

# Production of charged Higgs bosons associated with CP-even Higgs bosons at future multi–TeV muon colliders

Quang Hoang-Minh Pham<sup>a</sup>, Khoa Ngo-Thanh Ho<sup>a</sup>, Khiem Hong Phan<sup>b,c</sup>

<sup>a</sup>*VNUHCM-University of Science, 227 Nguyen Van Cu, District 5, Ho Chi Minh City 700000, Vietnam*

<sup>b</sup>*Institute of Fundamental and Applied Sciences, Duy Tan University, Ho Chi Minh City 70000, Vietnam*

<sup>c</sup>*Faculty of Natural Sciences, Duy Tan University, Da Nang City 50000, Vietnam*

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## Abstract

We report the first results for charged Higgs boson production associated with CP-even Higgses at future multi–TeV muon colliders within the scenario of the Type-Y Two-Higgs-Doublet Model (THDM). The valid parameter regions in the Type-Y THDM are first updated, and based on the allowed parameter space, all two-body decay channels of the charged Higgs boson are then computed with the help of H-COUP. Both production processes  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h, H^\pm H^\mp H \rightarrow t\bar{t}b\bar{b}h, t\bar{t}b\bar{b}H$  are scanned over the constrained parameter space of the considered model. The production cross sections can reach  $\sim 0.5$  fb in several regions of the parameter space. The results provide an opportunity to probe charged Higgs bosons produced in association with the heavy CP-even Higgses at the high integrated luminosities proposed for future multi–TeV muon colliders. Furthermore, we evaluate the signal significances for  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h \rightarrow t\bar{t}b\bar{b}h \rightarrow t\bar{t}b\bar{b}b\bar{b}$  with respect to the Standard Model backgrounds. We also estimate the significances by including the top quark decay into leptons and bottom quarks and considering  $b$ -tagging. Thanks to the high integrated luminosities anticipated at future multi–TeV muon colliders, we find that the signal significances can exceed  $\sim 3\sigma$  at several benchmark points selected in the viable parameter space of the Type-Y THDM.

*Keywords:* Charged Higgs boson phenomenology, physics beyond the Standard Model, and new physics at present and future multi–TeV muon colliders.

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## 1. Introduction

The scalar Higgs sector remains one of the least understood parts of the Standard Model. It is known that experimental programs at future colliders, such as the High-Luminosity Large Hadron Collider (HL-LHC), the High-Energy LHC (HE-LHC), as well as proposed future lepton colliders, including the International Linear Collider and multi–TeV muon colliders, are planned to explore the structure of the scalar Higgs sector. In pursuing the goals, the program involves not only precise measurements of the SM-like Higgs boson properties but also searches for additional scalar particles at future colliders. In many of the additional scalar production processes, searches for singly charged Higgs bosons are interest greatly at current and future colliders. We briefly summarize several representative works in this section. At  $pp$  collisions, the light charged Higgs mass region has been measured produced via top-quark decay channels, as reported in Refs. [1, 2, 3, 4]. Charged Higgs bosons associated with top-pair production have been investigated at  $\sqrt{s} = 7$  TeV at the LHC, including the sequential decay mode  $H^\pm \rightarrow \tau\nu_\tau$  in Refs. [5, 6, 7, 8], and the decay channel  $H^\pm \rightarrow c\bar{s}$  in Refs. [9, 10]. Searches for a heavy charged Higgs mass regions via

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*Email address:* phanhongkhiem@duytan.edu.vn (Khiem Hong Phan)

the decay  $H^\pm \rightarrow tb$  through the scattering process  $pp \rightarrow tH^\pm \rightarrow t\bar{t}b$  have also been reported at the LHC [11, 12, 13]. Furthermore, probing charged Higgs production via vector boson fusion, including the decay  $H^\pm \rightarrow W^\pm Z$  as published in Ref. [14, 15]. A rare decay mode of the charged Higgs into charm and bottom quarks in Ref. [16] and the decay channel  $W^\pm h$  in [17, 18] have been searched in proton–proton collisions.

Many theoretical computations for charged Higgs productions at the LHC within various extensions of the SM have been performed. In Refs. [19, 20], the charged Higgs productions together with a top quark and its decay into a top and bottom quark, including top polarization effects, have been studied at the LHC. Moreover, charged Higgs boson pair production including its bosonic decay channels at the HL-LHC as well as the HE-LHC have been reported in [21]. The computation of charged Higgs production within the Minimal Supersymmetric Standard Model has been considered in Ref. [22]. Furthermore, the implications for charged Higgs at the LHC in the different mass regions have been investigated as in Refs. [23, 24, 25, 26, 27, 28, 29]. Charged Higgs productions have also been studied at future lepton colliders (LCs), including  $e^-e^+$  collisions and multi–TeV muon colliders, as reported in Refs. [30, 31, 32, 33, 34, 35] and also in many previous papers [36, 37, 38]. Future LCs are proposed for high-precision tests and searches for additional scalar particles in many beyond the Standard Models (BSM). The LCs provide a cleaner environment than the LHC, which suffers from huge QCD backgrounds. Moreover, multi–TeV muon colliders are designed for higher frontier energy regimes for testing new physics. In comparison with  $e^-e^+$  collisions, a key advantage of multi–TeV muon colliders is that the muon mass is about 207 times larger than the electron mass. Subsequently, s-channel exchanges by new scalar particles may exist and can be enhanced due to resonance effects. Furthermore, the couplings of the scalar to muons are proportional to the mixing angle  $\beta$ . Depending on the type of THDM, these contributions can also be significant. For the above reasons, multi–TeV muon colliders offer a great opportunity to probe and distinguish different types of THDM. In many of our previous works [36, 37, 38], we have studied charged Higgs boson production in Type-I and Type-X THDM. In this work, we report the first results for charged Higgs boson production associated with CP-even Higgses at future multi–TeV muon colliders in the scenario of the Type-Y THDM. The valid parameter regions for the Type-Y THDM are updated in this study. Based on the allowed parameter space, we compute all two-body decay channels of the charged Higgs boson with the help of **H-COUP** [46, 47]. The production cross sections for charged Higgs boson production associated with CP-even Higgs states are scanned over the viable parameter space at 3 TeV and 5 TeV center-of-mass energies at future multi–TeV muon colliders. Furthermore, we compute the signal significances for  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h \rightarrow t\bar{t}b\bar{b}h$  in the sequential decay of  $h \rightarrow b\bar{b}$  with respect to the Standard Model background. We also estimate the significances by including the top quark decay into leptons and bottom quarks and considering  $b$ -tagging. Thanks to the high integrated luminosities anticipated at future multi–TeV muon colliders, we find that the signal significances can exceed  $\sim 3\sigma$  at several benchmark points selected in the viable parameter space of the Type-Y THDM.

The remaining sections of the paper are organized as follows. In Section 2, the THDM are briefly reviewed and its constraints are discussed in further detail. Section 3 presents detailed calculations and phenomenological results for  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h/H$  at multi–TeV muon colliders. Conclusions and outlook are presented in Section 4. The Appendix shows the amplitudes for the processes under investigation.

## 2. The Two-Higgs-Doublet Model and Its Constraints

In this section, we briefly review the THDM. We refer to our previous works [36, 37, 38] and Ref. [39] for a comprehensive review of the THDM. In the studied model, the matter and

gauge boson content remain the same as in the SM, while the scalar sector is extended by an additional scalar doublet with hypercharge  $Y = 1/2$ . Consequently, the scalar potential and Yukawa Lagrangian are enlarged compared with those of the SM. The scalar potential is given as follows in Ref. [36]:

$$\begin{aligned} \mathcal{V}(\Phi_1, \Phi_2) = & \sum_{j=1}^2 m_{jj}^2 \Phi_j^\dagger \Phi_j - \left( m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{H.c.} \right) + \frac{1}{2} \sum_{j=1}^2 \lambda_j \left( \Phi_j^\dagger \Phi_j \right)^2 \\ & + \lambda_3 \Phi_1^\dagger \Phi_1 \Phi_2^\dagger \Phi_2 + \lambda_4 \Phi_1^\dagger \Phi_2 \Phi_2^\dagger \Phi_1 + \left[ \frac{1}{2} \lambda_5 \left( \Phi_1^\dagger \Phi_2 \right)^2 + \text{H.c.} \right]. \end{aligned} \quad (1)$$

