

Instantaneous Sobolev Regularization for Dissipative Bosonic Dynamics

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Abstract

We investigate quantum Markov semigroups on bosonic Fock space and identify a broad class of infinite-dimensional dissipative evolutions that exhibit instantaneous Sobolev-regularization. Motivated by stability problems in quantum computation, we show that for certain Lindblad operators that are polynomials of creation and annihilation operators, the resulting dynamics immediately transform any initial state into one with finite expectation in all powers of the number operator. A key application is in the bosonic cat code, where we obtain explicit estimates in the trace norm for the speed of convergence. These estimates sharpen existing perturbative bounds at both short and long times, offering new analytic tools for assessing stability and error suppression in bosonic quantum information processing. For example, we improve the strong exponential convergence of the (shifted) 2-photon dissipation to its fixed point to the uniform topology.

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1 Introduction

Quantum dynamical semigroups provide the mathematical framework for describing the time evolution of open quantum systems under the physically natural Markov (memoryless) assumption. From operator semigroup theory, it is well known that such a quantum Markov semigroup (QMS) is uniquely characterized by its infinitesimal generator $\mathcal{L}$, i.e., by the master equation

$$\frac{\partial}{\partial t}\rho(t) = \mathcal{L}(\rho(t)), \quad \rho(0) = \rho_0, \quad (1)$$

for appropriate initial states ρ_0 .

Motivated by well-known examples such as the quantum harmonic oscillator ($\mathcal{L} = -i[a^\dagger a, \cdot]$), the Bose–Hubbard model ($\mathcal{L} = -i[\sum_{i,j} \lambda_{ij} a_i^\dagger a_j + u_i(a_i^\dagger)^2 a_i^2 + \mu_i a_i^\dagger a_i, \cdot]$), and two-photon dissipation ($\mathcal{L} = a^2 \cdot (a^\dagger)^2 - \frac{1}{2}\{(a^\dagger)^2 a^2, \cdot\}$), all expressed in terms of the bosonic annihilation a and creation $a^\dagger$ operators defined on the Fock space $\mathcal{H}$, a separable Hilbert space, it becomes evident that the class of unbounded generators $\mathcal{L}$ is central to physically relevant applications. As Lindblad emphasized,

[boundedness is] a condition which is not fulfilled in many applications (We may hope that this restriction can be ultimately removed using more powerful mathematics) [1, p. 120].

This remark originates from Lindblad’s seminal paper [1] on the classification of bounded generators — equivalently, of QMSs that are uniformly continuous in time — published concurrently with the work of Gorini, Kossakowski, and Sudarshan [2]. Together, these results yield the celebrated Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) theorem, which establishes that any bounded generator acting on the space $\mathcal{T}_1(\mathcal{H})$ of trace-class operators on a separable Hilbert space $\mathcal{H}$ takes the form

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{j=1}^K \left(L_j \rho L_j^\dagger - \frac{1}{2} \{L_j^\dagger L_j, \rho\} \right). \quad (2)$$

For unbounded Hamiltonians and Lindblad operators, such as those described by polynomial functions of bosonic creation and annihilation operators on infinite-dimensional continuous-variable (CV) systems considered in this work, a direct extension of the GKSL form is highly nontrivial. This difficulty is already apparent in the pure birth process, where

$$\mathcal{L} = (a^\dagger)^2 \cdot a^2 - \frac{1}{2} \{a^2 (a^\dagger)^2, \cdot\},$$

fails to generate a QMS [3, Example 3.3] due to failure in preservation of probability mass (the semigroup is not trace preserving). Consequently, one must impose additional, sufficient conditions to ensure the existence of a well-defined QMS [4, 5, 6, 7, 8, 9, 10, 11, 12].

Particularly, the works [12, 13] laid the groundwork for CV systems that have become indispensable in quantum information processing [14, 15, 16, 17, 18, 19, 20, 21] and quantum sensing, particularly in the development of bosonic quantum error-correcting codes such as cat codes [22, 23, 24, 25, 26, 27, 28, 29, 30].

In the recent work [12] (as well as the doctoral theses [31, 32]) two of the present authors have introduced a sufficient condition on the operator in (2) to guarantee a unique solution of the initial-value problem (1). This condition, we term the *Sobolev stability condition*

$$\mathrm{tr}[\mathcal{L}(\rho)(N + \mathbb{1})^k] \leq \omega_k \mathrm{tr}[\rho(N + \mathbb{1})^k] \quad (3)$$

for certain $k \in \mathbb{N}$, ω_k , and suitable states ρ , ensures both the well-posedness of the generated semigroup and that it is *Sobolev preserving*. Specifically, the semigroup restricted to the quantum

Sobolev spaces (defined in Equation (8)) retains its semigroup properties while satisfying

$$\mathrm{tr}[\rho(t)(N + \mathbb{1})^k] \leq e^{\omega_k t} \mathrm{tr}[\rho_0(N + \mathbb{1})^k],$$

for all $t \geq 0$. This property provides crucial a priori estimates for perturbation theory on QMSs, as employed in our prior works [33, 34, 35].

However, in many physically relevant examples, a stronger inequality holds:

$$\mathrm{tr}[\mathcal{L}(\rho)(N + \mathbb{1})^k] \leq -c_k \mathrm{tr}[\rho(N + \mathbb{1})^{k+\delta}] + \mu_k, \quad (4)$$

for constants $c_k > 0$ and $\mu_k \geq 0$. As conjectured in [12], this condition implies not only the boundedness of moments but also their improvement over time. The derivation of this enhanced stability was developed in parallel with the doctoral thesis [32]. The present work integrates these foundational single-mode results into a broader framework, extending the analysis to multi-mode systems and rigorously establishing the consequences for perturbative and dynamical behavior.

In more detail, we identify and analyze a broad class of unbounded GKSL generators satisfying (4) on the bosonic Fock space, whose associated QMSs exhibit a form of regularization, which we call *instantaneous Sobolev-regularization*. That is, for any strictly positive time $t > 0$, the evolution maps arbitrary trace-class inputs to states with finite moments of all orders, with explicit quantitative control of how these moments behave as $t \downarrow 0$. This phenomenon explains stability mechanisms in dissipative bosonic dynamics and introduces new analytic tools for quantifying error suppression in bosonic quantum information protocols.

This instantaneous regularization is of a similar nature as the regularization property of the heat flow (see for example [36]): dissipative polynomial Lindbladians immediately generate finite moments of arbitrary order, paralleling the Sobolev smoothing of the classical heat semigroup. Technically, our approach builds on the generation and a priori estimate framework of [12], together with its extensions in [31]. Although the single-mode proofs are presented in the different format of the dissertation [32], we generalize the approach here to the multimode setting. Unlike the well-understood Gaussian (quadratic) case [37, 9, 10], our analysis also accommodates higher-order monomials in a and $a^\dagger$.

The paper is organized as follows. In Section 2, we introduce the mathematical preliminaries. The main technical results are then developed in Section 3, first for single-mode systems in Section 3.1 and subsequently extended to multi-mode systems in Section 3.2, with particular attention to locality. Perturbative applications to the bosonic cat code are discussed in Section 4, and we conclude in Section 6. Technical proofs and auxiliary results are collected in Appendices A–D.

2 Preliminaries

We begin by recalling the basic setting and assumptions used throughout this work. Let $\mathcal{H}$ denote the bosonic Fock space obtained as the closure of the linear span of the orthonormal basis $\{|n\rangle\}_{n \in \mathbb{N}_0}$ with respect to the standard inner product $\langle \cdot, \cdot \rangle$. The annihilation and creation operators a and $a^\dagger$ act on the Fock basis as $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, and satisfy the canonical commutation relation $[a, a^\dagger] = \mathbb{1}$. The number operator is given by $N = a^\dagger a$. For systems with m bosonic modes, we analogously define a_i , $a_i^\dagger$ and the local number operator $N_i = a_i^\dagger a_i$ for any mode $i \in \{1, \dots, m\}$. We denote by $\mathcal{T}_1$ the Banach space of trace-class operators on $\mathcal{H}$ and by $\mathcal{T}_f = \mathrm{span}\{|n\rangle\langle m| : n, m \in \mathbb{N}_0\}$ the subspace of finite-rank operators in the Fock basis (see [38, 39, 40] for details).

A *quantum Markov semigroup* (QMS) $(T_t)_{t \geq 0}$ is a family of completely positive, trace-preserving maps, which is additionally C_0 -semigroup on $\mathcal{T}_1$, i.e.,

$$T_t \circ T_s = T_{t+s}, \quad T_0 = \text{id}, \quad \text{and} \quad \lim_{t \downarrow 0} T_t \rho = \rho \quad (5)$$

for all $\rho \in \mathcal{T}_1$ and $t, s \geq 0$. Such a semigroup is uniquely determined by its (possibly unbounded) generator $\mathcal{L}$, defined by

$$\mathcal{L}(X) = \lim_{t \rightarrow 0^+} \frac{1}{t} (T_t(X) - X),$$

for X in its domain $\mathcal{D}(\mathcal{L}) = \{X \in \mathcal{T}_1 : t \mapsto T_t(X) \text{ is differentiable on } \mathbb{R}_{\geq 0}\}$. The generator is in both the bounded and unbounded case densely defined and *closed*. Being a *closed* operator means that the graph $\{(X, \mathcal{L}(X)) : X \in \mathcal{D}(\mathcal{L})\}$ is closed in $\mathcal{T}_1 \times \mathcal{T}_1$ (see [36, Ch. III] and [41, Sec. 2–3]).

On finite Hilbert spaces, the GKSL theorem [2, 1] guarantees that $\mathcal{L}$ admits the structure presented in Equation (2). This representation motivates the definition of the unbounded operators considered in the applications of this work, given by

$$\mathcal{L}(\rho) := -i[H, \rho] + \sum_{i=1}^K \left(L_i \rho L_i^\dagger - \frac{1}{2} \{L_i^\dagger L_i, \rho\} \right), \quad (6)$$

where H and the jump operators L_i are polynomials in the creation and annihilation operators $\{a_j, a_j^\dagger\}_{j=1}^m$ of finite total degree d with the domain $\mathcal{D}(\mathcal{L}) := \mathcal{T}_f$. In the multi-mode setting, we restrict to local polynomial interactions of the form $(a_i^\dagger)^{k_i} a_i^{\ell_i} (a_j^\dagger)^{k_j} a_j^{\ell_j}$, corresponding to nearest-neighbour couplings on an interaction graph (V, E) with $|V| = m$ and $(i, j) \in E$. Note, however, that for such unbounded coefficients, $\mathcal{L}$ need not generate a well-defined QMS on $\mathcal{T}_1$.

A general framework for verifying when operators of the form (6) generate a QMS was developed in [12]. Specifically, if for all $\rho \in \mathcal{T}_f$ and integers $k \geq 0$ there exists $\omega_k > 0$ such that

$$\text{tr} \left[\mathcal{L}(\rho) (N + \mathbb{1})^k \right] \leq \omega_k \text{tr} \left[\rho (N + \mathbb{1})^k \right], \quad (7)$$

then $(\mathcal{L}, \mathcal{T}_f)$ admits a unique closure generating a *k-Sobolev-preserving* QMS. More generally, we define $W^{k,1} \subseteq \mathcal{T}_1$ to be the weighted trace-class space

$$W^{k,1} := \left\{ X \in \mathcal{T}_1 : \|(N + \mathbb{1})^{k/2} X (N + \mathbb{1})^{k/2}\|_1 < \infty \right\},$$

equipped with the norm

$$\|X\|_{W^{k,1}} := \|(N + \mathbb{1})^{k/2} X (N + \mathbb{1})^{k/2}\|_1. \quad (8)$$

The notation $\|\cdot\|_{W^{k,1} \rightarrow W^{k',1}}$ denotes the corresponding operator norm.

Definition 2.1 We call a QMS *k-Sobolev preserving* if $e^{t\mathcal{L}}$ defines a semigroup on $W^{k,1}$ and

$$\|e^{t\mathcal{L}}\|_{W^{k,1} \rightarrow W^{k,1}} \leq e^{\omega_k t}, \quad t \geq 0.$$

Furthermore, many physically relevant unbounded models satisfy a stronger moment inequality. Namely, there exist constants $\mu_k > 0$, $c_k \geq 0$, and $\delta > 0$ such that

$$\text{tr} \left[\mathcal{L}(\rho) (N + \mathbb{1})^k \right] \leq -\mu_k \text{tr} \left[\rho (N + \mathbb{1})^{k+\delta} \right] + c_k, \quad \rho \in \mathcal{T}_f. \quad (9)$$

As shown in [12], inequality (9) implies the uniform bound

$$\|e^{t\mathcal{L}}\|_{W^{k,1} \rightarrow W^{k,1}} \leq \max\{1, c_k/\mu_k\}, \quad (10)$$

which holds independently of time. In the present work, we extend this result by showing that for generators satisfying (9), the associated semigroup exhibits *instantaneous improvement of Sobolev regularity*, that is, the production of higher moments at arbitrarily small positive times. Moreover, we extend the single-mode results to multi-modes and present several applications.

Unless otherwise stated, all operator equalities and estimates are understood on the common dense core $\mathcal{T}_f$, and extend by closure to the generated semigroup.

3 Sobolev-regularizing QMS

In this section, we present our main technical results establishing instantaneous Sobolev-regularization. We note that the fundamental derivation for the single-mode case (Section 3.1) was developed in parallel with the results that appeared in the dissertation of one of the authors, [32]. Here, we present these arguments in a self-contained manner before generalizing them to the multi-mode setting in Section 3.2 and applying them to stability problems.

