# McMullen's game for equicontinuously-twisted badly approximable points in continued fractions and beta expansions

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**Abstract**

In a $\beta$-transformation system for an integer $\beta > 0$ or the Gauss map system, given a sequence of functions $\left(f_n : [0,1] \to [0,1]\right)_{n=1}^{\infty}$ consider the set

$$\mathfrak{BA}\left((f_n)_n\right) := \left\{ x \in X : \liminf_{n\to\infty} \operatorname{dist}\left(T^n x, f_n(x)\right) > 0 \right\},$$

which generalizes the nonrecurrent set (by taking $f_n$'s to be identity functions) and the non-shrinking-target set (by taking $f_n$'s to be constant functions, which generalizes the set of badly approximable numbers). It has been shown if $(f_n)_n$ is a sequence of the same constant functions, then $\mathfrak{BA}\left((f_n)_n\right)$ is winning in the sense of McMullen's game, which implies it is dense and full Hausdorff dimensional everywhere. In this paper, we strengthen this result by extending $(f_n)_n$ to be any equicontinuous sequence.

## 0 Introduction

*Poincaré Recurrence Theorem* (see e.g. [EW11, Theorem 2.11]) is one of the most fundamental results in ergodic theory, and it asserts that for a metric measure preserving system $(X, \operatorname{dist}, \mathcal{B}, \mu, T)$ (i.e., $(X, \operatorname{dist})$ is a metric space, $\mathcal{B}$ is a Borel $\sigma$-algebra of the metric space $X$, $T : X \to X$ is a measurable function

$\mu$ is a $T$-invariant probability measure) with countable base, $\mu$-almost all points return close to themselves under the iterations of $T$ in the sense that

$$\mu(\mathfrak{R}) = 1, \quad \text{where } \mathfrak{R} := \left\{x \in X : \liminf_{n\to\infty} \operatorname{dist}(T^n x, x) = 0\right\}$$

(here and hereafter $\operatorname{dist}(\circ, \circ)$ denotes the usual Euclidean distance when $X = [0,1]$).

A closely related topic is the *shrinking target problem*, which studies the orbits accumulating near a given point. In any metric measure preserving system $(X, \operatorname{dist}, \mathcal{B}, \mu, T)$ where $T$ is ergodic, by Birkhoff's Ergodic Theorem (see e.g. [EW11, Theorem 2.30]), for any fixed target $y \in X$, the orbit of a $\mu$-generic point accumulates around $y$; namely,

$$\forall\, y \in X, \quad \mu\big(\mathfrak{A}(y)\big) = 1, \quad \text{where } \mathfrak{A}(y) := \left\{x \in X : \liminf_{n\to\infty} \operatorname{dist}(T^n x, y) = 0\right\}. \tag{1}$$

In particular, in [Phi67] Philipp[1] focused on the shrinking targets in the following systems

**System I.** Let $\gamma > 1$. $T : [0,1] \to [0,1], x \mapsto \gamma x \mod 1$.

**System II.** $T : [0,1] \to [0,1], x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} \mod 1 & \text{otherwise.} \end{cases}$

with the usual distances and invariant Borel measures $\mu$. The sets $\mathfrak{A}(y)$ of points whose orbits accumulate around a fixed point $y$ are of interests in Diophantine number theory. For example, it is well known (see e.g. [Khi64, Theorem 23] that the set of *badly approximable numbers*

$$\mathfrak{BA} := \left\{x \in [0,1] : \exists c > 0 \text{ such that } \left|x - \frac{p}{q}\right| > \frac{c}{q^2}\ \forall q \in \mathbb{N}, p \in \mathbb{Z}\right\}$$

is equal to the set of numbers whose partial denominators in their continued fraction expansions are bounded, namely $\mathfrak{A}(0)^{\mathsf{C}}$ in System II; i.e.,

$$\mathfrak{BA} = \left\{x \in X : \liminf_{n\to\infty} \operatorname{dist}(T^n x, 0) > 0\right\} = \mathfrak{A}(0)^{\mathsf{C}}, \quad \text{where } (X, T) \text{ is defined as in System II.}$$

Moreover, let $\gamma > 1$ be an integer. A real number $x$ is called *normal* to base $\gamma$ if for any interval $I \subset [0,1]$ with $\gamma$-adic rational endpoints and the Lebesgue

[1] The results in [Phi67] are stronger; it concerns sets $\overline{\mathfrak{A}}(y, \psi) := \limsup_{n\to\infty} \{x \in X : \operatorname{dist}(T^n x, y) < \psi(n)\}$ for a positive sequence $\psi$. An analogous results for recurrence concerning the sets $\overline{\mathfrak{R}}(\psi) := \limsup_{n\to\infty} \{x \in X : \operatorname{dist}(T^n x, x) < \psi(n)\}$ can be found in [HLSW22] and that for twisted recurrence can be found in [KZ23].

measure $\mathsf{Leb}$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{0\le k\le n-1 : T^k x\in I\right\}\right| = \mathsf{Leb}(I), \quad \text{where } (X,T) \text{ is defined as in System I.}$$

Equivalently, a number $x$ is normal to $\gamma$ iff every word of $n$ digits from $\{0,1,\dots,b-1\}$ occurs in the $\gamma$-expansion of $x$ with asymptotic frequency of $\frac{1}{\gamma^n}$ as $n\to\infty$. In System I, every number in $\mathfrak{A}(0)^{\mathsf{C}}$ has a uniform bound on the number of consecutive digit-0 in its base-$\gamma$ expansion. Clearly, such numbers are not normal, so $\mathfrak{A}(0)^{\mathsf{C}}\subset\mathfrak{B}\mathfrak{A}$.

Additionally, classic results (see [Sch80]) have shown that

**Theorem 1.** *1. The Hausdorff dimension of $\mathfrak{A}(y)^{\mathsf{C}}$ in System I is equal to 1 for all $y\in[0,1]$*

*2. The Hausdorff dimension of $\mathfrak{A}(0)^{\mathsf{C}}$ in System II is equal to 1.*

In particular, Theorem 1 implies that $\mathfrak{B}\mathfrak{A}$ and the collection of numbers not normal to a fixed integer base $\gamma$ have full Hausdorff dimension.

An upgrade to Theorem 1 is to consider the subset's winning/losing property in *Schmidt's game*, which was introduced in [Sch66]. Schmidt showed in [Sch66] that

**Theorem 2.** *1. For all integers $\gamma>1$, the set $\mathfrak{A}(y)^{\mathsf{C}}$ in System I is $\alpha$-winning for all $0<\alpha<1/2$.*

*2. The set $\mathfrak{A}(0)^{\mathsf{C}}$ in System II is $\alpha$-winning for all $0<\alpha<1/2$.*

We will define $\alpha$-winning in Section 1. This property is strictly stronger than full Hausdorff dimension, see [Sch66], so Theorem 2 strengthens Theorem 1. Note the fact proved by Schmidt [Sch66] that a countable intersection of $\alpha$-winning sets is also $\alpha$-winning, so the collection of numbers normal to no base is also $\alpha$-winning for all $0<\alpha<1/2$ and in particular it has Hausdorff dimension 1 as a consequence.

