.g., at NLO we encounter only single collinear ($p_i^\mu \parallel p_r^\mu$) and single soft ($p_r^\mu \to 0$) limits. Then, in order construct a single unresolved approximation that counts each singular region once, the inclusion-exclusion principle instructs us to sum all collinear and all soft limits and subtract the collinear-soft overlaps. To do this, we introduce the formal single unresolved limit operator $\boldsymbol{A}_1$ as follows,

$$\boldsymbol{A}_1 = \sum \Big(\boldsymbol{C}_{ir} + \boldsymbol{S}_r - \boldsymbol{C}_{ir} \cap \boldsymbol{S}_r\Big)\,, \tag{3.2}$$

where $\boldsymbol{C}_{ir}$ and $\boldsymbol{S}_r$ are generic operators that take the single collinear and single soft limits, while the overlapping limit is denoted by the $\cap$ symbol. We can then apply this formal operator to any specific object for which we wish to build a single unresolved approximation, such as $\mathrm{d}\sigma_{ab}^{\mathrm{RR}}$ and $\mathrm{d}\sigma_{ab}^{\mathrm{RV}}$ but also $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}}$, $\mathrm{d}\sigma_{ab}^{\mathrm{C_1}}$ and $\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}$. This construction generalizes to NNLO easily. Now we have four types of basic singular configurations: triple collinear ($p_i^\mu \parallel p_r^\mu \parallel p_s^\mu$), double collinear ($p_i^\mu \parallel p_r^\mu$ and $p_j^\mu \parallel p_s^\mu$), soft-collinear ($p_i^\mu \parallel p_r^\mu$ and $p_s^\mu \to 0$) and finally double soft ($p_r^\mu \to 0$ and $p_s^\mu \to 0$). Accordingly, the double unresolved limit operator $\boldsymbol{A}_2$ counting each singular region once is given by

$$\begin{aligned} \boldsymbol{A}_2 = \sum \Big[&\boldsymbol{C}_{irs} + \boldsymbol{C}_{ir,js} + \boldsymbol{CS}_{ir,s} + \boldsymbol{S}_{rs} \\ &- \Big(\boldsymbol{C}_{irs} \cap \boldsymbol{CS}_{ir,s} + \boldsymbol{C}_{irs} \cap \boldsymbol{S}_{rs} + \boldsymbol{C}_{ir,js} \cap \boldsymbol{CS}_{ir,s} + \boldsymbol{C}_{ir,js} \cap \boldsymbol{S}_{rs} + \boldsymbol{CS}_{ir,s} \cap \boldsymbol{S}_{rs}\Big) \\ &+ \Big(\boldsymbol{C}_{irs} \cap \boldsymbol{CS}_{ir,s} \cap \boldsymbol{S}_{rs} + \boldsymbol{C}_{ir,js} \cap \boldsymbol{CS}_{ir,s} \cap \boldsymbol{S}_{rs}\Big)\Big]\,. \end{aligned} \tag{3.3}$$

Here $\boldsymbol{C}_{irs}$, $\boldsymbol{C}_{ir,js}$, $\boldsymbol{CS}_{ir,s}$ and $\boldsymbol{S}_{rs}$ are limit operators for the four basic types of limits and overlaps are denoted as before. In writing eq. (3.3) we have used that the overlap of a triple and double collinear limit leads to a configuration where at least three partons are unresolved[5] so at NNLO we do not need to consider this overlap. Moreover, it is not necessarily true that each overlapping term in eq. (3.3) corresponds to a distinct limit formula. In particular, explicit calculations [10] show that at the level of symbolic operators, $\boldsymbol{C}_{ir,js} \cap \boldsymbol{CS}_{ir,s} \cap \boldsymbol{S}_{rs} = \boldsymbol{C}_{ir,js} \cap \boldsymbol{S}_{rs}$. Nevertheless, we cannot simply cancel these terms altogether, since the summation in eq. (3.3) is symbolic and it is not immediately clear if they both appear with the same coefficient once the summation is properly defined. In fact, as we will see in sec. 4.2, a complete cancellation does not occur.

Obviously eqs. (3.2) and (3.3) are symbolic since neither the summations nor the meaning of the symbol $\cap$ are precisely defined. Defining the summations properly is not difficult: one simply has to make sure that all basic unresolved configurations are counted precisely once. However, the meaning of the $\cap$ symbol is somewhat more involved, since it turns out that at the symbolic operator level, the overlap of limits as defined above is non-commutative. This non-commutativity arises from the formal definition of the soft and collinear limits and was investigated in detail in [10]. There it was also shown that the $\cap$ symbol in eqs. (3.2) and (3.3) should be understood to refer to the overlap of limits in the sense introduced above, i.e., as successive applications of limits. Hence, e.g., $\boldsymbol{C}_{ir} \cap \boldsymbol{S}_r$ is to be understood as an operator that computes the collinear limit of the soft limit, in this order.

3.2 From limit formulae to subtraction terms: general considerations

The extension of the IR limit formulae (both for direct and overlapping limits) over the full phase space requires two steps. First, the momenta entering the factorized squared matrix elements must be specified.

[4] Alternatively, one can choose to construct subtraction terms that smoothly interpolate between various limits as, e.g., in the case of dipole subtraction [18] or antenna subtraction [1].

[5] Consider the $\boldsymbol{C}_{irs} \cap \boldsymbol{C}_{ir,js}$ overlap, i.e., the limit where the $p_i^\mu \parallel p_r^\mu \parallel p_s^\mu$ and $p_i^\mu \parallel p_r^\mu$, $p_j^\mu \parallel p_s^\mu$ regions are approached simultaneously. In this region clearly $p_i^\mu \parallel p_r^\mu \parallel p_s^\mu \parallel p_j^\mu$, which is a quadruple collinear limit where three partons are unresolved.

Second, all kinematic quantities appearing in the IR factorization formulae, such as momentum fractions and transverse momenta in collinear parton splitting, as well as eikonal factors in soft emission, must be explicitly defined as functions of the original momenta of the event. Clearly, both the momentum mappings and the definitions of momentum fractions, etc., must be chosen in a way that the structure of cancellations in all overlapping limits is respected. This is of course a non-trivial constraint on the entire construction. Here we provide some general considerations which motivate our approach to the

- definition of momentum mappings;
- choice of momentum fractions;
- parametrization of transverse momenta;
- treatment of soft eikonal factors.

3.2.1 Definition of momentum mappings

Starting with the momentum mappings, we first of all require transformations of the original set of momenta, $(p_a, p_b; \{p\}_{m+X+k})$, to sets of reduced momenta, $(\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\}_{m+X+k-j})$ (here $j \leq k$ is the number of unresolved particles) which respect momentum conservation and the mass-shell conditions.[6] In addition, the mapped momenta must also have the proper behavior in IR limits as spelled out in the IR factorization theorems. Obviously, the choice of momentum mappings is not unique, and we opt to use mappings where the recoil is distributed over the entire event rather than being absorbed by some specific spectator parton(s). This choice makes it more straightforward to construct a single unresolved approximate cross section such that both $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}$ as well as its integrated form $\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}}$ have universal IR limits. In fact, this feature is not guaranteed by QCD factorization alone [52], and relies on the specific definitions of the subtraction terms. We note that for initial-state radiation at NNLO, we find it is enough to introduce just five elementary momentum mappings, whose precise definitions will be given below. (These definitions are also collected in appendix B for the readers' convenience.) In particular, as we will see, subtraction terms corresponding to overlapping limits do not require dedicated momentum mappings, while subtraction terms for iterated single unresolved limits can be defined using convolutions of single unresolved ones.

3.2.2 Choice of momentum fractions

Turning to the definition of kinematic quantities that appear in the IR limit formulae, let us begin with the momentum fractions in collinear splitting. Consider first the case of final-final single collinear splitting, $(ir) \to i + r$, where we seek to define the momentum fractions z_i and z_r of the daughter partons. In the strict collinear limit, the parent momentum is simply $p_{ir}^\mu = p_i^\mu + p_r^\mu$ and p_i^μ as well as p_r^μ are strictly proportional to p_{ir}^μ. The factors of proportionality define the momentum fractions z_i and z_r. In the strictly collinear configuration, these may be computed by contracting the parent and daughter momenta with some auxiliary four-vector v^μ (with $v \cdot p_{ir} \neq 0$) and taking the appropriate ratios, i.e., $z_i = (p_i \cdot v)/[(p_i + p_r) \cdot v]$ and $z_r = (p_r \cdot v)/[(p_i + p_r) \cdot v]$. Then, a rather obvious way of specifying the momentum fractions away from the strict limit is to adopt these expressions as the *definitions* of momentum fractions. These definitions can be made Lorentz-invariant by choosing v^μ to be the total (partonic) four-momentum of the event, $v^\mu = Q^\mu = (p_a + p_b)^\mu$. Here we have introduced $Q^\mu = p_a^\mu + p_b^\mu$ to denote the total incoming *partonic*

[6] This set of tilded momenta, $(\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\}_{m+X+k-j})$, is *different* from the set of barred momenta, $(\bar{p}_a, \bar{p}_b; \{\bar{p}\}_{m+X+k-j})$, that appears in the IR factorization formula of eq. (3.1). As explained above, the latter may violate momentum conservation and/or the mass-shell conditions away from the strict limit.

momentum. Thus, for *final-state* partons i and r we *define* the quantity

$$z_{i,r} = \frac{s_{iQ}}{s_{(ir)Q}}\,, \tag{3.4}$$

which is then interpreted as the momentum fraction of parton i in the $(ir) \to i + r$ final-state splitting. Obviously, the momentum fraction for parton r is then $z_{r,i} = 1 - z_{i,r}$. In eq. (3.4) we have taken the occasion to introduce some notation that we will use throughout. First, s_{jk} will denote twice the dot-product of any two momenta p_j and p_k. Second, indices in parentheses denote sums of the corresponding momenta so that, e.g.,

$$s_{jk} = 2p_j \cdot p_k\,, \qquad s_{j(k\ell)} = 2p_j \cdot (p_k + p_\ell)\,, \qquad s_{(jk)(\ell m)} = 2(p_j + p_k) \cdot (p_\ell + p_m)\,, \tag{3.5}$$

and so on. Importantly, eq. (3.4) should be understood to define the *function* $z_{i,r}$ whose arguments are the two final-state momenta with indices i and r, i.e., p_i^μ and p_r^μ. Thus, any expression of this form is to be interpreted as the ratio of the invariants s_{iQ} and $s_{(ir)Q}$, where i and r are indices of final-state momenta from some particular set. This is significant, since we will often need to define momentum factions for momenta obtained after some mapping, i.e., when the momenta are elements of the set $(\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\}_{m+X+k-j})$ discussed above. In such cases, the indices will inherit the tilde (or corresponding) notation. E.g., an expression of the form$z_{\tilde{i},r}$ will be understood to be defined by eq. (3.4),

$$z_{\tilde{i},r} = \frac{s_{\tilde{i}Q}}{s_{(\tilde{i}r)Q}} = \frac{2\tilde{p}_i \cdot (p_a + p_b)}{2(\tilde{p}_i + p_r) \cdot (p_a + p_b)}\,, \tag{3.6}$$

where for full clarity, we have also resolved our notation for invariants. Notice in particular that the dot-products are always computed with the total partonic momentum $Q^\mu = (p_a + p_b)^\mu$, irrespective of the fact that $\tilde{p}_i^\mu$ is an element of a set of mapped momenta. We will make heavy use of this notation throughout.

The definition in eq. (3.4) has several nice features. First, the momenta p_i and p_r appear symmetrically and the momentum fractions sum to one by construction, $z_{i,r} + z_{r,i} = 1$. Second, the generalization to final-state n-fold splitting is obvious. E.g., for final-state triple collinear $(irs) \to i + r + s$ splitting, we define

$$z_{i,rs} = \frac{s_{iQ}}{s_{(irs)Q}}\,. \tag{3.7}$$

This relation is again to be understood as defining the function $z_{i,rs}$ of the three final-state momenta whose indices appear in the subscript.

Turning to initial-final splitting, let us begin once more by considering the case of the single collinear limit $a \to (ar) + r$. We now look to define the momentum fractions x_a and x_r of the parton entering the reduced matrix element and the parton in the final-state. Note the slight abuse of notation in using x_a to denote the momentum fraction of parton (ar). In the strict collinear limit, the momentum of parton (ar) is $p_{ar}^\mu = p_a^\mu - p_r^\mu$ and p_{ar}^μ as well as p_r^μ are strictly proportional to p_a^μ. The factors of proportionality are the momentum fractions x_a and x_r. As in the final-state case, in the strictly collinear configuration, these may be computed by contracting the momenta with some auxiliary four-vector v^μ (with $v \cdot p_a \neq 0$) and taking appropriate ratios, i.e., $x_r = (p_r \cdot v)/(p_a \cdot v)$ and $x_a = [(p_a - p_r) \cdot v]/(p_a \cdot v) = 1 - x_r$. Thus, a rather straightforward way of specifying the momentum fractions away from the strict limit is to *define* them by these expressions. Once again, we can choose v^μ to be the total (partonic) four-momentum of the event, $v^\mu = Q^\mu = (p_a + p_b)^\mu$, which leads to a Lorentz-invariant definition of momentum fractions. Thus, for an *initial-state* parton a and *final-state* parton r, we *define* the quantities

$$x_{r,a} = \frac{s_{rQ}}{s_{aQ}}\,, \qquad x_{a,r} = 1 - x_{r,a}\,, \tag{3.8}$$

which are then interpreted as momentum fractions for partons r and (ar) in the $a \to (ar) + r$ initial-state splitting. Once more we stress that eq. (3.8) is to be understood as defining the *functions* $x_{r,a}$

and $x_{a,r}$ whose arguments are the final-state momentum p_r^μ and the initial-state momentum p_a^μ. Notice that the definitions in eq. (3.8) are not symmetric between initial-state and final-state momenta and the precise definition depends on whether the first index is that of a final-state parton or an initial-state parton. We emphasize furthermore that although $s_{aQ} = s_{ab}$, the right-hand side of $x_{r,a}$ in eq. (3.8) must be written precisely as given above and the subscripts refer to the indices of momenta in the numerator and denominator which *always* multiply the four-vector $Q^\mu = (p_a + p_b)^\mu$. This becomes significant when defining momentum fractions for mapped momenta. As with the final-state momentum fractions discussed above, in such cases the indices will inherit the tilde (or corresponding) notation of the mapped momenta, but dot-products must always be computed with $Q^\mu = (p_a + p_b)^\mu$, such that, e.g., $x_{r,\tilde{a}}$ is

$$x_{r,\tilde{a}} = \frac{s_{rQ}}{s_{\tilde{a}Q}} = \frac{2p_r \cdot (p_a + p_b)}{2\tilde{p}_a \cdot (p_a + p_b)}\,, \tag{3.9}$$

where we have spelled out all definitions explicitly for complete clarity. Such notation will also be used prominently.

The generalization to n-fold initial-state splitting is again simple. E.g., for initial-state $a \to (ars)+r+s$ triple collinear splitting, the obvious generalization of eq. (3.8) reads,

$$x_{r,a} = \frac{s_{rQ}}{s_{aQ}}\,, \qquad x_{s,a} = \frac{s_{sQ}}{s_{aQ}}\,, \qquad x_{a,rs} = 1 - x_{r,a} - x_{s,a}\,. \tag{3.10}$$

At the risk of belaboring the point, we stress one more time that eq. (3.10) defines *functions* of the initial-state momentum p_a^μ and the final-state momenta p_r^μ and p_s^μ, and dot-products are always computed with the four-vector $Q^\mu = (p_a + p_b)^\mu$. Thus, the number of indices in the subscript reflects the number of arguments of these functions and in particular the definitions of the momentum fractions of the final-state partons r and s are unchanged from to eq. (3.8). As discussed above, when one or more of the arguments is from a set which was obtained through some momentum mapping, the corresponding index inherits the notation of the mapped set. As for initial-final single collinear splitting, the definitions of momentum fractions in the n-fold case are not symmetric between initial-state and final-state partons, however the final-state partons do appear symmetrically.

We note in passing that the definitions in eq. (3.8) can also be obtained from eq. (3.4) by a crossing transformation, $x_{a,r} = 1/z_{i,r}|_{p_i \to -p_a}$ and $x_{r,a} = -z_{r,i}/z_{i,r}|_{p_i \to -p_a}$, see the discussion in appendix D.1. This transformation also relates momentum fractions for multiple-collinear splitting. Indeed, it is easy to check that $x_{a,rs}$ of eq. (3.10) can be obtained from eq. (3.7) as $x_{a,rs} = 1/z_{i,rs}|_{p_i \to -p_a}$ and moreover $x_{r,a} = -z_{r,is}/z_{i,rs}|_{p_i \to -p_a}$ and $x_{s,a} = -z_{s,ir}/z_{i,rs}|_{p_i \to -p_a}$. With regards to the last two relations, notice that the ratio $z_{r,is}/z_{i,rs}$ depends only on the momenta p_r and p_i, but not on p_s (similarly $z_{s,ir}/z_{i,rs}$ depends only on the momenta p_s and p_i, but not on p_r). Thus, ratios of momentum fractions for final-state triple collinear splitting actually define functions of just two momenta. Hence, the number of indices in the subscripts differ on the two sides of these particular crossing relations.

We finish the discussion of momentum fractions by observing that somewhat surprisingly, the definition in eq. (3.10) raises the following subtlety. It is not very difficult to see that $x_{a,rs}$ can take both positive and negative values. This implies that it vanishes at certain points of the double real emission phase space and these points turn out not to correspond to any IR limit (i.e., they are ordinary points inside the double real emission phase space). This alone would not be an issue, but it can be checked that the initial-final-final triple collinear splitting formulae have pieces with $x_{a,rs}$ in the denominator.[7] Thus, this definition of $x_{a,rs}$ introduces spurious non-integrable singularities. One way of addressing this issue is of course to search for new definitions of momentum fractions that do not vanish inside the real emission phase space. This is possible, but the obtained formulae are quite cumbersome and lead to complications when computing the integrated versions of the triple collinear subtraction terms. However, we may also deal with the spurious

[7] In fact, negative powers of $x_{a,rs}$ up to $1/x_{a,rs}^2$ can appear.

singularities by introducing appropriate damping functions around these singularities. In this work, we have chosen to pursue the latter option and keep the definition of the momentum fraction in eq. (3.10).

3.2.3 Parametrization of transverse momenta

Next we discuss the choice of transverse momenta. We start by observing that at the level of IR factorization formulae, there is a relation between momentum fractions, invariants and the squares of transverse momenta. E.g., for final-final single collinear $(ir) \to i + r$ splitting, standard Sudakov parametrization implies (see, e.g., ref. [18])

$$k_\perp^2 = -z_i z_r s_{ir}\,, \tag{3.11}$$

while the appropriate relations for triple parton splitting can be found in ref. [21] and are recalled in appendix D.1, see eqs. (D.44) and (D.45). Thus, the collinear splitting functions can be written in different ways that are nonetheless equivalent in the IR limit, e.g.,

$$P^{\mu\nu} = -Ag^{\mu\nu} + Bk_\perp^\mu k_\perp^\nu \qquad \text{and} \qquad P^{\mu\nu} = -Ag^{\mu\nu} - Bz_i z_r s_{ir}\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \tag{3.12}$$

are equivalent for any A and B. However, the two forms will not necessarily be equal once some explicit definitions of the momentum fractions and transverse momenta are adopted. We find it most convenient to always write the azimuthally correlated terms in the splitting functions as $k_\perp^\mu k_\perp^\nu / k_\perp^2$ such that the overall scale of $k_\perp$ cancels and substitute the definitions of momentum fractions and transverse momenta into those expressions. This choice renders the computation of azimuthally averaged splitting function trivial and simplifies their integration.

Regarding the actual definition of transverse momenta, let us consider first the case of final-final collinear splitting $(ir) \to i + r$. The momenta naturally available to construct the transverse momentum are p_i^μ, p_r^μ and the mapped momentum $\hat{p}_{ir}^\mu$ which enters the reduced matrix element (to be defined precisely in eq. (5.6) below). Thus for *final-state* partons i and r we define[8]

$$k_{\perp i,r}^\mu = \zeta_{i,r} p_r^\mu - \zeta_{r,i} p_i^\mu + Z_{ir}\hat{p}_{ir}^\mu\,. \tag{3.13}$$

The coefficients $\zeta_{i,r}$, $\zeta_{r,i}$ and Z_{ir} in eq. (3.13) can be fixed (up to an overall factor which is irrelevant as explained above) by requiting that $k_{\perp i,r}^\mu$ be orthogonal to $\hat{p}_{ir}^\mu$ and the total partonic momentum $Q^\mu = (p_a + p_b)^\mu$. These choices simplify the integration of the collinear subtraction terms. We will spell out our specific definitions below in sec. 5.1.1. As before, eq. (3.13) should be considered to define the *function* $k_{\perp i,r}^\mu$ of final-state momenta p_i and p_r and we will understand expressions such as $k_{\perp r,s}^\mu$ as being given by eq. (3.13) in a fashion very similar to the definition of momentum fractions with arbitrary indices above. We note that eq. (3.13) can be generalized to n-fold collinear splitting in a reasonably straightforward way. We will not use this generalization in this paper, however the explicit definition for the case of final-state triple collinear splitting was presented in ref. [12].

Turning to the case of initial-final collinear splitting $a \to (ar) + r$, it is most straightforward to define the transverse momentum of the final-state parton to be simply orthogonal to the direction of p_a^μ. Thus for a general *final-state* parton r we define

$$k_{\perp r,a}^\mu = p_{r,\perp}^\mu\,, \qquad k_{\perp a,r}^\mu = -k_{\perp r,a}^\mu\,, \tag{3.14}$$

where the 'transverse momentum' of the initial-state hard parton was formally defined by requiring that the transverse momenta in the splitting sum to zero.[9] Similarly to the case of initial-state momentum

[8] We have chosen the notation for the coefficients and signs for later convenience.

[9] Although the definition of $k_{\perp r,a}^\mu$, eq. (3.14), appears to involve only the momentum of parton r, notice that in fact p_a^μ must also be known in order to define the perpendicular direction. Hence, a also appears in the subscript.

fractions, $k^\mu_{\perp a,r}$ actually denotes the transverse momentum of parton (ar). Clearly, eq. (3.14) generalizes immediately to initial-state n-fold collinear splitting. E.g., for the initial-final-final $a \to (ars) + r + s$ triple collinear splitting, the transverse momenta of the final-state partons r and s are still defined as in eq. (3.14), while the 'transverse momentum' of the initial-state hard parton (ars) is once again given by the requirement that all transverse momenta sum to zero,

$$k^\mu_{\perp r,a} = p^\mu_{r,\perp}\,, \qquad k^\mu_{\perp s,a} = p^\mu_{s,\perp}\,, \qquad k^\mu_{\perp a,rs} = -k^\mu_{\perp r,a} - k^\mu_{\perp s,a}\,. \tag{3.15}$$

At this point, it is hopefully clear that eqs. (3.14) and (3.15) should once again be viewed as *functions* of the momenta whose indices appear in the subscripts, such that expressions like $k^\mu_{\perp\tilde{r},\tilde{a}}$ are understood to be defined by these equations. We note in passing that for a final-state momentum p^μ_r it is possible to give a Lorentz-invariant definition of $p^\mu_{r,\perp}$ as follows,

$$p^\mu_{r,\perp} = p^\mu_r - \frac{s_{br}}{s_{ab}} p^\mu_a - \frac{s_{ar}}{s_{ab}} p^\mu_b\,. \tag{3.16}$$

However, in actual numerical calculations, the direction of p^μ_a is usually fixed and there is no need to use this representation.

3.2.4 Treatment of soft eikonal factors

Last we briefly comment on the choice of eikonal functions in soft formulae. Clearly, the eikonal function for the emission of a soft gluon r,

$$\mathcal{S}_{jk}(r) = \frac{2s_{jk}}{s_{jr}s_{kr}}\,, \tag{3.17}$$

is well-defined as it stands over the full real emission phase space in terms of the original momenta of the event. However, we are free to use also the mapped hard momenta, i.e., make the replacement

$$\frac{2s_{jk}}{s_{jr}s_{kr}} \to \frac{2s_{\tilde{j}\tilde{k}}}{s_{\tilde{j}r}s_{\tilde{k}r}}\,. \tag{3.18}$$

At the level of IR limits, the two expressions are identical, as the transformed hard momenta go to the original ones in the soft limit. On the other hand, we find that using the form involving the mapped momenta leads to easier integrals when computing the integrated subtractions and at the same time does not lead to any complications for defining the unintegrated subtraction terms.

We have collected the definitions of all kinematic quantities introduced above in appendix C for the readers' convenience.

3.3 The general structure of subtraction terms

The upshot of the previous discussion is that we can find definitions of momentum mappings, momentum fractions, and so on that allow us promote the IR limit formulae of eq. (3.1) to true subtraction terms that are unambiguously defined over the full real emission phase space,

$$\boldsymbol{U}_j \big|\mathcal{M}^{(0)}_{ab,m+X+k}(p_a,p_b;\{p\}_{m+X+k})\big|^2 \to \mathcal{U}^{(0,0)}_j\,, \tag{3.19}$$

where we have set

$$\mathcal{U}^{(0,0)}_j = \left(\frac{\alpha_s}{2\pi}\right)^j \widetilde{\mathrm{Sing}}^{(0)}_j \big|\mathcal{M}^{(0)}_{\tilde{a}\tilde{b},m+X+k-j}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X+k-j})\big|^2\,. \tag{3.20}$$

Notice that the reduced matrix element is evaluated over the set of mapped momenta $(\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\}_{m+X+k-j})$, while $\widetilde{\mathrm{Sing}}_j^{(0)}$ denotes the expression for the appropriate singular structure, incorporating the precise definitions of momentum fractions, eikonal factors and so on. The $(0,0)$ superscript reminds us that these subtraction terms were derived from IR limit formulae that involve zero-loop singular structures and zero-loop squared matrix elements (as compared to the Born matrix element, see footnote 2 above).

Finally, the subtraction terms of eq. (3.20) can be used to build approximate cross sections. These are essentially sums over subtraction terms, constructed such that all relevant singular limits, including overlapping ones, are counted precisely once as dictated by the inclusion-exclusion principle. In sec. 4, we present the precise definitions of all approximate cross sections that appear in eq. (2.6) in terms of subtraction terms which are in turn defined in secs. 5–7.

4 Approximate cross sections for double real emission

We start by defining precisely the Born, real emission and double real emission partonic cross sections. Thus, let us consider the partonic process $a + b \to m$ jets $+ X$ (recall X denotes collectively the set of all colorless final-state particles). Clearly each cross section involves a sum over the various partonic subprocesses, hence

$$\mathrm{d}\sigma_{ab}^{\mathrm{B}}(p_a, p_b) = \sum_{\{m\}} \frac{1}{S_{\{m\}}} \mathrm{d}\sigma_{ab,m+X}^{\mathrm{B}}(p_a, p_b)\,, \tag{4.1}$$

$$\mathrm{d}\sigma_{ab}^{\mathrm{R}}(p_a, p_b) = \sum_{\{m+1\}} \frac{1}{S_{\{m+1\}}} \mathrm{d}\sigma_{ab,m+X+1}^{\mathrm{R}}(p_a, p_b)\,, \tag{4.2}$$

$$\mathrm{d}\sigma_{ab}^{\mathrm{RR}}(p_a, p_b) = \sum_{\{m+2\}} \frac{1}{S_{\{m+2\}}} \mathrm{d}\sigma_{ab,m+X+2}^{\mathrm{RR}}(p_a, p_b)\,, \tag{4.3}$$

where $\sum_{\{n\}}$ denotes the summation over the $a + b \to n$ partons $+ X$ partonic subprocesses and $S_{\{n\}}$ is the corresponding Bose symmetry factor for identical particles in the final state. Then the cross sections for the individual Born, real and double real subprocesses read

$$\begin{aligned}
\mathrm{d}\sigma_{ab,m+X}^{\mathrm{B}}(p_a, p_b) &= \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a \cdot p_b)} \\
&\quad \times \mathrm{d}\phi_{m+X}(\{p\}_{m+X}; p_a + p_b) \big|\mathcal{M}_{ab,m+X}^{(0)}(p_a, p_b; \{p\}_{m+X})\big|^2\,,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\mathrm{d}\sigma_{ab,m+X+1}^{\mathrm{R}}(p_a, p_b) &= \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a \cdot p_b)} \\
&\quad \times \mathrm{d}\phi_{m+X+1}(\{p\}_{m+X+1}; p_a + p_b) \big|\mathcal{M}_{ab,m+X+1}^{(0)}(p_a, p_b; \{p\}_{m+X+1})\big|^2\,,
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\mathrm{d}\sigma_{ab,m+X+2}^{\mathrm{RR}}(p_a, p_b) &= \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a \cdot p_b)} \\
&\quad \times \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b) \big|\mathcal{M}_{ab,m+X+2}^{(0)}(p_a, p_b; \{p\}_{m+X+2})\big|^2\,.
\end{aligned} \tag{4.6}$$

Here $\mathcal{N}$ collects all factors that are independent of QCD while $\Phi(p_a \cdot p_b)$ is the flux factor and $\{p\}_{m+X+k}$, $(k = 0, 1, 2)$ denotes the set of momenta of the final-state particles. The $(m + X + k)$-particle phase space appearing in eqs. (4.4)–(4.6) reads

$$\mathrm{d}\phi_{m+X+k}(\{p\}_{m+X+k};p_a+p_b) = \\ = \prod_{i=1}^{m+k} \frac{\mathrm{d}^d p_i}{(2\pi)^{d-1}}\delta_+(p_i^2)\frac{\mathrm{d}^d p_X}{(2\pi)^{d-1}}\delta_+(p_X^2-M_X^2)(2\pi)^d\delta^{(d)}\left(p_a+p_b-\sum_{i=1}^{m+k}p_i-p_X\right). \tag{4.7}$$

Finally, $\omega(a)$ and $\omega(b)$ count the number of colors and spins of the incoming partons and account for the necessary averaging over these in the cross section. They depend only on parton flavor and in conventional dimensional regularization we have

$$\omega(q)=\omega(\bar{q})=2N_{\mathrm{c}}\,,\qquad \omega(g)=2(1-\varepsilon)(N_{\mathrm{c}}^2-1)\,. \tag{4.8}$$

We can now spell out the precise definitions of the approximate cross sections relevant for regularizing the IR divergences present in the double real emission contribution.

4.1 Single unresolved emission

The approximate cross section regularizing single unresolved emission, $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_1}$ reads

$$\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_1} = \sum_{\{m+2\}}\frac{1}{S_{\{m+2\}}}\mathrm{d}\sigma_{ab,m+X+2}^{\mathrm{RR,A}_1}\,, \tag{4.9}$$

where the approximate cross section for any particular subprocess can be written symbolically as

$$\mathrm{d}\sigma_{ab,m+X+2}^{\mathrm{RR,A}_1} = \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{A}_1^{(0)}\,, \tag{4.10}$$

with

$$\mathcal{A}_1^{(0)} = \sum_{r\in F}\left[\mathcal{S}_r^{(0,0)}+\sum_{\substack{i\in F\\ i\neq r}}\left(\frac{1}{2}\mathcal{C}_{ir}^{FF(0,0)}-\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}\right)+\sum_{c\in I}\left(\mathcal{C}_{cr}^{IF(0,0)}-\mathcal{C}_{cr}^{IF}\mathcal{S}_r^{(0,0)}\right)\right]. \tag{4.11}$$

Here I and F denote the sets of initial-state and final-state partons. The various subtraction terms are explicit realizations of the generic subtraction term $\mathcal{U}_j^{(0,0)}$ of eq. (3.20) with $j=1$ and have the following physical origin: $\mathcal{S}_r^{(0,0)}$ denotes the subtraction term that regularizes the emission of a single soft gluon, while $\mathcal{C}_{ir}^{FF(0,0)}$ and $\mathcal{C}_{cr}^{IF(0,0)}$ are subtraction terms regularizing final-final and initial-final collinear singularities. Finally, as explained in detail in sec. 3.1, the overlapping terms $\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}$ and $\mathcal{C}_{cr}^{IF}\mathcal{S}_r^{(0,0)}$ remove the double subtraction in collinear-soft regions. Thus eq. (4.11) is simply the explicit realization of eq. (3.2), where we have carefully spelled out the summations such that each singular region is counted precisely once. Indeed, the factor of $\frac{1}{2}$ appears with $\mathcal{C}_{ir}^{FF(0,0)}$ for exactly this reason: it accounts for the fact that the double summation over $i,r\in F$ in eq. (4.11) counts this term twice. Finally, we remark that the subscripts on the concrete subtraction terms do not simply give the number of unresolved partons ($j=1$) in contrast to the generic notation in eq. (3.20). Instead they specify the actual limit from which the term derives. For example, a single unresolved collinear limit is specified by the indices of the two partons whose momenta become collinear, and these then appear in the subscripts of the corresponding subtraction terms in eq. (4.11). We will follow similar conventions for the rest of the subtraction terms throughout this paper.

We emphasize that the momentum mappings used to define the five types of subtraction terms in eq. (4.11) are not all different. Hence, the number of distinct configurations of reduced momenta, i.e., the number of counter-events, will be less than the number of subtraction terms. In fact, we find that the subtraction terms for overlapping limits can be defined using the momentum mapping appropriate to

Momentum mapping	Subtraction terms		
C_{ir}^{FF}	$\mathcal{C}_{ir}^{FF(0,0)}$		
$C_{cd,r}^{II,F}$	$\mathcal{C}_{cr}^{IF(0,0)}$		
S_r	$\mathcal{S}_r^{(0,0)}$	$\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}$	$\mathcal{C}_{cr}^{IF}\mathcal{S}_r^{(0,0)}$

Table 1: Single unresolved subtraction terms grouped according to momentum mappings. Note that collinear-soft overlapping terms are defined using the single soft momentum mapping.

the *innermost* limit, which can be read off as the rightmost one in our notation for the subtraction terms. Thus, the subtraction terms corresponding to collinear-soft overlaps, $\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}$ and $\mathcal{C}_{ar}^{IF}\mathcal{S}_r^{(0,0)}$, both employ the soft momentum mapping. This fact can be used to optimize the implementation of the approximate cross section, hence we present the grouping of the subtraction terms according to momentum mappings in table 1. The precise definitions of all momentum mappings and subtraction terms are provided in sec. 5.

4.2 Double unresolved emission

The approximate cross section $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_2}$ regularizing double unresolved emission for initial state radiation is

$$\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_2} = \sum_{\{m+2\}} \frac{1}{S_{\{m+2\}}} \mathrm{d}\sigma_{ab,m+X+2}^{\mathrm{RR,A}_2}\,, \tag{4.12}$$

where the approximate cross section for any given subprocess can be written symbolically as

$$\mathrm{d}\sigma_{ab,m+X+2}^{\mathrm{RR,A}_2} = \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{A}_2^{(0)}\,, \tag{4.13}$$

with

$$\begin{aligned}\mathcal{A}_2^{(0)} = \sum_{r\in F}\sum_{\substack{s\in F\\ s\neq r}}\Bigg\{&\frac{1}{2}\mathcal{S}_{rs}^{(0,0)}\\
&+\sum_{c\in I}\Bigg[\frac{1}{2}\mathcal{C}_{crs}^{IFF(0,0)}+\mathcal{CS}_{cr,s}^{IF(0,0)}-\mathcal{C}_{crs}^{IFF}\mathcal{CS}_{cr,s}^{IF(0,0)}-\frac{1}{2}\mathcal{C}_{crs}^{IFF}\mathcal{S}_{rs}^{(0,0)}-\mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)}+\mathcal{C}_{crs}^{IFF}\mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)}\\
&\quad+\sum_{\substack{d\in I\\ d\neq c}}\Bigg(\frac{1}{2}\mathcal{C}_{cr,ds}^{IF,IF(0,0)}-\mathcal{C}_{cr,ds}^{IF,IF}\mathcal{CS}_{cr,s}^{IF(0,0)}-\frac{1}{2}\mathcal{C}_{cr,ds}^{IF,IF}\mathcal{S}_{rs}^{(0,0)}+\mathcal{C}_{cr,ds}^{IF,IF}\mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)}\Bigg)\Bigg]\Bigg\}\,.\end{aligned} \tag{4.14}$$

The various subtraction terms appearing above correspond to the limits implied by the notation and in fact eq. (4.14) is just the explicit realization of the symbolic expression in eq. (3.3). As in eq. (4.11), various symmetry factors of $\frac{1}{2}$ have been inserted to ensure that each singular limit is counted precisely once. However, the expression in eq. (4.14) can be simplified, since the equivalence of the $\boldsymbol{C}_{ir,js}\cap\boldsymbol{CS}_{ir,s}\cap\boldsymbol{S}_{rs}$ and $\boldsymbol{C}_{ir,js}\cap\boldsymbol{S}_{rs}$ limits discussed below eq. (3.3) implies

$$\mathcal{C}_{cr,ds}^{IF,IF}\mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)} = \mathcal{C}_{cr,ds}^{IF,IF}\mathcal{S}_{rs}^{(0,0)}\,. \tag{4.15}$$

Hence we can write

$$\mathcal{A}_2^{(0)} = \sum_{r\in F}\sum_{\substack{s\in F\\ s\neq r}}\Bigg\{\frac{1}{2}\,\mathcal{S}_{rs}^{(0,0)} + \sum_{c\in I}\Bigg[\frac{1}{2}\,\mathcal{C}_{crs}^{IFF(0,0)} + \mathcal{CS}_{cr,s}^{IF(0,0)} - \mathcal{C}_{crs}^{IFF}\mathcal{CS}_{cr,s}^{IF(0,0)} - \frac{1}{2}\,\mathcal{C}_{crs}^{IFF}\,\mathcal{S}_{rs}^{(0,0)} - \mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)} + \mathcal{C}_{crs}^{IFF}\mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)} + \sum_{\substack{d\in I\\ d\neq c}}\left(\frac{1}{2}\,\mathcal{C}_{cr,ds}^{IF,IF(0,0)} - \mathcal{C}_{cr,ds}^{IF,IF}\mathcal{CS}_{cr,s}^{IF(0,0)} + \frac{1}{2}\,\mathcal{C}_{cr,ds}^{IF,IF}\,\mathcal{S}_{rs}^{(0,0)}\right)\Bigg]\Bigg\}\,. \tag{4.16}$$

Obviously this is a more economical representation of $\mathcal{A}_2^{(0)}$. Nevertheless, the original form in eq. (4.14) has the advantage that it makes certain further simplifications which emerge specifically in the color-singlet case more apparent. Indeed, as we will now discuss, all terms in eq. (4.14) that involve the soft-collinear limit cancel among each other for color-singlet production. To see this, consider first the soft-collinear IR limit formula for the configuration $p_c^\mu \parallel p_r^\mu$ and $p_s^\mu \to 0$, where c is an initial- state parton ($c = a, b$),

$$\boldsymbol{CS}_{cr,s}^{IF}\big|\mathcal{M}_{ab,m+X+2}^{(0)}(p_a,p_b;\{p\}_{m+X+2})\big|^2 = -(8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\sum_{\substack{j,k\in I\cup F\\ j,k\neq r,s}}\frac{1}{2}\mathcal{S}_{jk}(s)\boldsymbol{T}_j\boldsymbol{T}_k\frac{1}{x_{c,r}}\frac{1}{s_{cr}} \times \hat{P}_{(cr)r}(x_{c,r},k_{\perp r,c};\varepsilon)\big|\mathcal{M}_{(cr)d,m+X}^{(0)}(\hat{p}_c,\hat{p}_d;\{\hat{p}\}_{m+X})\big|^2\,, \tag{4.17}$$

where $\hat{P}_{(cr)r}$ is the Altarelli-Parisi splitting kernel for the initial-state splitting $c \to (cr) + r$ and $d \neq c$ is the other initial-state parton. In eq. (4.17), whenever j or k is equal to c in the sum, we must evaluate the appropriate term with the color charge operator $\boldsymbol{T}_{(cr)} = \boldsymbol{T}_c + \boldsymbol{T}_r$, while the eikonal factor must be computed with the reduced momentum $\hat{p}_c$ that appears in the factorized matrix element. However, in this limit $\hat{p}_c$ is strictly proportional to p_c, while the eikonal factor is degree-zero homogeneous in hard momenta (i.e., $\mathcal{S}_{\hat{c}k}(s) = \mathcal{S}_{ck}(s)$), thus in the color-singlet case ($j, k = a, b$ only), we find

$$\boldsymbol{CS}_{cr,s}^{IF}\big|\mathcal{M}_{ab,m+X+2}^{(0)}(p_a,p_b;\{p\}_{m+X+2})\big|^2 = -(8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\frac{2s_{ab}}{s_{as}s_{bs}}\boldsymbol{T}_a\boldsymbol{T}_b\frac{1}{x_{c,r}}\frac{1}{s_{cr}} \times \hat{P}_{(cr)r}(x_{c,r},k_{\perp r,c};\varepsilon)\big|\mathcal{M}_{(cr)d,m+X}^{(0)}(\hat{p}_c,\hat{p}_d;\{\hat{p}\}_{m+X})\big|^2\,. \tag{4.18}$$

Let us now compute the triple collinear limit of eq. (4.18), when $p_r \to x_{r,c}p_c$ and $p_s \to x_{s,c}p_c$. A quick computation yields

$$\boldsymbol{C}_{crs}^{IFF}\boldsymbol{CS}_{cr,s}^{IF}\big|\mathcal{M}_{ab,m+X+2}^{(0)}(p_a,p_b;\{p\}_{m+X+2})\big|^2 = (8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\frac{1}{s_{cs}}\frac{2}{x_{s,c}}\boldsymbol{T}_c^2\frac{1}{x_{c,r}}\frac{1}{s_{cr}} \times \hat{P}_{(cr)r}(x_{c,r},k_{\perp r,c};\varepsilon)\big|\mathcal{M}_{(cr)d,m+X}^{(0)}(\hat{p}_c,\hat{p}_d;\{\hat{p}\}_{m+X})\big|^2\,, \tag{4.19}$$

where we have used color conservation ($\boldsymbol{T}_a + \boldsymbol{T}_b = 0$). Similarly, in the double collinear limit, when $p_r = x_{r,c}p_c$ and $p_s = x_{s,d}p_d$ (where $c, d = a, b$ with $c \neq d$) we find

$$\boldsymbol{C}_{cr,ds}^{IF,IF}\boldsymbol{CS}_{cr,s}^{IF}\big|\mathcal{M}_{ab,m+X+2}^{(0)}(p_a,p_b;\{p\}_{m+X+2})\big|^2 = (8\pi\alpha_{\rm s}\mu^{2\varepsilon})^2\frac{1}{s_{ds}}\frac{2}{x_{s,d}}\boldsymbol{T}_d^2\frac{1}{x_{c,r}}\frac{1}{s_{cr}} \times \hat{P}_{(cr)r}(x_{c,r},k_{\perp r,c};\varepsilon)\big|\mathcal{M}_{(cr)d,m+X}^{(0)}(\hat{p}_c,\hat{p}_d;\{\hat{p}\}_{m+X})\big|^2\,. \tag{4.20}$$

But the three terms in eqs. (4.18)–(4.20) arise in eq. (4.14) in the combination

$$\begin{aligned}
&\left[\boldsymbol{CS}_{cr,s}^{IF} - \boldsymbol{C}_{crs}^{IFF}\boldsymbol{CS}_{cr,s}^{IF} - \boldsymbol{C}_{cr,ds}^{IF,IF}\boldsymbol{CS}_{cr,s}^{IF}\right] \left|\mathcal{M}_{ab,m+X+2}^{(0)}(p_a,p_b;\{p\}_{m+X+2})\right|^2 \\
&\quad = (8\pi\alpha_s\mu^{2\varepsilon})^2 \left[-\frac{2s_{ab}}{s_{as}s_{bs}}\boldsymbol{T}_a\boldsymbol{T}_b - \frac{1}{s_{cs}}\frac{2}{x_{s,c}}\boldsymbol{T}_c^2 - \frac{1}{s_{ds}}\frac{2}{x_{s,d}}\boldsymbol{T}_d^2\right]\frac{1}{x_{c,r}}\frac{1}{s_{cr}}\hat{P}_{(cr)r}(x_{c,r},k_{\perp r,c};\varepsilon) \\
&\quad \times \left|\mathcal{M}_{(cr)d,m+X}^{(0)}(\hat{p}_c,\hat{p}_d;\{\hat{p}\}_{m+X})\right|^2 .
\end{aligned} \tag{4.21}$$

The color-charge algebra is trivial in the color-singlet case, $\boldsymbol{T}_a\boldsymbol{T}_b = -\boldsymbol{T}_a^2 = -\boldsymbol{T}_b^2$, (recall $c,d \in I = \{a,b\}$) so the color-charge operators in eq. (4.21) can be factored and the combination of terms becomes proportional to

$$\left[\frac{s_{ab}}{s_{as}s_{bs}} - \frac{1}{s_{cs}}\frac{1}{x_{s,c}} - \frac{1}{s_{ds}}\frac{1}{x_{s,d}}\right] . \tag{4.22}$$

However, if we define the momentum fractions $x_{s,c}$ and $x_{s,d}$ as in eq. (3.8), then the expression in eq. (4.22) vanishes for both $c = a$, $d = b$ and $c = b$, $d = a$. Thus, with this choice of momentum fractions, we find the following cancellation among the subtraction terms appearing in eq. (4.14),[10]

$$\mathcal{CS}_{cr,s}^{IF(0,0)} - \mathcal{C}_{crs}^{IFF}\mathcal{CS}_{cr,s}^{IF(0,0)} - \sum_{\substack{d\in I\\ d\neq c}}\mathcal{C}_{cr,ds}^{IF,IF}\mathcal{CS}_{cr,s}^{IF(0,0)} = 0 . \tag{4.23}$$

Very similar arguments also establish that

$$\mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)} - \mathcal{C}_{crs}^{IFF}\mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)} - \sum_{\substack{d\in I\\ d\neq c}}\mathcal{C}_{cr,ds}^{IF,IF}\mathcal{CS}_{cr,s}^{IF}\mathcal{S}_{rs}^{(0,0)} = 0 , \tag{4.24}$$

hence all terms involving the soft-collinear limit cancel in eq. (4.14) as promised and we find the simplified form of $\mathcal{A}_2^{(0)}$ from ref. [53], appropriate for color-singlet production,

$$\mathcal{A}_2^{(0)} = \frac{1}{2}\sum_{r\in F}\sum_{\substack{s\in F\\ s\neq r}}\left\{\mathcal{S}_{rs}^{(0,0)} + \sum_{c\in I}\left[\mathcal{C}_{crs}^{IFF(0,0)} - \mathcal{C}_{crs}^{IFF}\mathcal{S}_{rs}^{(0,0)} + \sum_{\substack{d\in I\\ d\neq c}}\left(\mathcal{C}_{cr,ds}^{IF,IF(0,0)} - \mathcal{C}_{cr,ds}^{IF,IF}\mathcal{S}_{rs}^{(0,0)}\right)\right]\right\} . \tag{4.25}$$

Obviously, this result can also be reached starting from eq. (4.16) by using eq. (4.15). However, the nature of the cancellations is then somewhat obscured.