We begin with an extension of Grönwall's inequality [42], which serves as a key analytical tool for handling the assumption

$$\mathrm{tr} \left[\mathcal{L}(\rho)(N + \mathbb{1})^k \right] \leq -\mu_k \mathrm{tr} \left[\rho(N + \mathbb{1})^{k+\delta} \right] + c_k ,$$

through an application of Jensen's inequality. This approach leads to our principal finding: quantum Markov semigroups satisfying the above inequality are *Sobolev-regularizing*, that is,

$$\|e^{t\mathcal{L}}(\rho)\|_{W^{k,1}} \leq c(t)\|\rho\|_1 \quad (11)$$

for a time-dependent constant $c(t) > 0$ and any state ρ . Moreover, in Section 3.2, we extend this result to multimode systems in a locality-preserving manner. For readability, the detailed proofs of the technical results are moved to Section A. We begin with the following auxiliary lemma.

Lemma 3.1 *Let $y : [0, \infty) \rightarrow [0, \infty)$ be a continuously differentiable function satisfying the differential inequality*

$$\frac{dy}{dt}(t) \leq -a y(t)^p + b$$

for constants $a, b > 0$ and exponent $p > 1$. Then,

$$y(t) \leq \left(\max\{y(0) - \left(\frac{b}{a}\right)^{1/p}, 0\}^{1-p} + a(p-1)t \right)^{-\frac{1}{p-1}} + \left(\frac{b}{a}\right)^{\frac{1}{p}} =: z(t) .$$

Proof. The proof can be found in Theorem A.1. □

3.1 Single-mode regularizing QMS

In this section, we apply the tools developed above to establish instantaneous Sobolev-regularization. Define

$$y(t) = \|e^{t\mathcal{L}}(\rho)\|_{W^{k,1}} .$$

Our goal is to apply the assumption (9). Since this assumption does not exactly fit the form required by Theorem 3.1, we first state and prove the following variant of Jensen's inequality adapted to number-operator moments.

Lemma 3.2 (Jensen's inequality for moments) *Let $\rho \in \mathcal{T}_f$ be a state. Then, for all $p \geq q > 0$,*

$$\mathrm{tr} [\rho(N + \mathbb{1})^p] \geq \left(\mathrm{tr} [\rho(N + \mathbb{1})^q] \right)^{\frac{p}{q}} . \quad (12)$$

Proof. Since ρ has finite rank K in the Fock basis, Jensen's inequality shows

$$\mathrm{tr} [\rho(N + \mathbb{1})^p] := \sum_{n=0}^{K-1} p_n (n+1)^{q \frac{p}{q}} \geq \left(\sum_n p_n (n+1)^q \right)^{\frac{p}{q}} = \left(\mathrm{tr} [\rho(N + \mathbb{1})^q] \right)^{\frac{p}{q}}.$$

Here, we used that $\{p_n = \langle n, \rho n \rangle\}_{n=0}^{K-1}$ is a probability distribution and $x \mapsto x^{\frac{p}{q}}$ is convex for $p \geq q > 0$. $\square$

Combining [Theorem 3.1](#) with the above Jensen-type inequality yields the following result.

Theorem 3.3 *Let $(\mathcal{L}, \mathcal{T}_f)$ generate a Sobolev-preserving quantum Markov semigroup satisfying*

$$\mathrm{tr} [\mathcal{L}(\rho)(N + \mathbb{1})^k] \leq -\mu_k \mathrm{tr} [\rho(N + \mathbb{1})^{k+\delta}] + c_k \quad (13)$$

for constants $k, \mu_k, \delta, c_k > 0$. Then, for all $t > 0$,

$$\|e^{t\mathcal{L}}\|_{1 \rightarrow W^{k,1}} \leq \left(\frac{k}{\delta \mu_k t} \right)^{\frac{k}{\delta}} + \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}}. \quad (14)$$

We call such a QMS k -Sobolev-regularizing.

Proof. Since $\mathcal{T}_f$ is dense in $\mathcal{T}_1$ and $e^{t\mathcal{L}}$ is completely positive and trace-preserving, it suffices to prove (14) for $\rho \in \mathcal{T}_f$ a state. Let $\rho(t) = e^{t\mathcal{L}}(\rho)$. Differentiating with respect to t and applying (13) yields

$$\frac{d}{dt} \mathrm{tr} [\rho(t)(N + \mathbb{1})^k] = \mathrm{tr} [\mathcal{L}(\rho(t))(N + \mathbb{1})^k] \leq -\mu_k \mathrm{tr} [\rho(t)(N + \mathbb{1})^{k+\delta}] + c_k.$$

By [Theorem 3.2](#) with $p = k + \delta$ and $q = k$, we obtain

$$\frac{d}{dt} \mathrm{tr} [\rho(t)(N + \mathbb{1})^k] \leq -\mu_k \left(\mathrm{tr} [\rho(t)(N + \mathbb{1})^k] \right)^{\frac{k+\delta}{k}} + c_k.$$

Setting $y(t) := \mathrm{tr} [\rho(t)(N + \mathbb{1})^k]$, we have

$$y'(t) \leq -\mu_k y(t)^{1+\frac{\delta}{k}} + c_k.$$

This is of the form required by [Theorem 3.1](#) with parameters $a = \mu_k$, $b = c_k$, and $p = 1 + \frac{\delta}{k} > 1$. Applying that lemma gives

$$y(t) \leq \left(\max\{y(0) - \left(\frac{c_k}{\mu_k}\right)^{\frac{k}{k+\delta}}, 0\}^{-\frac{\delta}{k}} + \frac{\delta}{k} \mu_k t \right)^{-\frac{k}{\delta}} + \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}}.$$

Using the estimate

$$\left(\max\{y(0) - \left(\frac{c_k}{\mu_k}\right)^{\frac{k}{k+\delta}}, 0\}^{-\frac{\delta}{k}} + \frac{\delta}{k} \mu_k t \right)^{-\frac{k}{\delta}} \leq \left(\frac{k}{\delta \mu_k t} \right)^{\frac{k}{\delta}},$$

we obtain the claimed bound (14). $\square$

Remark 1. The argument above relies solely on Jensen's inequality and the structure of the Sobolev-type norm $\|\cdot\|_{W^{k,1}}$. Therefore, analogous moment-production and regularization results hold for any functional that satisfies an inequality of the form [Theorem 3.1](#).

Remark 2. [Theorem 3.3](#) establishes that if (13) holds for some $\delta > 0$, we can study the dynamics of the whole set of quantum states, by extending the previous known case on $W^{k,1}$ to $\mathcal{T}_1$. By comparing the bounds (10) and (14), we notice that there exists $t_0 > 0$ such that for every $t \geq t_0$

$$\left(\frac{k}{\delta\mu_k t}\right)^{\frac{k}{\delta}} + \left(\frac{c_k}{\mu_k}\right)^{\frac{k}{k+\delta}} \leq \max\left\{1, \frac{c_k}{\mu_k}\right\}. \quad (15)$$

In other words, this new estimate presents a better behavior for large enough t .

3.2 Multit-mode: concentrated moment propagation bounds

Next, we extend the above result to multi-mode systems defined on a D -dimensional lattice (V, E) with $|V| = m$. As a first step, we introduce a locally concentrated Sobolev space. To that end, we define the following reference operator centered at a mode $v \in V$ with a decay constant $\kappa > 0$:

$$\mathcal{W}_v^k(x) = \sum_{i \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} (N_i + \mathbb{1})^{k/2} x (N_i + \mathbb{1})^{k/2} \quad (16)$$

for all $x \in \mathcal{T}_f$ and normalization constant $Z_v = \sum_{j \in V} e^{-\kappa \text{dist}(v,j)}$, which admits lower and upper bounds independent of m (see [Thm. B.1](#)). The spectral decomposition of $N_i + \mathbb{1}$ on a single mode directly translates to multi-modes and the eigenvectors are given by the Fock basis. In this basis and for $a_i = Z_v^{-1} e^{-\kappa \text{dist}(v,i)} > 0$, it is easy to see that

$$\|(\mathcal{W}_v^k)^{-1}(x)\|_1 \leq (a_1 + \dots + a_n)^{-1} \|x\|_1 = \|x\|_1,$$

for $x \in \mathcal{T}_f$, due to the fact that $N_i + \mathbb{1} \geq \mathbb{1}$. By the uniform boundedness theorem and the fact that $\mathcal{T}_f$ is dense in $\mathcal{T}_1$, we conclude that $(\mathcal{W}_v^k)^{-1}$ is a bounded linear operator. If we then define the local Sobolev space $W_v^{k,1} := \mathcal{D}(\mathcal{W}_v^k)$, equipped with the norm

$$\|x\|_{W_v^{k,1}} = \|\mathcal{W}_v^k(x)\|_1$$

for all $x \in \mathcal{D}(\mathcal{W}_v^k)$, we obtain by [12, Lem. 2.2] that $(\mathcal{D}(\mathcal{W}_v^k), \|\cdot\|_{W_v^{k,1}})$ is a Banach space.

One key motivation for the structure of the norm was that control on local moments directly imply control of global moments without any dependence on system size. For example, for any $\alpha \in \mathbb{C}^m$ (see [Thm. B.2](#))

$$\|\alpha\rangle\langle\alpha|\|_{W_v^{k,1}} \leq \max_{i \in \{1, \dots, m\}} \text{tr} \left[|\alpha_i\rangle\langle\alpha_i| (N_i + \mathbb{1})^k \right] \leq \left(\frac{2k}{\ln(k / \max_{i \in \{1, \dots, m\}} |\alpha_i|^2 + 1)} \right)^k.$$

With this notion of a concentrated Sobolev space, we can define k -Sobolev preserving semigroups as follows:

Definition 3.4 We call a QMS (k, v) -Sobolev preserving if $e^{t\mathcal{L}}$ defines a semigroup on $W_v^{k,1}$ and there are constants $\omega, C > 0$ such that

$$\|e^{t\mathcal{L}}\|_{W_v^{k,1} \rightarrow W_v^{k,1}} \leq C e^{\omega t}, \quad t \geq 0.$$

Note that in the above definition, we do not require any independence of the constants of the system size. This definition is only above well-definedness, the constants are improved later on.

Then, analogous to the single-mode setting discussed in [Section 3.1](#), we assume the existence of $k > 0$ and constants $c_k, \mu_k \geq 0$ such that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ satisfies the inequality

$$\mathrm{tr} \left[\mathcal{W}_v^k(\mathcal{L}(\rho)) \right] \leq -\mu_k \mathrm{tr} \left[\mathcal{W}_v^{k+\delta}(\rho) \right] + c_k \quad (17)$$

for all states $\rho \in \mathcal{T}_f$, we set

$$y(t) = \|e^{t\mathcal{L}}(\rho)\|_{W_v^{k,1}},$$

and aim to apply [Theorem 3.1](#). However, due to the summation structure in the definition of $\|\cdot\|_{W_v^{k,1}}$, the Jensen-type inequality for number-operator moments must be modified accordingly.

Lemma 3.5 (Jensen's inequality for multi-mode moments) *Let $\rho \in \mathcal{T}_f$ be a state. Then, for all $p \geq q > 0$,*

$$\mathrm{tr} [\mathcal{W}_v^p(\rho)] \geq \left(\mathrm{tr} [\mathcal{W}_v^q(\rho)] \right)^{\frac{p}{q}}. \quad (18)$$

Proof. The proof follows the lines of the proof of [Theorem 3.2](#), with slightly more general coefficients p_n . Details can be found in [Theorem B.3](#) in [Section B](#). $\square$

As in [Section 3.1](#), we are now able to combine [Theorem 3.1](#) with the above Jensen-type inequality:

Theorem 3.6 *Let $(\mathcal{L}, \mathcal{T}_f)$ generate a (k, v) -Sobolev-preserving QMS satisfying*

$$\mathrm{tr} \left[\mathcal{W}_v^k(\mathcal{L}(\rho)) \right] \leq -\mu_k \mathrm{tr} \left[\mathcal{W}_v^{k+\delta}(\rho) \right] + c_k$$

for constants $\mu_k, \delta, c_k, k > 0$. Then, for all $t > 0$,

$$\|e^{t\mathcal{L}}\|_{1 \rightarrow W_v^{k,1}} \leq \left(\frac{k}{\delta \mu_k t} \right)^{\frac{k}{\delta}} + \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}}.$$

We call such a QMS, (k, v) -Sobolev-regularizing.

Proof. The proof follows the proof of [Theorem 3.3](#) and is presented in [Theorem B.4](#). $\square$

This result completes our abstract analysis of instantaneous Sobolev-regularization in both single and multi-mode settings. Next, we present several examples that satisfy the assumptions of the theorems and demonstrate their direct implications for perturbation theory and simulation.

