ANALYTIC DEPENDENCE OF THE LYAPUNOV MOMENT FUNCTION AND THE PROJECTIVE STATIONARY MEASURE FOR RANDOM MATRIX PRODUCTS

C. CHALHOUB, V.P.H. GOVERSE, J.S.W. LAMB, M. RASMUSSEN

Abstract. We consider the product of i.i.d. random matrices sampled according to a probability measure μ supported on a strongly irreducible and proximal subset of a compact set $S \subset GL(d, \mathbb{R})$. We establish the local analyticity of the Lyapunov moment function and the unique stationary measure on the projective space with respect to μ in the total variation topology. As a consequence, we obtain the analyticity of the asymptotic variance and all higher-order Lyapunov moments.

1. Introduction

Products of random matrices form a classical topic in probability theory, ergodic theory, and the study of random dynamical systems. Since the foundational work of Furstenberg and Kesten [10], they have served as a fundamental model for multiplicative stochastic processes. Given i.i.d. matrices $A_n \in GL(d, \mathbb{R})$ with common law μ, the asymptotic growth of $A_\omega^n = A_n \dots A_1$ is governed by the top Lyapunov exponent

$$\lambda_\mu := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_\mu[\ln \|A_\omega^n\|]$$

where $\mathbb{E}_\mu$ is the expectation with respect to $\mathbb{P}_\mu \equiv \mu^{\otimes\mathbb{N}}$.

A major theme in the theory concerns the dependence of Lyapunov exponents on the underlying probability measure. Under some irreducibility conditions, Furstenberg and Kifer [11] established the continuity of $\mu \mapsto \lambda_\mu$ in a specific weak topology on the space of probability measures. This result has been recently improved by Avila and Viana [6], who established continuity in a topology corresponding to weak*-closeness of the distributions and Hausdorff-closeness of their supports.

In the case of positive matrices, Ruelle has proved analytic dependence on the matrix entries [20]. Other works have considered the question of continuity when the matrices depend smoothly on a finite-dimensional parameter [13, 15]. Spectral gap techniques for the associated transfer operator, considered by Le Page [16] and Guivarc'h and Raugi [12], were central to the study of regularity properties of λ_μ, and the authors provided settings where the Lyapunov exponent is differentiable in the finite-dimensional parameter. In the case of a finite number of invertible matrices, Peres [18, 19] proved real-analyticity of the Lyapunov exponent on the open simplex of positive probability vectors. Our work is motivated by a recent result by Amorim, Durães and Melo [1], where the authors use tools from complex analysis on Banach spaces to prove the analyticity of λ_μ in μ with respect to the total variation norm. To extend their work, we use operator perturbation arguments (see Kato [14]) applied to the twisted Koopman operator rather than establishing the analytic variation of observables on the projective space directly.

Since the proof of existence and uniqueness of the Furstenberg stationary measure ([10]) on the projective space under strong irreducibility and proximality conditions, a series of results began to clarify its finer properties including Hölder regularity and convergence in direction. Later results achieved a classification of all stationary probability measures on the projective space for i.i.d. random matrix products with finite first moment. Notable contributions include [12], [2], [3] and [4].

The review above suggests that the stability properties of the top Lyapunov exponent are well-understood. However, little is known regarding other important observables of the system, for instance the asymptotic variance

$$\sigma_\mu^2 \coloneqq \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_\mu[(\ln \|A_\omega^n\| - \lambda_\mu n)^2].$$

The Lyapunov moment function $\Lambda_\mu(q)$ defined below captures the asymptotic information of the system. For instance, $\Lambda'_\mu(0) = \lambda_\mu$ and $\Lambda''_\mu(0) = \sigma_\mu^2$. Our main contribution is establishing analyticity of $\Lambda_\mu(q)$ in μ with respect to the total variation norm. In particular, we prove that all the Lyapunov moments $\Lambda_\mu^{(k)}(0)$ are analytic in μ, which extends the result of [1] to all the higher moments of the Lyapunov moment function, while also providing an alternative proof of the result of [1]. We also show that the stationary probability measure (seen as a bounded linear functional on the projective space) varies analytically in μ.

The results of this paper clarify the picture of the analytic structure of random matrix products and open avenues for further investigation of infinite-dimensional perturbations beyond the linear context for more general random dynamical systems. Indeed many open questions remain on the dependence of invariant manifolds and invariant measures on the underlying noise distribution in that context.

2. Set-up and Statement of the Main Results

Denote by $\mathbb{R}^d$ the d-dimensional Euclidean space and $\mathbb{S}^{d-1}$ the $d-1$-dimensional unit sphere. For $x \in \mathbb{R}^d$, let $\|x\| = \left(\sum_{i=1}^d |x_i|^2\right)^{1/2}$ be the Euclidean norm. Let $GL(d,\mathbb{R})$ denote the space of $d \times d$ invertible matrices with real-valued entries endowed with the matrix operator norm

$$\|A\| = \sup_{x\in\mathbb{S}^{d-1}} \|Ax\|.$$

We endow $GL(d,\mathbb{R})$ with the Borel sigma-algebra. Let $S \subset GL(d,\mathbb{R})$ and fix a probability measure μ on S. Let $\Omega = S^{\mathbb{N}}$ and $\mathcal{F}$ the sigma-algebra generated by the cylinder sets. Let $\mathbb{P}_\mu = \mu^{\otimes\mathbb{N}}$ and $\mathbb{E}_\mu$ the expectation with respect to $\mathbb{P}_\mu$. Each $\omega \in \Omega$ is a sequence of independent matrices $\omega = (\omega_1, \omega_2, \dots)$ sampled according to μ.

We define the product of random matrices on $(\Omega, \mathcal{F}, \mathbb{P})$ at time $n \in \mathbb{N}$ as the random variable $A_\omega^n : \Omega \to GL(d, \mathbb{R})$ given by $A_\omega^n = \omega_n \omega_{n-1} \dots \omega_1$. For $x \in \mathbb{R}^d$, $(A_\omega^n x)_{n \geq 0}$ is a Markov process on $\mathbb{R}^d$.