As in Ref. [38], we also focus on the CP-conserving version of the THDM, in which all parameters, such as  $m_{11}^2$ ,  $m_{22}^2$ ,  $m_{12}^2$ ,  $\lambda_1, \dots, \lambda_5$ , are taken to be real variables. As indicated in many previous works, a  $Z_2$  discrete symmetry is introduced into the scalar potential (up to the soft-breaking term) to avoid tree-level flavor-changing neutral currents. Consequently, the THDM is classified into four distinct types, which have different Yukawa couplings as shown in Ref. [40]. The Yukawa Lagrangian is written in its most general parametrized form, as presented in Ref. [36]:

$$\begin{aligned} -\mathcal{L}_Y = & \sum_{f=u,d,\ell} \left( \sum_{\phi_j=h,H} \frac{m_f}{v} \xi_{\phi_j}^f \phi_j \bar{f} f - i \frac{m_f}{v} \xi_A^f \bar{f} \gamma_5 f A \right) \\ & + \frac{\sqrt{2}}{v} [\bar{u}_i V_{ij} (m_{u_i} \xi_A^u P_L + \xi_A^d m_{d_j} P_R) d_j H^+] + \frac{\sqrt{2}}{v} \bar{\nu}_L \xi_A^\ell m_{\ell_R} H^+ + \text{H.c.} \end{aligned} \quad (2)$$

The elements of the CKM matrix are denoted by  $V_{ij}$ , and the left- and right-handed leptons are represented as  $\ell_{L/R}$ . The projection operators are defined as  $P_{L/R} = \frac{1 \mp \gamma_5}{2}$ . All coupling coefficients appearing in Eq. 2 are summarized in Table 2 of Ref. [40]. After electroweak symmetry breaking (EWSB), the additional scalar particles in the considered model include a CP-even Higgs ( $H$ ), a CP-odd Higgs ( $A$ ), and two singly charged Higgs bosons ( $H^\pm$ ). The independent parameters of the THDM used in our analysis consist of the mixing angles, expressed in terms of  $s_{\beta-\alpha}$  and  $t_\beta$ , the scalar masses  $m_H$ ,  $m_A$ ,  $m_{H^\pm}$ , and the soft-breaking parameter  $m_{12}^2$ .

Before presenting the calculations and phenomenological results of this work, we first update the parameter constraints for the Type-Y THDM in the following paragraphs. Detailed explanations of the theoretical and experimental constraints were given in our previous studies, especially for Type-I in Ref. [36] and for Type-X in Ref. [37, 38]. The present work focuses on scanning the parameter space of the Type-Y THDM. The parameter space of the Type-Y THDM is scanned over the same ranges as in Refs. [36]:  $s_{\beta-\alpha} \in [0.97, 1]$ ,  $t_\beta \in [0.5, 45]$ ,  $m_H \in [130, 1000]$  GeV,  $m_{A, H^\pm} \in [130, 1000]$  GeV, and  $m_{12}^2 \in [0, 10^6]$  GeV<sup>2</sup>, with the SM-like Higgs mass fixed at  $m_h = 125.09$  GeV. It is emphasized that the scalar masses in the Type-Y THDM should be scanned up to 1500 GeV. However, in this study, we focus on charged Higgs bosons with masses up to 1000 GeV. Heavier charged Higgs states generally yield smaller production cross sections, making them difficult to probe at future colliders. For these reasons, we restrict our parameter space to the mass ranges specified above. The steps for imposing the parameter constraints can be summarized as follows. First, the sampled points are checked against the theoretical conditions. In this step, the model under consideration must satisfy unitarity, perturbativity, and vacuum stability of the scalar potential. Second, the surviving parameter points are tested against the experimental limits on the  $S$ ,  $T$ , and  $U$  parameters at the 95% CL. In the next step, all allowed points are confronted with the LHC data. We note that the computer program 2HDMC-1.8.0 [41] is used for all the above tests. In the third stage of constraints,

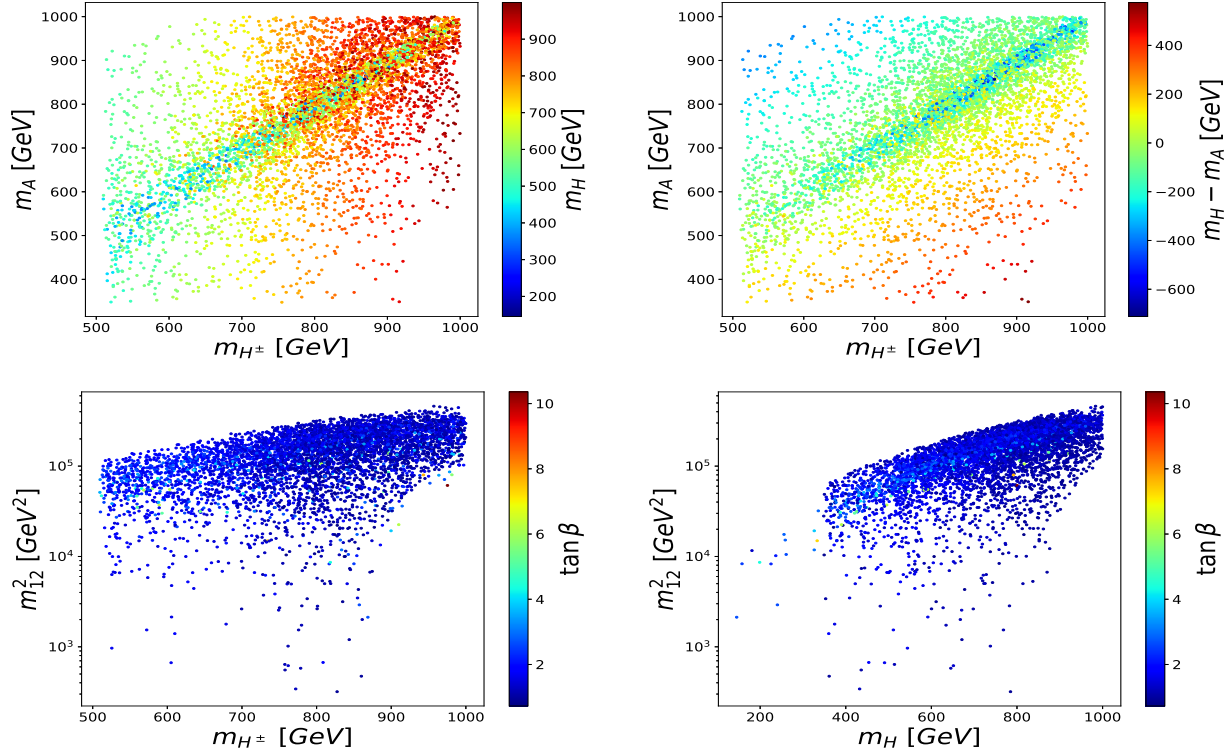


Figure 1: The scatter plots show the correlations among  $m_A$ ,  $m_{H^\pm}$ , and  $m_H$  in the upper-left panel, and those among  $m_A$ ,  $m_{H^\pm}$ , and  $m_H - m_A$  in the upper-right panel. The lower-left plot shows the correlations between  $m_{12}^2$ ,  $m_{H^\pm}$ , and  $\tan\beta$ , while the lower-right panel illustrates the correlations among  $m_{12}^2$ ,  $m_H$ , and  $\tan\beta$ .

both **HiggsSignals** [42] and **HiggsBounds** [43] are taken into account for matching the theoretical predictions with the corresponding experimental data. Their implementations are already included in **2HDMC-1.8.0**. Finally, all the remaining points are further constrained by the flavor data, which can be evaluated using **SuperISO** [44].

After obtaining the valid parameter space for the Type-Y THDM, we proceed to discuss the physical regions through the following scatter plots. We first focus on the scatter plot showing the correlations among  $m_A$ ,  $m_{H^\pm}$ , and  $m_H$  in the upper-left panel, and those among  $m_A$ ,  $m_{H^\pm}$ , and  $m_H - m_A$  in the upper-right panel, as shown in Fig. 1. From these plots, we observe that in the region where  $m_A \simeq m_{H^\pm}$ , the CP-even Higgs  $H$  can span a wide range from approximately 200 GeV to 1000 GeV. The second favored regime corresponds to  $m_H \simeq m_{H^\pm}$ , in which the CP-odd Higgs  $A$  is allowed to take any value from roughly 200 GeV to 1000 GeV. Flavor constraints require the charged Higgs mass to be larger than  $\sim 520$  GeV. In the upper-right panel plot, we find that for  $m_A - m_H \sim -200$  GeV, the CP-odd Higgs  $A$  has a mass in the range of approximately 800 GeV to 1000 GeV, while  $520 \text{ GeV} \leq m_{H^\pm} \leq 700 \text{ GeV}$ . Following the data shown in the left plot, if  $m_A \simeq m_{H^\pm}$ , the degenerate scalar mass difference  $m_A - m_H$  can span a wide range. Finally, one finds that in the region  $m_{H^\pm} \leq m_A$ , the degenerate scalar mass difference ranges from 0 GeV to approximately +200 GeV. The scatter plots show the correlations between  $m_{12}^2$ ,  $m_{H^\pm}$ , and  $\tan\beta$  in the lower-left panel, while the correlations among  $m_{12}^2$ ,  $m_H$ , and  $\tan\beta$  are illustrated in the lower-right panel. Our finding is that the updated parameter space is concentrated in the region  $10^4 \lesssim m_{12}^2 \lesssim 5 \cdot 10^5$  and  $m_H \gtrsim 400$  GeV. The favored data points concentrate in the region  $t_\beta \leq 3$ . Based on the resulting physical parameter regions, we proceed to study the production of charged Higgs bosons in association with CP-even Higgs bosons at future multi-TeV muon colliders in the following sections.



