

Iterative Contact-resolving Hybrid Methods for Multiscale Contact Mechanics

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Abstract

Modeling contact mechanics with high contrast coefficients presents significant mathematical and computational challenges, especially in achieving strongly symmetric stress approximations. Due to the inherent nonlinearity of contact problems, conventional methods that treat the entire domain as a monolithic system often lead to high global complexity. To address this, we develop an iterative contact-resolving hybrid method by localizing nonlinear contact constraints within a smaller subdomain, while the larger subdomain is governed by a linear system. Our system employs variational inequality theory, minimization principles, and penalty methods. More importantly, we propose four discretization types within the two-subdomain framework, ranging from applying standard/mixed FEM across the entire domain to combining standard/mixed multiscale methods in the larger subdomain with standard/mixed FEM in the smaller one. By employing a multiscale reduction technique, the method avoids excessive degrees of freedom inherent in conventional methods in the larger domain, while the mixed formulation enables direct stress computation, ensures local momentum conservation, and resists locking in nearly incompressible materials. Convergence analysis and the corresponding algorithms are provided for all cases. Extensive numerical experiments are presented to validate the effectiveness of the approaches.

Keywords— contact mechanics, high contrast coefficients, multiscale method, mixed formulation

1 Introduction

Contact mechanics with high contrast coefficients arise in numerous engineering and geophysical applications where materials with significantly different mechanical properties interact. Typical examples include rubber seals pressing against metal surfaces, tire-road contact, and geological faults between dissimilar rock strata. Such problems are characterized by a large disparity in material parameters (e.g., Young's modulus, Lamé coefficients) across the contacting interfaces, leading to complex, localized deformation patterns and challenging numerical simulation. The high contrast coefficients coupled with nonlinear contact conditions poses significant challenges, including solution ill-conditioning and boundary layers, which require robust discretization and solution techniques. Kikuchi and Oden [26] established a foundational framework for elasticity contact problems using variational inequalities. Computational methods essential for solving these nonlinear and potentially ill-conditioned systems are discussed in modern works by Wriggers [34] and Laursen [27]. Lagrangian elements are commonly employed to discretize the displacement field. Contact constraints can be enforced via various methods, including Lagrange multipliers, penalty approaches, and augmented Lagrangian techniques. Many numerical analyses have been developed for elasticity contact

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problems, such as optimized Schwarz methods [35], mixed finite element method (FEM) [2, 7, 14, 20, 24], augmented Lagrangian method [30, 37], least squares method [5], discontinuous Petrov–Galerkin methods (DPG) [19] and so on.

The resulting system of equations in contact mechanics is typically a large-scale, nonlinear, and ill-conditioned problem, especially in cases of high material contrast. To tackle these challenges, we propose an efficient iterative contact-resolving hybrid framework for the penalized contact problem. This approach employs a Robin boundary condition as the transmission condition, combined with a derivative-free technique for updating transmission data on the interfaces (see [15, 17]). In our setting, the contact boundary is contained entirely within the smaller subdomain and combined with penalty approach, while the larger subdomain allows for flexible discretization choices. For instance, the constraint energy minimizing generalized multiscale finite element method (CEM-GMsFEM) [10, 12] can be employed to significantly reduce computational costs associated with high-contrast features in the larger domain, while still preserving coarse-mesh convergence. More specifically, we introduce four types of discretizations within the iterative contact-resolving hybrid framework. These include combinations where both subdomains use standard/mixed finite element methods (FEM), or where the larger subdomain uses standard/mixed CEM-GMsFEM and the smaller one uses standard/mixed FEM. Corresponding algorithms are provided for each case, forming efficient strategies for designing scalable solvers and preconditioners.

Our main contributions are threefold. Firstly, we develop a novel iterative contact-resolving hybrid method for multiscale contact mechanics that isolates nonlinear contact constraints within a smaller subdomain, while the larger subdomain retains a linear system (see Algorithms 3.1-3.4). Conventional methods typically treat the nonlinear contact problem across the entire domain, leading to a globally complex system. In our approach, the nonlinear subproblem, which is restricted to a very small region with a width of one coarse mesh element, is efficiently solved by means of a semismooth Newton method [23]. This localized treatment significantly shortens iteration time. Meanwhile, the larger subdomain supports flexible discretization choices without the need to account for nonlinearity. Secondly, we introduce mixed formulations (see Algorithms 3.2 and 3.4) for solving linear elasticity contact problems involving highly heterogeneous and high-contrast coefficients. By adopting a stress-displacement mixed formulation, our method enables direct computation of stress fields, avoiding post-processing and enforcing local momentum conservation for enhanced physical consistency. Moreover, this formulation inherently prevents Poisson’s ratio locking, enabling robust simulations of nearly incompressible materials. Lastly, we incorporate the CEM-GMsFEM framework within the larger subdomain (see Algorithms 3.3-3.4). It is well-known that obtaining strongly symmetric stress approximations is challenging. While established methods [1, 3, 4, 18, 38] require excessive degrees of freedom to enforce such symmetry, our multiscale model reduction technique substantially lowers computational cost while preserving accuracy.

This paper is structured as follows. In Section 2, we introduce the model problem and some notation. The contact problem in elasticity is reformulated using a minimization theorem and penalty methods. Section 3 presents an iterative contact-resolving hybrid framework along with various discretization schemes. Specifically, the methods based on standard FEM and mixed FEM formulations in the whole domain are detailed in Sections 3.1 and 3.2, respectively, while those incorporating standard and mixed CEM-GMsFEM in the larger subdomain are discussed in Sections 3.3 and 3.4. Corresponding algorithms are provided for all four cases (i.e. Algorithms 3.1-3.4). In Section 4, we give some analyses for the convergence of the iterative contact-resolving hybrid methods proposed in Section 3. Section 5 reports numerical experiments on two test models to demonstrate the performance of the proposed method. Finally, a summary of the conclusions is presented in Section 6.

2 Model problem

We consider the elasticity contact problem in the domain $\Omega \subset \mathbb{R}^N$:

$$\mathcal{A}\underline{\sigma} = \underline{\epsilon}(\mathbf{u}) \quad \text{in } \Omega, \quad (1a)$$

$$-\nabla \cdot (\underline{\sigma}(\mathbf{u})) = \mathbf{f} \quad \text{in } \Omega, \quad (1b)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (1c)$$

$$\mathbf{u} \cdot \mathbf{n}_c \leq 0, \quad \mathbf{n}_c \cdot (\underline{\boldsymbol{\sigma}} \cdot \mathbf{n}_c) \leq 0, \quad (\mathbf{n}_c \cdot (\underline{\boldsymbol{\sigma}} \cdot \mathbf{n}_c)) \mathbf{u} \cdot \mathbf{n}_c = 0 \quad \text{on } \Gamma_C, \quad (1d)$$

where $\underline{\boldsymbol{\sigma}} : \Omega \rightarrow \mathbb{R}^{n \times n}$ is the symmetric stress tensor, $\mathbf{u}$ denotes the displacement and $\underline{\boldsymbol{\epsilon}}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)$ the linearized strain tensor. $\mathbf{n}_c$ is the unit outward normal vector to Γ_C . $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) is a bounded and connected Lipschitz polyhedral domain occupied by an isotropic and linearly elastic solid. $\mathcal{A}$ is the inverse of the elasticity operator, which is given by $\mathcal{A}\underline{\boldsymbol{\tau}} := \frac{1}{2\mu}\underline{\boldsymbol{\tau}}^D + \frac{1}{n(n\lambda+2\mu)}(tr\underline{\boldsymbol{\tau}})\mathbf{I}$, where λ and μ are the Lamé coefficients. $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{N \times N}$ and the deviatoric tensor $\underline{\boldsymbol{\tau}}^D := \underline{\boldsymbol{\tau}} - \frac{1}{n}(tr\underline{\boldsymbol{\tau}})\mathbf{I}$. For nearly incompressible materials, λ is large in comparison with μ . In this paper, λ and μ are highly heterogeneous in space and possibly high contrast.

The domain Ω can be decomposed into two subdomains based on the contact boundary Γ_C : a smaller subdomain that contains Γ_C , and a larger subdomain that is free of it. $\partial\Omega$ is the boundary of the domain Ω , which admits a disjoint partition $\partial\Omega = \Gamma_D \cup \Gamma_C$. Note that Γ_D has a positive measure. For simplicity, we consider a two dimensional model as depicted in Figure 2.1. In this simple case, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2 = \Omega_1 \cup \Omega_2 \cup \gamma_{12} \cup \partial\Omega$, $\gamma_{12} = \gamma_{21} = \partial\Omega_1 \cap \partial\Omega_2$. Ω_2 is the smaller subdomain that contains Γ_C while the larger domain Ω_1 devoid of it. Denote $\mathbf{n}_i$ as the unit outward normal vector to the boundary of Ω_i for $i = 1, 2$.

We further assume frictionless contact, i.e., $\mathbf{t}_c \cdot (\underline{\boldsymbol{\sigma}} \cdot \mathbf{n}_c) = 0$ on Γ_C , where $\mathbf{t}_c$ is the unit tangential vector to Γ_C . To simplify the notation, we define the normal displacement $u_c = \mathbf{u} \cdot \mathbf{n}_c$ and the normal stress $\sigma_c = \mathbf{n}_c \cdot (\underline{\boldsymbol{\sigma}} \cdot \mathbf{n}_c)$. Then the contact boundary condition (1d) can be reformulated as

$$u_c \leq 0, \quad \sigma_c \leq 0, \quad \sigma_c u_c = 0 \quad \text{on } \Gamma_C.$$

We define the following subsets of the Sobolev space $\mathbf{H}^1(\Omega)$ of vector valued functions:

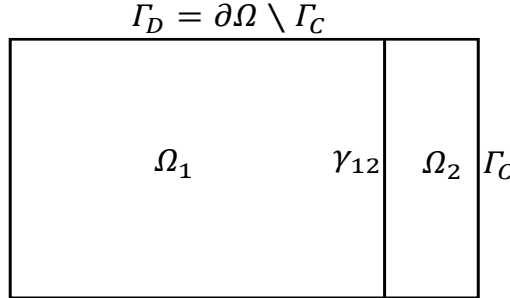


Figure 2.1: Simplified 2D model

$$\mathbf{H}_{\Gamma_D}^1(\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) | \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}, \quad \mathbf{V} := \{\mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega) | v_c \leq 0 \text{ a.e. on } \Gamma_C\}.$$

Following the analysis in [13], the exact solution to problem (1) is given by the variational inequality: find $\mathbf{u} \in \mathbf{V}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}) \quad (2)$$

for all $\mathbf{v} \in \mathbf{V}$. Here we define $a_D(\mathbf{u}, \mathbf{v}) = \int_D \underline{\boldsymbol{\sigma}}(\mathbf{u}) : \nabla \mathbf{v} dx = \int_D \mathcal{A}^{-1} \underline{\boldsymbol{\epsilon}}(\mathbf{u}) : \underline{\boldsymbol{\epsilon}}(\mathbf{v}) dx$, $(\mathbf{f}, \mathbf{v})_D = \int_D \mathbf{f} \cdot \mathbf{v} dx$ for all $D \subset \Omega$. We will drop the subscript D in $a_D(\cdot, \cdot)$ and $(\cdot, \cdot)_D$ when $D = \Omega$. According to [13], the weak problem (2) is well-posed and admits a unique solution. Furthermore, the solution can be equivalently characterized as the minimizer of a constrained minimization problem. Specifically, given the functional $F : \mathbf{H}_{\Gamma_D}^1(\Omega) \rightarrow \mathbb{R}$ as $F(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - (\mathbf{f}, \mathbf{v})$, find $\mathbf{u} \in \mathbf{V}$ such that

$$F(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{V}} F(\mathbf{v}).$$

Next, we employ the penalty method to transform the constrained minimization problem into an unconstrained one. This is achieved by introducing a penalty term into the objective function that effectively

suppresses constraint violations. We denote this penalty term as $P : \mathbf{H}_{\Gamma_D}^1(\Omega) \rightarrow \mathbb{R}$ and a new functional $F_\delta : \mathbf{H}_{\Gamma_D}^1(\Omega) \rightarrow \mathbb{R}$ depending on a real parameter $\delta > 0$ as follows

$$F_\delta(\mathbf{v}) = F(\mathbf{v}) + \frac{1}{\delta} P(\mathbf{v}),$$

where $P(\mathbf{v}) := \frac{1}{2} \int_{\Gamma_C} |v_c^+|^2 ds$, $v_c^+ = \max\{0, v_c\}$ and $v_c^- = -\min\{0, v_c\}$. Note that $v_c = v_c^+ - v_c^-$, $-v_c^- \cdot v_c = |v_c^-|^2$, $v_c^+ \cdot v_c = |v_c^+|^2$, $v_c^+ v_c^- = 0$. Then for all $\mathbf{v}, \mathbf{w} \in \mathbf{H}_{\Gamma_D}^1(\Omega)$, we have

$$\begin{aligned} (v_c^+ - w_c^+)(v_c - w_c) &= (v_c^+ - w_c^+)^2 + (v_c^+ - w_c^+)(w_c^- - v_c^-) \\ &= (v_c^+ - w_c^+)^2 + v_c^- w_c^+ + w_c^- v_c^+ \geq (v_c^+ - w_c^+)^2 \geq 0. \end{aligned} \quad (3)$$

The penalty term P is designed such that a candidate minimizer $\mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega)$ incurs a larger penalty with increasing violation of the constraint $v_c \leq 0$ on Γ_C . Consequently, for every $\delta > 0$, the functional F_δ admits a minimizer $\mathbf{u}_\delta \in V$. This leads to the constrained minimization problem: find $\mathbf{u} \in \mathbf{H}_{\Gamma_D}^1(\Omega)$ such that $F_\delta(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega)} F_\delta(\mathbf{v})$.

3 Iterative contact-resolving hybrid methods associated with various discretizations

In this section, we present various discretization schemes for the iterative contact-resolving hybrid framework. These include applying standard/mixed FEM across the entire domain (in Sections 3.1 and 3.2), as well as a combined approach that couples standard/mixed CEM-GMsFEM in the larger domain with standard/mixed FEM in the smaller domain (in Sections 3.3 and 3.4).

3.1 Iterative contact-resolving hybrid method associated with Standard FEM

This section firstly introduces the standard FEM formulation for the penalized contact problem across the entire domain in Section 3.1.1. Then Section 3.1.2 presents the corresponding iterative contact-resolving hybrid algorithm (Algorithm 3.1) and provides the associated convergence result (Theorem 3.2).

3.1.1 Standard FEM formulation for the penalized contact problem

Provided that the functionals F and P (introduced in Section 2) are Gateaux-differentiable, by [26, Chapter 6], the unconstrained minimization problem admits a characterization: find $\mathbf{u} \in \mathbf{H}_{\Gamma_D}^1(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) + \frac{1}{\delta} \int_{\Gamma_C} u_c^+ \cdot v_c ds = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega). \quad (4)$$

It is clear that (4) can be considered as the weak form of the following boundary value problem

$$\mathcal{A}\underline{\sigma} = \underline{\epsilon}(\mathbf{u}) \quad \text{in } \Omega \quad (5a)$$

$$-\nabla \cdot (\underline{\sigma}(\mathbf{u})) = \mathbf{f} \quad \text{in } \Omega = \Omega_1 \cup \Omega_2, \quad (5b)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (5c)$$

$$\sigma_c + \frac{1}{\delta} u_c^+ = 0 \quad \text{on } \Gamma_C. \quad (5d)$$

Here we assume that $\sigma_c \in L^2(\gamma_{ij})$ ($i, j = 1, 2, i \neq j$).