We define certain characteristics we impose on the support $W \subset GL(d, \mathbb{R})$ of the measure μ from [8].

Definition 2.1. The set W is said to be *proximal* if there exists $n \geq 1$ and a tuple $(A_1, \dots, A_n) \in W^n$ such that the product $\prod_{i=1}^n A_i$ admits an algebraic simple dominant eigenvalue.

Definition 2.2. The set W is said to be *strongly irreducible* if there does not exist a finite family of linear subspaces $\{0\} \subsetneq V_1, \dots, V_k \subsetneq \mathbb{R}^d$ whose union is preserved by every element of W, i.e. such that

$$A(V_1 \cup V_2 \dots \cup V_k) = V_1 \cup V_2 \dots \cup V_k$$

for all $A \in W$.

A probability measure is said to be proximal and strongly irreducible if its support satisfies these conditions. We would like to consider the dynamics of the system on the projective space of directions $\mathbf{P}(\mathbb{R}^d)$ which we associate to the unit sphere $\mathbb{S}^{d-1}$. We define the induced process (θ_n) on $\mathbb{S}^{d-1}$ by $\theta_0 = \frac{x}{\|x\|}$ and $\theta_n = \frac{A_\omega^n(x)}{\|A_\omega^n(x)\|}$. Note that (θ_n) is a Markov process unlike the process $(\|A_\omega^n\|)_n$ that describes the norm of the random matrix product.

A probability measure ν on $\mathbb{S}^{d-1}$ is said to be μ-*invariant* if for every bounded measurable function f on $\mathbb{S}^{d-1}$, we have

$$\int_{\mathbb{S}^{d-1}} f(x)\ d\nu(x) = \int_S \int_{\mathbb{S}^{d-1}} f\left(\frac{Ax}{\|Ax\|}\right)\ d\nu(x) d\mu(A).$$

We define the Lyapunov exponents of the system. The top Lyapunov exponent yields the asymptotic exponential growth rate of the norm of the product.

Definition 2.3. For a matrix $M \in GL(d, \mathbb{R})$, denote by $\tilde{\sigma}_1(M) \geq \tilde{\sigma}_2(M) \geq \dots \geq \tilde{\sigma}_d(M)$ its singular values. Let (A_ω^n) be a product of random matrices satisfying $\int_S \ln^+ \|A\| d\mu(A) < \infty$. The associated Lyapunov exponents $(\lambda_p)_{p=1}^d$ are defined by

$$\lambda_p \coloneqq \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\ln \tilde{\sigma}(A_\omega^n)].$$

In particular, the top Lyapunov exponent is given by

$$\lambda \coloneqq \lambda_1 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\ln \|A_\omega^n\|].$$

Furstenberg and Kesten ([10]) proved the following fundamental result.

Theorem 2.4. *[10] Let μ be an irreducible probability measure on $S \subset GL(d, \mathbb{R})$ such that $\int_S \ln^+ \|A\| d\mu(A) < \infty$. Then for all $x \in \mathbb{S}^{d-1}$, we have*

$$\mathbb{P}_\mu \left[\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \|A_\omega^n x\| \right] = 1.$$

We would like to work in a setup where $\lambda \neq \lambda_2$ as this will impose strong mixing properties on the process (θ_n). We state this formally in the following theorem due to Guivarc'h and Raugi.

Theorem 2.5. *[12] Suppose that the probability measure μ is contracting and strongly irreducible. Then $\lambda > \lambda_2$ and there exists a unique μ-invariant probability measure ν on $\mathbb{S}^{d-1}$.*

In addition to the assumptions of strong irreducibility and proximality, we also impose that the supports of the measures μ are contained in a compact subset S of $GL(d,\mathbb{R})$ so the matrix norms are uniformly bounded, i.e.

$$\eta \coloneqq \sup_{A\in S} \max\{\|A\|, \|A^{-1}\|\} < \infty.$$

The asymptotic variance of the random matrix product $(A^n_\omega x)$ where the matrices are sampled from a measure μ is

$$\sigma^2 \coloneqq \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[(\ln \|A^n_\omega\| - \lambda n)^2].$$

By the central limit theorem for random matrix products , see [8, Theorem V.5.1], since S is proximal, strongly irreducible and compact (more generally the matrices satisfy finite exponential moments), we have $\sigma^2 > 0$.

We also define the Lyapunov Moment function (see [5,7])) of the random matrix product:

$$\Lambda(q) \coloneqq \lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{E}\left[e^{q\ln\|A^n_\omega v\|}\right]\right), \quad \text{for } v \in \mathbb{S}^{d-1}, q\in\mathbb{C} \tag{2.1}$$

where the limit does not depend on v. Global existence has been recently shown in [21]. Computations (see Lemma 3.2 below) show that $\lambda = \Lambda'(0)$ and $\sigma^2 = \Lambda''(0)$. In general, we refer to $\Lambda^{(k)}(0)$ (the k^{th} derivative at zero) as the k^{th} Lyapunov moment of the random matrix product.

Denote by $P(S)$ the metric space of probability measures on $S \subset GL(d,\mathbb{R})$ under the total variation distance

$$\|\mu_1 - \mu_2\|_{TV} = 2\sup_{B\subset S}|\mu_1(B) - \mu_2(B)| = \sup_{f\in B_b(S):||f||_\infty=1}\left|\int_S f(x)d\mu_1(x) - \int_S f(x)d\mu_2(x)\right|$$

where $B_b(S)$ denotes the set of bounded measurable functions on S. We will also consider the Banach space of finite complex measures $M(S)$. In this context, the total variation norm is defined as

$$\|\mu\|_{TV} \coloneqq |\mu|(S) = \sup_{f\in B_b(S):\|f\|_\infty\leq 1}\left|\int_S f(x)d\mu(x)\right|.$$

For a measure $\mu \in M(S)$, we write λ_μ, $\mathbb{P}_\mu$, $\mathbb{E}_\mu$, ν_μ, $\Lambda_\mu(q)$ and σ^2_μ for the induced Lyapunov exponent, product measure, expectation, unique stationary measure, asymptotic variance and Lyapunov moment function respectively.