A modification of Schmidt's game was introduced by McMullen in [McM10] which is later been refered to as *McMullen's game*.

**Theorem 3.** *1. For all integers $\gamma>1$, the set $\mathfrak{A}(y)^{\mathsf{C}}$ in System I is $\beta$-absolute winning for all $\beta\in\left(0,\frac{1}{3}\right)$.*

*2. The set $\mathfrak{A}(0)^{\mathsf{C}}$ in System II is $\beta$-absolute winning for all $\beta\in\left(0,\frac{1}{3}\right)$.*

We will define $\beta$-absolute-winning in Section 1. $\beta$-absolute-winning for all $\beta\in\left(0,\frac{1}{3}\right)$ is strictly stronger than $\alpha$-winning for all $\alpha\in\left(0,\frac{1}{2}\right)$, see [McM10], so Theorem 3 strengthens Theorem 2.

Given a metric measure preserving system $(X,\mathrm{dist},\mathcal{B},\mu,T)$, an effort to unify the settings of recurrence and shrinking targets is to consider a *twist*, namely a Borel measurable function $f: X\to X$. For a sequence of functions

$\left(f_n : X \to X\right)_n$, define the set of $(f_n)_n$-*twisted-badly-approximable* points for $T$ by

$$\mathfrak{BA}\left((f_n)_n\right) := \left\{ x \in X : \liminf_{n\to\infty} \operatorname{dist}\left(T^n x, f_n(x)\right) > 0 \right\}.$$

The recurrence and shrinking targets settings correspond to the twists $\left(f_n : X \to X\right)_n$ being the identity functions and constant functions respectively. It was shown in [KZ23, Corollary 2.4] that if $T$ is ergodic and $\mu$ has full support, then $\mathfrak{BA}\left((f_n)_n\right)$ has $\mu$-measure 0. However, our result will show that $\mathfrak{BA}\left((f_n)_n\right)$ is "big" in terms of its fractal dimensions and games, despite being null in invariant measures. Here is our main result:

**Theorem 4.** *In Systems I and II, for any equicontinuous sequence of functions* $\left(f_n : [0,1] \to [0,1]\right)_n$*, the set* $\mathfrak{BA}\left((f_n)_n\right)$ *is* $\beta$*-absolute winning for all* $\beta \in \left(0, \frac{1}{3}\right)$.

The structure of the paper is as follows. In Section 1, we will introduce Schimdt's game and McMullen's game, and some important properties of winning sets in either game. In Section 2, we will prove the main result Theorem 4 by induction: in Section 2.1, we will set up the notations and definitions that we will use throughout the proof, and the inductive hypotheses will be established; in Section 2.2 and 2.3, we will construct the strategies for $\beta$-winning in McMullen's game for all $\beta \in \left(0, \frac{1}{3}\right)$ for System I and System II respectively, and the inductive hypothese will be verified; in Section 2.4, we will finish the proof.

# 1 Schmidt's game and McMullen's absolute game

For this article to be self-contained, we define *Schmidt's game* and *McMullen's game*, which are used to state Theorems 2 and 3 and our main result Theorem 4. Let $(X, d)$ be a complete metric space.

For any pair $(z, \rho) \in X \times (0, \infty)$, define the *closed ball* $B(z, \rho)$ to be the set

$$B(z, \rho) := \left\{ x \in X : d(z, r) \leq \rho \right\}.$$

Schmidt's game was first introduced by Schmidt in [Sch66] and since then this infinite game has been used as a notion of fractal dimensions (for example, see [Dan88, BFK11, KW10a, KW10b, BBF+10, Fis09, Tse09]). To define Schmidt's game, let $\alpha, \beta \in (0,1)$ be constants. Consider two players Alice and Bob. Bob and Alice take turns to pick elements $(z_k, r_k)$ and $(\zeta_k, \rho_k)$ of $X \times (0, \infty)$ respectively, such that

$$B(z_1, r_1) \supset B(\zeta_1, \rho_1) \supset B(z_2, r_2) \supset \cdots B(z_k, r_k) \supset B(\zeta_k, \rho_k) \supset B(z_{k+1}, r_{k+1}) \supset \cdots$$

and

$$\rho_k = \alpha r_k \quad \text{and} \quad r_{k+1} = \beta \rho_k.$$

A subset $S$ of $X$ is called

- $(\alpha,\beta)$-*winning* given constants $\alpha,\beta \in (0,1)$ if Alice can always set up her plays, regardless of how Bob plays, so that $\bigcap_{k=1}^{\infty} B(z_k,r_k) \cap S \neq \varnothing$, and

- $\alpha$-*winning* given a constant $\alpha \in (0,1)$ if $S$ is $(\alpha,\beta)$-winning for all $\beta \in (0,1)$, and

- *winning* if $S$ is $\alpha$-winning for some $\alpha \in (0,1)$.

It is easy to observe that

**Proposition 1.** *If $S \subset X$ is winning, then $S$ is dense in $X$.*

$\alpha$-winning in Schmidt's game is stable under taking countable intersections; specifically,

**Proposition 2.** *For all $\alpha \in (0,1)$, the countable intersection of $\alpha$-winning sets is $\alpha$-winning.*

For a proof of Proposition 2, see e.g., [Sch66, Theorem 2].

Winning properties in Schmidt's game also imply full Hausdorff dimensions in *Ahlfors regular* measure spaces. For a constant $s > 0$, a probability measure $\mu$ of $X$ is called *Ahlfors $s$-regular* if there is positive constants $\rho_{\max}$ and $K$ such that for all $(z,\rho) \in X \times (0,\rho_{\max})$

$$\frac{1}{K}\rho^s \le \mu\left(B(z,\rho)\right) \le K\rho^s.$$

The space $X$ is called *Ahlfors $s$-regular* if it is equal to the support of an Ahlfors $s$-regular probability measure. It is very well known (see e.g., [KW10b, Proposition 5.1]) that

**Proposition 3.** *Let $s > 0$. Suppose $(X,d)$ is a Ahlfors $s$-regular metric space. If $S \subset X$ is winning, then the Hausdorff dimension of $S$ is $s$.*

McMullen introduced a variation of an infinite game like Schmidt's game in [McM10]. To define McMullen's game, let $\beta \in (0,1/3)$ be a constant. Again, consider two players Alice and Bob. Bob and Alice take turns to pick elements $(z_k,r_k)$ and $(\zeta_k,\rho_k)$ of $X \times (0,\infty)$ respectively, such that

$$B(z_1,r_1) \supset \left[B(\zeta_1,\rho_1)\backslash B(z_1,r_1)\right] \supset B(z_2,r_2) \supset \cdots B(z_k,r_k) \supset \left[B(\zeta_k,\rho_k)\backslash B(z_k,r_k)\right] \supset B(z_{k+1},r_{k+1}) \supset \cdots$$

and

$$\rho_k \le \beta r_k \quad \text{and} \quad r_{k+1} \ge \beta r_k.$$

A subset $S$ of $X$ is called $\beta$-*absolute winning* given a constant $\beta \in (0,1/3)$ if Alice can always set up her plays, regardless of how Bob plays, so that $\bigcap_{k=1}^{\infty} B(z_k,r_k) \cap S \neq \varnothing$.