### 3. Production processes $\mu^- \mu^+ \rightarrow H^\pm H^\mp h/H$ at multi-TeV muon colliders

Detailed calculations for the production processes  $\mu^- \mu^+ \rightarrow H^\pm H^\mp h/H$  at multi-TeV muon colliders are presented in this section. The calculations are performed with the help of the programs `FeynArts/FormCalc` [45]. As stated in the introduction, the  $s$ -channel exchange of scalar particles may give significant contributions due to the resonance effects. This highlights the advantage of future multi-TeV muon colliders in comparison with  $e^- e^+$  colliders. In order to taking into account these contributions, we should consider all tree-level Feynman diagrams contributing to the production cross sections. It should be mentioned that the processes are generated in the general  $\mathcal{R}_\xi$  gauge. The advantage of this choice is that the results can be verified by self-consistency checks through their independence from the  $\xi$ -gauge parameter.

Before scanning the production cross sections over the valid parameter space of Type-Y THDM, we first present the evaluations of the charged Higgs branching fractions. The results from charged Higgs branching ratios allow us to select the appropriate decay modes for the significance simulations in the last subsection.

#### 3.1. Branching fractions

The viable parameter space after the constraints is transferred to `H-COUP` [46, 47] for evaluating all two-body decay channels of charged Higgs. Its branching fractions for all the decay channels are shown in the following paragraphs. First, the branching ratios are generated as functions of the charged Higgs mass at the benchmark first point (BP1) showing in Table 1 for an example, as indicated in the above-left plot of Fig. 2. Across the entire range of  $m_{H^\pm}$ , the decay  $H^\pm \rightarrow tb$  dominates in the lower mass region of the charged Higgs below  $\sim 750$  GeV. In the higher mass region ( $m_{H^\pm} \gtrsim 750$  GeV), the channels  $H^\pm \rightarrow W^\pm \phi_j$  with  $\phi_j = h, H, A$  open and they become the dominant contributions. Because the tree-level couplings of  $H^\pm W^\mp H$  and  $H^\pm W^\mp A$  differ only by the factor  $s_{\beta-\alpha}$ , and in this benchmark point we take  $s_{\beta-\alpha} = 0.99$ , the branching ratios of  $H^\pm \rightarrow W^\pm H$  and  $H^\pm \rightarrow W^\pm A$  become nearly equal. Furthermore, we find that the branching ratio of  $H^\pm \rightarrow W^\pm h$  is approximately  $10^{-1}$  over the entire range of charged Higgs masses. In the above-right plot of Fig. 2, the branching fraction of  $H^\pm \rightarrow tb$  is shown as a function of  $m_{H^\pm}$  and  $\tan\beta$ . This decay mode dominates at low values of  $\tan\beta$  and in the low-mass region of the charged Higgs. As pointed out above, the channels  $H^\pm \rightarrow W^\pm h/H/A$  provide large contributions in the high-mass region. This explains why the branching fraction of  $H^\pm \rightarrow tb$  is reduced in these regions. In the two lower-plots shown in Fig. 2, the decay channels  $H^\pm \rightarrow W^\pm h/H$  are displayed, and they become the dominant contributions as increasing of the charged Higgs masses.

In the above-left plot of Fig. 3, the branching fraction of  $H^\pm \rightarrow W^\pm A$  is shown as a function of  $m_{H^\pm}$  and  $\tan\beta$ . We observe behavior similar to that of the  $H^\pm \rightarrow W^\pm H$  channel. All remaining decay channels of the charged Higgs, such as  $H^\pm \rightarrow \tau\nu_\tau$ ,  $W^\pm Z$ , and  $W^\pm \gamma$ , are also shown in Fig. 3, and we find that their branching fractions are below  $10^{-3}$  for all value of  $t_\beta$  and charged Higgs masses. In general, the results clearly indicate that the decay modes  $H^\pm \rightarrow tb$  and  $H^\pm \rightarrow W^\pm \phi_j$  (with  $\phi_j = h, H, A$ ) provide significant contributions. In the scope of the current paper, we focus on the  $H^\pm \rightarrow tb$  channel in the calculations of the signal significances in the last subsection.

#### 3.2. Process $\mu^- \mu^+ \rightarrow H^\pm H^\mp h$

The process  $\mu^- \mu^+ \rightarrow H^\pm H^\mp h$  is first examined in this subsection. In the phenomenological analysis of this production process, we take into consideration the dominant decay channel of the charged Higgs boson, namely  $H^\pm \rightarrow tb$ . Since the final-state top quarks subsequently decay

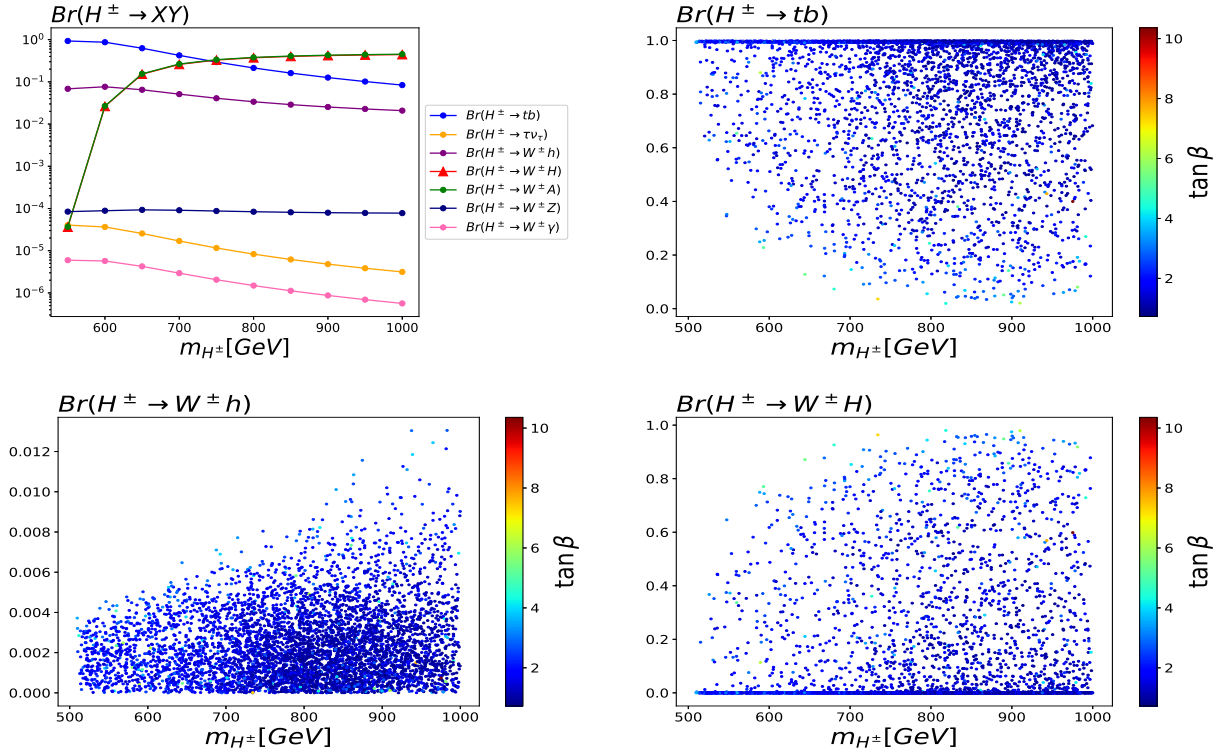


Figure 2: The upper-left plot displays the branching fractions of the charged Higgs for all decay channels at the benchmark first point presented in Table 1. The upper-right plot shows the decay rate of  $H^\pm \rightarrow tb$ , while the lower-left plot corresponds to  $H^\pm \rightarrow W^\pm h$ . The lower-right plot illustrates  $H^\pm \rightarrow W^\pm H$ . All plots are generated as functions of the charged Higgs mass and  $\tan \beta$ .

into lighter particles, we do not apply any cuts on the top quarks themselves. Meanwhile, we apply the following cuts for the final-state bottom and anti-bottom quarks as follows:

$$p_T(b) \geq 20 \text{ GeV}, \quad p_T(\bar{b}) \geq 20 \text{ GeV}, \quad |\eta(b)| \leq 2.4, \quad |\eta(\bar{b})| \leq 2.4. \quad (3)$$

The cross sections (in ab) are shown in the plane of the charged Higgs masses and the mixing angle  $\tan \beta$ , as presented in Fig. 7. The production cross sections are displayed at  $\sqrt{s} = 3 \text{ TeV}$  in the left panel and at  $\sqrt{s} = 5 \text{ TeV}$  in the right panel. For small values of  $\tan \beta$ , the cross section is of the order of  $\mathcal{O}(100)$  ab. With the high integrated luminosities expected at future multi-TeV colliders (up to  $\mathcal{L} = 3000 \text{ fb}^{-1}$ ), one expects to collect roughly  $\sim \mathcal{O}(100)$  events. The same data are scanned over the charged Higgs masses and the CP-even Higgs mass. The cross sections are of the same order as in the previous plots. The enhanced cross sections occur primarily in the region where  $m_H \sim m_{H^\pm}$ . The data shows that the production cross section reaches large values in the region of  $\tan \beta \lesssim 3$ , and the mass difference between the two scalar particles should be small ( $|\Delta m_\phi| \lesssim 50 \text{ GeV}$ ). There are two distinct regions for the CP-even Higgs where the production cross sections are enhanced. The first region corresponds to  $m_H \sim 500 \text{ GeV}$  when  $m_{H^\pm} \leq 750 \text{ GeV}$ . The second region appears for  $m_H \geq 800 \text{ GeV}$  when  $m_{H^\pm} \geq 750 \text{ GeV}$ . These scans provides useful information for selecting benchmark points to simulate the signal significances in the last subsection.