Let $\mathbf{V}_h \subset \mathbf{H}^1(\Omega)$ be the standard piecewise linear finite element space ($\mathbf{P}_1$). Define $\mathbf{V}_{h, \Gamma_D}$ to be the space consisting of the functions in $\mathbf{V}_h$ that takes the value $\mathbf{0}$ on Γ_D . Then the standard finite element method for (4) is to find $\mathbf{u}_h \in \mathbf{V}_{h, \Gamma_D}$ such that (denote $v_{hc} = \mathbf{v}_h \cdot \mathbf{n}_c$ for all $\mathbf{v}_h \in \mathbf{V}_{h, \Gamma_D}$)

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + \frac{1}{\delta} \int_{\Gamma_C} (u_{hc})^+ \cdot v_{hc} ds = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_{h, \Gamma_D}. \quad (6)$$

More precisely, $\mathbf{V}_h = \text{span} \{ \boldsymbol{\varphi}_j^1, \boldsymbol{\varphi}_j^2 \}_{j=1}^{N_o}$ is chosen to be a finite element space, where N_o denotes the finite element nodal point set (for simplicity, we consider the two dimensional case for the basis functions). In this case, the first component of $\boldsymbol{\varphi}_j^1$ (for each $1 \leq j \leq N_o$) is a nodal basis and the second component is 0. Similarly, the second component of $\boldsymbol{\varphi}_j^2$ (for each $1 \leq j \leq N_o$) is a nodal basis and the first component is 0. Since the standard $\mathbf{P}_1$ finite element is conforming in $\mathbf{H}^1(\Omega)$, we clearly have $a_h(\cdot, \cdot) = a(\cdot, \cdot)$. The corresponding a -norm (i.e. $\|\mathbf{v}\|_a^2 = \int_{\Omega} \mathcal{A}^{-1} \underline{\epsilon}(\mathbf{v}) : \underline{\epsilon}(\mathbf{v}) dx$ for all $\mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega)$) is also used in the following.

3.1.2 An iterative contact-resolving hybrid method for discrete scheme (6)

In this section, an iterative contact-resolving hybrid method for solving (6) is developed (similar to [17]). Following the method, we construct Algorithm 3.1 and provide the convergence result in Theorem 3.2.

Let $\mathbf{V}_{h,i} = \mathbf{V}_{h,\Gamma_D}|_{\Omega_i}$ for $i = 1, 2$, and $\mathbf{V}_h(\gamma_{ij}) = \left\{ \mathbf{v} \in \mathbf{V}_h \mid \mathbf{v} = \sum_{p \in \gamma_{ij} \cap N_o} (y_p^1 \boldsymbol{\varphi}_p^1 + y_p^2 \boldsymbol{\varphi}_p^2), y_p^1, y_p^2 \in \mathbb{R} \right\}$, where $\{ \boldsymbol{\varphi}_p^1, \boldsymbol{\varphi}_p^2 \}_{p \in N_o}$ is the nodal basis at the point p of the finite element space $\mathbf{V}_h$. We denote

$$\int_{\gamma_{ij}}^* \mathbf{u} \cdot \mathbf{v} ds = \sum_{p \in \gamma_{ij} \cap N_o} u_1(p) v_1(p) w_p^1 + u_2(p) v_2(p) w_p^2, \quad (7)$$

where $\mathbf{u} = (u_1, u_2)^t$, $\mathbf{v} = (v_1, v_2)^t$. w_p^1, w_p^2 are the weights at p obtained from Composite Newton-Cotes Quadrature. Note that the Lagrange basis function nodes on γ_{ij} and the integral nodes used in Composite Newton-Cotes Quadrature are consistent (see, for instance [21, 22, 32]). (7) means we have used numerical integration to compute the integrals on the interfaces. In our theoretical analysis, we employ the composite Newton-Cotes quadrature method due to its simplicity and the intuitive placement of its integration nodes compared to alternatives like Gaussian quadrature. For practical numerical simulations, however, we can adopt Gaussian quadrature to achieve higher computational accuracy and efficiency.

Similar to [16, Theorem 4.1], we will give an equivalent splitting subproblem form (i.e. Theorem 3.1) for the standard finite element problem (6) in order to prove the convergence of Algorithm 3.1 in Theorem 3.2. Detailed proofs of Theorems 3.1 and 3.2 will be given in Section 4. Before presenting the results, we need to define some notations. Let $\mathbf{u}_h \in \mathbf{V}_{h,\Gamma_D}$ be the solution of the finite element problem (6) and $\mathbf{u}_{h,i} = \mathbf{u}_h|_{\Omega_i}$. Let $\mathbf{u}_i^n$ ($i = 1, 2$) be the solutions of the subproblems (8)-(9) on the subdomain Ω_i at iterative step n . Define $a_i^h(\mathbf{u}, \mathbf{v}) = a_{\Omega_i}(\mathbf{u}, \mathbf{v}) = \int_{\Omega_i} \mathcal{A}^{-1} \underline{\epsilon}(\mathbf{u}) : \underline{\epsilon}(\mathbf{v}) dx$ for $i = 1, 2$. We then denote

$$\mathbf{u}^n = (\mathbf{u}_i^n)_{i=1,2} \in \prod_{i=1}^2 \mathbf{V}_{h,i}, \quad \mathbf{e}_h^n = (\mathbf{e}_{h,i}^n)_{i=1,2} := (\mathbf{u}_i^n - \mathbf{u}_{h,i})_{i=1,2} \in \prod_{i=1}^2 \mathbf{V}_{h,i},$$

and the norms as follows: for $i = 1, 2$,

$$\|\mathbf{e}_{h,i}^n\|_{a_i^h}^2 = a_i^h(\mathbf{e}_{h,i}^n, \mathbf{e}_{h,i}^n), \quad \|\mathbf{e}_h^n\|_e^2 = \sum_{i=1}^2 \|\mathbf{e}_{h,i}^n\|_{a_i^h}^2 + \frac{1}{\delta} \int_{\Gamma_C} \left| (u_{2,c}^n)^+ - (u_{h,c,2})^+ \right|^2 ds,$$

where $u_{2,c}^n = \mathbf{u}_2^n \cdot \mathbf{n}_c$, $u_{h,c,2} = \mathbf{u}_{h,2} \cdot \mathbf{n}_c$.

Theorem 3.1. *The problem (6) can be split into an equivalent splitting subproblem form. That is, there exist $\mathbf{g}_{ij}^* \in \mathbf{V}_h(\gamma_{ij})$ ($i, j = 1, 2, i \neq j$) such that $\mathbf{u}_{h,i} \in \mathbf{V}_{h,i}$ ($i = 1, 2$) satisfies*

$$a_1^h(\mathbf{u}_{h,1}, \mathbf{v}_{h,1}) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{u}_{h,1} \cdot \mathbf{v}_{h,1} ds = (\mathbf{f}, \mathbf{v}_{h,1})_{\Omega_1} + \int_{\gamma_{12}}^* \mathbf{g}_{12}^* \cdot \mathbf{v}_{h,1} ds \quad \forall \mathbf{v}_{h,1} \in \mathbf{V}_{h,1},$$

and

$$\begin{aligned} & a_2^h(\mathbf{u}_{h,2}, \mathbf{v}_{h,2}) + \frac{1}{\delta} \int_{\Gamma_C} (u_{h,c,2})^+ \cdot v_{h,c,2} ds + \alpha_{21} \int_{\gamma_{21}}^* \mathbf{u}_{h,2} \cdot \mathbf{v}_{h,2} ds \\ & = (\mathbf{f}, \mathbf{v}_{h,2})_{\Omega_2} + \int_{\gamma_{21}}^* \mathbf{g}_{21}^* \cdot \mathbf{v}_{h,2} ds \quad \forall \mathbf{v}_{h,2} \in \mathbf{V}_{h,2}, \end{aligned}$$

where $u_{h,c,2} = \mathbf{u}_{h,2} \cdot \mathbf{n}_c$, $v_{h,c,2} = \mathbf{v}_{h,2} \cdot \mathbf{n}_c$.

Algorithm 3.1 an iterative contact-resolving hybrid algorithm for solving (6)

1. Given $\mathbf{g}_{ij}^0 \in \mathbf{V}_h(\gamma_{ij})$, $i, j = 1, 2$, $i \neq j$, arbitrarily (i.e., given $\mathbf{g}_{12}^0 \in \mathbf{V}_h(\gamma_{12})$, $\mathbf{g}_{21}^0 \in \mathbf{V}_h(\gamma_{21})$).
2. Recursively find $(\mathbf{u}_1^n, \mathbf{u}_2^n) \in \mathbf{V}_{h,1} \times \mathbf{V}_{h,2}$ by solving the subproblems in parallel:

$$a_1^h(\mathbf{u}_1^n, \mathbf{v}_{h,1}) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{u}_1^n \cdot \mathbf{v}_{h,1} ds = (\mathbf{f}, \mathbf{v}_{h,1})_{\Omega_1} + \int_{\gamma_{12}}^* \mathbf{g}_{12}^n \cdot \mathbf{v}_{h,1} ds \quad (8)$$

for all $\mathbf{v}_{h,1} \in \mathbf{V}_{h,1}$; and

$$a_2^h(\mathbf{u}_2^n, \mathbf{v}_{h,2}) + \frac{1}{\delta} \int_{\Gamma_C} (u_{2,c}^n)^+ \cdot v_{hc,2} ds + \alpha_{21} \int_{\gamma_{21}}^* \mathbf{u}_2^n \cdot \mathbf{v}_{h,2} ds = (\mathbf{f}, \mathbf{v}_{h,2})_{\Omega_2} + \int_{\gamma_{21}}^* \mathbf{g}_{21}^n \cdot \mathbf{v}_{h,2} ds \quad (9)$$

for all $\mathbf{v}_{h,2} \in \mathbf{V}_{h,2}$. Here $u_{2,c}^n = \mathbf{u}_2^n \cdot \mathbf{n}_c$, $v_{hc,2} = \mathbf{v}_{h,2} \cdot \mathbf{n}_c$.

3. Update the data of the transmission condition on the interfaces:

$$\begin{aligned} \mathbf{g}_{12}^{n+1}(p) &= 2\alpha_{12}\mathbf{u}_2^n(p) - \mathbf{g}_{21}^n(p), \\ \mathbf{g}_{21}^{n+1}(p) &= 2\alpha_{21}\mathbf{u}_1^n(p) - \mathbf{g}_{12}^n(p), \end{aligned}$$

on $p \in \gamma_{12} \cap N_o$, where $\alpha_{12} = \alpha_{21} > 0$ are the coefficients of Robin-type transmission condition on the artificial boundaries (see [15] for more details)).

Then we give the following convergence result.

Theorem 3.2. *Let $\mathbf{u}_h \in \mathbf{V}_{h,\Gamma_D}$ be the solution of the standard finite element problem (6) and let $\mathbf{u}_i^n \in \mathbf{V}_{h,i}$ ($i = 1, 2$) be the solutions of subproblems (8) and (9) at iterative step n . Then we have*

$$\|\mathbf{e}_h^n\|_e = \left(\sum_{i=1}^2 \|\mathbf{u}_i^n - \mathbf{u}_{h,i}\|_{a_i^h}^2 + \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right]^2 ds \right)^{\frac{1}{2}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

3.2 Iterative Contact-resolving hybrid method associated with Mixed FEM

In this section, we begin by presenting the mixed FEM formulation for the penalized contact problem across the entire domain in Section 3.2.1. Building on this formulation, Section 3.2.2 then introduces the corresponding iterative contact-resolving hybrid algorithm (i.e. Algorithm 3.2).

3.2.1 Mixed formulation for the penalized contact problem

For the contact problem, designing iteration schemes becomes more flexible when considering the mixed formulation. We will introduce the mixed penalty problems in this section. In two dimension, by using Voigt notation, the inverse of the elasticity operator, $\mathcal{A}$, can be expressed in terms of the following coefficient matrix:

$$\mathcal{A} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix}^{-1}.$$

Denote $\mathbf{H}_D^{1/2}(\Gamma_C) := \{\boldsymbol{\mu} \in \mathbf{H}^{1/2}(\partial\Omega) : \boldsymbol{\mu} = \mathbf{0} \text{ on } \Gamma_D\}$. Let $\underline{\mathbf{Y}} := \{\boldsymbol{\tau} \in \underline{\mathbf{H}}(\text{div}; \Omega, \mathbb{S}) : \tau_c := \mathbf{n}_c \cdot (\boldsymbol{\tau} \cdot \mathbf{n}_c) \in L^2(\Gamma_C), \mathbf{t}_c \cdot (\boldsymbol{\tau} \cdot \mathbf{n}_c) = 0 \text{ on } \Gamma_C\}$, which is a subspace of $\underline{\mathbf{H}}(\text{div}; \Omega, \mathbb{S})$. Denote $\underline{\mathbf{K}}$ as a convex subset of $\underline{\mathbf{Y}}$:

$$\underline{\mathbf{K}} := \{\boldsymbol{\tau} \in \underline{\mathbf{Y}} : \int_{\Gamma_C} (\boldsymbol{\tau} \cdot \mathbf{n}_c) \cdot \boldsymbol{\mu} ds \leq 0, \forall \boldsymbol{\mu} \in \mathbf{H}_D^{1/2}(\Gamma_C), \mu_c := \boldsymbol{\mu} \cdot \mathbf{n}_c \geq 0 \text{ on } \Gamma_C\}.$$

By utilizing the standard decomposition of stress and displacement vectors on $\partial\Omega$ in a tangential and a normal component, we have $\underline{\sigma} \cdot \mathbf{n}_c = \underline{\sigma}_t + \sigma_c \mathbf{n}_c$, $\mathbf{u} = \mathbf{u}_t + u_c \mathbf{n}_c$. In the set $\underline{\mathbf{K}}$, the conditions $\mathbf{t}_c \cdot (\underline{\tau} \cdot \mathbf{n}_c) = 0$ and $\int_{\Gamma_C} (\underline{\tau} \cdot \mathbf{n}_c) \cdot \underline{\mu} ds \leq 0$, together with the standard decompositions, imply that $\underline{\tau}_t = \mathbf{0}$ and $\int_{\Gamma_C} (\underline{\tau} \cdot \mathbf{n}_c) \cdot \underline{\mu} ds = \int_{\Gamma_C} (\underline{\tau}_t + \tau_c \mathbf{n}_c) \cdot (\underline{\mu}_t + \mu_c \mathbf{n}_c) = \int_{\Gamma_C} \tau_c \mathbf{n}_c \cdot (\underline{\mu}_t + \mu_c \mathbf{n}_c) = \int_{\Gamma_C} \tau_c \mu_c \leq 0$. Thus, $\underline{\mathbf{K}}$ can be rewritten as

$$\underline{\mathbf{K}} := \{\underline{\tau} \in \underline{\mathbf{Y}} : \int_{\Gamma_C} \tau_c \mu_c ds \leq 0, \forall \underline{\mu} \in \mathbf{H}_D^{1/2}(\Gamma_C), \mu_c \geq 0 \text{ on } \Gamma_C\}. \quad (10)$$

We consider the following dual mixed formulation of problem (1): find $(\underline{\sigma}, \mathbf{u}) \in \underline{\mathbf{K}} \times \mathbf{L}^2(\Omega)$ such that

$$(\mathcal{A}\underline{\sigma}, \underline{\tau} - \underline{\sigma}) + (\operatorname{div}(\underline{\tau} - \underline{\sigma}), \mathbf{u}) \geq 0 \quad \forall \underline{\tau} \in \underline{\mathbf{K}}, \quad (11a)$$

$$(\operatorname{div}\underline{\sigma}, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{L}^2(\Omega). \quad (11b)$$