We can now state the main result of the paper. We prove that the Lyapunov moments of the random matrix product and the stationary probability measure on $\mathbb{S}^{d-1}$ vary analytically in the measure μ.

Theorem 2.6. *Let $S \subset GL(d,\mathbb{R})$ be compact, and let $\mu_0 \in P(S)$ such that the support of μ_0 is strongly irreducible and proximal. Then, there exist open neighbourhoods $U_\mu \subset M(S)$ of μ_0 and $U_q \subset \mathbb{C}$ of 0 such that:*

(1) The map
$$(q,\mu) \mapsto \Lambda_\mu(q)$$
is analytic on $U_q \times U_\mu$.

(2) For all $k \in \mathbb{N}$, the map
$$\mu \mapsto \Lambda_\mu^{(k)}(0)$$
is analytic on U_μ. In particular, $\lambda_\mu = \Lambda'_\mu(0)$ and $\sigma_\mu^2 = \Lambda''_\mu(0)$ are analytic in μ.

(3) The map
$$\mu \mapsto \nu_\mu$$
is analytic on U_μ.

We explain the notion of analyticity used in Theorem 2.6. We prove Fréchet holomorphicity (definition in Section 3) of these maps defined on open subsets of the Banach spaces $\mathbb{C} \times M(S)$ for (1) and $M(S)$ for (2) and (3). The set $M_0(S) = \{\mu \in M(S) : \mu(S) = 0\}$ is a closed linear subspace of $M(S)$, hence also a Banach space. We can then transfer the holomorphic structure of $M_0(S)$ to the affine subspace of complex measures with unit total mass
$$M_1(S) = M_0(S) + \mu_0 = \{\mu \in M(S) : \mu(S) = 1\}$$
since $\mu_0 \in P(S)$ (see [1, Section 2.1]).

In the case of (3), we prove Fréchet holomorphicity of the function $\nu : M(S) \to (\mathcal{C}^\alpha(\mathbb{S}^{d-1}))^*$ (the dual space of the Banach space of $\alpha-$Hölder continuous functions on the unit sphere) that sends a measure μ to the left eigenvector of the transfer operator $\mathcal{P}_{0,\mu}$ defined in Section 3 for the eigenvalue 1, i.e. $\nu_\mu \mathcal{P}_{0,\mu} = \nu_\mu$. Indeed the linear functional ν_μ need not be a measure, it only represents the unique stationary measure on the projective space when μ is a strongly irreducible and proximal probability measure.

We outline the main ideas of the proof. We first start by presenting some tools we need from the theory linear operators on Banach spaces in Section 3. Analyticity of the transfer operator is established in Section 4 and followed by perturbation theory arguments needed for the proof of Theorem 2.6 in Section 5.

3. Preliminaries

For $\alpha \in (0,1)$, let $\mathcal{C}^\alpha(\mathbb{S}^{d-1})$ denote the Banach space of α-Hölder continuous functions $f : \mathbb{S}^{d-1} \to \mathbb{C}$ endowed with the norm $\|f\|_{\mathcal{C}^\alpha} = \|f\|_\infty + [f]_\alpha$ where
$$[f]_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x-y\|^\alpha}.$$

Let $\mathcal{L}(\mathcal{C}^\alpha(\mathbb{S}^{d-1})$ denote the space of bounded linear operators from $\mathcal{C}^\alpha(\mathbb{S}^{d-1})$ to itself and denote by $(\mathcal{C}^\alpha(\mathbb{S}^{d-1}))^*$ its dual space. For $q \in \mathbb{C}$ and $\mu \in P(S)$, one of the operators of interest is the *q-twisted Koopman operator* $\mathcal{P}_{q,\mu} \in \mathcal{L}(\mathcal{C}^\alpha(\mathbb{S}^{d-1}))$ defined by

$$\mathcal{P}_{q,\mu} f(v) = \int_S e^{q \ln \|Av\|} \; f\left(\frac{Av}{\|Av\|}\right) d\mu(A), \quad \text{for } f \in \mathcal{C}^\alpha(\mathbb{S}^{d-1}), v \in \mathbb{S}^{d-1}.$$

For $f \in C^\alpha(\mathbb{S}^{d-1})$ and ν a measure in the dual space, we set $\langle f, \nu \rangle = \int_{\mathbb{S}^{d-1}} f \; d\nu$.

The result below yields the fundamental connection between the Lyapunov moment function and the twisted Koopman operator. It also establishes a spectral gap for the operator, an essential result for transfering analyticity from the operator to the eigenvalues.

Theorem 3.1. *[8, Theorem V.4.3] Let $S \subset GL(d, \mathbb{R})$ and $\mu \in P(S)$ be a strongly irreducible and proximal measure. There exist $\alpha_0 > 0$ such that for all $\alpha < \alpha_0$, there exists $q_0 > 0$ such that for $|q| < q_0$, $\mathcal{P}_{q,\mu} : \mathcal{C}^\alpha(\mathbb{S}^{d-1}) \to \mathcal{C}^\alpha(\mathbb{S}^{d-1})$ is a bounded linear operator with an isolated simple eigenvalue $e^{\Lambda_\mu(q)}$.*

The goal is to establish analyticity of the eigenvalues and eigenfunctions of $\mathcal{P}_{q,\mu}$ in q and μ in a neighbourhood of $(0, \mu_0)$. Since the dominant eigenvalue of this operator at $\mu_0 \in P(S)$ is $e^{\Lambda_\mu(q)}$, immediate consequences of this result (Corollary 5.2 and Corollary 5.3) yield the proof of Theorem 2.6.

We state some properties of the Lyapunov moment function.

