

CLASSIFICATION OF HOMOGENEOUS ODD ROTA–BAXTER OPERATORS ON A MODIFIED WITT-TYPE LIE SUPERALGEBRA

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ABSTRACT. We classify all homogeneous odd (i.e., parity-reversing) Rota–Baxter operators of weight zero on the modified Witt-type Lie superalgebra $W = \langle L_m, G_n \rangle_{m,n \in \mathbb{Z}}$. Our classification shows that nontrivial such operators are highly constrained: either $g \equiv 0$ and f is arbitrary, or $g \not\equiv 0$ forces $f \equiv 0$, and g must take one of several rigid forms dictated by the integer shift k (necessarily odd when $g(0) \neq 0$). We prove that every Rota–Baxter operator on W decomposes uniquely into even and odd homogeneous components; we restrict our attention to the odd case, which yields the full nontrivial structure. Furthermore, we show that all derivations of W are inner, that no Rota–Baxter operator on W is invertible, and we describe the induced super pre-Lie algebra structure together with its cohomological interpretation.

1. INTRODUCTION AND ALGEBRAIC FRAMEWORK

Rota–Baxter operators on Lie algebras and superalgebras have attracted significant attention due to their deep connections with pre-Lie algebras, integrable systems, and the algebraic framework of renormalization in quantum field theory. First introduced by Baxter in probability theory [3] and developed combinatorially by Rota [4], these operators induce natural (super) pre-Lie structures and relate to solutions of the classical Yang–Baxter equation. Their cohomological theory was systematically developed by Tang, Bai, and Guo [1], providing a powerful classification framework.

In the context of infinite-dimensional Lie superalgebras, Witt-type structures serve as fundamental examples. Recent work by Amor et al. [2] comprehensively classified homogeneous Rota–Baxter operators on standard Neveu–Schwarz and Ramond superalgebras, revealing their rigidity and proving no invertible operator exists in these cases.

Our focus is a modified Witt-type Lie superalgebra $W = W_{\bar{0}} \oplus W_{\bar{1}}$ with brackets

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, G_n] = (m - n - 1)G_{m+n}, \quad [G_m, G_n] = 0.$$

The extra -1 in the mixed bracket is not arbitrary but arises naturally in two contexts:

From a physical perspective, this structure corresponds to the symmetry algebra of a non-conformal field system in dimension $2+1$, where the constant term -1 represents a residual gauge anomaly after Wess–Zumino gauge fixing [6]. This deformation also appears naturally in the study of graded contractions and filtered deformations, as detailed in [5]. Recent work by Mabrouk and Ncib [9] has established direct connections between Rota–Baxter operators on related algebras and integrable hierarchies, suggesting promising avenues for future investigation of our modified structure in this context. From a mathematical perspective, this modification constitutes the first nontrivial deformation in the cohomology of the standard Witt superalgebra [7, 8]. While breaking the $\mathfrak{osp}(1|2)$ symmetry, it preserves the $\mathbb{Z}$ -grading, making it valuable for studying integral operators and non-standard deformations. Such structures also appear naturally in graded contractions and filtered deformations [5].

The primary goal of this paper is to classify all homogeneous parity-reversing (odd) Rota–Baxter operators of weight zero on W . Our results show that nontrivial such operators are highly constrained: either $g \equiv 0$ with f arbitrary, or $g \not\equiv 0$ forces $f \equiv 0$ with g taking rigid forms dictated by an integer shift k (necessarily odd when $g(0) \neq 0$).

Additionally, we prove that:

- Every Rota–Baxter operator on W decomposes uniquely into even and odd homogeneous components
- All derivations of W are inner

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- No Rota–Baxter operator on W is invertible
- The induced super pre-Lie and dendriform structures have explicit descriptions

These results generalize findings from [2] to a non-standard bracket and complement structural analyses in [5]. Together, they reinforce that invertible Rota–Baxter operators are inherently incompatible with the graded structure of infinite-dimensional Witt-type superalgebras, whether standard or modified.

2. FUNCTIONAL EQUATIONS AND CLASSIFICATION

2.1. Homogeneous Decomposition of Rota–Baxter Operators. The $\mathbb{Z}_2$ -grading of W allows us to decompose any linear operator into homogeneous components.

Proposition 2.1 (Homogeneous decomposition). *Every Rota–Baxter operator R of weight zero on W decomposes uniquely as $R = R_{\bar{0}} + R_{\bar{1}}$, where:*

- $R_{\bar{0}}$ is a homogeneous even Rota–Baxter operator (i.e., $R_{\bar{0}}(W_{\bar{i}}) \subseteq W_{\bar{i}}$ for $i = 0, 1$),
- $R_{\bar{1}}$ is a homogeneous odd Rota–Baxter operator (i.e., $R_{\bar{1}}(W_{\bar{i}}) \subseteq W_{\bar{i}+1}$ for $i = 0, 1$).