3.3 Intermediate and large time perturbation bounds

Under a slightly stronger assumption on the degree of dissipation, we can even extend known perturbation-theory bounds for both intermediate and large times to the $1 \rightarrow 1$ norm. To highlight the underlying difficulty, we begin by improving the standard integral equation using our Sobolev-regularization property. It is well known that

$$e^{t\mathcal{L}}(\rho) - e^{t(\mathcal{L}+\mathcal{E})}(\rho) = t \int_0^1 e^{(1-s)t(\mathcal{L}+\mathcal{E})} \mathcal{E} e^{st\mathcal{L}}(\rho) ds$$

holds for generators $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ and $(\mathcal{L} + \mathcal{E}, \mathcal{D}(\mathcal{L} + \mathcal{E}))$, and for ρ an element of the underlying Banach space — such as $\mathcal{T}_1$ — provided the integral is well-defined. Even under the relative boundedness assumptions

$$\|\mathcal{E}(\rho)\| \leq c_1 \|\mathcal{L}(\rho)\| + c_2 \|\rho\|$$

for $\rho \in \mathcal{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{E})$, one does not expect a substantially better bound than

$$\|e^{t\mathcal{L}}(\rho) - e^{t(\mathcal{L}+\mathcal{E})}(\rho)\| \leq t(c_1\|\mathcal{L}(\rho)\| + c_2\|\rho\|),$$

which depends on the output of an unbounded operator (see, e.g., [36, Chap. 9]). In our setting of Sobolev-regularizing quantum Markov semigroups (QMS), we improve the above bound to the uniform topology. We state the result directly in the multi-mode setup, but emphasize that it reduces immediately to the single-mode case used in Section 4.3.

To achieve the improved bounds, we assume that the perturbation is relatively bounded with respect to a power k of $\mathcal{W}_v^k$ with $k < \delta$.

Proposition 3.7 (Intermediate-time perturbation bound) *Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ be the generator of a (k, v) -Sobolev-regularizing QMS with $\delta > k$, and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be an unbounded operator on $\mathcal{T}_1(\mathcal{H}^{\otimes m})$ such that $(\mathcal{L} + \mathcal{E}, \mathcal{D}(\mathcal{L} + \mathcal{E}))$ generates a contractive semigroup and*

$$\|\mathcal{E}(\rho)\|_1 \leq c_1\|\mathcal{W}_v^k(\rho)\| + c_2\|\rho\|_1 = c_1\|\rho\|_{W_v^{k,1}} + c_2\|\rho\|_1.$$

Then,

$$\|e^{t\mathcal{L}} - e^{t(\mathcal{L}+\mathcal{E})}\|_{1 \rightarrow 1} \leq \varepsilon \left(c_1 t^{1-\frac{k}{\delta}} \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 t \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 t \right).$$

Proof. First, by the fundamental theorem of calculus, contractivity, and relative boundedness,

$$\begin{aligned} \|e^{t\mathcal{L}}(\rho) - e^{t(\mathcal{L}+\mathcal{E})}(\rho)\|_1 &= \left\| \varepsilon t \int_0^1 e^{(1-s)t(\mathcal{L}+\mathcal{E})} \mathcal{E} e^{st\mathcal{L}}(\rho) ds \right\|_1 \\ &\leq \varepsilon t \int_0^1 \|\mathcal{E} e^{st\mathcal{L}}(\rho)\|_1 ds \\ &\leq \varepsilon t \int_0^1 c_1 \|e^{st\mathcal{L}}(\rho)\|_{W_v^{k,1}} ds + \varepsilon t c_2 \|\rho\|_1 \end{aligned}$$

for $\rho \in W_v^{k,1}$. Note that

$$s \mapsto e^{(1-s)t(\mathcal{L}+\mathcal{E})} \mathcal{E} e^{st\mathcal{L}}(\rho)$$

is continuous, so the Bochner integral [43, Sec. 3.5–8] is well-defined. This follows from Lemma 2.1 in [33], the relative boundedness assumption, and the Sobolev-regularization property — specifically, the Sobolev-preserving property. Applying the (k, v) -Sobolev-regularization property yields

$$\begin{aligned} \|e^{t\mathcal{L}}(\rho) - e^{t(\mathcal{L}+\mathcal{E})}(\rho)\|_1 &\leq \varepsilon t \left(\int_0^1 c_1 \left(\frac{k}{\delta\mu_k s t} \right)^{\frac{k}{\delta}} ds + c_1 \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 \right) \|\rho\|_1 \\ &= \varepsilon \left(c_1 t^{1-\frac{k}{\delta}} \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 t \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 t \right) \|\rho\|_1. \end{aligned}$$

Since $W_v^{k,1}$ is dense in $\mathcal{T}_1$ and the norms are continuous, closure completes the proof. $\square$

One of our main applications, presented in Section 4–5, is the bosonic quantum error-correction code. A key ingredient is an underlying dynamics that converges exponentially fast to an invariant subspace — the codespace. In addition to the perturbation analysis above, a bound on the propagation of small errors for large times is crucial when the dynamics is not implemented perfectly. To this end, we extend the result of Theorem 6 in [44] to the setting of unbounded generators and then verify the corresponding assumptions used in Section 4 in the case of the bosonic cat code.

A key ingredient is a strengthened exponential convergence assumption enabled by the Sobolev-regularization property. Assuming a (k, v) -Sobolev-regularizing semigroup converges exponentially fast, that is, there exist a contractive projection $\mathcal{P}$ and constants $C, \gamma > 0$ such that

$$\|e^{t\mathcal{L}}(\rho) - \mathcal{P}(\rho)\|_1 \leq Ce^{-\gamma t} \|\rho\|_{W_v^{k,1}}$$

for states $\rho \in W_v^{k,1}$, we can lift this directly to a $1 \rightarrow 1$ trace norm bound:

$$\|e^{t\mathcal{L}}(\rho) - \mathcal{P}(\rho)\|_1 \leq \tilde{C} t^{1-\frac{\tilde{k}}{\delta}} e^{-\gamma t} \|\rho\|_1,$$

for some constant $\tilde{C}$, where δ is a Sobolev-regularization parameter from [Theorem 3.6](#). Importantly, $\tilde{k}$ need not be smaller than δ . A more detailed discussion in the same spirit appears in [Section 4.2](#).

We begin with the case of a unique fixed point/ steady state.

Proposition 3.8 (Perturbation bound for steady states) *Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ and $(\mathcal{L} + \mathcal{E}, \mathcal{D}(\mathcal{L} + \mathcal{E}))$ be generators of a (k, v) -Sobolev-regularizing QMS, where $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is an unbounded operator on $\mathcal{T}_1(\mathcal{H}^{\otimes m})$ satisfying*

$$\|\mathcal{E}(\rho)\|_1 \leq c_1 \|\mathcal{W}_v^k(\rho)\| + c_2 \|\rho\|_1 = c_1 \|\rho\|_{W_v^{k,1}} + c_2 \|\rho\|_1.$$

If $e^{t\mathcal{L}}$ converges to a steady state $\bar{\rho} \in \mathcal{T}_1$, i.e.,

$$\|e^{t\mathcal{L}}(\rho) - \bar{\rho}\|_{1 \rightarrow 1} \leq \tilde{C} t^{1-\frac{\tilde{k}}{\delta}} e^{-\gamma t} \|\rho\|_1$$

for any state $\rho \in \mathcal{T}_1$, $\tilde{k} \in \mathbb{N}$, and $\delta > k$, then for any states $\rho, \sigma \in \mathcal{T}_1$,

$$\|e^{t\mathcal{L}}(\sigma) - e^{t(\mathcal{L} + \varepsilon\mathcal{E})}(\rho)\|_1 \leq \begin{cases} \|\sigma - \rho\|_1 + \varepsilon t^{1-\frac{k}{\delta}} \hat{C}_1 \|\rho\|_1, & t \leq 1, \\ \tilde{C} e^{-\gamma t} \|\sigma - \rho\|_1 + \varepsilon \hat{C}_2 \hat{C}_1 \|\rho\|_1, & t \geq 1, \end{cases}$$

where $\hat{C}_1 = c_1 \left(\frac{k}{\delta\mu_k}\right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k}\right)^{\frac{k}{k+\delta}} + c_2$ and $\hat{C}_2 = \tilde{C} \frac{e^{-\gamma} - e^{-t\gamma}}{\gamma} + 1$.

Proof. We follow the proof of Theorem 6 in [\[44\]](#), combined with the newly developed techniques for (k, v) -Sobolev-regularizing QMS. Details are presented in [Theorem C.1](#). $\square$

Remark 3. The time $t^* = 1$, which separates the two regimes above, is chosen for convenience. If sharper estimates are needed, t^* can be optimized. The same applies to [Theorem 3.9](#).

Continuous quantum error correction relies on the idea that a quantum dynamical process continuously drives the state always back into an invariant subspace — the codespace. Since the codespace has usually dimension greater than one, we next consider QMS converging to an invariant subspace. Note that perturbation insight the codespace may lead to non-convergent rotations, which is why our error metric only considers the error outside the codespace.

Proposition 3.9 (Perturbation bound for invariant subsets) *Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ and $(\mathcal{L} + \mathcal{E}, \mathcal{D}(\mathcal{L} + \mathcal{E}))$ be generators of a (k, v) -Sobolev-regularizing QMS, where $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is an unbounded operator on $\mathcal{T}_1(\mathcal{H}^{\otimes m})$ satisfying*

$$\|\mathcal{E}(\rho)\|_1 \leq c_1 \|\mathcal{W}_v^k(\rho)\| + c_2 \|\rho\|_1 = c_1 \|\rho\|_{W_v^{k,1}} + c_2 \|\rho\|_1.$$

If $e^{t\mathcal{L}}$ converges to an invariant subset with contractive projection $\mathcal{P}$ (with $\mathcal{P}^\perp = \mathbb{1} - \mathcal{P}$), i.e.,

$$\|e^{t\mathcal{L}}(\rho) - \mathcal{P}(\rho)\|_1 \leq \tilde{C} t^{1-\frac{\tilde{k}}{\delta}} e^{-\gamma t} \|\rho\|_1$$

for some $\tilde{k} \in \mathbb{N}$ and $\delta > k$, then for any states $\rho, \sigma \in \mathcal{T}_1$,

$$\|\mathcal{P}^\perp e^{t\mathcal{L}}(\sigma) - \mathcal{P}^\perp e^{t(\mathcal{L} + \varepsilon\mathcal{E})}(\rho)\|_1 \leq \begin{cases} \|\mathcal{P}^\perp(\sigma - \rho)\|_1 + \varepsilon t^{1-\frac{k}{\delta}} \hat{C}_1 \|\rho\|_1, & t \leq 1, \\ \tilde{C} e^{-\gamma t} \|\mathcal{P}^\perp(\sigma - \rho)\|_1 + \varepsilon \hat{C}_2 \hat{C}_1 \|\rho\|_1, & t \geq 1, \end{cases}$$

where $\hat{C}_1 = c_1 \left(\frac{k}{\delta\mu_k}\right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k}\right)^{\frac{k}{k+\delta}} + c_2$ and $\hat{C}_2 = 2\tilde{C} \frac{e^{-\gamma} - e^{-t\gamma}}{\gamma} + 1$.

Proof. As in Theorem 5.2 of [12], combined with the (k, v) -Sobolev-regularizing techniques. Full details appear in Theorem C.2. $\square$

4 Single-mode applications

Supplementary to the above theoretical section, we now present several examples that satisfy the stated assumption. In this section, we focus on the single-mode results combined with exponentially fast convergence to the invariant subset, as well as large-time perturbation bounds in Section 4.1. We would like to emphasize that the obtained results are in the uniform topology and are not fixed to a specific energy level of the system, which is one of the crucial findings. In Section 5, we then extend the applications to multi-mode systems.

Note that many results are based on the work [12] and also part of [32].

4.1 Families of photon dissipation

A natural and simple example in quantum optics and superconductors is the 2-photon dissipation $\mathcal{L}[a^2]$, which satisfies for any $\rho \in \mathcal{T}_f$

$$\begin{aligned} \text{tr} [\mathcal{L}[a^2](\rho)(N + \mathbb{1})^k] &= -\text{tr} [\rho N(N - \mathbb{1})((N + \mathbb{1})^k - (N - \mathbb{1})^k)] \\ &\leq -2 \text{tr} [\rho(N + 1)^{k+1}] + 6 \text{tr} [\rho(N + 1)^k] \\ &\leq -\text{tr} [\rho(N + 1)^{k+1}] + 6^{k+1} \end{aligned}$$

by Theorem D.1-D.4. Therefore, the 2-photon dissipation is Sobolev-regularizing by Theorem 3.3. Motivated by this simple but important example, we shortly state the bounds for the shifted k -photon dissipation $\mathcal{L}_\ell := \mathcal{L}[a^\ell - \alpha^\ell]$ key for the bosonic error correction cat code. Due to the shift, it admits the following invariant subspace — the code space:

$$\mathcal{C}_\ell(\alpha) := \text{span} \left\{ |\alpha_1\rangle \langle \alpha_2| : \alpha_1, \alpha_2 \in \left\{ \alpha e^{\frac{i2\pi j}{\ell}} : j \in \{0, \dots, \ell - 1\} \right\} \right\}. \quad (19)$$

which is protected due to the exponential convergence analyzed in the following section Section 4.2. A detailed analysis of the QEC and construction of the above dynamics can be found in [45, Sec. 3] and [26, 12].