Having established the well-posedness of problem (11) (see, e.g., [31]), we now introduce the mixed penalty problem: find $(\underline{\sigma}, \mathbf{u}) \in \underline{\mathbf{Y}} \times \mathbf{L}^2(\Omega)$ such that (here, δ again denotes the penalty parameter)

$$(\mathcal{A}\underline{\sigma}, \underline{\tau}) + (\operatorname{div}\underline{\tau}, \mathbf{u}) + \frac{1}{\delta} \int_{\Gamma_C} \sigma_c^+ \tau_c ds = 0 \quad \forall \underline{\tau} \in \underline{\mathbf{Y}}, \quad (12a)$$

$$(\operatorname{div}\underline{\sigma}, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{L}^2(\Omega). \quad (12b)$$

It is clear that (12) can be considered as the weak form of the following boundary value problem

$$\mathcal{A}\underline{\sigma} = \underline{\epsilon}(\mathbf{u}) \quad \text{in } \Omega \quad (13a)$$

$$-\nabla \cdot (\underline{\sigma}(\mathbf{u})) = \mathbf{f} \quad \text{in } \Omega = \Omega_1 \cup \Omega_2, \quad (13b)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (13c)$$

$$u_c + \frac{1}{\delta} \sigma_c^+ = 0 \quad \text{on } \Gamma_C. \quad (13d)$$

Let $\underline{\Sigma}_h \subset \underline{\mathbf{Y}}$ and $\mathbf{U}_h \subset \mathbf{L}^2(\Omega)$ be conforming discrete spaces (which can be chosen in various ways, see e.g., [4, 12, 18, 25, 33]). Following the discretization approach in [12, 18], we consider the following discrete problem: find $(\underline{\sigma}_h, \mathbf{u}_h) \in \underline{\Sigma}_h \times \mathbf{U}_h$ such that

$$(\mathcal{A}\underline{\sigma}_h, \underline{\tau}_h) + (\operatorname{div}\underline{\tau}_h, \mathbf{u}_h) + \frac{1}{\delta} \int_{\Gamma_C} \sigma_{hc}^+ \tau_{hc} ds = 0 \quad \forall \underline{\tau}_h \in \underline{\Sigma}_h, \quad (14a)$$

$$-(\operatorname{div}\underline{\sigma}_h, \mathbf{v}_h) = s(\pi(\tilde{k}^{-1}\mathbf{f}), \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \quad (14b)$$

where $s(\cdot, \cdot)$ is a weighted L^2 inner product (see [12] or Section 3.3.2 for more details). Note that in the discrete scheme (14), we denote the right-hand side as the bilinear form $s(\cdot, \cdot)$ rather than a standard L^2 inner product. This choice is made for convenience in certain error analyses and does not fundamentally affect the overall stability or convergence (see [12] for details). The well-posedness of Eqs. (14) can be confirmed by combining the references [12, 24, 26, 31]. Similar to (8)-(9), by utilizing a Robin-type transmission condition on the interfaces, we can split (14) into an equivalent subproblem form in two subdomains: there exist $\mathbf{g}_{ij}^{mix} \in \mathbf{U}_h(\gamma_{ij})$ ($i, j = 1, 2, i \neq j$) such that $(\underline{\sigma}_{h,i}, \mathbf{u}_{h,i}) \in \underline{\Sigma}_{h,i} \times \mathbf{U}_{h,i} := \underline{\Sigma}_h|_{\Omega_i} \times \mathbf{U}_h|_{\Omega_i}$ ($i = 1, 2$) satisfies

$$(\mathcal{A}\underline{\sigma}_{h,1}, \underline{\tau}_{h,1})_{\Omega_1} + (\operatorname{div}\underline{\tau}_{h,1}, \mathbf{u}_{h,1})_{\Omega_1} + \beta_{12} \int_{\gamma_{12}} \sigma_{hn,1} \tau_{hn,1} ds = \int_{\gamma_{12}} \mathbf{g}_{12}^{mix} \cdot (\underline{\tau}_{h,1} \cdot \mathbf{n}_1) ds \quad \forall \underline{\tau}_{h,1} \in \underline{\Sigma}_{h,1}, \quad (15a)$$

$$(\operatorname{div}\underline{\sigma}_{h,1}, \mathbf{v}_{h,1})_{\Omega_1} = s(\pi(\tilde{k}^{-1}\mathbf{f}), \mathbf{v}_{h,1}) \quad \forall \mathbf{v}_{h,1} \in \mathbf{U}_{h,1}, \quad (15b)$$

$$(\mathcal{A}\underline{\sigma}_{h,2}, \underline{\tau}_{h,2})_{\Omega_2} + (\operatorname{div}\underline{\tau}_{h,2}, \mathbf{u}_{h,2})_{\Omega_2} + \beta_{21} \int_{\gamma_{21}} \sigma_{hn,2} \tau_{hn,2} ds + \frac{1}{\delta} \int_{\Gamma_C} \sigma_{hc,2}^+ \tau_{hc,2} ds$$

$$= \int_{\gamma_{21}} \mathbf{g}_{21}^{mix} \cdot (\boldsymbol{\tau}_{h,2} \cdot \mathbf{n}_2) ds \quad \forall \boldsymbol{\tau}_{h,2} \in \underline{\Sigma}_{h,2}, \quad (16a)$$

$$(\operatorname{div} \underline{\boldsymbol{\sigma}}_{h,2}, \mathbf{v}_{h,2})_{\Omega_2} = s(\pi(\tilde{k}^{-1} \mathbf{f}), \mathbf{v}_{h,2}) \quad \forall \mathbf{v}_{h,2} \in \mathbf{U}_{h,2}, \quad (16b)$$

where $\tau_{hn,i} := \mathbf{n}_i \cdot (\boldsymbol{\tau}_h \cdot \mathbf{n}_i)$ for $i = 1, 2$. Note that the values of the transmission coefficients β_{ij} in the mixed setting (i.e. (15)-(16)) differ significantly from those of α_{ij} in the standard setting (i.e. (8)-(9)).

3.2.2 An iterative contact-resolving hybrid method for the mixed discretization (15)-(16)

This section presents an iterative contact-resolving hybrid algorithm (Algorithm 3.2) for solving systems (15)–(16). Given the similarity of its convergence behavior to that established in Theorem 3.2 for Algorithm 3.1, the convergence statement and corresponding proof for Algorithm 3.2 are omitted.

Algorithm 3.2 an iterative contact-resolving hybrid algorithm for solving (15)–(16)

1. Given $\mathbf{g}_{ij}^0 \in \mathbf{V}_h(\gamma_{ij})$, $i, j = 1, 2$, $i \neq j$, arbitrarily (i.e., given $\mathbf{g}_{12}^0 \in \mathbf{V}_h(\gamma_{12})$, $\mathbf{g}_{21}^0 \in \mathbf{V}_h(\gamma_{21})$).
2. Recursively find $(\underline{\boldsymbol{\sigma}}_1^n, \mathbf{u}_1^n) \in \underline{\Sigma}_{h,1} \times \mathbf{U}_{h,1}$ by solving

$$\begin{aligned} (\mathcal{A} \underline{\boldsymbol{\sigma}}_1^n, \boldsymbol{\tau}_{h,1})_{\Omega_1} + (\operatorname{div} \boldsymbol{\tau}_{h,1}, \mathbf{u}_1^n)_{\Omega_1} + \beta_{12} \int_{\gamma_{12}}^* (\underline{\boldsymbol{\sigma}}_1^n)_n \tau_{hn,1} ds &= \int_{\gamma_{12}}^* \mathbf{g}_{12}^n \cdot (\boldsymbol{\tau}_{h,1} \cdot \mathbf{n}_1) ds \quad \forall \boldsymbol{\tau}_{h,1} \in \underline{\Sigma}_{h,1}, \\ (\operatorname{div} \underline{\boldsymbol{\sigma}}_1^n, \mathbf{v}_{h,1})_{\Omega_1} &= s(\pi(\tilde{k}^{-1} \mathbf{f}), \mathbf{v}_{h,1}) \quad \forall \mathbf{v}_{h,1} \in \mathbf{U}_{h,1}, \end{aligned}$$

and find $(\underline{\boldsymbol{\sigma}}_2^n, \mathbf{u}_2^n) \in \underline{\Sigma}_{h,2} \times \mathbf{U}_{h,2}$ by solving

$$\begin{aligned} (\mathcal{A} \underline{\boldsymbol{\sigma}}_2^n, \boldsymbol{\tau}_{h,2})_{\Omega_2} + (\operatorname{div} \boldsymbol{\tau}_{h,2}, \mathbf{u}_2^n)_{\Omega_2} + \beta_{21} \int_{\gamma_{21}}^* (\underline{\boldsymbol{\sigma}}_2^n)_n \tau_{hn,2} ds + \frac{1}{\delta} \int_{\Gamma_C} (\underline{\boldsymbol{\sigma}}_{2,c}^n)^+ \tau_{hc,2} ds \\ = \int_{\gamma_{21}}^* \mathbf{g}_{21}^n \cdot (\boldsymbol{\tau}_{h,2} \cdot \mathbf{n}_2) ds, \quad \forall \boldsymbol{\tau}_{h,2} \in \underline{\Sigma}_{h,2}, \\ (\operatorname{div} \underline{\boldsymbol{\sigma}}_2^n, \mathbf{v}_{h,2})_{\Omega_2} = (\mathbf{f}, \mathbf{v}_{h,2})_{\Omega_2}, \quad \forall \mathbf{v}_{h,2} \in \mathbf{U}_{h,2}, \end{aligned}$$

where $(\underline{\boldsymbol{\sigma}}_i^n)_n = \mathbf{n}_i \cdot (\underline{\boldsymbol{\sigma}}_i^n \cdot \mathbf{n}_i)$, $\underline{\boldsymbol{\sigma}}_{i,c}^n = \mathbf{n}_c \cdot (\underline{\boldsymbol{\sigma}}_i^n \cdot \mathbf{n}_c)$, $\tau_{hn,i} = \mathbf{n}_i \cdot (\boldsymbol{\tau}_{h,i} \cdot \mathbf{n}_i)$, $\tau_{hc,2} = \mathbf{n}_c \cdot (\boldsymbol{\tau}_{h,2} \cdot \mathbf{n}_c)$ for $i = 1, 2$.

3. Update the data of the transmission condition on the interfaces:

$$\begin{aligned} \mathbf{g}_{12}^{n+1}(p) &= -2\beta_{12} \underline{\boldsymbol{\sigma}}_2^n(p) \cdot \mathbf{n}_2 + \mathbf{g}_{21}^n(p), \\ \mathbf{g}_{21}^{n+1}(p) &= -2\beta_{21} \underline{\boldsymbol{\sigma}}_1^n(p) \cdot \mathbf{n}_1 + \mathbf{g}_{12}^n(p), \end{aligned}$$

on $p \in \gamma_{12} \cap N_o$ (in the computation, p can be taken to be the Gauss points), where $\beta_{12} = \beta_{21} > 0$ are the coefficients of Robin-type transmission condition on the artificial boundaries in the mixed setting.

3.3 Iterative contact-resolving hybrid method associated with standard CEM-GMsFEM

In this section, we consider the iterative contact-resolving hybrid method associated with standard CEM-GMsFEM [10, 12, 36]. Specifically, the method employs standard CEM-GMsFEM in the larger domain Ω_1 and standard FEM in the smaller domain Ω_2 , which contains the contact boundary. The splitting formulation for this setting is detailed in Section 3.3.1, while the construction of multiscale basis functions within the larger subdomain is described in Section 3.3.2. An iterative domain decomposition procedure

(Algorithm 3.3) is then developed in Section 3.3.3. A key aspect of the present framework is the careful treatment of boundary conditions (see different stages in Algorithm 3.3). The solution $\mathbf{u}_h$ obtained from (6) is regarded as the reference solution.

3.3.1 Splitting formulation for the method associated with standard CEM-GMsFEM

Analogous to (8)–(9), we develop the splitting subproblem formulation for the iterative contact-resolving hybrid method associated with standard CEM-GMsFEM in this section.

Utilizing the multiscale space $\mathbf{V}_{\text{ms}}$ (constructed in Section 3.3.2) for the displacement field in Ω_1 and the standard FEM space $\mathbf{V}_{h,2}$ for that in Ω_2 , we obtain the numerical solution $(\mathbf{u}_{\text{ms},1}, \mathbf{u}_{h,2}) \in \mathbf{V}_{\text{ms}} \times \mathbf{V}_{h,2}$ by solving

$$a_1^h(\mathbf{u}_{\text{ms},1}, \mathbf{v}_{h,1}) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{u}_{\text{ms},1} \cdot \mathbf{v}_{h,1} ds = (f, \mathbf{v}_{h,1})_{\Omega_1} + \int_{\gamma_{12}}^* \mathbf{g}_{12}^* \cdot \mathbf{v}_{h,1} ds \quad \forall \mathbf{v}_{h,1} \in \mathbf{V}_{\text{ms}}, \quad (17a)$$

$$a_2^h(\mathbf{u}_{h,2}, \mathbf{v}_{h,2}) + \frac{1}{\delta} \int_{\Gamma_C} (u_{hc,2})^+ \cdot v_{hc,2} ds + \alpha_{21} \int_{\gamma_{21}}^* \mathbf{u}_{h,2} \cdot \mathbf{v}_{h,2} ds = (\mathbf{f}, \mathbf{v}_{h,2})_{\Omega_2} + \int_{\gamma_{21}}^* \mathbf{g}_{21}^* \cdot \mathbf{v}_{h,2} ds \quad (17b)$$

for all $\mathbf{v}_{h,2} \in \mathbf{V}_{h,2}$. Note that we have utilized a Robin-type transmission condition on the interfaces in (17) similar to (8)–(9).

3.3.2 The construction of the multiscale basis functions in the larger subdomain

In this section, we will present the construction of the multiscale basis functions for the standard CEM-GMsFEM in Ω_1 (see [10, 12]). Let $\mathcal{T}_H := \cup_{i=1}^N \{K_i\}$ denote a conforming quasi-uniform partition of the domain Ω into quadrilateral elements, where H represents the coarse mesh size and N is the total number of coarse elements. We refer to $\mathcal{T}_H$ as the coarse grid, where each coarse element K_i is further subdivided into a connected union of fine-grid blocks. The corresponding fine grid, denoted by $\mathcal{T}_h := \cup_{i=1}^{N_h} \{T_i\}$ (with N_h being the number of fine elements), is constructed as a refinement of $\mathcal{T}_H$. For an illustrative example, see [12, Figure 3.1], which depicts the coarse mesh, fine mesh, and an oversampling domain.