As in the case of eq. (4.11), not all momentum mappings used to define the five types of subtraction terms in eq. (4.25) are distinct, since once again, subtraction terms corresponding to overlapping limits can be defined using the momentum mapping appropriate to the innermost limit, i.e., the rightmost one in our notation. The grouping of subtraction terms according to momentum mappings is presented in table 2 for the color-singlet case. The precise definition of each momentum mapping and subtraction term is given in sec. 6.

4.3 Iterated single unresolved emission

The approximate cross section regularizing iterated single unresolved emission, $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_{12}}$, reads

$$\mathrm{d}\sigma_{ab}^{\mathrm{RR,A}_{12}} = \sum_{\{m+2\}}\frac{1}{S_{\{m+2\}}}\mathrm{d}\sigma_{ab,m+X+2}^{\mathrm{RR,A}_{12}} , \tag{4.26}$$

[10]Notice that since $I = \{a,b\}$, the sum over d in eq. (4.23) constitutes only a single term: if $c = a$ then $d = b$, while if $c = b$ then $d = a$.

Momentum mapping	Subtraction terms		
$C^{II,FF}_{cd,rs}$	$\mathcal{C}^{IFF(0,0)}_{crs}$	$\mathcal{C}^{IF,IF(0,0)}_{cr,ds}$	
S_{rs}	$\mathcal{S}^{(0,0)}_{rs}$	$\mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0,0)}_{rs}$	$\mathcal{C}^{IF,IF}_{cr,ds}\mathcal{S}^{(0,0)}_{rs}$

Table 2: Double unresolved subtraction terms grouped according to momentum mappings. Note that collinear-soft overlapping terms are defined using the double soft momentum mapping.

where for any particular subprocess the approximate cross section can be written symbolically as

$$\mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{ab,m+X+2} = \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{A}^{(0)}_{12}\,, \tag{4.27}$$

with

$$\mathcal{A}^{(0)}_{12} = \sum_{s\in F}\left[\mathcal{A}^{(0)}_2\mathcal{S}_s + \sum_{\substack{r\in F\\ r\neq s}}\left(\frac{1}{2}\mathcal{A}^{(0)}_2\mathcal{C}^{FF}_{rs} - \mathcal{A}^{(0)}_2\mathcal{C}^{FF}_{rs}\mathcal{S}_s\right) + \sum_{c\in I}\left(\mathcal{A}^{(0)}_2\mathcal{C}^{IF}_{cs} - \mathcal{A}^{(0)}_2\mathcal{C}^{IF}_{cs}\mathcal{S}_s\right)\right]. \tag{4.28}$$

As anticipated above in sec. 3.1, $\mathcal{A}^{(0)}_{12}$ is formally obtained by applying the same construction as in eq. (4.11) to $\mathcal{A}^{(0)}_2$. It is then clearly more advantageous to consider the form of $\mathcal{A}^{(0)}_2$ in eq. (4.16) in order to decrease the number of terms we must consider and we find

$$\begin{aligned}
\mathcal{A}^{(0)}_2\mathcal{S}_s &= \sum_{\substack{r\in F\\ r\neq s}}\Bigg[\mathcal{S}^{(0)}_{rs}\mathcal{S}_s + \sum_{c\in I}\Bigg(\mathcal{C}^{IFF(0)}_{crs}\mathcal{S}_s + \mathcal{CS}^{IF(0)}_{cr,s}\mathcal{S}_s - \mathcal{C}^{IFF}_{crs}\mathcal{CS}^{IF(0)}_{cr,s}\mathcal{S}_s \\
&\qquad - \mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0)}_{rs}\mathcal{S}_s - \mathcal{CS}^{IF}_{cr,s}\mathcal{S}^{(0)}_{rs}\mathcal{S}_s + \mathcal{C}^{IFF}_{crs}\mathcal{CS}^{IF}_{cr,s}\mathcal{S}^{(0)}_{rs}\mathcal{S}_s\Bigg)\Bigg]\,,
\end{aligned}\tag{4.29}$$

$$\mathcal{A}^{(0)}_2\mathcal{C}^{FF}_{rs} = \mathcal{S}^{(0)}_{rs}\mathcal{C}^{FF}_{rs} + \sum_{c\in I}\left(\mathcal{C}^{IFF(0)}_{crs}\mathcal{C}^{FF}_{rs} - \mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0)}_{rs}\mathcal{C}^{FF}_{rs}\right), \tag{4.30}$$

$$\mathcal{A}^{(0)}_2\mathcal{C}^{FF}_{rs}\mathcal{S}_s = \mathcal{S}^{(0)}_{rs}\mathcal{C}^{FF}_{rs}\mathcal{S}_s + \sum_{c\in I}\left(\mathcal{C}^{IFF(0)}_{crs}\mathcal{C}^{FF}_{rs}\mathcal{S}_s - \mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0)}_{rs}\mathcal{C}^{FF}_{rs}\mathcal{S}_s\right), \tag{4.31}$$

$$\begin{aligned}
\mathcal{A}^{(0)}_2\mathcal{C}^{IF}_{cs} &= \sum_{\substack{r\in F\\ r\neq s}}\Bigg[\mathcal{C}^{IFF(0)}_{csr}\mathcal{C}^{IF}_{cs} + \mathcal{CS}^{IF(0)}_{cs,r}\mathcal{C}^{IF}_{cs} - \mathcal{C}^{IFF}_{csr}\mathcal{CS}^{IF(0)}_{cs,r}\mathcal{C}^{IF}_{cs} \\
&\qquad + \sum_{\substack{d\in I\\ d\neq c}}\left(\mathcal{C}^{IF,IF(0)}_{cs,dr}\mathcal{C}^{IF}_{cs} - \mathcal{C}^{IF,IF(0)}_{cs,dr}\mathcal{CS}^{IF(0)}_{cs,r}\mathcal{C}^{IF}_{cs}\right)\Bigg]\,,
\end{aligned}\tag{4.32}$$

$$\begin{aligned}
\mathcal{A}^{(0)}_2\mathcal{C}^{IF}_{cs}\mathcal{S}_s &= \sum_{\substack{r\in F\\ r\neq s}}\Bigg[\mathcal{S}^{(0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s + \mathcal{C}^{IFF(0)}_{csr}\mathcal{C}^{IF}_{cs}\mathcal{S}_s - \mathcal{C}^{IFF}_{csr}\mathcal{S}^{(0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s \\
&\qquad + \sum_{\substack{d\in I\\ d\neq c}}\left(\mathcal{CS}^{IF(0)}_{dr,s}\mathcal{C}^{IF}_{cs}\mathcal{S}_s - \mathcal{CS}^{IF}_{dr,s}\mathcal{S}^{(0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s\right)\Bigg]\,.
\end{aligned}\tag{4.33}$$

We note that in writing eqs. (4.29)–(4.33), we have made use of various cancellations that are manifest already at the level of iterated IR factorization formulae, since the direct calculation of the terms on the right-hand side of eq. (4.28) yields significantly larger expressions than these [10]. Concentrating on color-singlet production, it is obvious that the cancellations which lead from eqs. (4.14) and (4.16) to eq. (4.25) imply several relations among the terms in eqs. (4.29)–(4.33). In particular, we find

$$\mathcal{S}^{(0)}_{rs}\mathcal{C}^{FF}_{rs}\mathcal{S}_s - \sum_{c\in I}\mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0)}_{rs}\mathcal{C}^{FF}_{rs}\mathcal{S}_s = 0\,, \tag{4.34}$$

$$\sum_{c\in I}\left(\mathcal{CS}^{IF,(0)}_{cs,r}\mathcal{C}^{IF}_{cs} - \mathcal{C}^{IFF}_{csr}\mathcal{CS}^{IF,(0)}_{cs,r}\mathcal{C}^{IF}_{cs} - \sum_{\substack{d\in I\\ d\neq c}}\mathcal{C}^{IF,IF}_{cs,dr}\mathcal{CS}^{IF,(0)}_{cs,r}\mathcal{C}^{IF}_{cs}\right) = 0\,, \tag{4.35}$$

$$\sum_{c\in I}\left(\mathcal{CS}^{IF(0)}_{cr,s}\mathcal{S}_s - \mathcal{C}^{IFF}_{crs}\mathcal{CS}^{IF(0)}_{cr,s}\mathcal{S}_s - \sum_{\substack{d\in I\\ d\neq c}}\mathcal{CS}^{IF(0)}_{dr,s}\mathcal{C}^{IF}_{cs}\mathcal{S}_s\right) = 0\,, \tag{4.36}$$

$$\sum_{c\in I}\left(\mathcal{CS}^{IF}_{cr,s}\mathcal{S}^{(0)}_{rs}\mathcal{S}_s - \mathcal{C}^{IFF}_{crs}\mathcal{CS}^{IF}_{cr,s}\mathcal{S}^{(0)}_{rs}\mathcal{S}_s - \sum_{\substack{d\in I\\ d\neq c}}\mathcal{CS}^{IF}_{dr,s}\mathcal{S}^{(0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s\right) = 0\,, \tag{4.37}$$

which allows us to rewrite the various terms in eq. (4.28) as

$$\mathcal{A}^{(0)}_2\mathcal{S}_s = \sum_{\substack{r\in F\\ r\neq s}}\left[\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s + \sum_{c\in I}\left(\mathcal{C}^{IFF(0,0)}_{crs}\mathcal{S}_s - \mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s\right)\right], \tag{4.38}$$

$$\mathcal{A}^{(0)}_2\mathcal{C}^{FF}_{rs} = \mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{FF}_{rs} + \sum_{c\in I}\left(\mathcal{C}^{IFF(0,0)}_{crs}\mathcal{C}^{FF}_{rs} - \mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{FF}_{rs}\right), \tag{4.39}$$

$$\mathcal{A}^{(0)}_2\mathcal{C}^{FF}_{rs}\mathcal{S}_s = \sum_{c\in I}\mathcal{C}^{IFF(0,0)}_{crs}\mathcal{C}^{FF}_{rs}\mathcal{S}_s\,, \tag{4.40}$$

$$\mathcal{A}^{(0)}_2\mathcal{C}^{IF}_{cs} = \sum_{\substack{r\in F\\ r\neq s}}\left(\mathcal{C}^{IFF(0,0)}_{csr}\mathcal{C}^{IF}_{cs} + \sum_{\substack{d\in I\\ d\neq c}}\mathcal{C}^{IF,IF(0,0)}_{cs,dr}\mathcal{C}^{IF}_{cs}\right), \tag{4.41}$$

$$\mathcal{A}^{(0)}_2\mathcal{C}^{IF}_{cs}\mathcal{S}_s = \sum_{\substack{r\in F\\ r\neq s}}\left(\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s + \mathcal{C}^{IFF(0,0)}_{csr}\mathcal{C}^{IF}_{cs}\mathcal{S}_s - \mathcal{C}^{IFF}_{csr}\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s\right). \tag{4.42}$$

Clearly, eqs. (4.38)–(4.42) can also be derived starting from the simplified form of $\mathcal{A}^{(0)}_2$ in eq. (4.25), appropriate to color-singlet production.

Once more, the number of independent momentum mappings used to define the subtraction terms in eq. (4.28) is less than the total number of subtraction terms and the grouping of terms according to momentum mappings is presented in table 3 for the color-singlet case.[11] In fact, as anticipated above, iterated single unresolved subtraction terms can be defined using convolutions of single unresolved momentum mappings. The precise convolution for each term can be determined as follows. First, we perform a mapping of momenta appropriate to the single unresolved limit involved in the definition of the specific term. Thus, collinear limits imply that the first mapping is of single collinear type, while soft and collinear-soft limits indicate that the first mapping is a single soft one. We have chosen our notation such that the single limits stand to the right of the double limits in iterated single unresolved subtraction terms specifically so that this mapping can be read off as the rightmost limit, which then corresponds to the same prescription as for the overlaps of limits. The next mapping can then be inferred from the rightmost double unresolved limit appearing in the notation: it is of collinear or soft type depending on the nature of this double unresolved limit. By way of example, consider e.g., the subtraction term $\mathcal{C}^{IFF}_{csr}\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s$. The rightmost limit is the

[11] In table 3, we use the standard notation for the convolution of maps, so that, e.g., $S_{\tilde{r}} \circ S_s$ implies that the S_s mapping is performed first, followed by the $S_{\tilde{r}}$ mapping.

Momentum mapping	Subtraction terms			
$C^{II,F}_{\hat{c}\hat{d},\hat{r}\hat{s}} \circ C^{FF}_{rs}$	$\mathcal{C}^{IFF(0,0)}_{crs}\mathcal{C}^{FF}_{rs}$			
$C^{II,F}_{\hat{c}\hat{d},\hat{r}} \circ C^{II,F}_{cd,s}$	$\mathcal{C}^{IFF(0,0)}_{csr}\mathcal{C}^{IF}_{cs}$	$\mathcal{C}^{IF,IF(0,0)}_{cs,dr}\mathcal{C}^{IF}_{cs}$		
$C^{II,F}_{\tilde{c}\tilde{d},\tilde{r}} \circ S_s$	$\mathcal{C}^{IFF(0,0)}_{crs}\mathcal{S}_s$	$\mathcal{C}^{IFF(0,0)}_{crs}\mathcal{C}^{FF}_{rs}\mathcal{S}_s$	$\mathcal{C}^{IFF(0,0)}_{crs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s$	
$S_{\hat{r}\hat{s}} \circ C^{FF}_{rs}$	$\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{FF}_{rs}$	$\mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{FF}_{rs}$		
$S_{\tilde{r}} \circ S_s$	$\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s$	$\mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s$	$\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s$	$\mathcal{C}^{IFF}_{crs}\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{cs}\mathcal{S}_s$

Table 3: Iterated single unresolved subtraction terms grouped according to momentum mappings. The symbol $\circ$ denotes the standard convolution of maps.

single soft limit $\mathcal{S}_s$, so the first mapping is going to be the corresponding single soft one, which we denote by S_s. Then, the first double unresolved limit from the right is the double soft limit $\mathcal{S}^{(0,0)}_{rs}$. Thus, the second mapping is also of soft type. As the first mapping removes the soft momentum p_s and remaps the other soft momentum $p_r \to \tilde{p}_r$, this second single soft momentum mapping will correspond to removing the remapped momentum $\tilde{p}_r$ and is denoted by $S_{\tilde{r}}$. Thus the momentum mapping for this subtraction term is obtained by the convolution of these two mappings, $S_{\tilde{r}} \circ S_s$, as shown in table 3. Hence, in the case of both overlapping as well as iterated single unresolved subtraction terms, the appropriate mappings can be inferred from the notation by reading from right to left. The explicit definition of each subtraction term in eqs. (4.38)–(4.42) is given in sec. 7.

5 Subtraction terms for single unresolved emission

In sec. 4.1, we introduced the subtraction terms for single unresolved emission. In this section we provide their precise definitions.

5.1 Final-state single collinear-type subtraction term

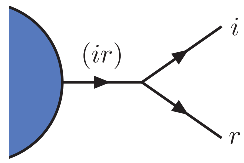
5.1.1 $\mathcal{C}^{FF(0,0)}_{ir}$

Subtraction term: For final-state partons i and r we define the final-final single collinear subtraction term

$$\mathcal{C}^{FF(0,0)}_{ir}(p_a, p_b; \{p\}_{m+X+2}) = 8\pi\alpha_s\mu^{2\varepsilon}\frac{1}{s_{ir}}\hat{P}^{(0)}_{ir}(z_{i,r}, k_{\perp i,r}; \varepsilon)\left|\mathcal{M}^{(0)}_{ab,m+X+1}(\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1})\right|^2, \quad (5.1)$$

where the reduced matrix element is obtained by removing the final-state partons i and r and replacing them with a single parton (ir) whose flavor is determined by the flavor summation rules

$$q + g = q\,, \qquad \bar{q} + g = \bar{q}\,, \qquad q + \bar{q} = g\,, \qquad g + g = g\,. \quad (5.2)$$



i	r	(ir)
$q/\bar{q}$	$\bar{q}/q$	g
$q/\bar{q}$	g	$q/\bar{q}$
g	$q/\bar{q}$	$q/\bar{q}$
g	g	g

Figure 1: The splitting configuration for $(ir) \to i + r$ final-state splitting. The arrows in the figure denote momentum and flavor flow. The precise flavor mapping for the $\mathcal{C}_{ir}^{FF(0,0)}$ counterterm is given in the table.

Fig. 1 shows the corresponding final-state splitting configuration and gives the precise flavor mapping for all cases. The set of mapped momenta entering the reduced matrix element is defined below in eq. (5.6). Furthermore, $\hat{P}_{ir}^{(0)}$ are the final-final tree-level Altarelli-Parisi splitting functions for $(ir) \to i + r$ splitting, whose explicit expressions are recalled in eqs. (D.1)–(D.4). The momentum fraction $z_{i,r}$ is given in eq. (3.4). The transverse momentum $k_{\perp i,r}^{\mu}$ is defined as in eq. (3.13). We choose to fix the coefficients appearing there as follows,

$$\zeta_{i,r} = z_{i,r} - \frac{s_{ir}}{\alpha_{ir} s_{(ir)Q}}\,, \qquad \zeta_{r,i} = z_{r,i} - \frac{s_{ir}}{\alpha_{ir} s_{(ir)Q}}\,, \tag{5.3}$$

and

$$Z_{ir} = \frac{s_{ir}}{\alpha_{ir} s_{\widehat{ir}Q}} (z_{r,i} - z_{i,r})\,, \tag{5.4}$$

where α_{ir} is defined by eq. (5.7) below, while $s_{\widehat{ir}Q} = 2\hat{p}_{ir} \cdot (p_a + p_b)$, with $\hat{p}_{ir}^{\mu}$ given in eq. (5.6). These choices ensure that $k_{\perp i,r}^{\mu}$ is orthogonal to both $\hat{p}_{ir}^{\mu}$ and to $Q^{\mu} = (p_a + p_b)^{\mu}$.

Momentum mapping: The final-state single collinear (FF) momentum mapping,

$$C_{ir}^{FF} : \ (p_a, p_b; \{p\}_{m+X+2}) \xrightarrow{C_{ir}^{FF}} (\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1})\,, \tag{5.5}$$

is defined as $(i, r \in F)$

$$\begin{aligned}
\hat{p}_a^{\mu} &= (1 - \alpha_{ir}) p_a^{\mu}\,, \\
\hat{p}_b^{\mu} &= (1 - \alpha_{ir}) p_b^{\mu}\,, \\
\hat{p}_{ir}^{\mu} &= p_i^{\mu} + p_r^{\mu} - \alpha_{ir} (p_a + p_b)^{\mu}\,, \\
\hat{p}_n^{\mu} &= p_n^{\mu}\,, \qquad n \in F\,, \quad n \neq i, r\,, \\
\hat{p}_X^{\mu} &= p_X^{\mu}\,.
\end{aligned} \tag{5.6}$$

The value of α_{ir} is fixed by requiring that the parent momentum, $\hat{p}_{ir}^{\mu}$, be massless, $\hat{p}_{ir}^2 = 0$, which gives

$$\alpha_{ir} = \frac{1}{2}\left[\frac{s_{(ir)Q}}{s_{ab}} - \sqrt{\frac{s_{(ir)Q}^2}{s_{ab}^2} - \frac{4 s_{ir}}{s_{ab}}}\,\right]. \tag{5.7}$$

Clearly, as p_i^{μ} and p_r^{μ} become collinear, $\alpha_{ir} \to 0$ and so $\hat{p}_{ir}^{\mu} \to p_i^{\mu} + p_r^{\mu}$ while the rest of the momenta remain unchanged. This corresponds precisely to the $p_i^{\mu} \parallel p_r^{\mu}$ final-final single collinear kinematic configuration. Throughout, we will denote momenta obtained via a collinear-type momentum mapping with hats, as in eq. (5.6).

Phase space convolution: By inserting the identity,

$$1=\frac{\mathrm{d}^dK}{(2\pi)^d}(2\pi)^d\delta^{(d)}(K-p_i-p_r)\,\mathrm{d}K^2\,\delta_+(K^2-s_{ir})\,, \tag{5.8}$$

the $(m+X+2)$-particle phase space of eq. (4.7) (with $k=2$) can be factored on partons i and r through the convolution formula,

$$\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)=\frac{\mathrm{d}K^2}{2\pi}\,\mathrm{d}\phi_{m+X+1}(p_a,\ldots,K,\ldots,p_{m+2},p_X;p_a+p_b)\,\mathrm{d}\phi_2(p_i,p_r;K)\,. \tag{5.9}$$

Moreover, using then the identity,

$$1=\mathrm{d}\alpha\,\delta(\alpha-\alpha_{ir})\,\mathrm{d}^d\hat{p}_{ir}\,\delta^{(d)}\left(\hat{p}_{ir}-K+\alpha(p_a+p_b)\right)\,, \tag{5.10}$$

and exchanging the momenta $\{p_n\}$ and p_X with the momenta $\{\hat{p}_n\}$ and $\hat{p}_X$ through the mapping in eq. (5.6), we can write the convolution of the $(m+X+2)$-particle phase space in eq. (5.9) as,

$$\begin{aligned}&\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)=\\&\quad=\int_{\alpha_{\min}}^{\alpha_{\max}}\mathrm{d}\alpha\,\mathrm{d}\phi_{m+X+1}\big(\{\hat{p}\}_{m+X+1};(1-\alpha)(p_a+p_b)\big)\frac{s_{\widehat{ir}Q}}{2\pi}\mathrm{d}\phi_2(p_i,p_r;K)\,,\end{aligned} \tag{5.11}$$

where K is fixed by eq. (5.10), i.e., $K=\hat{p}_{ir}+\alpha(p_a+p_b)$. The limits of the α integration are

$$\alpha_{\min}=0\qquad\text{and}\qquad\alpha_{\max}=1-\frac{M}{\sqrt{s_{ab}}}\,, \tag{5.12}$$

where M is the sum of the masses of all final-state particles (both partons and non-QCD particles). Finally, we note that the flux factor $\Phi(p_a\cdot p_b)$ written in terms of mapped momenta reads

$$\Phi(p_a\cdot p_b)=\frac{\Phi(\hat{p}_a\cdot\hat{p}_b)}{(1-\alpha)^2}\,, \tag{5.13}$$

where we have exploited eq. (5.10) to set $\alpha_{ir}=\alpha$.

Integrated subtraction term: Using the definitions of the subtraction term and the phase space convolution, eqs. (5.1) and (5.11), we write the integrated counterterm as

$$\begin{aligned}&\int_1\mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{C}_{ir}^{FF(0,0)}(p_a,p_b;\{p\}_{m+X+2})\\&\quad\mathcal{N}\int_{\alpha_{\min}}^{\alpha_{\max}}\mathrm{d}\alpha\,\frac{1}{\omega(a)\omega(b)\Phi(\hat{p}_a\cdot\hat{p}_b)}\mathrm{d}\phi_{m+X+1}\big(\{\hat{p}\}_{m+X+1};(1-\alpha)(p_a+p_b)\big)\\&\quad\times\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\left[\mathrm{C}_{ir}^{FF(0,0)}(\alpha;\varepsilon)\right]\left|\mathcal{M}_{ab,m+X+1}^{(0)}(\hat{p}_a,\hat{p}_b;\{\hat{p}\}_{m+X+1})\right|^2.\end{aligned} \tag{5.14}$$

Here and in the following, the notation $\int_1$ indicates that we are integrating only over the phase space measure of the single unresolved emission. Furthermore, using the delta function in eq. (5.10), the initial-state momenta in the reduced matrix element above are written as $\hat{p}_a=(1-\alpha)p_a$ and $\hat{p}_b=(1-\alpha)p_b$ and we introduced the integrated counterterm

$$\left[\mathrm{C}_{ir}^{FF(0,0)}(\alpha;\varepsilon)\right]=\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^\varepsilon\int\mathrm{d}\phi_2(p_i,p_r;K)(1-\alpha)^2\frac{s_{\widehat{ir}Q}}{2\pi}\frac{1}{s_{ir}}\hat{P}_{ir}^{(0)}(z_{i,r},k_{\perp i,r};\varepsilon)\,. \tag{5.15}$$

Recall that $K = \hat{p}_{ir} + \alpha(p_a + p_b)$ and we set

$$S_\varepsilon = \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}\,. \tag{5.16}$$

Notice that on the right hand side of eq. (5.14), we have rewritten the flux factor in terms of mapped momenta using eq. (5.13) and hence a factor of $\Phi(\hat{p}_a \cdot \hat{p}_b)/\Phi(p_a \cdot p_b) = (1-\alpha)^2$ appears explicitly in the definition of the integrated counterterm in eq. (5.15). With this rewriting, we readily recognize the expression of the real emission cross section for the appropriate partonic subprocess, $\mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}(\hat{p}_a, \hat{p}_b)$ defined in eq. (4.5), on the right hand side of eq. (5.14). Thus we can write

$$\begin{aligned}
&\int_1 \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a \cdot p_b)} \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b)\, \mathcal{C}^{FF(0,0)}_{ir}(p_a, p_b; \{p\}_{m+X+2}) \\
&\quad = \frac{\alpha_{\mathrm{s}}}{2\pi} S_\varepsilon \left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon \left(\left[\mathrm{C}^{FF(0,0)}_{ir}\right] \otimes \mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1} \right)
\end{aligned} \tag{5.17}$$

where the symbol $\otimes$ denotes the phase space convolution

$$\left[\mathrm{C}^{FF(0,0)}_{ir}\right] \otimes \mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1} = \int_{\alpha_{\min}}^{\alpha_{\max}} \mathrm{d}\alpha \left[\mathrm{C}^{FF(0,0)}_{ir}(\alpha;\varepsilon)\right] \mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}(\hat{p}_a, \hat{p}_b)\,. \tag{5.18}$$

As noted previously, the unintegrated subtraction term is generally a matrix in color and/or spin space and this property is inherited by the integrated counterterm.[12] Thus, the product of the integrated counterterm and the real emission cross section in eq. (5.18) is to be understood accordingly. In the rest of the paper, we will present all integrated counterterms immediately in the more compact form corresponding to eqs. (5.17) and (5.18). In order to evaluate $\left[\mathrm{C}^{FF(0,0)}_{ir}(\alpha;\varepsilon)\right]$, we start by using $k_{\perp i,r} \cdot \hat{p}_{ir} = 0$ and perform the azimuthal integration by simply passing in eq. (5.15) to the azimuthally averaged splitting functions, $\hat{P}^{(0)}_{ir}(z_{i,r}, k_{\perp i,r};\varepsilon) \to P^{(0)}_{ir}(z_{i,r};\varepsilon)$, which are given in eqs. (D.11)–(D.14).[13] Hence we have

$$\left[\mathrm{C}^{FF(0,0)}_{ir}(\alpha;\varepsilon)\right] = \frac{8\pi}{S_\varepsilon} s_{ab}^\varepsilon \int \mathrm{d}\phi_2(p_i, p_r; K)(1-\alpha)^2 \frac{s_{\widehat{ir}Q}}{s_{ir}} P^{(0)}_{ir}(z_{i,r};\varepsilon)\,. \tag{5.19}$$

In this paper, we will limit ourselves to setting up the definitions of the integrated subtraction terms as in eq. (5.19) and defer their explicit calculation to separate dedicated publications.

5.2 Initial-state single collinear-type subtraction term

5.2.1 $\mathcal{C}^{IF(0,0)}_{ar}$

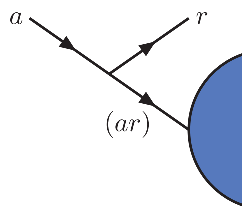
Subtraction term: For an initial-state parton a and a final-state parton r we define the initial-final single collinear subtraction term

$$\mathcal{C}^{IF(0,0)}_{ar}(p_a, p_b; \{p\}_{m+X+2}) = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon} \frac{1}{x_{a,r} s_{ar}} \hat{P}^{(0)}_{(ar)r}(x_{a,r}, k_{\perp r,a};\varepsilon) \left|\mathcal{M}^{(0)}_{(ar)b,m+X+1}(\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1})\right|^2, \tag{5.20}$$

where the reduced matrix element is obtained by removing the final-state parton r and replacing the initial-state parton a by the parton (ar) whose flavor is determined by the flavor summation rules of eq. (5.2) by requiring that $a = (ar) + r$. Clearly the flavor of the initial-state parton that is involved in

[12] In practice it turns out that all integrated counterterms become proportional to the unit matrix in spin space. However, the same is not true in color space.

[13] Thus, as noted above, the non-trivial spin correlations generally present in eq. (5.1) disappear upon integration.



a	r	(ar)
$q/\bar{q}$	$q/\bar{q}$	g
$q/\bar{q}$	g	$q/\bar{q}$
g	$q/\bar{q}$	$\bar{q}/q$
g	g	g

Figure 2: The splitting configuration for $a \to (ar) + r$ initial-state splitting. The arrows in the figure denote momentum and flavor flow. The precise flavor mapping for the $\mathcal{C}_{ar}^{IF(0,0)}$ counterterm is given in the table.

the splitting is generally different in the original double real emission matrix element and in the reduced matrix element. The initial-state splitting configuration and precise flavor mapping are shown in fig. 2. The set of momenta entering the reduced matrix element is defined below in eq. (5.22). The tree-level initial-final Altarelli-Parisi splitting functions for $a \to (ar) + r$ splitting, $\hat{P}^{(0)}_{(ar)r}$, can be obtained from those for final-final splitting by the crossing relation discussed in appendix D.1, see eq. (D.21), and are given explicitly in eqs. (D.23)–(D.26). As usual, we have chosen to label the splitting function with the flavor of the initial-state parton in the *reduced* matrix element, (ar), and the flavor of the final-state parton r. The momentum fraction $x_{a,r}$ and transverse momentum $k_{\perp r,a}$ are defined by eqs. (3.8) and (3.14).

Momentum mapping: The initial-state single collinear (IF) momentum mapping,

$$C_{ab,r}^{II,F} \,:\, (p_a, p_b; \{p\}_{m+X+2}) \xrightarrow{C_{ab,r}^{II,F}} (\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1}) \,, \tag{5.21}$$

is defined as ($a, b \in I$ and $r \in F$)

$$\begin{aligned} \hat{p}_a^\mu &= \xi_{a,r} p_a^\mu \,, \\ \hat{p}_b^\mu &= \xi_{b,r} p_b^\mu \,, \\ \hat{p}_n^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu \, p_n^\nu \,, \qquad n \in F, \quad n \neq r \,, \\ \hat{p}_X^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu \, p_X^\nu \,. \end{aligned} \tag{5.22}$$

Here $\Lambda(P,\hat{P})^\mu{}_\nu$ is a proper Lorentz transformation that takes the massive momentum P^μ into a momentum $\hat{P}^\mu$ of the same mass, with

$$P^\mu = (p_a + p_b)^\mu - p_r^\mu \,, \qquad \hat{P}^\mu = (\hat{p}_a + \hat{p}_b)^\mu = (\xi_{a,r} p_a + \xi_{b,r} p_b)^\mu \,. \tag{5.23}$$

One specific representation of $\Lambda(P,\hat{P})^\mu{}_\nu$ is

$$\Lambda(P,\hat{P})^\mu{}_\nu = g^\mu{}_\nu - 2\frac{(P+\hat{P})^\mu (P+\hat{P})_\nu}{(P+\hat{P})^2} + 2\frac{\hat{P}^\mu P_\nu}{P^2} \,. \tag{5.24}$$

Requiring that $P^2 = \hat{P}^2$ only fixes the value of the product of $\xi_{a,r}$ and $\xi_{b,r}$,

$$\xi_{a,r}\xi_{b,r} = 1 - \frac{s_{rQ}}{s_{ab}} \,, \tag{5.25}$$

so there is some freedom in their explicit definition. We find the choice of ref. [54] convenient and we set

$$\xi_{a,r} = \sqrt{\frac{s_{ab} - s_{br}}{s_{ab} - s_{ar}} \frac{s_{ab} - s_{rQ}}{s_{ab}}}\,, \qquad \xi_{b,r} = \sqrt{\frac{s_{ab} - s_{ar}}{s_{ab} - s_{br}} \frac{s_{ab} - s_{rQ}}{s_{ab}}}\,, \tag{5.26}$$

which fulfills eq. (5.25). Notice that $\xi_{a,r}$ and $\xi_{b,r}$ are related by the exchange $a \leftrightarrow b$. It is not difficult to see that as, e.g., p_a^μ and p_r^μ become collinear such that $p_r^\mu \to x_r p_a^\mu$, we have $\xi_{a,r} \to 1 - x_r$ while $\xi_{b,r} \to 1$, hence $\hat{P}^\mu \to P^\mu$ and the Lorentz-transformation in eq. (5.24) becomes the identity. Thus we obtain precisely the $p_a^\mu \parallel p_r^\mu$ initial-final single collinear kinematic configuration. Due to the $a \leftrightarrow b$ symmetry of the mapping, the mapped momenta also have the correct behavior to describe the $p_b^\mu \parallel p_r^\mu$ initial-final collinear limit.

Phase space convolution: Using the identity,

$$1 = \mathrm{d}\xi_a\, \delta(\xi_a - \xi_{a,r})\, \mathrm{d}\xi_b\, \delta(\xi_b - \xi_{b,r})\,, \tag{5.27}$$

and shifting the momenta as in eq. (5.22), the $(m + X + 2)$-particle phase space of eq. (4.7) (with $k = 2$) can be written as the convolution,

$$\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b) = \int \mathrm{d}\xi_a\, \mathrm{d}\xi_b\, \mathrm{d}\phi_{m+X+1}\big(\{\hat{p}\}_{m+X+1}; \xi_a p_a + \xi_b p_b\big)\, \mathrm{d}\phi_{II,F}(p_r, \xi_a, \xi_b)\,, \tag{5.28}$$

with

$$\mathrm{d}\phi_{II,F}(p_r, \xi_a, \xi_b) = \frac{\mathrm{d}^d p_r}{(2\pi)^{d-1}}\, \delta_+(p_r^2)\, \delta(\xi_a - \xi_{a,r})\, \delta(\xi_b - \xi_{b,r})\,, \tag{5.29}$$

where in eq. (5.28) the limits of integration for both ξ_a and ξ_b are 0 and 1, with the constraints

$$(\xi_a \xi_b)_{\min} = \frac{M^2}{s_{ab}}\,, \qquad (\xi_a \xi_b)_{\max} = 1\,, \tag{5.30}$$

where M was defined below eq. (5.12). We note that the phase space measure $\mathrm{d}\phi_{II,F}(p_r, \xi_a, \xi_b)$ can also be expressed in the following form,

$$\mathrm{d}\phi_{II,F}(p_r, \xi_a, \xi_b) = \mathrm{d}\phi_2(p_r, K; p_a + p_b)\, \mathcal{J}_{II,F}\, \delta[\xi_a(s_{ab} - s_{ar}) - \xi_b(s_{ab} - s_{br})]\,, \tag{5.31}$$

with the Jacobian

$$\mathcal{J}_{II,F} = \frac{s_{ab}[\xi_a(s_{ab} - s_{ar}) + \xi_b(s_{ab} - s_{br})]}{2\pi}\,, \tag{5.32}$$

where the momentum K appearing in the two-particle phase space above is massive with $K^2 = \xi_a \xi_b s_{ab}$. Finally, we note that the flux factor $\Phi(p_a \cdot p_b)$ can be written in terms of mapped momenta as

$$\Phi(p_a \cdot p_b) = \frac{\Phi(\hat{p}_a \cdot \hat{p}_b)}{\xi_a \xi_b}\,, \tag{5.33}$$

where we have used eq. (5.27) to set $\xi_{a,r} = \xi_a$ and $\xi_{b,r} = \xi_b$.

Integrated subtraction term: Using the definitions of the subtraction term and the phase space convolution, eqs. (5.20) and (5.28), we write the integrated counterterm as

$$\begin{aligned} &\int_1 \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a \cdot p_b)} \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b) \mathcal{C}_{ar}^{IF(0,0)}(p_a, p_b; \{p\}_{m+X+2}) \\ &\quad = \frac{\alpha_\mathrm{s}}{2\pi} S_\varepsilon \left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon \Big(\Big[\mathrm{C}_{ar}^{IF(0,0)} \Big] \otimes \mathrm{d}\sigma^{\mathrm{R}}_{(ar)b,m+X+1} \Big)\,, \end{aligned} \tag{5.34}$$

where

$$\left[\mathrm{C}_{ar}^{IF(0,0)}\right]\otimes \mathrm{d}\sigma^{\mathrm{R}}_{(ar)b,m+X+1}=\int \mathrm{d}\xi_a\,\mathrm{d}\xi_b\,\left[\mathrm{C}_{ar}^{IF(0,0)}(\xi_a,\xi_b;\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{R}}_{(ar)b,m+X+1}(\hat{p}_a,\hat{p}_b)\,. \tag{5.35}$$

In eq. (5.35) the initial-state momenta entering the real emission cross section are written as $\hat{p}_a=\xi_a p_a$ and $\hat{p}_b=\xi_b p_b$ using the delta functions in eq. (5.29) and we introduced the integrated counterterm

$$\left[\mathrm{C}_{ar}^{IF(0,0)}(\xi_a,\xi_b;\varepsilon)\right]=\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{\varepsilon}\int \mathrm{d}\phi_{II,F}(p_r,\xi_a,\xi_b)\,\xi_a\xi_b\frac{1}{x_{a,r}s_{ar}}\frac{\omega(ar)}{\omega(a)}\hat{P}^{(0)}_{(ar)r}(x_{a,r},k_{\perp r,a};\varepsilon)\,. \tag{5.36}$$

Note in particular the factor of $\omega(ar)/\omega(r)$ in eq. (5.36). This is present because, similarly to the flux factor, the initial-state averaging factor $\omega(a)$ on the left-hand side of eq. (5.34) must be rewritten in terms of $\omega(ar)$, since it is the latter that appears in $\mathrm{d}\sigma^{\mathrm{R}}_{(ar)b,m+X+1}(\hat{p}_a,\hat{p}_b)$, see eq. (4.5). Hence, the integrated counterterm includes the appropriate ratio of ωs. Such factors appear in integrated counterterms whenever the flavor of the initial-state partons is different in the original double real emission matrix element and in the reduced matrix element in the subtraction term. In order to evaluate $\left[\mathrm{C}_{ar}^{IF(0,0)}(\xi_a,\xi_b;\varepsilon)\right]$, we use $k_{\perp r,a}\cdot\hat{p}_a=0$ to perform the azimuthal integration by simply passing to the azimuthally averaged splitting functions, $\hat{P}^{(0)}_{(ar)r}(x_{a,r},k_{\perp r,a};\varepsilon)\to P^{(0)}_{(ar)r}(x_{a,r};\varepsilon)$, given in eqs. (D.27)–(D.30). However, after azimuthal integration, the rest of the phase space measure $\mathrm{d}\phi_{II,F}(p_r,\xi_a,\xi_b)$ is completely fixed by the delta functions in eq. (5.29). In fact, inverting the relations in eq. (5.26), we find

$$s_{ar}=s_{ab}\frac{\xi_{a,r}(1-\xi_{b,r}^2)}{\xi_{a,r}+\xi_{b,r}}\,,\qquad s_{br}=s_{ab}\frac{\xi_{b,r}(1-\xi_{a,r}^2)}{\xi_{a,r}+\xi_{b,r}}\,, \tag{5.37}$$

which implies

$$x_{a,r}=\xi_{a,r}\xi_{b,r}\,. \tag{5.38}$$

Moreover, working out the Jacobian of the transformation from the remaining components of p_r to $\xi_{a,r}$ and $\xi_{b,r}$, the phase space measure $\mathrm{d}\phi_{II,F}(p_r,\xi_a,\xi_b)$ can be written as [54]

$$\begin{aligned}\mathrm{d}\phi_{II,F}&(p_r,\xi_a,\xi_b)\\&=\frac{S_\varepsilon}{8\pi^2}\frac{\mathrm{d}\Omega_{d-2}}{\Omega_{d-2}}\mathrm{d}\xi_{a,r}\,\mathrm{d}\xi_{b,r}\,s_{ab}^{1-\varepsilon}\left[\frac{\xi_a\xi_b(1-\xi_a^2)(1-\xi_b^2)}{(\xi_a+\xi_b)^2}\right]^{-\varepsilon}\frac{\xi_a\xi_b(1+\xi_a\xi_b)}{(\xi_a+\xi_b)^2}\delta(\xi_{a,r}-\xi_a)\delta(\xi_{b,r}-\xi_b)\,,\end{aligned} \tag{5.39}$$

where $\mathrm{d}\Omega_{d-2}$ represents the measure over angles on which the integrand does not depend and S_ε is given in eq. (5.16). Then, we can write the integrated subtraction term in eq. (5.36) as

$$\left[\mathrm{C}_{ar}^{IF(0,0)}(\xi_a,\xi_b;\varepsilon)\right]=2s_{ab}\left[\frac{\xi_a\xi_b(1-\xi_a^2)(1-\xi_b^2)}{(\xi_a+\xi_b)^2}\right]^{-\varepsilon}\frac{\xi_a^2\xi_b^2(1+\xi_a\xi_b)}{(\xi_a+\xi_b)^2}\frac{1}{x_{a,r}s_{ar}}\frac{\omega(ar)}{\omega(a)}P^{(0)}_{(ar)r}(x_{a,r};\varepsilon)\,, \tag{5.40}$$

where we have used the delta functions in eq. (5.39) to perform the integrations over $\xi_{a,r}$ and $\xi_{b,s}$. In particular, this implies that in eq. (5.40) above, $x_{a,r}$ and s_{ar} are expressed in terms of ξ_a and ξ_b as in eqs. (5.37) and (5.38) with the replacements $\xi_{a,r}\to\xi_a$ and $\xi_{b,r}\to\xi_b$.

5.3 Single soft-type subtraction terms

5.3.1 $\mathcal{S}_r^{(0,0)}$

Subtraction term: For a final-state gluon r, we define the single soft subtraction term as follows,

$$\mathcal{S}_r^{(0,0)}(p_a,p_b;\{p\}_{m+X+2})=-8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\sum_{\substack{j,k\in I\cup F\\ j,k\neq r}}\frac{1}{2}\mathcal{S}_{\tilde{j}\tilde{k}}(r)\boldsymbol{T}_j\boldsymbol{T}_k\left|\mathcal{M}^{(0)}_{ab,m+X+1}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X+1})\right|^2\,, \tag{5.41}$$

where the reduced matrix element is obtained simply by removing the final-state gluon r. The set of mapped momenta entering the reduced matrix element is defined below in eq. (5.44). Further, $\mathcal{S}_{\tilde{j}\tilde{k}}(r)$ denotes the eikonal factor, evaluated with the mapped hard momenta, as explained in sec. 3.2,

$$\mathcal{S}_{\tilde{j}\tilde{k}}(r) = \frac{2s_{\tilde{j}\tilde{k}}}{s_{\tilde{j}r}s_{\tilde{k}r}}\,, \qquad j,k \in I \cup F\,, \quad j,k \neq r\,. \tag{5.42}$$

The explicit factor of $1/2$ in eq. (5.41) comes from the sum over j and k not being ordered, hence each term is counted twice.

Momentum mapping: The single soft (S) mapping,

$$S_r:\ (p_a, p_b; \{p\}_{m+X+2}) \to (\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\}_{m+X+1})\,, \tag{5.43}$$

is defined as in ref. [55] ($r \in F$)

$$\begin{aligned} \tilde{p}_a^\mu &= \lambda_r p_a^\mu\,, \\ \tilde{p}_b^\mu &= \lambda_r p_b^\mu\,, \\ \tilde{p}_n^\mu &= \Lambda(P,\tilde{P})^\mu{}_\nu\, p_n^\nu\,, \qquad n \in F, \quad n \neq r\,, \\ \tilde{p}_X^\mu &= \Lambda(P,\tilde{P})^\mu{}_\nu\, p_X^\nu\,, \end{aligned} \tag{5.44}$$

where $\Lambda(P,\tilde{P})^\mu{}_\nu$ is the proper Lorentz transformation of eq. (5.24) that takes the massive momentum P^μ into a momentum $\tilde{P}^\mu$ of the same mass, with

$$P^\mu = (p_a + p_b)^\mu - p_r^\mu \qquad \text{and} \qquad \tilde{P}^\mu = (\tilde{p}_a + \tilde{p}_b)^\mu = \lambda_r (p_a + p_b)^\mu\,. \tag{5.45}$$

The value of λ_r is fixed by requiring that $P^2 = \tilde{P}^2$,

$$\lambda_r = \sqrt{1 - \frac{s_{rQ}}{s_{ab}}}\,. \tag{5.46}$$

Since the initial-state momenta p_a^μ and p_b^μ are rescaled by λ_r in eq. (5.44), so is the total momentum. The advantage of this mapping over the momentum-conserving soft mapping of ref. [56] is that eq. (5.44) can also be used in the presence of massive final-state particles, as well as in the color-singlet case, for which the mapped set $\{\tilde{p}\}_X$ consists of only massive non-QCD particles in the final state [55]. In the $p_r^\mu \to 0$ limit clearly $\lambda_r \to 1$ which implies $\tilde{P}^\mu \to P^\mu$, so the Lorentz-transformation goes to the identity and we recover the single soft $p_r^\mu \to 0$ kinematic configuration. We will use tildes to denote momenta obtained via soft-type momentum mappings throughout, as in eq. (5.44).