Lemma 4.1 (Pure photon dissipation) *For any $k \geq 1$, $\ell \geq 2$, $\alpha \in \mathbb{C}$ and any state $\rho \in \mathcal{T}_f$,*

$$\text{tr} [\mathcal{L}_\ell(\rho)(N + \mathbb{1})^k] \leq -\frac{\ell}{2} \text{tr} [\rho(N + \mathbb{1})^{k+\ell-1}] + \frac{\ell}{2} \mu_k^{(\ell)},$$

where $\mu_k^{(\ell)} = \Delta_\ell^\nu \left(\frac{(\nu-1)^{\nu-1}}{\nu^\nu} \right)$ with $\nu = \ell + k - 1$ and $\Delta_\ell = (\ell + 1)\ell + 4|\alpha|^\ell k \ell^{k-1} \sqrt{\ell!}$. Therefore, $\mathcal{L}_\ell$ generates a Sobolev-regularizing QMS satisfying

$$\|e^{t\mathcal{L}_\ell}\|_{1 \rightarrow W^{k,1}} \leq \left(\frac{2k}{(\ell-1)\ell\mu_k^{(\ell)}t} \right)^{\frac{k}{\ell-1}} + \left(\frac{1}{\mu_k^{(\ell)}} \right)^{\frac{k}{k+\ell-1}}.$$

Proof. The proof is a direct application of [Theorem 3.3](#) to Lemma 4.3 in [\[12\]](#). $\square$

In the context of the bosonic error correction cat code, these models normally come with a Hamiltonian under certain degree assumption. The Hamiltonian is of the following structure

$$H = \sum_{\substack{i \leq j \\ i+j \leq d_H}} \lambda_{i,j} a^i (a^\dagger)^j + \overline{\lambda_{i,j}} a^j (a^\dagger)^i. \quad (20)$$

for coefficients λ_i , $i \in \{1, \dots, m\}$. With the assumption that $d_H \leq 2(\ell-1)$, we achieve the following Sobolev-regularization result.

Lemma 4.2 *Let $k \geq 1$, $\ell \geq 2$, $\alpha \in \mathbb{C}$, and H the Hamiltonian (20) with $d_H \leq 2(\ell-1)$. Then, for all states $\rho \in \mathcal{T}_f$*

$$\text{tr} [(-i[H, \cdot] + \mathcal{L}_\ell)(\rho)(N + \mathbb{1})^k] \leq -\frac{\ell}{2} \text{tr} [\rho(N + \mathbb{1})^{\ell+k-1}] + \frac{\ell}{2} \mu_k^{(\ell)}. \quad (21)$$

for $\mu_k, \nu \geq 1$ defined by

$$\mu_k^{(\ell)} = c^\nu \left(\frac{(\nu-1)^{\nu-1}}{\nu^\nu} \right) \quad \text{with} \quad c = (\ell+1)\ell + 4|\alpha|^\ell k \ell^{k-1} \sqrt{\ell!} + \Lambda(2\ell)^k \sqrt{(2\ell)!}, \quad \nu = \ell + k - 1.$$

Therefore, $-i[H, \cdot] + \mathcal{L}_\ell$ generates a Sobolev-regularizing QMS, which satisfies

$$\|e^{t\mathcal{L}_\ell}\|_{1 \rightarrow W^{k,1}} \leq \left(\frac{2k}{(\ell-1)\ell\mu_k^{(\ell)}t} \right)^{\frac{k}{\ell-1}} + \left(\frac{1}{\mu_k^{(\ell)}} \right)^{\frac{k}{k+\ell-1}}.$$

Proof. The proof is a direct application of [Theorem 3.3](#) to Lemma 4.6 in [\[12\]](#). $\square$

Remark 4. Following the same strategies as in [Theorem 4.1](#) and [4.2](#), the modified photon dissipation $\mathcal{L}[a^{\ell-1}(a - \alpha)]$, which is key for the learning scheme in [\[35\]](#), admits the same bounds as achieved for the photon dissipation $\mathcal{L}_\ell$ up to constants.

4.2 Exponential convergence in $1 \rightarrow 1$ norm

As mentioned above, one key property of the shifted ℓ -photon dissipation is its rapid convergence to the code space. However, this was first proven by [\[26\]](#) in the following topology:

$$\text{tr} [L_\ell e^{t\mathcal{L}_\ell}(\rho) L_\ell^\dagger] \leq e^{-t\ell!} \text{tr} [L_\ell \rho L_\ell^\dagger],$$

where $L_\ell = a^\ell - \alpha^\ell$. Then, some of the authors improved the convergence in [\[35, Prop. A.2\]](#) to the strong topology:

$$\|(e^{t\mathcal{L}_\ell} - \mathcal{P}_\ell)(\rho)\|_1 \leq 6e^{-\frac{t}{2}\ell!} \left(\left(1 + \frac{|\alpha|^\ell}{\sqrt{\ell!}} \right)^2 \|\rho\|_{W^{\ell,1}} + \|\rho\|_1 \right)$$

for all $\rho \in W^{\ell,1}$ and $\mathcal{P}_\ell$ is the projection onto the codespace $\mathcal{C}_\ell$. By a simple trick and using all the machinery above, we can prove the following result:

Lemma 4.3 Let $\ell \geq 2$ and $\mathcal{P}_\ell$ be the ON projection onto $\mathcal{C}_\ell(\alpha)$ (19). Then, for all $t \geq 0$

$$\|(e^{t\mathcal{L}_\ell} - \mathcal{P}_\ell)(\rho)\|_{1 \rightarrow 1} \leq 6e^{-\frac{t}{4}\ell!} \left(\left(1 + \frac{|\alpha|^\ell}{\sqrt{\ell!}}\right)^2 \left(\frac{16}{((\ell-1)\mu_\ell^{(\ell)}t)^2} + \frac{1}{\mu_\ell^{(\ell)}} \right) + 1 \right),$$

where $\mu_\ell^{(\ell)}$ is given by Theorem 4.1.

Proof. First, we assume $\rho \in W^{\ell,1}$. Then Proposition A.2 in [35] applied to $e^{\frac{t}{2}\mathcal{L}}(\rho_{\frac{t}{2}})$ shows

$$\|(e^{\frac{t}{2}\mathcal{L}_\ell} - \mathcal{P}_\ell)(\rho_{\frac{t}{2}})\|_1 \leq 6e^{-\frac{t}{4}\ell!} \left(\left(1 + \frac{|\alpha|^\ell}{\sqrt{\ell!}}\right)^2 \|\rho_{\frac{t}{2}}\|_{W^{\ell,1}} + \|\rho_{\frac{t}{2}}\|_1 \right).$$

Here, we also used that $\rho \in W^{\ell,1}$ implies $e^{\frac{t}{2}\mathcal{L}}(\rho) \in W^{\ell,1}$ due to Theorem 4.1. Then, a final application of Theorem 4.1, i.e.,

$$\|e^{\frac{t}{2}\mathcal{L}_\ell}\|_{1 \rightarrow W^{\ell,1}} \leq \left(\frac{4}{(\ell-1)\mu_\ell^{(\ell)}t} \right)^{\frac{\ell}{\ell-1}} + \left(\frac{1}{\mu_\ell^{(\ell)}} \right)^{\frac{\ell}{2\ell-1}} \leq \left(\frac{4}{(\ell-1)\mu_\ell^{(\ell)}t} \right)^2 + \frac{1}{\mu_\ell^{(\ell)}}$$

shows

$$\|(e^{t\mathcal{L}_\ell} - \mathcal{P}_\ell)(\rho)\|_1 \leq 6e^{-\frac{t}{4}\ell!} \left(\left(1 + \frac{|\alpha|^\ell}{\sqrt{\ell!}}\right)^2 \left(\frac{16}{((\ell-1)\mu_\ell^{(\ell)}t)^2} + \frac{1}{\mu_\ell^{(\ell)}} \right) + 1 \right) \|\rho\|_1,$$

which finishes the proof. $\square$

4.3 Perturbation theory in operator norm

With the results above, we are now able to directly apply the results of Section 3.3, which address several aspects of perturbing the bosonic cat code. We begin with a standard result concerning finite-time perturbations.

Lemma 4.4 Let $\ell \geq 2$, $\alpha \in \mathbb{C}$, and let H be the Hamiltonian (20) with $d_H \leq \ell - 2$. Then,

$$\|e^{t\mathcal{L}_\ell} - e^{t(\mathcal{L}_\ell - \varepsilon i[H, \cdot])}\|_{1 \rightarrow 1} \leq \varepsilon \left(c_1 t^{\frac{1}{\ell-1}} \left(\frac{\ell-2}{(\ell-1)\mu_k^{(\ell)}} \right)^{\frac{\ell-2}{\ell-1}} + c_1 t \left(\frac{c_k}{\mu_k^{(\ell)}} \right)^{\frac{\ell-1}{2\ell-3}} \right).$$

Proof. This is a direct application of Theorem 3.7. The assumptions are satisfied due to Theorem 4.2, which shows that $-i[H, \cdot] + \mathcal{L}_\ell$ generates a Sobolev-regularizing QMS with $\delta = \ell - 1$. Furthermore, Lemma 18 [31, p. 292] implies the existence of a constant c_1 such that

$$\|i[H, \rho]\| \leq c_1 \|\rho\|_{W_v^{k,1}},$$

with $k = d_H \leq \ell - 2$. Substituting these bounds into Theorem 3.7 completes the proof. $\square$

Next, we extend the large-time perturbation bound proven in Theorem 5.2 of [12] using the tools established above.

Lemma 4.5 Let $\ell \geq 2$, $\alpha \in \mathbb{C}$, and H the Hamiltonian (20) with $d_H \leq \ell - 2$. Then,

$$\|\mathcal{P}_\ell^\perp e^{t\mathcal{L}_\ell}(\sigma) - \mathcal{P}_\ell^\perp e^{t(\mathcal{L}_\ell + \varepsilon \mathcal{E})}(\rho)\|_1 \leq \begin{cases} \|\mathcal{P}_\ell^\perp(\sigma - \rho)\|_1 + \varepsilon t^{\frac{1}{\ell-1}} \hat{C}_1 \|\rho\|_1, & t \leq 1, \\ \tilde{C} e^{-t\frac{\ell!}{4}} \|\mathcal{P}_\ell^\perp(\sigma - \rho)\|_1 + \varepsilon \hat{C}_2 \hat{C}_1 \|\rho\|_1, & t \geq 1, \end{cases}$$

for states $\rho, \sigma \in \mathcal{T}_1$. The constants $\tilde{C}, \hat{C}_1, \hat{C}_2 \geq 0$ are defined in Theorem 3.9, and $\mathcal{P}_\ell^\perp = \mathbb{1} - \mathcal{P}_\ell$.

Proof. The result follows by direct application of [Theorem 3.9](#). The required assumptions hold due to [Theorem 4.2](#), which ensures that $-i[H, \cdot] + \mathcal{L}_\ell$ generates a Sobolev-regularizing QMS with $\delta = \ell - 1$, and by [Theorem 4.3](#), which gives

$$\|(e^{t\mathcal{L}_\ell} - \mathcal{P}_\ell)(\rho)\|_{1 \rightarrow 1} \leq 6e^{-\frac{t}{4}\ell!} \left[\left(1 + \frac{|\alpha|^\ell}{\sqrt{\ell!}}\right)^2 \left(\frac{16}{((\ell-1)\mu_\ell^{(\ell)}t)^2} + \frac{1}{\mu_\ell^{(\ell)}} \right) + 1 \right].$$

Moreover, Lemma C.1 implies the existence of c_1 such that

$$\|i[H, \rho]\| \leq c_1 \|\rho\|_{\mathcal{W}_v^{k,1}},$$

with $k = d_H \leq \ell - 2$. Substituting these expressions into [Theorem 3.9](#) finishes the proof. $\square$

Remark 5. In both results above, Lindbladian or even non-Markovian noise can also be accommodated, provided it is relatively upper bounded by $\mathcal{W}_v^k$. One may then follow the argument of the corresponding perturbation theorem, and the same proofs as presented above carry through.

Remark 6. Following the construction in [\[35\]](#), namely

$$\mathcal{L}[a - \alpha] + \mathcal{L}[a^\ell - \alpha^\ell],$$

the resulting QMS converges exponentially fast to the state $|\alpha\rangle\langle\alpha|$, as shown in [\[26\]](#) using the generation theory of [\[12\]](#). For such QMS, [Theorem 3.8](#) applies under the same assumptions used in [Theorem 4.5](#).

5 Multi-mode applications

In this section, we extend the single-mode examples presented in [Section 4](#) to multi-mode systems and show that local perturbations perturb expectations only locally. One motivation for this analysis is the study of multi-mode bosonic error-correcting cat-code extensions, which are discussed, for example, in [\[30\]](#). Although we focus on photon-loss dissipation, many other forms of dissipation should satisfy similar properties when analyzed using the arguments developed here.

As in the previous sections, detailed calculations are provided in [Section D](#) for the sake of clarity.