It is very obvious that winning in McMullen's game is stronger than winning in Schmidt's game, as in McMullen's game, one may view Alice's moves as merely blocking one of Bob's possible moves, so Alice has very little control over the states of the game. In particular,

**Proposition 4.** *If a subset $S$ is $\beta$-absolute winning in McMullen's game for all $\beta \in \left(0, \frac{1}{3}\right)$, then $S$ is $\alpha$-winning in Schmidt's game for all $\alpha \in \left(0, \frac{1}{2}\right)$.*

Therefore, from Theorem 4, it immediately follows that

**Corollary 1.** *In Systems I and II, for any equicontinuous sequence of functions $\left(f_n : [0,1] \to [0,1]\right)_n$, the set $\mathfrak{BA}\left((f_n)_n\right)$ is $\alpha$-winning (in Schmidt's game) for all $\alpha \in \left(0, \frac{1}{2}\right)$.*

Consequently,

**Corollary 2.** *In Systems I and II, for any equicontinuous sequence of functions $\left(f_n : [0,1] \to [0,1]\right)_n$, the set $\mathfrak{BA}\left((f_n)_n\right)$ has local Hausdorff dimension 1.*

# 2 Proof of Theorem 4

## 2.1 Setup and inductive hypothesis

Let $\beta \in \left(0, \frac{1}{3}\right]$. We will supply Alice with a winning strategy for each system.

First, we define the following terms:

- A point $z \in [0,1]$ is called a $n$th-order **vertex** (also denoted by vertex$^{(n)}$) if $z \in T^{-n}(0)$.
- An interval is called an $n$th-order **cylinder** (also denoted by cylinder$^{(n)}$) if its endpoints are two consecutive (ordered by the natural order of real numbers) vertices$^{(n)}$.

  *Remark.* Note that

  - $T^n$ is injective on any cylinder$^{(n)}$, and
  - given a subset of $[0,1]$, one can count how many cylinders$^{(n)}$ it intersects by counting how many vertices$^{(n)}$ are contained in it; for example, if a set does not contain any vertex$^{(n)}$, then it is included in a cylinder$^{(n)}$.

- For all $n \geq 1$, a point $z \in [0,1]$ is called a $n$th-order ***-vertex** (denoted by *-vertex$^{(n)}$) if $z \in T^{-(n-1)}\left(f_{n-1}\left(c^{(n-1)}\right)\right)$ where $c^{(n-1)}$ is the center of $B_1^{(n-1)}$.
- Denote the collection of the *-vertices of order less than or equal to $n$ by $\mathcal{S}^{(n)}$.
- Denote the diameter of $B_k^{(n)}$ by $\tau_k^{(n)}$, denote the radius of $B_k^{(n)}$ by $r_k^{(n)}$, and denote the diameter of $T^{n-1}B_k^{(n)}$ by $\rho_k^{(n)}$.

Denote the balls chosen by $\mathsf{Bob}$ by $\underbrace{B_1^{(1)}}_{\subset[0,1]} \to \underbrace{B_2^{(1)}}_{\subset B_1^{(1)}\setminus A_1^{(1)}} \to B_3^{(1)} \to \cdots$ and those chosen by $\mathsf{Alice}$ by $\underbrace{A_1^{(1)}}_{\subset B_1^{(1)}} \to A_2^{(1)} \to \cdots$. We will allow *zero and negative indices* for these balls; for example, $B_0^{(n)}$ is the ball one round before $B_1^{(n)}$ and $B_{-5}^{(n)}$ is the ball six rounds before $B_1^{(n)}$.

In both proofs, we will verify that the following hypotheses hold for all $n \in \mathbb{N}$. Let $\rho > 0$ and $\lambda > 0$ be constants that will be specified in the strategies.

**Statement 1.**

- In System I, a natural number $n$ (which will be later referred to as the "$n$th level") can be categorized into one of the three types:
  - **Type I:** Level $n$ is said to be of **Type I** if $B_1^{(n)}$ does not contain any vertex$^{(n)}$.
  - **Type II:** Level $n$ is said to be of **Type II** if the interior of $B_1^{(n)}$ does not contain any vertex$^{(n-1)}$, $T^{n-1}B_1^{(n)}$ contains exactly one vertex$^{(1)}$ and diam $\left(T^{n-1}B_1^{(n)}\right) < \frac{\rho}{\gamma}$.
  - **Type III:** Level $n$ is said to be of **Type III** if The interior of $B_1^{(n)}$ does not contain any vertex$^{(n-1)}$, $T^{n-1}B_1^{(n)}$ contains at least one vertex$^{(1)}$ and diam $\left(T^{n-1}B_1^{(n)}\right) \geq \frac{\rho}{\gamma}$.
- In System II, level $n$ belongs to one of the three types:
  - **Type I:** Level $n$ is said to be of **Type I** if $B_1^{(n)}$ does not contain any vertex$^{(n)}$.
  - **Type II:** Level $n$ is said to be of **Type II** if the interior of $B_1^{(n)}$ does not contain any vertex$^{(n-1)}$, $T^{n-1}B_1^{(n)}$ contains exactly one vertex$^{(1)}$, and diam$(T^{n-1}B_1^{(n)}) < \frac{\beta^2}{4(N^{(n)})^2}$ where $\frac{1}{N^{(n)}}$ is the vertex$^{(1)}$ in $T^{n-1}B_1^{(n)}$.
  - **Type III:** Level $n$ is said to be of **Type III** if the interior of $B_1^{(n)}$ does not contain any vertex$^{(n-1)}$, $T^{n-1}B_1^{(n)}$ contains at least one vertex$^{(1)}$, and diam$(T^{n-1}B_1^{(n)}) \geq \frac{\beta^2}{4(N^{(n)})^2}$ where $\frac{1}{N^{(n)}}$ is the rightmost vertex$^{(1)}$ in $T^{n-1}B_1^{(n)}$.

Denote by $J^{(n)}$ the last appearance of a Type-III level before level $n$; i.e.,

$$J^{(n)} = \begin{cases} \max\left\{k:\ k<n \text{ and level}^{(k)} \text{ is Type-III}\right\} & \text{if } \left\{k:\ k<n \text{ and level}^{(k)} \text{ is Type-III}\right\} \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

**Statement 2.** $\rho_1^{(n)} \geq \begin{cases} \rho & \text{if there is a Type-II level after the last Type-III level and before Level } n; \\ \beta\rho & \text{otherwise.} \end{cases}$

**Statement 3.** For all Type-III levels $n, m$ and $m - n \geq \lambda$, there must exist some $n < k < m$ in such that level $k$ is of Type III.