The construction of the basis functions are developed on the two-scale mesh. Let K_i be the i -th coarse block and let $\mathbf{V}_{h,i}(K_i)$ be the restriction of $\mathbf{V}_{h,i}$ on K_i . Define $K_{i,m}$ as the region which is a m coarse-grid layer extension of K_i . We solve a local spectral problem: for each K_i , find a real number $\lambda_j^i \in \mathbb{R}$ and a function $\phi_j^i \in \mathbf{V}_{h,i}(K_i)$ such that

$$a_{1,i}^h(\phi_j^i, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12} \cap \partial K_i}^* \phi_j^i \cdot \mathbf{v} ds = \lambda_j^i s_i(\phi_j^i, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_{h,i}(K_i) \quad (18)$$

where

$$a_{1,i}^h(\mathbf{w}, \mathbf{v}) = \int_{K_i} \mathcal{A}^{-1} \underline{\epsilon}(\mathbf{w}) : \underline{\epsilon}(\mathbf{v}) dx, \quad s_i(\mathbf{w}, \mathbf{v}) = \int_{K_i} \tilde{k} \mathbf{w} \mathbf{v} dx, \quad \tilde{k} = kH^{-2}, \quad k = \lambda + 2\mu.$$

Assume that $s_i(\phi_j^i, \phi_j^i) = 1$ and we arrange the eigenvalues of (18) in non-decreasing order $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{L_i}$, where L_i is the dimension of the space $\mathbf{V}_{h,i}(K_i)$. For each $i \in \{1, 2, \dots, N\}$, choose the first l_i ($1 \leq l_i \leq L_i$) eigenfunctions $\{\phi_j^i\}_{j=1}^{l_i}$ corresponding the first l_i smallest eigenvalues. Then, we define the local auxiliary multiscale space $\mathbf{V}_{\text{aux}}(K_i)$ for the displacement field as $\mathbf{V}_{\text{aux}}(K_i) := \text{span}\{\phi_j^i : 1 \leq j \leq l_i\}$. The global auxiliary multiscale finite element space V_{aux} is defined by $\mathbf{V}_{\text{aux}} = \oplus_i \mathbf{V}_{\text{aux}}(K_i)$. Define the projection operator $\pi_i : \mathbf{L}^2(K_i) \rightarrow \mathbf{V}_{\text{aux}}(K_i)$ with respect to the inner product $s_i(\cdot, \cdot)$. Specifically, for any $\mathbf{q} \in \mathbf{L}^2(K_i)$, the operator π_i is given by

$$\pi_i(\mathbf{q}) = \sum_{j=1}^{l_i} s_i(\mathbf{q}, \mathbf{p}_j^i) \mathbf{p}_j^i.$$

Similarly, we define the global projection operator $\pi: \mathbf{L}^2(\Omega) \rightarrow \mathbf{V}_{\text{aux}}$ with respect to the inner product $s(\cdot, \cdot)$. For any $\mathbf{q} \in \mathbf{L}^2(\Omega)$, this operator takes the form $\pi(\mathbf{q}) = \sum_{i=1}^N \sum_{j=1}^{l_i} s_i(\mathbf{q}, \mathbf{p}_j^i) \mathbf{p}_j^i$. Clearly $\pi = \sum_{i=1}^N \pi_i$.

Next we present the construction of the multiscale basis functions. We directly consider the relaxed constraint energy minimizing generalized multiscale finite element method [11]. We solve the following unconstrained minimization problem: find $\psi_{j,\text{ms}}^i \in \mathbf{V}_{h,1}(K_{i,m})$ such that

$$\psi_{j,\text{ms}}^i = \operatorname{argmin} \left\{ a_1^h(\psi, \psi) + \alpha_{12} \int_{\gamma_{12}}^* \psi \cdot \psi ds + s(\pi\psi - \phi_j^i, \pi\psi - \phi_j^i) : \psi \in \mathbf{V}_{i,h}^m \right\},$$

where $\mathbf{V}_{i,h}^m = \{\mathbf{v} \in \mathbf{V}_{h,1}(K_{i,m}) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \cap \partial K_{i,m} \text{ or } \Omega \cap \partial K_{i,m}\}$. Clearly $\mathbf{V}_{i,h}^m \subset \mathbf{V}_{h,1}$. It can be shown that $\psi_{j,\text{ms}}^i$ satisfies the following variational form

$$a_1^h(\psi_{j,\text{ms}}^i, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12}}^* \psi_{j,\text{ms}}^i \cdot \mathbf{v} ds + s(\pi\psi_{j,\text{ms}}^i, \pi\mathbf{v}) = s(\phi_j^i, \pi\mathbf{v}) \quad (19)$$

for all $\mathbf{v} \in \mathbf{V}_{i,h}^m$. The global multiscale basis function $\psi_j^i \in \mathbf{V}_{h,1}$ is defined in a similar way, namely,

$$\psi_j^i = \operatorname{argmin} \left\{ a_1^h(\psi, \psi) + \alpha_{12} \int_{\gamma_{12}}^* \psi \cdot \psi ds + s(\pi\psi - \phi_j^i, \pi\psi - \phi_j^i) : \psi \in \mathbf{V}_{h,1} \right\}$$

and the corresponding variational form is

$$a_1^h(\psi_j^i, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12}}^* \psi_j^i \cdot \mathbf{v} ds + s(\pi\psi_j^i, \pi\mathbf{v}) = s(\phi_j^i, \pi\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_{h,1}. \quad (20)$$

Denote $\mathbf{V}_{\text{ms}} := \operatorname{span}\{\psi_{j,\text{ms}}^i : 1 \leq j \leq l_i, 1 \leq i \leq N\}$, $\mathbf{V}_{\text{glo}} := \operatorname{span}\{\psi_j^i : 1 \leq j \leq l_i, 1 \leq i \leq N\}$. Following the methodology in [36], we compute $\mathbf{u}_{\text{ms},1}$ in (17a) using a computational procedure comprising the following four steps:

Step 1: Find $\mathcal{N}_i^m \mathbf{g}_{12}^*$ such that for all $\mathbf{v} \in \mathbf{V}_{i,h}^m$,

$$a_1^h(\mathcal{N}_i^m \mathbf{g}_{12}^*, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12}}^* \mathcal{N}_i^m \mathbf{g}_{12}^* \cdot \mathbf{v} ds + s(\pi(\mathcal{N}_i^m \mathbf{g}_{12}^*), \pi\mathbf{v}) = \int_{\partial K_i \cap \gamma_{12}}^* \mathbf{g}_{12}^* \cdot \mathbf{v} ds.$$

Then we obtain $\mathcal{N}^m \mathbf{g}_{12}^* = \sum_{i=1}^N \mathcal{N}_i^m \mathbf{g}_{12}^*$.

Step 2: Prepare the multiscale space $\mathbf{V}_{\text{ms}}$ via (18) and (19).

Step 3: Solve $\mathbf{w}^m$ such that for all $\mathbf{v} \in \mathbf{V}_{\text{ms}}$,

$$a_1^h(\mathbf{w}^m, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{w}^m \cdot \mathbf{v} ds = (\mathbf{f}, \mathbf{v})_{\Omega_1} + \int_{\gamma_{12}}^* \mathbf{g}_{12}^* \cdot \mathbf{v} ds - \left[a_1^h(\mathcal{N}^m \mathbf{g}_{12}^*, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12}}^* \mathcal{N}^m \mathbf{g}_{12}^* \cdot \mathbf{v} ds \right].$$

Step 4: Construct the numerical solution $\mathbf{u}_{\text{ms},1}$ to as

$$\mathbf{u}_{\text{ms},1} \approx \mathbf{w}^m + \mathcal{N}^m \mathbf{g}_{12}^*.$$

3.3.3 An iterative contact-resolving hybrid method for problem (17)

In this section, we define the iterative procedure for problem (17) as Algorithm 3.3. Due to the similarity of its convergence behavior to that of Algorithm 3.1 (established in Theorem 3.2), the convergence analysis for Algorithm 3.3 is omitted.

Algorithm 3.3 an iterative contact-resolving hybrid algorithm for solving (17)

1. Given $\mathbf{g}_{ij}^0 \in \mathbf{V}_h(\gamma_{ij})$, $i, j = 1, 2$, $i \neq j$, arbitrarily (i.e., given $\mathbf{g}_{12}^0 \in \mathbf{V}_h(\gamma_{12})$, $\mathbf{g}_{21}^0 \in \mathbf{V}_h(\gamma_{21})$).
2. Recursively find $(\mathbf{u}_1^n, \mathbf{u}_2^n) \in \mathbf{V}_{\text{ms}} \times \mathbf{V}_{h,2}$ by solving the subproblems in parallel:

(i) To obtain $\mathbf{u}_1^n \in \mathbf{V}_{\text{ms}}$, we consider the following three stages:

Stage a: Find $\mathcal{N}_i^m \mathbf{g}_{12}^n$ such that for all $\mathbf{v} \in \mathbf{V}_{i,h}^m$,

$$a_1^h(\mathcal{N}_i^m \mathbf{g}_{12}^n, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12}}^* \mathcal{N}_i^m \mathbf{g}_{12}^n \cdot \mathbf{v} ds + s(\pi(\mathcal{N}_i^m \mathbf{g}_{12}^n), \pi \mathbf{v}) = \int_{\partial K_i \cap \gamma_{12}}^* \mathbf{g}_{12}^n \cdot \mathbf{v} ds.$$

Then we obtain $\mathcal{N}^m \mathbf{g}_{12}^n = \sum_{i=1}^N \mathcal{N}_i^m \mathbf{g}_{12}^n$.

Stage b: Solve $\mathbf{w}_n^m$ such that of for all $\mathbf{v} \in \mathbf{V}_{\text{ms}}$,

$$a_1^h(\mathbf{w}_n^m, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{w}_n^m \cdot \mathbf{v} ds = (\mathbf{f}, \mathbf{v})_{\Omega_1} + \int_{\gamma_{12}}^* \mathbf{g}_{12}^n \cdot \mathbf{v} ds - \left[a_1^h(\mathcal{N}^m \mathbf{g}_{12}^n, \mathbf{v}) + \alpha_{12} \int_{\gamma_{12}}^* \mathcal{N}^m \mathbf{g}_{12}^n \cdot \mathbf{v} ds \right].$$

Note that we utilize the multiscale space $\mathbf{V}_{\text{ms}}$ produced in Step 2 of the computational method listed in section 3.3.2, which means we do not need update multiscale bases in this algorithm.

Stage c: Construct the numerical solution $\mathbf{u}_1^n$ as

$$\mathbf{u}_1^n \approx \mathbf{w}_n^m + \mathcal{N}^m \mathbf{g}_{12}^n.$$

(ii) Solve $\mathbf{u}_2^n \in \mathbf{V}_{h,2}$ by

$$\begin{aligned} & a_2^h(\mathbf{u}_2^n, \mathbf{v}_{h,2}) + \frac{1}{\delta} \int_{\Gamma_C} (u_{2,c}^n)^+ \cdot v_{hc,2} ds + \alpha_{21} \int_{\gamma_{21}}^* \mathbf{u}_2^n \cdot \mathbf{v}_{h,2} ds \\ & = (\mathbf{f}, \mathbf{v}_{h,2})_{\Omega_2} + \int_{\gamma_{21}}^* \mathbf{g}_{21}^n \cdot \mathbf{v}_{h,2} ds \end{aligned}$$

for all $\mathbf{v}_{h,2} \in \mathbf{V}_{h,2}$. Here $u_{2,c}^n = \mathbf{u}_2^n \cdot \mathbf{n}_c$, $v_{hc,2} = \mathbf{v}_{h,2} \cdot \mathbf{n}_c$.

3. Update the data of the transmission condition on the interfaces:

$$\begin{aligned} \mathbf{g}_{12}^{n+1}(p) &= 2\alpha_{12} \mathbf{u}_2^n(p) - \mathbf{g}_{21}^n(p), \\ \mathbf{g}_{21}^{n+1}(p) &= 2\alpha_{21} \mathbf{u}_1^n(p) - \mathbf{g}_{12}^n(p), \end{aligned}$$

on $p \in \gamma_{12} \cap N_o$, where $\alpha_{12} = \alpha_{21} > 0$ are the coefficients of Robin-type transmission condition on the artificial boundaries.

3.4 Iterative contact-resolving hybrid method associated with mixed CEM-GMsFEM

In this section, we develop an iterative contact-resolving hybrid method based on a combination of mixed CEM-GMsFEM and mixed FEM. The method applies the mixed CEM-GMsFEM to the larger domain Ω_1 and the standard mixed FEM to the smaller domain Ω_2 containing the contact boundary. The splitting formulation is presented in Section 3.4.1, and the construction of multiscale basis functions is described in Section 3.4.2. Building on this foundation, Section 3.4.3 devises the iterative domain decomposition procedure (Algorithm 3.4). The numerical solution $(\underline{\sigma}_h, \mathbf{u}_h)$ computed from (14) is taken as the reference solution for the mixed multiscale formulation.

3.4.1 Splitting formulation for the method associated with mixed CEM-GMsFEM

Similar to (15)–(16), this section develops the splitting subproblem formulation for the iterative contact-resolving hybrid method associated with standard CEM-GMsFEM. The method employs the multiscale spaces Σ_{ms} and U_{aux} (to be constructed in Section 3.4.2) to approximate the stress and displacement in Ω_1 . The numerical solution $(\underline{\sigma}_{\text{ms},1}, \mathbf{u}_{\text{ms},1})$ in Ω_1 and $(\underline{\sigma}_{h,2}, \mathbf{u}_{h,2})$ in Ω_2 , are obtained by solving the following

$$(\mathcal{A}\underline{\sigma}_{\text{ms},1}, \underline{\tau}_1)_{\Omega_1} + (\text{div } \underline{\tau}_1, \mathbf{u}_{\text{ms},1})_{\Omega_1} + \beta_{12} \int_{\gamma_{12}} \sigma_{\text{msn},1} \tau_{n,1} ds = \int_{\gamma_{12}} \mathbf{g}_{12}^{\text{mix}} \cdot (\underline{\tau}_1 \cdot \mathbf{n}_1) ds, \quad \forall \underline{\tau}_1 \in \Sigma_{\text{ms}}, \quad (21a)$$

$$(\text{div } \underline{\sigma}_{\text{ms},1}, \mathbf{v}_1)_{\Omega_1} = (\mathbf{f}, \mathbf{v}_1)_{\Omega_1}, \quad \forall \mathbf{v}_1 \in U_{\text{aux}}, \quad (21b)$$

$$\begin{aligned} & (\mathcal{A}\underline{\sigma}_{h,2}, \underline{\tau}_{h,2})_{\Omega_2} + (\text{div } \underline{\tau}_{h,2}, \mathbf{u}_{h,2})_{\Omega_2} + \beta_{21} \int_{\gamma_{21}} \sigma_{hn,2} \tau_{hn,2} ds + \frac{1}{\delta} \int_{\Gamma_C} (\sigma_{hc,2})^+ \tau_{hc,2} ds \\ &= \int_{\gamma_{21}} \mathbf{g}_{21}^{\text{mix}} \cdot (\underline{\tau}_{h,2} \cdot \mathbf{n}_2) ds, \quad \forall \underline{\tau}_{h,2} \in \underline{\Sigma}_{h,2}, \end{aligned} \quad (22a)$$

$$(\text{div } \underline{\sigma}_{h,2}, \mathbf{v}_{h,2})_{\Omega_2} = (\mathbf{f}, \mathbf{v}_{h,2})_{\Omega_2}, \quad \forall \mathbf{v}_{h,2} \in \mathbf{U}_{h,2}, \quad (22b)$$

where $\sigma_{\text{msn},1} = \mathbf{n}_1 \cdot (\underline{\sigma}_{\text{ms},1} \cdot \mathbf{n}_1)$, $\tau_{n,1} = \mathbf{n}_1 \cdot (\underline{\tau}_1 \cdot \mathbf{n}_1)$. Note that we have utilized a Robin-type transmission condition on the interfaces in (21)–(22) analogous to (15)–(16).

3.4.2 The construction of the multiscale basis functions for (21)

This section presents the construction of multiscale basis functions for the mixed formulation (21) in Ω_1 , following the methodology of [12] and adopting all notations from Section 3.3.2. The basis functions are constructed over the mesh illustrated in [12, Figure 3.1]. The computation of the multiscale basis functions is divided into two stages. The first stage consists of constructing the multiscale space for the displacement $\mathbf{u}$. In the second stage, we will use the multiscale space for displacement to construct a multiscale space for the stress $\underline{\sigma}$. We point out that the supports of displacement basis are the coarse elements. For stress basis functions, the support is an oversampled region containing the support of displacement basis functions.

Stage I:

We will construct a set of auxiliary multiscale basis functions for displacement on each coarse element K_i by solving a local spectral problem. First, we define some notation. For a general set R , we define $\mathbf{U}_{h,1}(R)$ as the restriction of $\mathbf{U}_{h,1}$ on $R \subset \Omega_1$ and $\underline{\Sigma}_{h,1}(R) := \{\underline{\tau}_h \in \underline{\Sigma}_{h,1} : \underline{\tau}_h \mathbf{n} = \mathbf{0} \text{ on } \Omega_1 \cap \partial R\}$. Note that $\underline{\Sigma}_{h,1}(R)$ is with homogeneous traction boundary condition on R , which confirms the conforming property of the multiscale bases in the construction process.