5.1 Multi-mode photon dissipation

In this section, we show (k, v) -Sobolev regularization for multi-mode ℓ -photon dissipation in the presence of nearest-neighbor Hamiltonians consisting of polynomials in creation and annihilation operators. The interaction structure of the model is given by a D -dimensional lattice defined by a graph (V, E) with $V \subset \mathbb{Z}^D$ and $|V| = m$. In the sake of notation, we allow cycles denoting the on-mode terms $V \subset E$. The dissipation is taken to be the sum of (shifted) photon-dissipation terms acting on single-modes, i.e.

$$\mathcal{L}_\ell(\rho) = \sum_{j \in V} \mathcal{L}[a_j^\ell - \alpha_j^\ell], \quad (22)$$

for a vector $\alpha \in \mathbb{C}^m$. Recall that $\mathcal{L}[L] = L \cdot L^\dagger - \frac{1}{2}\{L^\dagger L, \cdot\}$ for any operator L . Alongside this Lindbladian, we also consider a 2-local Hamiltonian of the form

$$H = \sum_{i \sim j} H_{ij} = \sum_{i \sim j} \sum_{\substack{u \in \mathbb{N}_0^4 \\ \|u\|_1 \leq d_H}} \lambda_{ij}^{(u)} a_i^{u_1} a_j^{u_3} (a_i^*)^{u_2} (a_j^*)^{u_4} + \bar{\lambda}_{ij}^{(u)} a_j^{u_4} a_i^{u_2} (a_j^*)^{u_3} (a_i^*)^{u_1}, \quad (23)$$

where $i \sim j$ means $(i, j) \in E$, corresponding to nearest neighbors on the D -dimensional lattice. Together, this defines the operator

$$\mathcal{L}_\ell^{(H)} := -i[H, \cdot] + \mathcal{L}_\ell = \sum_{i \sim j} -i[H_{ij}, \cdot] + \frac{1}{\gamma_i} \mathcal{L}[a_i^\ell - \alpha_i^\ell] + \frac{1}{\gamma_j} \mathcal{L}[a_j^\ell - \alpha_j^\ell] =: \sum_{i \sim j} \mathcal{L}_{i,j} \quad (24)$$

with $\gamma_i = |\{i \sim j \mid j \in V\}|$ — the connectivity of the graph-node $i \in V$. A direct consequence of the k -Sobolev preservation of single-mode photon loss (see [Thm. 4.1](#)) is that the multi-mode extension (22) combined with single-mode Hamiltonians is (k, v) -Sobolev preserving for every $v \in V$.

Corollary 5.1 *Let $k \geq 1$, $\ell \geq 2$, $\alpha \in \mathbb{C}$, and $\rho \in \mathcal{T}_f$. Then for every Hamiltonian (23) only acting on single-modes, i.e. $H = \sum_j H_{jj}$, with $d_H \leq 2(\ell - 1)$*

$$\text{tr} \left[\mathcal{W}_v^k(-i[H, \cdot] + \mathcal{L}_\ell(\rho)) \right] \leq -\frac{\ell}{2} \text{tr} \left[\mathcal{W}_v^{k+\ell-1}(\rho) \right] + \frac{\ell}{2} \mu_k^{(\ell)},$$

where $\mu_k^{(\ell)} = c^\nu \left(\frac{(\nu-1)^{\nu-1}}{\nu^\nu} \right)$ with $\nu = \ell + k - 1$, and $c = (\ell + 1)\ell + 4|\alpha|^\ell k \ell^{k-1} \sqrt{\ell!} + \Lambda(2\ell)^k \sqrt{(2\ell)!}$. Consequently, $-i[H, \cdot] + \mathcal{L}_\ell$ generates a Sobolev-regularizing QMS satisfying

$$\|e^{t(-i[H, \cdot] + \mathcal{L}_\ell)}\|_{1 \rightarrow W_v^{k,1}} \leq \left(\frac{2k}{(\ell-1)\ell \mu_k^{(\ell)} t} \right)^{\frac{k}{\ell-1}} + \left(\frac{1}{\mu_k^{(\ell)}} \right)^{\frac{k}{k+\ell-1}}. \quad (25)$$

Proof. Since not only the generators $\mathcal{L}[a_j^\ell - \alpha_j^\ell]$ commute for different $j \in V$, but also the generators $-i[H_{jj}, \cdot]$ commute for different $j \in V$, we have $\prod_{j \in V} e^{t(\mathcal{L}[a_j^\ell - \alpha_j^\ell] + -i[H_{jj}, \cdot])} = e^{t(\mathcal{L}_\ell - i[H_{jj}, \cdot])}$ for any ordering of the product. Thus, [Theorem 4.2](#) directly yields the required estimate for each $j \in V$, and [Theorem 3.6](#) then proves the (k, v) -Sobolev stated bound. $\square$

Next, we extend the single-mode result to the concentrated Sobolev norms by assuming differentiability of the moments. Then, we achieve the two bounds in [Equation \(26\)](#) and [\(27\)](#).

Theorem 5.2 *Let $\rho \in \mathcal{T}_f$, $\mathcal{L}_\ell$ be a Limbladian of the form (22) associated to ℓ -photon-dissipation and H a 2-local Hamiltonian of the form (23). Assume that there exists a $k \geq 2$ such that $t \mapsto \text{tr} [\mathcal{W}_v^{k,1}(\rho(t))]$ is differentiable for every $t \geq 0$.*

1. *If we consider a coupling of the form $-i[H, \cdot] + \eta \mathcal{L}_\ell$ with $d_H \leq 2(\ell - 1)$, then there exist constants $\eta_k, C_k, \mu_k > 0$ such that for every $\eta \geq \eta_k$,*

$$\text{tr} [(-i[H, \cdot] + \eta \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t))] \leq -\frac{C_k}{2} \text{tr} [\rho(t) \mathcal{W}_v^{\ell+k-1}] + \mu_k. \quad (26)$$

2. *If $d_H \leq 2(\ell - 2)$, then there exists a constant μ_k such that*

$$\text{tr} [(-i[H, \cdot] + \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t))] \leq -\frac{\ell}{4} \text{tr} [\rho(t) \mathcal{W}_v^{\ell+k-1}] + \mu_k. \quad (27)$$

Note that the constants C_k , μ_k , and η_k depend on the defining parameters of the generator and can be improved by model specific calculations.

Proof. To prove this result, we split the Hamiltonian $H = H^{(1)} + H^{(2)}$, where $H^{(1)}$ corresponds to the diagonal terms in the decomposition (23) and $H^{(2)}$ to the off-diagonal terms. In first place, we make use of [Theorem 5.1](#) to establish a bound on the Limbladian $\mathcal{L}$ and $\mathcal{H}(H^{(1)})$. Afterwards, we obtain upper bounds for the off-diagonal Hamiltonian $\mathcal{H}(H^{(2)})$ under our specific assumptions on η or on the degree of the polynomial. Finally, we make use of [Theorem D.5](#) and [Theorem 3.6](#) to obtain the desired result. The details can be found in [Theorem D.6](#). $\square$

5.2 Local perturbation perturb locally

In the following short subsection, we discuss the effect of local perturbations on the expectation values, which can be measured experimentally. The concept presented plays a crucial role in the perturbation analysis of local expectations of evolved states and a first step in the direction of stability of these systems (see [46, 47] for more details).

Corollary 5.3 *Let $\mathcal{L}_\ell^{(H)}$ be as defined in Equation (24), and let H_B be any 2-body interacting Hamiltonian of the form (23) with degree $d_H \leq 2(\ell-2)$ acting non-trivially on $B \subset V$. Here (V, E) denotes the graph on the lattice on which the operators are defined. Then, if $\delta > 2d|B|$, the closure of $\mathcal{L}_\ell^{(H)}$, as well as the closure of its sum with $-\varepsilon[H_B, \cdot] + \mathcal{L}^{(\ell)}$, is a QMS satisfying*

$$\left| \text{tr} \left[O_A e^{t\mathcal{L}_\ell^{(H)}}(\rho) \right] - \text{tr} \left[O_A e^{t\mathcal{L}}(\rho) \right] \right| \leq \varepsilon \|O_A\|_\infty \left(c_1 t^{1-\frac{d_H}{\ell-1}} \left(\frac{d_H}{(\ell-1)\mu} \right)^{\frac{d_H}{\ell-1}} + c_1 t \left(\frac{c}{\mu} \right)^{\frac{d_H}{d_H+\ell-1}} \right)$$

for constants c_1, μ, c appearing in the proof, all independent of the system size m .

Proof. Due to the relative boundedness established in [31, p. 292] and applying the Young's inequality, there exists a constant c such that

$$\begin{aligned} \|\mathcal{H}_B(\rho)\|_1 &\leq c \left\| \bigotimes_{i \in B} (N_i + \mathbb{1})^d \rho \bigotimes_{i \in B} (N_i + \mathbb{1})^d \bigotimes_{i \in B} (N_i + \mathbb{1})^{-d} \right\|_1 \\ &\leq c \text{tr} \left[\bigotimes_{i \in B} (N_i + \mathbb{1})^{\frac{1}{|B|} 2d|B|} \rho \right] \\ &\leq \frac{c}{|B|} \sum_{i \in B} \text{tr} \left[(N_i + \mathbb{1})^{d|B|} \rho (N_i + \mathbb{1})^{d|B|} \right] \\ &\leq \frac{c}{|B|} \frac{1}{\min_i e^{-\kappa \text{dist}(v,i)}} \|\mathcal{W}_v^{2d|B|}(\rho)\|_1 \end{aligned}$$

Then, since $\delta > 2d|B|$, by Theorem 3.7,

$$\begin{aligned} \left| \text{tr} \left[O_A e^{t(\mathcal{L} + \varepsilon \mathcal{H}_B)}(\rho) \right] - \text{tr} \left[O_A e^{t\mathcal{L}}(\rho) \right] \right| &\leq \|O_A\|_\infty \left\| e^{t(\mathcal{L} + \varepsilon \mathcal{H}_B)}(\rho) - e^{t\mathcal{L}}(\rho) \right\|_1 \\ &\leq \varepsilon \|O_A\|_\infty \left(c_1 t^{1-\frac{d}{\ell-1}} \left(\frac{d}{(\ell-1)\mu} \right)^{\frac{d}{\ell-1}} + c_1 t \left(\frac{c}{\mu} \right)^{\frac{d}{d+\ell-1}} \right), \end{aligned}$$

where $c_1 = \frac{c}{|B|} \frac{1}{\min_i e^{-\kappa \text{dist}(v,i)}}$ and for the constants defined in Theorem 5.2. $\square$

6 Conclusion

In this work, we identified a broad class of unbounded GKSL generators on bosonic Fock space whose associated quantum Markov semigroups exhibit instantaneous Sobolev regularization. We showed that dissipative evolutions generated by polynomial creation and annihilation operators not only preserve finite moments but also immediately improve them, mapping arbitrary trace-class states into highly regular ones for every $t > 0$. This provides a structural explanation for stability phenomena in bosonic continuous-variable systems and yields explicit convergence bounds for arbitrary input states. In particular, our bounds sharpen existing perturbative results at both

short and long times, offering new analytic tools for assessing robustness in bosonic quantum information processing.

The multi-mode framework developed here points toward several promising directions. A central open question is whether instantaneous Sobolev regularization can be exploited to improve bosonic information-propagation bounds, such as Lieb–Robinson–type estimates. Another natural direction for future work is the explicit analysis of particular bosonic quantum error-correcting codes and the implications of our theory for their performance.

Acknowledgments We would like to thank Robert Salzmann, Cambyse Rouzé, and Marius Lemm for their valuable discussions on the topic. This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 470903074 - TRR 352 (PG, TM) and Germany’s Excellence Strategy-EXC2111-390814868 (PCR), QuantERA II Programme that has received funding from the EU’s H2020 research and innovation programme under the GA No 101017733 (TM).

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A ODE comparison lemma

In the following section, we prove [Theorem 3.1](#). For clarity of presentation, we begin by restating the lemma:

Lemma A.1 *Let $y : [0, \infty) \rightarrow [0, \infty)$ be a continuously differentiable function satisfying the differential inequality*

$$\frac{dy}{dt}(t) \leq -a y(t)^p + b$$

for constants $a, b > 0$ and exponent $p > 1$. Then,

$$y(t) \leq \left(\max\{y(0) - \left(\frac{b}{a}\right)^{1/p}, 0\}^{1-p} + (p-1)at \right)^{-\frac{1}{p-1}} + \left(\frac{b}{a}\right)^{\frac{1}{p}} =: z(t).$$

Proof. In a first step, we define

$$t_c := \inf\{t \geq 0 : y(t) \leq (b/a)^{1/p}\}.$$

For $t > t_c$, it follows immediately that $y(t) \leq (b/a)^{1/p}$, and hence $y(t) \leq z(t)$. We verify this by contradiction. Suppose there exists $t_v > t_c$ such that $y(t_v) > (b/a)^{1/p}$. By continuity of y , the intermediate value theorem implies that there exist t_0, t_1 with $t_0 < t_1$ such that

$$y(t_0) = (b/a)^{1/p}, \quad y(t_1) > (b/a)^{1/p}, \quad \text{and} \quad y(t) \geq (b/a)^{1/p} \quad \text{for all } t \in [t_0, t_1].$$

By the mean value theorem, there exists $t_m \in (t_0, t_1)$ such that

$$y'(t_m) = \frac{y(t_1) - y(t_0)}{t_1 - t_0} > 0.$$

However, from the differential inequality and the fact that $y(t_m) \geq (b/a)^{1/p}$, we have

$$y'(t_m) \leq -a y(t_m)^p + b \leq -a(b/a) + b = 0,$$

a contradiction. Therefore, $y(t) \leq (b/a)^{1/p}$ for all $t \geq t_c$.