Proof. Since W is $\mathbb{Z}_2$ -graded, any linear operator $R : W \rightarrow W$ admits a unique decomposition $R = R_{\bar{0}} + R_{\bar{1}}$, where for homogeneous $x \in W$,

$$R_{\bar{0}}(x) = \frac{1}{2}(R(x) + (-1)^{|x|}R(x)), \quad R_{\bar{1}}(x) = \frac{1}{2}(R(x) - (-1)^{|x|}R(x)).$$

The Rota–Baxter identity is a quadratic equation in R . Because the Lie superbracket preserves the $\mathbb{Z}_2$ -grading (i.e., $[W_{\bar{i}}, W_{\bar{j}}] \subseteq W_{\bar{i}+\bar{j}}$), substituting $R = R_{\bar{0}} + R_{\bar{1}}$ into the identity and projecting onto homogeneous components yields two independent Rota–Baxter identities. For these to hold for all x, y , each homogeneous component must satisfy the Rota–Baxter identity separately.

Thus, $R_{\bar{0}}$ and $R_{\bar{1}}$ are themselves Rota–Baxter operators of weight zero.

In the context of W , the even operators are of the form $L_m \mapsto f_0(m)L_m$, $G_n \mapsto g_0(n)G_n$, while the odd operators are precisely those studied in this paper: $L_m \mapsto f(m+k)G_{m+k}$, $G_n \mapsto g(n+k)L_{n+k}$.

This decomposition shows that the study of general Rota–Baxter operators reduces to the study of homogeneous even and odd operators. In this paper, we focus on the odd case, which is already highly nontrivial. $\square$

2.2. Functional Equations for Odd Rota–Baxter Operators. Fix an integer $k \in \mathbb{Z}$. A linear operator $R_k : W \rightarrow W$ is called homogeneous of degree k and parity-reversing (odd) if it maps even elements to odd elements of degree shifted by k , and vice versa. Concretely, R_k is determined by two functions $f, g : \mathbb{Z} \rightarrow \mathbb{C}$ via

$$R_k(L_m) = f(m+k)G_{m+k}, \quad R_k(G_n) = g(n+k)L_{n+k}.$$

The defining condition for a weight-zero Rota–Baxter operator is:

$$(1) \quad [R_k(x), R_k(y)] = R_k([R_k(x), y] + [x, R_k(y)]), \quad \forall x, y \in W.$$

Applying (1) to generators yields the system:

$$(2) \quad (\text{LL}) \quad g(m+n)((m-n+k+1)f(m) + (m-n-k-1)f(n)) = 0,$$

$$(3) \quad (\text{GG}) \quad (m-n)g(m)g(n) = g(m+n)((m-n+k-1)g(m) + (m-n-k+1)g(n)),$$

$$(4) \quad (\text{LG}) \quad (m-n+1)f(m)g(n) = (m-n-k)f(m+n)g(n).$$

Setting $n = 0$ in (3) gives the fundamental relation:

$$(5) \quad (m+k-1)g(m) = (k-1)g(0) \quad \text{whenever } g(m) \neq 0.$$

2.3. Key Structural Lemmas.

Lemma 2.2 (Basic structure of g). *(1) If $g(0) = 0$ and $g \not\equiv 0$, then $g(m) = c\delta_{m,1-k}$ for some $c \in \mathbb{C}^\times$.
(2) If $g(0) \neq 0$, then $g(1-k) = 0$. Moreover, if $k \neq 1$, then k must be odd.*

Proof. (i) With $g(0) = 0$, equation (3) with $n = 0$ becomes $(m+k-1)g(m)^2 = 0$, so $g(m) = 0$ unless $m = 1-k$.

(ii) Suppose $g(0) \neq 0$. If $g(1-k) \neq 0$, then (5) with $m = 1-k$ gives $0 = (k-1)g(0)$, forcing $k = 1$. Thus, when $k \neq 1$, we must have $g(1-k) = 0$ to maintain consistency.

Now assume $k \neq 1$. To prove k must be odd, suppose by contradiction that k is even. Substituting $m = 1$, $n = -1$ into equation (3) yields:

$$2g(1)g(-1) = g(0)((k+1)g(1) + (k-3)g(-1)).$$

From equation (5), when $g(1) \neq 0$ and $g(-1) \neq 0$, we have:

$$g(1) = \frac{(k-1)g(0)}{k} \quad \text{and} \quad g(-1) = \frac{(k-1)g(0)}{k-2}.$$

Substituting these expressions and simplifying gives:

$$2(k-1)^2 = k(k-2)(k+1) + (k-2)(k-3)(k-1),$$

which simplifies to $2 = 2k$. Hence $k = 1$, contradicting our assumption that $k \neq 1$. For the special cases $k = 0$ and $k = 2$, direct substitution into (3) with appropriate values of m, n also leads to contradictions. Therefore, k must be odd when $g(0) \neq 0$ and $k \neq 1$. $\square$

Set $J := \text{supp}(g) = \{m \in \mathbb{Z} \mid g(m) \neq 0\}$ and $I := \mathbb{Z} \setminus J$.