### 3.3. Process $\mu^- \mu^+ \rightarrow H^\pm H^\mp H$

We next examine the production process  $\mu^- \mu^+ \rightarrow H^\pm H^\mp H \rightarrow t\bar{t}b\bar{b}H$  at future multi-TeV muon colliders. We first scan the production cross sections over the parameter space of  $m_{H^\pm}$

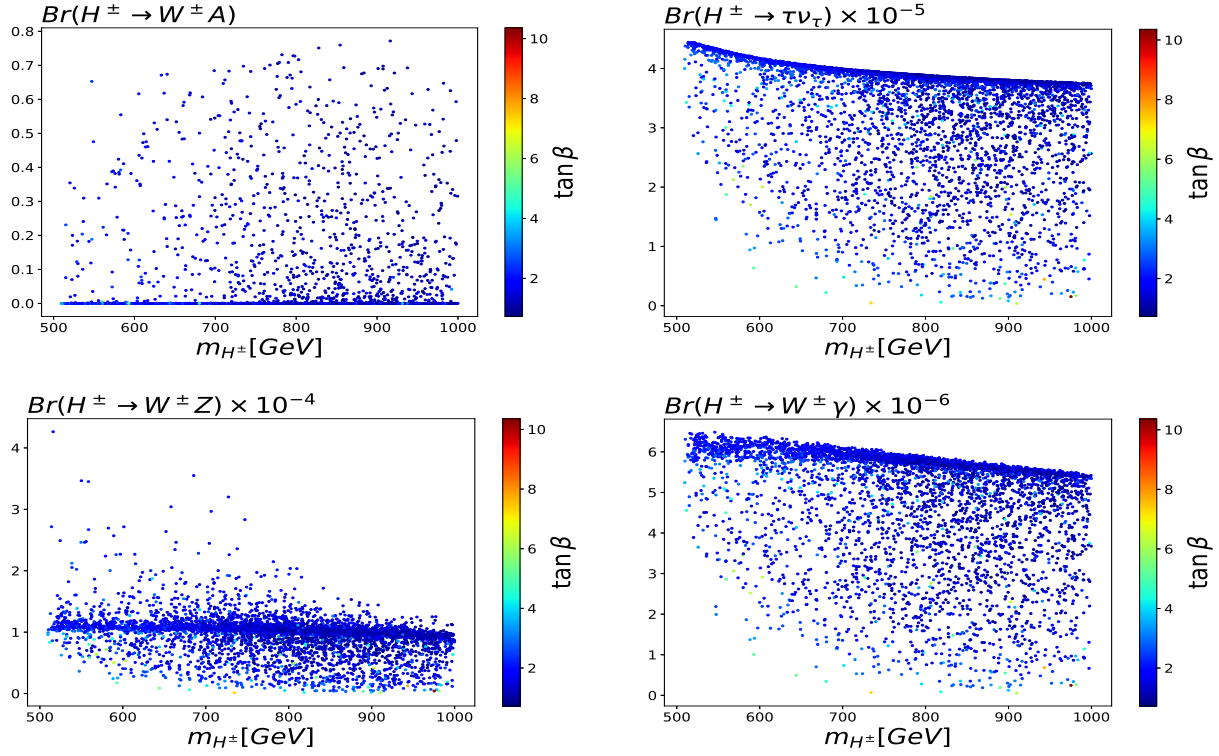


Figure 3: The upper-left plot shows for the branching fractions of the charged Higgs boson for the decay  $H^\pm \rightarrow W^\pm A$ , and the upper-right plot shows the decay rate of  $H^\pm \rightarrow \tau\nu_\tau$ . Meanwhile, the lower-left plot corresponds to  $H^\pm \rightarrow W^\pm Z$  and the lower-right plot illustrates  $H^\pm \rightarrow W^\pm \gamma$ . It is noted that all plots are generated as functions of the charged Higgs mass and  $t_\beta$ .

and  $\tan\beta$ , as shown in the two upper plots of Fig. 5. In particular, the left plot shows the cross sections at a 3 TeV of center-of-mass energy, and the right plot is for a 5 TeV of center-of-mass energy. Within the entire range of  $\sim 1 \leq t_\beta \leq 10$ , the cross sections decrease with increasing charged Higgs masses. Due to the fact that the branching ratio of  $H^\pm \rightarrow tb$  is dominant in the region of small values of  $t_\beta$ , we find that the cross sections focus mainly on the small values of  $t_\beta$ . Meanwhile, the production cross sections are presented in terms of  $m_{H^\pm}$  and  $m_H$  in the two middle plots of Fig. 5. The left plot shows the cross section at 3 TeV of center-of-mass energy, and the right plot is for a 5 TeV of center-of-mass energy. The cross sections decrease as the charged Higgs and CP-even Higgs masses increase. The cross sections focus mainly on two regions of the CP-even Higgs. The first regime is  $m_{H^\pm} \leq 750$  GeV and  $m_H \sim 500$  GeV, and the second one is  $m_{H^\pm} \geq 750$  GeV and  $m_H \geq 800$  GeV. Finally, the same quantities are plotted as a function of  $m_{H^\pm}$  and  $m_A - m_H$  (at 3 TeV for the left panel and at 5 TeV for the right panel). The cross sections mainly focus on the region  $|\Delta m_\phi| = |m_A - m_H| \lesssim 50$  GeV. From the data, we find that the production cross section reaches large values in the region of  $\tan\beta \lesssim 3$ , where the mass difference between the two scalar particles is small ( $|\Delta m_\phi| \lesssim 50$  GeV) and the charged Higgs mass is low ( $m_{H^\pm} \leq 600$  GeV). In other regions the cross sections become negligible.

### 3.4. Significances

In this subsection, the significances for  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h$  are computed at several selected benchmark points. Before presenting the results, we note that initial-state radiation (ISR) is important at future lepton colliders, as reported in many of our previous works. Therefore, we first consider the two-loop ISR corrections to the process  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h$  as performed in