For $0 \leq t < t_c$, define $g(t) := y(t) - (b/a)^{1/p} > 0$. Then $g'(t) = y'(t)$, and the differential inequality yields

$$g'(t) \leq -a g(t)^p.$$

Separating variables and integrating from 0 to t gives

$$\frac{g(t)^{1-p} - g(0)^{1-p}}{1-p} = \int_0^t \frac{g'(s)}{g(s)^p} ds \leq \int_0^t -a ds = -at.$$

Rearranging terms, we obtain

$$g(t)^{1-p} \geq g(0)^{1-p} + (p-1)at.$$

Taking both sides to the power of $\frac{1}{1-p}$ (which reverses the inequality since $\frac{1}{1-p} < 0$), we find

$$g(t) \leq \frac{1}{(g(0)^{1-p} + (p-1)at)^{\frac{1}{p-1}}}.$$

Substituting back $g(t) = y(t) - (b/a)^{1/p}$ yields

$$y(t) \leq \frac{1}{((y(0) - (b/a)^{1/p})^{1-p} + (p-1)at)^{\frac{1}{p-1}}} + (b/a)^{1/p}.$$

Using the conventions $1/\infty = 0$ and $1/0 = \infty$ completes the proof. $\square$

B Multi-mode extension

In this appendix, we restate the multi-mode extensions previously introduced in [Section 3.2](#) and provide detailed proofs. In a first step, we shortly discuss upper and lower bounds on the normalization constant Z_v , which show that the normalization admits lower and upper bounds which are system size independent.

Lemma B.1 *Let (V, E) define a D -dimensional lattice. Then, for $v \in V$, we have for $K = \max_{i \in V} \text{dist}(i, v)$*

$$1 \leq \frac{1 - e^{-(\kappa-1)(K+1)}}{1 - e^{-(\kappa-1)}} \leq Z_v \leq \frac{e^{2D-1}}{1 - e^{-(\kappa-1)}}$$

Proof. In a first step, we reshape the sum with $K = \max_{i \in V} \text{dist}(i, v)$ as follows

$$Z_v = \sum_{i=1}^m e^{-\kappa \text{dist}(v, i)} = \sum_{\ell=0}^K |\{i \in V \mid \text{dist}(i, v) = \ell\}| e^{-\kappa \ell}.$$

Then, we use a simple upper bound on the cardinality of a sphere in D -dimensional lattice (see [\[48, Appx. B.2\]](#)). Combined with the geometric series, we achieve

$$Z_v \leq \sum_{\ell=0}^K 2^D \frac{(D-1+\ell)^{D-1}}{(D-1)!} e^{-\kappa \ell} \leq e^D \sum_{\ell=0}^K \sum_{j=0}^{\infty} \frac{(D-1+\ell)^j}{j!} e^{-\kappa \ell} \leq e^{2D-1} \sum_{\ell=0}^{\infty} e^{-(\kappa-1)\ell} = \frac{e^{2D-1}}{1 - e^{-(\kappa-1)}}$$

For the lower bound, we just use 1 as lower bound for the cardinalities of the spheres and again the geometric sum so that

$$Z_v \geq \sum_{\ell=0}^K e^{-\kappa \ell} \geq \frac{1 - e^{-(\kappa-1)(K+1)}}{1 - e^{-(\kappa-1)}} \geq 1$$

finishes the result. $\square$

Next, we prove the following inequality, which motivates the definition of concentrated Sobolev norms — local moments bound the global but concentrated Sobolev norm.

Lemma B.2 *Let $\alpha \in \mathbb{C}^m$ define the coherent state $|\alpha\rangle$, then for $k > 0$*

$$\| |\alpha\rangle\langle\alpha| \|_{W_v^{k,1}} \leq \max_{i \in \{1, \dots, m\}} \text{tr} \left[|\alpha_i\rangle\langle\alpha_i| (N_i + \mathbb{1})^k \right] \leq \left(\frac{2k}{\ln(k / \max_{i \in \{1, \dots, m\}} |\alpha_i|^2 + 1)} \right)^k.$$

Proof. This proof follows the proof of Lemma 7 in [34]. By definition of the concentrated Sobolev norm, the norm reduces to

$$\| |\alpha\rangle\langle\alpha| \|_{W_v^{k,1}} = \sum_{i=1}^m \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \langle \alpha_i | (N_i + \mathbb{1})^k | \alpha_i \rangle,$$

and by that to a single mode. Using a simple bound on the k^{th} moment of the Poisson distribution [49], one can show

$$\begin{aligned} \langle \alpha_i | (N_i + \mathbb{1})^k | \alpha_i \rangle &= e^{-|\alpha_i|^2} \sum_{n_i=0}^{\infty} \frac{|\alpha_i|^{2n_i}}{n_i!} (n_i + 1)^k \\ &= \sum_{l=0}^k \binom{k}{l} \sum_{n_i=0}^{\infty} n_i^l \frac{|\alpha_i|^{2n_i}}{n_i!} e^{-|\alpha_i|^2} \\ &\leq \left(\frac{2k}{\ln(k / |\alpha_i|^2 + 1)} \right)^k. \end{aligned}$$

Therefore,

$$\| |\alpha\rangle\langle\alpha| \|_{W_v^{k,1}} \leq \sum_{i=1}^m \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \left(\frac{2k}{\ln(k / |\alpha_i|^2 + 1)} \right)^k \leq \left(\frac{2k}{\ln(k / \max_{i \in \{1, \dots, m\}} |\alpha_i|^2 + 1)} \right)^k$$

by Hölder's inequality and the normalization constant. $\square$

We begin with a Jensen-type inequality for multi-mode bosonic moments:

Lemma B.3 (Jensen's inequality for multi-mode moments) *Let $\rho \in \mathcal{T}_f$ be a state. Then, for all $p \geq q > 0$,*

$$\text{tr} [\mathcal{W}_v^p(\rho)] \geq \left(\text{tr} [\mathcal{W}_v^q(\rho)] \right)^{\frac{p}{q}}. \quad (28)$$

Proof. The proof follows the lines of the proof of Theorem 3.2, with slightly more general coefficients p_n . Similar as before, the state ρ has finite rank K in the Fock basis, so that we can write

$$\text{tr} [\mathcal{W}_v^p(\rho)] = \sum_{i \in V} \sum_{n_i=0}^{K-1} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \langle n_i, \rho_{n_i} \rangle (n_i + 1)^p = \sum_{i \in V} \sum_{n_i=0}^{K-1} p_{n_i,i} (n_i + 1)^{q \frac{p}{q}}.$$

Due to the normalization Z_v , $p_{n_i,i} = \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \langle n_i, \rho_{n_i} \rangle$ is a probability distribution, so that the convexity of the function $x \mapsto x^{\frac{p}{q}}$ for $p \geq q > 0$ shows by Jensen's inequality that

$$\text{tr} [\mathcal{W}_v^p(\rho)] \geq \left(\sum_{i \in V} \sum_{n_i=0}^{K-1} p_{n_i,i} (n_i + 1)^q \right)^{\frac{p}{q}} = \left(\text{tr} [\mathcal{W}_v^q(\rho)] \right)^{\frac{p}{q}},$$

which completes the proof. $\square$

With that result in hand, and in conjunction with [Theorem 3.1](#), we now prove [Theorem 3.6](#).

Theorem B.4 *Let $(\mathcal{L}, \mathcal{T}_f)$ generate a (k, v) -Sobolev-preserving quantum Markov semigroup satisfying*

$$\mathrm{tr} \left[\mathcal{W}_v^k(\mathcal{L}(\rho)) \right] \leq -\mu_k \mathrm{tr} \left[\mathcal{W}_v^{k+\delta}(\rho) \right] + c_k$$

for constants $k, \mu_k, \delta, c_k > 0$. Then, for all $t > 0$,

$$\|e^{t\mathcal{L}}\|_{1 \rightarrow W_v^{k,1}} \leq \left(\frac{k}{\delta \mu_k t} \right)^{\frac{k}{\delta}} + \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}}.$$

Proof. Since $\mathcal{T}_f$ is dense in $\mathcal{T}_1$ and $e^{t\mathcal{L}}$ is completely positive and trace-preserving, it suffices to prove (14) for $\rho \in \mathcal{T}_f$ a state. Let $\rho(t) = e^{t\mathcal{L}}(\rho)$. Differentiating with respect to t and applying (13) yields

$$\frac{d}{dt} \mathrm{tr} \left[\mathcal{W}_v^k(\rho(t)) \right] = \mathrm{tr} \left[\mathcal{W}_v^k(\mathcal{L}(\rho(t))) \right] \leq -\mu_k \mathrm{tr} \left[\mathcal{W}_v^{k+\delta}(\rho(t)) \right] + c_k.$$

By [Theorem B.3](#) with $p = k + \delta$ and $q = k$, we obtain

$$\frac{d}{dt} \mathrm{tr} \left[\mathcal{W}_v^k(\rho(t)) \right] \leq -\mu_k \left(\mathrm{tr} \left[\mathcal{W}_v^k(\rho(t)) \right] \right)^{\frac{k+\delta}{k}} + c_k.$$

Setting $y(t) := \mathrm{tr} \left[\mathcal{W}_v^k(\rho(t)) \right]$, we have

$$y'(t) \leq -\mu_k y(t)^{1+\frac{\delta}{k}} + c_k.$$

This is of the form required by [Theorem 3.1](#) with parameters $a = \mu_k$, $b = c_k$, and $p = 1 + \frac{\delta}{k} > 1$. Applying that lemma gives

$$y(t) \leq \left(\max\{y(0) - \left(\frac{c_k}{\mu_k}\right)^{\frac{k}{k+\delta}}, 0\}^{-\frac{\delta}{k}} + \frac{\delta}{k} \mu_k t \right)^{-\frac{k}{\delta}} + \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}}.$$

Using the estimate

$$\left(\max\{y(0) - \left(\frac{c_k}{\mu_k}\right)^{\frac{k}{k+\delta}}, 0\}^{-\frac{\delta}{k}} + \frac{\delta}{k} \mu_k t \right)^{-\frac{k}{\delta}} \leq \left(\frac{k}{\delta \mu_k t} \right)^{\frac{k}{\delta}},$$

we obtain the claimed bound (14). □

C Perturbation theory

In this section, we outline the details of [Section 3.3](#), which present various results in the direction of perturbation theory exploiting our (k, v) -Sobolev-regularization property. We start with [Theorem 3.8](#):

Proposition C.1 (Perturbation bound for steady states) *Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ and $(\mathcal{L} + \mathcal{E}, \mathcal{D}(\mathcal{L} + \mathcal{E}))$ be generators of a (k, v) -Sobolev-regularizing QMS, where $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is an unbounded operator on $\mathcal{T}_1(\mathcal{H}^{\otimes m})$ satisfying*

$$\|\mathcal{E}(\rho)\|_1 \leq c_1 \|\mathcal{W}_v^k(\rho)\| + c_2 \|\rho\|_1 = c_1 \|\rho\|_{W_v^{k,1}} + c_2 \|\rho\|_1.$$

If $e^{t\mathcal{L}}$ converges to a steady state $\bar{\rho} \in \mathcal{T}_1$, i.e.,

$$\|e^{t\mathcal{L}}(\rho) - \bar{\rho}\|_{1 \rightarrow 1} \leq \tilde{C} t^{1-\frac{k}{\delta}} e^{-\gamma t} \|\rho\|_1$$

for any state $\rho \in \mathcal{T}_1$, $\tilde{k} \in \mathbb{N}$, and $\delta > k$, then for any states $\rho, \sigma \in \mathcal{T}_1$,

$$\|e^{t\mathcal{L}}(\sigma) - e^{t(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 \leq \begin{cases} \|\sigma - \rho\|_1 + \varepsilon t^{1-\frac{k}{\delta}} \hat{C}_1 \|\rho\|_1, & t \leq 1, \\ \tilde{C} e^{-\gamma t} \|\sigma - \rho\|_1 + \varepsilon \hat{C}_2 \hat{C}_1 \|\rho\|_1, & t \geq 1, \end{cases}$$

where $\hat{C}_1 = c_1 \left(\frac{k}{\delta\mu_k}\right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k}\right)^{\frac{k}{k+\delta}} + c_2$ and $\hat{C}_2 = \tilde{C} \frac{e^{-\gamma} - e^{-t\gamma}}{\gamma} + 1$.