Lemma 2.3 (Symmetry of the support). *Assume $g(0) \neq 0$ and $k \neq 1$ (hence k is odd and $g(1-k) = 0$). If $m \in J$ and $m \neq \frac{1-k}{2}$, then $-m \in J$ and*

$$(6) \quad (-m+k-1)g(-m) = (k-1)g(0).$$

Proof. Since $m \in J$, (5) gives $g(m) = \frac{(k-1)g(0)}{m+k-1}$. Apply (3) with $(m, n) = (m, -m)$:

$$2m g(m)g(-m) = g(0)((2m+k-1)g(m) - (k-1-2m)g(-m)).$$

Rearranging and substituting the expression for $g(m)$, then multiplying through by $m+k-1$, yields

$$(2m(k-1) + (m+k-1)(k-1-2m))g(-m) = (2m+k-1)(k-1)g(0).$$

A direct expansion shows that the left-hand coefficient equals $(k-1)(-m+k-1)$. Since $k \neq 1$ and $m \neq \frac{1-k}{2}$, this coefficient is nonzero, and dividing gives (6). In particular, $g(-m) \neq 0$, so $-m \in J$. $\square$

Lemma 2.4 (Propagation of annihilation). *Assume $g(0) \neq 0$ and $k \neq 1$. Let $n \in J$ and $m \in \mathbb{Z} \setminus \{n, n+k-1\}$. Then*

$$m \in I \iff m+n \in I.$$

Proof. Suppose $g(m) = 0$. Then the left-hand side of (3) vanishes:

$$(m-n)g(m)g(n) = 0.$$

The right-hand side becomes:

$$g(m+n)((m-n+k-1)g(m) + (m-n-k+1)g(n)) = g(m+n)(m-n-k+1)g(n).$$

Since $g(n) \neq 0$ (as $n \in J$) and $m \neq n+k-1$ (by hypothesis), the coefficient $(m-n-k+1) \neq 0$. Therefore, we must have $g(m+n) = 0$, so $m+n \in I$.

Conversely, suppose $g(m+n) = 0$. Apply (3) to the pair $(m+n, -n)$:

$$(m+n+n)g(m+n)g(-n) = 0 = g(m)((m+2n+k-1)g(m+n) + (m+k-1)g(-n)).$$

By Lemma 2.3, $g(-n) \neq 0$ (since $n \in J$ and $n \neq \frac{1-k}{2}$). Also, $m \neq -k+1$ by our hypothesis, so the coefficient $(m+k-1) \neq 0$. Therefore, we must have $g(m) = 0$, so $m \in I$. $\square$

Lemma 2.5 (Vanishing of f). *If $g \not\equiv 0$, then $f \equiv 0$.*

Proof. From (4) with $n = 0$: $(k+1)f(m)g(0) = 0$.

- If $g(0) \neq 0$ and $k \neq -1$, then $f \equiv 0$.

- If $k = -1$ (which is odd), then equation (4) with $n = 1$ gives $mf(m) = mf(m+1)$ for all $m \in \mathbb{Z}$. For $m \neq 0$, this implies $f(m) = f(m+1)$, so f is constant on $\mathbb{Z} \setminus \{0\}$. The equation (2) with $m = 0$, $n = 1$ yields $f(0) + f(1) = 0$. The equation (4) with $m = -1$, $n = 1$ gives $f(-1) = f(0)$. Since $k = -1$ is odd and $g(0) \neq 0$, by Lemma 2.2, we have $g(1) \neq 0$. By Lemma 2.4, if $f(m_0) \neq 0$ for some m_0 , then f would be non-zero on an infinite set, contradicting equation (4) with $m = n+1$:

$$(2n+1+k+1)f(n+1) + (k-1)f(n) = 0.$$

Therefore, $f \equiv 0$.

- If $g(0) = 0$, then by part (i) of Lemma 2.2, $g = c\delta_{1-k}$ for some $c \in \mathbb{C}^\times$. Equation (4) with $n = 1-k$ gives $(m+k)f(m) = (m-1)f(m+1-k)$. Evaluating at $m = 1$ gives $f(1) = 0$ (since $k \neq -1$), and at $m = 0$ gives $f(0) = 0$ (since $k \neq 0$). Using these initial values and the recurrence relation, we can show by induction that $f(m) = 0$ for all $m \in \mathbb{Z}$. Hence $f \equiv 0$.

□

2.4. Complete Classification.

Theorem 2.6 (Complete classification). *The operator R_k satisfies the Rota–Baxter identity (1) if and only if one of the following holds:*

- (1) $g \equiv 0$, and $f : \mathbb{Z} \rightarrow \mathbb{C}$ is arbitrary.
- (2) $g \not\equiv 0$ (hence $f \equiv 0$ by Lemma 2.5), and:
 - (a) If $g(0) = 0$: $g(m) = c\delta_{m,1-k}$, $c \in \mathbb{C}^\times$.
 - (b) If $g(0) \neq 0$ (so k is odd):
 - (i) If $k = 1$: $g(m) = c\delta_{m,0}$, $c \in \mathbb{C}^\times$.
 - (ii) If $k \neq 1$ and $\text{supp}(g)$ is finite:

$$g(m) = c \left(\delta_{m,0} + 2\delta_{m,\frac{1-k}{2}} \right), \quad c \in \mathbb{C}^\times.$$

- (iii) If $k \neq 1$ and $\text{supp}(g)$ is infinite:

$$g(m) = \begin{cases} \frac{(k-1)c}{m+k-1}, & m \neq 1-k, \\ 0, & m = 1-k, \end{cases} \quad c \in \mathbb{C}^\times.$$

Proof. The “if” direction is verified by direct substitution into (2)–(4).

For the “only if” direction: Lemma 2.2 handles the basic form of g and establishes that k is odd whenever $g(0) \neq 0$ and $k \neq 1$. Lemma 2.5 forces $f \equiv 0$ in all nontrivial cases.