Next, we define the local spectral problem. For each coarse element $K_i \subset \Omega_1$, we solve the eigenvalue problem: find $(\underline{\phi}_j^i, \mathbf{p}_j^i) \in \underline{\Sigma}_{h,1}(K_i) \times \mathbf{U}_{h,1}(K_i)$ and $\lambda_j^i \in \mathbb{R}$ such that

$$(\mathcal{A}\underline{\phi}_j^i, \underline{\tau}_h)_{K_i} + (\text{div } \underline{\tau}_h, \mathbf{p}_j^i)_{K_i} + \beta_{12} \int_{\gamma_{12} \cap \partial K_i} \phi_{j,n}^i \tau_{hn} ds = 0 \quad \forall \underline{\tau}_h \in \underline{\Sigma}_{h,1}(K_i), \quad (23a)$$

$$-(\operatorname{div} \underline{\phi}_j^i, \mathbf{v}_h)_{K_i} = \lambda_j^i s_i(\mathbf{p}_j^i, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{U}_{h,1}(K_i), \quad (23b)$$

where $\phi_{j,n}^i = \mathbf{n}_1 \cdot (\underline{\phi}_j^i \cdot \mathbf{n}_1)$, $\tau_{hn} = \mathbf{n}_1 \cdot (\underline{\tau}_h \cdot \mathbf{n}_1)$. We arrange the eigenvalues of (23) in non-decreasing order $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{L'_i}$, where L'_i is the dimension of the space $\mathbf{U}_{h,1}(K_i)$. For each $i \in \{1, 2, \dots, N\}$, choose the first l_i ($1 \leq l_i \leq L'_i$) eigenfunctions $\{\mathbf{p}_j^i\}_{j=1}^{l_i}$ corresponding the first l_i smallest eigenvalues. Then, we define the local auxiliary multiscale space $U_{\text{aux}}(K_i)$ for displacement as $U_{\text{aux}}(K_i) := \operatorname{span}\{\mathbf{p}_j^i | 1 \leq j \leq l_i\}$. The global auxiliary multiscale space U_{aux} is defined by $U_{\text{aux}} = \oplus_i U_{\text{aux}}(K_i)$. Note that the space U_{aux} will be used as the approximation space for the displacement.

Stage II:

Then we present the construction of the stress basis functions. We directly present the relaxed version of stress multiscale basis functions (see [12]). Let $\mathbf{p}_j^i \in U_{\text{aux}}$ be a given displacement basis function supported in K_i . We will define a stress basis function $\underline{\psi}_{j,\text{ms}}^i \in \underline{\Sigma}_h(K_{i,m})$ by solving (24). The multiscale space is defined as $\Sigma_{\text{ms}} := \operatorname{span}\{\underline{\psi}_{j,\text{ms}}^i\}$. Note that the basis function is supported in $K_{i,m}$, which is a union of connected coarse elements and contains K_i . We define J_i as the set of indices such that if $k \in J_i$, then $K_k \in K_{i,m}$. We also define $U_{\text{aux}}(K_{i,m}) = \operatorname{span}\{\mathbf{p}_j^k | 1 \leq j \leq l_k, k \in J_i\}$.

We find $\underline{\psi}_{j,\text{ms}}^i \in \underline{\Sigma}_{h,1}(K_{i,m})$ and $\mathbf{q}_{j,\text{ms}}^i \in \mathbf{U}_{h,1}(K_{i,m})$ such that

$$(\mathcal{A}\underline{\psi}_{j,\text{ms}}^i, \underline{\tau}_h)_{\Omega_1} + (\operatorname{div} \underline{\tau}_h, \mathbf{q}_{j,\text{ms}}^i)_{\Omega_1} + \beta_{12} \int_{\gamma_{12}} \psi_{j,\text{msn}}^i \tau_{hn} ds = 0 \quad \forall \underline{\tau}_h \in \underline{\Sigma}_{h,1}(K_{i,m}), \quad (24a)$$

$$s(\pi \mathbf{q}_{j,\text{ms}}^i, \pi \mathbf{v}_h) - (\operatorname{div} \underline{\psi}_{j,\text{ms}}^i, \mathbf{v}_h)_{\Omega_1} = s(\mathbf{p}_j^i, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{U}_{h,1}(K_{i,m}), \quad (24b)$$

where $\psi_{j,\text{msn}}^i = \mathbf{n}_1 \cdot (\underline{\psi}_{j,\text{ms}}^i \cdot \mathbf{n}_1)$. Similarly, the global basis function $\underline{\psi}_j^i \in \underline{\Sigma}_{h,1}$ is constructed by solving the following problem: find $\underline{\psi}_j^i \in \underline{\Sigma}_{h,1}$ and $\mathbf{q}_j^i \in \mathbf{U}_{h,1}$ such that

$$(\mathcal{A}\underline{\psi}_j^i, \underline{\tau}_h)_{\Omega_1} + (\operatorname{div} \underline{\tau}_h, \mathbf{q}_j^i)_{\Omega_1} + \beta_{12} \int_{\gamma_{12}} \psi_{j,n}^i \tau_{hn} ds = 0 \quad \forall \underline{\tau}_h \in \underline{\Sigma}_{h,1}, \quad (25a)$$

$$s(\pi \mathbf{q}_j^i, \pi \mathbf{v}_h) - (\operatorname{div} \underline{\psi}_j^i, \mathbf{v}_h)_{\Omega_1} = s(\mathbf{p}_j^i, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{U}_{h,1}. \quad (25b)$$

The global multiscale space is defined as $\Sigma_{\text{glo}} = \operatorname{span}\{\underline{\psi}_j^i\}$.

Similar to section 3.3.2, to estimate $(\underline{\sigma}_{\text{ms},1}, \mathbf{u}_{\text{ms},1})$ in (21), we can consider the computational method consisting of the following four steps:

Step i: Find $(\mathcal{Q}_i^m \mathbf{g}_{12}^{mix}, \mathcal{N}_i^m \mathbf{g}_{12}^{mix})$ such that for all $(\underline{\tau}_h, \mathbf{v}_h) \in \underline{\Sigma}_{h,1}(K_{i,m}) \times \mathbf{U}_{h,1}(K_{i,m})$,

$$(\mathcal{A}(\mathcal{Q}_i^m \mathbf{g}_{12}^{mix}), \underline{\tau}_h)_{\Omega_1} + (\operatorname{div} \underline{\tau}_h, \mathcal{N}_i^m \mathbf{g}_{12}^{mix})_{\Omega_1} + \beta_{12} \int_{\gamma_{12}}^* (\mathcal{Q}_i^m \mathbf{g}_{12}^{mix})_n \tau_{hn} ds = \int_{\gamma_{12} \cap \partial K_i}^* \mathbf{g}_{12}^{mix} \cdot (\underline{\tau}_h \cdot \mathbf{n}_1) ds, \quad (26a)$$

$$s(\pi(\mathcal{Q}_i^m \mathbf{g}_{12}^{mix}), \pi \mathbf{v}_h) - (\operatorname{div}(\mathcal{Q}_i^m \mathbf{g}_{12}^{mix}), \mathbf{v}_h)_{\Omega_1} = 0. \quad (26b)$$

Here $\tau_{hn} = \mathbf{n}_1 \cdot (\underline{\tau}_h \cdot \mathbf{n}_1)$. Then we obtain $\mathcal{Q}^m \mathbf{g}_{12}^{mix} = \sum_{i=1}^N \mathcal{Q}_i^m \mathbf{g}_{12}^{mix}$, $\mathcal{N}^m \mathbf{g}_{12}^{mix} = \sum_{i=1}^N \mathcal{N}_i^m \mathbf{g}_{12}^{mix}$.

Step ii: Prepare the multiscale space $U_{\text{aux}}, \Sigma_{\text{ms}}$ by solving (23) and (24).

Step iii: Solve $(\underline{\mathbf{r}}^m, \mathbf{w}^m)$ such that for all $(\underline{\tau}, \mathbf{v}) \in \Sigma_{\text{ms}} \times U_{\text{aux}}$,

$$\begin{aligned} (\mathcal{A}\underline{\mathbf{r}}^m, \underline{\tau})_{\Omega_1} + (\operatorname{div} \underline{\tau}, \mathbf{w}^m)_{\Omega_1} + \beta_{12} \int_{\gamma_{12}}^* (\underline{\mathbf{r}}^m)_n \tau_n ds &= \int_{\gamma_{12}}^* \mathbf{g}_{12}^{mix} \cdot (\underline{\tau} \cdot \mathbf{n}_1) ds \\ &- ((\mathcal{A}(\mathcal{Q}^m \mathbf{g}_{12}^{mix}), \underline{\tau})_{\Omega_1} + (\operatorname{div} \underline{\tau}, \mathcal{N}^m \mathbf{g}_{12}^{mix})_{\Omega_1} + \beta_{12} \int_{\gamma_{12}}^* (\mathcal{Q}^m \mathbf{g}_{12}^{mix})_n \tau_n ds), \end{aligned} \quad (27a)$$

$$(\operatorname{div} \underline{\mathbf{r}}^m, \mathbf{v})_{\Omega_1} = (\mathbf{f}, \mathbf{v})_{\Omega_1} - (\operatorname{div}(\mathcal{Q}^m \mathbf{g}_{12}^{mix}), \mathbf{v})_{\Omega_1}. \quad (27b)$$

Here $(\underline{r}^m)_n = \mathbf{n}_1 \cdot (\underline{r}^m \cdot \mathbf{n}_1)$, $\tau_n = \mathbf{n}_1 \cdot (\underline{\tau} \cdot \mathbf{n}_1)$.

Step iv: Construct the numerical solution $(\underline{\sigma}_{\text{ms},1}, \mathbf{u}_{\text{ms},1})$ as

$$\underline{\sigma}_{\text{ms},1} \approx \underline{r}^m + \mathcal{Q}^m \mathbf{g}_{12}^{mix}, \quad \mathbf{u}_{\text{ms},1} \approx \mathbf{w}^m + \mathcal{N}^m \mathbf{g}_{12}^{mix}.$$

3.4.3 An iterative contact-resolving hybrid method for problem (21)-(22)

In this section, analogous to Section 3.3.3, we define the iterative procedure as Algorithm 3.4. Owing to the similarity in convergence behavior between this algorithm and Algorithm 3.1 (cf. Theorem 3.2), the convergence analysis for Algorithm 3.4 is omitted accordingly.

4 Analysis of the iterative contact-resolving hybrid method for discrete elastic contact problem

In this section, we present convergence analyses for the algorithms proposed in Section 3. Since Algorithms 3.1–3.4 share the same iterative contact-resolving hybrid framework and differ only in their choice of finite element methods, their convergence analyses are analogous. Therefore, we provide detailed proofs for the theorems in Section 3.1 and omit the repetitive arguments for those in Sections 3.2–3.4.

4.1 Proof of Theorem 3.1

In this section, we prove the equivalence of splitting discrete subproblems and global problems used in the algorithms proposed in Section 3 (i.e. Theorem 3.1).

proof of Theorem 3.1. It is clear that (6) is equivalent to the following equation (note that $\{\varphi_p^1, \varphi_p^2\}_{p \in N_o}$ is a nodal basis of $\mathbf{V}_h$)

$$a_h(\mathbf{u}_h, \varphi_p^l) + \frac{1}{\delta} \int_{\Gamma_C} (u_{hc})^+ \varphi_{p,c}^l ds = (\mathbf{f}, \varphi_p^l) \quad \forall p \in N_o, l = 1, 2.$$

where $\varphi_{p,c}^l = \varphi_p^l \cdot \mathbf{n}_c$. Since $\varphi_p^1(p) = (1, 0)$, $\varphi_p^2(p) = (0, 1)$ and $\varphi_p^i(p') = \mathbf{0}$ for all $p' \in N_o$, $p' \neq p$ and $l = 1, 2$, then $a_1^h(\mathbf{u}_h, \varphi_p^l) + a_2^h(\mathbf{u}_h, \varphi_p^l) + \frac{1}{\delta} \int_{\Gamma_C} (u_{hc,2})^+ \varphi_{p,c}^l ds = (\mathbf{f}, \varphi_p^l)_{\Omega_1} + (\mathbf{f}, \varphi_p^l)_{\Omega_2}$ for all $p \in \gamma_{12} \cap N_o$, $l = 1, 2$. That is, we have

$$a_1^h(\mathbf{u}_h, \varphi_p^l) - (\mathbf{f}, \varphi_p^l)_{\Omega_1} = -(a_2^h(\mathbf{u}_h, \varphi_p^l) + \frac{1}{\delta} \int_{\Gamma_C} (u_{hc,2})^+ \varphi_{p,c}^l ds - (\mathbf{f}, \varphi_p^l)_{\Omega_2}) \quad (28)$$

for all $p \in \gamma_{12} \cap N_o$, and $l = 1, 2$. For γ_{12} , we define G_{12}^p and G_{21}^p as follows:

$$G_{12}^{p,l} = -\frac{1}{w_p} (a_2^h(\mathbf{u}_h, \varphi_p^l) - (\mathbf{f}, \varphi_p^l)_{\Omega_2} + \frac{1}{\delta} \int_{\Gamma_C} (u_{hc,2})^+ \varphi_{p,c}^l ds),$$

$$G_{21}^{p,l} = -\frac{1}{w_p} (a_1^h(\mathbf{u}_h, \varphi_p^l) - (\mathbf{f}, \varphi_p^l)_{\Omega_1}).$$

Then we construct $\mathbf{g}_{12}^*, \mathbf{g}_{21}^* \in \mathbf{V}_h(\gamma_{12})$, $\text{meas}(\gamma_{12}) > 0$, such that

$$\begin{aligned} \mathbf{g}_{12}^*(x) &= \sum_{p \in \gamma_{ij} \cap N_o, l=1,2} \left(\alpha_{12} u_{h,1}^l(p) + G_{12}^{p,l} \right) \varphi_p^l(x), \\ \mathbf{g}_{21}^*(x) &= \sum_{p \in \gamma_{ij} \cap N_o, l=1,2} \left(\alpha_{21} u_{h,2}^l(p) + G_{21}^{p,l} \right) \varphi_p^l(x), \end{aligned} \quad (29)$$

Algorithm 3.4 an iterative contact-resolving hybrid algorithm for solving (21)-(22)

1. Given $\mathbf{g}_{ij}^0 \in \mathbf{U}_h(\gamma_{ij})$, $i, j = 1, 2$, $i \neq j$, arbitrarily (i.e., given $\mathbf{g}_{12}^0 \in \mathbf{V}_h(\gamma_{12})$, $\mathbf{g}_{21}^0 \in \mathbf{V}_h(\gamma_{21})$).
2. Recursively find $(\underline{\boldsymbol{\sigma}}_1^n, \mathbf{u}_1^n) \in \Sigma_{\text{ms}} \times U_{\text{aux}}$, $(\underline{\boldsymbol{\sigma}}_2^n, \mathbf{u}_2^n) \in \underline{\Sigma}_{h,2} \times \mathbf{U}_{h,2}$ by solving the subproblems in parallel:
 - (i) To obtain $(\underline{\boldsymbol{\sigma}}_1^n, \mathbf{u}_1^n) \in \Sigma_{\text{ms}} \times U_{\text{aux}}$, we consider the following three stages:

Stage a: Find $(\mathcal{Q}_i^m \mathbf{g}_{12}^n, \mathcal{N}_i^m \mathbf{g}_{12}^n)$ such that for all $(\underline{\boldsymbol{\tau}}_h, \mathbf{v}_h) \in \underline{\Sigma}_{h,1}(K_{i,m}) \times \mathbf{U}_{h,1}(K_{i,m})$,

$$(\mathcal{A}(\mathcal{Q}_i^m \mathbf{g}_{12}^n), \underline{\boldsymbol{\tau}}_h)_{\Omega_1} + (\text{div } \underline{\boldsymbol{\tau}}_h, \mathcal{N}_i^m \mathbf{g}_{12}^n)_{\Omega_1} + \beta_{12} \int_{\gamma_{12}}^* (\mathcal{Q}_i^m \mathbf{g}_{12}^n)_n \tau_{hn} ds = \int_{\gamma_{12} \cap \partial K_i}^* \mathbf{g}_{12}^n \cdot (\underline{\boldsymbol{\tau}}_h \cdot \mathbf{n}_1) ds,$$

$$s(\pi(\mathcal{Q}_i^m \mathbf{g}_{12}^n), \pi \mathbf{v}_h) - (\text{div } (\mathcal{Q}_i^m \mathbf{g}_{12}^n), \mathbf{v}_h)_{\Omega_1} = 0.$$

Here $\tau_{hn} = \mathbf{n}_1 \cdot (\underline{\boldsymbol{\tau}}_h \cdot \mathbf{n}_1)$, $(\mathcal{Q}_i^m \mathbf{g}_{12}^n)_n = \mathbf{n}_1 \cdot (\mathcal{Q}_i^m \mathbf{g}_{12}^n \cdot \mathbf{n}_1)$. Then we obtain $\mathcal{Q}^m \mathbf{g}_{12}^n = \sum_{i=1}^N \mathcal{Q}_i^m \mathbf{g}_{12}^n$, $\mathcal{N}^m \mathbf{g}_{12}^n = \sum_{i=1}^N \mathcal{N}_i^m \mathbf{g}_{12}^n$.