**Lemma 1.** *Statements 1, 2 and 3 hold for all natural numbers $m, n$.*

To prove Statement 3 holds for all $m, n$, we will need the following lemma:

**Lemma 2.** *For all natural numbers $n, m$ with $1 \leq n < m$, if levels $n, m$ belong to Type II, then there exists some natural number $k$ in $(n, m)$ such that level $k$ is of Type III.*

We claim that under the given strategy, the following property holds:

**Proposition 5.** *There exists some $\delta > 0$ such that for all $n > 1$ and for all $x \in B_1^{(n+1)}$, one has that*

$$\operatorname{dist}\left(T^n x, f_n\left(c^{(n)}\right)\right) \geq \delta.$$

## 2.2 Construction of strategy, I

Let $\rho = \beta \cdot \min\left\{\rho_1^{(1)}, \rho_{\#}\right\}$, where $\rho_{\#}$ is a positive solution to

$$\frac{-\log \rho_{\#} - \log \beta}{\log \gamma} = -4, \tag{2}$$

and $\lambda = \frac{-\log \rho - \log \beta}{\log \gamma} + 1$.

The base case for Lemma 1 easily holds.

Let $M \in \mathbb{N}\backslash\{0\}$. Assume the following:

**Inductive Hypothesis.** Statements 1, 2 and 3 hold for all natural numbers $0 < m, n < M$.

Next, we state the strategy for Alice:

*Case I:* If the $n$th level is of Type I, then Alice relabels $B_1^{(n)}$ by $B_1^{(n+1)}$. Note that this "action" of relabeling by Alice does not alter the state of the game.

*Case II:* If the $n$th level is of Type II, then Alice takes $A_1^{(n)} = B(v^{(n)}, \beta \cdot \tau_1^{(n)})$, where $v^{(n)}$ is the vertex$^{(n)}$ in $B_1^{(n)}$, and then Alice relabels $B_2^{(n)}$ by $B_1^{(n+1)}$.

*Case III:* Suppose that the $n$th level is of Type III. Denote the collection of the *-vertices of order less than or equal to $n$ by $\mathcal{S}^{(n)}$. We will see that by induction $\left|\mathcal{S}^{(n-1)} \cap B_1^{(n)}\right| \leq \lambda$. For all $k \geq 1$, *we ask* Alice *to take the* *center of* $A_k^{(n)}$ *to be* $\begin{cases} \textit{an element of } \mathcal{S}^{(n-1)} \cap B_k^{(n)} & \textit{if } B_k^{(n)} \cap \mathcal{S}^{(n-1)} \neq \varnothing, \\ \textit{the center of } B_k^{(n)} & \textit{if } B_k^{(n)} \cap \mathcal{S}^{(n-1)} = \varnothing, \end{cases}$ *with the maximal allowed radii* $\beta \cdot r_k^{(n)}$, until the time

$$H^{(n)} := \min \left\{ k : \; B_k^{(n)} \cap \mathcal{S}^{(n-1)} = \varnothing \text{ and } \operatorname{diam}(B_k^{(n)}) < \frac{\rho}{\gamma} \right\};$$

we denote the smallest $k$ such that $\operatorname{diam}(B_k^{(n)}) < \frac{\rho}{\gamma}$ by $L^{(1)}$. Note that it takes at most $\lceil \frac{\log \lambda}{\log 2} \rceil$ rounds for Alice to remove all the points in $\mathcal{S}^{(n-1)} \cap B_1^{(n)}$, so

$$H^{(n)} \leq L^{(n)} + \lceil \frac{\log \lambda}{\log 2} \rceil. \tag{3}$$

Note that all cylinders contained $T^{n-1} B_1^{(n)} \subset [0,1]$ have diameter $\frac{1}{\gamma}$, so $B_{H^{(n)}}^{(n)}$ contains at most one vertex$^{(n)}$ and two *-vertex$^{(n)}$. Alice remove these in two turns. Then

$$\begin{aligned} \operatorname{dist}(T^{n-1}(B_{H^{(n)}+2}^{(n)} \backslash A_{H^{(n)}+2}^{(n)}), \mathcal{S}^{(n)}) \geq & \operatorname{radius}(T^{n-1} A_{H^{(n)}+2}) \\ \geq & \beta^{\lceil \frac{\log \lambda}{\log 2} \rceil + 3} \cdot \operatorname{radius}(T^{n-1} B_{L^{(n)}-1}^{(n)}) \\ \geq & \frac{1}{2} \beta^{\lceil \frac{\log \lambda}{\log 2} \rceil + 3} \cdot \frac{\beta}{2\gamma}, \end{aligned}$$

so as a consequence, for all $x \in B_1^{(n+1)} \subset B_{H^{(n)}+2}^{(n)} \backslash A_{H^{(n)}+2}^{(n)})$, one has

$$\operatorname{dist}(T^k x, f\left(c^{(k)}\right)) \geq \gamma^{-\lambda} \frac{1}{2} \beta^{\lceil \frac{\log \lambda}{\log 2} \rceil + 3} \cdot \frac{\beta}{2\gamma} \quad \text{for all } J^{(n)} < k \leq n. \tag{4}$$

Additionally, we have

$$\begin{aligned} \rho_1^{(n+1)} =& \operatorname{diam}\left(T^n B_1^{(n+1)}\right) \\ =& \gamma^n \cdot \operatorname{diam}\left(B_1^{(n+1)}\right) \\ \geq& \gamma^n \cdot \beta \cdot \operatorname{diam}\left(B_{H^{(n)}+2}^{(n)}\right) \\ \overset{(3)}{\geq}& \gamma^n \cdot \beta \cdot \operatorname{diam}\left(B_{(L^{(n)}-1) + \lceil \frac{\log \lambda}{\log 2} \rceil + 3}\right) \\ \geq& \gamma^n \cdot \beta^{\lceil \frac{\log \lambda}{\log 2} \rceil + 4} \cdot \operatorname{diam}\left(B_{L^{(n)}-1}^{(n)}\right) \\ =& \gamma \cdot \beta^{\lceil \frac{\log \lambda}{\log 2} \rceil + 4} \cdot \operatorname{diam}\left(T^{n-1} B_{L^{(n)}-1}^{(n)}\right) \\ \geq& \gamma \cdot \beta^{\lceil \log_2 \frac{-\log \rho - \log \beta}{\log \gamma} \rceil + 4} \cdot \frac{\rho}{\gamma} \\ \overset{(2)}{=}& \rho. \end{aligned} \tag{5}$$

Now we verify our hypotheses hold in the progression.

*Proof of Lemma 1 for System I.* For Statement 1, note that if a ball $B$ contains two vertices$^{(n)}$, then the diameter of $T^{n-1}B$ is at least $\frac{1}{\gamma}$, so if $B_1^{(n)}$ contains a vertex$^{(n)}$ (not of Type I), then either it contains exactly one vertex$^{(n)}$ (of Type II) or $T^{n-1}B_1^{(n)} \geq \frac{1}{\gamma}$ (of Type III).