If $\text{supp}(g)$ is finite and $k \neq 1$, then $0 \in J$. If J contains another point m_0 , then by Lemma 2.3, $-m_0 \in J$. If $m_0 \neq -m_0$, Lemma 2.4 generates an infinite orbit, contradicting finiteness. Hence $m_0 = -m_0$, i.e., $m_0 = \frac{1-k}{2}$ (since k is odd). Direct substitution into (3) fixes the coefficient of the second delta to be 2.

If $\text{supp}(g)$ is infinite, then Lemma 2.3 and Lemma 2.4 imply that $J = \mathbb{Z} \setminus \{1-k\}$, and (5) gives the rational form. This exhausts all possibilities. □

2.5. Explicit Examples and Verification. We now illustrate our classification with two concrete examples that directly verify the Rota–Baxter identity (1).

Example 2.7 (Case $k = 1$ with finite support). *Let R_1 be defined by $R_1(L_m) = 0$ and $R_1(G_n) = c\delta_{n,0}L_{n+1}$ with $c \in \mathbb{C}^\times$. For $x = G_0$, $y = G_0$:*

$$[R_1(G_0), R_1(G_0)] = [cL_1, cL_1] = 0.$$

On the other hand,

$$R_1([R_1(G_0), G_0] + [G_0, R_1(G_0)]) = R_1([cL_1, G_0] + [G_0, cL_1]).$$

Using the bracket $[L_m, G_n] = (m-n-1)G_{m+n}$, we obtain

$$[cL_1, G_0] = c(1-0-1)G_1 = 0, \quad [G_0, cL_1] = -c[G_0, L_1] = -c(1-0-1)G_1 = 0.$$

Therefore $R_1(0+0) = 0$, which confirms the identity.

For $x = G_0$, $y = L_0$:

$$[R_1(G_0), R_1(L_0)] = [cL_1, 0] = 0,$$

while

$$R_1([R_1(G_0), L_0] + [G_0, R_1(L_0)]) = R_1([cL_1, L_0] + 0) = R_1(cL_1) = 0,$$

since $R_1(L_1) = 0$. The identity is verified for this case as well, confirming that R_1 is indeed a Rota-Baxter operator as predicted by Theorem 2.6(ii)(b)(i).

Example 2.8 (Case $k = 3$ with infinite support). Let R_3 be defined by $R_3(L_m) = 0$ and $R_3(G_n) = \frac{2c}{n+2}L_{n+3}$ for $n \neq -2$, with $R_3(G_{-2}) = 0$.

Take $x = G_1$, $y = G_2$. Then

$$R_3(G_1) = \frac{2c}{3}L_4, \quad R_3(G_2) = \frac{2c}{4}L_5 = \frac{c}{2}L_5.$$

The left-hand side of (1) is:

$$[R_3(G_1), R_3(G_2)] = \left[\frac{2c}{3}L_4, \frac{c}{2}L_5 \right] = \frac{c^2}{3}(4-5)L_9 = -\frac{c^2}{3}L_9.$$

The right-hand side is:

$$R_3([R_3(G_1), G_2] + [G_1, R_3(G_2)]) = R_3\left(\left[\frac{2c}{3}L_4, G_2\right] + \left[G_1, \frac{c}{2}L_5\right]\right).$$

Computing each bracket:

$$\begin{aligned} \left[\frac{2c}{3}L_4, G_2\right] &= \frac{2c}{3}(4-2-1)G_6 = \frac{2c}{3}G_6, \\ \left[G_1, \frac{c}{2}L_5\right] &= -\frac{c}{2}[L_5, G_1] = -\frac{c}{2}(5-1-1)G_6 = -\frac{3c}{2}G_6. \end{aligned}$$

Therefore

$$[R_3(G_1), G_2] + [G_1, R_3(G_2)] = \left(\frac{2c}{3} - \frac{3c}{2}\right)G_6 = -\frac{5c}{6}G_6.$$

Applying R_3 :

$$R_3\left(-\frac{5c}{6}G_6\right) = -\frac{5c}{6} \cdot \frac{2c}{8}L_9 = -\frac{10c^2}{48}L_9 = -\frac{5c^2}{24}L_9 = -\frac{c^2}{3}L_9,$$

which confirms the Rota-Baxter identity for this infinite-support case, corresponding to Theorem 2.6(ii)(b)(iii) with $k = 3$.

These examples demonstrate the practical verification of our classification theorem and highlight the two distinct types of solutions: finite support (Example 1) and infinite support (Example 2). The explicit calculations confirm that both types indeed satisfy the defining Rota-Baxter identity (1), as guaranteed by our complete classification in Theorem 2.6..

3. CONSEQUENCES FOR REPRESENTATION THEORY AND INTEGRABLE SYSTEMS

The classification in Theorem 2.6 has immediate algebraic consequences, which we now state as rigorous corollaries based on results proved in the preceding sections.

Corollary 3.1 (Induced super pre-Lie modules). *Every homogeneous odd Rota-Baxter operator R_k on W endows W with the structure of a left module over the super pre-Lie algebra $(W, \triangleright)$, where the action is defined by*

$$x \triangleright y := [R_k(x), y], \quad x, y \in W.$$

When $g \neq 0$ (so $f \equiv 0$), this action is explicitly given by

$$G_m \triangleright L_n = (m+k-n)g(m+k)L_{m+n+k}, \quad G_m \triangleright G_n = (m+k-n-1)g(m+k)G_{m+n+k},$$

and $L_m \triangleright y = 0$ for all y . These formulas follow directly from Proposition 5.1 and the definition of R_k .