Phase space convolution: By inserting the identity,

$$1 = \frac{\mathrm{d}^d K}{(2\pi)^d}(2\pi)^d \delta^{(d)}\left(K + p_r - p_a - p_b\right) \mathrm{d}K^2\, \delta_+(K^2 - s_{ab} + s_{rQ})\,, \tag{5.47}$$

with $\delta_+(K^2 - s_{ab} + s_{rQ}) = \delta_+(K^2 - \lambda_r^2 s_{ab})$, the $(m+X+2)$-particle phase space in eq. (4.7) (with $k=2$) can be factored on parton r and momentum K through the convolution formula,

$$\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b) = \int \frac{\mathrm{d}K^2}{2\pi}\, \mathrm{d}\phi_{m+X+1}(p_a, \ldots, p_{m+2}, p_X; K)\, \mathrm{d}\phi_2(p_r, K; p_a + p_b)\,. \tag{5.48}$$

Then using the identity,

$$1 = \mathrm{d}\lambda\, \delta(\lambda - \lambda_r)\,, \tag{5.49}$$

and exchanging the momenta $\{p_n\}$ and p_X with the momenta $\{\tilde{p}_n\}$ and $\tilde{p}_X$ through the mapping of eq. (5.44) lets us write the convolution in eq. (5.48) as,

$$\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b) = \int_{\lambda_{\min}}^{\lambda_{\max}} \mathrm{d}\lambda\, \mathrm{d}\phi_{m+X+1}\big(\{\tilde{p}\}_{m+X+1};\lambda(p_a+p_b)\big)\,\frac{s_{ab}}{\pi}\lambda\,\mathrm{d}\phi_2(p_r,K;p_a+p_b)\,, \tag{5.50}$$

where momentum K is massive with $K^2 = \lambda^2 s_{ab}$. The limits of the λ integration are

$$\lambda_{\min} = \frac{M}{\sqrt{s_{ab}}}\,, \qquad \lambda_{\max} = 1\,, \tag{5.51}$$

where M has been defined below eq. (5.12). Finally, we note that the flux factor $\Phi(p_a\cdot p_b)$ written in terms of mapped momenta reads

$$\Phi(p_a\cdot p_b) = \frac{\Phi(\tilde{p}_a\cdot\tilde{p}_b)}{\lambda^2}\,, \tag{5.52}$$

where we have exploited eq. (5.49) to set $\lambda_r = \lambda$.

Integrated subtraction term: Using the definitions of the subtraction term and the phase space convolution, eqs. (5.41) and (5.50), we write the integrated counterterm as

$$\begin{aligned}&\int_1 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{S}_r^{(0,0)}(p_a,p_b;\{p\}_{m+X+2})\\ &\quad= \frac{\alpha_\mathrm{s}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\left(\left[\mathrm{S}_r^{(0,0)}\right]\otimes \mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}\right),\end{aligned} \tag{5.53}$$

where

$$\left[\mathrm{S}_r^{(0,0)}\right]\otimes\mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1} = \int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda \sum_{\substack{j,k\in I\cup F\\ j,k\neq r}}\frac{1}{2}\left[\mathrm{S}_r^{(0,0)}(\lambda;\varepsilon)\right]^{(j,k)}\boldsymbol{T}_j\boldsymbol{T}_k\mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}(\tilde{p}_a,\tilde{p}_b)\,. \tag{5.54}$$

The initial-state momenta entering the real emission cross section in eq. (5.54) are $\tilde{p}_a = \lambda p_a$ and $\tilde{p}_b = \lambda p_b$, where we have exploited the delta function in eq. (5.49), and we introduced the integrated counterterm

$$\left[\mathrm{S}_r^{(0,0)}(\lambda;\varepsilon)\right]^{(j,k)} = -\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{\varepsilon}\int \mathrm{d}\phi_2(p_r,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^3\mathcal{S}_{\tilde{j}\tilde{k}}(r)\,. \tag{5.55}$$

5.3.2 $\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}$

Subtraction term: For a final-state parton i and a final-state gluon r we define the single final-final collinear-soft overlapping subtraction term as follows,

$$\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}(p_a,p_b;\{p\}_{m+X+2}) = 8\pi\alpha_\mathrm{s}\mu^{2\varepsilon}\frac{1}{s_{\tilde{i}r}}\frac{2z_{\tilde{i},r}}{z_{r,\tilde{i}}}\boldsymbol{T}_i^2\left|\mathcal{M}^{(0)}_{ab,m+X+1}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X+1})\right|^2, \tag{5.56}$$

where the reduced matrix element is obtained by removing the final-state gluon r and the momenta of the partons in this reduced matrix element are given in eq. (5.44).[14] The momentum fractions $z_{\tilde{i},r}$ and $z_{r,\tilde{i}}$

[14] Note that the flavor of parton i remains unchanged in the reduced matrix element. This is of course consistent with the cancellation between $\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}$ and $\mathcal{C}_{ir}^{FF(0,0)}$ in the soft limit, because whenever parton r is a gluon, the flavor of the emitting parton (ir) in $\mathcal{C}_{ir}^{FF(0,0)}$ is the same as that of the original parton i. Similar comments apply to the rest of the collinear-soft type terms that appear in $\mathcal{A}_1^{(0)}$, $\mathcal{A}_2^{(0)}$ as well as $\mathcal{A}_{12}^{(0)}$.

appearing in eq. (5.56) are given by eq. (3.4). Notice that the momentum fractions $z_{\tilde{i},r}$ and $z_{r,\tilde{i}}$, as well as the invariant $s_{\tilde{i}r}$ in eq. (5.56) must be defined with the mapped momentum $\tilde{p}_i^\mu$ in order to correctly cancel the $\tilde{p}_i^\mu \parallel \tilde{p}_r^\mu$ collinear limit of the soft subtraction term $\mathcal{S}_r^{(0,0)}$. This is because in this collinear limit $\tilde{p}_i^\mu$ does not go to p_i^μ and so the mapped momenta must be retained in the definition of $\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}$. However, in the $p_r \to 0$ soft limit we have $\tilde{p}_i^\mu \to p_i^\mu$, hence $\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}$ correctly cancels $\mathcal{C}_{ir}^{FF(0,0)}$ in this limit.

Momentum mapping and phase space factorization: This subtraction term is defined using the single soft momentum mapping of sec. 5.3.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (5.50) and (5.56), we write the integrated counterterm as

$$\begin{aligned}
&\int_1 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}(p_a,p_b;\{p\}_{m+X+2}) \\
&\quad = \frac{\alpha_\mathrm{s}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\left(\left[\mathrm{C}_{ir}^{FF}\mathrm{S}_r^{(0,0)}\right]\otimes \mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}\right),
\end{aligned} \tag{5.57}$$

where

$$\left[\mathrm{C}_{ir}^{FF}\mathrm{S}_r^{(0,0)}\right]\otimes \mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1} = \int_{\lambda_\mathrm{min}}^{\lambda_\mathrm{max}}\mathrm{d}\lambda\,\left[\mathrm{C}_{ir}^{FF}\mathrm{S}_r^{(0,0)}(\lambda;\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}(\tilde{p}_a,\tilde{p}_b)\,. \tag{5.58}$$

The initial-state momenta entering the real emission cross section in eq. (5.58) are once again $\tilde{p}_a = \lambda p_a$ and $\tilde{p}_b = \lambda p_b$ and we introduced the integrated counterterm

$$\left[\mathrm{C}_{ir}^{FF}\mathrm{S}_r^{(0,0)}(\lambda;\varepsilon)\right] = \frac{(4\pi)^2}{S_\varepsilon}s_{ab}^\varepsilon\int \mathrm{d}\phi_2(p_r,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^3\frac{1}{s_{\tilde{i}r}}\frac{2z_{\tilde{i},r}}{z_{r,\tilde{i}}}\boldsymbol{T}_i^2\,. \tag{5.59}$$

5.3.3 $\mathcal{C}_{ar}^{IF}\mathcal{S}_r^{(0,0)}$

Subtraction term: For an initial-state parton a and a final-state gluon r we define the initial-final collinear-soft overlapping subtraction term as follows,

$$\mathcal{C}_{ar}^{IF}\mathcal{S}_r^{(0,0)}(p_a,p_b;\{p\}_{m+X+2}) = 8\pi\alpha_\mathrm{s}\mu^{2\varepsilon}\frac{1}{s_{\tilde{a}r}}\frac{2}{x_{r,\tilde{a}}}\boldsymbol{T}_a^2\left|\mathcal{M}^{(0)}_{ab,m+X+1}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X+1})\right|^2, \tag{5.60}$$

where the reduced matrix element is obtained once more by removing the final-state gluon r, and the momenta entering the reduced matrix element are given in eq. (5.44). The momentum fraction $x_{r,\tilde{a}}$ appearing in eq. (5.60) is defined by eq. (3.8). Using this definition of $x_{r,\tilde{a}}$ and the fact that the single soft momentum mapping simply rescales p_a to obtain $\tilde{p}_a$, we find the equality

$$\frac{1}{s_{\tilde{a}r}}\frac{2}{x_{r,\tilde{a}}} = \frac{1}{s_{ar}}\frac{2}{x_{r,a}}\,. \tag{5.61}$$

Nevertheless, we choose to retain the definition as given in eq. (5.60) for the following reason. Using eqs. (3.4) and (3.8), we have immediately that

$$\frac{1}{s_{\tilde{i}r}}\frac{2z_{\tilde{i},r}}{z_{r,\tilde{i}}} = \frac{1}{s_{\tilde{i}r}}\frac{2s_{\tilde{i}Q}}{s_{rQ}} \qquad \text{and} \qquad \frac{1}{s_{\tilde{a}r}}\frac{2}{x_{r,\tilde{a}}} = \frac{1}{s_{\tilde{a}r}}\frac{2s_{\tilde{a}Q}}{s_{rQ}}\,. \tag{5.62}$$

Hence, the collinear-soft counterterms of eqs. (5.56) and (5.60) have the same form once the momentum fractions are written explicitly in terms of mapped momenta. This fact can be exploited to optimize the implementation of these subtraction terms. We will follow this convention of defining initial-final collinear-soft factors in terms of mapped momenta throughout.

Momentum mapping and phase space factorization: This subtraction term is defined using the single soft momentum mapping of sec. 5.3.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (5.50) and (5.60), we write the integrated counterterm as

$$\begin{aligned}\int_1 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{C}_{ar}^{IF}\mathcal{S}_r^{(0,0)}(p_a,p_b;\{p\}_{m+X+2})\\ =\frac{\alpha_\mathrm{s}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\Big(\Big[\mathrm{C}_{ar}^{IF}\mathrm{S}_r^{(0,0)}\Big]\otimes\mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}\Big)\,,\end{aligned}\tag{5.63}$$

where

$$\Big[\mathrm{C}_{ar}^{IF}\mathrm{S}_r^{(0,0)}\Big]\otimes\mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}=\int_{\lambda_\mathrm{min}}^{\lambda_\mathrm{max}}\mathrm{d}\lambda\,\Big[\mathrm{C}_{ar}^{IF}\mathrm{S}_r^{(0,0)}(\lambda;\varepsilon)\Big]\,\mathrm{d}\sigma^{\mathrm{R}}_{ab,m+X+1}(\tilde p_a,\tilde p_b)\,.\tag{5.64}$$

Recall that in eq. (5.64) we have $\tilde p_a=\lambda p_a$ and $\tilde p_b=\lambda p_b$ and we introduced the integrated subtraction term

$$\Big[\mathrm{C}_{ar}^{IF}\mathrm{S}_r^{(0,0)}(\lambda;\varepsilon)\Big]=\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^\varepsilon\int\mathrm{d}\phi_2(p_r,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^3\frac{1}{s_{\tilde a r}}\frac{2}{x_{r,\tilde a}}\boldsymbol{T}_a^2\,.\tag{5.65}$$

6 Subtraction terms for double unresolved emission

In this section, we provide the explicit definitions of the individual double unresolved subtraction terms introduced in sec. 4.2.

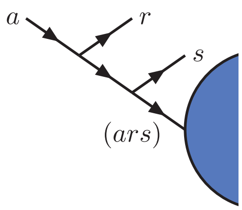
6.1 Initial-state double collinear-type subtraction terms

6.1.1 $\mathcal{C}_{ars}^{IFF(0,0)}$

Subtraction term: For an initial-state parton a and final-state partons r and s we define the initial-final-final triple collinear subtraction term as

$$\begin{aligned}\mathcal{C}_{ars}^{IFF(0,0)}(p_a,p_b;\{p\}_{m+X+2})&=(8\pi\alpha_\mathrm{s}\mu^{2\varepsilon})^2\frac{1}{x_{a,rs}(s_{ar}+s_{as}-s_{rs})^2}\\ &\times\hat P^{(0)}_{(ars)rs}(x_{a,rs},x_{r,a},x_{s,a},s_{ar},s_{as},s_{rs},k_{\perp a,rs},k_{\perp r,a},k_{\perp s,a};\varepsilon)\\ &\times\big|\mathcal{M}^{(0)}_{(ars)b,m+X}(\hat p_a,\hat p_b;\{\hat p\}_{m+X})\big|^2\mathcal{F}(x_{a,rs},\xi_{a,rs}\xi_{b,rs})\,,\end{aligned}\tag{6.1}$$

where the reduced matrix element is obtained by removing the final-state partons r and s and replacing the initial-state parton a with a parton (ars) whose flavor is determined by the flavor summation rules of eq. (5.2) by the requirement that $a=(ars)+r+s$. Hence, the flavor of the initial-state parton that



a	r	s	(ars)	a	r	s	(ars)
$q/\bar{q}$	$q/\bar{q}$	g	g	g	$q/\bar{q}$	g	$\bar{q}/q$
$q/\bar{q}$	g	$q/\bar{q}$	g	g	g	$q/\bar{q}$	$\bar{q}/q$
$q/\bar{q}$	$q/\bar{q}$	$\bar{q}/q$	$q/\bar{q}$	g	$q/\bar{q}$	$\bar{q}/q$	g
$q/\bar{q}$	g	g	$q/\bar{q}$	g	g	g	g

Figure 3: The splitting configuration for $a \to (ars)+r+s$ initial-state triple collinear splitting. The arrows in the figure denote momentum and flavor flow. The precise flavor mapping for the $\mathcal{C}_{ars}^{IFF(0,0)}$ counterterm is given in the table.

is involved in the splitting is in general different in the original double real emission matrix element and in the reduced matrix element. The initial-state triple collinear splitting configuration is shown in fig. 3, together with the precise flavor mapping. The set of momenta entering the reduced matrix element is defined below in eq. (6.5). The tree-level initial-final-final splitting functions for $a \to (ars)+r+s$ splitting, $\hat{P}^{(0)}_{(ars)rs}$, can be obtained from the final-state triple collinear splitting functions of refs. [20, 21], recalled here in eqs. (D.32)–(D.41), by the crossing relation discussed in appendix D.1, see eq. (D.54). We have once more chosen to label the splitting function with the flavor of the initial-state parton in the *reduced* matrix element, (ars), and the flavors of the final-state partons r and s. The momentum fractions $x_{a,rs}$, $x_{r,a}$ and $x_{s,a}$ are defined by eq. (3.10), while the transverse momenta $k_{\perp a,rs}$, $k_{\perp r,a}$ and $k_{\perp s,a}$ are given by eq. (3.15). Finally, the function $\mathcal{F}$ in eq. (6.1) is defined as

$$\mathcal{F}(x,y) = \left(\frac{x}{y}\right)^2 , \tag{6.2}$$

so that

$$\mathcal{F}(x_{a,rs}, \xi_{a,rs}\xi_{b,rs}) = \left(\frac{x_{a,rs}}{\xi_{a,rs}\xi_{b,rs}}\right)^2 = \left(\frac{s_{ab} - s_{(rs)Q}}{s_{ab} - s_{(rs)Q} + s_{rs}}\right)^2 , \tag{6.3}$$

where $\xi_{a,rs}$ and $\xi_{b,rs}$ are given in eq. (6.8) below and their product is recorded in eq. (6.7). Clearly in the triple collinear $p_a^\mu \parallel p_r^\mu \parallel p_s^\mu$ configuration $\mathcal{F}(x_{a,rs}, \xi_{a,rs}\xi_{b,rs}) \to 1$, so the subtraction term in eq. (6.1) is a correct regulator of the double real emission matrix element in this limit. The reason for including $\mathcal{F}(x_{a,rs}, \xi_{a,rs}\xi_{b,rs})$ in eq. (6.1) was discussed in sec. 3.2: its role is to cancel the unphysical singularity associated with the vanishing of $x_{a,rs}$ inside the double real emission phase space. Indeed, using the expressions in appendix D.1, one can check that the subtraction term contains at most factors of $1/x_{a,rs}^2$, so the spurious singularities may be canceled by simply including the factor of $\mathcal{F}(x_{a,rs}, \xi_{a,rs}\xi_{b,rs})$ in eq. (6.1). On the other hand, the rest of the subtraction terms, especially the iterated single unresolved ones, can be defined in a way such that all internal cancellation between the various subtraction terms are maintained in every unresolved limit.

Momentum mapping: The initial-state double collinear momentum mapping,

$$C_{ab,rs}^{II,FF} : \; (p_a, p_b; \{p\}_{m+X+2}) \xrightarrow{C_{ab,rs}^{II,FF}} (\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X}) , \tag{6.4}$$

is defined as ($a, b \in I$ and $r, s \in F$)

$$\begin{aligned}
\hat{p}_a^\mu &= \xi_{a,rs} p_a^\mu \,, \\
\hat{p}_b^\mu &= \xi_{b,rs} p_b^\mu \,, \\
\hat{p}_n^\mu &= \Lambda(P, \hat{P})^\mu{}_\nu \, p_n^\nu \,, \qquad n \in F, \quad n \neq r \,, \\
\hat{p}_X^\mu &= \Lambda(P, \hat{P})^\mu{}_\nu \, p_X^\nu \,,
\end{aligned} \tag{6.5}$$

where $\Lambda(P, \hat{P})^\mu{}_\nu$ is the proper Lorentz transformation of eq. (5.24) that takes the massive momentum P^μ into a momentum $\hat{P}^\mu$ of the same mass, with

$$P^\mu = (p_a + p_b)^\mu - p_r^\mu - p_s^\mu \,, \qquad \hat{P}^\mu = (\hat{p}_a + \hat{p}_b)^\mu = (\xi_{a,rs} p_a + \xi_{b,rs} p_b)^\mu \,. \tag{6.6}$$

The value of the product $\xi_{a,rs}\xi_{b,rs}$ is fixed by requiring that $P^2 = \hat{P}^2$,

$$\xi_{a,rs}\xi_{b,rs} = 1 - \frac{s_{(rs)Q}}{s_{ab}} + \frac{s_{rs}}{s_{ab}} \,, \tag{6.7}$$

but this does not fix $\xi_{a,rs}$ and $\xi_{b,rs}$ uniquely. Following ref. [54], we set

$$\xi_{a,rs} = \sqrt{\frac{s_{ab} - s_{b(rs)}}{s_{ab} - s_{a(rs)}} \frac{s_{ab} - s_{(rs)Q} + s_{rs}}{s_{ab}}} \,, \qquad \xi_{b,rs} = \sqrt{\frac{s_{ab} - s_{a(rs)}}{s_{ab} - s_{b(rs)}} \frac{s_{ab} - s_{(rs)Q} + s_{rs}}{s_{ab}}} \,, \tag{6.8}$$

which fulfills eq. (6.7). Clearly this mapping is simply a generalization of the initial-state single collinear momentum mapping of eq. (5.22). Note that $\xi_{a,rs}$ and $\xi_{b,rs}$ are related by exchanging $a \leftrightarrow b$. Furthermore, they are symmetric under the $r \leftrightarrow s$ interchange. The mapping of eq. (6.5) has the advantage that it reproduces both the initial-final-final triple collinear kinematic configuration, as well as the initial-final, initial-final double collinear kinematic configuration in the appropriate limit. Indeed, when e.g., $p_a^\mu \parallel p_r^\mu \parallel p_s^\mu$ so that $p_r^\mu \to x_r p_a^\mu$ and $p_s^\mu \to x_s p_a^\mu$ we find $\xi_{a,rs} \to 1 - x_r - x_s$ and $\xi_{b,rs} \to 1$. Then, $\hat{P}^\mu \to P^\mu$ implying that the Lorentz transformation becomes the identity, and we recover the $p_a^\mu \parallel p_r^\mu \parallel p_s^\mu$ initial-final-final triple collinear kinematic configuration. Likewise, in the $p_a^\mu \parallel p_r^\mu$, $p_b^\mu \parallel p_s^\mu$ limit when $p_r^\mu \to x_r p_a^\mu$ and $p_s^\mu \to x_s p_b^\mu$, we obtain $\xi_{a,rs} \to 1 - x_r$ and $\xi_{b,rs} \to 1 - x_s$. This again leads to $\hat{P}^\mu \to P^\mu$ and the Lorentz transformation going to the identity, and we obtain the $p_a^\mu \parallel p_r^\mu$, $p_b^\mu \parallel p_s^\mu$ initial-final, initial-final double collinear kinematic configuration. Due to the $a \leftrightarrow b$ symmetry of the mapping, the mapped momenta have the correct behavior also in the $p_b^\mu \parallel p_r^\mu \parallel p_s^\mu$, as well as the $p_b^\mu \parallel p_r^\mu$, $p_a^\mu \parallel p_s^\mu$ limits.

Phase space convolution: Using the identity,

$$1 = \mathrm{d}\xi_a \, \delta(\xi_a - \xi_{a,rs}) \, \mathrm{d}\xi_b \, \delta(\xi_b - \xi_{b,rs}) \,, \tag{6.9}$$

and shifting the momenta as in eq. (6.5), the $(m + X + 2)$-particle phase space of eq. (4.7) (with $k = 2$) can be written as the convolution,

$$\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b) = \int \mathrm{d}\xi_a \, \mathrm{d}\xi_b \, \mathrm{d}\phi_{m+X}\big(\{\hat{p}\}_{m+X}; \xi_a p_a + \xi_b p_b\big) \, \mathrm{d}\phi_{II,FF}(p_r, p_s, \xi_a, \xi_b) \,, \tag{6.10}$$

with

$$\mathrm{d}\phi_{II,FF}(p_r, p_s, \xi_a, \xi_b) = \frac{\mathrm{d}^d p_r}{(2\pi)^{d-1}} \, \delta_+(p_r^2) \, \frac{\mathrm{d}^d p_s}{(2\pi)^{d-1}} \, \delta_+(p_s^2) \, \delta(\xi_a - \xi_{a,rs}) \, \delta(\xi_b - \xi_{b,rs}) \,. \tag{6.11}$$

The limits of the ξ_a and ξ_b integrations are as in eq. (5.30). We note that the phase space measure $\mathrm{d}\phi_{II,FF}(p_r, p_s, \xi_a, \xi_b)$ can also be expressed in the following form,

$$\mathrm{d}\phi_{II,FF}(p_r, p_s, \xi_a, \xi_b) = \mathrm{d}\phi_3(p_r, p_s, K; p_a + p_b) \, \mathcal{J}_{II,FF} \, \delta[\xi_a(s_{ab} - s_{a(rs)}) - \xi_b(s_{ab} - s_{b(rs)})] \,, \tag{6.12}$$

with the Jacobian

$$\mathcal{J}_{II,FF} = \frac{s_{ab}[\xi_a(s_{ab} - s_{a(rs)}) + \xi_b(s_{ab} - s_{b(rs)})]}{2\pi} \,. \tag{6.13}$$

The momentum K appearing in the three-particle phase space above is massive with $K^2 = \xi_a\xi_b s_{ab}$. Finally, we note that the flux factor $\Phi(p_a \cdot p_b)$ can be written in terms of mapped momenta as

$$\Phi(p_a \cdot p_b) = \frac{\Phi(\hat{p}_a \cdot \hat{p}_b)}{\xi_a\xi_b} \,, \tag{6.14}$$

where we have used eq. (6.9) to set $\xi_{a,rs} = \xi_a$ and $\xi_{b,rs} = \xi_b$. Note that eq. (6.14) is formally identical to eq. (5.33), although the precise meaning of the mapped momenta in these equations is different.

Integrated subtraction term: Using the definitions of the subtraction term and the phase space convolution, eqs. (6.1) and (6.10), we write the integrated counterterm as

$$\begin{aligned}
&\int_2 \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a \cdot p_b)} \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b)\, \mathcal{C}^{IFF(0,0)}_{ars}(p_a, p_b; \{p\}_{m+X+2}) \\
&\quad = \left[\frac{\alpha_\mathrm{s}}{2\pi} S_\varepsilon \left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2 \left(\left[\mathrm{C}^{IFF(0,0)}_{ars}\right] \otimes \mathrm{d}\sigma^\mathrm{B}_{(ars)b,m+X}\right),
\end{aligned} \tag{6.15}$$

where

$$\left[\mathrm{C}^{IFF(0,0)}_{ars}\right] \otimes \mathrm{d}\sigma^\mathrm{B}_{(ars)b,m+X} = \int \mathrm{d}\xi_a\, \mathrm{d}\xi_b\, \left[\mathrm{C}^{IFF(0,0)}_{ars}(\xi_a, \xi_b; \varepsilon)\right] \mathrm{d}\sigma^\mathrm{B}_{(ars)b,m+X}(\hat{p}_a, \hat{p}_b) \,. \tag{6.16}$$

Here and below, the notation $\int_2$ indicates that the phase space integration is performed only over the two unresolved emissions. Furthermore, exploiting the delta functions in eq. (6.9), the initial-state momenta entering the Born cross section are written as $\hat{p}_a = \xi_a p_a$ and $\hat{p}_b = \xi_b p_b$ and we introduced the integrated counterterm

$$\begin{aligned}
\left[\mathrm{C}^{IFF(0,0)}_{ars}(\xi_a, \xi_b; \varepsilon)\right] &= \left(\frac{(4\pi)^2}{S_\varepsilon} s^\varepsilon_{ab}\right)^2 \int \mathrm{d}\phi_{II,FF}(p_r, p_s, \xi_a, \xi_b)\, \xi_a\xi_b \frac{1}{x_{a,rs}(s_{ar} + s_{as} - s_{rs})^2} \\
&\times \frac{\omega(ars)}{\omega(a)} \hat{P}^{(0)}_{(ars)rs}(x_{a,rs}, x_{r,a}, x_{s,a}, s_{ar}, s_{as}, s_{rs}, k_{\perp a,rs}, k_{\perp r,a}, k_{\perp s,a}; \varepsilon)\mathcal{F}(x_{a,rs}, \xi_a\xi_b) \,.
\end{aligned} \tag{6.17}$$

In order to evaluate $\left[\mathrm{C}^{IFF(0,0)}_{ars}(\xi_a, \xi_b; \varepsilon)\right]$, we use the representation of the factorized phase space measure in eq. (6.12) and exploit $k_{\perp a,rs} \cdot \hat{p}_a = k_{\perp r,a} \cdot \hat{p}_a = k_{\perp s,a} \cdot \hat{p}_a = 0$ to perform the azimuthal integration by passing to the azimuthally averaged splitting functions,

$$\hat{P}^{(0)}_{(ars)rs}(x_{a,rs}, x_{r,a}, x_{s,a}, s_{ar}, s_{as}, s_{rs}, k_{\perp a,rs}, k_{\perp r,a}, k_{\perp s,a}; \varepsilon) \to P^{(0)}_{(ars)rs}(x_{a,rs}, x_{r,a}, x_{s,a}, s_{ar}, s_{as}, s_{rs}; \varepsilon) \,,$$

see the discussion in appendix D.1. We then find

$$\begin{aligned}
\left[\mathrm{C}^{IFF(0,0)}_{ars}(\xi_a, \xi_b; \varepsilon)\right] &= \left(\frac{(4\pi)^2}{S_\varepsilon} s^\varepsilon_{ab}\right)^2 \int \mathrm{d}\phi_3(p_r, p_s, K; p_a + p_b)\, \mathcal{J}_{II,FF}\, \delta[\xi_a(s_{ab} - s_{a(rs)}) - \xi_b(s_{ab} - s_{b(rs)})] \\
&\times \frac{\xi_a\xi_b}{x_{a,rs}(s_{ar} + s_{as} - s_{rs})^2} \frac{\omega(ars)}{\omega(a)} P^{(0)}_{(ars)rs}(x_{a,rs}, x_{r,a}, x_{s,a}, s_{ar}, s_{as}, s_{rs}; \varepsilon)\mathcal{F}(x_{a,rs}, \xi_a\xi_b) \,.
\end{aligned} \tag{6.18}$$

In contrast to the case of the integrated single unresolved initial-final subtraction term $\left[\mathrm{C}^{IF(0,0)}_{ar}(\xi_a, \xi_b; \varepsilon)\right]$ of eq. (5.36), this time the integration is no longer fully fixed by the delta functions in eq. (6.11).

6.1.2 $\mathcal{C}^{IF,IF(0,0)}_{ar,bs}$

Subtraction term: For initial-state partons a and b and final-state partons r and s we define the initial-final, initial-final double collinear subtraction term as

$$\begin{aligned}\mathcal{C}^{IF,IF(0,0)}_{ar,bs}(p_a,p_b;\{p\}_{m+X+2}) = (8\pi\alpha_s\mu^{2\varepsilon})^2\frac{1}{x_{a,r}s_{ar}}\frac{1}{x_{b,s}s_{bs}}\hat{P}^{(0)}_{(ar)r}(x_{a,r},k_{\perp r,a};\varepsilon)\hat{P}^{(0)}_{(bs)s}(x_{b,s},k_{\perp s,b};\varepsilon)\\ \times\left|\mathcal{M}^{(0)}_{(ar)(bs),m+X}(\hat{p}_a,\hat{p}_b;\{\hat{p}\}_{m+X})\right|^2\mathcal{F}(x_{a,r}x_{b,s},\xi_{a,rs}\xi_{b,rs})\,,\end{aligned} \tag{6.19}$$

where the reduced matrix element is obtained by removing the final-state partons r and s and replacing the initial-state partons a and b by partons of flavors (ar) and (bs) using the flavor summation rules in eq. (5.2). The flavors of partons (ar) and (bs) are fixed by requiring that $a=(ar)+r$ and $b=(bs)+s$, see fig. 2. Clearly, the flavors of both initial-state partons in the reduced matrix element are generally different from those in the original double real emission matrix element. The set of momenta entering the reduced matrix element was defined in eq. (6.5), while the tree-level initial-final Altarelli-Parisi splitting functions $\hat{P}^{(0)}_{(ar)r}$ and $\hat{P}^{(0)}_{(bs)s}$ are the same as those appearing in the single unresolved initial-final collinear subtraction term of eq. (5.20). Their explicit expressions are given in eqs. (D.23)–(D.26). The momentum fractions and transverse momenta entering eq. (6.19) are defined by eqs. (3.8) and (3.14). Note that these definitions are identical to the ones in the single unresolved initial-final collinear subtraction term of sec. 5.2.1 for the (ar) pair, while the corresponding expressions for the (bs) pair are obtained by the simple relabeling of $a\to b$ and $r\to s$. Finally, the function $\mathcal{F}$ which appears in eq. (6.19) was defined in eq. (6.2), hence

$$\mathcal{F}(x_{a,r}x_{b,s},\xi_{a,rs}\xi_{b,rs}) = \left(\frac{x_{a,r}x_{b,s}}{\xi_{a,rs}\xi_{b,rs}}\right)^2 = \left(\frac{(s_{ab}-s_{rQ})(s_{ab}-s_{sQ})}{s_{ab}(s_{ab}-s_{(rs)Q}+s_{rs})}\right)^2\,. \tag{6.20}$$

In the double collinear $p_a^\mu \parallel p_r^\mu$, $p_b^\mu \parallel p_s^\mu$ configuration $\mathcal{F}(x_{a,r}x_{b,s},\xi_{a,rs}\xi_{b,rs})\to 1$, so the subtraction term in eq. (6.19) is a correct regulator of the double real emission matrix element in this limit. The function $\mathcal{F}(x_{a,r}x_{b,s},\xi_{a,rs}\xi_{b,rs})$ is included in eq. (6.19) in order to preserve the structure of internal cancellations between the subtraction terms in all unresolved limits, see the discussion in sec. 7.2.2.

Momentum mapping and phase space factorization: This subtraction term is defined using the initial-state double collinear momentum mapping of sec. 6.1.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (6.10) and (6.19), we write the integrated counterterm as

$$\begin{aligned}\int_2\mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{C}^{IF,IF(0,0)}_{ar,bs}(p_a,p_b;\{p\}_{m+X+2})\\ = \left[\frac{\alpha_s}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2\left(\left[\mathrm{C}^{IF,IF(0,0)}_{ar,bs}\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ar)(bs),m+X}\right),\end{aligned} \tag{6.21}$$

where

$$\left[\mathrm{C}^{IF,IF(0,0)}_{ar,bs}\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ar)(bs),m+X} = \int\mathrm{d}\xi_a\,\mathrm{d}\xi_b\,\left[\mathrm{C}^{IF,IF(0,0)}_{ar,bs}(\xi_a,\xi_b;\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{B}}_{(ar)(bs),m+X}(\hat{p}_a,\hat{p}_b)\,. \tag{6.22}$$

In eq. (6.22), we once again have $\hat{p}_a=\xi_a p_a$ and $\hat{p}_b=\xi_b p_b$ and we introduced the integrated counterterm

$$\begin{aligned}\left[\mathrm{C}^{IF,IF(0,0)}_{ar,bs}(\xi_a,\xi_b;\varepsilon)\right] = \left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^\varepsilon\right)^2\int\mathrm{d}\phi_{II,FF}(p_r,p_s,\xi_a,\xi_b)\,\xi_a\xi_b\frac{1}{x_{a,r}s_{ar}}\frac{1}{x_{b,s}s_{bs}}\\ \times\frac{\omega(ar)}{\omega(a)}\frac{\omega(bs)}{\omega(b)}\hat{P}^{(0)}_{(ar)r}(x_{a,r},k_{\perp r,a};\varepsilon)\hat{P}^{(0)}_{(bs)s}(x_{b,s},k_{\perp s,b};\varepsilon)\mathcal{F}(x_{a,r}x_{b,s},\xi_a\xi_b)\,.\end{aligned} \tag{6.23}$$

In order to evaluate $\left[\mathrm{C}^{IF,IF(0,0)}_{ar,bs}(\xi_a,\xi_b;\varepsilon)\right]$, we use the representation of the factorized phase space measure in eq. (6.12) and exploit $k_{\perp r,a}\cdot\hat{p}_a=0$ and $k_{\perp s,b}\cdot\hat{p}_b=0$ to perform the azimuthal integration by passing to the azimuthally averaged splitting functions, $\hat{P}^{(0)}_{(ar)r}(x_{a,r},k_{\perp r,a};\varepsilon)\to P^{(0)}_{(ar)r}(x_{a,r};\varepsilon)$ and $\hat{P}^{(0)}_{(bs)s}(x_{b,s},k_{\perp s,b};\varepsilon)\to P^{(0)}_{(bs)s}(x_{b,s};\varepsilon)$, given in eqs. (D.27)–(D.30). We then find

$$\begin{aligned}\left[\mathrm{C}^{IF,IF(0,0)}_{ar,bs}(\xi_a,\xi_b;\varepsilon)\right]&=\left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{\varepsilon}\right)^2\int\mathrm{d}\phi_3(p_r,p_s,K;p_a+p_b)\,\mathcal{J}_{II,FF}\,\delta[\xi_a(s_{ab}-s_{a(rs)})-\xi_b(s_{ab}-s_{b(rs)})]\\&\quad\times\xi_a\xi_b\frac{1}{x_{a,r}s_{ar}}\frac{1}{x_{b,s}s_{bs}}\frac{\omega(ar)}{\omega(a)}\frac{\omega(bs)}{\omega(b)}P^{(0)}_{(ar)r}(x_{a,r};\varepsilon)P^{(0)}_{(bs)s}(x_{b,s};\varepsilon)\mathcal{F}(x_{a,r}x_{b,s},\xi_a\xi_b)\,.\end{aligned}\tag{6.24}$$

Similarly to the case of the integrated triple collinear subtraction term $\left[\mathrm{C}^{IFF(0,0)}_{ars}(\xi_a,\xi_b;\varepsilon)\right]$ of eq. (6.18), the integration is not fully fixed by the delta functions in eq. (6.11).

6.2 Double soft-type subtraction terms

6.2.1 $\mathcal{S}^{(0,0)}_{rs}$

Subtraction term: For two final-state partons r and s, either two gluons or a same-flavor quark-antiquark pair, we define the double soft subtraction term as follows,

$$\begin{aligned}\mathcal{S}^{(0,0)}_{r_gs_g}(p_a,p_b;\{p\}_{m+X+2})&=(8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\\&\times\Bigg[\frac{1}{8}\sum_{\substack{i,k,j,\ell\in I\cup F\\ i,k,j,\ell\neq r,s}}\mathcal{S}_{\tilde{i}\tilde{k}}(r)\mathcal{S}_{\tilde{j}\tilde{\ell}}(s)\{\boldsymbol{T}_i\boldsymbol{T}_k,\boldsymbol{T}_j\boldsymbol{T}_\ell\}\big|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X})\big|^2\\&\qquad-\frac{1}{4}C_{\mathrm{A}}\sum_{\substack{i,k\in I\cup F\\ i,k\neq r,s}}\mathcal{S}_{\tilde{i}\tilde{k}}(r,s)\boldsymbol{T}_i\boldsymbol{T}_k\big|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X})\big|^2\Bigg]\,,\end{aligned}\tag{6.25}$$

and

$$\begin{aligned}&\mathcal{S}^{(0,0)}_{r_qs_{\bar{q}}}(p_a,p_b;\{p\}_{m+X+2})=(8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\\&\quad\times\frac{1}{s_{rs}^2}T_{\mathrm{R}}\sum_{\substack{i,k\in I\cup F\\ i,k\neq r,s}}\left(\frac{s_{\tilde{i}r}s_{\tilde{k}s}+s_{\tilde{i}s}s_{\tilde{k}r}-s_{\tilde{i}\tilde{k}}s_{rs}}{s_{\tilde{i}(rs)}s_{\tilde{k}(rs)}}-\frac{s_{\tilde{i}r}s_{\tilde{i}s}}{s^2_{\tilde{i}(rs)}}-\frac{s_{\tilde{k}r}s_{\tilde{k}s}}{s^2_{\tilde{k}(rs)}}\right)\boldsymbol{T}_i\boldsymbol{T}_k\big|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X})\big|^2\,,\end{aligned}\tag{6.26}$$

where the reduced matrix elements are obtained by removing the final-state partons r and s. The set of mapped momenta entering the reduced matrix elements is defined below in eq. (6.36). In eq. (6.25), the eikonal factors $\mathcal{S}_{\tilde{i}\tilde{k}}(r)$ and $\mathcal{S}_{\tilde{j}\tilde{\ell}}(s)$ are again written in terms of mapped hard momenta as in eq. (5.42),

$$\mathcal{S}_{\tilde{i}\tilde{k}}(r)=\frac{2s_{\tilde{i}\tilde{k}}}{s_{\tilde{i}r}s_{\tilde{k}r}}\,,\qquad\mathcal{S}_{\tilde{j}\tilde{\ell}}(s)=\frac{2s_{\tilde{j}\tilde{\ell}}}{s_{\tilde{j}s}s_{\tilde{\ell}s}}\,,\qquad i,k,j,\ell\in I\cup F\,,\quad i,k,j,\ell\neq r,s\,.\tag{6.27}$$

The non-abelian two-gluon soft function in eq. (6.25) is given by ($i,k\in I\cup F$, $i,k\neq r,s$)

$$\mathcal{S}_{\tilde{i}\tilde{k}}(r,s)=\mathcal{S}^{(\mathrm{s.o.})}_{\tilde{i}\tilde{k}}(r,s)+4\frac{s_{\tilde{i}r}s_{\tilde{k}s}+s_{\tilde{i}s}s_{\tilde{k}r}}{s_{\tilde{i}(rs)}s_{\tilde{k}(rs)}}\left[\frac{1-\varepsilon}{s_{rs}^2}-\frac{1}{8}\mathcal{S}^{(\mathrm{s.o.})}_{\tilde{i}\tilde{k}}(r,s)\right]-\frac{8s_{\tilde{i}\tilde{k}}}{s_{rs}s_{\tilde{i}(rs)}s_{\tilde{k}(rs)}}\,,\tag{6.28}$$

where

$$\mathcal{S}^{(\mathrm{s.o.})}_{\tilde{i}\tilde{k}}(r,s) = \mathcal{S}_{\tilde{i}\tilde{k}}(s)\Big(\mathcal{S}_{\tilde{i}s}(r) + \mathcal{S}_{\tilde{k}s}(r) - \mathcal{S}_{\tilde{i}\tilde{k}}(r)\Big) \tag{6.29}$$

is the form of $\mathcal{S}_{\tilde{i}\tilde{k}}(r,s)$ in the strongly-ordered limit (gluon s is much softer than gluon r). Note that for $i = k$, eq. (6.28) does not vanish. In fact,

$$\mathcal{S}_{\tilde{i}\tilde{i}}(r,s) = 8\frac{s_{\tilde{i}r}s_{\tilde{i}s}}{s^2_{\tilde{i}(rs)}}\frac{1-\varepsilon}{s^2_{rs}}\,. \tag{6.30}$$

We note that in the color-singlet case ($m = 0$), all sums in eqs. (6.25) and (6.26) run only over the two initial-state partons a and b. Moreover, the color-charge algebra is trivial, since color conservation ($\boldsymbol{T}_a + \boldsymbol{T}_b = 0$) implies

$$\boldsymbol{T}^2_a = \boldsymbol{T}^2_b \equiv \boldsymbol{T}^2_{\mathrm{ini}}\,, \qquad \boldsymbol{T}_a\boldsymbol{T}_b = -\boldsymbol{T}^2_{\mathrm{ini}} \tag{6.31}$$

and

$$\{\boldsymbol{T}_a\boldsymbol{T}_b, \boldsymbol{T}_a\boldsymbol{T}_b\} = 2\left(\boldsymbol{T}^2_{\mathrm{ini}}\right)^2\,. \tag{6.32}$$

Thus, in the color-singlet case, we find

$$\begin{aligned}\mathcal{S}^{(0,0)}_{r_g s_g}&(p_a,p_b;\{p\}_{X+2}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\\ &\times\bigg\{\mathcal{S}_{\tilde{a}\tilde{b}}(r)\mathcal{S}_{\tilde{a}\tilde{b}}(s)\boldsymbol{T}^2_{\mathrm{ini}} + \frac{1}{2}\bigg[\mathcal{S}_{\tilde{a}\tilde{b}}(r,s) - \frac{1}{2}\Big(\mathcal{S}_{\tilde{a}\tilde{a}}(r,s) + \mathcal{S}_{\tilde{b}\tilde{b}}(r,s)\Big)\bigg]C_{\mathrm{A}}\bigg\}\boldsymbol{T}^2_{\mathrm{ini}}\big|\mathcal{M}^{(0)}_{ab,X}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_X)\big|^2\,,\end{aligned} \tag{6.33}$$

and

$$\begin{aligned}\mathcal{S}^{(0,0)}_{r_q s_{\bar{q}}}&(p_a,p_b;\{p\}_{X+2}) = -(8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\\ &\times\frac{2}{s^2_{rs}}T_{\mathrm{R}}\left(\frac{s_{\tilde{a}r}s_{\tilde{b}s} + s_{\tilde{a}s}s_{\tilde{b}r} - s_{\tilde{a}\tilde{b}}s_{rs}}{s_{\tilde{a}(rs)}s_{\tilde{b}(rs)}} - \frac{s_{\tilde{a}r}s_{\tilde{a}s}}{s^2_{\tilde{a}(rs)}} - \frac{s_{\tilde{b}r}s_{\tilde{b}s}}{s^2_{\tilde{b}(rs)}}\right)\boldsymbol{T}^2_{\mathrm{ini}}\big|\mathcal{M}^{(0)}_{ab,X}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_X)\big|^2\,.\end{aligned} \tag{6.34}$$

Momentum mapping: The double soft mapping,

$$S_{rs}:\ (p_a,p_b;\{p\}_{m+X+2}) \xrightarrow{S_{rs}} (\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X})\,, \tag{6.35}$$

is defined as ($r,s \in F$),

$$\begin{aligned}\tilde{p}^\mu_a &= \lambda_{rs}p^\mu_a\,,\\ \tilde{p}^\mu_b &= \lambda_{rs}p^\mu_b\,,\\ \tilde{p}^\mu_n &= \Lambda(P,\hat{P})^\mu{}_\nu\,p^\nu_n\,, \qquad n\in F, \quad n\neq r\,,\\ \tilde{p}^\mu_X &= \Lambda(P,\hat{P})^\mu{}_\nu\,p^\nu_X\,,\end{aligned} \tag{6.36}$$

where $\Lambda(P,\hat{P})^\mu{}_\nu$ is the proper Lorentz transformation of eq. (5.24) that takes the massive momentum P^μ into a momentum $\tilde{P}^\mu$ of the same mass, where

$$P^\mu = (p_a+p_b)^\mu - p^\mu_r - p^\mu_s\,, \qquad \tilde{P}^\mu = (\tilde{p}_a+\tilde{p}_b)^\mu = \lambda_{rs}(p_a+p_b)^\mu\,. \tag{6.37}$$

The value of λ_{rs} is fixed by requiring that $P^2 = \tilde{P}^2$,

$$\lambda_{rs} = \sqrt{1 - \frac{s_{(rs)Q}}{s_{ab}} + \frac{s_{rs}}{s_{ab}}}\,. \tag{6.38}$$

Clearly this mapping is simply a generalization of the single soft mapping of eq. (5.44). As in the single soft case, eq. (6.36) can be used in the presence of massive final-state particles as well as in the color-singlet case. Moreover, in the $p^\mu_r \to 0$, $p^\mu_s \to 0$ limit we have $\lambda_{rs} \to 1$ which implies $\tilde{P}^\mu \to P^\mu$. Then, the Lorentz transformation becomes the identity and we obtain the $p^\mu_r \to 0$, $p^\mu_s \to 0$ double soft kinematic configuration.