By assuming Statements 2 and 3 for $n < M$, if Level $(M-1)$ is Type-III, then Statement 2 for $n = M$ follows immediately from (5). Recall that

$$J^{(n)} = \begin{cases} \max\left\{k : \, k < n \text{ and level}^{(k)} \text{ is Type-III}\right\} & \text{if } \left\{k : \, k < n \text{ and level}^{(k)} \text{ is Type-III}\right\} \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

We now prove Lemma 2 for this system and then use it to show Statement 3 holds.

*Proof of Lemma 2 for System I.* Assume the contrary; suppose there are two consecutive Type-II levels $n, m$ with $n < m$ without any intermediate Type-III level. Then $B_1^{(n)}$ contains a vertex$^{(n)}$ $v_1$ and a vertex$^{(m)}$ $v_2$, which are distinct. Then

$$\operatorname{diam}\left(B_1^{(m)}\right) = \beta \cdot \operatorname{diam}\left(B_1^{(n)}\right) \geq \beta \cdot \operatorname{dist}\left(v_1, v_2\right) \geq \beta \cdot \frac{1}{\gamma^m},$$

contradicting that Level $m$ is Type-II. □

Then by Lemma 2, there is at most one Type-II level between Level $J^{(M)}$ and Level $M$. Then

$$\rho_1^{(M)} = \operatorname{diam}\left(T^{M-1}B_1^{(M)}\right) \geq \beta \cdot \operatorname{diam}\left(T^{M-1}B_1^{(J^{(M)}+1)}\right) \geq \beta \cdot \operatorname{diam}\left(T^{J^{(M)}}B_1^{(J^{(M)}+1)}\right) \overset{(5)}{\geq} \beta \cdot \rho.$$

Since Statement 3 is assumed for all $0 < n < M$, by the expanding property of $T$, note that for all $0 < n < M$,

$$\operatorname{diam}(T^{n+\lambda-1}B_1^{(n+\lambda)}) \geq \beta \cdot \gamma^{\lambda-1} \operatorname{diam}(T^n B_1^{(n)}) = \frac{1}{\gamma} \cdot \frac{\gamma}{\rho \cdot \beta} \cdot \rho = 1,$$

then $T^{\lceil \lambda \rceil} B_1^{(n)} = [0,1]$ since its diameter exceeds 1, meaning that within $\lambda$ levels any Type-III level $(n)$, there must be another Type-III level and proving Statement 3 for all $n$. □

From (4), we finally obtain the most crucial result: Proposition 5. We will finish our proof of Theorem 4 in Section 2.4.

## 2.3 Construction of strategy, II

Before we start our statement of the strategy, we state two well-known properties of System II.

**Lemma 3** (Expanding property for System II)**.** *There exists a constant $R > 0$ such that for all positive $n \in \mathbb{N}$ and for all $x \in (0, 1)$,*

$$|D_x T^n| \geq R \cdot \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^2 \right]^n .$$

For the remaining of the paper, we denote $\psi := \left( \frac{1+\sqrt{5}}{2} \right)^2$.

*Proof.* From [Khi64, Chapter 1, §2], in view of the notations in this book, the denominators of the congregants satisfy the recurrence relation:

$$\begin{cases} q_0 = 1, q_1 = a_1, \\ q_n = a_n q_{n-1} + q_{n-2} \quad \forall \text{ integers } n > 1. \end{cases}$$

together with the facts that $a_n \geq 1$ for all $n$, so $q_n$ is greater than or equal to the $n$th Fibonacci number, and hence $q_n \geq \psi^{n/2}$. By the Mean Value Theorem, diameter of a cylinder$^{(n)}$ cannot be less than $\frac{1}{|D_x T^n|}$. The diameter of a cylinder$^{(n)}$ is $\frac{1}{q_n \left( q_n + q_{n-1} \right)}$, so the lemma is proved. □

**Lemma 4** (Bounded distortion for System II)**.** *For all positive $n \in \mathbb{N}$ and for all $x, y$ in an nth order cylinder,*

$$\frac{1}{4} \leq \frac{|D_x T^n|}{|D_y T^n|} \leq 4.$$

Lemma 4 is also well known (see e.g., [CFSi82, Section 7.4, Lemma 2]).

Let $\rho_\#, \lambda$ be positive solutions to the system

$$\begin{cases} (\beta/4)^{\lceil \frac{\log \lambda}{\log 2} \rceil + 6} \geq 2\rho_\#, \\ \frac{R}{4} \beta \psi^{\lambda - 1} \geq \frac{1}{\rho_\#}. \end{cases} \tag{6}$$

One can simply take $\lambda$ large enough such that

$$\frac{4}{R\beta} \cdot \left( \frac{1}{\psi} \right)^{\lambda - 1} \leq \frac{1}{2} \cdot (\beta/4)^{\frac{\log \lambda}{\log 2} + 6},$$

and $\rho_\#$ to be any number in $\left( \frac{4}{R\beta} \cdot \psi^{1-\lambda}, \frac{1}{32} \cdot (\beta/4)^{7 + \frac{\log \lambda}{\log 2}} \right)$. Let $\rho = \min \left\{ \rho_1^{(1)}, \rho_\# \right\}$.

The base case for Lemma 1 easily holds.

Let $M \in \mathbb{N} \backslash \{0\}$. Assume the following:

**Inductive Hypothesis.** Statements 1, 2 and 3 hold for all natural numbers $0 < m, n < M$.

Next, we state the strategy for Alice:

*Case I:* If the $n$th level is of Type I, then Alice relabels $B_1^{(n)}$ by $B_1^{(n+1)}$.

*Case II:* If the $n$th level is of Type II, then Alice takes $A_1^{(n)} = B(v^{(n)}, \beta \cdot r_1^{(n)})$, where $v^{(n)}$ is the vertex$^{(n)}$ in $B_1^{(n)}$ and $r_1^{(n)}$ is the radius of $B_1^{(n)}$, and then Alice relabels $B_2^{(n)}$ by $B_1^{(n+1)}$.

*Case III:* Suppose that the $n$th level is of Type III.

Alice first select $A_1^{(n)}$ with radius $\beta \cdot \text{radius}(B_1^{(n)})$ so that $T^{n-1}A_1^{(n)}$ is placed as left as possible in $T^{n-1}B_1^{(n)}$. Note that $\rho_1^{(n)} \begin{cases} < \frac{1}{N^{(n)}-1} & \text{if } N^{(1)} \neq 1, \\ \leq \frac{1}{N^{(n)}} & \text{if } N^{(1)} = 1; \end{cases}$ in particular, we can conclude that

$$\rho_1^{(n)} < \frac{1}{N^{(n)}}. \tag{7}$$

In Case III, we need to consider the following:

- *Subcase 1*: <u>*Suppose $B_1^{(n)} \backslash A_1^{(n)}$ contains zero or one vertex$^{(n)}$, say $v^{(n)}$.*</u> Then $B_2^{(n)}$ contains at most $\lambda$ *-vertices from $\mathcal{S}^{(n-1)}$, where we denote the collection of the *-vertices of order less than or equal to $n$ by $\mathcal{S}^{(n)}$. By induction, $\left|\mathcal{S}^{(n-1)} \cap B_1^{(n)}\right| \leq \lambda$. Alice then selects each $A_k^{(n)}$ in $A_2^{(n)}, \ldots, A_{1+\lceil \log_2(\lambda+2) \rceil}^{(n)}$ with the maximal radii allowed and first centered at $v^{(n)}$ and then centered at the *-vertices in $\mathcal{S}^{(n-1)} \cap B_2^{(n)}$ if $\mathcal{S}^{(n-1)} \cap B_k^{(n)} \neq \varnothing$, and centered at the center of $B_k^{(n)}$ if $\mathcal{S}^{(n-1)} \cap B_k^{(n)} = \varnothing$. Consequently, for all $J^{(n)} < k < n$ where $J^{(n)}$ is the last appeared Type III level, we have

$$\begin{aligned} \text{dist}\left(T^{k-1}B_1^{(n+1)}, T^{-1}(f_k(c^{(k)}))\right) =& \text{dist}\left(T^{k-1}B_{2+\lceil \log_2(\lambda+2) \rceil}^{(n)}, T^{-1}(f_k(c^{(k)}))\right) \qquad (8) \\ \geq& \frac{1}{4} \cdot \text{radius}(T^{k-1}A_{1+\lceil \log_2(\lambda+2) \rceil}^{(n)}) \\ \geq& \frac{1}{4} \cdot \beta^{1+\lceil \log_2(\lambda+2) \rceil} \cdot \rho_1^{(k)} \geq \frac{1}{4} \cdot \beta^{1+\lceil \log_2(\lambda+2) \rceil} \cdot \rho. \end{aligned}$$

  Hence consequently, for all $x \in B_1^{(n)} = B_{2+\lceil \log_2(\lambda+2) \rceil}^{(n)}$, one has

$$\text{dist}\left(T^k x, f_n\left(c^{(k)}\right)\right) \geq \frac{1}{4} \cdot \beta^{1+\lceil \log_2(\lambda+2) \rceil} \cdot \rho. \tag{9}$$

Then Alice relabels $B^{(n)}_{2+\lceil \log_2(\lambda+2)\rceil}$ by $B^{(n+1)}_1$. Note that

$$\begin{aligned}\operatorname{diam}(T^{n-1}B^{(n+1)}_1) =&\operatorname{diam}\left(T^{n-1}B^{(n)}_{2+\lceil \log_2(\lambda+2)\rceil}\right)\\ \geq&\operatorname{diam}(T^{n-1}B^{(n)}_1)\cdot \beta^{2+\lceil \log_2(\lambda+2)\rceil}\\ \geq&\frac{1}{4\left(N^{(n)}\right)^2}\cdot \beta^{2+\lceil \log_2(\lambda+2)\rceil} \end{aligned}\tag{10}$$

On the other hand, since the ball $T^{n-1}B^{(n+1)}_1$ does not contain any vertices$^{(n)}$, $T^{n-1}B^{(n+1)}_1$ is included in a cylinder $J^{(1)}$ of order 1. Since $J^{(1)}$ is to the left of the cylinder$^{(1)}$ $\left[\frac{1}{N^{(n)}},\frac{1}{N^{(n)}-1}\right]$, its diameter will not exceed that of $\left[\frac{1}{N^{(n)}},\frac{1}{N^{(n)}-1}\right]$; hence $\operatorname{diam}\left(J^{(1)}\right)\leq \frac{1}{N^{(n)}\left(N^{(n)}-1\right)}\leq \frac{2}{\left(N^{(n)}\right)^2}$, so together with the bounded distortion property (Lemma 4) and (10), we have that

$$\begin{aligned}\rho^{(n)}_1 = \frac{\operatorname{diam}(T^nB^{(n+1)}_1)}{\operatorname{diam}\left([0,1]\right)} \geq& \left[\frac{\operatorname{diam}\left(T^{n-1}B^{(n+1)}_1\right)}{\operatorname{diam}\left(J^{(n)}\right)}\right]\cdot\frac{1}{4}\\ \overset{(10)}{\geq}& \left[\frac{1}{4\left(N^{(n)}\right)^2}\cdot \beta^{2+\lceil \log_2(\lambda+2)\rceil}\Big/ \operatorname{diam}\left(J^{(1)}\right)\right]\cdot\frac{1}{4}\\ \geq& \left[\frac{1}{4\left(N^{(n)}\right)^2}\cdot \beta^{2+\lceil \log_2(\lambda+2)\rceil}\Big/ \frac{2}{\left(N^{(n)}\right)^2}\right]\cdot\frac{1}{4} \qquad (11)\\ =&\frac{\beta^{2+\lceil \log_2(\lambda+2)\rceil}}{32}\\ \overset{(6)}{\geq}&\rho. \end{aligned}\tag{12}$$

- *Subcase 2: Suppose $B^{(n)}_1\backslash A^{(1)}_1$ contains more than one vertex$^{(n)}$.* Let $\frac{1}{K^{(n)}}$ be the leftmost vertex$^{(1)}$ in $T^{n-1}(B^{(n)}_1\backslash A^{(n)}_1)$. Since $T^{n-1}(A^{(n)}_1)$ is the leftmost subinterval of $T^{n-1}(B^{(n)}_1)$ and the length of $T^{n-1}(A^{(n)}_1)$ is at least $\frac{\beta}{4}\cdot\rho^{(n)}_1$ by Lemma 4, so $\frac{1}{K^{(1)}}>\frac{\beta}{4}\cdot\rho^{(n)}_1$. Consequently, we have

$$\frac{1}{N^{(n)}}-\frac{1}{K^{(n)}}<\operatorname{diam}(T^{n-1}B^{(n)}_1)=\rho^{(n)}_1<\frac{\beta}{4}\cdot\frac{1}{K^{(n)}},$$

so

$$\frac{1}{N^{(n)}}<(1+\beta/2)\cdot\frac{1}{K^{(n)}}.\tag{13}$$

Since $B_1^{(n)}$ does not contain any vertex$^{(n-1)}$, it contains at most $\lambda$ ($\lambda$ is defined in Lemma) *-vertices of order less than $n$. Then Alice takes the radius of $A_k^{(n)}$ to be $\beta \cdot r_k^{(n)}$ (maximal allowed radius) and its center to be

$$\begin{cases} \text{an element of } \mathcal{S}^{(n-1)} \cap B_k^{(n)} & \text{if } B_k^{(n)} \cap \mathcal{S}^{(n-1)} \neq \varnothing, \\ \text{the center of } B_k^{(n)} & \text{if } B_k^{(n)} \cap \mathcal{S}^{(n-1)} = \varnothing, \end{cases}$$

until the time (until Alice's turn $A_{H^{(n)}}^{(n)}$)

$$H^{(n)} := \min \left\{ k : \, k > \left| \mathcal{S}^{(n)} \right| + 1 \text{ and } \operatorname{diam}(T^{n-1} B_k^{(n)}) < \frac{\beta^2}{4(N^{(n)})^2} \right\}; \tag{14}$$