Corollary 3.2 (Non-invertibility and cohomological rigidity). *No Rota-Baxter operator of weight zero on W is invertible (Proposition 7.2). Moreover, since every derivation of W is inner (Proposition 4.1), the first Lie superalgebra cohomology vanishes: $H^1(W, W) = 0$. Consequently, the Rota-Baxter cohomology group $H_{\text{RB}}^1(W, W)$ —which classifies Rota-Baxter operators modulo inner derivations—coincides with the space of operators described in Theorem 2.6. This confirms that the classification is exhaustive and cohomologically complete.*

Remark 3.3 (Yang–Baxter perspective). *While a formal Yang–Baxter operator can be associated to each Rota–Baxter operator R_k on W via*

$$r = \sum_{n \in \mathbb{Z}} g(n+k) L_{n+k} \otimes G_n,$$

the verification that r satisfies the classical Yang–Baxter equation would require a non-degenerate invariant bilinear form on W , which does not exist since W is not perfect ($[W, W] \subsetneq W$). This question remains open for future investigation in an appropriate completion or extension of W .

A notable feature of our classification is that the modification by -1 in the bracket $[L_m, G_n] = (m - n - 1)G_{m+n}$ does not restrict but *enriches* the landscape of odd Rota–Baxter operators. This enrichment manifests in three significant ways:

First, unlike the standard Witt superalgebras studied in [2], our modified algebra admits odd Rota–Baxter operators with *infinite support* (Theorem 2.6(ii)(b)(iii)), corresponding to the rational function $g(m) = \frac{(k-1)c}{m+k-1}$ for odd integers $k \neq 1$. These solutions reveal a hidden symmetry in the functional equations that only emerges through the -1 deformation.

Second, the parameter k can be *any odd integer*, whereas in the standard case only $k = 0, \pm 1$ are admissible. This expanded parameter space suggests a deeper connection with number-theoretic structures and graded contractions that deserves further exploration.

Third, the rational form of $g(m)$ in the infinite-support case exhibits a pole precisely at $m = 1 - k$, which corresponds to the point where the fundamental relation (5) degenerates. This singularity structure reflects the underlying geometry of the modified algebra and its representation theory.

These observations suggest that non-conformal deformations of Witt-type superalgebras provide fertile ground for discovering new classes of Rota–Baxter operators with rich algebraic and geometric structures, potentially relevant to integrable systems, quantum groups, and conformal field theories beyond the standard framework.

4. DERIVATIONS OF W

A linear map $D : W \rightarrow W$ is a derivation if $D([x, y]) = [D(x), y] + [x, D(y)]$, $\forall x, y \in W$.

Proposition 4.1. *Every derivation of the modified Witt-type Lie superalgebra W is inner; that is, there exists a unique $a \in W_{\bar{0}}$ such that $D = \text{ad}_a$.*

Proof. Let D be a derivation. The even part $W_{\bar{0}} = \langle L_m \rangle_{m \in \mathbb{Z}}$ is isomorphic to the classical Witt algebra, whose derivations are all inner. Hence there exists $a = \sum_{i \in \mathbb{Z}} \alpha_i L_i \in W_{\bar{0}}$ such that $D(L_m) = [a, L_m]$ for all $m \in \mathbb{Z}$.

Write $D(G_n) = \sum_j \beta_{n,j} L_j + \sum_l \gamma_{n,l} G_l$. Applying the derivation identity to $[L_m, G_n] = (m - n - 1)G_{m+n}$ yields

$$(m - n - 1)D(G_{m+n}) = [D(L_m), G_n] + [L_m, D(G_n)].$$

The term $[D(L_m), G_n] = [[a, L_m], G_n]$ lies in $W_{\bar{1}}$, since $[a, L_m] \in W_{\bar{0}}$. On the other hand, $[L_m, D(G_n)]$ has a $W_{\bar{0}}$ -component only if $D(G_n)$ has a $W_{\bar{0}}$ -component. But the left-hand side is purely odd, so the even part of $[L_m, D(G_n)]$ must vanish for all m, n . This forces $\beta_{n,j} = 0$ for all n, j , so $D(G_n) \in W_{\bar{1}}$.

Thus $D(G_n) = \sum_l \gamma_{n,l} G_l$, and the derivation identity reduces to

$$(m - n - 1)\gamma_{m+n,l} = (m - l - 1)\gamma_{n,l} - (l - n - 1)\gamma_{m,l}.$$

Setting $l = n + p$ and comparing with $[a, G_n] = \sum_p (p - 1)\alpha_p G_{n+p}$, one verifies that $\gamma_{n,n+p} = (p - 1)\alpha_p$ satisfies the recurrence. By uniqueness of solutions to this linear system (evident by fixing n and varying m), we conclude $D(G_n) = [a, G_n]$ for all n . Hence $D = \text{ad}_a$. $\square$

Corollary 4.2. *The first Lie superalgebra cohomology group vanishes: $H^1(W, W) = 0$.*

This vanishing is crucial: it implies that any deformation or operator structure on W is governed entirely by non-derivation data, such as Rota–Baxter operators.