Phase space convolution: Using the identity,

$$1=\frac{\mathrm{d}^dK}{(2\pi)^d}(2\pi)^d\delta^{(d)}\left(K+p_r+p_s-p_a-p_b\right)\,\mathrm{d}K^2\,\delta_+(K^2-s_{ab}+s_{(rs)Q}-s_{rs})\,, \tag{6.39}$$

with $\delta_+(K^2-s_{ab}+s_{(rs)Q}-s_{rs})=\delta_+(K^2-\lambda_{rs}^2s_{ab})$, the $(m+X+2)$-particle phase space of eq. (4.7) (with $k=2$) can be factored on partons r and s and momentum K through the convolution formula,

$$\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)=\int\frac{\mathrm{d}K^2}{2\pi}\,\mathrm{d}\phi_{m+X}(p_a,\ldots,p_{m+2},p_X;K)\,\mathrm{d}\phi_3(p_r,p_s,K;p_a+p_b)\,. \tag{6.40}$$

Inserting then the identity,

$$1=\mathrm{d}\lambda\,\delta(\lambda-\lambda_{rs})\,, \tag{6.41}$$

and exchanging the momenta $\{p_n\}$ and p_X with the momenta $\{\tilde{p}_n\}$ and $\tilde{p}_X$ through the mapping of eq. (6.36) lets us write the convolution in eq. (6.40) as,

$$\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)=\int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\,\mathrm{d}\phi_{m+X}\big(\{\tilde{p}\}_{m+X};\lambda(p_a+p_b)\big)\,\frac{s_{ab}}{\pi}\lambda\,\mathrm{d}\phi_3(p_r,p_s,K;p_a+p_b)\,, \tag{6.42}$$

where momentum K is massive with $K^2=\lambda^2s_{ab}$, and the integration limits are given in eq. (5.51). Finally, we note that the flux factor $\Phi(p_a\cdot p_b)$ written in terms of mapped momenta reads

$$\Phi(p_a\cdot p_b)=\frac{\Phi(\tilde{p}_a\cdot\tilde{p}_b)}{\lambda^2}\,, \tag{6.43}$$

where we have exploited eq. (6.41) to set $\lambda_{rs}=\lambda$. Notice that eq. (6.43) is formally the same as eq. (5.52), however the precise meaning of the mapped momenta is different in the two equations.

Integrated subtraction term for double soft gluon emission: Using the definitions of the subtraction term and the phase space convolution, eqs. (6.25) and (6.42), we write the integrated counterterm for double soft gluon emission as

$$\begin{aligned}&\int_2\mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{S}^{(0,0)}_{r_gs_g}(p_a,p_b;\{p\}_{m+X+2})\\&\quad=\left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2\left(\left[\mathrm{S}^{(0,0)}_{gg}\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}\right)\,,\end{aligned} \tag{6.44}$$

where

$$\begin{aligned}\left[\mathrm{S}^{(0,0)}_{gg}\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}=\int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\,\Bigg[&\frac{1}{8}\sum_{\substack{i,k,j,\ell\in I\cup F\\ i,k,j,\ell\neq r,s}}\left[\mathrm{S}^{(0,0)}_{gg}(\lambda;\varepsilon)\right]^{(ik,j\ell)}\{\boldsymbol{T}_i\boldsymbol{T}_k,\boldsymbol{T}_j\boldsymbol{T}_\ell\}\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{p}_a,\tilde{p}_b)\\&-\frac{1}{4}\sum_{\substack{i,k\in I\cup F\\ i,k\neq r,s}}\left[\mathrm{S}^{(0,0)}_{gg}(\lambda;\varepsilon)\right]^{(i,k)}C_{\mathrm{A}}\boldsymbol{T}_i\boldsymbol{T}_k\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{p}_a,\tilde{p}_b)\Bigg]\,.\end{aligned} \tag{6.45}$$

The initial-state momenta entering the Born cross sections in eq. (6.45) are $\tilde{p}_a=\lambda p_a$ and $\tilde{p}_b=\lambda p_b$ (using the delta function in eq. (6.41)) and we have introduced the integrated counterterms

$$\left[\mathrm{S}^{(0,0)}_{gg}(\lambda;\varepsilon)\right]^{(ik,j\ell)}=\left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^\varepsilon\right)^2\int\mathrm{d}\phi_3(p_r,p_s,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^3\mathcal{S}_{\tilde{i}\tilde{k}}(r)\mathcal{S}_{\tilde{j}\tilde{\ell}}(s)\,, \tag{6.46}$$

$$\left[\mathrm{S}^{(0,0)}_{gg}(\lambda;\varepsilon)\right]^{(i,k)}=\left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^\varepsilon\right)^2\int\mathrm{d}\phi_3(p_r,p_s,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^3\mathcal{S}_{\tilde{i}\tilde{k}}(r,s)\,. \tag{6.47}$$

Of course, for color-singlet production, we only need to evaluate eqs. (6.46) and (6.47) for $i,j,k,\ell = a,b$. Using eqs. (6.31) and (6.32), in the color-singlet case eq. (6.45) becomes

$$\begin{aligned}\left[\mathrm{S}_{gg}^{(0,0)}\right] \otimes \mathrm{d}\sigma_{ab,X}^{\mathrm{B}} = \int_{\lambda_{\min}}^{\lambda_{\max}} \mathrm{d}\lambda \Bigg\{ \left[\mathrm{S}_{gg}^{(0,0)}(\lambda;\varepsilon)\right]^{(ab,ab)} \boldsymbol{T}_{\mathrm{ini}}^2 \\ + \frac{1}{2}\Bigg(\left[\mathrm{S}_{gg}^{(0,0)}(\lambda;\varepsilon)\right]^{(a,b)} - \frac{1}{2}\left[\mathrm{S}_{gg}^{(0,0)}(\lambda;\varepsilon)\right]^{(a,a)} - \frac{1}{2}\left[\mathrm{S}_{gg}^{(0,0)}(\lambda;\varepsilon)\right]^{(b,b)} \Bigg) C_{\mathrm{A}} \Bigg\} \boldsymbol{T}_{\mathrm{ini}}^2 \mathrm{d}\sigma_{ab,X}^{\mathrm{B}}(\tilde{p}_a,\tilde{p}_b)\,.\end{aligned} \tag{6.48}$$

Integrated subtraction term for double soft quark-antiquark emission: Using the definitions of the subtraction term and the phase space convolution, eqs. (6.26) and (6.42), we write the integrated counterterm for double soft quark-antiquark emission as

$$\begin{aligned}&\int_2 \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)} \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a+p_b) \mathcal{S}_{r_q s_{\bar{q}}}^{(0,0)}(p_a,p_b;\{p\}_{m+X+2}) \\ &= \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\varepsilon \left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\right]^2 \left(\left[\mathrm{S}_{q\bar{q}}^{(0,0)}\right] \otimes \mathrm{d}\sigma_{ab,m+X}^{\mathrm{B}} \right),\end{aligned} \tag{6.49}$$

where

$$\left[\mathrm{S}_{q\bar{q}}^{(0,0)}\right] \otimes \mathrm{d}\sigma_{ab,m+X}^{\mathrm{B}} = \int_{\lambda_{\min}}^{\lambda_{\max}} \mathrm{d}\lambda \sum_{\substack{i,k\in I\cup F\\ i,k\neq r,s}} \left[\mathrm{S}_{q\bar{q}}^{(0,0)}(\lambda;\varepsilon)\right]^{(i,k)} T_{\mathrm{R}} \boldsymbol{T}_i \boldsymbol{T}_k \mathrm{d}\sigma_{ab,m+X}^{\mathrm{B}}(\tilde{p}_a,\tilde{p}_b)\,. \tag{6.50}$$

Once more, the initial-state momenta are $\tilde{p}_a = \lambda p_a$ and $\tilde{p}_b = \lambda p_b$ and we introduced the integrated counterterm

$$\begin{aligned}\left[\mathrm{S}_{q\bar{q}}^{(0,0)}(\lambda;\varepsilon)\right]^{(i,k)} &= \left(\frac{(4\pi)^2}{S_\varepsilon} s_{ab}^{\varepsilon}\right)^2 \int \mathrm{d}\phi_3(p_r,p_s,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^3 \\ &\times \frac{1}{s_{rs}^2}\left(\frac{s_{\tilde{i}r}s_{\tilde{k}s} + s_{\tilde{i}s}s_{\tilde{k}r} - s_{\tilde{i}\tilde{k}}s_{rs}}{s_{\tilde{i}(rs)}s_{\tilde{k}(rs)}} - \frac{s_{\tilde{i}r}s_{\tilde{i}s}}{s_{\tilde{i}(rs)}^2} - \frac{s_{\tilde{k}r}s_{\tilde{k}s}}{s_{\tilde{k}(rs)}^2} \right).\end{aligned} \tag{6.51}$$

In the color-singlet case, we need to evaluate eq. (6.51) only for $i,k = a,b$ and exploiting eq. (6.31) we find

$$\left[\mathrm{S}_{q\bar{q}}^{(0,0)}\right] \otimes \mathrm{d}\sigma_{ab,X}^{\mathrm{B}} = -2\int_{\lambda_{\min}}^{\lambda_{\max}} \mathrm{d}\lambda \left[\mathrm{S}_{q\bar{q}}^{(0,0)}(\lambda;\varepsilon)\right]^{(a,b)} T_{\mathrm{R}} \boldsymbol{T}_{\mathrm{ini}}^2 \mathrm{d}\sigma_{ab,X}^{\mathrm{B}}(\tilde{p}_a,\tilde{p}_b)\,. \tag{6.52}$$

6.2.2 $\mathcal{C}_{ars}^{IFF}\mathcal{S}_{rs}^{(0,0)}$

Subtraction term: For an initial-state parton a and final-state partons r and s which can be either two gluons or a same-flavor quark-antiquark pair, we define the triple collinear-double soft overlapping subtraction term as follows,

$$\begin{aligned}\mathcal{C}_{ars}^{IFF}\mathcal{S}_{rs}^{(0,0)}(p_a,p_b;\{p\}_{m+X+2}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 C_{ars}^{IFF} S_{rs}^{(0)}(x_{\tilde{a},rs}, x_{r,\tilde{a}}, x_{s,\tilde{a}}, s_{\tilde{a}r}, s_{\tilde{a}s}, s_{rs};\varepsilon) \\ \times \boldsymbol{T}_a^2 \left|\mathcal{M}_{ab,m+X}^{(0)}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X})\right|^2,\end{aligned} \tag{6.53}$$

where the reduced matrix element is obtained by removing the final-state partons r and s, while the set of mapped momenta is defined in eq. (6.36). The function $C_{ars}^{IFF} S_{rs}^{(0)}(x_{\tilde{a},rs}, x_{r,\tilde{a}}, x_{s,\tilde{a}}, s_{\tilde{a}r}, s_{\tilde{a}s}, s_{rs};\varepsilon)$ in eq. (6.53) depends on the flavors of partons r and s. When both r and s are gluons, we have

$$\begin{aligned}
C^{IFF}_{ar_gs_g}S^{(0)}_{r_gs_g}&(x_{\tilde{a},rs},x_{r,\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}r},s_{\tilde{a}s},s_{rs};\varepsilon) = \boldsymbol{T}_a^2\frac{4}{s_{\tilde{a}r}s_{\tilde{a}s}x_{r,\tilde{a}}x_{s,\tilde{a}}} \\
&+ C_{\mathrm{A}}\Bigg[\frac{(1-\varepsilon)}{s_{\tilde{a}(rs)}s_{rs}}\frac{(s_{\tilde{a}r}x_{s,\tilde{a}}-s_{\tilde{a}s}x_{r,\tilde{a}})^2}{s_{\tilde{a}(rs)}s_{rs}(x_{r,\tilde{a}}+x_{s,\tilde{a}})^2} - \frac{1}{s_{\tilde{a}(rs)}s_{rs}}\left(\frac{4}{x_{r,\tilde{a}}+x_{s,\tilde{a}}}-\frac{1}{x_{r,\tilde{a}}}\right) - \frac{1}{s_{\tilde{a}(rs)}s_{\tilde{a}r}}\frac{2}{x_{r,\tilde{a}}(x_{r,\tilde{a}}+x_{s,\tilde{a}})} \\
&\qquad - \frac{1}{s_{\tilde{a}(rs)}s_{\tilde{a}s}}\frac{1}{x_{r,\tilde{a}}(x_{r,\tilde{a}}+x_{s,\tilde{a}})} + \frac{1}{s_{\tilde{a}r}s_{rs}}\left(\frac{1}{x_{s,\tilde{a}}}+\frac{1}{x_{r,\tilde{a}}+x_{s,\tilde{a}}}\right) + (r\leftrightarrow s)\Bigg]\,,
\end{aligned} \tag{6.54}$$

while for a same-flavor quark-antiquark pair we find

$$C^{IFF}_{ar_qs_{\bar{q}}}S^{(0)}_{r_qs_{\bar{q}}}(x_{\tilde{a},rs},x_{r,\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}r},s_{\tilde{a}s},s_{rs};\varepsilon) = T_{\mathrm{R}}\frac{2}{s_{\tilde{a}(rs)}s_{rs}}\left[\frac{1}{x_{r,\tilde{a}}+x_{s,\tilde{a}}} - \frac{(s_{\tilde{a}r}x_{s,\tilde{a}}-s_{\tilde{a}s}x_{r,\tilde{a}})^2}{s_{\tilde{a}(rs)}s_{rs}(x_{r,\tilde{a}}+x_{s,\tilde{a}})^2}\right]. \tag{6.55}$$

The momentum fractions $x_{\tilde{a},rs}$, $x_{r,\tilde{a}}$ and $x_{s,\tilde{a}}$ are defined by eq. (3.10).

Momentum mapping and phase space factorization: This subtraction term is defined using the double soft momentum mapping of sec. 6.2.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (6.42) and (6.53), we write the integrated counterterm as

$$\begin{aligned}
\int_2 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}&\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{C}^{IFF}_{ars}\mathcal{S}^{(0,0)}_{rs}(p_a,p_b;\{p\}_{m+X+2}) \\
&= \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2\left(\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}\right),
\end{aligned} \tag{6.56}$$

where

$$\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X} = \int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\,\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}(\lambda;\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{p}_a,\tilde{p}_b)\,. \tag{6.57}$$

Once more, in the Born cross section $\tilde{p}_a=\lambda p_a$ and $\tilde{p}_b=\lambda p_b$ and the integrated counterterm is defined as

$$\begin{aligned}
\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}(\lambda;\varepsilon)\right] = \left(\frac{(4\pi)^2}{S_\varepsilon}s^\varepsilon_{ab}\right)^2&\int\mathrm{d}\phi_3(p_r,p_s,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^3 \\
&\times C^{IFF}_{ars}S^{(0)}_{rs}(x_{\tilde{a},rs},x_{r,\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}r},s_{\tilde{a}s},s_{rs};\varepsilon)\boldsymbol{T}_a^2\,.
\end{aligned} \tag{6.58}$$

6.2.3 $\mathcal{C}^{IF,IF}_{ar,bs}\mathcal{S}^{(0,0)}_{rs}$

Subtraction term: For initial-state partons a and b and final-state gluons r and s, we define the double collinear-double soft overlapping subtraction term as follows,[15]

$$\mathcal{C}^{IF,IF}_{ar,bs}\mathcal{S}^{(0,0)}_{rs}(p_a,p_b;\{p\}_{m+X+2}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{2}{s_{\tilde{a}r}x_{r,\tilde{a}}}\boldsymbol{T}_a^2\frac{2}{s_{\tilde{b}s}x_{s,\tilde{b}}}\boldsymbol{T}_b^2\left|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X})\right|^2, \tag{6.59}$$

where the reduced matrix element is yet again obtained by dropping the final-state partons r and s and the set of mapped momenta is given in eq. (6.36). The momentum fractions $x_{r,\tilde{a}}$ and $x_{s,\tilde{b}}$ are defined by eq. (3.8).

[15]Note that the double soft quark-antiquark subtraction term $\mathcal{S}^{(0,0)}_{r_qs_{\bar{q}}}$ in eq. (6.26) does not have a leading singularity in the $p_a^\mu \parallel p_r^\mu$, $p_b^\mu \parallel p_s^\mu$ double collinear limit.

Momentum mapping and phase space factorization: This subtraction term is defined using the double soft momentum mapping of sec. 6.2.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (6.42) and (6.59), we write the integrated counterterm as

$$\begin{aligned}&\int_2 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{C}^{IF,IF}_{ar,bs}\,\mathcal{S}^{(0,0)}_{rs}(p_a,p_b;\{p\}_{m+X+2})\\&\quad=\left[\frac{\alpha_\mathrm{s}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\right]^2\left(\left[\mathrm{C}^{IF,IF}_{ar,bs}\,\mathrm{S}^{(0,0)}_{rs}\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}\right),\end{aligned}\tag{6.60}$$

where

$$\left[\mathrm{C}^{IF,IF}_{ar,bs}\,\mathrm{S}^{(0,0)}_{rs}\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}=\int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\,\left[\mathrm{C}^{IF,IF}_{ar,bs}\,\mathrm{S}^{(0,0)}_{rs}(\lambda;\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{p}_a,\tilde{p}_b)\,.\tag{6.61}$$

As before, $\tilde{p}_a=\lambda p_a$ and $\tilde{p}_b=\lambda p_b$ and the integrated subtraction term is defined as

$$\left[\mathrm{C}^{IF,IF}_{ar,bs}\,\mathrm{S}^{(0,0)}_{rs}(\lambda;\varepsilon)\right]=\left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{\varepsilon}\right)^2\int\mathrm{d}\phi_3(p_r,p_s,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^3\frac{2}{s_{\tilde{a}r}x_{r,\tilde{a}}}\boldsymbol{T}_a^2\frac{2}{s_{\tilde{b}s}x_{s,\tilde{b}}}\boldsymbol{T}_b^2\,.\tag{6.62}$$

7 Subtraction terms for iterated single unresolved emission

In this section, we provide the explicit definitions of all subtraction terms entering $\mathcal{A}_{12}^{(0)}$, which was defined in sec. 4.3.

7.1 Final-state collinear limit of double collinear-type subtractions

7.1.1 $\mathcal{C}^{IFF(0,0)}_{ars}\mathcal{C}^{FF}_{rs}$

Subtraction term: For an initial-state parton a and final-state partons r and s, we define

$$\begin{aligned}&\mathcal{C}^{IFF(0,0)}_{ars}\mathcal{C}^{FF}_{rs}(p_a,p_b;\{p\}_{m+X+2})=(8\pi\alpha_\mathrm{s}\mu^{2\varepsilon})^2\frac{1}{s_{rs}}\frac{1}{x_{\hat{a},\widehat{rs}}s_{\hat{a}\widehat{rs}}}\\&\quad\times\hat{P}^{(\mathrm{C}),(0)}_{(ars)rs}(z_{r,s},k_{\perp r,s},\hat{k}_{\perp r,s},x_{\hat{a},\widehat{rs}},k_{\perp\widehat{rs},\hat{a}},\hat{p}_a;\varepsilon)\left|\mathcal{M}^{(0)}_{(ars)b,m+X}(\hat{\hat{p}}_a,\hat{\hat{p}}_b;\{\hat{\hat{p}}\}_{m+X})\right|^2,\end{aligned}\tag{7.1}$$

where the reduced matrix element is obtained by removing the final-state partons r and s and replacing the initial-state parton a with a parton of flavor (ars) using the flavor summation rules of eq. (5.2) and requiring that $a=(ars)+r+s$, see fig. 3. Hence, the flavor of the initial-state parton involved in the splitting is generally different in the original double real emission matrix element and in the reduced matrix element. The set of mapped momenta entering the reduced matrix element is defined below in eq. (7.5). The strongly-ordered initial-final-final splitting functions for the $a\to a+(rs)\to a+r+s$ splitting, $\hat{P}^{(\mathrm{C}),(0)}_{(ars)rs}$, can be obtained from the corresponding final- state strongly-ordered splitting functions introduced in ref. [10] and recalled in eqs. (D.55)–(D.60) by the crossing relation discussed in appendix D.2, see eq. (D.61). The splitting function is labeled with the flavor (ars) of the initial-state parton in the reduced matrix element and the flavors r and s of the final-state partons. In particular, the last two indices correspond to those of the strongly-ordered pair. The momentum fractions entering eq. (7.1) are defined by eqs. (3.4) and (3.8). The various transverse momenta are defined as follows. First, $k^{\mu}_{\perp r,s}$ is defined as in eq. (3.13), with coefficients given in eqs. (5.3) and (5.4) with the relabeling $i \to r$ and $r \to s$. Hence, $k^{\mu}_{\perp r,s}$ is orthogonal to both $\hat{p}^{\mu}_{rs}$ and $(p_a + p_b)^{\mu}$. Note that $k_{\perp r,s}$ appears in eqs. (D.55)–(D.60) only in the invariant $s_{\hat{i}k_{\perp r,s}} = 2\hat{p}_i \cdot k_{\perp r,s}$ or squared as $k^2_{\perp r,s}$, but never with a free index to be contracted with the factorized matrix element. After crossing parton i to the initial state, $\hat{P}^{(\mathrm{C}),(0)}_{(ars)rs}$ will depend on the invariant $s_{\hat{a}k_{\perp r,s}} = 2\hat{p}_a \cdot k_{\perp r,s}$. This is of note, because $k^{\mu}_{\perp r,s}$ is not orthogonal to the parent momentum $\hat{p}^{\mu}_a$ in the factorized matrix element. Hence, whenever it would appear in the strongly ordered splitting function of ref. [10] with open indices[16], it is replaced by $\hat{k}^{\mu}_{\perp r,s}$, which is orthogonal to $\hat{\hat{p}}_a$ by construction,

$$\hat{k}^{\mu}_{\perp r,s} = k^{\mu}_{\perp r,s} - \frac{k_{\perp r,s} \cdot \hat{\hat{p}}_a}{\hat{\hat{p}}_a \cdot (p_a + p_b)} (p_a + p_b)^{\mu} \,. \tag{7.2}$$

Such a modification was justified in detail in ref. [57] (see their appendix D.1) for the case of final-state splitting and the arguments presented there remain valid also after crossing. Finally, $k^{\mu}_{\perp \widehat{rs},\hat{a}}$ is given by eq. (3.14).

Momentum mapping: The IF–FF iterated momentum mapping is defined by the successive application of the final-state single collinear momentum mapping of eq. (5.6) and the initial-state single collinear momentum mapping of eq. (5.22),

$$C^{II,F}_{\hat{a}\hat{b},\widehat{rs}} \circ C^{FF}_{rs} : \; (p_a, p_b; \{p\}_{m+X+2}) \xrightarrow{C^{FF}_{rs}} (\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1}) \xrightarrow{C^{II,F}_{\hat{a}\hat{b},\widehat{rs}}} (\hat{\hat{p}}_a, \hat{\hat{p}}_b; \{\hat{\hat{p}}\}_{m+X}) \,. \tag{7.3}$$

Specifically, we first define the set of momenta $(\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1})$ via the C^{FF}_{rs} mapping of eq. (5.6)

$$\begin{aligned}
\hat{p}^{\mu}_a &= (1 - \alpha_{rs}) p^{\mu}_a \,, \\
\hat{p}^{\mu}_b &= (1 - \alpha_{rs}) p^{\mu}_b \,, \\
\hat{p}^{\mu}_{rs} &= p^{\mu}_r + p^{\mu}_s - \alpha_{rs} (p_a + p_b)^{\mu} \,, \\
\hat{p}^{\mu}_n &= p^{\mu}_n \,, \qquad n \in F \,, \quad n \neq r, s \,, \\
\hat{p}^{\mu}_X &= p^{\mu}_X \,,
\end{aligned} \tag{7.4}$$

with α_{rs} given in eq. (5.7) with the relabeling $i \to r$ and $r \to s$. Then, the set $(\hat{\hat{p}}_a, \hat{\hat{p}}_b; \{\hat{\hat{p}}\}_{m+X})$ is constructed using the $C^{II,F}_{\hat{a}\hat{b},\widehat{rs}}$ mapping of eq. (5.22), starting from the momenta in eq. (7.4),

$$\begin{aligned}
\hat{\hat{p}}^{\mu}_a &= \xi_{\hat{a},\widehat{rs}} \hat{p}^{\mu}_a \,, \\
\hat{\hat{p}}^{\mu}_b &= \xi_{\hat{b},\widehat{rs}} \hat{p}^{\mu}_b \,, \\
\hat{\hat{p}}^{\mu}_n &= \Lambda(\hat{P}, \hat{\hat{P}})^{\mu}{}_{\nu} \, \hat{p}^{\nu}_n \,, \qquad n \in F, \quad n \neq \widehat{rs} \,, \\
\hat{\hat{p}}^{\mu}_X &= \Lambda(\hat{P}, \hat{\hat{P}})^{\mu}{}_{\nu} \, \hat{p}^{\nu}_X \,,
\end{aligned} \tag{7.5}$$

where $\Lambda(\hat{P}, \hat{\hat{P}})^{\mu}{}_{\nu}$ is the proper Lorentz transformation in eq. (5.24), while

$$\hat{P}^{\mu} = (\hat{p}_a + \hat{p}_b)^{\mu} - \hat{p}^{\mu}_{rs} \,, \qquad \hat{\hat{P}}^{\mu} = (\hat{\hat{p}}_a + \hat{\hat{p}}_b)^{\mu} = (\xi_{\hat{a},\widehat{rs}} \hat{p}_a + \xi_{\hat{b},\widehat{rs}} \hat{p}_b)^{\mu} = (1 - \alpha_{rs})(\xi_{\hat{a},\widehat{rs}} p_a + \xi_{\hat{b},\widehat{rs}} p_b)^{\mu} \,. \tag{7.6}$$

The quantities $\xi_{\hat{a},\widehat{rs}}$ and $\xi_{\hat{b},\widehat{rs}}$ are computed as in eq. (5.26) but with hatted momenta, i.e., with the replacement $p_a \to \hat{p}_a$, $p_b \to \hat{p}_b$, $Q \to \hat{Q} = \hat{p}_a + \hat{p}_b$ and $p_r \to \hat{p}_{rs}$. Notice that we use the double hat notation to indicate that the final set of momenta are obtained via the successive application of two collinear-type mappings.

[16]The strongly ordered triple collinear splitting functions as originally derived in [10] naturally depend only on two transverse momenta: the one of the two partons that constitute the strongly ordered pair and the one of the strongly ordered pair with respect to the third parton.

Phase space convolution: Using the appropriate convolution formulae for the two mappings, eqs. (5.11) and (5.28), we obtain the following representation of the $(m+X+2)$-particle phase space of eq. (4.7) (with $k=2$),

$$\begin{aligned} & \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a+p_b) = \\ & \quad = \int_{\alpha_{\min}}^{\alpha_{\max}} \mathrm{d}\alpha \int \mathrm{d}\hat{\xi}_a\, \mathrm{d}\hat{\xi}_b\, \mathrm{d}\phi_{m+X}\big(\{\hat{p}\}_{m+X}; (1-\alpha)(\hat{\xi}_a p_a + \hat{\xi}_b p_b)\big) \mathrm{d}\phi_{II,F}(\hat{p}_{rs}, \hat{\xi}_a, \hat{\xi}_b) \frac{s_{\widehat{rs}Q}}{2\pi} \mathrm{d}\phi_2(p_r, p_s; K)\,, \end{aligned} \tag{7.7}$$

where $\mathrm{d}\phi_{II,F}$ is defined by eq. (5.29) and $K = \hat{p}_{rs} + \alpha(p_a+p_b)$. The limits for the α integration are the same as in eq. (5.12), while the $\hat{\xi}_a$ and $\hat{\xi}_b$ integrations run from 0 to 1 with the constraints

$$(\hat{\xi}_a\hat{\xi}_b)_{\min} = \frac{M^2}{(1-\alpha)^2 s_{ab}}\,, \qquad (\hat{\xi}_a\hat{\xi}_b)_{\max} = 1\,. \tag{7.8}$$

Finally, the flux factor $\Phi(p_a \cdot p_b)$ can be written in terms of the mapped momenta as follows,

$$\Phi(p_a \cdot p_b) = \frac{\Phi(\hat{\hat{p}}_a \cdot \hat{\hat{p}}_b)}{(1-\alpha)^2 \hat{\xi}_a \hat{\xi}_b}\,. \tag{7.9}$$

Integrated subtraction term: Using the definitions of the subtraction term and the phase space convolution, eqs. (7.1) and (7.7), we write the integrated counterterm as

$$\begin{aligned} & \int_2 \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a \cdot p_b)} \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a+p_b)\, \mathcal{C}_{ars}^{IFF(0,0)} \mathcal{C}_{rs}^{FF}(p_a, p_b; \{p\}_{m+X+2}) \\ & \quad = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\varepsilon \left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\right]^2 \Big(\Big[\mathrm{C}_{ars}^{IFF(0,0)} \mathrm{C}_{rs}^{FF}\Big] \otimes \mathrm{d}\sigma^{\mathrm{B}}_{(ars)b,m+X}\Big)\,, \end{aligned} \tag{7.10}$$

where

$$\Big[\mathrm{C}_{ars}^{IFF(0,0)} \mathrm{C}_{rs}^{FF}\Big] \otimes \mathrm{d}\sigma^{\mathrm{B}}_{(ars)b,m+X} = \int \mathrm{d}\alpha \int \mathrm{d}\hat{\xi}_a\, \mathrm{d}\hat{\xi}_b\, \Big[\mathrm{C}_{ars}^{IFF(0,0)} \mathrm{C}_{rs}^{FF}(\alpha, \hat{\xi}_a, \hat{\xi}_b; \varepsilon)\Big]\, \mathrm{d}\sigma^{\mathrm{B}}_{(ars)b,m+X}(\hat{\hat{p}}_a, \hat{\hat{p}}_b)\,. \tag{7.11}$$

The initial-state momenta in the Born cross section are $\hat{\hat{p}}_a = (1-\alpha)\hat{\xi}_a p_a$ and $\hat{\hat{p}}_b = (1-\alpha)\hat{\xi}_b p_b$ and we defined the integrated counterterm,

$$\begin{aligned} \Big[\mathrm{C}_{ars}^{IFF(0,0)} \mathrm{C}_{rs}^{FF}(\alpha, \hat{\xi}_a, \hat{\xi}_b; \varepsilon)\Big] &= \left(\frac{(4\pi)^2}{S_\varepsilon} s_{ab}^{\varepsilon}\right)^2 \int \mathrm{d}\phi_{II,F}(\hat{p}_{rs}, \hat{\xi}_a, \hat{\xi}_b) \frac{s_{\widehat{rs}Q}}{2\pi} \mathrm{d}\phi_2(p_r, p_s; K) \\ & \times \frac{(1-\alpha)^2 \hat{\xi}_a \hat{\xi}_b}{s_{rs} x_{\hat{a},\widehat{rs}} s_{\hat{a}\widehat{rs}}} \frac{\omega(ars)}{\omega(a)} \hat{P}^{(\mathrm{C}),(0)}_{(ars)rs}(z_{r,s}, k_{\perp r,s}, \hat{k}_{\perp r,s}, x_{\hat{a},\widehat{rs}}, k_{\perp \widehat{rs},\hat{a}}, \hat{p}_a; \varepsilon)\,. \end{aligned} \tag{7.12}$$

In order to evaluate $\Big[\mathrm{C}_{ars}^{IFF(0,0)} \mathrm{C}_{rs}^{FF}(\alpha, \hat{\xi}_a, \hat{\xi}_b; \varepsilon)\Big]$, we exploit that $\hat{k}_{\perp r,s} \cdot \hat{p}_a = k_{\perp \widehat{rs},\hat{a}} \cdot \hat{p}_a = 0$ and perform the azimuthal integration by passing to the azimuthally averaged splitting functions,[17]

$$\hat{P}^{(\mathrm{C}),(0)}_{(ars)rs}(z_{r,s}, k_{\perp r,s}, \hat{k}_{\perp r,s}, x_{\hat{a},\widehat{rs}}, k_{\perp \widehat{rs},\hat{a}}, \hat{p}_a; \varepsilon) \to P^{(\mathrm{C}),(0)}_{(ars)rs}(z_{r,s}, x_{\hat{a},\widehat{rs}}, s_{\hat{a}k_{\perp r,s}}; \varepsilon)\,,$$

[17] As noted above, $k_{\perp r,s} \cdot \hat{p}_a \neq 0$, so the integration of terms involving $s_{\hat{a}k_{\perp r,s}}$ requires care. Nevertheless, the free indices can all be treated in the usual way, through azimuthal averaging.

see the discussion in appendix D.2. After azimuthal integration, the phase space measure $\mathrm{d}\phi_{II,F}(\hat{p}_{rs},\hat{\xi}_a,\hat{\xi}_b)$ is fully fixed by the delta functions in eq. (5.29) and we have (with an obvious relabeling of eq. (5.39))

$$\begin{aligned}&\mathrm{d}\phi_{II,F}(\hat{p}_{rs},\hat{\xi}_a,\hat{\xi}_b)\\&\quad=\frac{S_\varepsilon}{8\pi^2}\frac{\mathrm{d}\Omega_{d-2}}{\Omega_{d-2}}\mathrm{d}\xi_{\hat{a},\widehat{rs}}\,\mathrm{d}\xi_{\hat{b},\widehat{rs}}\,s_{\hat{a}\hat{b}}^{1-\varepsilon}\left[\frac{\hat{\xi}_a\hat{\xi}_b(1-\hat{\xi}_a^2)(1-\hat{\xi}_b^2)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\right]^{-\varepsilon}\frac{\hat{\xi}_a\hat{\xi}_b(1+\hat{\xi}_a\hat{\xi}_b)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\delta(\xi_{\hat{a},\widehat{rs}}-\hat{\xi}_a)\delta(\xi_{\hat{b},\widehat{rs}}-\hat{\xi}_b)\,.\end{aligned}\tag{7.13}$$

Then, using $s_{\hat{a}\hat{b}}=(1-\alpha)^2 s_{ab}$, we find

$$\begin{aligned}\left[\mathrm{C}_{ars}^{IFF(0,0)}\mathrm{C}_{rs}^{FF}(\alpha,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\right]&=\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{1+\varepsilon}\int\mathrm{d}\phi_2(p_r,p_s;K)\,\frac{s_{\widehat{r}\widehat{s}Q}}{\pi}(1-\alpha)^{4-2\varepsilon}\\&\times\left[\frac{\hat{\xi}_a\hat{\xi}_b(1-\hat{\xi}_a^2)(1-\hat{\xi}_b^2)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\right]^{-\varepsilon}\frac{\hat{\xi}_a^2\hat{\xi}_b^2(1+\hat{\xi}_a\hat{\xi}_b)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\frac{1}{s_{rs}x_{\hat{a},\widehat{rs}}s_{\hat{a}\widehat{rs}}}\frac{\omega(ars)}{\omega(a)}P^{(\mathrm{C}),(0)}_{(ars)rs}(z_{r,s},x_{\hat{a},\widehat{rs}},s_{\hat{a}k_{\perp r,s}};\varepsilon)\,,\end{aligned}\tag{7.14}$$

where we have used the delta functions in eq. (7.13) to perform the integrations over $\xi_{\hat{a},\widehat{rs}}$ and $\xi_{\hat{b},\widehat{rs}}$. This implies that $x_{\hat{a},\widehat{rs}}$ and $s_{\hat{a}\widehat{rs}}$ in eq. (7.14) above are understood to be expressed with $\hat{\xi}_a$ and $\hat{\xi}_b$. In particular, we find

$$s_{\hat{a}\widehat{rs}}=(1-\alpha)^2 s_{ab}\frac{\hat{\xi}_a(1-\hat{\xi}_b^2)}{\hat{\xi}_a+\hat{\xi}_b}\,,\qquad s_{\hat{b}\widehat{rs}}=(1-\alpha)^2 s_{ab}\frac{\hat{\xi}_b(1-\hat{\xi}_a^2)}{\hat{\xi}_a+\hat{\xi}_b}\,,\tag{7.15}$$

which implies

$$x_{\hat{a},\widehat{rs}}=\hat{\xi}_a\hat{\xi}_b\,.\tag{7.16}$$

7.2 Initial-state collinear limit of double collinear-type subtractions

7.2.1 $\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}$

Subtraction term: For an initial-state parton a and final-state partons r and s, we define

$$\begin{aligned}\mathcal{C}_{asr}^{IFF(0,0)}\mathcal{C}_{as}^{IF}(p_a,p_b;\{p\}_{m+X+2})&=(8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{1}{x_{a,s}s_{as}x_{\hat{a},\hat{r}}s_{\hat{a}\hat{r}}}\hat{P}^{(\mathrm{C}),(0)}_{(ars)(as)s}(x_{a,s},k_{\perp s,a},\hat{k}_{\perp s,a},x_{\hat{a},\hat{r}},k_{\perp\hat{r},\hat{a}},\hat{p}_r;\varepsilon)\\&\times\left|\mathcal{M}^{(0)}_{(ars)b,m+X}(\hat{\hat{p}}_a,\hat{\hat{p}}_b;\{\hat{\hat{p}}\}_{m+X})\right|^2\mathcal{F}(x_{\hat{a},\hat{r}},\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})\,,\end{aligned}\tag{7.17}$$

where once again, the reduced matrix element is obtained by dropping the final-state partons r and s and replacing the initial-state parton a by a parton of flavor (ars), given by the flavor summation rules of eq. (5.2) such that $a=(ars)+r+s$, see fig. 3. Thus, the flavor of the initial-state parton involved in the splitting is generally not the same in the original double real emission matrix element and the reduced matrix element. The momenta entering the reduced matrix element are defined below in eq. (7.23). The strongly-ordered initial-final-final splitting functions for the $a\to(as)+r\to a+s+r$ splitting, $\hat{P}^{(\mathrm{C}),(0)}_{(ars)(as)s}$, can be obtained from the corresponding final-state strongly-ordered splitting functions introduced in ref. [10] and recalled in eqs. (D.55)–(D.60) by the crossing relation discussed in appendix D.2, see eq. (D.62). The strongly-ordered splitting function is indexed by the flavors of the initial-state partons (ars) and (as), as well as the final-state parton s. Once more, the last two indices correspond to those of the strongly-ordered pair. The momentum fractions entering eq. (7.17) are defined by eq. (3.8), while the transverse momenta are defined as follows. First, $k_{\perp s,a}$ and $k_{\perp\hat{r},\hat{a}}$ are given by eq. (3.14). Then, since the mapping in eq. (7.23) is such that $\hat{\hat{p}}_a$ is related to $\hat{p}_a$ by a simple rescaling, $k_{\perp s,a}$ is also orthogonal to $\hat{\hat{p}}_a$. Hence we can simply set

$$\hat{k}^\mu_{\perp s,a} = k^\mu_{\perp s,a} \,. \tag{7.18}$$

Finally, the function $\mathcal{F}$ in eq. (7.17) is defined as in eq. (6.2), so that we find

$$\mathcal{F}(x_{\hat{a},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}) = \left(\frac{x_{\hat{a},\hat{r}}}{\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}}\right)^2 = \left(\frac{s_{\hat{a}\hat{Q}}(s_{\hat{a}Q} - s_{\hat{r}Q})}{s_{\hat{a}Q}(s_{\hat{a}\hat{Q}} - s_{\hat{r}\hat{Q}})}\right)^2 \,. \tag{7.19}$$

This function must be included in eq. (7.17) for two reasons. On the one hand, it is needed in order to reproduce the $p^\mu_a \parallel p^\mu_s$ collinear behavior of the triple collinear subtraction term $\mathcal{C}^{IFF(0,0)}_{ars}$ of eq. (6.1). Indeed, it is easy to see that in this limit, the $\mathcal{F}(x_{a,rs}, \xi_{a,rs}\xi_{b,rs})$ function in eq. (6.3) does not go to one. However, it is not difficult to show that when $p^\mu_a \parallel p^\mu_s$, the values of $\mathcal{F}(x_{\hat{a},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})$ and $\mathcal{F}(x_{a,rs}, \xi_{a,rs}\xi_{b,rs})$ approach the same limit. Hence $\mathcal{C}^{IFF(0,0)}_{asr}\mathcal{C}^{IF}_{as}$ in eq. (7.17) is a correct regulator of the triple collinear subtraction term $\mathcal{C}^{IFF(0,0)}_{ars}$ in the $p^\mu_a \parallel p^\mu_s$ collinear limit. On the other hand, $\mathcal{F}(x_{\hat{a},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})$ also plays a role in cancelling an unphysical singularity, this time associated with the vanishing of $x_{\hat{a},\hat{r}}$ inside the double real emission phase space. Using eqs. (D.55)–(D.60), one can check that this subtraction term contains at most factors of $1/x^2_{\hat{a},\hat{r}}$, so the spurious singularities may be canceled by including the factor of $\mathcal{F}(x_{\hat{a},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})$ in eq. (7.17). We note in passing that as $\hat{p}^\mu_a \parallel \hat{p}^\mu_r$, we find $\mathcal{F}(x_{\hat{a},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}) \to 1$, and $\mathcal{C}^{IFF(0,0)}_{asr}\mathcal{C}^{IF}_{as}$ correctly regulates the single unresolved subtraction term $\mathcal{C}^{IF(0,0)}_{as}$ in this limit.

Momentum mapping: The IF–IF iterated momentum mapping is defined by successive application of the initial-state single collinear momentum mapping of eq. (5.22),

$$C^{II,F}_{\hat{a}\hat{b},\hat{r}} \circ C^{II,F}_{ab,s} \,:\, (p_a, p_b; \{p\}_{m+X+2}) \xrightarrow{C^{II,F}_{ab,s}} (\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1}) \xrightarrow{C^{II,F}_{\hat{a}\hat{b},\hat{r}}} (\hat{\hat{p}}_a, \hat{\hat{p}}_b; \{\hat{\hat{p}}\}_{m+X}) \,. \tag{7.20}$$

Specifically, we first define the set of momenta $(\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1})$ via the $C^{II,F}_{ab,s}$ mapping,

$$\begin{aligned}
&\hat{p}^\mu_a = \xi_{a,s} p^\mu_a \,, \\
&\hat{p}^\mu_b = \xi_{b,s} p^\mu_b \,, \\
&\hat{p}^\mu_n = \Lambda(P, \hat{P})^\mu{}_\nu \, p^\nu_n \,, \qquad n \in F, \quad n \neq s \,, \\
&\hat{p}^\mu_X = \Lambda(P, \hat{P})^\mu{}_\nu \, p^\nu_X \,,
\end{aligned} \tag{7.21}$$

where $\Lambda(P, \hat{P})^\mu{}_\nu$ is the proper Lorentz transformation in eq. (5.24) and

$$P^\mu = (p_a + p_b)^\mu - p^\mu_s \,, \qquad \hat{P}^\mu = (\hat{p}_a + \hat{p}_b)^\mu = (\xi_{a,s} p_a + \xi_{b,s} p_b)^\mu \,, \tag{7.22}$$

with $\xi_{a,s}$ and $\xi_{b,r}$ defined by eq. (5.26) with the relabeling $r \to s$. Then, the set $(\hat{\hat{p}}_a, \hat{\hat{p}}_b; \{\hat{\hat{p}}\}_{m+X})$ is constructed using the $C^{II,F}_{\hat{a}\hat{b},\hat{r}}$ mapping of eq. (5.22), starting from the momenta in eq. (7.21),

$$\begin{aligned}
&\hat{\hat{p}}^\mu_a = \xi_{\hat{a},\hat{r}} \hat{p}^\mu_a \,, \\
&\hat{\hat{p}}^\mu_b = \xi_{\hat{b},\hat{r}} \hat{p}^\mu_b \,, \\
&\hat{\hat{p}}^\mu_n = \Lambda(\hat{P}, \hat{\hat{P}})^\mu{}_\nu \, \hat{p}^\nu_n \,, \qquad n \in F, \quad n \neq \hat{r} \,, \\
&\hat{\hat{p}}^\mu_X = \Lambda(\hat{P}, \hat{\hat{P}})^\mu{}_\nu \, \hat{p}^\nu_X \,,
\end{aligned} \tag{7.23}$$

where now

$$\hat{P}^\mu = (\hat{p}_a + \hat{p}_b)^\mu - \hat{p}^\mu_r \,, \qquad \hat{\hat{P}}^\mu = (\hat{\hat{p}}_a + \hat{\hat{p}}_b)^\mu = (\xi_{\hat{a},\hat{r}} \hat{p}_a + \xi_{\hat{b},\hat{r}} \hat{p}_b)^\mu = (\xi_{\hat{a},\hat{r}} \xi_{a,s} p_a + \xi_{\hat{b},\hat{r}} \xi_{b,s} p_b)^\mu \,. \tag{7.24}$$

The quantities $\xi_{\hat{a},\hat{r}}$ and $\xi_{\hat{b},\hat{r}}$ are computed as in eq. (5.26) but with hatted momenta, i.e., with the replacement $p_a \to \hat{p}_a$, $p_b \to \hat{p}_b$, $Q \to \hat{Q} = \hat{p}_a + \hat{p}_b$ and $p_r \to \hat{p}_r$.