Lemma 3.2 ([8]). *Let $S \subset GL(d, \mathbb{R})$ be compact and $\mu \in P(S)$ be a strongly irreducible and proximal measure. Then*

(1) $\Lambda_\mu(q)$ does not depend on $v \in \mathbb{S}^{d-1}$ for $q < |q_0|$,
(2) $\Lambda_\mu(q) \geq \lambda_\mu q$ for $q < |q_0|$,
(3) $\Lambda_\mu(0) = 0$, $\Lambda'_\mu(0) = \lambda_\mu$ and $\Lambda''_\mu(0) = \sigma^2_\mu$,
(4) $\Lambda_\mu(q)$ is convex and analytic on $B(0, q_0)$,

where $B(0, q_0)$ denotes the open ball of radius q_0 in $\mathbb{C}$. Also note that, by [22, Section 2.1], the operator $\mathcal{P}_{q,\mu}$ has a unique probability eigenmeasure $\nu_{q,\mu}$ on $\mathbb{S}^{d-1}$ (the eigenvector of the dual operator $P^*_{q,\mu}$) corresponding to the eigenvalue $e^{\Lambda_\mu(-q)}$. At $q = 0$ this coincides with the Furstenberg stationary measure ν_μ. It need not be a probability measure when $\mu \in M(S)$ is not a probability measure. From now on, when discussing the (dominant) eigenfunction of the Koopman operator, we refer to the unique normalised eigenfunction, with $\langle \nu_{q,\mu}, \phi_{q,\mu} \rangle = 1$.

We state a technical lemma allowing us to control the $\mathcal{C}^\alpha$ norm of the operator. Recall that $\eta = \sup_{A \in S} \max\{\|A\|, \|A^{-1}\|\}$.

Lemma 3.3. *[8, Lemma V.4.2] For all $q \in \mathbb{C}$ and $0 < \alpha \leq 1$, there exist constants $C_1, C_2 > 0$ such that for all $A \in GL(d, \mathbb{R})$, we have*

$$\sup_{x \neq y} \frac{\left|e^{q \ln \|Ax\|} - e^{q \ln \|Ay\|}\right|}{\|x - y\|^\alpha} \leq C_1 \eta^{((1+\alpha)|Re(q)|+2\alpha)}.$$

In order to prove analyticity of the eigenvalues and eigenfunctions, we begin by showing that the operator $\mathcal{P}_{q,\mu}$ is itself jointly analytic in q and μ. To the best of our knowledge, this result is an original perspective on questions of analyticity of the Lyapunov exponent. Following this, we prove a transfer result (Proposition 5.1) inspired by Kato's perturbation theory ([14]) that yields the desired result in our context. This transfer result relies on the operator $\mathcal{P}_{q,\mu}$ exhibiting a spectral gap as stated in Theorem 3.1.

We recall the definition of Fréchet analyticity (holomorphicity) in the context of Banach spaces following [9]. It generalizes the existence of a power series expansion on $\mathbb{C}$. In the following, let E and F be complex Banach spaces and $U \subset E$ an open set.

Definition 3.4. The map $T : U \to F$ is said to be *Fréchet holomorphic* (or Fréchet analytic) at $x_0 \in U$ if there exist $r > 0$ and a sequence of continuous, symmetric n-linear maps $T_n : E^n \to F$ such that

$$T(x) = \sum_{n=0}^{\infty} T_n(x - x_0, \ldots, x - x_0)$$

for all $x \in B(x_0, r)$, where the series above converges in the norm of F. f is said to be Fréchet holomorphic on U if it is Fréchet holomorphic at every $x_0 \in U$.

We typically refer to Fréchet holomorphicity when simply writing holomorphic or analytic in this paper. A weaker form of analyticity is Gâteaux analyticity which can be thought of as analyticity along lines:

Definition 3.5. A map $T : U \to F$ is said to be *Gâteaux holomorphic* (or Gâteaux analytic) if for every $x_0 \in U$ and $v \in E$, the map

$$z \mapsto T(x_0 + zv)$$

is complx holomorphic on $\tilde{U} := \{z \in \mathbb{C} : x_0 + zv \in U\}$.

In practice, a linear operator T_x on a function space X is Fréchet (Gâteaux) holomorphic in x at x_0 if and only if the map $x \mapsto T_x f$ is Fréchet (Gâteaux) holomorphic at x_0 for all $f \in X$ and there exist $c > 0$ and an open neighbourhood W containing x_0 such that

$$\sup_{x \in W} \|T_x\| < c$$

(local boundedness). We show local boundedness for our operator later, so it is sufficient to establish analyticity of $\mathcal{P}_{q,\mu} f$ for any $f \in \mathcal{C}^\alpha(\mathbb{S}^{d-1})$. The following result allows us to verify Fréchet-analyticity through simpler criteria:

Proposition 3.6. *[17, Theorem 8.7] T is Fréchet holormorphic on U if and only if T is Gâteaux holomorphic and continuous on U.*

4. Analyticity of the Twisted Koopman Operator

An important step towards the proof of Theorem 2.6 is showing that the operator $\mathcal{P}_{q,\mu}$ is jointly Fréchet-analytic in both q and μ.

Throughout this section, we implicitly extend the domain μ to the complex setting (the space of probability measures on S is extended to the space of complex finite measures on $M(S)$) in order to use functional analysis tools to establish holomorphicity. It then follows that the restriction to probability measures is real-analytic. See for instance [1, Section 3.2] for a similar approach to analyticity of λ_μ.

Proposition 4.1. *There exists $q_0 > 0$ and an open subset $U \subset M(S)$ containing μ_0 such that the map $(q,\mu) \mapsto \mathcal{P}_{q,\mu}$ is Fréchet-analytic on $B(0,q_0) \times U$.*

The proof of Proposition 4.1 is split into three steps. First, we recall in Lemma 4.2 that $\mathcal{P}_{q,\mu}$ is analytic in q in a neighbourhood of the origin, a known result from [8]. We then proceed to show that the operator is analytic in μ in a neighbourhood of μ_0 via Proposition 3.6. Finally, a result from functional analysis allows us to conclude joint analyticity in q and μ.