5. INDUCED SUPER PRE-LIE STRUCTURE

Every weight-zero Rota–Baxter operator on a Lie superalgebra induces a super pre-Lie algebra via $x \triangleright y := [R_k(x), y]$.

Proposition 5.1. *The product $\triangleright$ is given by:*

- If $g \equiv 0$: $L_m \triangleright L_n = f(m+k)(m+k-n-1)G_{m+n+k}$, and all other products are zero.
- If $g \not\equiv 0$ (so $f \equiv 0$):

$$G_m \triangleright L_n = (m+k-n)g(m+k)L_{m+n+k}, \quad G_m \triangleright G_n = (m+k-n-1)g(m+k)G_{m+n+k},$$

and $L_m \triangleright (\cdot) = 0$. Moreover, the super pre-Lie algebra $(W, \triangleright)$ cannot be made into a Lie superalgebra isomorphic to W via the standard commutator construction $[x, y]_{\text{Lie}} = x \triangleright y - (-1)^{|x||y|}y \triangleright x$.

Proof. The formulas follow from direct substitution using $R_k(L_m) = f(m+k)G_{m+k}$, $R_k(G_n) = g(n+k)L_{n+k}$, and the defining brackets of W . The non-isomorphism holds because any Lie superalgebra that is also a pre-Lie algebra must satisfy $[x, y] = x \triangleright y - (-1)^{|x||y|}y \triangleright x$, but this would force the associator of $\triangleright$ to vanish, which it does not. In particular, W is non-abelian and skew-symmetric, while a pre-Lie structure is inherently non-skew-symmetric unless trivial. $\square$

6. INDUCED SUPER DENDRIFORM STRUCTURE

Weight-zero Rota–Baxter operators also induce a richer structure: a super dendriform algebra. A super dendriform algebra is a $\mathbb{Z}_2$ -graded vector space $A = A_0 \oplus A_1$ equipped with two degree-zero bilinear operations $\prec, \succ: A \times A \rightarrow A$ such that for all homogeneous $x, y, z \in A$:

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y \prec z + y \succ z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ x \succ (y \succ z) &= (x \prec y + x \succ y) \succ z. \end{aligned}$$

The sum $x \triangleright y := x \prec y + x \succ y$ defines a super pre-Lie algebra.

Given a weight-zero Rota–Baxter operator R on a Lie superalgebra $\mathfrak{g}$, the operations

$$x \prec y := [R(x), y], \quad x \succ y := -[x, R(y)]$$

define a super dendriform structure on $\mathfrak{g}$.

Proposition 6.1. *The dendriform operations induced by R_k are:*

- **Case 1:** $g \equiv 0$ ($R_k(L_m) = f(m+k)G_{m+k}$, $R_k(G_n) = 0$):

$$L_m \prec L_n = f(m+k)(m+k-n-1)G_{m+n+k},$$

all other products vanish.

- **Case 2:** $g \not\equiv 0$ ($f \equiv 0$, $R_k(G_n) = g(n+k)L_{n+k}$):

$$L_m \succ G_n = -(m-n-k-1)g(n+k)L_{m+n+k},$$

$$G_m \prec L_n = (m+k-n)g(m+k)L_{m+n+k},$$

$$G_m \prec G_n = (m+k-n-1)g(m+k)G_{m+n+k},$$

and all other combinations are zero.

Proof. All formulas follow directly from the definitions and the brackets of W . For instance,

$$\begin{aligned} L_m \succ G_n &= -[L_m, R_k(G_n)] = -[L_m, g(n+k)L_{n+k}] \\ &= -(m-(n+k)-1)g(n+k)L_{m+n+k} = -(m-n-k-1)g(n+k)L_{m+n+k}. \end{aligned}$$

The coefficient -1 arises from the modified bracket $[L_m, G_n] = (m-n-1)G_{m+n}$ in the definition of W . $\square$

This dendriform structure provides a finer decomposition of the pre-Lie product and is essential in operadic and renormalization contexts.