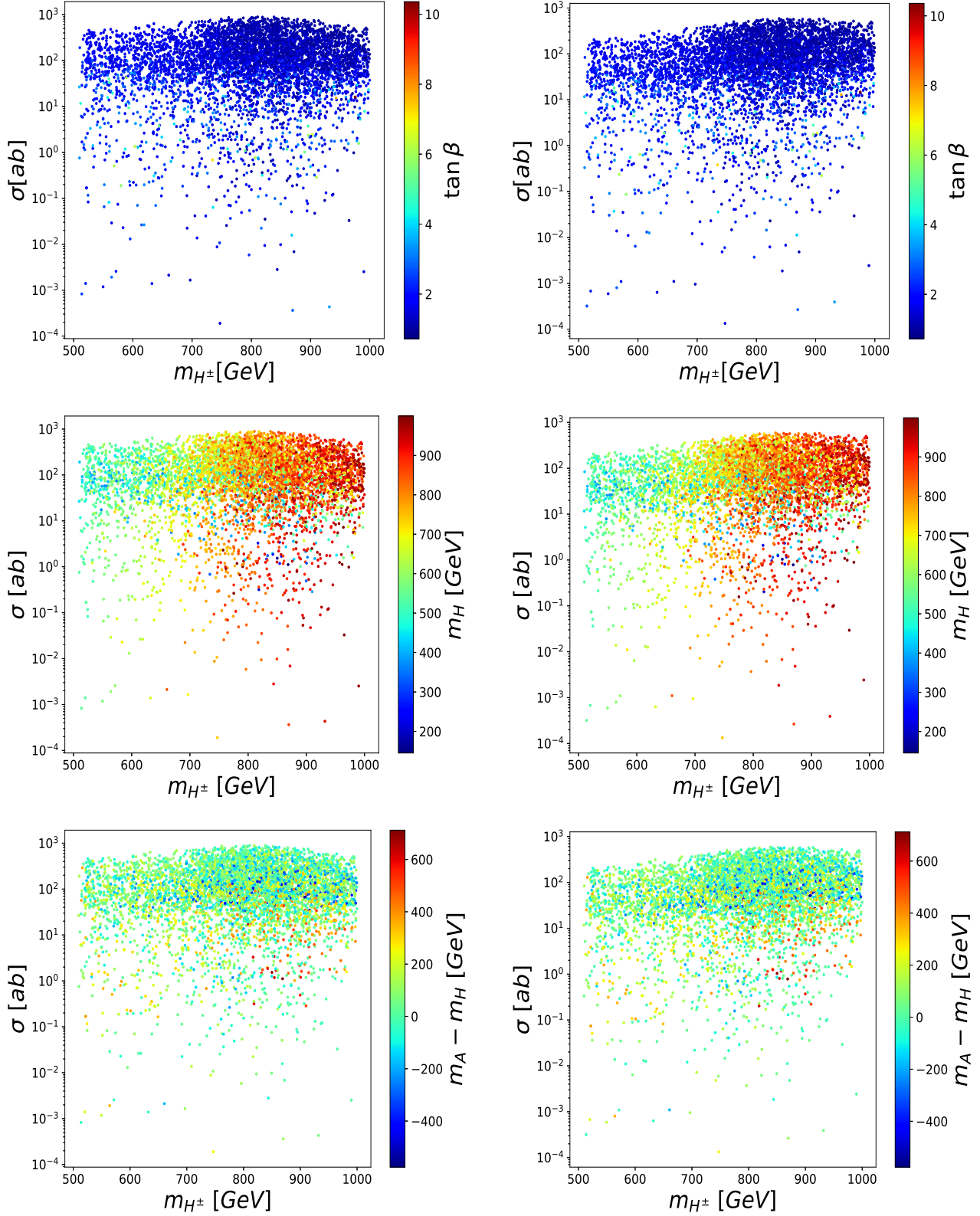


Figure 4: The production cross sections for  $\mu^- \mu^+ \rightarrow H^\pm H^\mp h \rightarrow t\bar{t} b\bar{b} h$  at  $\sqrt{s} = 3$  TeV (left) and  $\sqrt{s} = 5$  TeV (right) are shown in the parameter space of  $m_{H^\pm}$  and  $t_\beta$  (upper plots),  $m_{H^\pm}$  and  $m_H$  (middle plots), and  $m_{H^\pm}$  and  $m_A - m_H$  (lower plots).

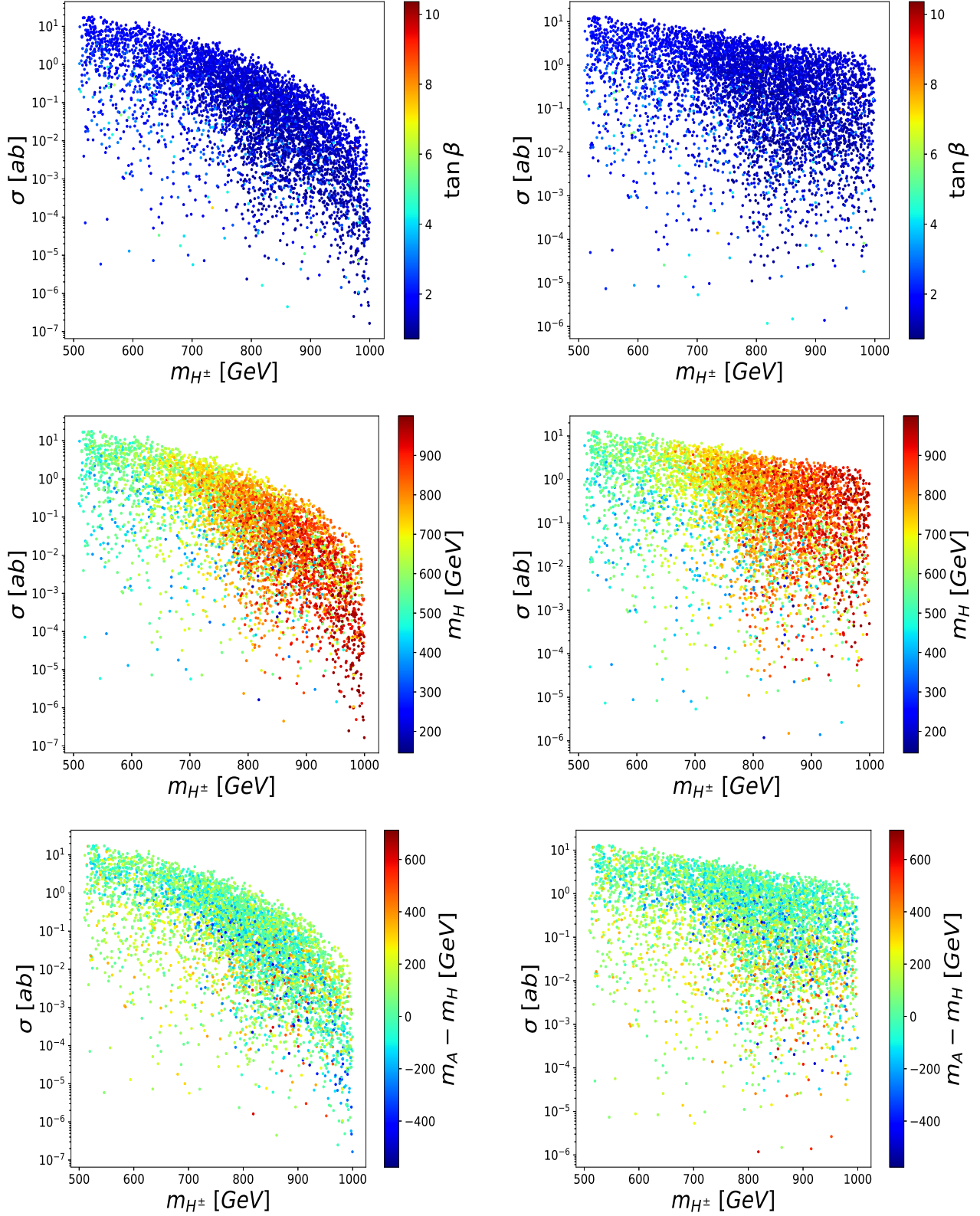


Figure 5: The production cross sections for  $\mu^-\mu^+ \rightarrow H^\pm H^\mp H \rightarrow t\bar{t}b\bar{b}H$  at  $\sqrt{s} = 3$  TeV (left) and  $\sqrt{s} = 5$  TeV (right) are shown in the parameter space of  $m_{H^\pm}$  and  $t_\beta$  (upper plots),  $m_{H^\pm}$  and  $m_H$  (middle plots), and  $m_{H^\pm}$  and  $m_A - m_H$  (lower plots).



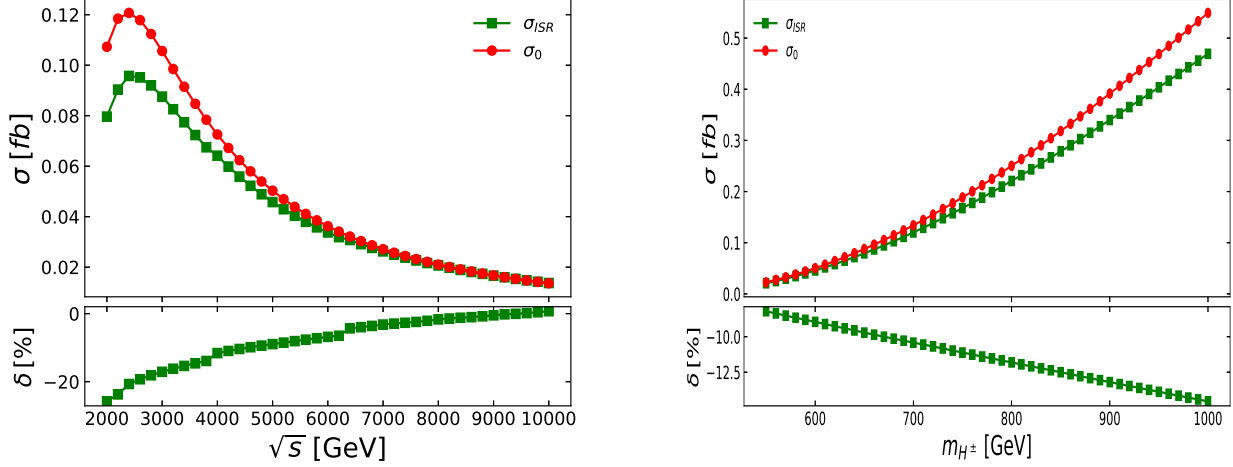


Figure 6: The ISR corrections to the process  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h$  are studied at different center-of-mass energies  $\sqrt{s} = 2$  TeV to 10 TeV (left) and at fixing  $\sqrt{s} = 5$  TeV with varying charged Higgs masses  $m_{H^\pm} = 600$  GeV to 1000 GeV (right).

| Benchmark | $m_H$ (GeV) | $m_A$ (GeV) | $t_\beta$ | $s_{\beta-\alpha}$ | $m_{12}^2$ (GeV <sup>2</sup> ) | $m_{H^\pm}$ (GeV) |
|-----------|-------------|-------------|-----------|--------------------|--------------------------------|-------------------|
| BP1       | 500         | 500         | 1.5       | 0.99               | $10^5$                         | 550–1000          |
| BP2       | 800         | 800         | 1.5       | 0.99               | $10^5$                         | 550–1000          |

Table 1: Benchmark points selected in the analysis.