Proof. Similar to the proof of Theorem 6 in [44], we first apply the fundamental theorem of calculus:

$$\|e^{t\mathcal{L}}(\sigma) - e^{t(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 \leq \|e^{t\mathcal{L}}(\sigma - \rho)\|_1 + \varepsilon \int_0^t \|e^{s\mathcal{L}} \mathcal{E} e^{(t-s)(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 ds.$$

Note that the Bochner integral is well-defined by the same argument given in the proof of [Theorem 3.7](#). We then split the integral into the cases $0 \leq t \leq t^*$ and $1 \leq t^* \leq t$ for $t^* > 0$. For the case $0 \leq t \leq t^*$, we first use the contractivity of QMS, then the relative boundedness assumption, and finally the (k, v) -Sobolev regularization:

$$\begin{aligned} \|e^{t\mathcal{L}}(\sigma) - e^{t(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 &\leq \|\sigma - \rho\|_1 + \varepsilon \int_0^t \|\mathcal{E} e^{(t-s)(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 ds \\ &\leq \|\sigma - \rho\|_1 + \varepsilon \int_0^t \left(c_1 \|e^{(t-s)(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_{W_v^{k,1}} + c_2 \|\rho\|_1 \right) ds \\ &\leq \|\sigma - \rho\|_1 + \varepsilon \left(\int_0^t c_1 \left(\frac{k}{\delta\mu_k(t-s)} \right)^{\frac{k}{\delta}} ds + c_1 t \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 t \right) \|\rho\|_1 \\ &\leq \|\sigma - \rho\|_1 + \varepsilon \left(c_1 t^{1-\frac{k}{\delta}} \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 t \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 t \right) \|\rho\|_1. \end{aligned}$$

Next, we consider the case $1 \leq t^* \leq t$. Before analyzing this case, we recall the following bound also used in [44]. For a traceless self-adjoint operator with trace norm 1, the spectral decomposition yields

$$\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$$

for two states σ^+ and σ^- . Then

$$\|e^{t\mathcal{L}}(\sigma)\|_1 = \frac{1}{2} \left(\|e^{t\mathcal{L}}(\sigma^+) - \mathcal{P}(\sigma^+)\|_1 + \|e^{t\mathcal{L}}(\sigma^-) - \mathcal{P}(\sigma^-)\|_1 \right) \leq C e^{-\gamma t}$$

implies that for $1 \leq t^* \leq t$,

$$\begin{aligned}
& \|e^{t\mathcal{L}}(\sigma) - e^{t(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 \\
& \leq \|e^{t\mathcal{L}}(\sigma - \rho)\|_1 + \varepsilon \left(\int_0^{t^*} \|e^{s\mathcal{L}} \mathcal{E} e^{(t-s)(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 ds + \int_{t^*}^t \|e^{s\mathcal{L}} \mathcal{E} e^{(t-s)(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 ds \right) \\
& \leq \tilde{C} t^{1-\frac{\tilde{k}}{\delta}} e^{-\gamma t} \|\sigma - \rho\|_1 + \varepsilon \left(c_1 (t^*)^{1-\frac{k}{\delta}} \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 t^* \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 t^* \right) \|\rho\|_1 \\
& \quad + \varepsilon \int_{t^*}^t \tilde{C} s^{1-\frac{\tilde{k}}{\delta}} e^{-s\gamma} \left(c_1 \left(\frac{k}{\delta\mu_k s} \right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 \right) \|\rho\|_1 ds \\
& \leq \tilde{C} t^{1-\frac{\tilde{k}}{\delta}} e^{-\gamma t} \|\sigma - \rho\|_1 + \varepsilon \left(c_1 (t^*)^{1-\frac{k}{\delta}} \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 t^* \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 t^* \right) \|\rho\|_1 \\
& \quad + \varepsilon \int_{t^*}^t \tilde{C} e^{-s\gamma} \left(c_1 \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 \right) \|\rho\|_1 ds \\
& \leq \tilde{C} t^{1-\frac{\tilde{k}}{\delta}} e^{-\gamma t} \|\sigma - \rho\|_1 + \varepsilon \left(c_1 (t^*)^{1-\frac{k}{\delta}} \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 t^* \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 t^* \right) \|\rho\|_1 \\
& \quad + \varepsilon \tilde{C} \frac{e^{-t^*\gamma} - e^{-t\gamma}}{\gamma} \left(c_1 \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 \right) \|\rho\|_1.
\end{aligned}$$

For simplicity, we choose $t^* = 1$ and by that $t^{1-\frac{\tilde{k}}{\delta}} \leq 1$ for $t \leq t^*$, yielding

$$\|e^{t\mathcal{L}}(\sigma) - e^{t(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 \leq \tilde{C} e^{-\gamma t} \|\sigma - \rho\|_1 + \varepsilon \hat{C} \left(c_1 \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 \right) \|\rho\|_1,$$

where $\hat{C} = \tilde{C} \frac{e^{-\gamma} - e^{-t\gamma}}{\gamma} + 1$, which completes the proof. $\square$

Next, we prove [Theorem 3.9](#) in detail.

Proposition C.2 (Perturbation bound for invariant subsets) *Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ and $(\mathcal{L} + \mathcal{E}, \mathcal{D}(\mathcal{L} + \mathcal{E}))$ be generators of a (k, v) -Sobolev-regularizing QMS, where $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is an unbounded operator on $\mathcal{T}_1(\mathcal{H}^{\otimes m})$ satisfying*

$$\|\mathcal{E}(\rho)\|_1 \leq c_1 \|\mathcal{W}_v^k(\rho)\| + c_2 \|\rho\|_1 = c_1 \|\rho\|_{W_v^{k,1}} + c_2 \|\rho\|_1.$$

If $e^{t\mathcal{L}}$ converges to an invariant subset with contractive projection $\mathcal{P}$ (with $\mathcal{P}^\perp = \mathbb{1} - \mathcal{P}$), i.e.,

$$\|e^{t\mathcal{L}}(\rho) - \mathcal{P}(\rho)\|_1 \leq \tilde{C} t^{1-\frac{\tilde{k}}{\delta}} e^{-\gamma t} \|\rho\|_1$$

for some $\tilde{k} \in \mathbb{N}$ and $\delta > k$, then for any states $\rho, \sigma \in \mathcal{T}_1$,

$$\|\mathcal{P}^\perp e^{t\mathcal{L}}(\sigma) - \mathcal{P}^\perp e^{t(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 \leq \begin{cases} \|\mathcal{P}^\perp(\sigma - \rho)\|_1 + \varepsilon t^{1-\frac{k}{\delta}} \hat{C}_1 \|\rho\|_1, & t \leq 1, \\ \tilde{C} e^{-\gamma t} \|\mathcal{P}^\perp(\sigma - \rho)\|_1 + \varepsilon \hat{C}_2 \hat{C}_1 \|\rho\|_1, & t \geq 1, \end{cases}$$

where $\hat{C}_1 = c_1 \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2$ and $\hat{C}_2 = 2\tilde{C} \frac{e^{-\gamma} - e^{-t\gamma}}{\gamma} + 1$.

Proof. For the case $t \leq 1$, we follow the proof of [Theorem 3.8](#), which shows

$$\|\mathcal{P}^\perp e^{t\mathcal{L}}(\sigma) - \mathcal{P}^\perp e^{t(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 \leq \|\mathcal{P}^\perp(\sigma - \rho)\|_1 + \varepsilon 2 \left(c_1 t^{1-\frac{k}{\delta}} \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 t \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 t \right) \|\rho\|_1.$$

For the case $1 \leq t^* \leq t$, we again follow the proof of [Theorem 3.8](#), but now directly apply

$$\|\mathcal{P}^\perp e^{t\mathcal{L}}(\sigma)\|_{1 \rightarrow 1} = \frac{1}{2} \|\mathcal{P}^\perp(e^{t\mathcal{L}} - \mathcal{P})(\sigma^+ - \sigma^-)\|_{1 \rightarrow 1} \leq 2\tilde{C} t^{1-\frac{\tilde{k}}{\delta}} e^{-\gamma t} \|\sigma\|_1,$$

for σ as described in [Theorem 3.8](#).

Combining these bounds with the argument in the proof of [Theorem 3.8](#), we obtain

$$\|e^{t\mathcal{L}}(\sigma) - e^{t(\mathcal{L}+\varepsilon\mathcal{E})}(\rho)\|_1 \leq \tilde{C} e^{-\gamma t} \|\sigma - \rho\|_1 + \varepsilon \hat{C} \left(c_1 \left(\frac{k}{\delta\mu_k} \right)^{\frac{k}{\delta}} + c_1 \left(\frac{c_k}{\mu_k} \right)^{\frac{k}{k+\delta}} + c_2 \right) \|\rho\|_1,$$

where $\hat{C} = 2\tilde{C} \frac{e^{-\gamma} - e^{-t\gamma}}{\gamma} + 1$. This completes the proof. $\square$

D Technical results for the applications

Before diving into the proofs of different applications, we first recall, for completeness, some standard tools used to establish the following moment stability bounds (see [\[12, 34, 31\]](#)).

First, we begin with the commutation relations between the number operator N and a or $a^\dagger$, which follow directly from the CCR and the definitions.

Lemma D.1 ([\[34, Eq. 17/18\]](#)) *Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a real-valued function. Then,*

$$\begin{aligned} af(N + j\mathbb{1}) &= f(N + (j+1)\mathbb{1})a, & a^\dagger 1_{>j} f(N - j\mathbb{1}) &= f(N - (j+1)\mathbb{1})a^\dagger 1_{>j}, \\ f(N - j\mathbb{1})a 1_{>j} &= af(N - (j+1)\mathbb{1})1_{>j}, & f(N + j\mathbb{1})a^\dagger &= a^\dagger f(N + (j+1)\mathbb{1}) \end{aligned}$$

and

$$\begin{aligned} (a^\dagger)^\ell a^\ell &= (N - (\ell-1)\mathbb{1})(N - (\ell-2)\mathbb{1}) \cdots (N - \mathbb{1})N, \\ a^\ell (a^\dagger)^\ell &= (N + \mathbb{1})(N + 2\mathbb{1}) \cdots (N + (\ell-1)\mathbb{1})(N + \ell\mathbb{1}). \end{aligned}$$

Next, we define the following difference, which is in many calculations the key step to achieve the result: For $\ell \in \mathbb{N}$, $k > 0$ and $f(x) = (x+1)^k 1_{x \geq -1}$, we define

$$g_\ell(x) = \begin{cases} f(x) - f(x-\ell) & x \geq \ell-1; \\ f(x) & \ell-1 > x \geq 0; \\ 0 & 0 > x. \end{cases} \quad (29)$$

Lemma D.2 ([\[12, Lem. C.1\]](#)) *Let g_ℓ be defined in [Equation \(29\)](#) for $\ell \in \mathbb{N}$. Then, for all $k \geq 2$ and $x \in \mathbb{R}$*

$$\begin{aligned} g_\ell(x) &\leq g_{\ell+1}(x), \\ g_\ell(x-\ell) &\leq g_\ell(x). \end{aligned}$$

Beyond monotonicity, the functions approaches the following lower and upper bounds.

Lemma D.3 ([12, Lem. C.2]) *Let $g_\ell : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be defined in Equation (29) for $\ell \in \mathbb{N}$. Then, for all $x \in \mathbb{R}$ and $k \geq 1$,*

$$\left. \begin{array}{ll} x \geq \ell - 1 & (x+1)^{k-1}\ell \\ \ell - 1 > x \geq 0 & (x+1)^k \\ 0 > x & 0 \end{array} \right\} \leq g_\ell(x)$$

and

$$g_\ell(x) \leq \begin{cases} \frac{k\ell}{2} (1 + \mathbf{1}_{k=1}) (x+1)^{k-1} & x \geq 0 \\ (x+1)^k & x \geq 0 \\ 0 & 0 > x \end{cases}.$$

Next, we recall the following upper and lower bounds on increasing product formulas.

Lemma D.4 ([12, Lem. C.3]) *Let $\ell \in \mathbb{N}$ and $x \geq \ell$. Then,*

$$\begin{aligned} (x+1)^\ell - \frac{(\ell+1)\ell}{2} (x+1)^{\ell-1} &\leq ((x+1) - \ell) \cdots ((x+1) - 1) \leq (x+1)^\ell, \\ (x+1)^\ell &\leq (x+1) \cdots (x+1 + \ell - 1) \leq \ell! (x+1)^\ell. \end{aligned}$$

The final tool is a simple optimization over a polynomial with a negative leading term:

Lemma D.5 ([34, Lem. 11]) *For $\alpha, \beta > 0$ and $a > b > 0$, the polynomial $p(X) = -\alpha X^a + \beta X^b$ satisfies*

$$\max_{X \geq 0} p(X) \leq \left(\frac{\beta b}{\alpha a} \right)^{\frac{b}{a-b}} \beta.$$

With these three tools at hand, we now proceed to prove the moment stability bound of Theorem 5.2. For presentation, we repeat the result:

Theorem D.6 *Let $\rho \in \mathcal{T}_f$, $\mathcal{L}_l$ be a Limbladian of the form (22) associated to l -photon-dissipation and H a 2-local Hamiltonian of the form (23). Assume that there exists a $k \geq 2$ such that $t \mapsto \text{tr} [\mathcal{W}_v^{k,1}(\rho(t))]$ is differentiable for every $t \geq 0$.*

1. *If we consider a coupling of the form $-i[H, \cdot] + \eta \mathcal{L}_\ell$ with $d_H \leq 2(\ell - 1)$, then there exist constants $\eta_k, C_k, \mu_k > 0$ such that for every $\eta \geq \eta_k$,*

$$\text{tr} [(-i[H, \cdot] + \eta \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t))] \leq -\frac{C_k}{2} \text{tr} [\rho(t) \mathcal{W}_v^{\ell+k-1}] + \mu_k.$$

2. *If $d_H \leq 2(\ell - 2)$, then there exists a constant μ_k such that*

$$\text{tr} [(-i[H, \cdot] + \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t))] \leq -\frac{\ell}{4} \text{tr} [\rho(t) \mathcal{W}_v^{\ell+k-1}] + \mu_k.$$

Note that the constants C_k, μ_k , and η_k depend on the defining parameters of the generator and can be improved by model specific calculations.