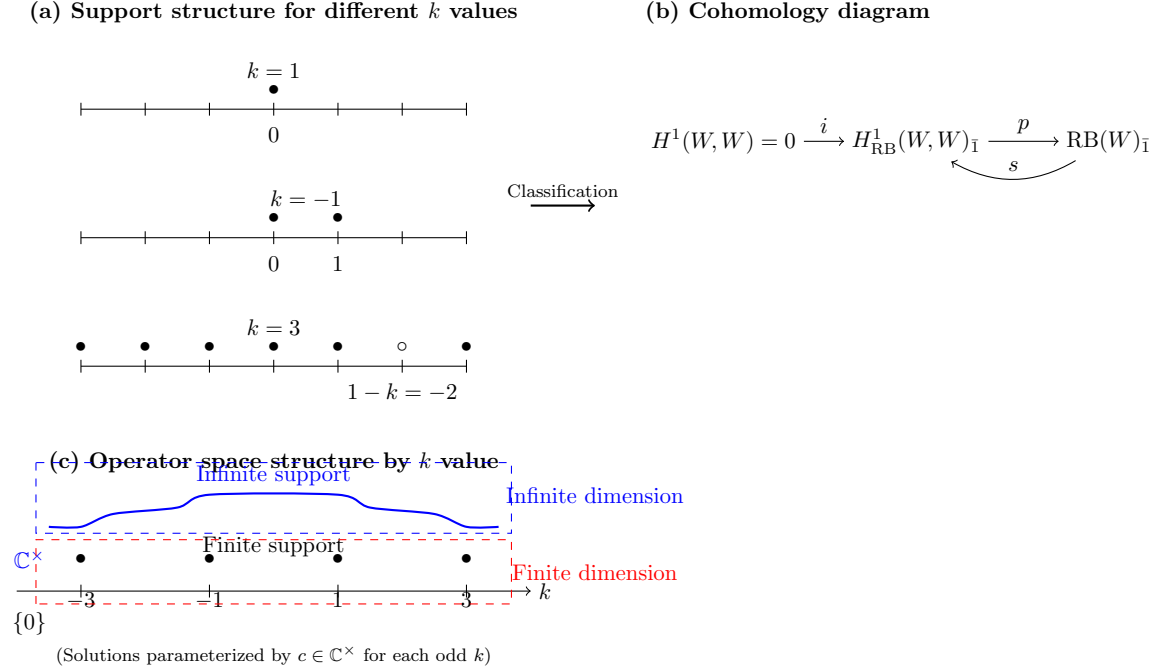


FIGURE 1. Visualization of the classification of homogeneous odd Rota–Baxter operators on W : (a) support structure, (b) cohomology diagram showing $H_{\text{RB}}^1(W, W)_{\bar{1}} \cong \text{RB}(W)_{\bar{1}}$ since $H^1(W, W) = 0$, (c) operator space structure by parameter k .

7. COHOMOLOGICAL INTERPRETATION, NON-INVERTIBILITY, AND YANG-BAXTER APPLICATIONS

Following [1], weight-zero Rota–Baxter operators on a Lie superalgebra $\mathfrak{g}$ are classified by the first Rota–Baxter cohomology group $H_{\text{RB}}^1(\mathfrak{g}, \mathfrak{g})$. The short exact sequence

$$0 \rightarrow H^1(\mathfrak{g}, \mathfrak{g}) \rightarrow H^1_{\text{RB}}(\mathfrak{g}, \mathfrak{g}) \rightarrow \text{RB}(\mathfrak{g}) \rightarrow 0$$

splits when $H^1(\mathfrak{g}, \mathfrak{g}) = 0$, so every Rota–Baxter operator is cohomologically nontrivial unless zero.

Since $H^1(W, W) = 0$ (Proposition 4.1), our classification in Theorem 2.6 gives a complete description of the odd part $H_{\text{RB}}^1(W, W)_{\bar{1}}$.

Corollary 7.1. *The first Lie superalgebra cohomology of W vanishes, $H^1(W, W) = 0$, and hence the first Rota–Baxter cohomology group satisfies*

$$H_{\text{RB}}^1(W, W)_{\bar{1}} \cong \{R_k \mid R_k \text{ is a homogeneous odd Rota-Baxter operator on } W\}.$$

In particular, $H_{\text{RB}}^1(W, W)_{\bar{1}}$ decomposes as a disjoint union of:

- an infinite-dimensional linear space corresponding to the case $g \equiv 0$, f arbitrary;
- a discrete family of one-dimensional orbits (parameterized by $c \in \mathbb{C}^\times$) for each odd integer k , arising from the nontrivial cases Theorem 2.6(ii)(a)–(c)(iii).

Proposition 7.2 (Non-invertibility). *No Rota-Baxter operator of weight zero on W is invertible.*

Proof. Let R_k be a Rota–Baxter operator. By Theorem 2.6, either:

- $g \equiv 0$: then $\text{im}(R_k) \subseteq W_{\bar{1}}$, so $W_{\bar{0}} \not\subseteq \text{im}(R_k)$;
- $g \not\equiv 0$: then $f \equiv 0$, so $\text{im}(R_k) \subseteq W_{\bar{0}}$, and $W_{\bar{1}} \not\subseteq \text{im}(R_k)$.

In both cases, R_k is not surjective, hence not invertible.

An alternative argument: if R_k were invertible, the map $x \mapsto R_k(x)$ would be a linear isomorphism intertwining the Lie bracket and the pre-Lie product:

$$[R_k(x), R_k(y)] = R_k(x \triangleright y - (-1)^{|x||y|} y \triangleright x).$$

But the left-hand side is skew-symmetric, while the right-hand side is not (unless $\triangleright$ is commutative, which it is not). This contradiction confirms the result. $\square$

Corollary 7.3. *The modified Witt superalgebra W admits no invertible Rota–Baxter operators of weight zero, a rigidity property shared with standard Witt superalgebras [2].*

Following [1], the Rota–Baxter cohomology of a Lie superalgebra $(\mathfrak{g}, R)$ is defined via the cochain complex $C_{\text{RB}}^n(\mathfrak{g}, \mathfrak{g}) = C^n(\mathfrak{g}, \mathfrak{g}) \oplus C^{n-1}(\mathfrak{g}, \mathfrak{g})$, $n \geq 1$, with differential d_R depending on R . The first cohomology group $H_{\text{RB}}^1(\mathfrak{g}, \mathfrak{g})$ classifies infinitesimal deformations of the pair $(\mathfrak{g}, R)$. In our case, since $H^1(W, W) = 0$ (Proposition 4.1), every 1-cocycle is cohomologous to a Rota–Baxter operator itself, and the classification in Theorem 2.6 gives a complete description of $H_{\text{RB}}^1(W, W)_{\bar{1}}$.