Phase space convolution: Using the convolution formula for the initial-state single collinear mapping, eq. (5.28), we obtain the following representation of the $(m+X+2)$-particle phase space of eq. (4.7) (with $k=2$),

$$\begin{aligned}
&\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b) = \\
&\quad = \int \mathrm{d}\xi_a\,\mathrm{d}\xi_b \int \mathrm{d}\hat{\xi}_a\,\mathrm{d}\hat{\xi}_b\,\mathrm{d}\phi_{m+X}\big(\{\hat{\hat{p}}\}_{m+X};\hat{\xi}_a\xi_a p_a+\hat{\xi}_b\xi_b p_b\big)\,\mathrm{d}\phi_{II,F}(\hat{p}_r,\hat{\xi}_a,\hat{\xi}_b)\,\mathrm{d}\phi_{II,F}(p_s,\xi_a,\xi_b)\,,
\end{aligned} \tag{7.25}$$

where $\mathrm{d}\phi_{II,F}$ is defined by eq. (5.29). The limits of integration for ξ_a and ξ_b are the same as in eq. (5.30), while the $\hat{\xi}_a$ and $\hat{\xi}_b$ integrations run from 0 to 1 with the constraints

$$(\hat{\xi}_a\hat{\xi}_b)_{\min} = \frac{M^2}{\xi_a\xi_b s_{ab}}\,, \qquad (\hat{\xi}_a\hat{\xi}_b)_{\max} = 1\,. \tag{7.26}$$

Finally, the flux factor $\Phi(p_a\cdot p_b)$ can be written in terms of the mapped momenta as follows,

$$\Phi(p_a\cdot p_b) = \frac{\Phi(\hat{\hat{p}}_a\cdot\hat{\hat{p}}_b)}{\xi_a\xi_b\hat{\xi}_a\hat{\xi}_b}\,. \tag{7.27}$$

Integrated subtraction term: Using the definitions of the subtraction term and the phase space convolution, eqs. (7.17) and (7.25), we write the integrated counterterm as

$$\begin{aligned}
&\int_2 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{C}^{IFF(0,0)}_{asr}\mathcal{C}^{IF}_{as}(p_a,p_b;\{p\}_{m+X+2}) \\
&\quad = \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\right]^2\Big(\Big[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\Big]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ars)b,m+X}\Big)\,,
\end{aligned} \tag{7.28}$$

where

$$\begin{aligned}
&\Big[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\Big]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ars)b,m+X} \\
&\quad = \int \mathrm{d}\xi_a\,\mathrm{d}\xi_b \int \mathrm{d}\hat{\xi}_a\,\mathrm{d}\hat{\xi}_b\,\Big[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\Big]\,\mathrm{d}\sigma^{\mathrm{B}}_{(ars)b,m+X}(\hat{\hat{p}}_a,\hat{\hat{p}}_b)\,.
\end{aligned} \tag{7.29}$$

Above, the initial-state momenta appearing in the Born cross section are given by $\hat{\hat{p}}_a = \xi_a\hat{\xi}_a p_a$ and $\hat{\hat{p}}_b = \xi_b\hat{\xi}_b p_b$ and we introduced the integrated counterterm,

$$\begin{aligned}
\Big[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\Big] &= \left(\frac{(4\pi)^2}{S_\varepsilon}s^{\varepsilon}_{ab}\right)^2\int \mathrm{d}\phi_{II,F}(\hat{p}_r,\hat{\xi}_a,\hat{\xi}_b)\,\mathrm{d}\phi_{II,F}(p_s,\xi_a,\xi_b)\frac{\xi_a\xi_b\hat{\xi}_a\hat{\xi}_b}{x_{a,s}s_{as}x_{\hat{a},\hat{r}}s_{\hat{a}\hat{r}}} \\
&\quad\times\frac{\omega(ars)}{\omega(a)}\hat{P}^{(\mathrm{C})(0)}_{(ars)(as)s}(x_{a,s},k_{\perp s,a},\hat{k}_{\perp s,a},x_{\hat{a},\hat{r}},k_{\perp\hat{r},\hat{a}},\hat{p}_r;\varepsilon)\mathcal{F}(x_{\hat{a},\hat{r}},\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})\,.
\end{aligned} \tag{7.30}$$

In order to evaluate $\Big[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\Big]$, we exploit that $\hat{k}_{\perp s,a}\cdot\hat{\hat{p}}_a = k_{\perp\hat{r},\hat{a}}\cdot\hat{\hat{p}}_a = 0$ and perform the azimuthal integration by passing to the azimuthally averaged splitting functions,[18]

$$\hat{P}^{(\mathrm{C})(0)}_{(ars)(as)s}(x_{a,s},k_{\perp s,a},\hat{k}_{\perp s,a},x_{\hat{a},\hat{r}},k_{\perp\hat{r},\hat{a}},\hat{p}_r;\varepsilon)\to P^{(\mathrm{C})(0)}_{(ars)(as)s}(x_{a,s},x_{\hat{a},\hat{r}},s_{\hat{r}k_{\perp s,a}};\varepsilon)\,,$$

[18]The transverse momentum, $k^{\mu}_{\perp s,a}$ appears in the integrand in the invariant $s_{\hat{r}k_{\perp s,a}}$ and the integration of the corresponding terms once more requires care. However, the free indices can all be treated in the usual way, through azimuthal averaging.

see the discussion in appendix D.2. After azimuthal integration, the phase space measures $\mathrm{d}\phi_{II,F}(p_s,\xi_a,\xi_b)$ and $\mathrm{d}\phi_{II,F}(\hat{p}_r,\hat{\xi}_a,\hat{\xi}_b)$ are fully fixed by the delta functions in eq. (5.29) as in eq. (5.39) (with appropriate relabeling when necessary) and we can write the complete integration measure appearing in eq. (7.30) above as

$$\begin{aligned}
\mathrm{d}\phi_{II,F}(p_s,\xi_a,\xi_b)\,\mathrm{d}\phi_{II,F}(\hat{p}_r,\hat{\xi}_a,\hat{\xi}_b) &= \left(\frac{S_\varepsilon}{8\pi^2}\right)^2 \frac{\mathrm{d}\Omega_{d-2}}{\Omega_{d-2}}\mathrm{d}\xi_a\,\mathrm{d}\xi_b\,\frac{\mathrm{d}\Omega_{d-2}}{\Omega_{d-2}}\mathrm{d}\hat{\xi}_a\,\mathrm{d}\hat{\xi}_b\,s_{ab}^{1-\varepsilon}s_{\hat{a}\hat{b}}^{1-\varepsilon} \\
&\times \left[\frac{\xi_a\xi_b(1-\xi_a^2)(1-\xi_b^2)}{(\xi_a+\xi_b)^2}\right]^{-\varepsilon}\frac{\xi_a\xi_b(1+\xi_a\xi_b)}{(\xi_a+\xi_b)^2}\left[\frac{\hat{\xi}_a\hat{\xi}_b(1-\hat{\xi}_a^2)(1-\hat{\xi}_b^2)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\right]^{-\varepsilon}\frac{\hat{\xi}_a\hat{\xi}_b(1+\hat{\xi}_a\hat{\xi}_b)}{(\hat{\xi}_a+\hat{\xi}_b)^2} \\
&\times \delta(\xi_{a,s}-\xi_a)\delta(\xi_{b,s}-\xi_b)\delta(\xi_{\hat{a},\hat{r}}-\hat{\xi}_a)\delta(\xi_{\hat{b},\hat{r}}-\hat{\xi}_b)\,.
\end{aligned} \tag{7.31}$$

Then, using $s_{\hat{a}\hat{b}} = \xi_{a,s}\xi_{b,s}s_{ab}$, we find

$$\begin{aligned}
\left[\mathrm{C}_{asr}^{IFF(0,0)}\mathrm{C}_{as}^{IF}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\right] &= 4s_{ab}^2\left[\frac{\xi_a^2\xi_b^2(1-\xi_a^2)(1-\xi_b^2)}{(\xi_a+\xi_b)^2}\right]^{-\varepsilon}\frac{\xi_a^3\xi_b^3(1+\xi_a\xi_b)}{(\xi_a+\xi_b)^2}\left[\frac{\hat{\xi}_a\hat{\xi}_b(1-\hat{\xi}_a^2)(1-\hat{\xi}_b^2)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\right]^{-\varepsilon} \\
&\times \frac{\hat{\xi}_a^2\hat{\xi}_b^2(1+\hat{\xi}_a\hat{\xi}_b)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\frac{1}{x_{a,s}s_{as}x_{\hat{a},\hat{r}}s_{\hat{a}\hat{r}}}\frac{\omega(ars)}{\omega(a)}P^{(\mathrm{C})(0)}_{(ars)(as)s}(x_{a,s},x_{\hat{a},\hat{r}},s_{\hat{r}k_{\perp s,a}};\varepsilon)\mathcal{F}(x_{\hat{a},\hat{r}},\hat{\xi}_a\hat{\xi}_b)\,,
\end{aligned} \tag{7.32}$$

where we have used the delta functions in eq. (7.31) to perform the integrations over $\xi_{a,s}$, $\xi_{b,s}$ as well as $\xi_{\hat{a},\hat{r}}$ and $\xi_{\hat{b},\hat{r}}$. Thus, s_{as}, $x_{a,s}$, $s_{\hat{a}\hat{r}}$ and $x_{\hat{a},\hat{r}}$ are all understood to be expressed with ξ_a, ξ_b, $\hat{\xi}_a$ and $\hat{\xi}_b$ in eq. (7.31). In particular, s_{as} and $x_{a,s}$ are given by eqs. (5.37) and (5.38) after a trivial $r \to s$ relabeling. The corresponding relations for $s_{\hat{a}\hat{r}}$ and $x_{\hat{a},\hat{r}}$ can be obtained through a straightforward, if somewhat tedious calculation and we find

$$s_{\hat{a}\hat{r}} = \xi_a\xi_b s_{ab}\frac{\hat{\xi}_a(1-\hat{\xi}_b^2)}{\hat{\xi}_a+\hat{\xi}_b}\,,\qquad s_{\hat{b}\hat{r}} = \xi_a\xi_b s_{ab}\frac{\hat{\xi}_b(1-\hat{\xi}_a^2)}{\hat{\xi}_a+\hat{\xi}_b} \tag{7.33}$$

and

$$x_{\hat{a},\hat{r}} = \frac{\xi_a(\hat{\xi}_a+\hat{\xi}_b)-\hat{\xi}_a\xi_b(1-\hat{\xi}_b^2)-\xi_a\hat{\xi}_b(1-\hat{\xi}_a^2)}{\xi_a(\hat{\xi}_a+\hat{\xi}_b)}\,. \tag{7.34}$$

Using this representation, it is not difficult to see that $x_{\hat{a},\hat{r}}$ does in fact vanish at certain points inside the double real emission phase space (i.e., $x_{\hat{a},\hat{r}}$ becomes zero for values of ξ_a, ξ_b, $\hat{\xi}_a$ and $\hat{\xi}_b$ that are strictly greater than zero and less than one).

7.2.2 $\mathcal{C}_{as,br}^{IF,IF(0,0)}\mathcal{C}_{as}^{IF}$

Subtraction term: For initial-state partons a and b and final-state partons r and s we define

$$\begin{aligned}
\mathcal{C}_{as,br}^{IF,IF(0,0)}\mathcal{C}_{as}^{IF}(p_a,p_b;\{p\}_{m+X+2}) &= (8\pi\alpha_s\mu^{2\varepsilon})^2\frac{1}{x_{a,s}s_{as}x_{\hat{b},\hat{r}}s_{\hat{b}\hat{r}}}\hat{P}^{(0)}_{(as)s}(x_{a,s},k_{\perp s,a};\varepsilon)\hat{P}^{(0)}_{(br)r}(x_{\hat{b},\hat{r}},k_{\perp\hat{r},\hat{b}};\varepsilon) \\
&\times \left|\mathcal{M}^{(0)}_{(as)(br),m+X}(\hat{\hat{p}}_a,\hat{\hat{p}}_b;\{\hat{\hat{p}}\}_{m+X})\right|^2\mathcal{F}(x_{\hat{b},\hat{r}},\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})\,,
\end{aligned} \tag{7.35}$$

where the reduced matrix element is obtained by removing the final-state partons r and s and replacing the initial-state partons a and b by partons of flavors (as) and (br). The flavors of partons (as) and (br) are fixed by requiring that $a = (as) + s$ and $b = (br) + r$ as in fig. 2. The set of mapped momenta entering the reduced matrix element is defined in eq. (7.23). The tree-level initial-final Altarelli-Parisi splitting functions are given in eqs. (D.23)–(D.26), while the momentum fractions and transverse momenta entering eq. (7.35) are defined by eqs. (3.8) and (3.14). Finally, the function $\mathcal{F}$ in eq. (7.35) is given in eq. (6.2), so that

$$\mathcal{F}(x_{\hat{b},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}) = \left(\frac{x_{\hat{b},\hat{r}}}{\xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}}\right)^2 = \left(\frac{s_{\hat{b}\hat{Q}}(s_{\hat{b}Q} - s_{\hat{r}Q})}{s_{\hat{b}Q}(s_{\hat{b}\hat{Q}} - s_{\hat{r}\hat{Q}})}\right)^2 . \tag{7.36}$$

In fact, $\mathcal{F}(x_{\hat{b},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})$ is the same as the function $\mathcal{F}(x_{\hat{a},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})$ appearing in eq. (7.19), up to a $\hat{a} \leftrightarrow \hat{b}$ exchange, and serves the same role. On the one hand, it ensures that $\mathcal{C}^{IF,IF(0,0)}_{as,br}\mathcal{C}^{IF}_{as}$ in eq. (7.35) is a proper regulator of the double collinear subtraction term $\mathcal{C}^{IF,IF(0,0)}_{as,br}$ of eq. (6.19) (with r and s exchanged). Indeed in the $p^\mu_a \parallel p^\mu_s$ limit, the $\mathcal{F}(x_{a,s}x_{b,r}, \xi_{a,rs}\xi_{b,rs})$ function in $\mathcal{C}^{IF,IF(0,0)}_{as,br}$ (see eq. (6.20) with r and s exchanged) does not go to one. However, it is straightforward to show that its value coincides with the value of $\mathcal{F}(x_{\hat{b},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})$ in this limit. On the other hand, the inclusion of $\mathcal{F}(x_{\hat{b},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}})$ in eq. (7.35) is necessary to cancel the unphysical singularity due to the vanishing of $x_{\hat{b},\hat{r}}$ inside the double real emission phase space. We also note that in the $\hat{p}^\mu_b \parallel \hat{p}^\mu_r$ limit we have $\mathcal{F}(x_{\hat{b},\hat{r}}, \xi_{\hat{a},\hat{r}}\xi_{\hat{b},\hat{r}}) \to 1$, and $\mathcal{C}^{IF,IF(0,0)}_{as,br}\mathcal{C}^{IF}_{as}$ correctly regulates the single unresolved subtraction term $\mathcal{C}^{IF(0,0)}_{as}$ in this limit.

Momentum mapping and phase space factorization: This subtraction term is defined using the IF–IF iterated momentum mapping of sec. 7.2.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (7.25) and (7.35), we write the integrated counterterm as

$$\begin{aligned}\int_2 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}&\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b)\,\mathcal{C}^{IF,IF(0,0)}_{as,br}\mathcal{C}^{IF}_{as}(p_a, p_b; \{p\}_{m+X+2})\\ &= \left[\frac{\alpha_s}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2\Big(\Big[\mathrm{C}^{IF,IF(0,0)}_{as,br}\mathrm{C}^{IF}_{as}\Big]\otimes \mathrm{d}\sigma^{\mathrm{B}}_{(as)(br),m+X}\Big),\end{aligned} \tag{7.37}$$

where

$$\begin{aligned}&\Big[\mathrm{C}^{IF,IF(0,0)}_{as,br}\mathrm{C}^{IF}_{as}\Big]\otimes \mathrm{d}\sigma^{\mathrm{B}}_{(ar)(bs),m+X}\\ &\quad= \int \mathrm{d}\xi_a\,\mathrm{d}\xi_b \int \mathrm{d}\hat{\xi}_a\,\mathrm{d}\hat{\xi}_b\,\Big[\mathrm{C}^{IF,IF(0,0)}_{as,br}\mathrm{C}^{IF}_{as}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\Big]\,\mathrm{d}\sigma^{\mathrm{B}}_{(as)(br),m+X}(\hat{p}_a,\hat{p}_b)\,.\end{aligned} \tag{7.38}$$

Once again, the initial-state momenta in the Born cross section are $\hat{p}_a = \xi_a\hat{\xi}_a p_a$ and $\hat{p}_b = \xi_b\hat{\xi}_b p_b$ and we introduced the integrated counterterm,

$$\begin{aligned}\Big[\mathrm{C}^{IF,IF(0,0)}_{as,br}\mathrm{C}^{IF}_{as}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\Big] &= \left(\frac{(4\pi)^2}{S_\varepsilon}s^\varepsilon_{ab}\right)^2\int \mathrm{d}\phi_{II,F}(\hat{p}_r,\hat{\xi}_a,\hat{\xi}_b)\,\mathrm{d}\phi_{II,F}(p_s,\xi_a,\xi_b)\,\frac{\xi_a\xi_b\hat{\xi}_a\hat{\xi}_b}{x_{a,s}s_{as}x_{\hat{b},\hat{r}}s_{\hat{b}\hat{r}}}\\ &\times\frac{\omega(as)}{\omega(a)}\frac{\omega(br)}{\omega(b)}\hat{P}^{(0)}_{(as)s}(x_{a,s},k_{\perp s,a};\varepsilon)\hat{P}^{(0)}_{(br)r}(x_{\hat{b},\hat{r}},k_{\perp\hat{r},\hat{b}};\varepsilon)\mathcal{F}(x_{\hat{b},\hat{r}},\xi_{\hat{a},\hat{r}},\xi_{\hat{b},\hat{r}})\,.\end{aligned} \tag{7.39}$$

In order to evaluate $\Big[\mathrm{C}^{IF,IF(0,0)}_{as,br}\mathrm{C}^{IF}_{as}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\Big]$, we exploit that $k_{\perp s,a}\cdot\hat{p}_a = k_{\perp\hat{r},\hat{b}}\cdot\hat{p}_b = 0$ and perform the azimuthal integration by passing to the azimuthally averaged splitting functions, $\hat{P}^{(0)}_{(as)s}(x_{a,s},k_{\perp s,a};\varepsilon) \to$

$P^{(0)}_{(as)s}(x_{a,s};\varepsilon)$ and $\hat{P}^{(0)}_{(br)r}(x_{\hat{b},\hat{r}},k_{\perp\hat{r},\hat{b}};\varepsilon) \to P^{(0)}_{(br)r}(x_{\hat{b},\hat{r}};\varepsilon)$, see eqs. (D.27)–(D.30). After azimuthal integration, the phase space measures $\mathrm{d}\phi_{II,F}(p_s,\xi_a,\xi_b)$ and $\mathrm{d}\phi_{II,F}(\hat{p}_r,\hat{\xi}_a,\hat{\xi}_b)$ are fully fixed by the delta functions in eq. (5.29) as in eq. (5.39) (with appropriate relabeling when necessary) and we can write the complete integration measure appearing in eq. (7.39) above as in eq. (7.31). Then, using $s_{\hat{a}\hat{b}} = \xi_{a,s}\xi_{b,s}s_{ab}$, we find

$$\begin{aligned}\Big[\mathrm{C}^{IF,IF(0,0)}_{as,br}\mathrm{C}^{IF}_{as}(\xi_a,\xi_b,\hat{\xi}_a,\hat{\xi}_b;\varepsilon)\Big] &= 4s^2_{ab}\left[\frac{\xi^2_a\xi^2_b(1-\xi^2_a)(1-\xi^2_b)}{(\xi_a+\xi_b)^2}\right]^{-\varepsilon}\frac{\xi^3_a\xi^3_b(1+\xi_a\xi_b)}{(\xi_a+\xi_b)^2}\left[\frac{\hat{\xi}_a\hat{\xi}_b(1-\hat{\xi}^2_a)(1-\hat{\xi}^2_b)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\right]^{-\varepsilon} \\ &\times\frac{\hat{\xi}^2_a\hat{\xi}^2_b(1+\hat{\xi}_a\hat{\xi}_b)}{(\hat{\xi}_a+\hat{\xi}_b)^2}\frac{1}{x_{a,s}s_{as}x_{\hat{b},\hat{r}}s_{\hat{b}\hat{r}}}\frac{\omega(as)}{\omega(a)}\frac{\omega(br)}{\omega(b)}P^{(0)}_{(as)s}(x_{a,s};\varepsilon)P^{(0)}_{(br)r}(x_{\hat{b},\hat{r}};\varepsilon)\mathcal{F}(x_{\hat{b},\hat{r}},\hat{\xi}_a\hat{\xi}_b)\,,\end{aligned} \tag{7.40}$$

where the delta functions in eq. (7.31) have been used to perform the integrations over $\xi_{a,s}$, $\xi_{b,s}$ as well as $\xi_{\hat{a},\hat{r}}$ and $\xi_{\hat{b},\hat{r}}$. Hence, in eq. (7.31), $x_{a,s}$, s_{as}, $x_{\hat{b},\hat{r}}$ and $s_{\hat{b}\hat{r}}$ are all expressed in terms of ξ_a, ξ_b, $\hat{\xi}_a$ and $\hat{\xi}_b$. In particular, s_{as} and $x_{a,s}$ can be obtained from eqs. (5.37) and (5.38) after a trivial $r \to s$ relabeling, while the corresponding relations for $s_{\hat{b}\hat{r}}$ and $x_{\hat{b},\hat{r}}$ are given in eqs. (7.33) and (7.34) after the replacement $a \to b$. Because of this, it is evident that in addition to $x_{\hat{a},\hat{r}}$, $x_{\hat{b},\hat{r}}$ can also vanish inside the double real emission phase space.

7.3 Soft-type limits of double collinear-type subtractions

7.3.1 $\mathcal{C}^{IFF(0,0)}_{ars}\mathcal{S}_s$

Subtraction term: For an initial-state parton a, final-state parton r and final-state gluon s we define

$$\begin{aligned}\mathcal{C}^{IFF(0,0)}_{ars}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2}) &= (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 P^{(\mathrm{S}),(0)}_{(ar)rs}(x_{\tilde{a},\tilde{r}s},x_{\tilde{r},\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}\tilde{r}},s_{\tilde{a}s},s_{\tilde{r}s};\varepsilon) \\ &\times\frac{1}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\big|\mathcal{M}^{(0)}_{(ar)b,m+X}(\hat{\tilde{p}}_a,\hat{\tilde{p}}_b;\{\hat{\tilde{p}}\}_{m+X})\big|^2\,,\end{aligned} \tag{7.41}$$

where the reduced matrix element is defined by removing the final-state parton r and gluon s and replacing the initial-state parton a by a parton of flavor (ar) such that $a = (ar) + r$, see fig. 2. The set of momenta entering the reduced matrix element is defined below in eq. (7.45). In eq. (7.41), $\hat{P}^{(0)}_{(ar)r}$ are once more the tree-level initial-final Altarelli-Parisi splitting functions, whose explicit expressions are given in eqs. (D.23)–(D.26). Moreover, the initial-final-final soft functions $P^{(\mathrm{S}),(0)}_{(ar)rs}$ can be obtained from the corresponding final-final-final soft functions introduced in ref. [10] and recalled here in eqs. (D.63)–(D.67) by the crossing relation discussed in appendix D.3, see eq. (D.69). The momentum fractions entering eq. (7.41) are defined by eqs. (3.8) and (3.10), while the transverse momentum $k_{\perp\tilde{r},\tilde{a}}$ is given by eq. (3.14).

Momentum mapping: The IF–S iterated momentum mapping is defined by the successive application of the single soft momentum mapping of eq. (5.44) and the initial-state single collinear momentum mapping of eq. (5.22),

$$C^{II,F}_{\hat{a}\hat{b},\hat{r}}\circ S_s:\ (p_a,p_b;\{p\}_{m+X+2})\xrightarrow{S_s}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X+1})\xrightarrow{C^{II,F}_{\hat{a}\hat{b},\hat{r}}}(\hat{\tilde{p}}_a,\hat{\tilde{p}}_b;\{\hat{\tilde{p}}\}_{m+X})\,. \tag{7.42}$$

Specifically, we first define the set of momenta $(\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\}_{m+X+1})$ via the S_s mapping,

$$
\begin{aligned}
\tilde{p}_a^\mu &= \lambda_s p_a^\mu \,, \\
\tilde{p}_b^\mu &= \lambda_s p_b^\mu \,, \\
\tilde{p}_n^\mu &= \Lambda(P,\tilde{P})^\mu{}_\nu \, p_n^\nu \,, \qquad n \in F, \quad n \neq s \,, \\
\tilde{p}_X^\mu &= \Lambda(P,\tilde{P})^\mu{}_\nu \, p_X^\nu \,,
\end{aligned}
\tag{7.43}
$$

where $\Lambda(P,\tilde{P})^\mu{}_\nu$ is the proper Lorentz transformation in eq. (5.24) and

$$
P^\mu = (p_a + p_b)^\mu - p_s^\mu \,, \qquad \tilde{P}^\mu = (\tilde{p}_a + \tilde{p}_b)^\mu = \lambda_s (p_a + p_b)^\mu \,, \tag{7.44}
$$

with λ_s defined by eq. (5.46) with the relabeling $r \to s$. Then, the set $(\hat{\tilde{p}}_a, \hat{\tilde{p}}_b; \{\hat{\tilde{p}}\}_{m+X})$ is constructed using the $C^{II,F}_{\hat{a}\hat{b},\hat{r}}$ mapping of eq. (5.22), starting from the momenta in eq. (7.43),

$$
\begin{aligned}
\hat{\tilde{p}}_a^\mu &= \xi_{\tilde{a},\tilde{r}} \tilde{p}_a^\mu \,, \\
\hat{\tilde{p}}_b^\mu &= \xi_{\tilde{b},\tilde{r}} \tilde{p}_b^\mu \,, \\
\hat{\tilde{p}}_n^\mu &= \Lambda(\tilde{P},\hat{\tilde{P}})^\mu{}_\nu \, \tilde{p}_n^\nu \,, \qquad n \in F, \quad n \neq \tilde{r} \,, \\
\hat{\tilde{p}}_X^\mu &= \Lambda(\tilde{P},\hat{\tilde{P}})^\mu{}_\nu \, \tilde{p}_X^\nu \,,
\end{aligned}
\tag{7.45}
$$

where once again $\Lambda(\tilde{P},\hat{\tilde{P}})^\mu{}_\nu$ is the proper Lorentz transformation in eq. (5.24), while

$$
\tilde{P}^\mu = (\tilde{p}_a + \tilde{p}_b)^\mu - \tilde{p}_r^\mu \,, \qquad \hat{\tilde{P}}^\mu = (\hat{\tilde{p}}_a + \hat{\tilde{p}}_b)^\mu = (\xi_{\tilde{a},\tilde{r}} \tilde{p}_a + \xi_{\tilde{b},\tilde{r}} \tilde{p}_b)^\mu = \lambda_s (\xi_{\tilde{a},\tilde{r}} p_a + \xi_{\tilde{b},\tilde{r}} p_b)^\mu \,. \tag{7.46}
$$

The quantities $\xi_{\tilde{a},\tilde{r}}$ and $\xi_{\tilde{b},\tilde{r}}$ are computed as in eq. (5.26) but with the tilded momenta of eq. (7.43), i.e., with the replacement $p_a \to \tilde{p}_a$, $p_b \to \tilde{p}_b$, $Q \to \tilde{Q} = \tilde{p}_a + \tilde{p}_b$ and $p_r \to \tilde{p}_r$. We use the tilde-hat notation to indicate that the final set of momenta is obtained by applying first a soft mapping, followed by a collinear one.

Phase space convolution: Using the appropriate convolution formulae for the two mappings, eqs. (5.28) and (5.50), we obtain the following representation of the $(m+X+2)$-particle phase space of eq. (4.7) (with $k = 2$),

$$
\begin{aligned}
&\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b) = \\
&\quad = \int \mathrm{d}\lambda \int \mathrm{d}\tilde{\xi}_a \, \mathrm{d}\tilde{\xi}_b \, \mathrm{d}\phi_{m+X}\big(\{\hat{\tilde{p}}\}_{m+X}; \lambda(\tilde{\xi}_a p_a + \tilde{\xi}_b p_b)\big) \mathrm{d}\phi_{II,F}(\tilde{p}_r, \tilde{\xi}_a, \tilde{\xi}_b) \frac{s_{ab}}{\pi} \lambda \, \mathrm{d}\phi_2(p_s, K; p_a + p_b) \,,
\end{aligned}
\tag{7.47}
$$

where $\mathrm{d}\phi_{II,F}$ is defined by eq. (5.29), while K is a massive momentum with mass $K^2 = \lambda^2 s_{ab}$. The limits of the λ integration are given in eq. (5.51), while the $\tilde{\xi}_a$ and $\tilde{\xi}_b$ integrations run from 0 to 1 with the constraints

$$
(\tilde{\xi}_a \tilde{\xi}_b)_{\min} = \frac{M^2}{\lambda^2 s_{ab}} \,, \qquad (\tilde{\xi}_a \tilde{\xi}_b)_{\max} = 1 \,. \tag{7.48}
$$

Finally, the flux factor $\Phi(p_a \cdot p_b)$ can be written in terms of the mapped momenta as follows,

$$
\Phi(p_a \cdot p_b) = \frac{\Phi(\hat{\tilde{p}}_a \cdot \hat{\tilde{p}}_b)}{\lambda^2 \tilde{\xi}_a \tilde{\xi}_b} \,. \tag{7.49}
$$

Integrated counterterm: Using the definitions of the subtraction term and the phase space convolution, eqs. (7.41) and (7.47), we write the integrated counterterm as

$$
\begin{aligned}
&\int_2 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{C}^{IFF(0,0)}_{ars}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2})\\
&\quad=\left[\frac{\alpha_\mathrm{s}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\right]^2\left(\left[\mathrm{C}^{IFF(0,0)}_{ars}\mathrm{S}_s\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X}\right),
\end{aligned}
\tag{7.50}
$$

where

$$
\left[\mathrm{C}^{IFF(0,0)}_{ars}\mathrm{S}_s\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X}=\int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int\mathrm{d}\tilde{\xi}_a\,\mathrm{d}\tilde{\xi}_b\,\left[\mathrm{C}^{IFF(0,0)}_{ars}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X}(\hat{\tilde{p}}_a,\hat{\tilde{p}}_b)\,. \tag{7.51}
$$

The initial-state momenta in the Born cross section are given by $\hat{\tilde{p}}_a=\lambda\tilde{\xi}_a p_a$ and $\hat{\tilde{p}}_b=\lambda\tilde{\xi}_b p_b$ and we define the integrated counterterm

$$
\begin{aligned}
\left[\mathrm{C}^{IFF(0,0)}_{ars}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\right]&=\left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{\varepsilon}\right)^2\int\mathrm{d}\phi_{II,F}(\tilde{p}_r,\tilde{\xi}_a,\tilde{\xi}_b)\frac{s_{ab}}{\pi}\mathrm{d}\phi_2(p_s,K;p_a+p_b)\frac{\lambda^3\tilde{\xi}_a\tilde{\xi}_b}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}\\
&\times\frac{\omega(ar)}{\omega(a)}P^{(\mathrm{S}),(0)}_{(ar)rs}(x_{\tilde{a},\tilde{r}s},x_{\tilde{r},\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}\tilde{r}},s_{\tilde{a}s},s_{\tilde{r}s};\varepsilon)\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\,.
\end{aligned}
\tag{7.52}
$$

In order to evaluate $\left[\mathrm{C}^{IFF(0,0)}_{ars}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\right]$, we exploit that $k_{\perp\tilde{r},\tilde{a}}\cdot\hat{\tilde{p}}_a=0$ and perform the azimuthal integration by passing to the azimuthally averaged splitting functions $\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\to P^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}};\varepsilon)$ of eqs. (D.27)–(D.30). After azimuthal integration, the phase space measure $\mathrm{d}\phi_{II,F}(\tilde{p}_r,\tilde{\xi}_a,\tilde{\xi}_b)$ is fully fixed by the delta functions in eq. (5.29) and can be written (with a trivial relabeling of eq. (5.39)) as

$$
\begin{aligned}
&\mathrm{d}\phi_{II,F}(\tilde{p}_r,\tilde{\xi}_a,\tilde{\xi}_b)\\
&\quad=\frac{S_\varepsilon}{8\pi^2}\frac{\mathrm{d}\Omega_{d-2}}{\Omega_{d-2}}\mathrm{d}\xi_{\tilde{a},\tilde{r}}\,\mathrm{d}\xi_{\tilde{b},\tilde{r}}\,s_{\tilde{a}\tilde{b}}^{1-\varepsilon}\left[\frac{\tilde{\xi}_a\tilde{\xi}_b(1-\tilde{\xi}_a^2)(1-\tilde{\xi}_b^2)}{(\tilde{\xi}_a+\tilde{\xi}_b)^2}\right]^{-\varepsilon}\frac{\tilde{\xi}_a\tilde{\xi}_b(1+\tilde{\xi}_a\tilde{\xi}_b)}{(\tilde{\xi}_a+\tilde{\xi}_b)^2}\delta(\xi_{\tilde{a},\tilde{r}}-\tilde{\xi}_a)\delta(\xi_{\tilde{b},\tilde{r}}-\tilde{\xi}_b)\,.
\end{aligned}
\tag{7.53}
$$

We then find

$$
\begin{aligned}
\left[\mathrm{C}^{IFF(0,0)}_{ars}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\right]&=2\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{1+\varepsilon}\int\mathrm{d}\phi_2(p_s,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^{5-2\varepsilon}\left[\frac{\tilde{\xi}_a\tilde{\xi}_b(1-\tilde{\xi}_a^2)(1-\tilde{\xi}_b^2)}{(\tilde{\xi}_a+\tilde{\xi}_b)^2}\right]^{-\varepsilon}\\
\times\frac{\tilde{\xi}_a^2\tilde{\xi}_b^2(1+\tilde{\xi}_a\tilde{\xi}_b)}{(\tilde{\xi}_a+\tilde{\xi}_b)^2}\frac{1}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}&\frac{\omega(ar)}{\omega(a)}P^{(\mathrm{S}),(0)}_{(ar)rs}(x_{\tilde{a},\tilde{r}s},x_{\tilde{r},\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}\tilde{r}},s_{\tilde{a}s},s_{\tilde{r}s};\varepsilon)P^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}};\varepsilon)\,,
\end{aligned}
\tag{7.54}
$$

where we used that $s_{\tilde{a}\tilde{b}}=\lambda^2 s_{ab}$ and performed the integrations over $\xi_{\tilde{a},\tilde{r}}$ and $\xi_{\tilde{b},\tilde{r}}$ using the delta functions in eq. (7.53). This means that in eq. (7.54) above, $x_{\tilde{a},\tilde{r}}$ and $s_{\tilde{a}\tilde{r}}$ are understood to be expressed with $\tilde{\xi}_a$ and $\tilde{\xi}_b$. In particular, we find

$$
s_{\tilde{a}\tilde{r}}=\lambda^2 s_{ab}\frac{\tilde{\xi}_a(1-\tilde{\xi}_b^2)}{\tilde{\xi}_a+\tilde{\xi}_b}\,,\qquad s_{\tilde{b}\tilde{r}}=\lambda^2 s_{ab}\frac{\tilde{\xi}_b(1-\tilde{\xi}_a^2)}{\tilde{\xi}_a+\tilde{\xi}_b}\,, \tag{7.55}
$$

which gives

$$
x_{\tilde{a},\tilde{r}}=\tilde{\xi}_a\tilde{\xi}_b\,. \tag{7.56}
$$

7.3.2 $\mathcal{C}_{ars}^{IFF(0,0)}\mathcal{C}_{rs}^{FF}\mathcal{S}_s$

Subtraction term: For an initial-state parton a, final-state parton r and final-state gluon s we define

$$\begin{aligned}\mathcal{C}_{ars}^{IFF(0,0)}\mathcal{C}_{rs}^{FF}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2}) &= (8\pi\alpha_s\mu^{2\varepsilon})^2\frac{2z_{\tilde{r},s}}{s_{\tilde{r}s}z_{s,\tilde{r}}}\boldsymbol{T}_r^2 \\ &\times\frac{1}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\big|\mathcal{M}^{(0)}_{(ar)b,m+X}(\hat{\tilde{p}}_a,\hat{\tilde{p}}_b;\{\hat{\tilde{p}}\}_{m+X})\big|^2\,,\end{aligned}\tag{7.57}$$

where again the reduced matrix element is obtained by removing the final-state parton r and gluon s and replacing the initial-state parton a with a parton of flavor (ar), where $a=(ar)+r$ as in fig. 2. The mapped momenta entering the factorized matrix element are given by eq. (7.45). In eq. (7.57), $\hat{P}^{(0)}_{(ar)r}$ are once again the tree-level initial-final Altarelli-Parisi splitting functions, whose explicit expressions are given in eqs. (D.23)–(D.26). The momentum fractions entering eq. (7.41) are defined by eqs. (3.4) and (3.8), while the transverse momentum $k_{\perp\tilde{r},\tilde{a}}$ is given by eq. (3.14).

Momentum mapping and phase space factorization: This subtraction term is defined using the IF–S iterated momentum mapping of sec. 7.3.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (7.47) and (7.57), we write the integrated counterterm as

$$\begin{aligned}\int_2\mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{C}_{ars}^{IFF(0,0)}\mathcal{C}_{rs}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2}) \\ =\left[\frac{\alpha_s}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2\Big(\Big[\mathrm{C}_{ars}^{IFF(0,0)}\mathrm{C}_{rs}\mathrm{S}_s\Big]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X}\Big)\,,\end{aligned}\tag{7.58}$$

where

$$\Big[\mathrm{C}_{ars}^{IFF(0,0)}\mathrm{C}_{rs}\mathrm{S}_s\Big]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X}=\int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int\mathrm{d}\tilde{\xi}_a\,\mathrm{d}\tilde{\xi}_b\,\Big[\mathrm{C}_{ars}^{IFF(0,0)}\mathrm{C}_{rs}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\Big]\,\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X}(\hat{\tilde{p}}_a,\hat{\tilde{p}}_b)\,.\tag{7.59}$$

Once again, the initial-state momenta in the Born cross section are $\hat{\tilde{p}}_a=\lambda\tilde{\xi}_a p_a$ and $\hat{\tilde{p}}_b=\lambda\tilde{\xi}_b p_b$ and we introduced the integrated counterterm

$$\begin{aligned}\Big[\mathrm{C}_{ars}^{IFF(0,0)}\mathrm{C}_{rs}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\Big]&=\left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^\varepsilon\right)^2\int\mathrm{d}\phi_{II,F}(\tilde{p}_r,\tilde{\xi}_a,\tilde{\xi}_b)\frac{s_{ab}}{\pi}\mathrm{d}\phi_2(p_s,K;p_a+p_b)\frac{1}{s_{\tilde{r}s}}\frac{2z_{\tilde{r},s}}{z_{s,\tilde{r}}}\boldsymbol{T}_r^2 \\ &\times\frac{\lambda^3\tilde{\xi}_a\tilde{\xi}_b}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}\frac{\omega(ar)}{\omega(a)}\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\,.\end{aligned}\tag{7.60}$$

In order to evaluate $\Big[\mathrm{C}_{ars}^{IFF(0,0)}\mathrm{C}_{rs}\mathrm{S}_r(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\Big]$, we exploit $k_{\perp\tilde{r},\tilde{a}}\cdot\hat{\tilde{p}}_a=0$ and perform the azimuthal integration by passing to the azimuthally averaged splitting functions, $\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\to P^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}};\varepsilon)$, see eqs. (D.27)–(D.30). After azimuthal integration, the phase space measure $\mathrm{d}\phi_{II,F}(\tilde{p}_r,\tilde{\xi}_a,\tilde{\xi}_b)$ is fully fixed by the delta functions in eq. (5.29) and is given in eq. (7.53). Then, using $s_{\tilde{a}\tilde{b}} = \lambda^2 s_{ab}$, we find

$$\begin{aligned}\left[\mathrm{C}^{IFF(0,0)}_{ars}\mathrm{C}_{rs}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\right] &= 2\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{1+\varepsilon}\int \mathrm{d}\phi_2(p_s,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^{5-2\varepsilon}\left[\frac{\tilde{\xi}_a\tilde{\xi}_b(1-\tilde{\xi}_a^2)(1-\tilde{\xi}_b^2)}{(\tilde{\xi}_a+\tilde{\xi}_b)^2}\right]^{-\varepsilon}\\ &\times\frac{\tilde{\xi}_a^2\tilde{\xi}_b^2(1+\tilde{\xi}_a\tilde{\xi}_b)}{(\tilde{\xi}_a+\tilde{\xi}_b)^2}\frac{2z_{\tilde{r},s}}{s_{\tilde{r}s}z_{s,\tilde{r}}}\boldsymbol{T}_r^2\frac{1}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}\frac{\omega(ar)}{\omega(a)}P^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}};\varepsilon)\,,\end{aligned} \tag{7.61}$$

where the integrations over $\xi_{\tilde{a},\tilde{r}}$ and $\xi_{\tilde{b},\tilde{r}}$ were performed using the delta functions in eq. (7.53). Then, $x_{\tilde{a},\tilde{r}}$ and $s_{\tilde{a}\tilde{r}}$ can be expressed with $\tilde{\xi}_a$ and $\tilde{\xi}_b$ as in eqs. (7.55) and (7.56).

7.3.3 $\mathcal{C}^{IFF(0,0)}_{asr}\mathcal{C}^{IF}_{as}\mathcal{S}_s$

Subtraction term: For an initial-state parton a, final-state parton r and final-state gluon s we define

$$\begin{aligned}\mathcal{C}^{IFF(0,0)}_{asr}\mathcal{C}^{IF}_{as}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2}) &= (8\pi\alpha_s\mu^{2\varepsilon})^2\frac{2}{x_{s,\tilde{a}}s_{\tilde{a}s}}\boldsymbol{T}_a^2\\ &\times\frac{1}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\left|\mathcal{M}^{(0)}_{(ar)b,m+X}(\hat{\tilde{p}}_a,\hat{\tilde{p}}_b;\{\hat{\tilde{p}}\}_{m+X})\right|^2,\end{aligned} \tag{7.62}$$

where once more we obtain the reduced matrix element by removing the final-state parton r and gluon s and replacing the initial-state parton a by a parton of flavor (ar). The flavor of parton (ar) is again fixed by requiring that $a = (ar)+r$ as in fig. 2. The mapped momenta entering the reduced matrix element are given in eq. (7.45). Again, $\hat{P}^{(0)}_{(ar)r}$ in eq. (7.62) are the tree-level initial-final Altarelli-Parisi splitting functions of eqs. (D.23)–(D.26), while the momentum fractions and transverse momentum entering eq. (7.41) are defined by eqs. (3.8) and (3.14).

Momentum mapping and phase space factorization: This subtraction term is defined using the IF–S iterated momentum mapping of sec. 7.3.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (7.47) and (7.62), we write the integrated counterterm as

$$\begin{aligned}&\int_2\mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{C}^{IFF(0,0)}_{asr}\mathcal{C}^{IF}_{as}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2})\\ &\quad=\left[\frac{\alpha_s}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2\left(\left[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\mathrm{S}_s\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X}\right),\end{aligned} \tag{7.63}$$

where

$$\left[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\mathrm{S}_s\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X} = \int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int\mathrm{d}\tilde{\xi}_a\,\mathrm{d}\tilde{\xi}_b\,\left[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{B}}_{(ar)b,m+X}(\hat{\tilde{p}}_a,\hat{\tilde{p}}_b)\,. \tag{7.64}$$

The initial-state momenta in the Born cross section once again read $\hat{\tilde{p}}_a = \lambda\tilde{\xi}_a p_a$ and $\hat{\tilde{p}}_b = \lambda\tilde{\xi}_b p_b$ and the integrated counterterm is defined as

$$\Big[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\Big] = \left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{\varepsilon}\right)^2 \int \mathrm{d}\phi_{II,F}(\tilde{p}_r,\tilde{\xi}_a,\tilde{\xi}_b)\,\frac{s_{ab}}{\pi}\mathrm{d}\phi_2(p_s,K;p_a+p_b)\,\frac{2}{x_{s,\tilde{a}}s_{\tilde{a}s}}\boldsymbol{T}_a^2 \\ \times \frac{\lambda^3\tilde{\xi}_a\tilde{\xi}_b}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}\,\frac{\omega(ar)}{\omega(a)}\,\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\,. \tag{7.65}$$

In order to evaluate $\Big[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\Big]$, we exploit $k_{\perp\tilde{r},\tilde{a}}\cdot\hat{\tilde{p}}_a = 0$ and perform the azimuthal integration by passing to the azimuthally averaged splitting functions, $\hat{P}^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}},k_{\perp\tilde{r},\tilde{a}};\varepsilon)\to P^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}};\varepsilon)$, see eqs. (D.27)–(D.30). After azimuthal integration, the phase space measure $\mathrm{d}\phi_{II,F}(\tilde{p}_r,\tilde{\xi}_a,\tilde{\xi}_b)$ is fully fixed by the delta functions in eq. (5.29) and can be written as in eq. (7.53). Then, using $s_{\tilde{a}\tilde{b}} = \lambda^2 s_{ab}$, we find

$$\Big[\mathrm{C}^{IFF(0,0)}_{asr}\mathrm{C}^{IF}_{as}\mathrm{S}_s(\lambda,\tilde{\xi}_a,\tilde{\xi}_b;\varepsilon)\Big] = 2\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{1+\varepsilon}\int \mathrm{d}\phi_2(p_s,K;p_a+p_b)\,\frac{s_{ab}}{\pi}\lambda^{5-2\varepsilon}\left[\frac{\tilde{\xi}_a\tilde{\xi}_b(1-\tilde{\xi}_a^2)(1-\tilde{\xi}_b^2)}{(\tilde{\xi}_a+\tilde{\xi}_b)^2}\right]^{-\varepsilon} \\ \times \frac{\tilde{\xi}_a^2\tilde{\xi}_b^2(1+\tilde{\xi}_a\tilde{\xi}_b)}{(\tilde{\xi}_a+\tilde{\xi}_b)^2}\,\frac{2}{x_{s,\tilde{a}}s_{\tilde{a}s}}\boldsymbol{T}_a^2\,\frac{1}{x_{\tilde{a},\tilde{r}}s_{\tilde{a}\tilde{r}}}\,\frac{\omega(ar)}{\omega(a)}\,P^{(0)}_{(ar)r}(x_{\tilde{a},\tilde{r}};\varepsilon)\,. \tag{7.66}$$

As before, the integrations over $\xi_{\tilde{a},\tilde{r}}$ and $\xi_{\tilde{b},\tilde{r}}$ were performed using the delta functions in eq. (7.53), while $x_{\tilde{a},\tilde{r}}$ and $s_{\tilde{a}\tilde{r}}$ can be written in terms of $\tilde{\xi}_a$ and $\tilde{\xi}_b$ as in eqs. (7.55) and (7.56).