Ref. [38]. Detailed formulas for evaluating the ISR corrections are provided in Ref. [38]. In this subsection, we first present the effects of ISR corrections on the production process. In the left plot of Fig. 6, the ISR corrections are computed at center-of-mass energies ranging from 2 TeV to 10 TeV, taking the process  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h$  as a representative example for our survey. For this study, we use BP1 as the input parameters. In the right plot of Fig. 6, we show the ISR corrections at  $\sqrt{s} = 5$  TeV with varying charged Higgs masses. We find that the ISR corrections range from  $-25\%$  to  $-1\%$  for all center-of-mass energies from 2 TeV to 10 TeV, while the ISR corrections range from  $-4\%$  to  $-15\%$  for all charged Higgs masses at  $\sqrt{s} = 5$  TeV.

In this section, we compute the signal significances for the process  $\mu^-\mu^+ \rightarrow H^\pm H^\mp h \rightarrow t\bar{t}b\bar{b}h \rightarrow t\bar{t}b\bar{b}b\bar{b}$  evaluated at several selected benchmark points above for future multi-TeV muon colliders. The benchmark points in the allowed parameter space are shown in the Table 1. For the SM background, we consider two processes like  $\mu^-\mu^+ \rightarrow t\bar{t}b\bar{b}h \rightarrow t\bar{t}b\bar{b}b\bar{b}$  and  $\mu^-\mu^+ \rightarrow t\bar{t}b\bar{b}Z \rightarrow t\bar{t}b\bar{b}b\bar{b}$ . The background processes are calculated with the help of MadGraph5MC@NLO [48]. The significances are then computed as follows:

$$\mathcal{S} = \frac{N_S}{\sqrt{N_S + \varepsilon_B N_B}}. \quad (4)$$

Where  $N_{S/B} = \mathcal{L}\sigma_{S/B}$  is corresponding for number of event and background. The factor  $\varepsilon_B$  is for systematic uncertainty fraction on the background yield. We take  $\varepsilon_B = 1, 1.5$  in this work.

For both the signal and the SM backgrounds, we apply the following cuts on the final-state bottom and anti-bottom quarks originating from the charged Higgs decay as follows:

$$p_T(b) \geq 20 \text{ GeV}, \quad p_T(\bar{b}) \geq 20 \text{ GeV}, \quad |\eta(b)| \leq 2.4, \quad |\eta(\bar{b})| \leq 2.4. \quad (5)$$

For the signal events, the decay  $h \rightarrow b\bar{b}$  with a branching fraction of 53% is taken from particle data group [49].

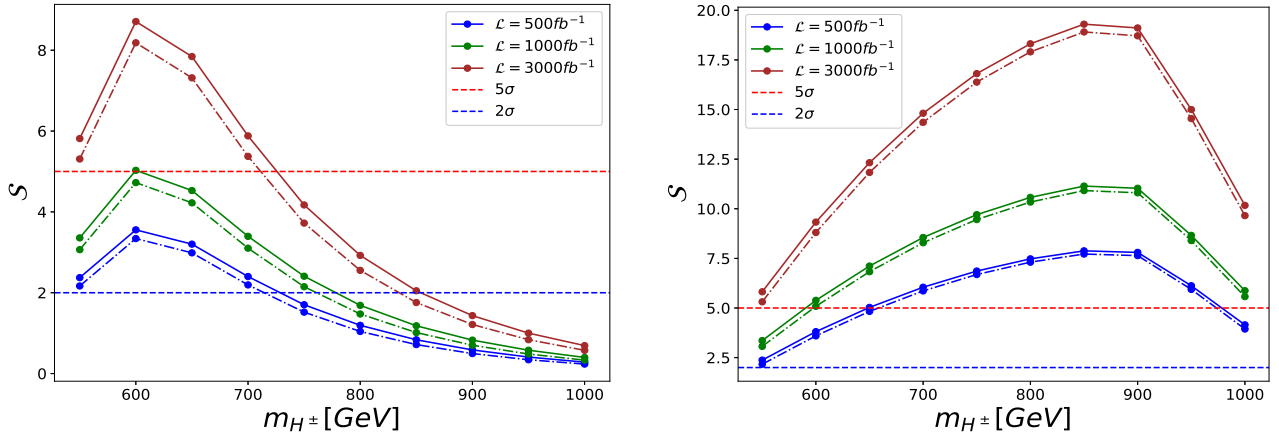


Figure 7: the significances are shown at  $\sqrt{s} = 3$  TeV, with BP1 displayed on the left panel and BP2 on the right.

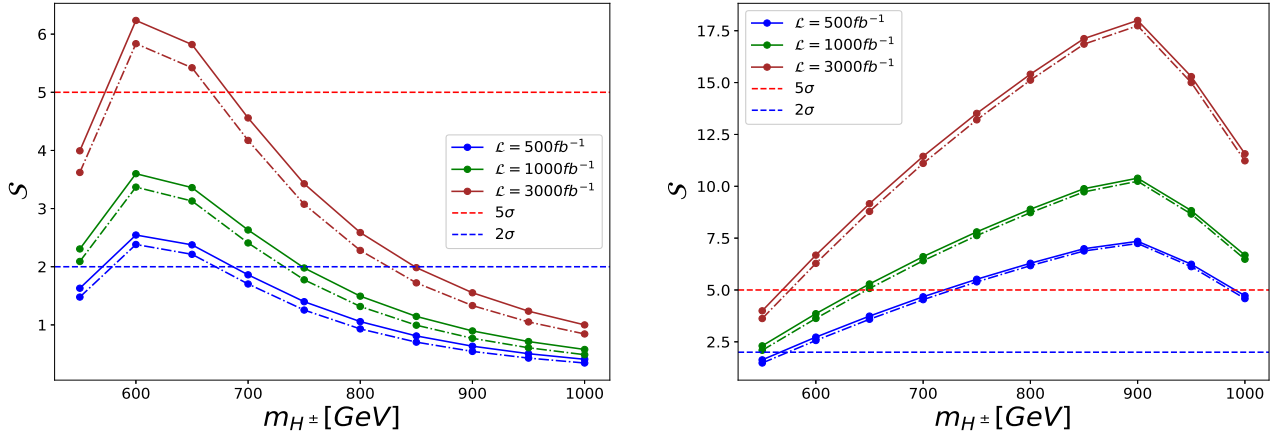


Figure 8: the significances are shown at  $\sqrt{s} = 5$  TeV, with BP1 displayed on the left panel and BP2 on the right.

In Fig. 7, the significances are computed at  $\sqrt{s} = 3$  TeV, with BP1 shown on the left panel and BP2 on the right. In all plots, the blue, red, and green curves correspond to the integrated luminosities  $\mathcal{L} = 0.5, 1$ , and  $3 \text{ ab}^{-1}$ , respectively. The dashed lines represent the case  $\varepsilon = 1$ , while the solid lines correspond to  $\varepsilon = 1.5$ . Our results indicate that the significances can reach 8 for BP1 and  $\sim 19$  for the second benchmark point BP2 at  $\mathcal{L} = 3 \text{ ab}^{-1}$ .

We next consider the significances at  $\sqrt{s} = 5$  TeV in which the results for BP1 displaying on the left panel and for BP2 on the right of Fig. 8. In all plots, we use the same notation for the lines as previous Figures. It is observed that the significances reach 6 for BP1 and  $\sim 17$  for the second benchmark point BP2 at  $\mathcal{L} = 3 \text{ ab}^{-1}$ .

Finally, the significances are also evaluated at  $\sqrt{s} = 10$  TeV, with BP1 displayed on the left panel and BP2 on the right of the scatter plots 9. We obtain nearly the same significances which are around 6 for BP1 and around 17 for BP2 at  $\mathcal{L} = 10 \text{ ab}^{-1}$ .