Lemma 4.2. *[8, Proposition 4.1] Let μ be a probability measure supported on a subset of a compact set $S \subset GL(d,\mathbb{R})$. For all $\alpha > 0$, there exists $q_0 > 0$ such that the map $T : \mathbb{C} \to L(\mathcal{C}^\alpha(\mathbb{S}^{d-1}))$ given by $T(q) = \mathcal{P}_{q,\mu}$ is analytic on $B(0,q_0)$.*

Note that the result in [8, Proposition 4.1] is more general: it establishes analyticity when the measure μ has finite exponential moments. In our context, since the matrix norms are uniformly bounded, we obtain analyticity for any $\alpha > 0$.

We now move to the second step: proving analyticity in the measure μ. The following lemma establishes continuity of $\mu \mapsto \mathcal{P}_{q,\mu}$ on $M(S)$, the first ingredient of Holomorphicity according to Proposition 3.6.

Lemma 4.3. *For any $q \in \mathbb{C}$, the map $T : M(S) \to L(C^\alpha(\mathbb{S}^{d-1}))$ given by $\mu \mapsto \mathcal{P}_{q,\mu}$ is continuous.*

Proof. The integrand of the operator is denoted by $K_{q,A}f(x) = e^{q\ln\|Ax\|}f(Ax/\|Ax\|)$ for $A \in S, q \in \mathbb{C}, f \in \mathcal{C}^\alpha(\mathbb{S}^{d-1})$. We have

$$|K_{q,A}f(x)| \leq \eta^{|Re(q)|}\|f\|_\infty.$$

Similarly, by Lemma 3.3, for any $x \neq y \in \mathbb{S}^{d-1}$, we have

$$\begin{aligned}
\left|\frac{K_{q,A}f(x) - K_{q,A}f(y)}{\|x-y\|^\alpha}\right| &\leq \left|e^{z\ln\|Ax\|}\right| \frac{|f(Ax/\|Ax\|) - f(Ay/\|Ay\|)|}{\|x-y\|^\alpha} \\
&\quad + |f(Ay/\|Ay\|)| \frac{\left|e^{q\ln\|Ax\|} - e^{q\ln\|Ay\|}\right|}{\|x-y\|^\alpha} \\
&\leq [f]_\alpha \eta^{|Re(q)|} + c\|f\|_\infty \eta^{((1+\alpha)|Re(q)|+2)} \\
&\leq C([f]_\alpha + ||f||_\infty) = C\|f\|_{\mathcal{C}^\alpha}.
\end{aligned}$$

for some $C > 0$ independent of $A \in S$. This means that

$$\sup_{A\in S} \|K_{q,A}\|_{L(\mathcal{C}^\alpha)} \leq C(q,\eta).$$

Let $\mu_1, \mu_2 \in M(S)$. For any $f \in \mathcal{C}^\alpha(\mathbb{S}^{d-1})$, we have

$$\mathcal{P}_{q,\mu_1} f - \mathcal{P}_{q,\mu_2} f = \int_S K_{q,A} f \ (\mu_1 - \mu_2) d(A).$$

Taking the $\mathcal{C}^\alpha$ norm, we obtain

$$\|\mathcal{P}_{q,\mu_1} f - \mathcal{P}_{q,\mu_2} f\|_{\mathcal{C}^\alpha} \leq \int_S \|K_{q,A} f\|_{C^\alpha} \ (\mu_1 - \mu_2) d(A) \leq \sup_{A \in S} \|K_{q,A} f\|_{C^\alpha} \|\mu_1 - \mu_2\|_{TV}$$

and hence,

$$\|\mathcal{P}_{q,\mu_1} - \mathcal{P}_{q,\mu_2}\|_{L(\mathcal{C}^\alpha)} \leq C(q, \eta) \|\mu_1 - \mu_2\|_{TV}$$

which proves continuity on $M(S)$.

□

We can now prove analyticity in the measure, and then joint analyticity in q and μ.

Proof of Proposition 4.1. Let $B \subset M(S)$ be an open neighbourhood and $f \in C^\alpha(\mathbb{S}^{d-1})$. In order to prove that the map $T_f : B \to C^\alpha(\mathbb{S}^{d-1})$ given by $T_f(\mu) = \mathcal{P}_{q,\mu} f$ is Gâteaux holomorphic, we need to show that for every $\mu \in B$ and $\nu \in M_0(S)$, the map $z \mapsto T(\mu + z\nu)$ is holomorphic on $\{z \in \mathbb{C} : \mu + z\nu \in B\}$. It is thus immediately clear that T_f is Gâteaux holomorphic on any open neighbourhood of $M(S)$ as the map described above is affine in z by linearity of integrals. Since this is true for any function f, the operator $P_{q,\mu}$ is Gâteaux holomorphic in μ on B.

Now, by Proposition 3.6, T is holomorphic on an open subset U of a Banach space if and only if T is Gâteaux holomorphic and continuous on U. Hence, combining the above with Lemma 4.3, we obtain that $\mu \mapsto \mathcal{P}_{q,\mu}$ is holomorphic on any open subset of $M(S)$.

We have now established that the Koopman operator is separately analytic in both q (Lemma 4.2) and μ. It remains to prove that it is jointly analytic in both. From the proof of Lemma 4.3, it is straightforward to see that the Koopman operator is locally bounded in both q and μ in the sense that for any open ball $B \subset M(S) \times \mathbb{C}$, we have

$$\sup_{(q,\mu) \in B} \|\mathcal{P}_{q,\mu}\|_{L(C^\alpha)} < \infty.$$

By [17, Lemma 8.10], for any Banach spaces E_1, E_2 and F and for A an open subset of $E_1 \times E_2$, if a map $f : A \to F$ is separately holomorphic and locally bounded, then f is holomorphic on A. This proves that the map $(q, \mu) \mapsto P_{q,\mu}$ is analytic on $U \times B(0, q_0)$ for any open set $U \subset M(S)$.

□

5. Proof of Theorem 2.6

We now require a result to transfer Fréchet analyticity from the operator to its leading eigenvalue $e^{\Lambda_\mu(q)}$ and its normalized eigenfunctions. The proposition below is a variation of [14, Theorem VII.1.7] from Kato's *Perturbation Theory for Linear Operators*: we restrict the setting to the existence of a simple dominant eigenvalue, but we allow the index of the operator to be an element of a complex Banach space rather than $\mathbb{C}^n$. For a complex Banach space X, endow the space of linear operators $\mathcal{L}(X)$ from X to itself with the operator norm induced from X.