Stage b: Solve $(\underline{\mathbf{r}}_n^m, \mathbf{w}_n^m)$ such that for all $(\underline{\boldsymbol{\tau}}, \mathbf{v}) \in \Sigma_{\text{ms}} \times U_{\text{aux}}$,

$$(\mathcal{A} \underline{\mathbf{r}}_n^m, \underline{\boldsymbol{\tau}})_{\Omega_1} + (\text{div } \underline{\boldsymbol{\tau}}, \mathbf{w}_n^m)_{\Omega_1} + \beta_{12} \int_{\gamma_{12}}^* (\underline{\mathbf{r}}_n^m)_n \tau_n ds = \int_{\gamma_{12}}^* \mathbf{g}_{12}^n \cdot (\underline{\boldsymbol{\tau}} \cdot \mathbf{n}_1) ds$$

$$- ((\mathcal{A}(\mathcal{Q}^m \mathbf{g}_{12}^n), \underline{\boldsymbol{\tau}})_{\Omega_1} + (\text{div } \underline{\boldsymbol{\tau}}, \mathcal{N}^m \mathbf{g}_{12}^n)_{\Omega_1} + \beta_{12} \int_{\gamma_{12}}^* (\mathcal{Q}^m \mathbf{g}_{12}^n)_n \tau_n ds)$$

$$(\text{div } \underline{\mathbf{r}}_n^m, \mathbf{v})_{\Omega_1} = (\mathbf{f}, \mathbf{v})_{\Omega_1} - (\text{div } (\mathcal{Q}^m \mathbf{g}_{12}^n), \mathbf{v})_{\Omega_1}.$$

Here $(\underline{\mathbf{r}}_n^m)_n = \mathbf{n}_1 \cdot (\underline{\mathbf{r}}_n^m \cdot \mathbf{n}_1)$, $\tau_n = \mathbf{n}_1 \cdot (\underline{\boldsymbol{\tau}} \cdot \mathbf{n}_1)$. Note that we utilize the multiscale spaces $\Sigma_{\text{ms}} \times U_{\text{aux}}$ for the stress and the displacement produced in Step ii of the computational method listed in section 3.4.2, which means we do not need update multiscale bases in this algorithm.

Stage c: Construct the numerical solution $(\underline{\boldsymbol{\sigma}}_1^n, \mathbf{u}_1^n)$ as

$$\underline{\boldsymbol{\sigma}}_1^n \approx \underline{\mathbf{r}}_n^m + \mathcal{Q}^m \mathbf{g}_{12}^n, \quad \mathbf{u}_1^n \approx \mathbf{w}_n^m + \mathcal{N}^m \mathbf{g}_{12}^n.$$

- (ii) Solve $(\underline{\boldsymbol{\sigma}}_2^n, \mathbf{u}_2^n) \in \underline{\Sigma}_{h,2} \times \mathbf{U}_{h,2}$ by

$$(\mathcal{A} \underline{\boldsymbol{\sigma}}_2^n, \underline{\boldsymbol{\tau}}_{h,2})_{\Omega_2} + (\text{div } \underline{\boldsymbol{\tau}}_{h,2}, \mathbf{u}_2^n)_{\Omega_2} + \beta_{21} \int_{\gamma_{21}}^* (\underline{\boldsymbol{\sigma}}_2^n)_n \tau_{hn,2} ds + \frac{1}{\delta} \int_{\Gamma_C} (\underline{\boldsymbol{\sigma}}_{2,c}^n)^+ \tau_{hc,2} ds$$

$$= \int_{\gamma_{21}}^* \mathbf{g}_{21}^n \cdot (\underline{\boldsymbol{\tau}}_{h,2} \cdot \mathbf{n}_2) ds, \quad \forall \underline{\boldsymbol{\tau}}_{h,2} \in \underline{\Sigma}_{h,2},$$

$$(\text{div } \underline{\boldsymbol{\sigma}}_2^n, \mathbf{v}_{h,2})_{\Omega_2} = (\mathbf{f}, \mathbf{v}_{h,2})_{\Omega_2}, \quad \forall \mathbf{v}_{h,2} \in \mathbf{U}_{h,2},$$

where $(\underline{\boldsymbol{\sigma}}_2^n)_n = \mathbf{n}_2 \cdot (\underline{\boldsymbol{\sigma}}_2^n \cdot \mathbf{n}_2)$, $\underline{\boldsymbol{\sigma}}_{2,c}^n = \mathbf{n}_c \cdot (\underline{\boldsymbol{\sigma}}_2^n \cdot \mathbf{n}_c)$, $\tau_{hn,2} = \mathbf{n}_2 \cdot (\underline{\boldsymbol{\tau}}_{h,2} \cdot \mathbf{n}_2)$, $\tau_{hc,2} = \mathbf{n}_c \cdot (\underline{\boldsymbol{\tau}}_{h,2} \cdot \mathbf{n}_c)$.

3. Update the data of the transmission condition on the interfaces:

$$\mathbf{g}_{12}^{n+1}(p) = -2\beta_{12} \underline{\boldsymbol{\sigma}}_2^n(p) \cdot \mathbf{n}_2 + \mathbf{g}_{21}^n(p),$$

$$\mathbf{g}_{21}^{n+1}(p) = -2\beta_{21} \underline{\boldsymbol{\sigma}}_1^n(p) \cdot \mathbf{n}_1 + \mathbf{g}_{12}^n(p),$$

on $p \in \gamma_{12} \cap N_o$ (in the computation, p can be taken to be the Gauss points), where $\beta_{12} = \beta_{21} > 0$ are the coefficients of Robin-type transmission condition on the artificial boundaries in the mixed setting.

where we note that $u_{h,1}^l(p)$ and $u_{h,2}^l(p)$ ($l = 1, 2$) are scalar values and they correspond to nodal values of the l -th component of the solutions $\mathbf{u}_{h,1}(x)$ and $\mathbf{u}_{h,2}(x)$, i.e., $\mathbf{u}_{h,k} = \sum_{p \in \gamma_{ij} \cap N_o, l=1,2} u_{h,k}^l(p) \boldsymbol{\varphi}_p^l(x)$, $k = 1, 2$.

Denote the m -th component of a vector function $\mathbf{v}$ as $(\mathbf{v})_m$. Clearly $(\boldsymbol{\varphi}_p^l(p))_m = \delta_{lm}$ for $l, m = 1, 2$. By (29), we know $(\mathbf{g}_{12}^*)_m, (\mathbf{g}_{21}^*)_m$ (for $m = 1, 2$) can be written as follows

$$(\mathbf{g}_{12}^*(x))_1 = \sum_{p \in \gamma_{ij} \cap N_o} \left(\alpha_{12} u_{h,1}^1(p) + G_{12}^{p,1} \right) (\boldsymbol{\varphi}_p^1(x))_1, \quad (30a)$$

$$(\mathbf{g}_{12}^*(x))_2 = \sum_{p \in \gamma_{ij} \cap N_o} \left(\alpha_{12} u_{h,1}^2(p) + G_{12}^{p,2} \right) (\boldsymbol{\varphi}_p^2(x))_2, \quad (30b)$$

$$(\mathbf{g}_{21}^*(x))_1 = \sum_{p \in \gamma_{ij} \cap N_o} \left(\alpha_{21} u_{h,2}^1(p) + G_{21}^{p,1} \right) (\boldsymbol{\varphi}_p^1(x))_1, \quad (30c)$$

$$(\mathbf{g}_{21}^*(x))_2 = \sum_{p \in \gamma_{ij} \cap N_o} \left(\alpha_{21} u_{h,2}^2(p) + G_{21}^{p,2} \right) (\boldsymbol{\varphi}_p^2(x))_2. \quad (30d)$$

Therefore, it follows from (28) and (30a) that, for all $p \in \overline{\Omega}_1 \cap N_o$, $\mathbf{u}_{h,1} = \mathbf{u}_h|_{\Omega_1}$ satisfies

$$\begin{aligned} a_1^h(\mathbf{u}_{h,1}, \boldsymbol{\varphi}_p^1) - (\mathbf{f}, \boldsymbol{\varphi}_p^1)_{\Omega_1} &= w_p G_{12}^{p,1} \\ &= w_p ((\mathbf{g}_{12}^*(p))_1 - \alpha_{12} \mathbf{u}_{h,1}^1(p)) = w_p (\mathbf{g}_{12}^*(p) - \alpha_{12} \mathbf{u}_{h,1}(p)) \cdot \boldsymbol{\varphi}_p^1(p) \\ &= \int_{\gamma_{12}}^* (\mathbf{g}_{12}^* - \alpha_{12} \mathbf{u}_{h,1}) \cdot \boldsymbol{\varphi}_p^1 ds, \end{aligned}$$

where $\boldsymbol{\varphi}_p^1(p) = (1, 0)$ has been used in the second line. Thus,

$$a_1^h(\mathbf{u}_{h,1}, \boldsymbol{\varphi}_p^1) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{u}_{h,1} \cdot \boldsymbol{\varphi}_p^1 ds = (\mathbf{f}, \boldsymbol{\varphi}_p^1)_{\Omega_1} + \int_{\gamma_{12}}^* \mathbf{g}_{12}^* \cdot \boldsymbol{\varphi}_p^1 ds.$$

Similarly, by (30b)-(30d), we obtain that

$$\begin{aligned} a_1^h(\mathbf{u}_{h,1}, \boldsymbol{\varphi}_p^2) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{u}_{h,1} \cdot \boldsymbol{\varphi}_p^2 ds &= (\mathbf{f}, \boldsymbol{\varphi}_p^2)_{\Omega_1} + \int_{\gamma_{12}}^* \mathbf{g}_{12}^* \cdot \boldsymbol{\varphi}_p^2 ds, \\ a_2^h(\mathbf{u}_{h,2}, \boldsymbol{\varphi}_p^l) - (\mathbf{f}, \boldsymbol{\varphi}_p^l)_{\Omega_2} + \frac{1}{\delta} \int_{\Gamma_C} (u_{hc,2})^+ \cdot \boldsymbol{\varphi}_{p,c}^l ds &= w_p G_{21}^{p,l} \\ &= w_p (\mathbf{g}_{21}^*(p) - \alpha_{21} \mathbf{u}_{h,2}(p)) \cdot \boldsymbol{\varphi}_p^l(p) = \int_{\gamma_{21}}^* (\mathbf{g}_{21}^* - \alpha_{21} \mathbf{u}_{h,2}) \cdot \boldsymbol{\varphi}_p^l ds \end{aligned}$$

for $l = 1, 2$. That is, we have,

$$\begin{aligned} a_1^h(\mathbf{u}_{h,1}, \boldsymbol{\varphi}_p^l) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{u}_{h,1} \cdot \boldsymbol{\varphi}_p^l ds &= (\mathbf{f}, \boldsymbol{\varphi}_p^l)_{\Omega_1} + \int_{\gamma_{12}}^* \mathbf{g}_{12}^* \cdot \boldsymbol{\varphi}_p^l ds, \\ a_2^h(\mathbf{u}_{h,2}, \boldsymbol{\varphi}_p^l) + \alpha_{21} \int_{\gamma_{21}}^* \mathbf{u}_{h,2} \cdot \boldsymbol{\varphi}_p^l ds + \frac{1}{\delta} \int_{\Gamma_C} (u_{hc,2})^+ \cdot \boldsymbol{\varphi}_{p,c}^l ds &= (\mathbf{f}, \boldsymbol{\varphi}_p^l)_{\Omega_2} + \int_{\gamma_{21}}^* \mathbf{g}_{21}^* \cdot \boldsymbol{\varphi}_p^l ds, \end{aligned}$$

for all $p \in \gamma_{12} \cap N_o$, and $l = 1, 2$. The proof is completed. $\square$

4.2 Convergence for Algorithm 3.1

This section is devoted to the proof of Theorem 3.2, which establishes the convergence of Algorithm 3.1. To this end, we first present two auxiliary lemmas and then proceed to the final proof.

Recalling the notations defined in Section 3.1, we clearly have that

$$a_1^h(\mathbf{e}_{h,1}^n, \mathbf{v}_{h,1}) + \alpha_{12} \int_{\gamma_{12}}^* \mathbf{e}_{h,1}^n \cdot \mathbf{v}_{h,1} ds = \int_{\gamma_{12}}^* \mathbf{g}_{12}^n \cdot \mathbf{v}_{h,1} ds \quad \forall \mathbf{v}_{h,1} \in \mathbf{V}_{h,1}. \quad (31)$$

$$\begin{aligned} & a_2^h(\mathbf{e}_{h,2}^n, \mathbf{v}_{h,2}) + \frac{1}{\delta} \int_{\Gamma_C} \left((u_{2,c}^n)^+ - (u_{hc,2})^+ \right) v_{hc,2} ds + \alpha_{21} \int_{\gamma_{21}}^* \mathbf{e}_{h,2}^n \cdot \mathbf{v}_{h,2} ds \\ &= \int_{\gamma_{21}}^* \mathbf{g}_{21}^n \cdot \mathbf{v}_{h,2} ds \quad \forall \mathbf{v}_{h,2} \in \mathbf{V}_{h,2}. \end{aligned} \quad (32)$$

$$\mathbf{g}_{12}^{n+1}(p) = 2\alpha_{12}\mathbf{e}_{h,2}^n(p) - \mathbf{g}_{21}^n(p), \quad \mathbf{g}_{21}^{n+1}(p) = 2\alpha_{21}\mathbf{e}_{h,1}^n(p) - \mathbf{g}_{12}^n(p) \quad (33)$$

on $p \in \gamma_{12} \cap N_o$. Here we have used $\mathbf{g}_{ij}^n$ to replace $\mathbf{g}_{ij}^n - \mathbf{g}_{ij}^*$ for $i, j = 1, 2, i \neq j$. Then we have Lemma 4.1 as follows.