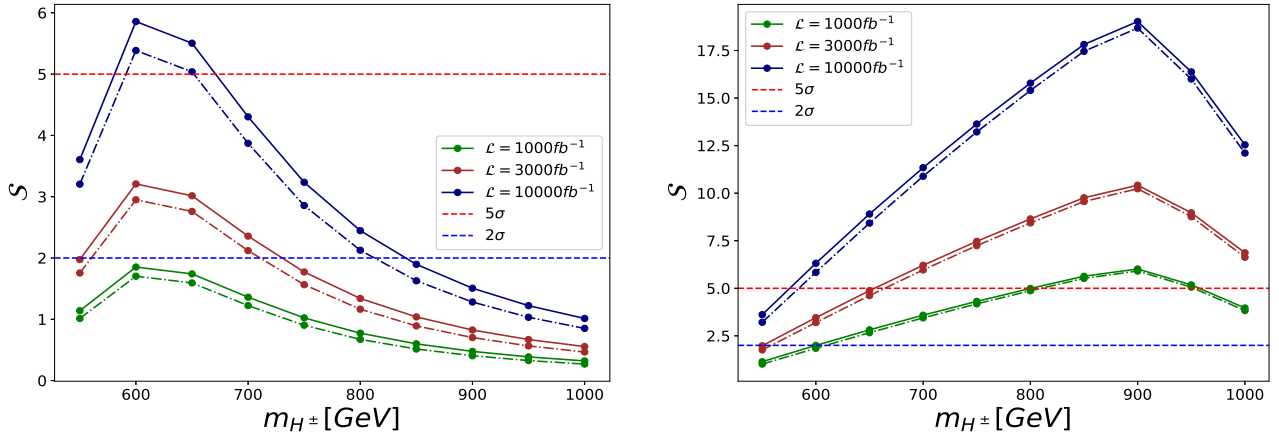


Figure 9: the significances are shown at  $\sqrt{s} = 10$  TeV, with BP1 displayed on the left panel and BP2 on the right.

We emphasize that the significances are enhanced when  $m_{H^\pm} \sim m_H + m_h$ , as their signal cross sections also peak under this condition, as shown in Fig. 5. The peaks arise from the exchange of  $s$ -channel CP-even Higgses. Furthermore, if we consider the top quark decay to leptons and bottom quark like  $t \rightarrow \ell \bar{\nu}_\ell b$  with  $\ell = e, \mu, \tau$ , using its branching ratio of 0.337 as given in particle data group [49], we then have six bottom quarks in the final state and apply a  $b$ -tagging factor of  $\sim 0.8$ . With these values, the significances for the process are scaled down by a factor of  $\sqrt{0.337^2 \times 0.8^6} \sim 0.2$ . Subsequently, the significance for BP1 will be reduced around  $\sim 1.3$ , and for BP2 around  $\sim 3.6$ . In comparison with our previous work [38], these production processes in the Type-Y THDM are difficult to probe at multi-TeV muon colliders. Last but not least, detector simulations for events with two leptons and six bottom quarks should be considered, as they also affect the significances. These topics will be addressed in our future work.

#### 4. Conclusions

We report the first results for charged Higgs boson production associated with CP-even Higgs states at future multi-TeV muon colliders in the scenario of the Type-Y Two-Higgs-Doublet Model. The updated parameter regions for the Type-Y Two-Higgs-Doublet Model are obtained in this study. Based on the allowed parameter space, we compute all two-body decay channels of the charged Higgs boson with the help of H-COUP. The production processes  $\mu^- \mu^+ \rightarrow H^\pm H^\mp h/H \rightarrow t\bar{t}b\bar{b} h/H$  are also scanned over the parameter space of the Type-Y THDM. In the allowed regions of the THDM, one can observe  $\mathcal{O}(100)$  events, providing an opportunity to probe the charged Higgs bosons produced in association with the heavy CP-even Higgs at future multi-TeV muon colliders. We compute the signal significances for  $\mu^- \mu^+ \rightarrow H^\pm H^\mp h \rightarrow t\bar{t}b\bar{b}h \rightarrow t\bar{t}b\bar{b}b\bar{b}$  with respect to the Standard Model background. We also estimate the significances by including the top quark decay into leptons and bottom quarks and considering  $b$ -tagging. Thanks to the high integrated luminosities anticipated at future multi-TeV muon colliders, we find that the signal significances can exceed  $\sim 3\sigma$  at several benchmark points selected in the viable parameter space of the Type-Y THDM.

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Process  $\mu^+\mu^- \rightarrow H^\pm H^\mp \phi_k$  for  $\phi_k = h, H$

In this appendix, we show a compact form for the tree-level amplitude of the process  $\mu^+\mu^- \rightarrow H^\pm H^\mp \phi_k$  for  $\phi_k = h, H$  in the context of the THDM. The results are computed with considering all Feynman diagrams with 't Hooft-Feynman gauge. Amplitude for processes  $\mu^+\mu^- \rightarrow H^\pm H^\mp h$  are given:

$$i\mathcal{M} = \sum_{k=1}^8 F_k \langle v | P_k | u \rangle, \quad (6)$$

where the tree-level form factors  $F_k$  are expressed in terms of the following kinematic variables such as  $s = s_{12} = (k_1 + k_2)^2$ ,  $s_{34} = (k_3 + k_4)^2$ ,  $t = t_{13} = (k_1 - k_3)^2$ ,  $t_{14} = (k_1 - k_4)^2$ ,  $t_{24} = (k_2 - k_4)^2$ , and  $u = u_{23} = (k_2 - k_3)^2$ . The explicit expressions for the form factors  $F_k$  are given by:

$$\begin{aligned}
F_1 = & g_{hff}^R \frac{g_{H^\pm H^\mp hh}}{s - m_h^2} + g_{Hff}^R \frac{g_{H^\pm H^\mp Hh}}{s - m_H^2} + \frac{g_{hH^\pm H^\mp}}{s_{34} - m_{H^\pm}^2} \left[ g_{hff}^R \frac{g_{hH^\pm H^\mp}}{s - m_h^2} + g_{Hff}^R \frac{g_{HH^\pm H^\mp}}{s - m_H^2} \right] \\
& + \frac{m_\mu g_{hH^\pm H^\mp} g_{H^\pm H^\mp V} \cdot g_{Vff}^{R-L}}{(s - m_Z^2)(s_{34} - m_{H^\pm}^2)} + \frac{\pi\alpha c_{\beta-\alpha}}{s_W^2 (s_{34} - m_W^2)} \left[ C_8 \cdot g_{Hff}^R \frac{s_{\beta-\alpha}}{s - m_H^2} - C_5 \cdot g_{hff}^R \frac{c_{\beta-\alpha}}{s - m_h^2} \right] \\
& + \frac{g_{hH^\pm H^\mp}}{3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24}} \left[ g_{hff}^R \frac{g_{hhh}}{s - m_h^2} + g_{Hff}^R \frac{g_{hHh}}{s - m_H^2} \right] \\
& + \frac{m_\mu g_{hH^\pm H^\mp} \cdot g_{hff}^R \cdot g_{hff}^{L+R}}{(t - m_\mu^2)(3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{4\pi\alpha \cdot g_{hff}^R \cdot (s_{34} + t - m_h^2 - m_{H^\pm}^2 - m_\mu^2)}{(t - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{g_{HH^\pm H^\mp}}{m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24}} \left[ g_{hff}^R \frac{g_{Hhh}}{s - m_h^2} + g_{Hff}^R \frac{g_{HHh}}{s - m_H^2} \right] \\
& + \frac{m_\mu g_{HH^\pm H^\mp} \cdot g_{Hff}^R \cdot g_{hff}^{L+R}}{(t - m_\mu^2)(m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{g_{H^\pm H^\mp V}}{m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24}} \left[ C_{10} \frac{m_\mu}{s - m_Z^2} + C_2 \frac{1}{t - m_\mu^2} \right] \\
& + \frac{g_{hH^\pm H^\mp}}{2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u} \left[ g_{hff}^R \frac{g_{hH^\pm H^\mp}}{s - m_h^2} + g_{Hff}^R \frac{g_{HH^\pm H^\mp}}{s - m_H^2} \right] \\
& + \frac{m_\mu g_{hH^\pm H^\mp} \cdot g_{H^\pm H^\mp V} \cdot g_{Vff}^{R-L}}{(s - m_Z^2)(2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u)} \\
& + \frac{\pi\alpha c_{\beta-\alpha}}{s_W^2 [2(m_h^2 + m_{H^\pm}^2 + m_\mu^2) - m_W^2 - s_{34} - t - u]} \times \\
& \quad \times \left[ \sqrt{2} g_{H^\pm ff} + C_4 \frac{c_{\beta-\alpha} g_{hff}^R}{s - m_h^2} - C_{11} \frac{s_{\beta-\alpha} g_{Hff}^R}{s - m_H^2} \right] \\
& + \frac{\sqrt{2}\pi\alpha c_{\beta-\alpha} (s_{34} - m_h^2 - m_{H^\pm}^2) g_{H^\pm ff}}{s_W^2 t_{14} [2(m_h^2 + m_{H^\pm}^2 + m_\mu^2) - m_W^2 - s_{34} - t - u]} \\
& - \frac{(s_{34} - m_h^2 - m_{H^\pm}^2) g_{hff}^R g_{H^\pm ff}^2}{2t_{14} (u - m_\mu^2)} + \frac{m_\mu g_{hH^\pm H^\mp} g_{hff}^R \cdot g_{hff}^{L+R}}{(u - m_\mu^2)(3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})}
\end{aligned}$$