7.4 Final-state collinear limit of double soft-type subtractions

7.4.1 $\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{FF}_{rs}$

Subtraction term: For two final-state partons r and s, either two gluons or a same-flavor quark-antiquark pair, we define

$$\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{FF}_{rs}(p_a,p_b;\{p\}_{m+X+2}) = (8\pi\alpha_s\mu^{2\varepsilon})^2 \\ \times \sum_{\substack{i,k\in I\cup F\\ i,k\neq r,s}} \frac{1}{2}\mathcal{S}^{\mu\nu}_{\tilde{\hat{i}}\tilde{\hat{k}}}(\widehat{rs})\frac{1}{s_{rs}}\langle\mu|\hat{P}^{(0)}_{rs}(z_{r,s},k_{\perp r,s};\varepsilon)|\nu\rangle\boldsymbol{T}_i\boldsymbol{T}_k\big|\mathcal{M}^{(0)}_{ab,m+X}(\hat{\tilde{p}}_a,\hat{\tilde{p}}_b;\{\hat{\tilde{p}}\}_{m+X})\big|^2\,, \tag{7.67}$$

where the reduced matrix element is obtained by simply removing the final-state partons r and s. The set of momenta entering the reduced matrix element is defined below in eq. (7.72). In eq. (7.67), the uncontracted eikonal factor $\mathcal{S}^{\mu\nu}_{\tilde{\hat{i}}\tilde{\hat{k}}}(\widehat{rs})$ is defined as

$$\mathcal{S}^{\mu\nu}_{\tilde{\hat{i}}\tilde{\hat{k}}}(\widehat{rs}) = 4\frac{\tilde{\hat{p}}_i^{\mu}\tilde{\hat{p}}_k^{\nu}}{s_{\tilde{\hat{i}}\widehat{rs}}s_{\tilde{\hat{k}}\widehat{rs}}}\,, \tag{7.68}$$

while $\hat{P}^{(0)}_{rs}(z_{r,s},k_{\perp r,s};\varepsilon)$ is the final-final single unresolved Altarelli-Parisi splitting kernel for $g\to q\bar{q}$ or $g\to gg$ splitting, given in eqs. (D.3) and (D.4). The momentum fraction $z_{r,s}$ is defined by eq. (3.4), while the transverse momentum $k_{\perp r,s}$ is constructed as in eq. (3.13), with coefficients given in eqs. (5.3) and (5.4) after the relabeling $i\to r$ and $r\to s$. Note that for $i=k$, eq. (7.67) does not vanish, since the product of the uncontracted eikonal factor with the transverse momentum-dependent part of the splitting kernels is non-zero. Of course, for color-singlet production, the sums in eq. (7.67) run only over a and b. Then using color conservation, eq. (6.31), in this case we have simply

$$\begin{aligned}
&\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{FF}_{rs}(p_a,p_b;\{p\}_{X+2}) = (8\pi\alpha_s\mu^{2\varepsilon})^2 \\
&\times\frac{1}{2}\frac{1}{s_{rs}}\left(\mathcal{S}^{\mu\nu}_{\tilde{\hat{a}}\tilde{\hat{a}}}(\widehat{rs})+\mathcal{S}^{\mu\nu}_{\tilde{\hat{b}}\tilde{\hat{b}}}(\widehat{rs})-2\mathcal{S}^{\mu\nu}_{\tilde{\hat{a}}\tilde{\hat{b}}}(\widehat{rs})\right)\langle\mu|\hat{P}^{(0)}_{rs}(z_{r,s},k_{\perp r,s};\varepsilon)|\nu\rangle\boldsymbol{T}^2_{\mathrm{ini}}\big|\mathcal{M}^{(0)}_{ab,X}(\{\tilde{\hat{p}}\}_X;\tilde{\hat{p}}_a,\tilde{\hat{p}}_b)\big|^2\,.
\end{aligned}\tag{7.69}$$

Momentum mapping: The S–IF iterated momentum mapping is defined by the successive application of the final-state single collinear momentum mapping of eq. (5.6) and the single soft momentum mapping of eq. (5.44),

$$S_{\widehat{rs}}\circ C^{FF}_{rs}\,:\,(p_a,p_b;\{p\}_{m+X+2})\xrightarrow{C^{FF}_{rs}}(\hat{p}_a,\hat{p}_b;\{\hat{p}\}_{m+X+1})\xrightarrow{S_{\widehat{rs}}}(\tilde{\hat{p}}_a,\tilde{\hat{p}}_b;\{\tilde{\hat{p}}\}_{m+X})\,.\tag{7.70}$$

Specifically, we first define the set of momenta $(\hat{p}_a,\hat{p}_b;\{\hat{p}\}_{m+X+1})$ via the C^{FF}_{rs} mapping given in eq. (5.6),

$$\begin{aligned}
\hat{p}^\mu_a &= (1-\alpha_{rs})p^\mu_a\,,\\
\hat{p}^\mu_b &= (1-\alpha_{rs})p^\mu_b\,,\\
\hat{p}^\mu_{rs} &= p^\mu_r+p^\mu_s-\alpha_{rs}(p_a+p_b)^\mu\,,\\
\hat{p}^\mu_n &= p^\mu_n\,,\qquad n\in F\,,\quad n\neq r,s\,,\\
\hat{p}^\mu_X &= p^\mu_X\,,
\end{aligned}\tag{7.71}$$

with α_{rs} as in eq. (5.7) after the relabeling $i\to r$ and $r\to s$. Then, the set $(\tilde{\hat{p}}_a,\tilde{\hat{p}}_b;\{\tilde{\hat{p}}\}_{m+X})$ is constructed using the $S_{\widehat{rs}}$ mapping defined by eq. (5.44), starting from the momenta in eq. (7.71),

$$\begin{aligned}
\tilde{\hat{p}}^\mu_a &= \lambda_{\widehat{rs}}\hat{p}^\mu_a\,,\\
\tilde{\hat{p}}^\mu_b &= \lambda_{\widehat{rs}}\hat{p}^\mu_b\,,\\
\tilde{\hat{p}}^\mu_n &= \Lambda(\hat{P},\tilde{\hat{P}})^\mu{}_\nu\,\hat{p}^\nu_n\,,\qquad n\in F,\quad n\neq\widehat{rs}\,,\\
\tilde{\hat{p}}^\mu_X &= \Lambda(\hat{P},\tilde{\hat{P}})^\mu{}_\nu\,\hat{p}^\nu_X\,,
\end{aligned}\tag{7.72}$$

where $\Lambda(\hat{P},\tilde{\hat{P}})^\mu{}_\nu$ is the proper Lorentz transformation in eq. (5.24), while

$$\hat{P}^\mu=(\hat{p}_a+\hat{p}_b)^\mu-\hat{p}^\mu_{rs}\,,\qquad\tilde{\hat{P}}^\mu=(\tilde{\hat{p}}_a+\tilde{\hat{p}}_b)^\mu=\lambda_{\widehat{rs}}(\hat{p}_a+\hat{p}_b)^\mu=\lambda_{\widehat{rs}}(1-\alpha_{rs})(p_a+p_b)^\mu\,.\tag{7.73}$$

The quantity $\lambda_{\widehat{rs}}$ is computed as in eq. (5.46) but with hatted momenta, i.e., with the replacement $p_a\to\hat{p}_a$, $p_b\to\hat{p}_b$, $Q\to\hat{Q}=\hat{p}_a+\hat{p}_b$ and $p_r\to\hat{p}_{rs}$. The hat-tilde notation is used to indicate that the final set of mapped momenta is obtained by applying a collinear mapping first, followed by a soft one.

Phase space convolution: Using the appropriate convolution formulae for the two mappings, eqs. (5.11) and (5.50), we obtain the following representation of the $(m+X+2)$-particle phase space of eq. (4.7) (with $k=2$),

$$\begin{aligned}
\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b) &= \int_{\alpha_{\min}}^{\alpha_{\max}}\mathrm{d}\alpha\int_{\hat{\lambda}_{\min}}^{\hat{\lambda}_{\max}}\mathrm{d}\hat{\lambda}\,\mathrm{d}\phi_{m+X}\big(\{\tilde{\hat{p}}\}_{m+X};\lambda(1-\alpha)(p_a+p_b)\big)\\
&\times\frac{s_{\hat{a}\hat{b}}}{\pi}\hat{\lambda}\,\mathrm{d}\phi_2(\hat{p}_{rs},\hat{K},\hat{p}_a+\hat{p}_b)\,\frac{s_{\widehat{rs}Q}}{2\pi}\mathrm{d}\phi_2(p_r,p_s;p_r+p_s)\,,
\end{aligned}\tag{7.74}$$

were the momentum $\hat{K}^\mu$ is massive with mass $\hat{K}^2=\hat{\lambda}^2s_{\hat{a}\hat{b}}=\hat{\lambda}^2(1-\alpha)^2s_{ab}$. The limits of the α integration are given in eq. (5.12), while the limits of the $\hat{\lambda}$ integration read

$$\hat{\lambda}_{\min}=\frac{M}{(1-\alpha)\sqrt{s_{ab}}}\,,\qquad\hat{\lambda}_{\max}=1\,.\tag{7.75}$$

Finally, the flux factor $\Phi(p_a \cdot p_b)$ can be written in terms of the mapped momenta as follows,

$$\Phi(p_a \cdot p_b) = \frac{\Phi(\tilde{\hat{p}}_a \cdot \tilde{\hat{p}}_b)}{(1-\alpha)^2 \hat{\lambda}^2} \,. \tag{7.76}$$

Integrated counterterm: Using the definitions of the subtraction term and the phase space convolution, eqs. (7.67) and (7.74), we write the integrated counterterm as

$$\begin{aligned} &\int_2 \mathcal{N} \frac{1}{\omega(a)\omega(b)\Phi(p_a \cdot p_b)} \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a + p_b)\, \mathcal{S}^{(0,0)}_{rs} \mathcal{C}^{FF}_{rs}(p_a, p_b; \{p\}_{m+X+2}) \\ &\quad = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\varepsilon \left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\right]^2 \left(\left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}\right] \otimes \mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X} \right) , \end{aligned} \tag{7.77}$$

where

$$\left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}\right] \otimes \mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X} = \int_{\alpha_{\min}}^{\alpha_{\max}} \mathrm{d}\alpha \int_{\hat{\lambda}_{\min}}^{\hat{\lambda}_{\max}} \mathrm{d}\hat{\lambda} \sum_{\substack{i,k \in I \cup F \\ i,k \neq r,s}} \frac{1}{2} \left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}(\alpha, \hat{\lambda}; \varepsilon)\right]^{(i,k)} \boldsymbol{T}_i \boldsymbol{T}_k \mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{\hat{p}}_a, \tilde{\hat{p}}_b) \,. \tag{7.78}$$

The initial-state momenta entering the Born cross section are $\tilde{\hat{p}}_a = (1-\alpha)\hat{\lambda} p_a$ and $\tilde{\hat{p}}_b = (1-\alpha)\hat{\lambda} p_b$ and the integrated counterterm is defined as

$$\begin{aligned} \left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}(\alpha, \hat{\lambda}; \varepsilon)\right]^{(i,k)} &= \left(\frac{(4\pi)^2}{S_\varepsilon} s^{\varepsilon}_{ab}\right)^2 \int \frac{s_{\hat{a}\hat{b}}}{\pi} \mathrm{d}\phi_2(\hat{p}_{rs}, \hat{K}; \hat{p}_a + \hat{p}_b) \frac{s_{\widehat{rs}Q}}{2\pi} \mathrm{d}\phi_2(p_r, p_s; p_r + p_s) \frac{\hat{\lambda}^3 (1-\alpha)^2}{s_{rs}} \\ &\quad \times \mathcal{S}^{\mu\nu}_{\tilde{\hat{i}}\tilde{\hat{k}}}(\widehat{rs}) \langle \mu | \hat{P}^{(0)}_{rs}(z_{r,s}, k_{\perp r,s}; \varepsilon) | \nu \rangle \,. \end{aligned} \tag{7.79}$$

The evaluation of $\left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}(\alpha, \hat{\lambda}; \varepsilon)\right]^{(i,k)}$ requires some care and will be discussed in a dedicated publication. Of course, in the color-singlet case, we must compute the integrated counterterm only for $i,k = a,b$. Then, using eq. (6.31), we have

$$\begin{aligned} \left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}\right] \otimes \mathrm{d}\sigma^{\mathrm{B}}_{ab,X} = \int_{\alpha_{\min}}^{\alpha_{\max}} \mathrm{d}\alpha \int_{\hat{\lambda}_{\min}}^{\hat{\lambda}_{\max}} \mathrm{d}\hat{\lambda} &\left\{ \frac{1}{2} \left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}(\alpha, \hat{\lambda}; \varepsilon)\right]^{(a,a)} + \frac{1}{2} \left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}(\alpha, \hat{\lambda}; \varepsilon)\right]^{(b,b)} \right. \\ &\left. - \left[\mathrm{S}^{(0,0)}_{rs} \mathrm{C}^{FF}_{rs}(\alpha, \hat{\lambda}; \varepsilon)\right]^{(a,b)} \right\} \boldsymbol{T}^2_{\mathrm{ini}} \mathrm{d}\sigma^{\mathrm{B}}_{ab,X}(\tilde{\hat{p}}_a, \tilde{\hat{p}}_b) \,. \end{aligned} \tag{7.80}$$

7.4.2 $\mathcal{C}^{IFF}_{ars} \mathcal{S}^{(0,0)}_{rs} \mathcal{C}^{FF}_{rs}$

Subtraction term: For an initial-state parton a and two final-state partons r and s, either two gluons or a same-flavor quark-antiquark pair, we define

$$\begin{aligned} &\mathcal{C}^{IFF}_{ar_g s_g} \mathcal{S}^{(0,0)}_{r_g s_g} \mathcal{C}^{FF}_{r_g s_g}(p_a, p_b; \{p\}_{m+X+2}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 \frac{2}{s_{rs} s_{\tilde{\hat{a}}\widehat{rs}}} \boldsymbol{T}^2_a C_{\mathrm{A}} \\ &\quad \times \left[\frac{2}{x_{\widehat{rs},\tilde{\hat{a}}}} \left(\frac{z_{r,s}}{1-z_{r,s}} + \frac{1-z_{r,s}}{z_{r,s}} \right) - (1-\varepsilon) z_{r,s}(1-z_{r,s}) \frac{s^2_{\tilde{\hat{a}} k_{\perp r,s}}}{k^2_{\perp r,s} s_{\tilde{\hat{a}}\widehat{rs}}} \right] \left| \mathcal{M}^{(0)}_{ab,m+X}(\tilde{\hat{p}}_a, \tilde{\hat{p}}_b; \{\tilde{\hat{p}}\}_{m+X}) \right|^2 , \end{aligned} \tag{7.81}$$

for two gluons, and

$$\begin{aligned}
&\mathcal{C}^{IFF}_{ar_q s_{\bar{q}}}\mathcal{S}^{(0,0)}_{r_q s_{\bar{q}}}\mathcal{C}^{FF}_{r_q s_{\bar{q}}}(p_a, p_b; \{p\}_{m+X+2}) = \\
&\quad = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 \frac{2}{s_{rs}s_{\tilde{\hat{a}}\widehat{rs}}} \boldsymbol{T}_a^2 T_{\mathrm{R}} \left[\frac{1}{x_{\widehat{rs},\tilde{\hat{a}}}} + z_{r,s}(1-z_{r,s})\frac{s^2_{\tilde{\hat{a}}k_{\perp r,s}}}{k^2_{\perp r,s}s_{\tilde{\hat{a}}\widehat{rs}}}\right] \left|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{\hat{p}}_a, \tilde{\hat{p}}_b; \{\tilde{\hat{p}}\}_{m+X})\right|^2 ,
\end{aligned} \tag{7.82}$$

for a quark-antiquark pair. The reduced matrix elements are obtained by dropping the final-state partons r and s, while the mapped momenta entering the reduced matrix elements are given in eq. (7.72). The momentum fraction $z_{r,s}$ is defined by eq. (3.4), while $x_{\widehat{rs},\tilde{\hat{a}}}$ is given by eq. (3.8). The transverse momentum $k^\mu_{\perp r,s}$ is given by eq. (3.13) with coefficients as in eqs. (5.3) and (5.4), after the relabeling $i \to r$ and $r \to s$. Moreover, we used the notation

$$s_{\tilde{\hat{a}}k_{\perp r,s}} = 2\tilde{\hat{p}}_a \cdot k_{\perp r,s} \,. \tag{7.83}$$

Momentum mapping and phase space factorization: This subtraction term is defined using the S–IF iterated momentum mapping of sec. 7.4.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution, eq. (7.74), and the subtraction terms, eqs. (7.81) and (7.82), we write the integrated counterterm as

$$\begin{aligned}
&\int_2 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)} \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2}; p_a+p_b)\, \mathcal{C}^{IFF}_{ars}\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{FF}_{rs}(p_a, p_b; \{p\}_{m+X+2}) \\
&\qquad = \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2 \left(\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\mathrm{C}^{FF}_{rs}\right]\otimes \mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}\right),
\end{aligned} \tag{7.84}$$

where

$$\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\mathrm{C}^{FF}_{rs}\right]\otimes \mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X} = \int_{\alpha_{\min}}^{\alpha_{\max}}\mathrm{d}\alpha \int_{\hat{\lambda}_{\min}}^{\hat{\lambda}_{\max}}\mathrm{d}\hat{\lambda}\, \left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\mathrm{C}^{FF}_{rs}(\alpha,\hat{\lambda};\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{\hat{p}}_a,\tilde{\hat{p}}_b)\,. \tag{7.85}$$

The initial-state momenta entering the Born cross section are $\tilde{\hat{p}}_a = (1-\alpha)\hat{\lambda}p_a$ and $\tilde{\hat{p}}_b = (1-\alpha)\hat{\lambda}p_b$ and the integrated counterterm is defined as

$$\begin{aligned}
\left[\mathrm{C}^{IFF}_{ar_g s_g}\mathrm{S}^{(0,0)}_{r_g s_g}\mathrm{C}^{FF}_{r_g s_g}(\alpha,\hat{\lambda};\varepsilon)\right] &= \left(\frac{(4\pi)^2}{S_\varepsilon}s^\varepsilon_{ab}\right)^2 \int \frac{s_{\hat{a}\hat{b}}}{\pi}\mathrm{d}\phi_2(\hat{p}_{rs},\hat{K};\hat{p}_a+\hat{p}_b)\frac{s_{\widehat{rs}Q}}{2\pi}\mathrm{d}\phi_2(p_r,p_s;p_r+p_s) \\
&\times \frac{2\hat{\lambda}^3(1-\alpha)^2}{s_{rs}s_{\tilde{\hat{a}}\widehat{rs}}}\boldsymbol{T}_a^2 C_{\mathrm{A}}\left[\frac{2}{x_{\widehat{rs},\tilde{\hat{a}}}}\left(\frac{z_{r,s}}{1-z_{r,s}}+\frac{1-z_{r,s}}{z_{r,s}}\right) - (1-\varepsilon)z_{r,s}(1-z_{r,s})\frac{s^2_{\tilde{\hat{a}}k_{\perp r,s}}}{k^2_{\perp r,s}s_{\tilde{\hat{a}}\widehat{rs}}}\right],
\end{aligned} \tag{7.86}$$

for r and s both gluons and

$$\begin{aligned}
\left[\mathrm{C}^{IFF}_{ar_q s_{\bar{q}}}\mathrm{S}^{(0,0)}_{r_q s_{\bar{q}}}\mathrm{C}^{FF}_{r_q s_{\bar{q}}}(\alpha,\hat{\lambda};\varepsilon)\right] &= \left(\frac{(4\pi)^2}{S_\varepsilon}s^\varepsilon_{ab}\right)^2 \int \frac{s_{\hat{a}\hat{b}}}{\pi}\mathrm{d}\phi_2(\hat{p}_{rs},\hat{K};\hat{p}_a+\hat{p}_b)\frac{s_{\widehat{rs}Q}}{2\pi}\mathrm{d}\phi_2(p_r,p_s;p_r+p_s) \\
&\times \frac{2\hat{\lambda}^3(1-\alpha)^2}{s_{rs}s_{\tilde{\hat{a}}\widehat{rs}}}\boldsymbol{T}_a^2 T_{\mathrm{R}}\left[\frac{1}{x_{\widehat{rs},\tilde{\hat{a}}}} + z_{r,s}(1-z_{r,s})\frac{s^2_{\tilde{\hat{a}}k_{\perp r,s}}}{k^2_{\perp r,s}s_{\tilde{\hat{a}}\widehat{rs}}}\right],
\end{aligned} \tag{7.87}$$

for r and s a same-flavor quark-antiquark pair. The evaluation of $\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\mathrm{C}^{FF}_{rs}(\alpha,\hat{\lambda};\varepsilon)\right]$ requires some care due to the presence of the structure $s_{\tilde{\hat{a}}k_{\perp r,s}}$ and will be discussed elsewhere.

7.5 Soft-type limits of double soft-type subtractions

7.5.1 $\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s$

Subtraction term: For final-state gluons r and s we define

$$
\begin{aligned}
\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2}) &= (8\pi\alpha_s\mu^{2\varepsilon})^2 \\
&\times\Bigg[\frac{1}{8}\sum_{\substack{i,k,j,\ell\in I\cup F\\ i,k,j,\ell\neq r,s}}\mathcal{S}_{\tilde{\tilde{i}}\tilde{\tilde{k}}}(\tilde{r})\mathcal{S}_{\tilde{j}\tilde{\ell}}(s)\{\boldsymbol{T}_i\boldsymbol{T}_k,\boldsymbol{T}_j\boldsymbol{T}_\ell\}\big|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b;\{\tilde{\tilde{p}}\}_{m+X})\big|^2 \\
&\quad -\frac{1}{4}C_{\mathrm{A}}\sum_{\substack{i,k\in I\cup F\\ i,k\neq r,s}}\mathcal{S}_{\tilde{\tilde{i}}\tilde{\tilde{k}}}(\tilde{r})\left(\mathcal{S}_{\tilde{i}\tilde{r}}(s)+\mathcal{S}_{\tilde{k}\tilde{r}}(s)-\mathcal{S}_{\tilde{i}\tilde{k}}(s)\right)\boldsymbol{T}_i\boldsymbol{T}_k\big|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b;\{\tilde{\tilde{p}}\}_{m+X})\big|^2\Bigg],
\end{aligned}
\tag{7.88}
$$

where the reduced matrix elements are obtained by simply removing the final-state gluons r and s. The set of momenta entering the reduced matrix elements is defined in eq. (7.93). In eq. (7.88), the eikonal factors are given by eq. (5.42) with appropriate relabeling as necessary. In the color-singlet case, the summations in eq. (7.88) run only over the initial-state partons a and b. Using color conservation (eqs. (6.31) and (6.32)) we then find

$$
\begin{aligned}
\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s(p_a,p_b;\{p\}_{X+2}) &= (8\pi\alpha_s\mu^{2\varepsilon})^2 \\
&\times\left[\mathcal{S}_{\tilde{\tilde{a}}\tilde{\tilde{b}}}(\tilde{r})\mathcal{S}_{\tilde{a}\tilde{b}}(s)\boldsymbol{T}^2_{\mathrm{ini}}+\frac{1}{2}\mathcal{S}_{\tilde{\tilde{a}}\tilde{\tilde{b}}}(\tilde{r})\left(\mathcal{S}_{\tilde{a}\tilde{r}}(s)+\mathcal{S}_{\tilde{b}\tilde{r}}(s)-\mathcal{S}_{\tilde{a}\tilde{b}}(s)\right)C_{\mathrm{A}}\right]\boldsymbol{T}^2_{\mathrm{ini}}\big|\mathcal{M}^{(0)}_{ab,X}(\{\tilde{\tilde{p}}\}_X;\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b)\big|^2\,.
\end{aligned}
\tag{7.89}
$$

Momentum mapping: The S–S momentum mapping is defined by successive application of the single soft momentum mapping of eq. (5.44),

$$
S_{\tilde{r}}\circ S_s:\ (p_a,p_b;\{p\}_{m+X+2})\xrightarrow{S_s}(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X+1})\xrightarrow{S_{\tilde{r}}}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b;\{\tilde{\tilde{p}}\}_{m+X})\,.
\tag{7.90}
$$

Specifically, we first define the set of momenta $(\tilde{p}_a,\tilde{p}_b;\{\tilde{p}\}_{m+X+1})$ via the S_s mapping of eq. (7.43),

$$
\begin{aligned}
\tilde{p}^\mu_a &= \lambda_s p^\mu_a\,,\\
\tilde{p}^\mu_b &= \lambda_s p^\mu_b\,,\\
\tilde{p}^\mu_n &= \Lambda(P,\tilde{P})^\mu{}_\nu\,p^\nu_n\,, \qquad n\in F,\quad n\neq s\,,\\
\tilde{p}^\mu_X &= \Lambda(P,\tilde{P})^\mu{}_\nu\,p^\nu_X\,,
\end{aligned}
\tag{7.91}
$$

where $\Lambda(P,\tilde{P})^\mu{}_\nu$ is the proper Lorentz transformation in eq. (5.24) and

$$
P^\mu = (p_a+p_b)^\mu - p^\mu_s\,,\qquad \tilde{P}^\mu = (\tilde{p}_a+\tilde{p}_b)^\mu = \lambda_s(p_a+p_b)^\mu\,,
\tag{7.92}
$$

with λ_s defined by eq. (5.46) with the relabeling $r\to s$. Then, the set $(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b;\{\tilde{\tilde{p}}\}_{m+X})$ is constructed using the $S_{\tilde{r}}$ mapping of eq. (5.44) once again, starting from the momenta in eq. (7.91),

$$
\begin{aligned}
\tilde{\tilde{p}}^\mu_a &= \lambda_{\tilde{r}}\tilde{p}^\mu_a\,,\\
\tilde{\tilde{p}}^\mu_b &= \lambda_{\tilde{r}}\tilde{p}^\mu_b\,,\\
\tilde{\tilde{p}}^\mu_n &= \Lambda(\tilde{P},\tilde{\tilde{P}})^\mu{}_\nu\,\tilde{p}^\nu_n\,, \qquad n\in F,\quad n\neq \tilde{r}\,,\\
\tilde{\tilde{p}}^\mu_X &= \Lambda(\tilde{P},\tilde{\tilde{P}})^\mu{}_\nu\,\tilde{p}^\nu_X\,,
\end{aligned}
\tag{7.93}
$$

where $\Lambda(\tilde{P},\tilde{\tilde{P}})^{\mu}{}_{\nu}$ is the proper Lorentz transformation in eq. (5.24), while

$$\tilde{P}^{\mu} = (\tilde{p}_a+\tilde{p}_b)^{\mu} - \tilde{p}_r^{\mu}\,, \qquad \tilde{\tilde{P}}^{\mu} = (\tilde{\tilde{p}}_a+\tilde{\tilde{p}}_b)^{\mu} = \lambda_{\tilde{r}}(\tilde{p}_a+\tilde{p}_b)^{\mu} = \lambda_s\lambda_{\tilde{r}}(p_a+p_b)^{\mu}\,. \tag{7.94}$$

The quantity $\lambda_{\tilde{r}}$ is computed as in eq. (5.46) but with tilded momenta, i.e., with the replacement $p_a \to \tilde{p}_a$, $p_b \to \tilde{p}_b$, $Q \to \tilde{Q} = \tilde{p}_a + \tilde{p}_b$ and $p_r \to \tilde{p}_r$. We employ the double tilde notation to indicate that the final set of mapped momenta are obtained by the successive application of two soft mappings.

Phase space convolution: Using the appropriate convolution formula for the single soft mapping, eq. (5.50), we obtain the following representation of the $(m+X+2)$-particle phase space of eq. (4.7) (with $k=2$),

$$\begin{aligned} \mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b) = \int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int_{\tilde{\lambda}_{\min}}^{\tilde{\lambda}_{\max}}\mathrm{d}\tilde{\lambda}\,\mathrm{d}\phi_{m+X}\big(\{\tilde{\tilde{p}}\}_{m+X};\lambda\tilde{\lambda}(p_a+p_b)\big) \\ \times\frac{s_{\tilde{a}\tilde{b}}}{\pi}\tilde{\lambda}\,\mathrm{d}\phi_2(\tilde{p}_r,\tilde{K};\tilde{p}_a+\tilde{p}_b)\,\frac{s_{ab}}{\pi}\lambda\,\mathrm{d}\phi_2(p_s,K;p_a+p_b)\,, \end{aligned} \tag{7.95}$$

where K^{μ} and $\tilde{K}^{\mu}$ are massive momenta with masses $K^2=\lambda^2 s_{ab}$ and $\tilde{K}^2=\tilde{\lambda}^2 s_{\tilde{a}\tilde{b}}=\lambda^2\tilde{\lambda}^2 s_{ab}$. The limits of the λ integration are given in eq. (5.51), while the limits of the $\hat{\lambda}$ integration read

$$\hat{\lambda}_{\min} = \frac{M}{\lambda\sqrt{s_{ab}}}\,, \qquad \hat{\lambda}_{\max} = 1\,. \tag{7.96}$$

Finally, the flux factor $\Phi(p_a\cdot p_b)$ can be written in terms of the mapped momenta as follows,

$$\Phi(p_a\cdot p_b) = \frac{\Phi(\tilde{\tilde{p}}_a\cdot\tilde{\tilde{p}}_b)}{\lambda^2\tilde{\lambda}^2}\,. \tag{7.97}$$

Integrated subtraction term: Using the definitions of the subtraction term and the phase space convolution, eqs. (7.88) and (7.95), we write the integrated counterterm as

$$\begin{aligned} &\int_2 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2}) \\ &\quad= \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_{\varepsilon}\left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\right]^2\Big(\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s\Big]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}\Big) \end{aligned} \tag{7.98}$$

where

$$\begin{aligned} &\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s\Big]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X} \\ &\quad=\int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int_{\tilde{\lambda}_{\min}}^{\tilde{\lambda}_{\max}}\mathrm{d}\tilde{\lambda}\Bigg\{\frac{1}{8}\sum_{\substack{i,k,j,\ell\in I\cup F\\ i,k,j,\ell\neq r,s}}\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\Big]^{(ik,j\ell)}\{\boldsymbol{T}_i\boldsymbol{T}_k,\boldsymbol{T}_j\boldsymbol{T}_\ell\}\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b) \\ &\qquad\qquad -\frac{1}{4}C_{\mathrm{A}}\sum_{\substack{i,k\in I\cup F\\ i,k\neq r,s}}\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\Big]^{(i,k)}\boldsymbol{T}_i\boldsymbol{T}_k\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b)\Bigg\}\,. \end{aligned} \tag{7.99}$$

The initial-state momenta entering the Born cross sections in eq. (7.99) are $\tilde{\tilde{p}}_a=\lambda\tilde{\lambda}p_a$ and $\tilde{\tilde{p}}_b=\lambda\tilde{\lambda}p_b$ and the integrated counterterms are defined as

$$\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\Big]^{(ik,j\ell)} = \left(\frac{s_{ab}}{\pi}\right)^2\left(\frac{(4\pi)^2}{S_{\varepsilon}}s_{ab}^{\varepsilon}\right)^2\int\mathrm{d}\phi_2(p_s,K;p_a+p_b)\mathrm{d}\phi_2(\tilde{p}_r,\tilde{K};\tilde{p}_a+\tilde{p}_b)\,\lambda^5\tilde{\lambda}^3\mathcal{S}_{\tilde{i}\tilde{k}}(\tilde{r})\mathcal{S}_{\tilde{j}\tilde{\ell}}(s)\,, \tag{7.100}$$

and

$$\left[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\right]^{(i,k)} = \left(\frac{s_{ab}}{\pi}\right)^2\left(\frac{(4\pi)^2}{S_\varepsilon}s^\varepsilon_{ab}\right)^2\int \mathrm{d}\phi_2(p_s,K;p_a+p_b)\mathrm{d}\phi_2(\tilde{p}_r,\tilde{K};\tilde{p}_a+\tilde{p}_b)$$
$$\times\,\lambda^5\tilde{\lambda}^3\mathcal{S}_{\tilde{\tilde{i}}\tilde{\tilde{k}}}(\tilde{r})\left(\mathcal{S}_{\tilde{\tilde{i}}\tilde{r}}(s)+\mathcal{S}_{\tilde{\tilde{k}}\tilde{r}}(s)-\mathcal{S}_{\tilde{\tilde{i}}\tilde{\tilde{k}}}(s)\right)\,. \tag{7.101}$$

Of course, for color-singlet production, we only need to evaluate eqs. (7.100) and (7.101) for $i,k,j,\ell = a,b$. Then, using eqs. (6.31) and (6.32) we obtain

$$\left[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,X} = \int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int_{\tilde{\lambda}_{\min}}^{\tilde{\lambda}_{\max}}\mathrm{d}\tilde{\lambda}\,\Bigg\{\left[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\right]^{(ab,ab)}\boldsymbol{T}^2_{\mathrm{ini}}$$
$$+\frac{1}{2}C_{\mathrm{A}}\left[\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\right]^{(a,b)}\Bigg\}\boldsymbol{T}^2_{\mathrm{ini}}\mathrm{d}\sigma^{\mathrm{B}}_{ab,X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b)\,. \tag{7.102}$$

7.5.2 $\mathcal{C}^{IFF}_{ars}\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s$

Subtraction term: For an initial-state parton a and final-state gluons r and s we define

$$\mathcal{C}^{IFF}_{ars}\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2})$$
$$= (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2 P^{(\mathrm{S}),(0)}_{(ar)rs}(x_{\tilde{a},\tilde{r}s},x_{\tilde{r},\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}\tilde{r}},s_{\tilde{a}s},s_{\tilde{r}s};\varepsilon)\frac{1}{s_{\tilde{\tilde{a}}\tilde{r}}}\frac{2}{x_{\tilde{r},\tilde{\tilde{a}}}}\boldsymbol{T}^2_a\left|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b;\{\tilde{\tilde{p}}\}_{m+X})\right|^2, \tag{7.103}$$

where once more the reduced matrix element is obtained by removing the final-state gluons r and s, while the mapped momenta are given in eq. (7.93). The initial-final-final soft function $P^{(\mathrm{S}),(0)}_{ag_rg_s}$ is the same as in sec. 7.3.1. When both r and s are gluons, it takes the form,

$$P^{(\mathrm{S}),(0)}_{ag_rg_s}(x_{\tilde{a},\tilde{r}s},x_{\tilde{r},\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}\tilde{r}},s_{\tilde{a}s},s_{\tilde{r}s};\varepsilon) = \boldsymbol{T}^2_a\frac{2}{x_{s,\tilde{a}}s_{\tilde{a}s}}+C_{\mathrm{A}}\left(\frac{s_{\tilde{a}\tilde{r}}}{s_{\tilde{a}s}s_{\tilde{r}s}}-\frac{1}{x_{s,\tilde{a}}s_{\tilde{a}s}}+\frac{x_{\tilde{r},\tilde{a}}}{x_{s,\tilde{a}}s_{\tilde{r}s}}\right)\,. \tag{7.104}$$

The momentum fractions entering eq. (7.103) are defined by eqs. (3.8) and (3.10).

Momentum mapping and phase space factorization: This subtraction term is defined using the S–S iterated momentum mapping of sec. 7.5.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (7.95) and (7.103), we write the integrated counterterm as

$$\int_2\mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{C}^{IFF}_{ars}\mathcal{S}^{(0,0)}_{rs}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2})$$
$$=\left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2\left(\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}\right), \tag{7.105}$$

where

$$\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s\right]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X} = \int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int_{\tilde{\lambda}_{\min}}^{\tilde{\lambda}_{\max}}\mathrm{d}\tilde{\lambda}\,\left[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\right]\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b)\,. \tag{7.106}$$

The initial-state momenta in the Born cross section are $\tilde{\tilde{p}}_a = \lambda\tilde{\lambda}p_a$ and $\tilde{\tilde{p}}_b = \lambda\tilde{\lambda}p_b$ as before and the integrated counterterm reads

$$\begin{aligned}\Big[\mathrm{C}^{IFF}_{ars}\mathrm{S}^{(0,0)}_{rs}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\Big] = \left(\frac{s_{ab}}{\pi}\right)^2\left(\frac{(4\pi)^2}{S_\varepsilon}s^\varepsilon_{ab}\right)^2 \int \mathrm{d}\phi_2(p_s,K;p_a+p_b)\mathrm{d}\phi_2(\tilde{p}_r,\tilde{K};\tilde{p}_a+\tilde{p}_b) \\ \times \lambda^5\tilde{\lambda}^3 P^{(\mathrm{S}),(0)}_{(ar)rs}(x_{\tilde{a},\tilde{r}s},x_{\tilde{r},\tilde{a}},x_{s,\tilde{a}},s_{\tilde{a}\tilde{r}},s_{\tilde{a}s},s_{\tilde{r}s};\varepsilon)\frac{1}{s_{\tilde{\tilde{a}}\tilde{r}}}\frac{2}{x_{\tilde{r},\tilde{\tilde{a}}}}\boldsymbol{T}^2_a\,.\end{aligned} \tag{7.107}$$

7.5.3 $\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{as}\mathcal{S}_s$

Subtraction term: For an initial-state parton a and final-state gluons r and s we define

$$\begin{aligned}&\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{as}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2}) \\ &\quad = -(8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{1}{s_{\tilde{a}s}}\frac{2}{x_{s,\tilde{a}}}\boldsymbol{T}^2_a\sum_{\substack{j,k\in I\cup F\\ j,k\neq r,s}}\frac{1}{2}\mathcal{S}_{\tilde{\tilde{j}}\tilde{\tilde{k}}}(\tilde{r})\boldsymbol{T}_j\boldsymbol{T}_k\big|\mathcal{M}^{(0)}_{ab,m+X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b;\{\tilde{\tilde{p}}\}_{m+X})\big|^2\,,\end{aligned} \tag{7.108}$$

where the reduced matrix element is obtained by removing the final-state gluons r and s. The mapped momenta appearing in eq. (7.108) are given in eq. (7.93). The momentum fraction $x_{s,\tilde{a}}$ is defined as in eq. (3.8), while the eikonal factor is given by eq. (5.42). Obviously, for color-singlet production we have $j,k=a,b$ and using color conservation (see eq. (6.31)) we find simply

$$\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{as}\mathcal{S}_s(p_a,p_b;\{p\}_{X+2}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{1}{s_{\tilde{a}s}}\frac{2}{x_{s,\tilde{a}}}\mathcal{S}_{\tilde{\tilde{a}}\tilde{\tilde{b}}}(\tilde{r})\left(\boldsymbol{T}^2_{\mathrm{ini}}\right)^2\big|\mathcal{M}^{(0)}_{ab,X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b;\{\tilde{\tilde{p}}\}_X)\big|^2\,. \tag{7.109}$$

Momentum mapping and phase space factorization: This subtraction term is defined using the S–S iterated momentum mapping of sec. 7.5.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (7.95) and (7.108), we write the integrated counterterm as

$$\begin{aligned}&\int_2\mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{S}^{(0,0)}_{rs}\mathcal{C}^{IF}_{as}\mathcal{S}_s(p_a,p_b;\{p\}_{m+X+2}) \\ &\quad = \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2\left(\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{C}^{IF}_{as}\mathrm{S}_s\Big]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}\right),\end{aligned} \tag{7.110}$$

where

$$\begin{aligned}&\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{C}^{IF}_{as}\mathrm{S}_s\Big]\otimes\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X} \\ &\quad = \int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int_{\tilde{\lambda}_{\min}}^{\tilde{\lambda}_{\max}}\mathrm{d}\tilde{\lambda}\sum_{\substack{j,k\in I\cup F\\ j,\ell\neq r,s}}\frac{1}{2}\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{C}^{IF}_{as}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\Big]^{(j,k)}\boldsymbol{T}_j\boldsymbol{T}_k\mathrm{d}\sigma^{\mathrm{B}}_{ab,m+X}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b)\,.\end{aligned} \tag{7.111}$$

The initial-state momenta in the Born cross section are $\tilde{\tilde{p}}_a = \lambda\tilde{\lambda}p_a$ and $\tilde{\tilde{p}}_b = \lambda\tilde{\lambda}p_b$ and the integrated counterterm is then defined as

$$\begin{aligned}\Big[\mathrm{S}^{(0,0)}_{rs}\mathrm{C}^{IF}_{as}\mathrm{S}_s(\lambda,\tilde{\lambda};\varepsilon)\Big]^{(j,k)} = -\left(\frac{s_{ab}}{\pi}\right)^2\left(\frac{(4\pi)^2}{S_\varepsilon}s^\varepsilon_{ab}\right)^2\int\mathrm{d}\phi_2(p_s,K;p_a+p_b)\mathrm{d}\phi_2(\tilde{p}_r,\tilde{K};\tilde{p}_a+\tilde{p}_b) \\ \times\lambda^5\tilde{\lambda}^3\frac{1}{s_{\tilde{a}s}}\frac{2}{x_{s,\tilde{a}}}\boldsymbol{T}^2_a\mathcal{S}_{\tilde{\tilde{j}}\tilde{\tilde{k}}}(\tilde{r})\,.\end{aligned} \tag{7.112}$$

For color-singlet production, the sums in eq. (7.111) run only over the initial-state partons a and b. Then, using color conservation, we find

$$\left[\mathrm{S}_{rs}^{(0,0)}\mathrm{C}_{as}^{IF}\mathrm{S}_{s}\right]\otimes \mathrm{d}\sigma_{ab,X}^{\mathrm{B}} = -\int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int_{\tilde{\lambda}_{\min}}^{\tilde{\lambda}_{\max}}\mathrm{d}\tilde{\lambda}\,\left[\mathrm{S}_{rs}^{(0,0)}\mathrm{C}_{as}^{IF}\mathrm{S}_{s}(\lambda,\tilde{\lambda};\varepsilon)\right]^{(a,b)}\boldsymbol{T}_{\mathrm{ini}}^{2}\mathrm{d}\sigma_{ab,X}^{\mathrm{B}}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b)\,. \tag{7.113}$$

7.5.4 $\mathcal{C}_{ars}^{IFF}\mathcal{S}_{rs}^{(0,0)}\mathcal{C}_{as}^{IF}\mathcal{S}_{s}$

Subtraction term: For an initial-state parton a and final-state gluons r and s we define

$$\mathcal{C}_{ars}^{IFF}\mathcal{S}_{rs}^{(0,0)}\mathcal{C}_{as}^{IF}\mathcal{S}_{s}(p_a,p_b;\{p\}_{m+X+2}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon})^2\frac{1}{s_{\tilde{a}s}}\frac{2}{x_{s,\tilde{a}}}\boldsymbol{T}_a^2\frac{1}{s_{\tilde{\tilde{a}}\tilde{r}}}\frac{2}{x_{\tilde{r},\tilde{\tilde{a}}}}\boldsymbol{T}_a^2\left|\mathcal{M}_{ab,m+X}^{(0)}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b;\{\tilde{\tilde{p}}\}_{m+X})\right|^2, \tag{7.114}$$

where once more we obtain the reduced matrix element by removing the final-state gluons r and s and the mapped momenta are defined in eq. (7.93). The momentum fractions $x_{s,\tilde{a}}$ and $x_{\tilde{r},\tilde{\tilde{a}}}$ entering eq. (7.114) are given by eq. (3.8).

Momentum mapping and phase space factorization: This subtraction term is defined using the S–S iterated momentum mapping of sec. 7.5.1, where the appropriate phase space convolution is also presented.

Integrated subtraction term: Using the definitions of the phase space convolution and the subtraction term, eqs. (7.95) and (7.114), we write the integrated counterterm as

$$\begin{aligned}\int_2 \mathcal{N}\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}\mathrm{d}\phi_{m+X+2}(\{p\}_{m+X+2};p_a+p_b)\,\mathcal{C}_{ars}^{IFF}\mathcal{S}_{rs}^{(0,0)}\mathcal{C}_{as}^{IF}\mathcal{S}_{s}(p_a,p_b;\{p\}_{m+X+2})\\ = \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{s_{ab}}\right)^{\varepsilon}\right]^2\left(\left[\mathrm{C}_{ars}^{IFF}\mathrm{S}_{rs}^{(0,0)}\mathrm{C}_{as}^{IF}\mathrm{S}_{s}\right]\otimes\mathrm{d}\sigma_{ab,m+X}^{\mathrm{B}}\right),\end{aligned} \tag{7.115}$$

where

$$\left[\mathrm{C}_{ars}^{IFF}\mathrm{S}_{rs}^{(0,0)}\mathrm{C}_{as}^{IF}\mathrm{S}_{s}\right]\otimes\mathrm{d}\sigma_{ab,m+X}^{\mathrm{B}} = \int_{\lambda_{\min}}^{\lambda_{\max}}\mathrm{d}\lambda\int_{\tilde{\lambda}_{\min}}^{\tilde{\lambda}_{\max}}\mathrm{d}\tilde{\lambda}\,\left[\mathrm{C}_{ars}^{IFF}\mathrm{S}_{rs}^{(0,0)}\mathrm{C}_{as}^{IF}\mathrm{S}_{s}(\lambda,\tilde{\lambda};\varepsilon)\right]\mathrm{d}\sigma_{ab,m+X}^{\mathrm{B}}(\tilde{\tilde{p}}_a,\tilde{\tilde{p}}_b)\,. \tag{7.116}$$

The initial-state momenta in the Born cross section are again $\tilde{\tilde{p}}_a=\lambda\tilde{\lambda}p_a$ and $\tilde{\tilde{p}}_b=\lambda\tilde{\lambda}p_b$ and the integrated counterterm is defined as

$$\begin{aligned}\left[\mathrm{C}_{ars}^{IFF}\mathrm{S}_{rs}^{(0,0)}\mathrm{C}_{as}^{IF}\mathrm{S}_{s}(\lambda,\tilde{\lambda};\varepsilon)\right] = \left(\frac{s_{ab}}{\pi}\right)^2\left(\frac{(4\pi)^2}{S_\varepsilon}s_{ab}^{\varepsilon}\right)^2\int\mathrm{d}\phi_2(p_s,K;p_a+p_b)\mathrm{d}\phi_2(\tilde{p}_r,\tilde{K};\tilde{p}_a+\tilde{p}_b)\\ \times\lambda^5\tilde{\lambda}^3\frac{1}{s_{\tilde{a}s}}\frac{2}{x_{s,\tilde{a}}}\boldsymbol{T}_a^2\frac{1}{s_{\tilde{\tilde{a}}\tilde{r}}}\frac{2}{x_{\tilde{r},\tilde{\tilde{a}}}}\boldsymbol{T}_a^2\,.\end{aligned} \tag{7.117}$$

8 Validation

In order to validate the definitions of the subtraction terms, we consider the production of a Higgs boson in gluon-gluon fusion in the Higgs effective field theory. In particular, we concentrate on the

$$g(p_1)+g(p_2)\to H(p_3)+g(p_4)+g(p_5) \tag{8.1}$$

fully gluonic subprocess which exhibits all IR singularities and spin correlations which may arise at NNLO accuracy in color-singlet production. This process has been implemented in the publicly available Monte Carlo program `NNLOCAL` [53], which includes a full set of dedicated phase space routines that allow the user to check the cancellation of kinematic singularities in any IR limit. However here we investigate these cancellations using an implementation of the matrix element and subtraction terms in the `Mathematica` computer algebra system. The arbitrary precision capabilities of the latter allow us to assess the numerical behavior of the subtractions without having to concern ourselves with questions of numerical stability. In the following we present some representative results and draw particular attention to the main features regarding the structure of cancellations among the various subtraction terms.