Lemma 4.1. *We have the following identities:*

$$\begin{aligned} a_1^h(\mathbf{e}_{h,1}^n, \mathbf{e}_{h,1}^n) &= \int_{\gamma_{12}}^* (\mathbf{g}_{12}^n - \alpha_{12}\mathbf{e}_{h,1}^n) \cdot \mathbf{e}_{h,1}^n ds, \\ a_2^h(\mathbf{e}_{h,2}^n, \mathbf{e}_{h,2}^n) &= -\frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right] (u_{2,c}^n - u_{hc,2}) ds + \int_{\gamma_{21}}^* (\mathbf{g}_{21}^n - \alpha_{21}\mathbf{e}_{h,2}^n) \cdot \mathbf{e}_{h,2}^n ds. \end{aligned}$$

Lemma 4.2. *There holds the following identity:*

$$\|\underline{\mathbf{g}}^{n+1}\|_*^2 = \|\underline{\mathbf{g}}^n\|_*^2 - 4 \left[\sum_{i=1}^2 a_i^h(\mathbf{e}_{h,i}^n, \mathbf{e}_{h,i}^n) + \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right] (u_{2,c}^n - u_{hc,2}) ds \right],$$

where $\|\underline{\mathbf{g}}^k\|_*^2 = \sum_{i,j=1}^2 \frac{1}{\alpha_{ij}} \int_{\gamma_{ij}}^* |\mathbf{g}_{ij}^k|^2 ds$, $\underline{\mathbf{g}}^k = (\mathbf{g}_{ij}^k)_{\substack{i,j=1,2 \\ i \neq j}}$, $k = n, n+1$.

Proof. By combining the definition of the norm $\|\cdot\|_*$ and Lemma 4.1, we have

$$\begin{aligned} \|\underline{\mathbf{g}}^{n+1}\|_*^2 &= \frac{1}{\alpha_{12}} \int_{\gamma_{12}}^* |\mathbf{g}_{12}^{n+1}|^2 ds + \frac{1}{\alpha_{21}} \int_{\gamma_{21}}^* |\mathbf{g}_{21}^{n+1}|^2 ds \\ &= \frac{1}{\alpha_{12}} \int_{\gamma_{12}}^* |2\alpha_{12}\mathbf{e}_{h,2}^n - \mathbf{g}_{21}^n|^2 ds + \frac{1}{\alpha_{21}} \int_{\gamma_{21}}^* |2\alpha_{21}\mathbf{e}_{h,1}^n - \mathbf{g}_{12}^n|^2 ds \\ &= \frac{1}{\alpha_{12}} \int_{\gamma_{12}}^* |\mathbf{g}_{21}^n|^2 ds - 4 \int_{\gamma_{12}}^* (\mathbf{g}_{21}^n - \alpha_{12}\mathbf{e}_{h,2}^n) \cdot \mathbf{e}_{h,2}^n ds \\ &\quad + \frac{1}{\alpha_{21}} \int_{\gamma_{21}}^* |\mathbf{g}_{12}^n|^2 ds - 4 \int_{\gamma_{21}}^* (\mathbf{g}_{12}^n - \alpha_{21}\mathbf{e}_{h,1}^n) \cdot \mathbf{e}_{h,1}^n ds \\ &= \|\underline{\mathbf{g}}^n\|_*^2 - 4a_1^h(\mathbf{e}_{h,1}^n, \mathbf{e}_{h,1}^n) - 4 \left[a_2^h(\mathbf{e}_{h,2}^n, \mathbf{e}_{h,2}^n) + \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right] (u_{2,c}^n - u_{hc,2}) ds \right] \\ &= \|\underline{\mathbf{g}}^n\|_*^2 - 4 \left[\sum_{i=1}^2 a_i^h(\mathbf{e}_{h,i}^n, \mathbf{e}_{h,i}^n) + \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right] (u_{2,c}^n - u_{hc,2}) ds \right]. \end{aligned}$$

Then the desired equality is proved. $\square$

Note that by (3), we have

$$\frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right] (u_{2,c}^n - u_{hc,2}) ds \geq \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right]^2 ds \geq 0. \quad (34)$$

Next we give the proof of the final convergence result.

proof of Theorem 3.2. By using Lemma 4.2, we have that for any positive integer M :

$$\begin{aligned} \sum_{n=0}^M \sum_{i=1}^2 a_i^h (\mathbf{e}_{h,i}^n, \mathbf{e}_{h,i}^n) &= \sum_{n=0}^M \left[\frac{1}{4} \left(\|\underline{\mathbf{g}}^n\|_*^2 - \|\underline{\mathbf{g}}^{n+1}\|_*^2 \right) - \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right] (u_{2,c}^n - u_{hc,2}) ds \right] \\ &= \frac{1}{4} \left(\|\underline{\mathbf{g}}^0\|_*^2 - \|\underline{\mathbf{g}}^{M+1}\|_*^2 \right) - \sum_{n=0}^M \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right] (u_{2,c}^n - u_{hc,2}) ds. \end{aligned}$$

That is,

$$\sum_{n=0}^M \left[\sum_{i=1}^2 a_i^h (\mathbf{e}_{h,i}^n, \mathbf{e}_{h,i}^n) + \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right] (u_{2,c}^n - u_{hc,2}) ds \right] = \frac{1}{4} \left(\|\underline{\mathbf{g}}^0\|_*^2 - \|\underline{\mathbf{g}}^{M+1}\|_*^2 \right) \leq \frac{1}{4} \|\underline{\mathbf{g}}^0\|_*^2. \quad (35)$$

In terms of (35) and (34), we have

$$\sum_{i=1}^2 a_i^h (\mathbf{e}_{h,i}^n, \mathbf{e}_{h,i}^n) + \frac{1}{\delta} \int_{\Gamma_C} \left[(u_{2,c}^n)^+ - (u_{hc,2})^+ \right]^2 ds \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (36)$$

This completes the proof. $\square$

5 Numerical experiments

In this section, we conduct numerical experiments to demonstrate the performance of the proposed iterative contact-resolving hybrid methods. The demonstration spans from the use of mixed FEM across the entire domain to a hybrid combination of mixed multiscale methods in the larger domain and mixed FEM in the smaller domain. All computations were performed using MATLAB 2021a on a Lenovo ThinkCentre M80q Gen 4 desktop computer equipped with an Intel Core i9-13900T processor and 32 GB of RAM.

The computational domain is set as $\Omega = [0, 1]^2$, with the subdomains defined by $\Omega_1 = [0, 1 - H] \times [0, 1]$ and $\Omega_2 = [1 - H, 1] \times [0, 1]$. We give two test models in Ω_1 as shown in Figure 5.1, where the Young's modulus E is taken as a piecewise constant with heterogeneity pattern. For clearer, we take the yellow part as E_1 and the blue part as E_2 . The Poisson ratio ν_1 (yellow region) and ν_2 (blue region) are both chosen as 0.35 in Sections 5.1-5.3. To simulate nearly incompressible materials, we will assign 0.49 to either ν_1 or ν_2 in Section 5.4. For simplicity, the Poisson's ratio and Young's modulus in Ω_2 are fixed at 0.35 and 1, respectively, throughout all numerical experiments. The Lamé coefficients λ, μ are denoted as

$$\lambda := \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu := \frac{E}{2(1+\nu)}.$$

In order to evaluate the accuracy of the iteration solution (at the last iteration time n), we use the following relative errors:

$$e_\sigma = \frac{\|\underline{\boldsymbol{\sigma}}_h - \underline{\boldsymbol{\sigma}}^n\|_{\mathcal{A}}}{\|\underline{\boldsymbol{\sigma}}_h\|_{\mathcal{A}}}, \quad e_u = \frac{\|\mathbf{u}_h - \mathbf{u}^n\|_{L^2}}{\|\mathbf{u}_h\|_{L^2}}.$$

In addition, the residual error is defined as $\|\underline{\boldsymbol{\sigma}}^n - \underline{\boldsymbol{\sigma}}^{n-1}\|_{L^\infty} + \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{L^\infty}$. The artificial Robin parameters is chosen as $\beta_{12} = \beta_{21} = \beta$. We set the tolerance condition (i.e. residual error) as 10^{-5} when $\beta = 1$ and 10^{-4} if we choose $\beta = \sqrt{h}$ (motivated by [29, 35]).

In Sections 5.1 and 5.2, we present the iteration behaviors, residual errors, and relative errors for both the stress and displacement fields using piecewise constant functions $\mathbf{f}$ in Test Models 1 and 2, respectively. The components of the stress and displacement fields, along with their values on the contact boundary Γ_C , are displayed. In Section 5.3, we report the relative errors and iteration numbers for stress and displacement as the oversampling size varies within the iterative framework associated with CEM-GMsFEM. Finally, Section 5.4 investigates the robustness of the iteration behavior for nearly incompressible materials.

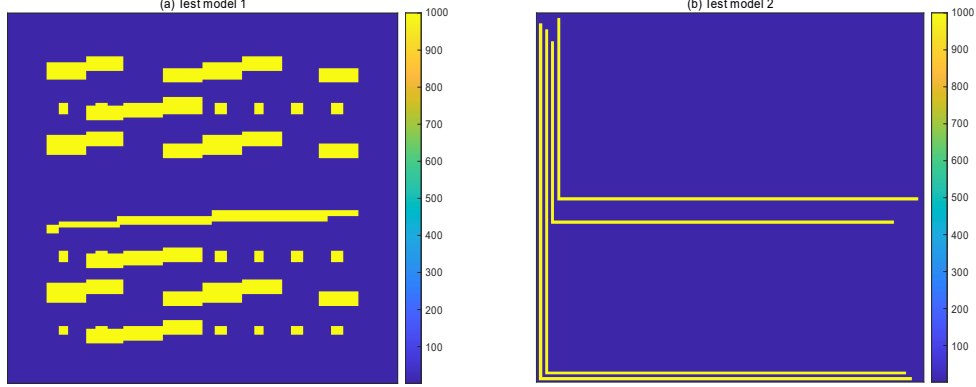


Figure 5.1: Young's Modulus of the test models in Ω_1

Table 5.1: Iterations and errors with different h, β for Algorithm 3.2 in Test model 1

h	β	Iterations	Residual error	e_σ	e_u
1/64	1	22	8.48e-05	0.066917	0.033849
1/128	1	20	9.57e-05	0.044427	0.014647
1/64	$\sqrt{h}$	8	7.69e-04	0.083024	0.050596
1/128	$\sqrt{h}$	9	9.62e-04	0.056787	0.052481

5.1 Test model 1

In the first numerical experiment, we test the iterative contact-resolving hybrid framework associated with mixed method (i.e. Algorithm 3.2). We consider Test Model 1 illustrated in Figure 5.1. Poisson's ratios are set equal with $\nu_1 = \nu_2 = 0.35$, while Young's moduli follow a contrast ratio with $E_2 = 1$ fixed and $E_1 = 10^3$ in Ω_1 . We take $\mathbf{f} = (f_1, f_2)$ with $f_2(x, y) = 0$ and $f_1(x, y)$ expressed as

$$f(x, y) = \begin{cases} -\frac{1}{2}, & \text{if } \frac{7}{8} \leq x \leq 1 \text{ and } \frac{1}{8} \leq y \leq \frac{1}{2} \\ 1, & \text{if } \frac{7}{8} \leq x \leq 1 \text{ and } \frac{5}{8} \leq y \leq \frac{7}{8} \\ 0, & \text{otherwise.} \end{cases}$$

We present the number of iterations, residual errors, and relative errors for the stress and displacement fields with different mesh sizes h in Table 5.1. The results show that for $\beta = 1$, the relative errors for both fields drop about 0.01 within approximately 20 iterations for $h = 1/64$ and $1/128$. In contrast, when $\beta = \sqrt{h}$, the relative error reaches 0.01 after about 10 iterations. These finding indicates that the value of β significantly influences the convergence speed. Iterations with $\beta = 1$ and different h for Test model 1 are presented in Figure 5.5.

A comparison of the stress field components is presented in Figure 5.2. The reference solution (panels (a)-(c)) and our iterative solution with $h = 1/64$, $\beta = 1$ (panels (d)-(f)) are in remarkable agreement, validating the method's accuracy. Excellent agreement is also evident for the displacement field in Figure 5.3. Furthermore, the behavior of the normal stress and displacement components on the contact boundary Γ_C is detailed in Figure 5.4. The results consistently satisfy the contact condition (1d): $\mathbf{u}^n \cdot \mathbf{n}_c = 0$ where $\mathbf{n}_c \cdot (\underline{\sigma}^n \cdot \mathbf{n}_c) < 0$, and $\mathbf{n}_c \cdot (\underline{\sigma}^n \cdot \mathbf{n}_c) = 0$ where $\mathbf{u}^n \cdot \mathbf{n}_c < 0$.

5.2 Test model 2

The second numerical experiment employs Test Model 2 (Figure 5.1) to evaluate Algorithm 3.2. The material parameters are specified with $\nu_1 = \nu_2 = 0.35$ and $E_1 = 10^3, E_2 = 1$. We take $\mathbf{f} = (f_1, f_2)$ with $f_2(x, y) = 0$

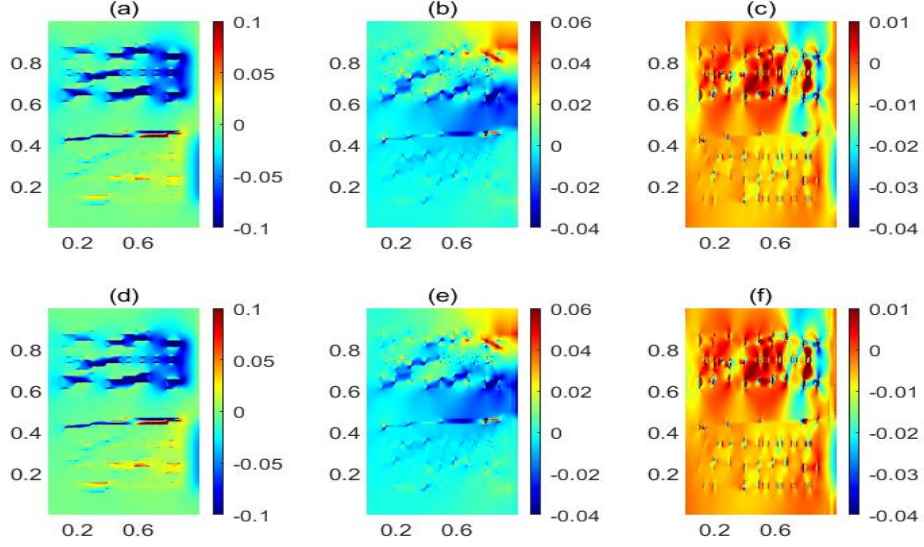


Figure 5.2: (a)-(c): components of stress reference solution in Test model 1 (i.e. $(\underline{\sigma}_h)_{11}, (\underline{\sigma}_h)_{12}, (\underline{\sigma}_h)_{22}$); (d)-(f): components of final stress iteration solution for Algorithm 3.2 in Test model 1 (i.e. $(\underline{\sigma}^n)_{11}, (\underline{\sigma}^n)_{12}, (\underline{\sigma}^n)_{22}$) with $h = 1/64, \beta = 1$.