we denote the smallest $k$ for which $\operatorname{diam}(T^{n-1} B_k^{(n)}) < \frac{\beta^2}{4(N^{(n)})^2}$ by $L^{(n)}$. It is easy to observe that $H^{(n)} \leq L^{(n)} + \lceil \frac{\log \lambda}{\log 2} \rceil$. Additionally, for all cylinders$^{(1)}$ $J$ contained in $T^{n-1}(B_1^{(n)} \backslash A_1^{(n)})$, the cylinder $J$ is contained in $\left[ \frac{1}{K^{(n)}}, \frac{1}{N^{(n)}} \right]$. By (13), in particular, $\frac{1}{N^{(n)}} < \left(1 + \beta/2\right) \cdot \frac{1}{K^{(n)}} < (1+1) \cdot \frac{1}{K^{(n)}}$ so $\left[ \frac{1}{K^{(n)}}, \frac{1}{N^{(n)}} \right] \subset \left[ \frac{1}{2N^{(n)}}, \frac{1}{N^{(n)}} \right]$, and hence

$$\frac{1}{2N^{(n)}} \cdot \frac{1}{2N^{(n)}} < \operatorname{diam}(J) < \frac{1}{N^{(n)}} \cdot \frac{1}{N^{(n)}},$$

and hence $B_{L^{(n)}}^{(n)}$ does not contain any cylinder$^{(n)}$. Therefore, $B_{L^{(n)} + \lceil \frac{\log \lambda}{\log 2} \rceil}^{(n)}$ contains at most one vertex$^{(n)}$, say $v^{(n)}$, and at most two *-vertices$^{(n)}$, say $s_1^{(n)}, s_2^{(n)}$, which Alice will removes them from the game as centers of $A_{H^{(n)}+1}$ and $A_{H^{(n)}+2}$; i.e., $A_{H^{(n)}+1} = B(v^{(n)}, \beta \rho_{H^{(n)}+1}^{(n)})$ and $A_{H^{(n)}+2} = B(s_i^{(n)}, \beta \rho_{H^{(n)}+2}^{(n)})$ where $s_i^{(n)} \in B_{H^{(n)}+1} \backslash A_{H^{(n)}+1}$.

Then Alice relabels $B^{(n)}_{H^{(n)}+3}$ by $B^{(n+1)}_1$. We find

$$\begin{aligned}
\operatorname{dist}\left(T^{n-1}B_1^{(n+1)}, f_n(c^{(n)})\right) \geq & \operatorname{dist}\left(T^{n-1}\left(B^{(n)}_{H^{(n)}+2}\setminus A^{(n)}_{H^{(n)}+2}\right), f_n\left(c^{(n)}\right)\right) \\
\geq & \frac{1}{4}\cdot \operatorname{radius}\left(T^{n-1}A_{L^{(n)}+\lceil\frac{\log\lambda}{\log 2}\rceil+2}\right) \\
\geq & \beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot \operatorname{radius}\left(T^{n-1}B^{(n)}_{L^{(n)}-1}\right) \\
\geq & \beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot \frac{\beta^2}{8(N^{(n)})^2} \\
\overset{(7)}{\geq} & \frac{\beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+6}}{8}\cdot (\rho_1^{(n)}/2)^2 \\
\overset{\text{Statement 2}}{\geq} & \frac{\beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+6}}{8}\cdot (\rho\beta/2)^2.
\end{aligned}$$

Additionally, for all $J^{(n)} < k < n$, it holds that

$$\begin{aligned}
\operatorname{dist}\left(T^{k-1}B_1^{(n+1)}, f_k\left(c^{(k)}\right)\right) \geq & \operatorname{dist}\left(T^{k-1}B^{(J^{(n)}+1)}_{\lceil\log\lambda/\log 2\rceil}, f_k\left(c^{(k)}\right)\right) \\
\geq & \operatorname{radius}\left(T^{k-1}A^{(J^{(n)}+1)}_{\lceil\log\lambda/\log 2\rceil+1}\right) \\
\geq & \frac{1}{4}\cdot \operatorname{radius}\left(T^{k-1}A_1^{(J^{(n)}+1)}\right)\cdot \beta^{\lceil\log\lambda/\log 2\rceil} \\
\geq & \frac{1}{4}\cdot \operatorname{radius}\left(T^{J^{(n)}}A_1^{(J^{(n)}+1)}\right)\cdot \beta^{\lceil\log\lambda/\log 2\rceil} \\
\geq & \frac{1}{16}\cdot \operatorname{radius}\left(T^{(J^{(n)})}B_1^{(J^{(n)}+1)}\right)\cdot \beta^{\lceil\log\lambda/\log 2\rceil+1} \\
\overset{\text{Statement 2}}{\geq} & \frac{1}{16}\cdot \beta\rho\cdot \beta^{\lceil\log\lambda/\log 2\rceil+1}.
\end{aligned}$$

Consequently, for all $x \in B_1^{(n+1)}$ and for all $J^{(n)} < k \leq n$, we have

$$\operatorname{dist}\left(T^{k-1}x, f_k\left(c^{(k)}\right)\right) \geq \min\left\{\frac{\beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+6}}{8}\cdot(\rho\beta/2)^2, \frac{1}{16}\cdot\beta\rho\cdot\beta^{\lceil\log\lambda/\log 2\rceil+1}\right\}. \tag{15}$$

Finally, note that

$$\begin{aligned}
\rho_1^{(n+1)} =& \operatorname{diam}\left(T^n B_1^{(n+1)}\right) \\
\geq & \frac{1}{4}\cdot\beta\cdot\operatorname{diam}\left(T^n B_{H^{(n)}+2}\right) \\
=& \frac{1}{4}\cdot\beta\cdot\operatorname{diam}\left(T^n B_{\left(L^{(n)}-1\right)+\lceil\frac{\log\lambda}{\log 2}\rceil+3}\right) \\
\geq & \frac{1}{4}\cdot\beta\cdot\operatorname{diam}\left(B_{\left(L^{(n)}-1\right)+\lceil\frac{\log\lambda}{\log 2}\rceil+3}\right)\cdot\frac{1}{R}\cdot\psi^{n-1} \\
\geq & 4^{-\lceil\frac{\log\lambda}{\log 2}\rceil+3}\cdot\beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot\operatorname{diam}\left(T^n B_{L^{(n)}-1}^{(n)}\right).
\end{aligned}$$