To begin, we generate sequences of phase space points tending to all unique singular kinematic configurations. All radiation variables that are not directly relevant for approaching the chosen singular configuration, such as azimuthal angles, are kept fixed. The approach to any given unresolved configuration is measured by the value of some suitably chosen kinematic invariants that vanish in the exact limit. In all cases, we scale the invariants by the total squared momentum of the event and introduce the notation

$$y_{jk} = \frac{s_{jk}}{s_{12}} = \frac{2p_j \cdot p_k}{2p_1 \cdot p_2}\,, \qquad y_{jQ} = \frac{s_{j(12)}}{s_{12}} = \frac{2p_j \cdot (p_1 + p_2)}{2p_1 \cdot p_2}\,, \qquad j,k = 1,\ldots,5\,. \tag{8.2}$$

Normalizing the invariants in this way makes comparisons between the various cases more straightforward. The double real emission squared matrix element of course diverges in all limits. Hence, we examine the behavior of the *ratios* of the various subtraction terms to the full $gg \to Hgg$ squared matrix element.

The results of such a study are reported in fig. 4 for the triple collinear, double collinear and double soft limits. In each case, we plot the ratio of the appropriate subtraction term to the squared matrix element (blue), the ratio of the total double unresolved subtraction term $\mathcal{A}_2^{(0)}$ to the squared matrix element (green), as well as the ratio of the iterated single unresolved subtraction term $\mathcal{A}_{12}^{(0)}$ to the single unresolved subtraction term $\mathcal{A}_1^{(0)}$ (yellow). Finally, the ratio of the sum of all subtraction terms, $\mathcal{A}^{(0)} = \mathcal{A}_1^{(0)} + \mathcal{A}_2^{(0)} - \mathcal{A}_{12}^{(0)}$ to the squared matrix element is also shown (red). Looking at the top panels, we observe first of all that in each case, the ratio of the appropriate subtraction term to the squared matrix element goes to one in the limit, establishing that the subtraction terms do indeed reproduce the singular behavior of the squared matrix element point-by-point in phase space. Moreover, we see that the ratio of the full double unresolved subtraction term $\mathcal{A}_2^{(0)}$ to the squared matrix element is also one in the limit. Hence, each double unresolved configuration is indeed counted precisely once and the appropriate cancellations necessary to avoid multiple subtraction take place as required. Furthermore, the ratio of the iterated single unresolved subtraction term $\mathcal{A}_{12}^{(0)}$ and the single unresolved subtraction term $\mathcal{A}_1^{(0)}$ approaches one in the limit as well. These two terms appear in the sum of all subtraction terms with opposite sign, so the fact that this ratio goes to one establishes that $\mathcal{A}_{12}^{(0)}$ correctly cancels $\mathcal{A}_1^{(0)}$ in double unresolved limits. This then implies that the sum of all subtraction terms properly cancels the singularities of the squared matrix element point-by-point in phase space in these limits. In order to make this cancellation more apparent as the limit is approached, in the bottom panels we plot the absolute distance of each ratio from one on a logarithmic scale. The 'spikes' visible in some of these graphs signal that the quantity $(1-R)$ has changed sign. We note that the structure of cancellations observed here between the squared matrix element and the various subtraction terms is expected to hold also for more general processes in our setup.

Before moving on to examine the single unresolved limits, let us highlight specifically the case of the double unresolved collinear-soft singularity. The same ratios that were plotted for the other double unresolved limits are shown in fig. 5. The notable difference with respect to the other cases discussed above is that in this case the collinear-soft subtraction term itself does not directly appear in $\mathcal{A}_2^{(0)}$, as explained in sec. 4.2 (see eq. (4.23) in particular). Hence, the ratio of the would-be collinear-soft subtraction term to the matrix element is represented by a dashed blue line. Nevertheless, as expected, we see that $\mathcal{A}_2^{(0)}$ does reproduce the behavior of the squared matrix element and $\mathcal{A}_{12}^{(0)}$ continues to cancel $\mathcal{A}_1^{(0)}$ also in this limit.

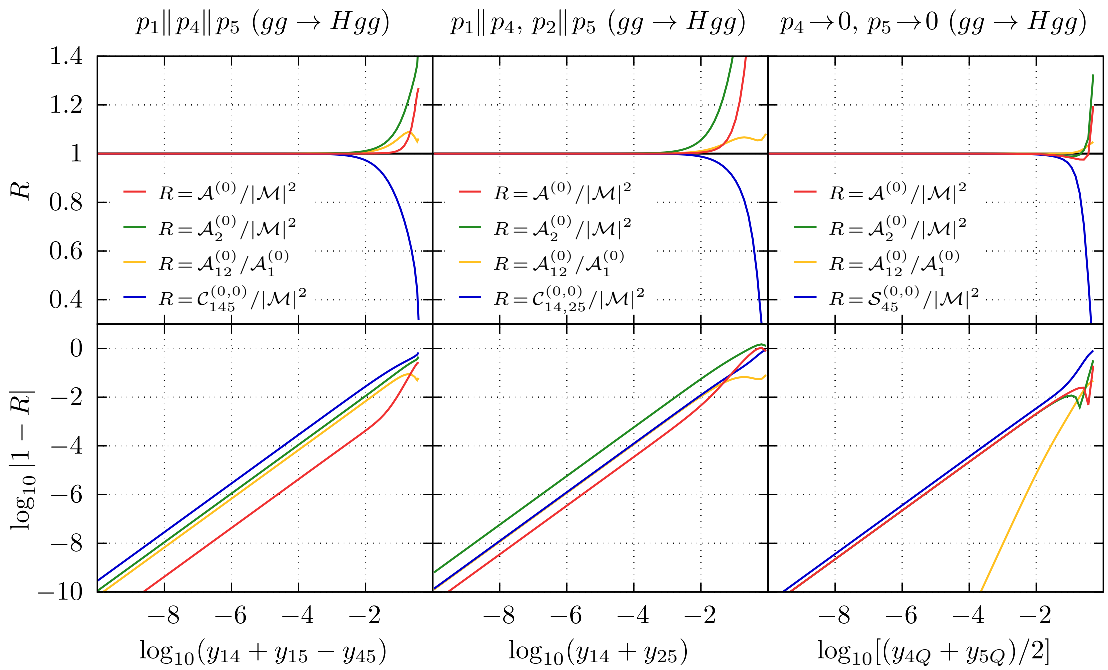


Figure 4: The behavior of ratios of subtraction terms and the squared matrix element in double unresolved limits. Here $|\mathcal{M}|^2$ denotes the tree-level squared matrix element for the $g(p_1)+g(p_2) \to H(p_3)+g(p_4)+g(p_5)$ subprocess, while $\mathcal{A}^{(0)} = \mathcal{A}_1^{(0)} + \mathcal{A}_2^{(0)} - \mathcal{A}_{12}$ is the sum of all subtraction terms.

Thus the sum of all subtraction terms has the correct behavior to cancel the double unresolved singularity of the squared matrix element also in the collinear-soft limit.

Finally, we turn to the case of single unresolved limits. In fig. 6, we present results for the final-final single collinear, the initial-final single collinear and the single soft limits. This time, the relevant quantities are the ratio of the appropriate subtraction term to the squared matrix element (blue), the ratio of the total single unresolved subtraction term $\mathcal{A}_1$ to the squared matrix element (green), the ratio of the iterated single unresolved subtraction term $\mathcal{A}_{12}^{(0)}$ to the double unresolved subtraction term $\mathcal{A}_2^{(0)}$ (yellow) and finally the ratio of the sum of all subtraction terms $\mathcal{A}^{(0)}$ to the squared matrix element (red). In the two collinear limits, we observe a behavior that is very much analogous to what we found in the double unresolved limits. The individual subtraction terms, as well as the total single unresolved subtraction term $\mathcal{A}_1$ reproduce the squared matrix element in the limits, while the full iterated single unresolved subtraction term $\mathcal{A}_{12}^{(0)}$ cancels the double unresolved subtraction term $\mathcal{A}_2^{(0)}$. Thus, the sum of all subtraction terms regularizes the single unresolved collinear singularities point-by-point in phase space. However, in the soft limit, a more intricate structure of cancellations is found. We see that while the individual soft subtraction term as well as the sum of all subtraction terms continue to correctly capture the singular structure of the squared matrix element in the limit, the ratio of the total single unresolved subtraction term $\mathcal{A}_1^{(0)}$ and the squared matrix element no longer goes to one. This behavior is however expected and arises due to singularities which appear in the *reduced matrix elements* entering certain single unresolved counterterms that are not relevant to canceling this soft limit. Similarly, the ratio of $\mathcal{A}_{12}^{(0)}$ to $\mathcal{A}_2^{(0)}$ also does not approach one, hence these two terms do not cancel in the single soft limit. Again, this is expected. In fact, in this limit $\mathcal{A}_{12}^{(0)}$ must not only regularize the single unresolved singularities of $\mathcal{A}_2^{(0)}$, but also those singularities in $\mathcal{A}_1^{(0)}$ that originate from the diverging behavior of reduced matrix elements. This behavior is the typical one expected in our setup

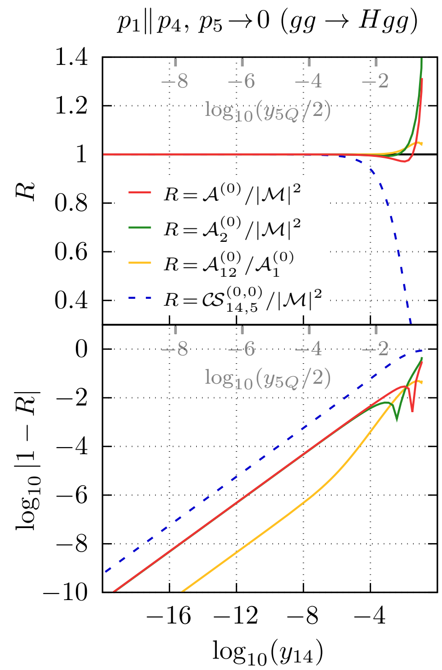


Figure 5: The behavior of ratios of subtraction terms and the squared matrix element in the double unresolved collinear-soft limit. $|\mathcal{M}|^2$ and $\mathcal{A}^{(0)}$ are the same as in fig. 4.

for single unresolved limits and the additional cancellations observed in the collinear cases (e.g., between $\mathcal{A}_{12}^{(0)}$ and $\mathcal{A}_2^{(0)}$) are more or less accidental and due to the simplicity of the underlying Born process.

We finish this section by noting that using arbitrary precision arithmetic in `Mathematica`, we have also checked the correct approach of the sum of all subtraction terms to the squared matrix element in very deep limits. We observe the required cancellation across several tens of orders of magnitude in all limits.

9 Conclusions and outlook

In this work, we have presented the complete set of local counterterms necessary to regularize the double real emission contribution for perturbative calculations of hadronic collisions involving color-singlet production in the CoLoRFulNNLO method. This constitutes a crucial step towards the realization of this scheme as a general and practical framework for computing next-to-next-to-leading order predictions in Quantum Chromodynamics.

Explicit formulae, along with the fully detailed momentum mappings, were derived following the CoLoRFulNNLO approach. This systematic methodology constructs the counterterms based on the known universal behavior of the real-emission matrix elements in the soft and/or collinear emission limits. All necessary ingredients for an independent, *de novo* implementation have been meticulously reported herein, and the formulae have already been incorporated into existing software. Indeed, the full set of counterterms and mappings have been successfully implemented within the publicly available, general-purpose Monte Carlo program `NNLOCAL`. This implementation provides a robust and flexible tool for the numerical evaluation of the regularized double real component of NNLO cross sections.

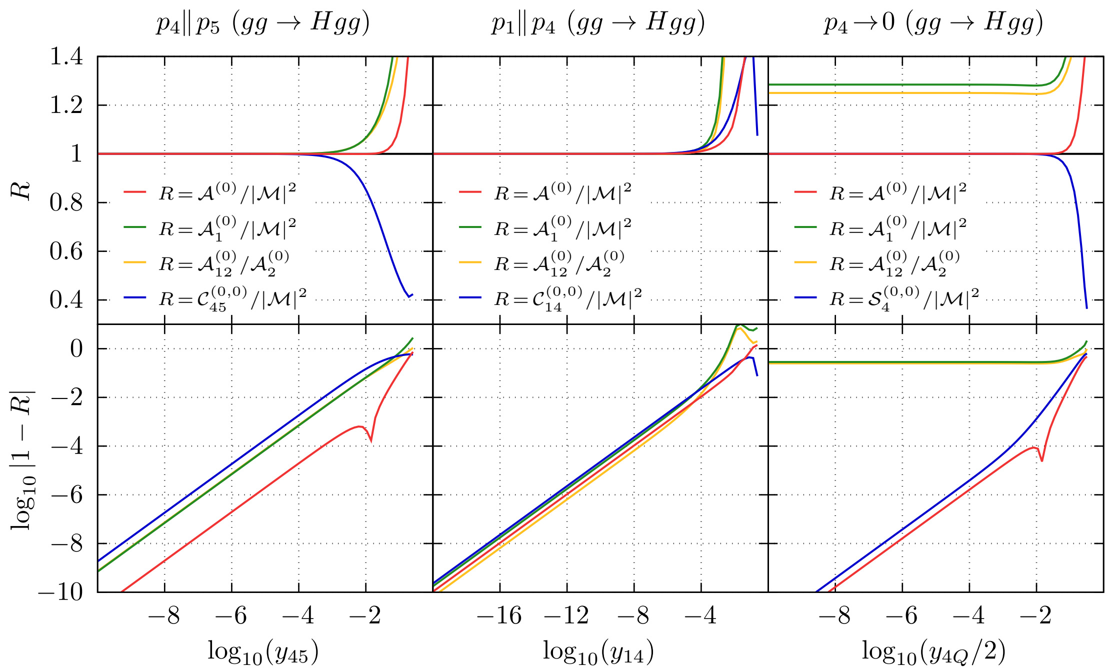


Figure 6: The behavior of ratios of subtraction terms and the squared matrix element in single unresolved limits. $|\mathcal{M}|^2$ and $\mathcal{A}^{(0)}$ are the same as in fig. 4.

The construction of a complete subtraction formalism also mandates the integration of the local counterterms over the phase space of the unresolved emission(s). While the integration itself is beyond the scope of this paper and will be presented elsewhere, we have leveraged the freedom in the definition of the real-emission subtractions and the momentum mappings to satisfy simultaneously two critical requirements. First, to obtain expressions that allow for *analytic integration* over the phase space of unresolved radiation and second, to achieve a *small number* of counter-events necessary to represent the approximate cross sections that locally cancel the singularities of the real radiation matrix elements. Such an optimized construction significantly streamlines both the analytical and numerical aspects of future NNLO calculations. Crucially, the subtraction terms defined here can be used to regularize initial-state radiation also in processes with final-state jets, whose treatment will however require the definition of additional subtraction terms that correspond to infrared limits which do not occur in color-singlet production. In this sense, the counterterms presented here constitute a self-contained subset of the set of double unresolved subtraction terms for general hadronic processes.

The methods employed in this paper, which underpin the CoLoRFulNNLO approach, are highly general. Following the same procedures, this framework can be systematically extended to address final-state radiation in hadronic collisions and can also be generalized to even higher perturbative orders. Those developments are left for future work.

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A Spin and color correlated matrix elements

Consider the generic process of partons a and b scattering to produce a final state with n partons and any number of non-QCD particles, collectively denoted by X, $a+b \to n$ partons $+X$. The corresponding tree-level matrix element is denoted by

$$\mathcal{M}_{f_a,f_b,f_1,\ldots,f_n}^{c_a,c_b,c_1,\ldots,c_n;s_a,s_b,s_1,\ldots,s_n}(p_a,p_b;\{p\}_{n+X})\,, \tag{A.1}$$

where $(c_a,c_b,c_1,\ldots,c_n)$, $(s_a,s_b,s_1,\ldots,s_n)$ and $(f_a,f_b,f_1,\ldots,f_n)$ are the color, spin and flavor indices of the partons, while $(p_a,p_b;\{p\}_{n+X})$ are the particle momenta. In particular, the set of final-state momenta is denoted by $\{p\}_{n+X}$. The squared matrix element, summed over colors and spins, is then

$$\begin{aligned}\left|\mathcal{M}^{(0)}_{ab,n+X}(p_a,p_b;\{p\}_{n+X})\right|^2 = \sum_{\text{spins}}\sum_{\text{colors}}&\left[\mathcal{M}_{f_a,f_b,f_1,\ldots,f_n}^{c_a,c_b,c_1,\ldots,c_n;s_a,s_b,s_1,\ldots,s_n}(p_a,p_b;\{p\}_{n+X})\right]^*\\ &\times\mathcal{M}_{f_a,f_b,f_1,\ldots,f_n}^{c_a,c_b,c_1,\ldots,c_n;s_a,s_b,s_1,\ldots,s_n}(p_a,p_b;\{p\}_{n+X})\,.\end{aligned} \tag{A.2}$$

Notice that $\left|\mathcal{M}^{(0)}_{ab,n+X}(p_a,p_b;\{p\}_{n+X})\right|^2$ does not carry any normalization factor related to the number of spins or colors of the initial-state partons.

In order to precisely define the product of Altarelli-Parisi splitting kernels and square matrix elements, we introduce the spin-polarization tensor as follows,

$$\begin{aligned}\mathcal{T}^{s_j s_j'}_{ab,n+X}(p_a,p_b;\{p\}_{n+X}) = \sum_{\text{spins}\,\neq s_j,s_j'}\sum_{\text{colors}}&\left[\mathcal{M}_{f_a,f_b,f_1,\ldots,f_n}^{c_a,c_b,c_1,\ldots,c_n;s_a,s_b,s_1,\ldots,s_j',\ldots,s_n}(p_a,p_b;\{p\}_{n+X})\right]^*\\ &\times\mathcal{M}_{f_a,f_b,f_1,\ldots,f_n}^{c_a,c_b,c_1,\ldots,c_n;s_a,s_b,s_1,\ldots,s_j,\ldots,s_n}(p_a,p_b;\{p\}_{n+X})\,.\end{aligned} \tag{A.3}$$

Thus, $\mathcal{T}^{s_j s_j'}_{ab,n+X}(p_a,p_b;\{p\}_{n+X})$ is defined similarly to the squared matrix element in eq. (A.2), however the sum over the spin polarizations of parton j is not carried out. Then, products between splitting kernels and squared matrix elements are defined as

$$\hat{P}^{(0)}_{ir}\left|\mathcal{M}^{(0)}_{ab,n+X}(p_a,p_b;\{p\}_{n+X})\right|^2 = \sum_{s_{ir},s_{ir}'}\hat{P}^{(0),s_{ir}s_{ir}'}_{ir}\mathcal{T}^{s_{ir}s_{ir}'}_{ab,n+X}(p_a,p_b;\{p\}_{n+X})\,, \tag{A.4}$$

where s_{ir} and s_{ir}' are the spin polarizations of the parent parton in the $(ir)\to i+r$ splitting. This definition generalizes easily to the case when the squared matrix element is multiplied by two splitting kernels as happens in, e.g., eqs. (6.19) and (7.35). First, the appropriate spin polarization tensor is defined as

$$\begin{aligned}\mathcal{T}^{s_j s_j', s_k s_k'}_{ab,n+X}(p_a,p_b;\{p\}_{n+X}) = \sum_{\substack{\text{spins}\,\neq\\ s_j,s_j',s_k,s_k'}}\sum_{\text{colors}}&\left[\mathcal{M}_{f_a,f_b,f_1,\ldots,f_n}^{c_a,c_b,c_1,\ldots,c_n;s_a,s_b,s_1,\ldots,s_j',\ldots,s_k',\ldots,s_n}(p_a,p_b;\{p\}_{n+X})\right]^*\\ &\times\mathcal{M}_{f_a,f_b,f_1,\ldots,f_n}^{c_a,c_b,c_1,\ldots,c_n;s_a,s_b,s_1,\ldots,s_j,\ldots,s_k,\ldots,s_n}(p_a,p_b;\{p\}_{n+X})\,,\end{aligned} \tag{A.5}$$

i.e., the summations over the spin polarizations of partons j and k are not carried out. Then we have

$$\hat{P}^{(0)}_{ir}\hat{P}^{(0)}_{js}\left|\mathcal{M}^{(0)}_{ab,n+X}(p_a,p_b;\{p\}_{n+X})\right|^2 = \sum_{s_{ir},s_{ir}'}\sum_{s_{js},s_{js}'}\hat{P}^{(0),s_{ir}s_{ir}'}_{ir}\hat{P}^{(0),s_{js}s_{js}'}_{js}\mathcal{T}^{s_{ir}s_{ir}',s_{js}s_{js}'}_{ab,n+X}(p_a,p_b;\{p\}_{n+X})\,, \tag{A.6}$$

where now s_{ir}, s'_{ir} and s_{js}, s'_{js} are the spin polarizations of the parent partons in the $(ir) \to i + r$ and $(js) \to j + s$ splittings.

Turning to the definition of products of color-charge operators with squared matrix elements, we consider first the case of two color-charge operators and define

$$\begin{aligned} \boldsymbol{T}_j \boldsymbol{T}_k \big|\mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X})\big|^2 = \sum_{\text{spins}} \sum_{\text{colors}} \Big[\mathcal{M}^{c_a,c_b,c_1,\dots,c'_j,\dots,c_k,\dots,c_n;s_a,s_b,s_1,\dots,s_n}_{f_a,f_b,f_1,\dots,f_n}(p_a, p_b; \{p\}_{n+X})\Big]^* \\ \times T^c_{c'_j c_j} T^c_{c_k c'_k} \mathcal{M}^{c_a,c_b,c_1,\dots,c_j,\dots,c'_k,\dots,c_n;s_a,s_b,s_1,\dots,s_n}_{f_a,f_b,f_1,\dots,f_n}(p_a, p_b; \{p\}_{n+X}) \,, \end{aligned} \tag{A.7}$$

where summation over repeated color indices is implied and the color-charge matrices are $T^c_{a_j b_j} = i f_{c a_j b_j}$ (the color-charge matrix in the adjoint representation) if parton j is a final-state gluon and $T^c_{\alpha_j \beta_j} = t^c_{\alpha_j \beta_j}$ (the color-charge matrix in the fundamental representation) if parton j is a final-state quark. For a final-state antiquark j, we have $T^c_{\alpha_j \beta_j} = \bar{t}^c_{\alpha_j \beta_j} = -t^c_{\beta_j \alpha_j}$. The color-charge matrices for initial-state partons can be obtained by crossing and we have $T^c_{a_j b_j} = i f_{a_j c b_j}$ for an initial-state gluon j, $T^c_{a_j b_j} = \bar{t}^c_{\alpha_j \beta_j} = -t^c_{\alpha_j \beta_j}$ for an initial-state quark j and $T^c_{a_j b_j} = t^c_{\alpha_j \beta_j}$ for an initial-state antiquark j. We must also consider the product of the squared matrix element with four color-charge operators as in eqs. (6.25) and (7.88). In this case we have

$$\begin{aligned} &\{\boldsymbol{T}_i \boldsymbol{T}_k, \boldsymbol{T}_j \boldsymbol{T}_\ell\} \big|\mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X})\big|^2 \\ &\quad = \sum_{\text{spins}} \sum_{\text{colors}} \Big[\mathcal{M}^{c_a,c_b,c_1,\dots,c'_i,\dots,c'_j,\dots,c_k,\dots,c_\ell,\dots,c_n;s_a,s_b,s_1,\dots,s_n}_{f_a,f_b,f_1,\dots,f_n}(p_a, p_b; \{p\}_{n+X})\Big]^* \Big[T^c_{c'_i c_i} T^c_{c_k c'_k} T^d_{c'_j c_j} T^d_{c_\ell c'_\ell} \\ &\qquad + T^d_{c'_j c_j} T^d_{c_\ell c'_\ell} T^c_{c'_i c_i} T^c_{c_k c'_k}\Big] \mathcal{M}^{c_a,c_b,c_1,\dots,c_i,\dots,c_j,\dots,c'_k,\dots,c'_\ell,\dots,c_n;s_a,s_b,s_1,\dots,s_n}_{f_a,f_b,f_1,\dots,f_n}(p_a, p_b; \{p\}_{n+X}) \,. \end{aligned} \tag{A.8}$$

Once more, repeated color indices are understood to be summed over. We note furthermore that the color-charge algebra for the product $\boldsymbol{T}_j \boldsymbol{T}_k$ is

$$\boldsymbol{T}_j \boldsymbol{T}_k = \boldsymbol{T}_k \boldsymbol{T}_j \,, \quad \text{if } j \neq k \qquad \text{and} \qquad \boldsymbol{T}_j^2 = C_j \,, \tag{A.9}$$

where C_j is the quadratic Casimir operator in the representation of parton j. Using the customary normalization of $T_{\mathrm{R}} = 1/2$, we have $C_j = C_{\mathrm{A}} = N_{\mathrm{c}}$ if j is a gluon, while $C_j = C_{\mathrm{F}} = (N_{\mathrm{c}}^2 - 1)/(2N_{\mathrm{c}})$ if j is a quark or antiquark.

Finally, in order to make contact with the color- and spin-space notation of ref. [18], we note that by introducing a basis $|c_a, c_b, c_1, \dots, c_n\rangle \otimes |s_a, s_b, s_1, \dots, s_n\rangle$ in color- and spin-space, the matrix element in eq. (A.1) can be represented by an abstract vector,

$$\begin{aligned} \big|\mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X})\big\rangle = \Big(|c_a, c_b, c_1, \dots, c_n\rangle \otimes |s_a, s_b, s_1, \dots, s_n\rangle\Big) \\ \times \mathcal{M}^{c_a,c_b,c_1,\dots,c_n;s_a,s_b,s_1,\dots,s_n}_{f_a,f_b,f_1,\dots,f_n}(p_a, p_b; \{p\}_{n+X}) \,. \end{aligned} \tag{A.10}$$

Notice that the flavors on the left-hand side are understood. Moreover, compared to ref. [18] (see their eq. (3.11)), we *do not* include factors of $1/\sqrt{N_{\mathrm{c}}(f)}$ for each initial-state parton of flavor f carrying $N_{\mathrm{c}}(f)$ colors. This corresponds to the normalization adopted in ref. [52]. Using this notation, the squared matrix element is simply

$$\big|\mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X})\big|^2 = \big\langle \mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X}) \big| \mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X}) \big\rangle \,. \tag{A.11}$$

Then, products between splitting functions and squared matrix can be defined as

$$\hat{P}^{(0)}_{ir} \big|\mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X})\big|^2 = \big\langle \mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X}) \big| \hat{P}^{(0)}_{ir} \big| \mathcal{M}^{(0)}_{ab,n+X}(p_a, p_b; \{p\}_{n+X}) \big\rangle \tag{A.12}$$

and

$$\hat{P}_{ir}^{(0)}\hat{P}_{js}^{(0)}\big|\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big|^2 = \big\langle\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big|\hat{P}_{ir}^{(0)}\hat{P}_{js}^{(0)}\big|\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big\rangle\,, \tag{A.13}$$

where the splitting kernels $\hat{P}_{ir}^{(0)}$ and $\hat{P}_{js}^{(0)}$ are matrices acting on the spin indices of the appropriate parent partons. In this notation, their matrix elements i.e., $\hat{P}_{ir}^{(0),s_{ir}s'_{ir}}$ and $\hat{P}_{js}^{(0),s_{js}s'_{js}}$, are written as $\langle s'_{ir}|\hat{P}_{ir}^{(0)}|s_{ir}\rangle$ and $\langle s'_{js}|\hat{P}_{js}^{(0)}|s_{js}\rangle$.[19] Products of color-charge operators and matrix elements may be defined in a similar way. First, if the color-charge operator associated to the emission of a gluon from parton i has color c, we can write

$$\boldsymbol{T}_i = T_i^c|c\rangle\,, \tag{A.14}$$

and the action of this operator in color space is defined as

$$\langle c'_a,c'_b,c'_1,\ldots,c'_i,\ldots,c'_n,c|\boldsymbol{T}_i|c_a,c_b,c_1,\ldots,c_i,\ldots,c_n\rangle = \delta_{c'_a c_a}\delta_{c'_b c_b}\ldots T^c_{c'_i,c_i}\ldots\delta_{c'_n c_n}\,. \tag{A.15}$$

Here the $T^c_{c'_i,c_i}$ are as explained above. Then, products between color-charge operators and matrix elements may be written as

$$\boldsymbol{T}_j\boldsymbol{T}_k\big|\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big|^2 = \big\langle\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big|\boldsymbol{T}_j\boldsymbol{T}_k\big|\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big\rangle\,, \tag{A.16}$$

and

$$\begin{aligned}&\{\boldsymbol{T}_i\boldsymbol{T}_k,\boldsymbol{T}_j\boldsymbol{T}_\ell\}\big|\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big|^2\\&\quad= \big\langle\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big|\{\boldsymbol{T}_i\boldsymbol{T}_k,\boldsymbol{T}_j\boldsymbol{T}_\ell\}\big|\mathcal{M}_{ab,n+X}^{(0)}(p_a,p_b;\{p\}_{n+X})\big\rangle\,,\end{aligned} \tag{A.17}$$

where $\{\boldsymbol{T}_i\boldsymbol{T}_k,\boldsymbol{T}_j\boldsymbol{T}_\ell\}$ denotes the anti-commutator of the products of color-charge operators.

B Momentum mappings

We gather here the definitions of the five elementary momentum mappings we have introduced.

In order to insure that this appendix is self-contained, we recall that throughout s_{jk} denotes twice the dot-product of momenta p_j and p_k. Also, indices in parentheses denote sums of the corresponding momenta so that, e.g.,

$$s_{jk} = 2p_j\cdot p_k\,,\qquad s_{j(k\ell)} = 2p_j\cdot(p_k+p_\ell)\,,\qquad s_{(jk)(\ell m)} = 2(p_j+p_k)\cdot(p_\ell+p_m)\,, \tag{B.1}$$

and so on. Furthermore, $Q^\mu = p_a^\mu + p_b^\mu$ always denotes the total incoming *partonic* momentum. The various momentum mappings are then defined as follows.

- The final-state single collinear (FF) momentum mapping,

$$C_{ir}^{FF}\,:\,(p_a,p_b;\{p\}_{m+X+2}) \xrightarrow{C_{ir}^{FF}} (\hat{p}_a,\hat{p}_b;\{\hat{p}\}_{m+X+1})\,, \tag{B.2}$$

[19]Here the generic symbol s denotes the spin of the parent parton. This notation is appropriate if the parent is a quark or antiquark. However, when the parent is a gluon it is customary to use Greek letters, e.g., μ and ν, to denote helicity indices. We will follow this convention below, see e.g., eqs. (D.1)–(D.4).

is defined as $(i, r \in F)$

$$\begin{aligned}
\hat{p}_a^\mu &= (1-\alpha_{ir}) p_a^\mu \,, \\
\hat{p}_b^\mu &= (1-\alpha_{ir}) p_b^\mu \,, \\
\hat{p}_{ir}^\mu &= p_i^\mu + p_r^\mu - \alpha_{ir}(p_a+p_b)^\mu \,, \\
\hat{p}_n^\mu &= p_n^\mu \,, \qquad n \in F \,, \quad n \neq i, r \,, \\
\hat{p}_X^\mu &= p_X^\mu \,.
\end{aligned} \tag{B.3}$$

The value of α_{ir} is fixed by requiring that the parent momentum, $\hat{p}_{ir}^\mu$, be massless, $\hat{p}_{ir}^2 = 0$, which gives

$$\alpha_{ir} = \frac{1}{2}\left[\frac{s_{(ir)Q}}{s_{ab}} - \sqrt{\frac{s_{(ir)Q}^2}{s_{ab}^2} - \frac{4 s_{ir}}{s_{ab}}}\,\right] . \tag{B.4}$$

- The initial-state single collinear (IF) momentum mapping,

$$C_{ab,r}^{II,F} \,:\, (p_a, p_b; \{p\}_{m+X+2}) \xrightarrow{C_{ab,r}^{II,F}} (\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X+1}) \,, \tag{B.5}$$

is defined as $(a, b \in I$ and $r \in F)$

$$\begin{aligned}
\hat{p}_a^\mu &= \xi_{a,r} p_a^\mu \,, \\
\hat{p}_b^\mu &= \xi_{b,r} p_b^\mu \,, \\
\hat{p}_n^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu \, p_n^\nu \,, \qquad n \in F, \quad n \neq r \,, \\
\hat{p}_X^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu \, p_X^\nu \,.
\end{aligned} \tag{B.6}$$

Here $\Lambda(P,\hat{P})^\mu{}_\nu$ is a proper Lorentz transformation that takes the massive momentum P^μ into a momentum $\hat{P}^\mu$ of the same mass,

$$\Lambda(P,\hat{P})^\mu{}_\nu = g^\mu{}_\nu - 2\frac{(P+\hat{P})^\mu (P+\hat{P})_\nu}{(P+\hat{P})^2} + 2\frac{\hat{P}^\mu P_\nu}{P^2} \,, \tag{B.7}$$

where

$$P^\mu = (p_a+p_b)^\mu - p_r^\mu \,, \qquad\qquad \hat{P}^\mu = (\hat{p}_a + \hat{p}_b)^\mu = (\xi_{a,r} p_a + \xi_{b,r} p_b)^\mu \,. \tag{B.8}$$

The variables $\xi_{a,r}$ and $\xi_{b,r}$ are defined as follows,

$$\xi_{a,r} = \sqrt{\frac{s_{ab}-s_{br}}{s_{ab}-s_{ar}} \frac{s_{ab}-s_{rQ}}{s_{ab}}} \,, \qquad \xi_{b,r} = \sqrt{\frac{s_{ab}-s_{ar}}{s_{ab}-s_{br}} \frac{s_{ab}-s_{rQ}}{s_{ab}}} \,. \tag{B.9}$$

These definitions ensure that indeed $P^2 = \hat{P}^2$.

- The single soft (S) mapping,

$$S_r \,:\, (p_a, p_b; \{p\}_{m+X+2}) \to (\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\}_{m+X+1}) \,, \tag{B.10}$$

is defined as follows $(r \in F)$

$$\begin{aligned}
\tilde{p}_a^\mu &= \lambda_r p_a^\mu \,, \\
\tilde{p}_b^\mu &= \lambda_r p_b^\mu \,, \\
\tilde{p}_n^\mu &= \Lambda(P,\tilde{P})^\mu{}_\nu \, p_n^\nu \,, \qquad n \in F, \quad n \neq r \,, \\
\tilde{p}_X^\mu &= \Lambda(P,\tilde{P})^\mu{}_\nu \, p_X^\nu \,,
\end{aligned} \tag{B.11}$$

where $\Lambda(P,\tilde{P})^\mu{}_\nu$ is the proper Lorentz transformation of eq. (B.7) that takes the massive momentum P^μ into a momentum $\tilde{P}^\mu$ of the same mass, with

$$P^\mu = (p_a + p_b)^\mu - p_r^\mu \qquad \text{and} \qquad \tilde{P}^\mu = (\tilde{p}_a + \tilde{p}_b)^\mu = \lambda_r (p_a + p_b)^\mu \,. \tag{B.12}$$

The value of λ_r is fixed by requiring that $P^2 = \tilde{P}^2$,

$$\lambda_r = \sqrt{1 - \frac{s_{rQ}}{s_{ab}}}\,. \tag{B.13}$$

- The initial-state double collinear momentum mapping,

$$C^{II,FF}_{ab,rs}:\ (p_a, p_b; \{p\}_{m+X+2}) \xrightarrow{C^{II,FF}_{ab,rs}} (\hat{p}_a, \hat{p}_b; \{\hat{p}\}_{m+X})\,, \tag{B.14}$$

is defined as ($a, b \in I$ and $r, s \in F$)

$$\begin{aligned}
\hat{p}_a^\mu &= \xi_{a,rs} p_a^\mu\,,\\
\hat{p}_b^\mu &= \xi_{b,rs} p_b^\mu\,,\\
\hat{p}_n^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu\, p_n^\nu\,, \qquad n \in F, \quad n \neq r\,,\\
\hat{p}_X^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu\, p_X^\nu\,,
\end{aligned} \tag{B.15}$$

where $\Lambda(P,\hat{P})^\mu{}_\nu$ is the proper Lorentz transformation of eq. (B.7) that takes the massive momentum P^μ into a momentum $\hat{P}^\mu$ of the same mass, with

$$P^\mu = (p_a + p_b)^\mu - p_r^\mu - p_s^\mu\,, \qquad \hat{P}^\mu = (\hat{p}_a + \hat{p}_b)^\mu = (\xi_{a,rs} p_a + \xi_{b,rs} p_b)^\mu\,. \tag{B.16}$$

The variables $\xi_{a,rs}$ and $\xi_{b,rs}$ are defined as follows,

$$\xi_{a,rs} = \sqrt{\frac{s_{ab} - s_{b(rs)}}{s_{ab} - s_{a(rs)}} \frac{s_{ab} - s_{(rs)Q} + s_{rs}}{s_{ab}}}\,, \qquad \xi_{b,rs} = \sqrt{\frac{s_{ab} - s_{a(rs)}}{s_{ab} - s_{b(rs)}} \frac{s_{ab} - s_{(rs)Q} + s_{rs}}{s_{ab}}}\,. \tag{B.17}$$

These definitions ensure that indeed $P^2 = \hat{P}^2$.

- The double soft mapping,

$$S_{rs}:\ (p_a, p_b; \{p\}_{m+X+2}) \xrightarrow{S_{rs}} (\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\}_{m+X})\,, \tag{B.18}$$

is defined as ($r, s \in F$),

$$\begin{aligned}
\tilde{p}_a^\mu &= \lambda_{rs} p_a^\mu\,,\\
\tilde{p}_b^\mu &= \lambda_{rs} p_b^\mu\,,\\
\tilde{p}_n^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu\, p_n^\nu\,, \qquad n \in F, \quad n \neq r\,,\\
\tilde{p}_X^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu\, p_X^\nu\,,
\end{aligned} \tag{B.19}$$

where $\Lambda(P,\hat{P})^\mu{}_\nu$ is the proper Lorentz transformation of eq. (B.7) that takes the massive momentum P^μ into a momentum $\tilde{P}^\mu$ of the same mass, where

$$P^\mu = (p_a + p_b)^\mu - p_r^\mu - p_s^\mu\,, \qquad \tilde{P}^\mu = (\tilde{p}_a + \tilde{p}_b)^\mu = \lambda_{rs} (p_a + p_b)^\mu\,. \tag{B.20}$$

The value of λ_{rs} is fixed by requiring that $P^2 = \tilde{P}^2$,

$$\lambda_{rs} = \sqrt{1 - \frac{s_{(rs)Q}}{s_{ab}} + \frac{s_{rs}}{s_{ab}}}\,. \tag{B.21}$$

C Momentum fractions, transverse momenta and eikonal factors

In this appendix we gather the definitions of the various kinematic quantities such as momentum fractions, transverse momenta and eikonal factors that we have introduced.

In order to ensure that this appendix is self-contained, we repeat our conventions for dot-products. Thus, throughout s_{jk} denotes twice the dot-product of momenta p_j and p_k, while indices in parentheses denote sums of the corresponding momenta so that, e.g.,

$$s_{jk} = 2p_j \cdot p_k \,, \qquad s_{j(k\ell)} = 2p_j \cdot (p_k + p_\ell) \,, \qquad s_{(jk)(\ell m)} = 2(p_j + p_k) \cdot (p_\ell + p_m) \,, \tag{C.1}$$

and so on. Moreover, $Q^\mu = p_a^\mu + p_b^\mu$ always denotes the total incoming *partonic* momentum. Then, momentum fractions are defined using the following functions of momenta.

- For *final-state* partons i and r we define the kinematic function $z_{i,r}$ of the momenta p_i^μ and p_r^μ as

$$z_{i,r} = \frac{s_{iQ}}{s_{(ir)Q}} \,, \qquad i, r \in F \,. \tag{C.2}$$

- For an *initial-state* parton a and a *final-state* parton r, we define the kinematic functions

$$x_{r,a} = \frac{s_{rQ}}{s_{aQ}} \,, \quad x_{a,r} = 1 - x_{r,a} = 1 - \frac{s_{rQ}}{s_{aQ}} \,, \qquad a \in I \,, \; r \in F \,. \tag{C.3}$$

- For an *initial-state* parton a and *final-state* partons r and s, we define the kinematic functions

$$x_{r,a} = \frac{s_{rQ}}{s_{aQ}} \,, \quad x_{s,a} = \frac{s_{sQ}}{s_{aQ}} \,, \quad x_{a,rs} = 1 - x_{r,a} - x_{s,a} = 1 - \frac{s_{rQ}}{s_{aQ}} - \frac{s_{sQ}}{s_{aQ}} \,, \qquad a \in I \,, \; r, s \in F \,. \tag{C.4}$$

Next, transverse momenta are defined as follows.

- For *final-state* partons i and r we set

$$k_{\perp i,r}^\mu = \zeta_{i,r} p_r^\mu - \zeta_{r,i} p_i^\mu + Z_{ir} \hat{p}_{ir}^\mu \,, \tag{C.5}$$

  where

$$\zeta_{i,r} = z_{i,r} - \frac{s_{ir}}{\alpha_{ir} s_{(ir)Q}} \,, \quad \zeta_{r,i} = z_{r,i} - \frac{s_{ir}}{\alpha_{ir} s_{(ir)Q}} \,, \quad Z_{ir} = \frac{s_{ir}}{\alpha_{ir} s_{\widehat{ir}Q}} (z_{r,i} - z_{i,r}) \,, \tag{C.6}$$

  with

$$\hat{p}_{ir}^\mu = p_i^\mu + p_r^\mu - \alpha_{ir} (p_a + p_b)^\mu \tag{C.7}$$

  and

$$\alpha_{ir} = \frac{1}{2} \left[\frac{s_{(ir)Q}}{s_{ab}} - \sqrt{\frac{s_{(ir)Q}^2}{s_{ab}^2} - \frac{4 s_{ir}}{s_{ab}}} \right] . \tag{C.8}$$

- For an *initial-state* parton a and a *final-state* parton r we set

$$k_{\perp r,a}^\mu = p_{r,\perp}^\mu \,, \quad k_{\perp a,r}^\mu = -k_{\perp r,a}^\mu = -p_{r,\perp}^\mu \,, \qquad a \in I \,, \; r \in F \,, \tag{C.9}$$

  where the orthogonal direction is defined with respect to the direction of p_a^μ.

- For an *initial-state* parton a and *final-state* partons r and s we set

$$k^{\mu}_{\perp r,a} = p^{\mu}_{r,\perp}\,, \quad k^{\mu}_{\perp s,a} = p^{\mu}_{s,\perp}\,, \quad k^{\mu}_{\perp a,rs} = -k^{\mu}_{\perp r,a} - k^{\mu}_{\perp s,a} = -p^{\mu}_{r,\perp} - p^{\mu}_{s,\perp}\,, \qquad a \in I\,,\ r,s \in F\,, \tag{C.10}$$

  where the orthogonal direction is defined with respect to the direction of p^{μ}_{a}.

Finally, for partons j and k (either initial-state or final-state) and *final-state* parton r, the eikonal factor is defined as

$$\mathcal{S}_{jk}(r) = \frac{2s_{jk}}{s_{jr}s_{kr}}\,, \qquad j,k \in I \cup F\,,\ r \in F\,. \tag{C.11}$$

Recall that when the momenta in the arguments of any of these functions is drawn from a set of mapped momenta, e.g., $(\tilde{p}_a, \tilde{p}_b; \{\tilde{p}\})$, the index inherits the tilde (or corresponding) notation.

D Collinear splitting functions

D.1 Splitting functions

The tree-level Altarelli-Parisi splitting kernels for final-final collinear splitting are [58]

$$\langle s|\hat{P}^{(0)}_{qg}(z, k_{\perp}; \varepsilon)|s'\rangle = C_{\mathrm{F}} \left[\frac{1+z^2}{1-z} - \varepsilon(1-z)\right] \delta_{ss'}\,, \tag{D.1}$$

$$\langle s|\hat{P}^{(0)}_{gq}(z, k_{\perp}; \varepsilon)|s'\rangle = C_{\mathrm{F}} \left[\frac{1+(1-z)^2}{z} - \varepsilon z\right] \delta_{ss'}\,, \tag{D.2}$$

$$\langle \mu|\hat{P}^{(0)}_{q\bar{q}}(z, k_{\perp}; \varepsilon)|\nu\rangle = T_{\mathrm{R}} \left[-g^{\mu\nu} + 4z(1-z)\frac{k^{\mu}_{\perp}k^{\nu}_{\perp}}{k^2_{\perp}}\right]\,, \tag{D.3}$$

$$\langle \mu|\hat{P}^{(0)}_{gg}(z, k_{\perp}; \varepsilon)|\nu\rangle = 2C_{\mathrm{A}} \left[-g^{\mu\nu}\left(\frac{z}{1-z} + \frac{1-z}{z}\right) - 2(1-\varepsilon)z(1-z)\frac{k^{\mu}_{\perp}k^{\nu}_{\perp}}{k^2_{\perp}}\right]\,, \tag{D.4}$$

where we have used the color- and spin-state notation of [18], recalled in appendix A above. To avoid any ambiguity, we note that z here refers to the momentum fraction of the first parton in the subscript. The azimuthally averaged splitting functions are obtained by contracting the splitting kernels with

$$\frac{1}{2}\delta_{ss'} \tag{D.5}$$

for fermion splitting or with

$$\frac{1}{d-2}d_{\mu\nu}(p,n) = \frac{1}{2(1-\varepsilon)}\left(-g_{\mu\nu} + \frac{p^{\mu}n^{\nu} + p^{\nu}n^{\mu}}{p \cdot n}\right) \tag{D.6}$$

for gluon splitting. Here p^{μ} is the on-shell momentum of the parent gluon and n is a reference vector required to fix the polarization vectors of the parent. Given that

$$\sum_{s,s'} \frac{1}{2}\delta_{ss'}\delta_{ss'} = 1 \tag{D.7}$$

and

$$\frac{1}{d-2}d_{\mu\nu}(p,n)(-g^{\mu\nu}) = 1\,, \qquad \frac{1}{d-2}d_{\mu\nu}(p,n)\frac{k^{\mu}_{\perp}k^{\nu}_{\perp}}{k^2_{\perp}} = -\frac{1}{2(1-\varepsilon)}\,, \tag{D.8}$$

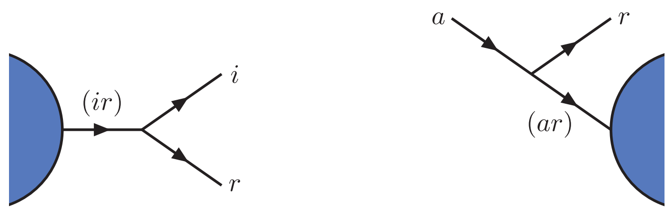

Figure 7: The splitting configurations for $(ir) \to i + r$ final-state splitting (left) and $a \to (ar) + r$ initial-state splitting (right). The arrows in the figure denote momentum and flavor flow.