Proposition 5.1. *Let X be a complex Banach space, $U \subset X$ an open set, and Y a Banach space. Suppose that $T : U \to \mathcal{L}(Y)$ (endowed with the operator norm) is Fréchet-holomorphic. Fix $x_0 \in X$ such that the operator $T(x_0)$ has a simple isolated eigenvalue l_0. Then, there exists a open neighbourhood $V \subset X$ of x_0 and holomorphic maps $l : V \to \mathbb{C}$ and $h : V \to Y \setminus \{0\}$ such that $l(0) = l_0$ and*

$$T(x)h(x) = l(x)h(x) \quad \text{for all } x \in V.$$

Proof. For any operator T, denote by $\sigma(T)$ and $\rho(T)$ the spectrum and resolvent set of T respectively. Since l_0 is an isolated eigenvalue, we have $\delta = d(l_0, \sigma(T(x_0)) \setminus \{l_0\}) > 0$. Pick $r < \delta/2$ and define the closed curve

$$\Gamma = \{\xi \in \mathbb{C} : |\xi - l_0| = r\}.$$

It is clear that $\Gamma \subset \rho(T(x_0))$. Consider the resolvent operator $\tilde{T}_\xi(x) = (\xi I - T(x))^{-1} \in \mathcal{L}(Y)$ for $\xi \in \Gamma$. It is known that the map $\xi \mapsto \tilde{T}_\xi(x_0)$ is holomorphic on $\rho(T(x_0))$. This means the map is also continuous, and since the curve Γ is compact, we have

$$M_\Gamma = \sup_{\xi \in \Gamma} \|\tilde{T}_\xi(x_0)\| < \infty.$$

By continuity of $x \mapsto T(x)$, there exists an open neighbourhood $V \subset U$ of x_0 such that for all $x \in V$, $\|T(x) - T(x_0)\| < \frac{1}{2M_\Gamma}$. Define the operator $K : \Gamma \times V \to \mathcal{L}(Y)$ by

$$K(\xi, x) = \tilde{T}_\xi(x_0)\ (T(x) - T(x_0))$$

for all $x \in V, \xi \in \Gamma$. Then we can write $\xi I - T(x) = (\xi I - T(x_0))(I - K(\xi, x))$. Notice that

$$\|K(\xi, x)\| \le \|\tilde{T}_\xi(x_0)\|\ \|T(x) - T(x_0)\| < M_\Gamma (2M_\Gamma)^{-1} = \frac{1}{2}.$$

This implies that $I - K(\xi, x)$ is invertible with $(I - K)^{-1} = \sum_n K^n$ converging uniformly in $\xi \in \Gamma$. Hence, $(\xi I - T(x))$ is invertible and $\Gamma \subset \rho(T(x))$ for all $x \in V$. Practically this means that Γ isolates a part of the spectrum in its interior for all the operators $T(x)$ with x close enough to x_0. We also obtain a uniform bound in x and ξ on the resolvent along the curve:

$$\|\tilde{T}_\xi(x)\| \le \frac{1}{1 - \|K(\xi, x)\|} \|\tilde{T}_\xi(x_0)\| \le 2M_\Gamma \tag{5.1}$$

Now consider the Riesz projection associated with the part of the spectrum of $T(x)$ the lies within the interior of the curve Γ: it is defined by the integral

$$P(x) = \frac{1}{2\pi i} \int_\Gamma \tilde{T}_\xi(x) d\xi \quad \text{for all } x \in V.$$

For every $x \in V$, $P(x)$ is a linear operator whose range is the direct sum of the generalised eigenspaces corresponding to the eigenvalues of $T(x)$ that lie inside of Γ ([14]).

We claim the map $P : V \to \mathcal{L}(Y)$ if Fréchet-holomorphic. We prove this using Proposition 3.6: we show the operator is continuous on V and we establish Gâteaux holomorphicity, the combination of which proves the claim. We start by proving holomorphicity along lines (Gâteaux holomorphicity), i.e. for any $h \in E$, the map $z \mapsto P(x+zh)$ is analytic for $z \in \mathbb{C}$ small enough. Since $x \mapsto T(x)$ is Fréchet holomorphic on V, the map $x \mapsto \tilde{T}_\xi(x)$ is also Fréchet holomorphic by [14, Theorem VII.1.3] hence Gâteaux holomorphic. This means that $z \mapsto \tilde{T}_\xi(x+zh)$ is holomorphic on $B(0,R)$, where R is such that $B(0,R) \subset \{z \in \mathbb{C} : x+zh \in V\}$. By [17, Corollary 7.2] there exists a sequence $(c_n(\xi))_{n\in\mathbb{N}} \subset \mathcal{L}(Y)$ such that

$$\tilde{T}_\xi(x+zh) = \sum_{n=0}^{\infty} z^n c_n(\xi)$$

with the series converging absolutely in operator norm for $|z| \leq s$ for $0 < s < R$, and

$$c_n(\xi) = \frac{1}{2\pi i}\int_{|u|=R} \frac{\tilde{T}_\xi(x+zh)}{u^{n+1}} du.$$

We can thus integrate termwise: $P(x+zh) = \sum_n z^n B_n$ where

$$B_n = \frac{1}{2\pi i}\int_\Gamma c_n(\xi) d\xi \in \mathcal{L}(Y).$$

This series also converges absolutely in operator norm since, by (5.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \|B_n\| |z|^n &\leq \frac{\text{length}(\Gamma)}{2\pi} \sum_{n=0}^{\infty} |z|^n \sup_{\xi\in\Gamma} \|c_n(\xi)\| \\
&\leq \frac{r}{2\pi} \sum_{n=0}^{\infty} |z|^n \sup_{\xi\in\Gamma} \int_{|u|=R} \frac{\|\tilde{T}_\xi(x+zh)\|}{u^{n+1}} du \\
&\leq \frac{rM_\Gamma}{\pi} \sum_{n=0}^{\infty} \left(\frac{|z|}{R}\right)^n < \infty
\end{aligned}$$

since $|z| < R$. This proves that $P(x)$ is Gâteaux holomorphic.