Note that $T^{n-1}B_{L^{(n)}-1}^{(n)}$ is included in a cylinder $J^{(1)}$ of order 1 with diameter not exceeding that of $\left[\frac{1}{N^{(n)}},\frac{1}{N^{(n)}-1}\right]$ and $\operatorname{diam}\left(J^{(1)}\right) = \frac{1}{N^{(n)}\left(N^{(n)}-1\right)} \leq \frac{2}{\left(N^{(n)}\right)^2}$. Hence by Lemma 4, we have

$$\begin{aligned}
\rho_1^{(n+1)} =& \frac{\operatorname{diam}\left(T^n B_1^{(n+1)}\right)}{\operatorname{diam}\left([0,1]\right)} \\
\geq & \left[\operatorname{diam}\left(T^{n-1}B_1^{(n+1)}\right)\Big/\operatorname{diam}\left(J^{(1)}\right)\right]\cdot\frac{1}{4} \\
\geq & \left[(1/4)^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot\beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot\operatorname{diam}\left(T^{n-1}B_{L^{(n)}-1}^{(n)}\right)\Big/\operatorname{diam}\left(J^{(1)}\right)\right]\cdot\frac{1}{4} \\
\geq & \left[\frac{\beta^2}{4\left(N^{(n)}\right)^2}\Bigg/\operatorname{diam}\left(J^{(1)}\right)\right]\cdot(1/4)^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot\beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot\frac{1}{4} \\
\geq & \left[\frac{\beta^2}{4\left(N^{(n)}\right)^2}\Bigg/\frac{2}{\left(N^{(n)}\right)^2}\right]\cdot(1/4)^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot\beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+4}\cdot\frac{1}{4} \qquad (16) \\
=& \frac{1}{2}\cdot 4^{-\lceil\frac{\log\lambda}{\log 2}\rceil-6}\cdot\beta^{\lceil\frac{\log\lambda}{\log 2}\rceil+6} \\
\overset{(6)}{\geq} & \rho. \qquad (17)
\end{aligned}$$

Now we verify our hypothesis hold in the progression.

We first prove Lemma 2 for this system and then use it to show Statement 3 holds.

*Proof of Lemma 1 for System II.* For Statement 1, note that if a ball $B$ contains two vertices$^{(n)}$, $T^{n-1}B$ contains a cylinder of order $n$, so $\operatorname{diam}\left(T^{n-1}B\right) \geq \frac{1}{N^{(n)}\left(N^{(n)}+1\right)}$ where $\frac{1}{N^{(n)}}$ is the rightmost vertex$^{(1)}$ in $T^{n-1}B$. Hence, if $B_1^{(n)}$ is not of Type I, then either it is Type II, or $\operatorname{diam}\left(T^{n-1}B_1^{(n)}\right) \geq \frac{1}{N^{(n)}\left(N^{(n)}+1\right)} \geq \frac{\beta^2}{4\left(N^{(n)}\right)^2}$ (of Type III).

By assuming Statements 2 and 3 for $n < M$, if Level $(M-1)$ is Type-III, then Statement 2 for $n = M$ follows immediately from (12) and (17). Next, we show that Lemma 2 also holds for System II.

*Proof of Lemma 2 for II.* Assume the contrary; suppose there are two consecutive Type-II levels, say $n$ and $m$ with $n < m$, without any intermediate Type-III level. Since $n, m$ are consecutive Type-II levels without any intermediate Type-III level, we know that $B_2^{(n)} = B_1^{(n+1)} = B_1^{(m)}$. Let $v_1$ be the vertex$^{(n)}$ in $B_1^{(n)}$ and let $v_2$ be the vertex$^{(m)}$ in $B_1^{(m)}$; denote $N_1 := \frac{1}{T^{n-1}v_1}$ and $N_2 := \frac{1}{T^{m-1}v_2}$. Then we have two cases to consider, depending on which connected component of $B_1^{(n)} \backslash A_1^{(n)} = B_1^{(n)} \backslash B\left(v_1, \beta \cdot \frac{\rho_1^{(n)}}{2}\right)$ Bob picks to place $B_2^{(n)}$.

- **Case 1: Suppose that $T^{n-1}B_2^{(n)}$ contains a neighborhood of $0$.** Then

$$\operatorname{diam}\left(T^{m-1}B_1^{(m)}\right) \geq \operatorname{diam}\left(\left(0, \frac{1}{N_2}\right]\right) \cdot \frac{\beta}{4} = \frac{1}{N_2} \cdot \frac{\beta}{4},$$

  but this contradicts that level $m$ is of Type II.

- **Case 2: Suppose that $T^{n-1}B_2^{(n)}$ contains a neighborhood of $1$.** Then

$$\operatorname{diam}\left(T^{m-1}B_1^{(m)}\right) \geq \operatorname{diam}\left(\left[\frac{1}{N_2}, 1\right)\right) \geq \operatorname{diam}\left(\left[\frac{1}{2}, 1\right)\right) \cdot \frac{\beta}{4} \geq \frac{1}{N_2} \cdot \frac{\beta}{4},$$

  but this contradicts that level $m$ is of Type II.

Therefore, we have proved the lemma by contradiction. $\square$

Then by Lemma 2 for System II, there is at most one Type-II level between level $J^{(M)}$ and level $M$, so

$$\rho_1^{(M)} \geq \beta \cdot \operatorname{diam}\left(T^{J^{(M)}} B_1^{(J^{(M)}+1)}\right) \overset{(12),(17)}{\geq} \beta \cdot \rho.$$

Since Statement 3 is assumed for all $0 < n < M$, by the expanding property of $T$ (Lemma 3), note that for all $0 < n < M$,

$$\operatorname{diam}(T^{n+\lambda-1}B_1^{(n+\lambda)}) \geq \beta \cdot \frac{1}{4} \cdot R \cdot \psi^{\lambda-1} \cdot \operatorname{diam}\left(T^n B_1^{(n)}\right) \geq \beta \cdot \frac{1}{4} \cdot R \cdot \psi^{\lambda-1} \cdot \rho \geq 1.$$

meaning that $T^{\lceil\lambda\rceil}B_1^{(n)} = [0,1]$ by its diameter, so within $\lambda$ levels any Type-III level $(n)$, there must be another Type-III level and proving Statement 3 for all $n$. $\square$

From (9) and (15), again we obtain the most crucial result: Proposition 5. We will finish our proof of Theorem 4 in the next section.

### 2.4 Completing the proofs

We are now ready to prove Theorem 4.

Let $\{w\} = \bigcap_{k,n} B_k^{(n)}$. Then by Proposition 5, there exists some $\delta > 0$ such that

$$\operatorname{dist}\left(T^n w, c^{(n)}\right) \geq \delta \quad \forall n \in \mathbb{N}.$$

On the other hand, since the sequence $(f_n)_n$ is equicontinuous, there exists some $\delta' > 0$ so that as long as $|x-y| < \delta'$, $\left|f_n(x) - f_n(y)\right| < \frac{\delta}{2}$.

Since the sequence $(c^n)_n$ converges to $w$, we may find $N$ large enough such that for all $n \geq N$, we have $\operatorname{dist}\left(c^{(n)}, w\right) < \delta'$. Hence for all $n \geq N$

$$\operatorname{dist}\left(T^n w, f_n(w)\right) \geq \operatorname{dist}\left(T^n w, f_n\left(c^{(n)}\right)\right) - \operatorname{dist}\left(f_n\left(c^{(n)}\right), f_n(w)\right) \geq \frac{\delta}{2} \quad \forall n \in \mathbb{N}.$$

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