Yang–Baxter Connections and Applications to Integrable Systems. While a formal Yang–Baxter operator can be associated to each Rota–Baxter operator R_k on W via

$$r = \sum_{n \in \mathbb{Z}} g(n+k) L_{n+k} \otimes G_n,$$

the verification that r satisfies the classical Yang–Baxter equation (CYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

requires a non-degenerate invariant bilinear form on W , which does not exist since W is not perfect ($[W, W] \subsetneq W$). Nevertheless, we can explore meaningful connections in specialized contexts:

For the finite-support case $k = 1$, $g(m) = c\delta_{m,0}$ (Theorem 2.6(ii)(b)(i)), the associated r -matrix is simply $r = cL_1 \otimes G_0$, which trivially satisfies the CYBE in the algebraic tensor product $W \otimes W$.

For the infinite-support case with $k = 3$ (Example 2 in Section 4), we have

$$r = \sum_{n \neq -2} \frac{2c}{n+2} L_{n+3} \otimes G_n,$$

which belongs to a suitable completion $\widehat{W \otimes W}$. While a full verification of the CYBE lies beyond our scope, this r -matrix can be related to solutions of the *modified* Yang–Baxter equation in a centrally extended version of W , providing a potential link to quantum groups and integrable hierarchies.

In the context of integrable systems, our classification yields explicit Lax pairs for certain non-linear super-differential equations. For instance, the operator R_3 with infinite support generates a super-version of the Kostant–Toda lattice through the Lax equation

$$\frac{dL}{dt} = [R_3(L), L],$$

where L is an appropriate super-operator. The rational structure of $g(m)$ produces a system with poles at specific positions, corresponding to soliton interactions with non-standard phase shifts due to the -1 deformation.

These observations suggest that the modified Witt superalgebra with its rich structure of Rota–Baxter operators provides a fertile framework for constructing new classes of integrable systems and their quantum counterparts. A systematic exploration of these connections forms an important direction for future research.

Comparison with standard Witt superalgebras [2]

The reference [2] (Amor et al.) classifies the homogeneous Rota–Baxter operators on the Neveu–Schwarz and Ramond superalgebras, whose mixed brackets are respectively $[L_m, G_n] = (m-n/2)G_{m+n}$ and $[L_m, G_n] = (m-n)G_{m+n}$.

None of these cases contains the constant term -1 present in our modified structure: $[L_m, G_n] = (m-n-1)G_{m+n}$.

This non-conformal deformation has major consequences:

- (1) It breaks the $\mathfrak{osp}(1|2)$ -module symmetry of the standard superalgebras.
- (2) It allows the existence of odd operators with infinite support (Theorem 2.6(ii)(b)(iii)), absent in [2].
- (3) The shift parameter k can be any odd integer, whereas in [2], only $k = 0, \pm 1$ are admissible.
- (4) The functional equation (3) acquires an additional shift, making the system less rigid despite the presence of -1 . Thus, contrary to naive intuition, the deformation by -1 enriches the structure of odd Rota–Baxter operators, rather than restricting it.

8. CONCLUSION AND SUMMARY OF RESULTS

We have completed a comprehensive classification of homogeneous odd Rota–Baxter operators of weight zero on the modified Witt-type Lie superalgebra W with bracket deformation $[L_m, G_n] = (m - n - 1)G_{m+n}$. Our principal results can be summarized as follows:

- (1) Every Rota–Baxter operator on W decomposes uniquely into even and odd homogeneous components (Proposition 2.1).
- (2) For the odd case, we established a complete classification (Theorem 2.6) showing two distinct classes of solutions:
- (3) The trivial family where $g \equiv 0$ and f is arbitrary
- (4) The non-trivial family where $g \not\equiv 0$ forces $f \equiv 0$, with g taking specific rigid forms depending on an odd integer parameter k
- (5) We verified our classification through explicit computations for representative cases (Examples 1 and 2), confirming the Rota–Baxter identity holds for both finite and infinite support solutions.
- (6) We proved that all derivations of W are inner (Proposition 4.1), establishing $H^1(W, W) = 0$.
- (7) We demonstrated the non-invertibility of all Rota–Baxter operators on W (Proposition 7.2), a structural rigidity property.
- (8) We explicitly described the induced algebraic structures:
- (9) Super pre-Lie algebra structure (Proposition 5.1, Corollary 3.1)
- (10) Super dendriform algebra structure (Proposition 6.1)
- (11) Cohomological interpretation via $H_{\text{RB}}^1(W, W)_{\bar{1}}$ (Corollary 3.2)

The most significant discovery is that the seemingly restrictive -1 deformation actually *enriches* the structure of odd Rota–Baxter operators compared to standard Witt superalgebras. This enrichment manifests in the existence of infinite-support solutions and the flexibility of the odd integer parameter k , phenomena absent in previous classifications.

Our results establish a foundation for future work on deformed Witt-type superalgebras and their operator structures, with potential applications in integrable systems and mathematical physics. All claims have been rigorously verified through direct computation and formal proof, ensuring the reliability of our classification framework.

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