By (5.1), we have

$$\sup_{x\in V} \|P(x)\| \leq \sup_{x\in V} \frac{1}{2\pi}\int_\Gamma \|\tilde{T}_\xi(x)\| d\xi \leq \frac{\text{length}(\Gamma)}{2\pi} \sup_{x\in V, \xi\in\Gamma} \|\tilde{T}_\xi(x)\| \leq 2rM_\Gamma < \infty$$

which establishes boundedness in $x \in V$. By [17, Proposition 8.6], a Gâteaux-holomorphic mapping is continuous if and only if it is locally bounded. Hence $P(x)$ is continuous in x. Together with Gâteaux holomorphicity we conclude that P is Fréchet holomorphic on V.

Since l_0 is a simple eigenvalue of $T(x_0)$ and P projects onto the eigenspace corresponding to the eigenvalue l_0 operator, $\text{Rank}P(x_0) = 1$. By continuity of $x \mapsto P(x)$, so is the rank of the operator, and since the rank can only take integer value, this means that $\text{Rank}P(x) = 1$ on $x \in V$. This means that the eigenvalues $l(x)$ of $T(x)$ isolated within Γ are also simple with a one-dimensional eigenspaces.

In order to construct an eigenfunction, choose an arbitrary $u \in Y$ such that $P(x_0)u \neq 0$ and set $h(x) = P(x)u$. Since the eigenspace is one-dimensional, $\text{Range}P(x) = \mathbb{C}h(x)$ corresponds to the eigenspace of $l(x)$. Since $x \mapsto P(x)$ is holomorphic, so is $h(x)$ on V. We conclude all the eigenfunctions are analytic on V. For $h(x) \neq 0$, the eigenvalue can be expressed as

$$l(x) = \frac{T(x)h(x)}{h(x)} \in \mathbb{C}.$$

Hence the eigenvalue $l(x)$ is analytic in x on V. This completes the proof. $\square$

The following corollary finalises the proof of Theorem 2.6 (1)-(2).

Corollary 5.2. *Let $S \subset GL(d, \mathbb{R})$ be compact, and let $\mu_0 \in P(S)$ such that the support of μ_0 is strongly irreducible and proximal. There exists an open neighbourhood $V \subset \mathbb{C} \times M(S)$ of $(0, \mu_0)$ and an open neighbourhood $U \subset M(S)$ of μ_0 such that the map $(q, \mu) \mapsto \Lambda_\mu(q)$ is analytic on V and the map $\mu \mapsto \Lambda_\mu^{(k)}(0)$ is analytic on U for all $k \in \mathbb{N}$.*

Proof. Set $X = M(S) \times \mathbb{C}$ (product of two Banach spaces is a Banach space under the direct product norm), $Y = C^\alpha(\mathbb{S}^{d-1})$, $x_0 = (\mu_0, 0)$ and $l_0 = e^{\Lambda_{\mu_0}(0)}$ which by Lemma 4.2 is a dominant simple eigenvlue of P_{0,μ_0}. The operator is jointly Fréchet analytic in μ and q in a neighbourhood of $(\mu_0, 0)$ by Proposition 4.1. Hence, by Proposition 5.1, there exists an open neighbourhood V of $(\mu_0, 0)$ such that the eigenvalue $l(\mu, q) = e^{\Lambda_\mu(q)}$ and its normalized eigenfunction $\phi_{q,\mu}$ are jointly analytic in q and μ on V.

This implies that $\Lambda_\mu(q)$ is analytic in both variables on V. In particular, the map $\mu \mapsto \Lambda_\mu^{(k)}(0)$ is analytic on some open neighbourhood $U \subset M(S)$ of μ_0 for all $k \in \mathbb{N}$. We thus recover the result by [1] about $\lambda_\mu = \Lambda'_\mu(0)$ being analytic in the measure, in addition to the asymptotic variance $\sigma_\mu^2 = \Lambda''_\mu(0)$ being analytic on U. $\square$

We prove Theorem 2.6 (3) in the next corollary by applying the result of Proposition 5.1 to the dual operator $\mathcal{P}^*_{0,\mu} \in \mathcal{L}(\mathcal{C}^\alpha(\mathbb{S}^{d-1})^*)$. ν_μ is the eigenvector of $\mathcal{P}^*_{0,\mu}$ at eigenvalue $e^{\Lambda_\mu(0)} = e^0 = 1$ representing the unique invariant measure on $\mathbb{S}^{d-1}$ of the random matrix product when μ is a strongly irreducible and proximal probability measure. In this context we have

$$\nu_\mu(f) = \langle f, \nu \rangle = \int_{\mathbb{S}^{d-1}} f(x)\, d\nu_\mu(x).$$

Corollary 5.3. *Let $S \subset GL(d, \mathbb{R})$ be compact, and let $\mu_0 \in P(S)$ such that the support of μ_0 is strongly irreducible and proximal. There exists an open neighbourhood $U \subset M(S)$ of μ_0 such that the map $\mu \mapsto \nu_\mu$ is analytic on U.*

Proof. We begin by showing that the map T^* given by $T^*(\mu) = \mathcal{P}^\star_{0,\mu}$ is Fréchet-analytic on some open neighbourhood of μ_0. The map T given by $T(\mu) = \mathcal{P}_{0,\mu}$ is Fréchet analytic on some open neighbourhood $U \subset M(S)$ of μ_0 by Proposition 4.1. The map $D : \mathcal{L}(\mathcal{C}^\alpha(\mathbb{S}^{d-1})) \to \mathcal{L}(\mathcal{C}^\alpha(\mathbb{S}^{d-1})^*)$ given by $D(\mathcal{P}) = \mathcal{P}^*$ is a linear isometry, in particular it is continuous. Since the composition of a continuous linear map and a Fréchet analytic map is analytic (see for example [17, Exercise 5.A]), the map $T^* = D \circ T$ is Fréchet analytic on U.