(we used $p \cdot k_\perp = 0$) we see that the average of the splitting kernel $\hat{P}^{(0)}_{ir}$ over the polarizations of the parent parton can be computed by a very simple prescription. For quark splitting we simply set

$$\delta_{ss'} \to 1\,, \tag{D.9}$$

while for gluon splitting we replace

$$-g^{\mu\nu} \to 1\,, \qquad \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} \to -\frac{1}{2(1-\varepsilon)}\,. \tag{D.10}$$

Thus, the averaged splitting functions $P^{(0)}_{ir}$ read

$$P^{(0)}_{qg}(z;\varepsilon) = C_\mathrm{F}\left[\frac{1+z^2}{1-z} - \varepsilon(1-z)\right], \tag{D.11}$$

$$P^{(0)}_{gq}(z;\varepsilon) = C_\mathrm{F}\left[\frac{1+(1-z)^2}{z} - \varepsilon z\right], \tag{D.12}$$

$$P^{(0)}_{q\bar{q}}(z;\varepsilon) = T_\mathrm{R}\left[1 - \frac{2z(1-z)}{1-\varepsilon}\right], \tag{D.13}$$

$$P^{(0)}_{gg}(z;\varepsilon) = 2C_\mathrm{A}\left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right]. \tag{D.14}$$

The splitting kernels for initial-final collinear splitting can be obtained from eqs. (D.1) and (D.4) by crossing. In order to derive the appropriate rules, consider crossing parton i to the initial state, i.e., $i \to a$. Clearly, the parent parton (ir) is then also crossed to the initial state, $(ir) \to (ar)$, and we have the corresponding reversal of momenta, $p_i^\mu \to -p_a^\mu$ and $p_{ir}^\mu \to -p_{ar}^\mu$. The appropriate splitting configurations and momentum flows were presented in fig. 1 and 2, but are reproduced here in fig. 7 for convenience. Then we seek relations between the momentum fractions and transverse momenta describing the two collinear configurations. These can be obtained by noting that on the one hand, starting from the parametrization relevant to final-final splitting and reversing the momenta of partons i and (ir) gives

$$p_i^\mu = z_i p_{ir}^\mu + k_{\perp i}^\mu + \ldots \xrightarrow{\text{cross}} p_a^\mu = z_i p_{ar}^\mu - k_{\perp i}^\mu + \ldots\,, \tag{D.15}$$

$$p_r^\mu = z_r p_{ir}^\mu + k_{\perp r}^{(F),\mu} + \ldots \xrightarrow{\text{cross}} p_r^\mu = -z_r p_{ar}^\mu + k_{\perp r}^{(F),\mu} + \ldots\,, \tag{D.16}$$

where the $\ldots$ denote terms that vanish faster than $k_\perp^\mu$ in the collinear limit and which do not play a role in the following discussion. On the other hand, the corresponding parametrization in initial-final splitting reads

$$p^{\mu}_{ar} = x_a p^{\mu}_a + k^{\mu}_{\perp a} + \ldots \quad \Rightarrow \quad p^{\mu}_a = \frac{1}{x_a} p^{\mu}_{ar} - \frac{1}{x_a} k^{\mu}_{\perp a} + \ldots\,, \tag{D.17}$$

$$p^{\mu}_{r} = x_r p_a + k^{(I),\mu}_{\perp r} + \ldots \quad \Rightarrow \quad p^{\mu}_r = \frac{x_r}{x_a} p^{\mu}_{ar} - \frac{x_r}{x_a} k^{\mu}_{\perp a} + k^{(I),\mu}_{\perp r} + \ldots\,. \tag{D.18}$$

Notice that we have used the superscripts (F) and (I) to distinguish the transverse momentum for parton r in final-state splitting, $k^{(F),\mu}_{\perp r}$, versus initial-state splitting, $k^{(I),\mu}_{\perp r}$. Then, comparing the $\mathcal{O}(k^0_\perp)$ and $\mathcal{O}(k_\perp)$ pieces of eqs. (D.15) and (D.17) as well as eqs. (D.16) and (D.18), we find

$$z_i \to \frac{1}{x_a}\,, \qquad z_r \to -\frac{x_r}{x_a} \tag{D.19}$$

and

$$k^{\mu}_{\perp i} \to \frac{1}{x_a} k^{\mu}_{\perp a}\,, \qquad k^{(F),\mu}_{\perp r} \to \frac{1}{x_a} k^{(I),\mu}_{\perp r}\,, \tag{D.20}$$

where we used $k^{\mu}_{\perp a} = -k^{(I),\mu}_{\perp r}$. Notice that the relations $z_i + z_r = 1$ and $k^{\mu}_{\perp i} + k^{(F),\mu}_{\perp r} = 0$ are preserved under crossing as expected. Then, comparing eqs. (5.1) and (5.20) and accounting for the signs associated with crossing a fermion from the final to the initial state, we find the relation

$$\hat{P}^{(0)}_{(ar)r}(x, k_\perp; \varepsilon) = -(-1)^{F(a)+F(ar)} x \hat{P}_{\bar{a}r}(1/x, k_\perp/x; \varepsilon)\,, \tag{D.21}$$

where $\bar{a}$ is determined by $a + \bar{a} = g$ using the flavor summation rules of eq. (5.2), while

$$F(q) = F(\bar{q}) = 1\,, \qquad F(g) = 0\,. \tag{D.22}$$

Note that factors which would account for the number of colors and spins of the incoming partons are not included because we define our matrix elements without those factors. Furthermore, the overall factor of $(-x)$ in eq. (D.21) accounts for the difference of kinematic factors that appear in eqs. (5.1) and (5.20). Indeed, the factor of $1/s_{ir}$ in eq. (5.1) becomes $-1/s_{ar}$ under crossing, however in eq. (5.20) we find instead $1/(x_a s_{ar})$. The ratio of these, i.e., $-x_a$, then appears in eq. (D.21). Finally, we note that in eqs. (D.1) and (D.4) the transverse momentum only appears in the form $k^{\mu}_{\perp} k^{\nu}_{\perp}/k^2_{\perp}$, hence the overall scale of $k^{\mu}_{\perp}$ is immaterial. Thus, the crossing rule in eq. (D.20) and consequently eq. (D.21) could be simplified by setting $k^{\mu}_{\perp}/x \to k^{\mu}_{\perp}$ for the initial-state transverse momenta. The splitting functions for initial-final collinear splitting are given explicitly in eqs. (D.23)–(D.26) below.

$$\langle s|\hat{P}^{(0)}_{qg}(x, k_\perp; \varepsilon)|s'\rangle = C_{\mathrm{F}} \left[\frac{1+x^2}{1-x} - \varepsilon(1-x)\right] \delta_{ss'}\,, \tag{D.23}$$

$$\langle \mu|\hat{P}^{(0)}_{gq}(x, k_\perp; \varepsilon)|\nu\rangle = T_{\mathrm{R}} \left[-g^{\mu\nu} x - 4\frac{1-x}{x} \frac{k^{\mu}_{\perp} k^{\nu}_{\perp}}{k^2_{\perp}}\right]\,, \tag{D.24}$$

$$\langle s|\hat{P}^{(0)}_{q\bar{q}}(x, k_\perp; \varepsilon)|s'\rangle = C_{\mathrm{F}} \left[1 - \varepsilon - 2x(1-x)\right] \delta_{ss'}\,, \tag{D.25}$$

$$\langle \mu|\hat{P}^{(0)}_{gg}(x, k_\perp; \varepsilon)|\nu\rangle = 2C_{\mathrm{A}} \left[-g^{\mu\nu}\left(x(1-x) + \frac{x}{1-x}\right) - 2(1-\varepsilon)\frac{1-x}{x} \frac{k^{\mu}_{\perp} k^{\nu}_{\perp}}{k^2_{\perp}}\right]\,. \tag{D.26}$$

Their azimuthally averaged counterparts read

$$P^{(0)}_{qg}(x; \varepsilon) = C_{\mathrm{F}} \left[\frac{1+x^2}{1-x} - \varepsilon(1-x)\right]\,, \tag{D.27}$$

$$P^{(0)}_{gq}(x; \varepsilon) = T_{\mathrm{R}} \left[x + 2\frac{1-x}{(1-\varepsilon)x}\right]\,, \tag{D.28}$$

$$P^{(0)}_{q\bar{q}}(z; \varepsilon) = C_{\mathrm{F}} \left[1 - \varepsilon - 2x(1-x)\right]\,, \tag{D.29}$$

$$P^{(0)}_{gg}(z; \varepsilon) = 2C_{\mathrm{A}} \left[x(1-x) + \frac{x}{1-x} + \frac{1-x}{x}\right]\,. \tag{D.30}$$

These expressions can be derived starting from eqs. (D.23)–(D.26) simply by using the replacements in eqs. (D.9) and (D.10).

Turning to the case of triple parton splitting, we first list the tree-level Altarelli-Parisi splitting functions for the $(irs) \to i + r + s$ final-state splitting. Introducing the notation,

$$s_{irs} = s_{ir} + s_{is} + s_{rs} \qquad \text{and} \qquad t_{jk,l} = 2\frac{z_j s_{kl} - z_k s_{jl}}{z_j + z_k} + \frac{z_j - z_k}{z_j + z_k} s_{jk}\,, \tag{D.31}$$

the splitting function for $q \to q\bar{q}'q'$ is

$$\begin{aligned}
&\langle s|\hat{P}^{(0)}_{q_i\bar{q}'_r q'_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle = \\
&= \frac{1}{2} C_{\mathrm{F}} T_{\mathrm{R}} \frac{s_{irs}}{s_{rs}} \left[-\frac{t^2_{rs,i}}{s_{rs}s_{irs}} + \frac{4z_i + (z_r - z_s)^2}{z_r + z_s} + (1 - 2\varepsilon)\left(z_r + z_s - \frac{s_{rs}}{s_{irs}} \right) \right] \delta_{ss'}\,.
\end{aligned} \tag{D.32}$$

If all quarks have identical flavors, we have

$$\begin{aligned}
&\langle s|\hat{P}^{(0)}_{q_i\bar{q}_r q_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle = \\
&\quad = \langle s|\hat{P}^{(0)}_{q_i\bar{q}'_r q'_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle \\
&\quad + \langle s|\hat{P}^{(0)}_{q_i\bar{q}'_r q'_s}(z_s, z_r, z_i, s_{rs}, s_{is}, s_{ir}, k_{\perp s}, k_{\perp r}, k_{\perp i}; \varepsilon)|s'\rangle \\
&\quad + \langle s|\hat{P}^{(0),(\mathrm{id})}_{q_i\bar{q}_r q_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle\,,
\end{aligned} \tag{D.33}$$

where

$$\begin{aligned}
&\langle s|\hat{P}^{(0),(\mathrm{id})}_{q_i\bar{q}_r q_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle = \\
&\quad = C_{\mathrm{F}} \left(C_{\mathrm{F}} - \frac{C_{\mathrm{A}}}{2} \right) \left\{ (1-\varepsilon)\left(\frac{2s_{is}}{s_{rs}} - \varepsilon \right) \right. \\
&\quad + \frac{s_{irs}}{s_{rs}} \left[\frac{1 + z_r^2}{1 - z_s} - \frac{2z_s}{1 - z_i} - \varepsilon\left(\frac{(1 - z_i)^2}{1 - z_s} + 1 + z_r - \frac{2z_s}{1 - z_i} \right) - \varepsilon^2(1 - z_i) \right] \\
&\quad \left. - \frac{s^2_{irs}}{s_{rs}s_{ir}} \frac{z_r}{2} \left[\frac{1 + z_r^2}{(1 - z_s)(1 - z_i)} - \varepsilon\left(1 + 2\frac{1 - z_s}{1 - z_i} \right) - \varepsilon^2 \right] + (i \leftrightarrow s) \right\}.
\end{aligned} \tag{D.34}$$

For $q \to qgg$ splitting we have

$$\begin{aligned}
&\langle s|\hat{P}^{(0)}_{q_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle = \\
&\quad = C_{\mathrm{F}}^2 \langle s|\hat{P}^{(0),(\mathrm{ab})}_{q_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle \\
&\quad + C_{\mathrm{A}} C_{\mathrm{F}} \langle s|\hat{P}^{(0),(\mathrm{nab})}_{q_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle\,,
\end{aligned} \tag{D.35}$$

where

$$\begin{aligned}
&\langle s|\hat{P}^{(0),(\mathrm{ab})}_{q_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|s'\rangle = \\
&\quad = \left\{ \frac{s^2_{irs}}{2s_{ir}s_{is}} z_i \left[\frac{1 + z_i^2}{z_r z_s} - \varepsilon\frac{z_r^2 + z_s^2}{z_r z_s} - \varepsilon(1 + \varepsilon) \right] \right. \\
&\quad + \frac{s_{irs}}{s_{ir}} \left[\frac{z_i(1 - z_r) + (1 - z_s)^3}{z_r z_s} + \varepsilon^2(1 + z_i) - \varepsilon\left(z_r^2 + z_r z_s + z_s^2 \right) \frac{1 - z_s}{z_r z_s} \right] \\
&\quad \left. + (1 - \varepsilon)\left[\varepsilon - (1 - \varepsilon)\frac{s_{is}}{s_{ir}} \right] \right\} + (r \leftrightarrow s)\,,
\end{aligned} \tag{D.36}$$

and

$$\begin{aligned}
&\langle s|\hat{P}^{(0),(\mathrm{nab})}_{q_i g_r g_s}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs},k_{\perp i},k_{\perp r},k_{\perp s};\varepsilon)|s'\rangle = \\
&\quad = \Bigg\{(1-\varepsilon)\left(\frac{t^2_{rs,i}}{4s^2_{rs}}+\frac{1}{4}-\frac{\varepsilon}{2}\right)+\frac{s^2_{irs}}{2s_{rs}s_{ir}}\Bigg[\frac{(1-z_i)^2(1-\varepsilon)+2z_i}{z_s} \\
&\quad + \frac{z_s^2(1-\varepsilon)+2(1-z_s)}{1-z_i}\Bigg]-\frac{s^2_{irs}}{4s_{ir}s_{is}}z_i\left[\frac{(1-z_i)^2(1-\varepsilon)+2z_i}{z_r z_s}+\varepsilon(1-\varepsilon)\right] \\
&\quad + \frac{s_{irs}}{2s_{rs}}\left[(1-\varepsilon)\frac{z_r(2-2z_r+z_r^2)-z_s(6-6z_s+z_s^2)}{z_s(1-z_i)}+2\varepsilon\frac{z_i(z_r-2z_s)-z_s}{z_s(1-z_i)}\right] \\
&\quad + \frac{s_{irs}}{2s_{ir}}\Bigg[(1-\varepsilon)\frac{(1-z_s)^3+z_i^2-z_s)}{z_s(1-z_i)}-\varepsilon\left(\frac{2(1-z_s)(z_s-z_i)}{z_s(1-z_i)}-z_r+z_s\right) \\
&\quad - \frac{z_i(1-z_r)+(1-z_s)^3}{z_r z_s}+\varepsilon(1-z_s)\left(\frac{z_r^2+z_s^2}{z_r z_s}-\varepsilon\right)\Bigg]\Bigg\}+(r\leftrightarrow s)\,.
\end{aligned} \tag{D.37}$$

For $g \to gq\bar{q}$ splitting we have

$$\begin{aligned}
&\langle \mu|\hat{P}^{(0)}_{g_i q_r \bar{q}_s}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs},k_{\perp i},k_{\perp r},k_{\perp s};\varepsilon)|\nu\rangle = \\
&\quad = C_{\mathrm{F}}T_{\mathrm{R}}\langle \mu|\hat{P}^{(0),(\mathrm{ab})}_{g_i q_r \bar{q}_s}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs},k_{\perp i},k_{\perp r},k_{\perp s};\varepsilon)|\nu\rangle \\
&\quad + C_{\mathrm{A}}T_{\mathrm{R}}\langle \mu|\hat{P}^{(0),(\mathrm{nab})}_{g_i q_r \bar{q}_s}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs},k_{\perp i},k_{\perp r},k_{\perp s};\varepsilon)|\nu\rangle\,,
\end{aligned} \tag{D.38}$$

where

$$\begin{aligned}
&\langle \mu|\hat{P}^{(0),(\mathrm{ab})}_{g_i q_r \bar{q}_s}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs},k_{\perp i},k_{\perp r},k_{\perp s};\varepsilon)|\nu\rangle = \\
&\quad = \Bigg\{-g^{\mu\nu}\left[-2+\frac{2s_{irs}s_{rs}+(1-\varepsilon)(s_{irs}-s_{rs})^2}{s_{ir}s_{is}}\right]+\frac{4s_{irs}}{s_{ir}s_{is}}\Bigg(\big(s_{is}-s_{irs}(1-z_r)\big)z_s\frac{k^\mu_{\perp r}k^\nu_{\perp r}}{k^2_{\perp r}} \\
&\quad + \big(s_{ir}-s_{irs}(1-z_s)\big)z_r\frac{k^\mu_{\perp s}k^\nu_{\perp s}}{k^2_{\perp s}}+s_{rs}\frac{k^\mu_{\perp rs}k^\nu_{\perp rs}}{k^2_{\perp rs}}-(1-\varepsilon)\big(s_{rs}-s_{irs}(1-z_i)\big)z_i\frac{k^\mu_{\perp i}k^\nu_{\perp i}}{k^2_{\perp i}}\Bigg)\Bigg\}\,,
\end{aligned} \tag{D.39}$$

and

$$\begin{aligned}
&\langle \mu|\hat{P}^{(0),(\mathrm{nab})}_{g_i q_r \bar{q}_s}(z_i,z_r,z_s,s_{ir},s_{is},s_{rs},k_{\perp i},k_{\perp r},k_{\perp s};\varepsilon)|\nu\rangle = \\
&\quad = \frac{1}{4}\Bigg\{\frac{s_{irs}}{s^2_{rs}}\left[g^{\mu\nu}\frac{t^2_{rs,i}}{s_{irs}}-16\frac{z_r^2 z_s^2}{z_i(1-z_i)}\left(-\frac{s_{rs}}{z_r z_s}\frac{k^\mu_{\perp rs}k^\nu_{\perp rs}}{k^2_{\perp rs}}\right)\right]+\frac{s_{irs}}{s_{ir}s_{is}}\Bigg[2s_{irs}g^{\mu\nu} \\
&\quad - 4\Bigg(\big(s_{is}-s_{irs}(1-z_r)\big)z_s\frac{k^\mu_{\perp r}k^\nu_{\perp r}}{k^2_{\perp r}}+\big(s_{ir}-s_{irs}(1-z_s)\big)z_r\frac{k^\mu_{\perp s}k^\nu_{\perp s}}{k^2_{\perp s}} \\
&\quad + s_{rs}\frac{k^\mu_{\perp rs}k^\nu_{\perp rs}}{k^2_{\perp rs}}-(1-\varepsilon)\big(s_{rs}-s_{irs}(1-z_i)\big)z_i\frac{k^\mu_{\perp i}k^\nu_{\perp i}}{k^2_{\perp i}}\Bigg)\Bigg] \\
&\quad - g^{\mu\nu}\left[-(1-2\varepsilon)+2\frac{s_{irs}}{s_{ir}}\frac{1-z_s}{z_i(1-z_i)}+2\frac{s_{irs}}{s_{rs}}\frac{1-z_i+2z_i^2}{z_i(1-z_i)}\right] \\
&\quad + \frac{s_{irs}}{s_{ir}s_{rs}}\Bigg[-2s_{irs}g^{\mu\nu}\frac{z_r(1-2z_i)}{z_i(1-z_i)}-16\big(s_{ir}-s_{irs}(1-z_s)\big)z_s\frac{k^\mu_{\perp s}k^\nu_{\perp s}}{k^2_{\perp s}}\frac{z_r^2}{z_i(1-z_i)} \\
&\quad + 8(1-\varepsilon)\big(s_{is}-s_{irs}(1-z_r)\big)z_r\frac{k^\mu_{\perp r}k^\nu_{\perp r}}{k^2_{\perp r}}+4\Bigg(\big(s_{is}-s_{irs}(1-z_r)\big)z_s\frac{k^\mu_{\perp r}k^\nu_{\perp r}}{k^2_{\perp r}} \\
&\quad + \big(s_{ir}-s_{irs}(1-z_s)\big)z_r\frac{k^\mu_{\perp s}k^\nu_{\perp s}}{k^2_{\perp s}}+s_{rs}\frac{k^\mu_{\perp rs}k^\nu_{\perp rs}}{k^2_{\perp rs}}\Bigg)\left(\frac{2z_r(z_s-z_i)}{z_i(1-z_i)}+(1-\varepsilon)\right)\Bigg]\Bigg\}+(r\leftrightarrow s)\,.
\end{aligned} \tag{D.40}$$

Finally for $g \to ggg$ splitting, we have

$$\begin{aligned}
&\langle\mu|\hat{P}^{(0)}_{g_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}, k_{\perp i}, k_{\perp r}, k_{\perp s}; \varepsilon)|\nu\rangle = \\
&\quad = C_{\mathrm{A}}^2 \Bigg\{ \frac{(1-\varepsilon)}{4s_{ir}^2} \left[-g^{\mu\nu} t_{ir,s}^2 + 16 s_{irs} \frac{z_i^2 z_r^2}{z_s(1-z_s)} \left(-\frac{s_{ir}}{z_i z_r} \frac{k_{\perp ir}^\mu k_{\perp ir}^\nu}{k_{\perp ir}^2} \right) \right] \\
&\quad - \frac{3}{4}(1-\varepsilon) g^{\mu\nu} + \frac{s_{irs}}{s_{ir}} g^{\mu\nu} \frac{1}{z_s} \left[\frac{2(1-z_s) + 4z_s^2}{1-z_s} - \frac{1-2z_s(1-z_s)}{z_i(1-z_i)} \right] \\
&\quad + \frac{s_{irs}(1-\varepsilon)}{s_{ir}s_{is}} \Bigg[2z_i \Bigg(\big(s_{is} - s_{irs}(1-z_r)\big) z_r \frac{k_{\perp r}^\mu k_{\perp r}^\nu}{k_{\perp r}^2} \frac{1-2z_s}{z_s(1-z_s)} \\
&\quad + \big(s_{ir} - s_{irs}(1-z_s)\big) z_s \frac{k_{\perp s}^\mu k_{\perp s}^\nu}{k_{\perp s}^2} \frac{1-2z_r}{z_r(1-z_r)} \Bigg) + \frac{s_{irs}}{2(1-\varepsilon)} g^{\mu\nu} \Bigg(\frac{4z_r z_s + 2z_i(1-z_i) - 1}{(1-z_r)(1-z_s)} \\
&\quad - \frac{1-2z_i(1-z_i)}{z_r z_s} \Bigg) + \Bigg(\big(s_{is} - s_{irs}(1-z_r)\big) z_s \frac{k_{\perp r}^\mu k_{\perp r}^\nu}{k_{\perp r}^2} + \big(s_{ir} - s_{irs}(1-z_s)\big) z_r \frac{k_{\perp s}^\mu k_{\perp s}^\nu}{k_{\perp s}^2} \\
&\quad + s_{rs} \frac{k_{\perp rs}^\mu k_{\perp rs}^\nu}{k_{\perp rs}^2} \Bigg) \left(\frac{2z_r(1-z_r)}{z_s(1-z_s)} - 3 \right) \Bigg] \Bigg\} + (5 \text{ permutations})\,.
\end{aligned} \tag{D.41}$$

In eqs. (D.39)–(D.41) we used the notation,

$$k_{\perp jk}^\mu = \frac{k_{\perp j}^\mu}{z_j} - \frac{k_{\perp k}^\mu}{z_k}\,. \tag{D.42}$$

As discussed in 3.2, we prefer to write gluon splitting functions such that any transverse momentum $k_\perp^\mu$ only appears in the combination $k_\perp^\mu k_\perp^\nu / k_\perp^2$. Thus, the gluon splitting functions reported above are obtained from the corresponding expressions in ref. [21] by the following prescription:

1. First, all occurrences of $k_{\perp j}^\mu k_{\perp k}^\nu + k_{\perp k}^\mu k_{\perp j}^\nu$ are rewritten with the help of eq. (D.42) such that only terms of the form $k_{\perp l}^\mu k_{\perp l}^\nu$ $(l = j, k, jk)$ appear,

$$k_{\perp j}^\mu k_{\perp k}^\nu + k_{\perp k}^\mu k_{\perp j}^\nu = \frac{z_k}{z_j} k_{\perp j}^\mu k_{\perp k}^\nu + \frac{z_j}{z_k} k_{\perp k}^\mu k_{\perp j}^\nu - z_j z_k k_{\perp jk}^\mu k_{\perp jk}^\nu\,. \tag{D.43}$$

2. Next, terms of the form $k_{\perp l}^\mu k_{\perp l}^\nu$ $(l = j, k, jk)$ are replaced according to the rules,

$$k_{\perp j}^\mu k_{\perp j}^\nu \to \big(s_{kl} - s_{jkl}(1-z_j)\big) z_j \frac{k_{\perp j}^\mu k_{\perp j}^\nu}{k_{\perp j}^2}\,, \tag{D.44}$$

$$k_{\perp jk}^\mu k_{\perp jk}^\nu \to -\frac{s_{jk}}{z_j z_k} \frac{k_{\perp jk}^\mu k_{\perp jk}^\nu}{k_{\perp jk}^2}\,. \tag{D.45}$$

These rules ensure that the collinear behavior of the triple collinear subtraction can be matched with that of the single collinear subtraction in single unresolved regions.

Splitting functions for initial-state triple collinear splitting can be obtained once more by crossing. In order to derive the appropriate rules, we again consider the crossing of parton i to the initial state, which implies also that the parent parton (irs) is crossed, $i \to a$ and $(irs) \to (ars)$. As before, the corresponding momenta are reversed, $p_i^\mu \to -p_a^\mu$ and $p_{irs}^\mu \to -p_{ars}^\mu$. The appropriate splitting configurations and momentum flows are shown in fig. 8. Then repeating the analysis leading to eqs. (D.19) and (D.20),

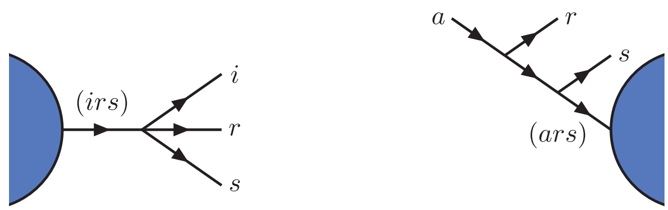

Figure 8: The triple collinear splitting configurations for $(irs) \to i + r + s$ final-state splitting (left) and $a \to (ars) + r + s$ initial-state splitting (right). The arrows in the figure denote momentum and flavor flow.

we write first the crossed form of the final-final-final collinear parametrization,

$$p_i^\mu = z_i p_{irs}^\mu + k_{\perp i}^\mu + \ldots \xrightarrow{\text{cross}} p_a^\mu = z_i p_{ars}^\mu - k_{\perp i}^\mu + \ldots\,, \tag{D.46}$$

$$p_r^\mu = z_r p_{irs}^\mu + k_{\perp r}^{(F),\mu} + \ldots \xrightarrow{\text{cross}} p_r^\mu = -z_r p_{ars}^\mu + k_{\perp r}^{(F),\mu} + \ldots\,, \tag{D.47}$$

$$p_s^\mu = z_s p_{irs}^\mu + k_{\perp s}^{(F),\mu} + \ldots \xrightarrow{\text{cross}} p_s^\mu = -z_s p_{ars}^\mu + k_{\perp s}^{(F),\mu} + \ldots\,. \tag{D.48}$$

The corresponding parametrization for initial-final-final splitting is

$$p_{ars}^\mu = x_a p_a^\mu + k_{\perp a}^\mu + \ldots \;\Rightarrow\; p_a^\mu = \frac{1}{x_a} p_{ars}^\mu - \frac{1}{x_a} k_{\perp a}^\mu + \ldots\,, \tag{D.49}$$

$$p_r^\mu = x_r p_a + k_{\perp r}^{(I),\mu} + \ldots \;\Rightarrow\; p_r^\mu = \frac{x_r}{x_a} p_{ars}^\mu - \frac{x_r}{x_a} k_{\perp a}^\mu + k_{\perp r}^{(I),\mu} + \ldots\,. \tag{D.50}$$

$$p_s^\mu = x_s p_a + k_{\perp s}^{(I),\mu} + \ldots \;\Rightarrow\; p_s^\mu = \frac{x_s}{x_a} p_{ars}^\mu - \frac{x_s}{x_a} k_{\perp a}^\mu + k_{\perp s}^{(I),\mu} + \ldots\,. \tag{D.51}$$

Once more, the superscripts (F) and (I) distinguish transverse momenta in final-state and initial-state splitting. Then, comparing the $\mathcal{O}(k_\perp^0)$ and $\mathcal{O}(k_\perp)$ pieces of eqs. (D.46)–(D.48) and eqs. (D.49)–(D.51), we find

$$z_i \to \frac{1}{x_a}\,, \qquad z_r \to -\frac{x_r}{x_a}\,, \qquad z_s \to -\frac{x_s}{x_a} \tag{D.52}$$

and

$$k_{\perp i}^\mu \to \frac{1}{x_a} k_{\perp a}^\mu\,, \qquad k_{\perp r}^{(F),\mu} \to \frac{1-x_s}{x_a} k_{\perp r}^{(I),\mu} + \frac{x_r}{x_a} k_{\perp s}^{(I),\mu}\,, \qquad k_{\perp s}^{(F),\mu} \to \frac{1-x_r}{x_a} k_{\perp s}^{(I),\mu} + \frac{x_s}{x_a} k_{\perp r}^{(I),\mu}\,, \tag{D.53}$$

where we used $k_{\perp a}^\mu = -k_{\perp r}^{(I),\mu} - k_{\perp s}^{(I),\mu}$. The relations $z_i + z_r + z_s = 1$ and $k_{\perp i}^\mu + k_{\perp r}^{(F),\mu} + k_{\perp s}^{(F),\mu} = 0$ are preserved under crossing as expected. Then, accounting for the signs associated with crossing a fermion from the final to the initial state, we find the relation

$$\begin{aligned} &\hat{P}^{(0)}_{(ars)rs}(x_a, x_r, x_s, s_{ar}, s_{as}, s_{rs}, k_{\perp a}, k_{\perp r}, k_{\perp s}; \varepsilon) = (-1)^{F(a)+F(ars)} x_a \\ &\quad \times \hat{P}_{\bar{a}rs}\left(\frac{1}{x_a}, -\frac{x_r}{x_a}, -\frac{x_s}{x_a}, -s_{ar}, -s_{as}, s_{rs}, \frac{k_{\perp a}}{x_a}, \frac{1-x_s}{x_a} k_{\perp r} + \frac{x_r}{x_a} k_{\perp s}, \frac{1-x_r}{x_a} k_{\perp s} + \frac{x_s}{x_a} k_{\perp r}; \varepsilon\right). \end{aligned} \tag{D.54}$$

Comments similar to those below eq. (D.21) apply to eq. (D.54) as well. In particular, factors which would account for the numbers of colors and spins of the incoming partons are not included. Further, the overall factor of x_a in eq. (D.54) appears because we choose to extract an overall factor of $1/x_a$ in eq. (6.1). Last, since transverse momenta only appear in the combination $k_\perp^\mu k_\perp^\nu / k_\perp^2$, the crossing rule in eq. (D.53) and thus eq. (D.54) could be simplified by dropping the x_a which appear in denominators in the definitions of initial-state transverse momenta.

Finally, we note that the azimuthally averaged triple splitting kernels can be obtained in the same way as before. In fact, the replacement rules of eqs. (D.9) and (D.10) obviously continue to hold also in this case.

D.2 Strongly-ordered splitting functions

Next, we recall the tree-level strongly-ordered final-state splitting functions for $(irs) \to i+(rs) \to i+r+s$ splitting from refs. [10,12]. For quark splitting we have

$$\langle s|\hat{P}^{(\mathrm{C}),(0)}_{q_i\bar{q}'_r q'_s}(z_{r,s}, k_{\perp rs}, \tilde{k}_{\perp rs}, z_{\hat{i},\widehat{rs}}, k_{\perp\hat{i}\widehat{rs}}, \hat{p}_i; \varepsilon)|s'\rangle \tag{D.55}$$
$$= T_\mathrm{R}\left[P^{(0)}_{q_{\hat{i}}g_{\widehat{rs}}}(z_{\hat{i},\widehat{rs}};\varepsilon) - 2C_\mathrm{F} z_{r,s}(1-z_{r,s})\left(1 - z_{\hat{i},\widehat{rs}} - \frac{s^2_{\hat{i}k_{\perp rs}}}{k^2_{\perp rs}s_{\hat{i}\widehat{rs}}}\right)\right]\delta_{ss'}\,,$$

$$\langle s|\hat{P}^{(\mathrm{C}),(0)}_{q_i g_r g_s}(z_{r,s}, k_{\perp rs}, \tilde{k}_{\perp rs}, z_{\hat{i},\widehat{rs}}, k_{\perp\hat{i}\widehat{rs}}, \hat{p}_i; \varepsilon)|s'\rangle \tag{D.56}$$
$$= 2C_\mathrm{A}\left[P^{(0)}_{q_{\hat{i}}g_{\widehat{rs}}}(z_{\hat{i},\widehat{rs}};\varepsilon)\left(\frac{z_{r,s}}{1-z_{r,s}} + \frac{1-z_{r,s}}{z_{r,s}}\right) + C_\mathrm{F}(1-\varepsilon)z_{r,s}(1-z_{r,s})\left(1 - z_{\hat{i},\widehat{rs}} - \frac{s^2_{\hat{i}k_{\perp rs}}}{k^2_{\perp rs}s_{\hat{i}\widehat{rs}}}\right)\right]\delta_{ss'}\,,$$

$$\langle s|\hat{P}^{(\mathrm{C}),(0)}_{g_i q_r g_s}(z_{r,s}, k_{\perp rs}, \tilde{k}_{\perp rs}, z_{\hat{i},\widehat{rs}}, k_{\perp\hat{i}\widehat{rs}}, \hat{p}_i; \varepsilon)|s'\rangle = P^{(0)}_{q_{\widehat{rs}}g_{\hat{i}}}(z_{\widehat{rs},\hat{i}};\varepsilon)P^{(0)}_{q_r g_s}(z_{r,s};\varepsilon)\delta_{ss'}\,. \tag{D.57}$$

Notice that the flavors of the partons involved in the most collinear splitting, i.e., r and s, always appear as the second and third indices in the subscript. We will keep this convention throughout. For gluon splitting we have

$$\langle\mu|\hat{P}^{(\mathrm{C}),(0)}_{\bar{q}_i q_r g_s}(z_{r,s}, k_{\perp rs}, \tilde{k}_{\perp rs}, z_{\hat{i},\widehat{rs}}, k_{\perp\hat{i}\widehat{rs}}, \hat{p}_i; \varepsilon)|\nu\rangle = \langle\mu|\hat{P}^{(0)}_{\bar{q}_{\hat{i}}q_{\widehat{rs}}}(z_{\widehat{rs},\hat{i}}, k_{\perp\widehat{rs}\hat{i}};\varepsilon)|\nu\rangle P^{(0)}_{q_r g_s}(z_{r,s};\varepsilon)\,, \tag{D.58}$$

$$\langle\mu|\hat{P}^{(\mathrm{C}),(0)}_{g_i q_r \bar{q}_s}(z_{r,s}, k_{\perp rs}, \tilde{k}_{\perp rs}, z_{\hat{i},\widehat{rs}}, k_{\perp\hat{i}\widehat{rs}}, \hat{p}_i; \varepsilon)|\nu\rangle \tag{D.59}$$
$$= 2C_\mathrm{A}T_\mathrm{R}\Bigg[-g^{\mu\nu}\left(\frac{z_{\hat{i},\widehat{rs}}}{1-z_{\hat{i},\widehat{rs}}} + \frac{1-z_{\hat{i},\widehat{rs}}}{z_{\hat{i},\widehat{rs}}} + z_{r,s}(1-z_{r,s})\frac{s^2_{\hat{i}k_{\perp rs}}}{k^2_{\perp rs}s_{\hat{i}\widehat{rs}}}\right)$$
$$+ 4z_{r,s}(1-z_{r,s})\frac{1-z_{\hat{i},\widehat{rs}}}{z_{\hat{i},\widehat{rs}}}\frac{\tilde{k}^\mu_{\perp rs}\tilde{k}^\nu_{\perp rs}}{\tilde{k}^2_{\perp rs}}\Bigg] - 4C_\mathrm{A}(1-\varepsilon)z_{\hat{i},\widehat{rs}}(1-z_{\hat{i},\widehat{rs}})P^{(0)}_{q_r\bar{q}_s}(z_{r,s};\varepsilon)\frac{k^\mu_{\perp\hat{i}\widehat{rs}}k^\nu_{\perp\hat{i}\widehat{rs}}}{k^2_{\perp\hat{i}\widehat{rs}}}\,,$$

$$\langle\mu|\hat{P}^{(\mathrm{C}),(0)}_{g_i g_r g_s}(z_{r,s}, k_{\perp rs}, \tilde{k}_{\perp rs}, z_{\hat{i},\widehat{rs}}, k_{\perp\hat{i}\widehat{rs}}, \hat{p}_i; \varepsilon)|\nu\rangle \tag{D.60}$$
$$= 4C^2_\mathrm{A}\Bigg[-g^{\mu\nu}\left(\frac{z_{\hat{i},\widehat{rs}}}{1-z_{\hat{i},\widehat{rs}}} + \frac{1-z_{\hat{i},\widehat{rs}}}{z_{\hat{i},\widehat{rs}}}\right)\left(\frac{z_{r,s}}{1-z_{r,s}} + \frac{1-z_{r,s}}{z_{r,s}}\right)$$
$$+ g^{\mu\nu}z_{r,s}(1-z_{r,s})\frac{1-\varepsilon}{2}\frac{s^2_{\hat{i}k_{\perp rs}}}{k^2_{\perp rs}s_{\hat{i}\widehat{rs}}} - 2(1-\varepsilon)z_{r,s}(1-z_{r,s})\frac{1-z_{\hat{i},\widehat{rs}}}{z_{\hat{i},\widehat{rs}}}\frac{\tilde{k}^\mu_{\perp rs}\tilde{k}^\nu_{\perp rs}}{\tilde{k}^2_{\perp rs}}\Bigg]$$
$$- 4C_\mathrm{A}(1-\varepsilon)z_{\hat{i},\widehat{rs}}(1-z_{\hat{i},\widehat{rs}})P^{(0)}_{g_r g_s}(z_{r,s};\varepsilon)\frac{k^\mu_{\perp\hat{i}\widehat{rs}}k^\nu_{\perp\hat{i}\widehat{rs}}}{k^2_{\perp\hat{i}\widehat{rs}}}\,,$$

where the azimuthally averaged tree-level splitting functions $P^{(0)}$ are given in eqs. (D.11)–(D.14).

The strongly-ordered splitting functions involving initial-state splitting can be obtained from those for final-state splitting, eqs. (D.55)–(D.60), by simple crossing relations. Consider first the case of $a \to a+(rs) \to a+r+s$ splitting. The appropriate strongly-ordered splitting function which appears in eq. (7.1) is obtained by the crossing relation,

$$\hat{P}^{(\mathrm{C}),(0)}_{(ars)rs}(z_r, k_{\perp r}, \hat{k}_{\perp rs}, x_{\hat{a}}, k_{\perp\widehat{rs}}, \hat{p}_a;\varepsilon)$$
$$= -(-1)^{F(a)+F(ars)}x_{\hat{a}}\hat{P}^{(\mathrm{C}),(0)}_{\bar{a}rs}(z_r, k_{\perp r}, \hat{k}_{\perp rs}, 1/x_{\hat{a}}, k_{\perp\widehat{rs}}/x_{\hat{a}}, -\hat{p}_a;\varepsilon)\,, \tag{D.61}$$

where the sign factors $F(a)$ and $F(ars)$ are given by eq. (D.22). Next, the $a \to (as) + r \to a + r + s$ strongly-ordered splitting appearing function appearing in eq. (7.17) is given by the following crossing formula,

$$\begin{aligned} &\hat{P}^{(\mathrm{C}),(0)}_{(ars)(as)s}(x_a, k_{\perp s}, \hat{k}_{\perp s}, x_{\hat{a}}, k_{\perp \hat{r}}, \hat{p}_r; \varepsilon) \\ &\qquad = (-1)^{F(as)+F(ars)} x_a x_{\hat{a}} \hat{P}^{(\mathrm{C}),(0)}_{r\bar{a}s}(1/x_a, k_{\perp s}/x_a, \hat{k}_{\perp s}/x_a, 1/x_{\hat{a}}, k_{\perp \hat{r}}/x_{\hat{a}}, \hat{p}_r; \varepsilon)\,, \end{aligned} \tag{D.62}$$

where the sign factors $F(as)$ and $F(ars)$ are given by eq. (D.22). Factors that would account for the numbers of colors and spins of the incoming partons are not included in eqs. (D.61) and (D.62). We recall that the flavors of the partons involved in the most collinear splitting, i.e., (as) and s, always appear as the second and third index in the subscript.

Finally, we mention that the azimuthally averaged forms of the strongly ordered splitting functions are computed in the same way as their non-strongly ordered counterparts. In particular, one may simply use the replacement rules in eqs. (D.9) and (D.10).

D.3 Soft limit of the splitting functions

Finally, we recall the tree-level soft functions $P^{(\mathrm{S}),(0)}_{irs}$. These represent the single soft $p_s^\mu \to 0$ limit of a set of n collinear final-state partons [32], for $n = 3$ and read [10],

$$P^{(\mathrm{S}),(0)}_{f_i f_r q_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}) = 0\,, \tag{D.63}$$

$$P^{(\mathrm{S}),(0)}_{q_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs},) = C_\mathrm{F} \frac{2}{s_{is}} \frac{z_i}{z_s} + C_\mathrm{A} \left(\frac{s_{ir}}{s_{is} s_{rs}} - \frac{1}{s_{is}} \frac{z_i}{z_s} + \frac{1}{s_{rs}} \frac{z_r}{z_s} \right), \tag{D.64}$$

$$P^{(\mathrm{S}),(0)}_{g_i q_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}) = C_\mathrm{F} \frac{2}{s_{rs}} \frac{z_r}{z_s} + C_\mathrm{A} \left(\frac{s_{ir}}{s_{is} s_{rs}} + \frac{1}{s_{is}} \frac{z_i}{z_s} - \frac{1}{s_{rs}} \frac{z_r}{z_s} \right), \tag{D.65}$$

$$P^{(\mathrm{S}),(0)}_{q_i \bar{q}_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}) = C_\mathrm{F} \frac{2 s_{ir}}{s_{is} s_{rs}} + C_\mathrm{A} \left(-\frac{s_{ir}}{s_{is} s_{rs}} + \frac{1}{s_{is}} \frac{z_i}{z_s} + \frac{1}{s_{rs}} \frac{z_r}{z_s} \right), \tag{D.66}$$

$$P^{(\mathrm{S}),(0)}_{g_i g_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}) = C_\mathrm{A} \left(\frac{s_{ir}}{s_{is} s_{rs}} + \frac{1}{s_{is}} \frac{z_i}{z_s} + \frac{1}{s_{rs}} \frac{z_r}{z_s} \right). \tag{D.67}$$

Notice that the index of the soft parton s appears last in the subscript. We note in passing that eqs. (D.65)–(D.67) can be written in a uniform manner [57],

$$\begin{aligned} &P^{(\mathrm{S}),(0)}_{f_i f_r g_s}(z_i, z_r, z_s, s_{ir}, s_{is}, s_{rs}) \\ &\qquad = (\boldsymbol{T}_i^2 + \boldsymbol{T}_r^2 - \boldsymbol{T}_{ir}^2) \frac{s_{ir}}{s_{is} s_{rs}} + (\boldsymbol{T}_{ir}^2 + \boldsymbol{T}_i^2 - \boldsymbol{T}_r^2) \frac{1}{s_{is}} \frac{z_i}{z_s} + (\boldsymbol{T}_{ir}^2 - \boldsymbol{T}_i^2 + \boldsymbol{T}_r^2) \frac{1}{s_{rs}} \frac{z_r}{z_s}\,. \end{aligned} \tag{D.68}$$

The corresponding soft function for initial-state splitting appears in eq. (7.41) and can be obtained by the following simple crossing relation,

$$P^{(\mathrm{S}),(0)}_{(ar)rs}(x_a, x_r, x_s, s_{ar}, s_{as}, s_{rs}; \varepsilon) = -(-1)^{F(a)+F(ar)} P^{(\mathrm{S}),(0)}_{\bar{a}rs} \left(\frac{1}{x_a}, -\frac{x_r}{x_a}, -\frac{x_s}{x_a}, -s_{ar}, -s_{as}, s_{rs} \right), \tag{D.69}$$

where the sign factors $F(a)$ and $F(ar)$ are given by eq. (D.22) and we have exploited the fact that parton s is a gluon.

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