Proof. 1. We split the Hamiltonian in two parts $H = H^{(1)} + H^{(2)}$, where $H^{(1)}$ is the diagonal part in terms of the representation matrices $(\lambda_{ij}^{(u)})_{ij \in V}$ and $H^{(2)}$ is the off-diagonal part. On the one hand, Theorem 5.1 shows the existence of a constant c such that

$$\text{tr} [(\mathcal{H}(H^{(1)}) + \eta \mathcal{L}) \mathcal{W}_v^k(\rho(t))] \leq -\eta \frac{\ell}{2} \text{tr} [\rho(t) (N_i + \mathbf{1})^{\ell+k-1}] + c \frac{\ell}{2} \text{tr} [\rho(t) (N_i + \mathbf{1})^{\ell+k-2}]$$

On the other hand, we need to consider the off-diagonal Hamiltonian $H^{(2)}$, i.e. H_{js} for $j \sim s$, $j \neq s$, and upper bound the quantity $\text{tr} [\mathcal{H}(H^{(2)})(\rho(t))f((N_i + \mathbb{1}))]$ for $i \in V$. Since the commutator is zero for $i \neq j, s$, choose w.l.o.g. $i = j$, define $g_t(x) = f(x) - f(x - t)$ for $x \geq t - 1$, and compute for $u_2 \geq u_1$

$$\begin{aligned}
i[f(N_j), H_{js}] &= \sum_{\substack{u \in (\mathbb{N}_0)^4 \\ \|u\|_1 \leq 2(\ell-1) \\ u_1 \leq u_2}} \lambda_{js}^{(u)} [f(N_j), a_j^{u_1} a_s^{u_3} (a_j^*)^{u_2} (a_s^*)^{u_4}] + \bar{\lambda}_{js}^{(u)} [f(N_j), a_s^{u_4} a_j^{u_2} (a_s^{u_3})^* (a_j^{u_1})^*] \\
&= \sum_{\substack{u \in (\mathbb{N}_0)^4 \\ \|u\|_1 \leq 2(\ell-1) \\ u_1 \leq u_2}} \lambda_{js}^{(u)} a_s^{u_3} (a_s^*)^{u_4} [f(N_j), a_j^{u_1} (a_j^*)^{u_2}] + \bar{\lambda}_{js}^{(u)} a_s^{u_4} (a_s^{u_3})^* [f(N_j), a_j^{u_2} (a_j^{u_1})^*] \\
&= \sum_{\substack{u \in (\mathbb{N}_0)^4 \\ \|u\|_1 \leq 2(\ell-1) \\ u_1 \leq u_2}} \lambda_{js}^{(u)} a_s^{u_3} (a_s^*)^{u_4} g_{u_2-u_1}(N_j) a_j^{u_1} (a_j^{u_2})^* - \bar{\lambda}_{js}^{(u)} a_s^{u_4} (a_s^{u_3})^* a_j^{u_2} (a_j^{u_1})^* g_{u_2-u_1}(N_j)
\end{aligned}$$

where we have used the relations provided in [Theorem D.1](#). An analogous computation can be done for $u_1 > u_2$.

Let $r_1 = \min\{u_1, u_2\}$, $r_3 = \min\{u_3, u_4\}$, $m_1 = |u_1 - u_2|$ and $m_3 = |u_3 - u_4|$. If we denote $N_j[r : k] = (N_j + r \mathbb{1}) \dots (N_j + k \mathbb{1})$, then

$$i[f(N_j), H_{js}] \leq 2 \sum_{\substack{u \in (\mathbb{N}_0)^4 \\ \|u\|_1 \leq 2(\ell-1)}} |\lambda_{js}^{(u)}| N_s[1 : r_3] \sqrt{N_s[1 : m_3]} g_{m_1}(N_j) \sqrt{N_j[1 : m_1]} N_j[1 : r_1].$$

Using the inequality $(x+1) \dots (x+u) \leq u!(x+1)^u$ provided by [Theorem D.4](#),

$$\sqrt{N_j[1 : m_1]} N_j[1 : r_1] \leq \sqrt{m_1!} (r_1!) (N_j + \mathbb{1})^{\frac{m_1}{2}} (N_j + \mathbb{1})^{r_1} \leq ((2\ell+1)!)^{3/2} (N_j + \mathbb{1})^{\frac{u_1+u_2}{2}}.$$

Similarly, we can bound the norm supported on the mode s . In addition, using [Theorem D.3](#) we can upper and lower bound g_u and obtain

$$\begin{aligned}
i[f(N_i), H_{js}] &\leq 2k \sum_{\substack{u \in (\mathbb{N}_0)^4 \\ \|u\|_1 \leq 2(\ell-1)}} |\lambda_{js}^{(u)}| ((2\ell+1)!)^3 |u_2 - u_1| (N_j + \mathbb{1})^{k-1} (N_j + \mathbb{1})^{\frac{u_1+u_2}{2}} (N_s + \mathbb{1})^{\frac{u_3+u_4}{2}} \\
&\leq 4k(\ell-1)((2\ell+1)!)^3 \|\lambda\|_\infty \sum_{\substack{u \in (\mathbb{N}_0)^4 \\ \|u\|_1 \leq 2(\ell-1)}} (N_j + \mathbb{1})^{\frac{u_1+u_2}{2}+k-1} (N_s + \mathbb{1})^{\frac{u_3+u_4}{2}}.
\end{aligned}$$

Let $x = \frac{u_1+u_2}{2}$, $y = \frac{u_3+u_4}{2}$, which satisfy $0 \leq x + y \leq \ell - 1$. For $k \geq 2$, let us consider $p, q > 1$

$$p = \frac{\ell + k - 1}{x + k - 1}, \quad q = \frac{\ell + k - 1}{\ell - x}, \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (30)$$

Applying Young's inequality and using that $|\{u \in (\mathbb{N}_0)^4 : \|u\|_1 \leq 2(\ell-1)\}| = \binom{2\ell+2}{4}$,

$$i[f(N_i), H_{js}] \leq 4k(\ell-1)((2\ell+1)!)^3 \binom{2\ell+2}{4} \|\lambda\|_\infty \left[(N_j + \mathbb{1})^{\ell+k-1} + (N_s + \mathbb{1})^{\ell+k-1} \right]. \quad (31)$$

Notice that the case $x = 0$ is also contained in the latter bound. We can now write for $i \in V$,

$$\begin{aligned}
\text{tr} \left[\mathcal{H}(H^{(2)})(\rho)(N_i + \mathbb{1})^{k/2} \right] &= i \sum_{\substack{j,s \in V \\ j < s}} \text{tr} [\rho(t)[f(N_i), H_{js}]] \\
&\leq \Gamma_{\ell,k} \sum_{\substack{s \in V \\ i \neq s}} \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-1} \right] + \text{tr} \left[\rho(t)(N_s + \mathbb{1})^{\ell+k-1} \right] \\
&\leq 2D\Gamma_{\ell,k} \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-1} \right] + \Gamma_{\ell,k} \sum_{\substack{s \in V \\ i \sim s}} \text{tr} \left[\rho(s)(N_s + \mathbb{1})^{\ell+k-1} \right]
\end{aligned}$$

since there are only $2D$ elements interacting with i . Putting all together, we can upper bound

$$\begin{aligned}
\text{tr} \left[(\mathcal{H}(H) + \eta \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t)) \right] &= \sum_{i,j \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \text{tr} \left[\rho(t)(\mathcal{H}(H + \eta \mathcal{L}(L_j))^*(N_i + \mathbb{1})^k \right] \\
&\leq \sum_{i \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \left(\left(2D\Gamma_{\ell,k} - \eta \frac{\ell}{2} \right) \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-1} \right] + c \frac{\ell}{2} \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-2} \right] \right. \\
&\quad \left. + \Gamma_{\ell,k} \sum_{\substack{s \in V \\ i \sim s}} \text{tr} \left[\rho(s)(N_s + \mathbb{1})^{\ell+k-1} \right] \right),
\end{aligned}$$

and regrouping now the terms,

$$\begin{aligned}
\text{tr} \left[(\mathcal{H}(H) + \eta \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t)) \right] &\leq \\
&\leq \sum_{i \in V} \left(\frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \left(2D\Gamma_{\ell,k} - \eta \frac{\ell}{2} \right) + \Gamma_{\ell,k} \sum_{\substack{s \in V \\ s \sim i}} \frac{e^{-\kappa \text{dist}(v,s)}}{Z_v} \right) \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-1} \right] \\
&\quad + \sum_{i \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \frac{\ell}{2} c \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-2} \right] \\
&= \sum_{i \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \left(\left(2D\Gamma_{\ell,k} - \eta \frac{\ell}{2} \right) + \Gamma_{\ell,k} \sum_{\substack{s \in V \\ s \sim i}} e^{-\kappa \text{dist}(v,s) + \kappa \text{dist}(v,i)} \right) \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-1} \right] \\
&\quad + \sum_{i \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \frac{\ell}{2} c \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-2} \right]
\end{aligned}$$

Finally, using that $-\text{dist}(v,s) + \text{dist}(v,i) \leq 1$ for $i \sim s$, if we choose $\eta_k > 0$ such that

$$-C_{\ell,k,\kappa,\eta} := 2D\Gamma_{\ell,k}(1 + 2De^\kappa) - \eta_k \frac{\ell}{2} < 0, \quad (32)$$

making use of [Theorem D.5](#) with $X = (N_i + \mathbb{1})$, $\alpha = -\frac{1}{2}C_{\ell,k,\kappa,\eta}$, $\beta = \frac{c\ell}{2}$, $a = \ell + k - 1$ and $b = \ell + k - 2$ we obtain that for every $\eta \geq \eta_k$

$$\begin{aligned}
\text{tr} \left[(\mathcal{H}(H) + \eta \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t)) \right] &\leq \\
&\leq \sum_{i \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \left(-\frac{C_{\ell,k,\kappa,\eta}}{2} \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-1} \right] + \frac{c\ell}{2} \left(\frac{c\ell}{C_{\ell,k,\kappa,\eta}} \right)^{\ell+k-2} \right) \\
&\leq -\frac{C_{\ell,k,\kappa,\eta}}{2} \text{tr} \left[\rho(t) \mathcal{W}_v^{k+\ell-1} \right] + \frac{c\ell}{2} \left(\frac{c\ell}{C_{\ell,k,\kappa,\eta}} \right)^{\ell+k-2},
\end{aligned}$$

and the result follows.

2. For the second part, notice that in the previous expression the leading term is not negative anymore, but we have to force η to make this term negative. We show that if in (30) we take $k \geq 2$ and the polynomial in the creation and annihilation operators of the Hamiltonian has degree at most $2(\ell - 2)$, we can obtain again the negativity of the leading term. Start with

$$i[f(N_i), H_{js}] \leq 4k(\ell - 1)((2\ell + 1)!)^3 \|\lambda\|_\infty \sum_{\substack{u \in (\mathbb{N}_0)^4 \\ \|u\|_1 \leq 2(\ell - 2)}} (N_j + \mathbb{1})^{\frac{u_1 + u_2}{2} + k - 1} (N_s + \mathbb{1})^{\frac{u_3 + u_4}{2}}. \quad (33)$$

Let $x = \frac{u_1 + u_2}{2}$, $y = \frac{u_3 + u_4}{2}$ satisfying $0 \leq x + y \leq \ell - 2$. For $k \geq 2$, we can set

$$p = \frac{\ell + k - 2}{x + k - 1}, \quad q = \frac{\ell + k - 2}{\ell - x - 1}, \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (34)$$

and we could then bound

$$\text{tr} \left[\mathcal{H}(H^{(2)})(\rho)(N_i + \mathbb{1})^{k/2} \right] \leq 2D\Gamma_{\ell,k} \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-2} \right] + \Gamma_{\ell,k} \sum_{\substack{s \sim V \\ i \neq s}} \text{tr} \left[\rho(t)(N_s + \mathbb{1})^{\ell+k-2} \right] \quad (35)$$

and therefore we can upper bound

$$\begin{aligned} \text{tr} \left[(\mathcal{H}(H) + \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t)) \right] &= \sum_{i,j \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \text{tr} [\rho(t)(\mathcal{H}(H + \mathcal{L}(L_j))^*(f(N_i)))] \\ &\leq \sum_{i \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \left(-\frac{\ell}{2} \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-1} \right] + \left(2D\Gamma_{\ell,k} + \frac{\ell}{2}c \right) \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-2} \right] \right. \\ &\quad \left. + \Gamma_{\ell,k} \sum_{\substack{s \in V \\ s \sim i}} \text{tr} \left[(N_s + \mathbb{1})^{\ell+k-2} \right] \right). \end{aligned}$$

Regrouping terms,

$$\begin{aligned} \text{tr} \left[(\mathcal{H}(H) + \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t)) \right] &\leq \sum_{i \in V} -\frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \frac{\ell}{2} \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-1} \right] \\ &\quad + \sum_{i \in V} \frac{e^{-\kappa \text{dist}(v,i)}}{Z_v} \left(2D\Gamma_{\ell,k} + \frac{\ell}{2}c + \Gamma_{\ell,k} \sum_{\substack{s \in V \\ s \sim i}} e^{-\kappa \text{dist}(v,s) + \kappa \text{dist}(v,i)} \right) \text{tr} \left[\rho(t)(N_i + \mathbb{1})^{\ell+k-2} \right]. \end{aligned}$$

Finally, using again that $-\text{dist}(v, s) + \text{dist}(v, i) \leq 1$ and letting

$$C_{\ell,k,\kappa,\alpha} = \left(2D\Gamma_{\ell,k}(1 + 2De^\kappa) + \frac{\ell}{2}c \right), \quad (36)$$

if we make use of [Theorem D.5](#) for $X = (N_i + \mathbb{1})$, $\alpha = \frac{\ell}{4}$, $\beta = C_{\ell,k,\kappa,\alpha}$, $a = \ell + k - 1$ and $b = \ell + k - 2$,

$$\text{tr} \left[(\mathcal{H}(H) + \mathcal{L}_\ell) \mathcal{W}_v^k(\rho(t)) \right] \leq -\frac{\ell}{4} \text{tr} \left[\rho(t) \mathcal{W}_v^{\ell+k-1} \right] + C_{\ell,k,\kappa,\alpha} \left(\frac{4C_{\ell,k,\kappa,\alpha}}{\ell} \right)^{\ell+k-2}. \quad (37)$$

and the result follows. $\square$