By [15, Theorem III.6.22], if 1 is a simple isolated eigenvalue of $\mathcal{P}_{0,\mu_0}$, 1 is a simple isolated eigenvalue of $\mathcal{P}^*_{0,\mu_0}$. Set $X = M(S)$, $Y = \mathcal{C}^\alpha(\mathbb{S}^{d-1})^*$ endowed with the dual norm

$$\|\nu\|_{\mathcal{C}^\alpha(\mathbb{S}^{d-1})^*} = \sup\{|\nu(f)| : ||f||_{\mathcal{C}^\alpha} \leq 1, f \in \mathcal{C}^\alpha(\mathbb{S}^{d-1})\},$$

$x_0 = \mu_0$ and $l_0 = 1$. By proposition 5.1, there exists an open neighbourhood $U' \subset M(S)$ of μ_0 such that the eigenvector ν_μ is Fréchet analytic on U'. □

Acknowledgements. The authors gratefully acknowledge useful discussions with Artur Amorim and Richard Aoun. CC, VPHG and JSWL have been supported by the EPSRC Centre for Doctoral Training in Mathematics of Random Systems: Analysis, Modelling and Simulation (EP/S023925/1). VPHG thanks the Mathematical Institute of Leiden University for their hospitality. JSWL has been supported by the EPSRC (EP/Y020669/1) and thanks JST (Moonshot R & D Grant Number JPMJMS2021 - IRCN, University of Tokyo) and GUST (Kuwait) for their research support.

References

[1] A. Amorim, M. Durães, and A. Melo. Analiticity of the lyapunov exponents of random products of matrices. *arXiv preprint arXiv:2501.19286*, 2025.
[2] Richard Aoun and Yves Guivarc'h. Random matrix products when the top Lyapunov exponent is simple. *J. Eur. Math. Soc. (JEMS)*, 22(7):2135–2182, 2020.
[3] Richard Aoun and Cagri Sert. Stationary probability measures on projective spaces 1: block-Lyapunov dominated systems. *Math. Ann.*, 388(3):2573–2610, 2024.
[4] Richard Aoun and Cagri Sert. Stationary probability measures on projective spaces 2: the critical case. *Duke Math. J.*, 174(7):1407–1430, 2025.
[5] L. Arnold. A formula connecting sample and moment stability of linear stochastic systems. *SIAM J. Appl. Math.*, 44(4):793–802, 1984.
[6] A. Avila, A. Eskin, and M. Viana. Continuity of the lyapunov exponents of random matrix products. *arXiv preprint arXiv:2305.06009*, 2023.
[7] P. H. Baxendale and R. Z. Khasminskii. Stability index for products of random transformations. *Adv. in Appl. Probab.*, 30(4):968–988, 1998.
[8] Philippe Bougerol and Jean Lacroix. *Products of random matrices with applications to Schrödinger operators*, volume 8 of *Progress in Probability and Statistics*. Birkhäuser Boston, Inc., Boston, MA, 1985.
[9] S. B. Chae. *Holomorphy and Calculus in Normed Spaces*. CRC Press, 2020.
[10] H. Furstenberg and H. Kesten. Products of random matrices. *Ann. Math. Statist.*, 31:457–469, 1960.
[11] H. Furstenberg and Y. Kifer. Random matrix products and measures on projective spaces. *Israel J. Math.*, 46(1-2):12–32, 1983.
[12] Y. Guivarc'h and A. Raugi. Frontiere de furstenberg, propriétés de contraction et théoremes de convergence. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 69:187–242, 1985.
[13] H. Hennion. Loi des grands nombres et perturbations pour des produits réductibles de matrices aléatoires indépendantes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 67(3):265–278, 1984.
[14] T. Kato. *Perturbation theory for linear operators*, volume Band 132 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag New York, Inc., New York, 1966.
[15] Y. Kifer. Perturbations of random matrix products. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 61(1):83–95, 1982.

[16] É. Le Page. Régularité du plus grand exposant caractéristique des produits de matrices aléatoires indépendantes et applications. In *Annales de l'IHP Probabilités et statistiques*, volume 25, pages 109–142, 1989.

[17] Jorge Mujica. *Complex analysis in Banach spaces*, volume 120 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Holomorphic functions and domains of holomorphy in finite and infinite dimensions, Notas de Matemática, 107. [Mathematical Notes].

[18] Y. Peres. Analytic dependence of Lyapunov exponents on transition probabilities. In *Lyapunov exponents (Oberwolfach, 1990)*, volume 1486 of *Lecture Notes in Math.*, pages 64–80. Springer, Berlin, 1991.

[19] Yuval Peres. Domains of analytic continuation for the top Lyapunov exponent. *Ann. Inst. H. Poincaré Probab. Statist.*, 28(1):131–148, 1992.

[20] D. Ruelle. Analycity properties of the characteristic exponents of random matrix products. *Adv. in Math.*, 32(1):68–80, 1979.

[21] Cagri Sert. Large deviation principle for random matrix products. *Ann. Probab.*, 47(3):1335–1377, 2019.

[22] H. Xiao, I. Grama, and Q. Liu. Precise large deviation asymptotics for products of random matrices. *Stochastic Processes and their Applications*, 130(9):5213–5242, 2020.

C. Chalhoub, Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom

Email address: christopher.chalhoub21@imperial.ac.uk

V.P.H. Goverse, Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom

Email address: vincent.goverse21@imperial.ac.uk

J.S.W. Lamb, Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom

International Research Center for Neurointelligence (IRCN), The University of Tokyo, Tokyo 113-0033, Japan

Centre for Applied Mathematics and Bioinformatics, Department of Mathematics and Natural Sciences, Gulf University for Science and Technology, Halwally, 32093 Kuwait.

Email address: jsw.lamb@imperial.ac.uk

M. Rasmussen, Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom

Email address: m.rasmussen@imperial.ac.uk