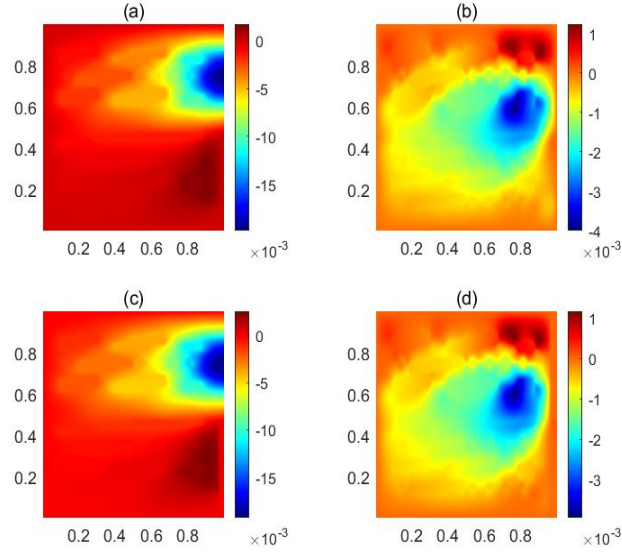


Figure 5.3: (a)-(b): components of displacement reference solution in Test model 1 (i.e. $(\mathbf{u}_h)_1, (\mathbf{u}_h)_2$); (c)-(d): components of final displacement iteration solution for Algorithm 3.2 in Test model 1 (i.e. $(\mathbf{u}^n)_1, (\mathbf{u}^n)_2$) with $h = 1/64, \beta = 1$

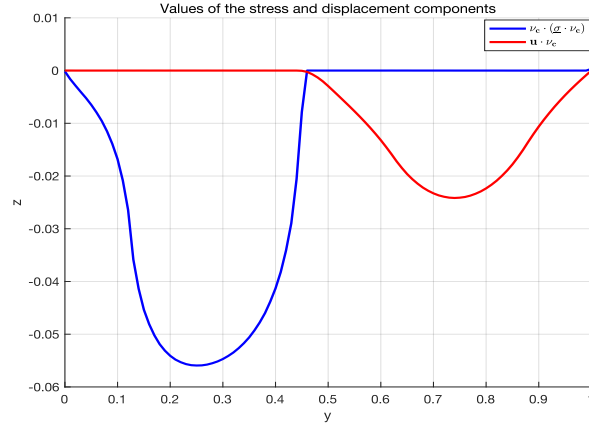


Figure 5.4: Contact values of final stress and displacement iteration solutions for Algorithm 3.2 in Test model 1

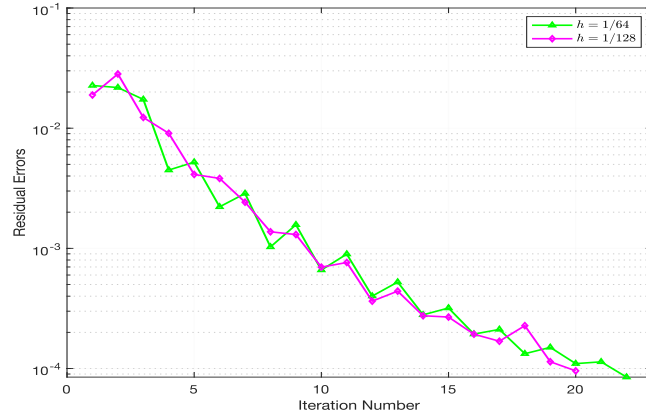


Figure 5.5: Iterations with $\beta = 1$ and different h for Algorithm 3.2 in Test model 1

Table 5.2: Iterations and errors with different h, β for Algorithm 3.2 in Test model 2

h	β	Iterations	Residual error	e_σ	e_u
1/64	1	16	8.82e-05	0.059892	0.034436
1/128	1	16	9.00e-05	0.040873	0.019060
1/64	$\sqrt{h}$	6	8.21e-04	0.068317	0.049629
1/128	$\sqrt{h}$	5	9.79e-04	0.060632	0.048185

and $f_1(x, y)$ expressed as

$$f(x, y) = \begin{cases} 1/2, & \text{if } \frac{7}{8} \leq x \leq 1 \text{ and } \frac{1}{8} \leq y \leq \frac{1}{4} \\ -1/4, & \text{if } \frac{7}{8} \leq x \leq 1 \text{ and } \frac{3}{8} \leq y \leq \frac{5}{8} \\ 1/2, & \text{if } \frac{7}{8} \leq x \leq 1 \text{ and } \frac{7}{8} \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

The number of iterations, residual errors, and relative errors in the stress and displacement fields with different mesh sizes h and Robin parameters β are presented in Table 5.2. We observe that for both β (1 or $\sqrt{h}$), the relative errors for both fields drop about 0.01 for $h = 1/64$ and $1/128$. However, the iteration numbers for $\beta = \sqrt{h}$ is less than that for $\beta = 1$. This indicates that the value of β significantly affects convergence speed. Iterations with $\beta = 1$ and different h for Test model 2 are presented in Figure 5.9.

Visual comparison of solution components provides additional validation of the method's accuracy. Figure 5.6 (a)-(c) display the reference stress solution components, while (d)-(f) present the iteration approximation. The remarkable agreement between these solutions is immediately apparent. Similar agreement is observed for the displacement field in Figure 5.7, further confirming the method's effectiveness. Moreover, our observations in Figure 5.8 confirm the contact boundary condition (1d): when $\mathbf{n}_c \cdot (\underline{\sigma}^n \cdot \mathbf{n}_c) < 0$, we have $\mathbf{u}^n \cdot \mathbf{n}_c = 0$; conversely, $\mathbf{n}_c \cdot (\underline{\sigma}^n \cdot \mathbf{n}_c) = 0$ when $\mathbf{u}^n \cdot \mathbf{n}_c < 0$.

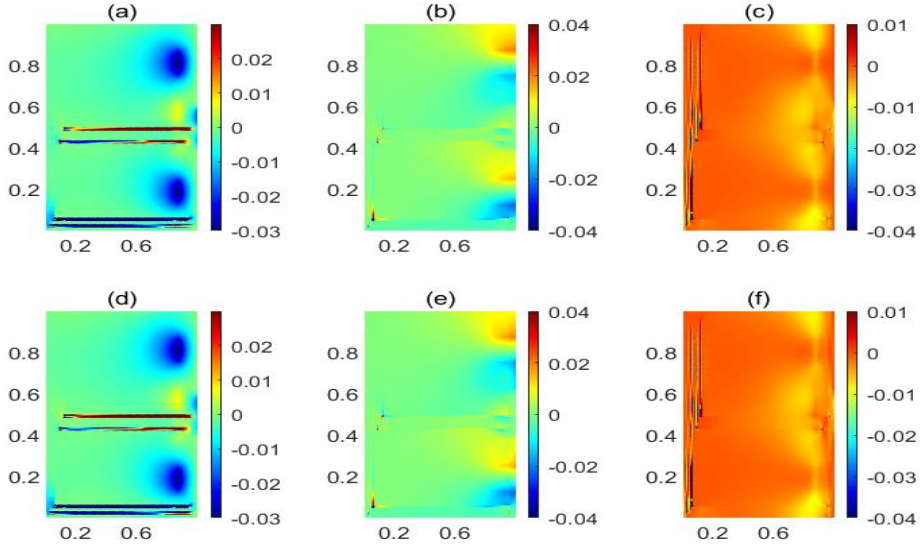


Figure 5.6: (a)-(c): components of stress reference solution in Test model 2 (i.e. $(\underline{\sigma}_h)_{11}, (\underline{\sigma}_h)_{12}, (\underline{\sigma}_h)_{22}$); (d)-(f): components of final stress iteration solution for Algorithm 3.2 in Test model 2 (i.e. $(\underline{\sigma}^n)_{11}, (\underline{\sigma}^n)_{12}, (\underline{\sigma}^n)_{22}$) with $h = 1/64, \beta = 1$.

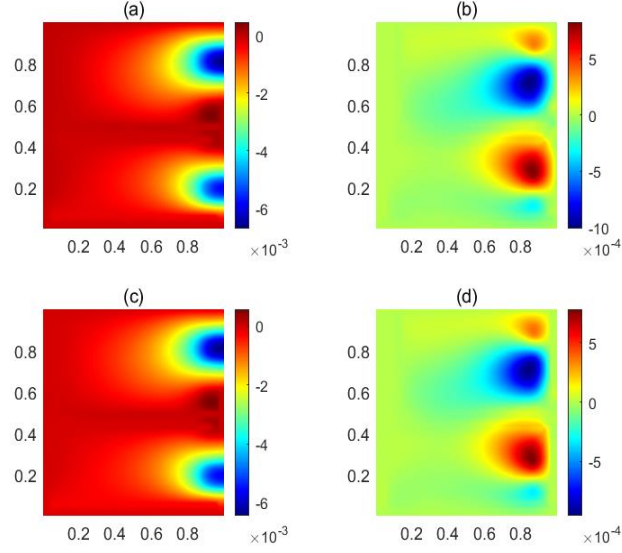


Figure 5.7: (a)-(b): components of displacement reference solution in Test model 2 (i.e. $(\mathbf{u}_h)_1, (\mathbf{u}_h)_2$); (c)-(d): components of final displacement iteration solution for Algorithm 3.2 in Test model 2 (i.e. $(\mathbf{u}^n)_1, (\mathbf{u}^n)_2$) with $h = 1/64, \beta = 1$

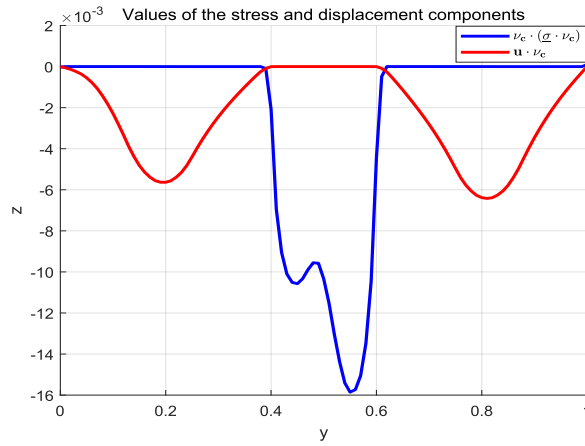


Figure 5.8: Contact values of final stress and displacement iteration solutions for Algorithm 3.2 in Test model 2

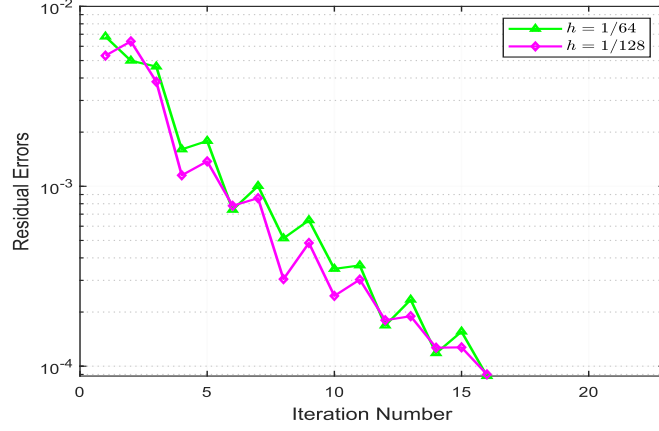


Figure 5.9: Iterations with $\beta = 1$ and different h for Algorithm 3.2 in Test model 2

5.3 Test for the iterative framework associated with CEM-GMsFEM

In this section, we test the iterative contact-resolving hybrid framework associated with mixed CEM-GMsFEM (Algorithm 3.4); the simulation uses Test Model 2 with all material parameters (ν_1 , ν_2 , E_1 , E_2) and the source term f consistent with Section 5.2. The parameters β , h , and H are fixed at 1, $1/64$, and $1/16$, respectively.

Table 5.3 presents the iteration counts, residual errors, and relative errors for the stress and displacement fields in Test Model 2 for different numbers of oversampling layers (denoted as *osly*). The key findings are: (i) The relative errors decrease as *osly* increases. (ii) For *osly* ≥ 3 , the relative error for the stress and displacement fields reach approximately 0.01. These results confirm the accuracy of the proposed method associated with CEM-GMsFEM.

Table 5.3: Iterations and errors with different *osly* for Algorithm 3.4 in Test model 2 ($\beta = 1$, $h = 1/64$, $H = 1/16$)

<i>osly</i>	Iterations	Residual error	e_σ	e_u
1	28	7.52e-05	0.270749	0.268115
2	20	9.43e-05	0.118002	0.041421
3	18	6.19e-05	0.075131	0.038070
4	16	7.73e-05	0.061885	0.035272

5.4 Test for nearly incompressible material

In this section, we examine the performance of the proposed methods in mixed formulations (Algorithms 3.2 and 3.4) applied to nearly incompressible materials via Test Model 1. The parameters are fixed as $\beta = 1$, $h = 1/64$, and $H = 1/16$. The following three parameter cases are studied: (i) $E_1/E_2 = 0.0001$, $\nu_1 = 0.35$, $\nu_2 = 0.49$; (ii) $E_1/E_2 = 10000$, $\nu_1 = 0.49$, $\nu_2 = 0.35$; (iii) $E_1/E_2 = 1$, $\nu_1 = \nu_2 = 0.49$.

Table 5.4 reveals that the relative errors for both stress and displacement fields uniformly converge to approximately 0.01 after a number of iterations, thereby confirming the locking robustness of the mixed FEM approach. The robustness of the CEM-GMsFEM-combined method is similarly verified in Tables 5.5–5.7, where both stress and displacement relative errors attain 0.01 under *osly* = 4 for all cases.

Table 5.4: Iterations and errors with different material parameters for Algorithm 3.2 in Test model 1

E_1/E_2	ν_1, ν_2	Iterations	Residual error	e_σ	e_u
10000	0.49, 0.35	19	8.27e-04	0.088950	0.044466
0.0001	0.35, 0.49	20	9.90e-04	0.095768	0.050037
1	0.49, 0.49	8	6.15e-04	0.065571	0.055915

Table 5.5: Iterations and errors under different *osly* for Algorithm 3.4 with $E_1/E_2 = 0.0001, \nu_1 = 0.35, \nu_2 = 0.49$ in Test model 1

<i>osly</i>	Iterations	Residual error	e_σ	e_u
1	29	9.80e-04	0.239235	0.172521
2	23	9.72e-04	0.151264	0.059722
3	21	9.86e-04	0.104981	0.056348
4	20	9.99e-04	0.098538	0.051538

Table 5.6: Iterations and errors under different *osly* for Algorithm 3.4 with $E_1/E_2 = 10000, \nu_2 = 0.35, \nu_1 = 0.49$ in Test model 1

<i>osly</i>	Iterations	Residual error	e_σ	e_u
1	29	9.45e-04	0.273976	0.360000
2	18	9.99e-04	0.126476	0.080269
3	15	9.70e-04	0.095766	0.047861
4	14	9.95e-04	0.089603	0.046969

Table 5.7: Iterations and errors under different *osly* for Algorithm 3.4 with $E_1/E_2 = 1, \nu_1 = \nu_2 = 0.49$ in Test model 1

<i>osly</i>	Iterations	Residual error	e_σ	e_u
1	27	9.86e-04	0.364996	0.286200
2	10	9.07e-04	0.135061	0.062167
3	10	7.74e-04	0.069805	0.059432
4	8	5.94e-04	0.067812	0.058886

6 Conclusions

In this work, we present an efficient iterative contact-resolving hybrid method tailored for multiscale contact mechanics involving high-contrast material coefficients. The proposed approach localizes the nonlinear contact constraints within a smaller subdomain, while the larger subdomain is described by a linear system. Within this framework, we introduce four distinct discretization strategies. These combine standard (mixed) finite element methods applied over the entire domain, or standard (mixed) multiscale methods in the larger subdomain coupled with standard (mixed) finite element methods in the smaller one. The use of the standard finite element method offers simplicity and ease of implementation. In contrast, the multiscale reduction technique employed in the larger subdomain effectively circumvents the excessive degrees of freedom typically associated with conventional approaches. Furthermore, the incorporation of mixed formulations across the framework enhances robustness against locking effects, even in nearly incompressible material regimes. Convergence analysis and corresponding algorithms are provided for all proposed schemes. Finally, a comprehensive set of numerical experiments demonstrates the accuracy and robustness of the presented methodology.

CRedit authorship contribution statement

Eric T. Chung: Writing – review & editing, Supervision, Resources, Methodology, Funding acquisition, Conceptualization. **Hyea Hyun Kim:** Writing – review & editing, Methodology, Software, Conceptualization. **Xiang Zhong:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT in order to improve readability and language. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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