$$\begin{aligned}
& -\frac{4\pi\alpha (s_{34} - m_h^2 - m_{H^\pm}^2 - m_\mu^2) g_{hff}^R}{(u - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{m_\mu g_{HH^\pm H^\mp} \cdot g_{Hff}^R \cdot g_{hff}^{L+R}}{(u - m_\mu^2) (m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& - C_{14} \frac{g_{H^\pm H^\mp V}}{(u - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& - \frac{4\pi\alpha u g_{hff}^R}{(u - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})}, \tag{7} \\
F_2 = & g_{hff}^L \frac{g_{H^\pm H^\mp hh}}{s - m_h^2} + g_{Hff}^L \frac{g_{H^\pm H^\mp Hh}}{s - m_H^2} + \frac{g_{hH^\pm H^\mp}}{s_{34} - m_{H^\pm}^2} \left[ g_{hff}^L \frac{g_{hH^\pm H^\mp}}{s - m_h^2} + g_{Hff}^L \frac{g_{HH^\pm H^\mp}}{s - m_H^2} \right] \\
& + \frac{m_\mu g_{hH^\pm H^\mp} g_{H^\pm H^\mp V} \cdot g_{Vff}^{L-R}}{(s - m_Z^2) (s_{34} - m_{H^\pm}^2)} + \frac{\pi\alpha c_{\beta-\alpha}}{s_W^2 (s_{34} - m_W^2)} \left[ C_8 \cdot g_{Hff}^L \frac{s_{\beta-\alpha}}{s - m_H^2} - C_5 \cdot g_{hff}^L \frac{c_{\beta-\alpha}}{s - m_h^2} \right] \\
& + \frac{g_{hH^\pm H^\mp}}{3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24}} \left[ g_{hff}^L \frac{g_{hhh}}{s - m_h^2} + g_{Hff}^L \frac{g_{hhH}}{s - m_H^2} \right] \\
& + \frac{m_\mu g_{hH^\pm H^\mp} g_{hff}^R \cdot g_{hff}^{L+R}}{(t - m_\mu^2) (3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{4\pi\alpha (s_{34} + t - m_h^2 - m_{H^\pm}^2 - m_\mu^2) g_{hff}^L}{(t - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{g_{HH^\pm H^\mp}}{m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24}} \left[ g_{hff}^L \frac{g_{hhH}}{s - m_h^2} + g_{Hff}^L \frac{g_{HHH}}{s - m_H^2} \right] \\
& + \frac{m_\mu g_{HH^\pm H^\mp} g_{Hff}^L \cdot g_{hff}^{L+R}}{(t - m_\mu^2) (m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{g_{H^\pm H^\mp V}}{m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24}} \left[ C_3 \frac{1}{t - m_\mu^2} - C_{10} \frac{m_\mu}{s - m_Z^2} \right] \\
& + \frac{g_{hH^\pm H^\mp}}{2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u} \left[ g_{hff}^L \frac{g_{hH^\pm H^\mp}}{s - m_h^2} + g_{Hff}^L \frac{g_{HH^\pm H^\mp}}{s - m_H^2} \right] \\
& + \frac{m_\mu g_{hH^\pm H^\mp} g_{H^\pm H^\mp V} \cdot g_{Vff}^{L-R}}{(s - m_Z^2) (2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u)} \\
& + \frac{m_\mu g_{hH^\pm H^\mp} g_{H^\pm ff}^2}{t_{14} (2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u)} \\
& + \frac{\pi\alpha c_{\beta-\alpha}}{s_W^2 [2(m_h^2 + m_{H^\pm}^2 + m_\mu^2) - m_W^2 - s_{34} - t - u]} \times \\
& \quad \times \left[ C_4 \frac{c_{\beta-\alpha} g_{hff}^L}{s - m_h^2} - C_{11} \frac{s_{\beta-\alpha} g_{Hff}^L}{s - m_H^2} - C_6 \frac{\sqrt{2} m_\mu^2 g_{H^\pm ff}}{t_{14}} \right] \\
& + \frac{m_\mu g_{hH^\pm H^\mp} g_{H^\pm ff}^2}{(s_{34} - m_{H^\pm}^2) (m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)} \\
& - C_{12} \frac{\sqrt{2}\alpha\pi c_{\beta-\alpha} g_{H^\pm ff}}{s_W^2 (s_{34} - m_{H^\pm}^2) (m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{g_{H^\pm ff}^2}{(t - m_\mu^2)(m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)} \left[ m_\mu^2 g_{hff}^L + m_\mu^2 g_{hff}^R + C_{13} \frac{g_{hff}^L}{2} \right] \\
& + \frac{m_\mu^2 g_{H^\pm ff}^2 \cdot g_{hff}^{L+R}}{t_{14}(u - m_\mu^2)} + \frac{m_\mu g_{hH^\pm H^\mp} g_{hff}^L \cdot g_{hff}^{L+R}}{(u - m_\mu^2)(3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& - \frac{4\pi\alpha(s_{34} - m_h^2 - m_{H^\pm}^2 - m_\mu^2) g_{hff}^L}{(u - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{m_\mu g_{HH^\pm H^\mp} g_{hff}^L \cdot g_{hff}^{L+R}}{(u - m_\mu^2)(m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& - C_7 \frac{g_{H^\pm H^\mp V}}{(u - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{4\pi\alpha g_{hff}^L u}{(u - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})}, \tag{8} \\
F_3 = & - \frac{2g_{hH^\pm H^\mp} g_{Vff}^R g_{H^\pm H^\mp V}}{(s - m_Z^2)(s_{34} - m_{H^\pm}^2)} + \frac{g_{hH^\pm H^\mp} g_{hff}^L g_{hff}^R}{(t - m_\mu^2)(3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{8\pi\alpha g_{hH^\pm H^\mp}}{s(s_{34} - m_{H^\pm}^2)} + \frac{g_{HH^\pm H^\mp} g_{hff}^R g_{hff}^L}{(t - m_\mu^2)(m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{g_{hVV} g_{Vff}^R g_{H^\pm H^\mp V}}{(s - m_Z^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{m_\mu g_{hff}^R g_{H^\pm H^\mp V} \cdot g_{Vff}^{L-R}}{(t - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{g_{hH^\pm H^\mp} g_{H^\pm ff}^2}{(s_{34} - m_{H^\pm}^2)(m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)} \\
& - C_1 \frac{\sqrt{2}\pi\alpha m_\mu c_{\beta-\alpha} g_{H^\pm ff}}{s_W^2(s_{34} - m_W^2)(m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)} \\
& + \frac{m_\mu g_{hff}^R g_{H^\pm ff}^2}{(t - m_\mu^2)(m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)} \\
& - \frac{g_{hH^\pm H^\mp} g_{hff}^L g_{hff}^R}{(u - m_\mu^2)(3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& - \frac{g_{HH^\pm H^\mp} g_{hff}^L g_{hff}^R}{(u - m_\mu^2)(m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{m_\mu g_{hff}^L g_{H^\pm H^\mp V} \cdot g_{Vff}^{L-R}}{(u - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})}, \tag{9} \\
F_4 = & - \frac{2g_{hH^\pm H^\mp} g_{Vff}^L g_{H^\pm H^\mp V}}{(s - m_Z^2)(s_{34} - m_{H^\pm}^2)} + \frac{8\pi\alpha m_\mu \cdot g_{hff}^{L+R}}{(t - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{8\pi\alpha g_{hH^\pm H^\mp}}{s(s_{34} - m_{H^\pm}^2)} - \frac{2g_{hVV} g_{Vff}^L g_{H^\pm H^\mp V}}{(s - m_Z^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& - \frac{2m_\mu g_{Vff}^L g_{H^\pm H^\mp V} \cdot g_{hff}^{L+R}}{(t - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})}
\end{aligned}$$

$$\begin{aligned}
& + \frac{g_{hH^\pm H^\mp}}{(2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u)} \left[ \frac{8\pi\alpha}{s} - \frac{2g_{Vff}^L g_{H^\pm H^\mp V}}{(s - m_Z^2)} \right] \\
& + \frac{8\pi\alpha m_\mu \cdot g_{hff}^{L+R}}{(u - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& - \frac{2m_\mu g_{Vff}^L g_{H^\pm H^\mp V} \cdot g_{hff}^{L+R}}{(u - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})}, \tag{10}
\end{aligned}$$

$$\begin{aligned}
F_5 = & - \frac{2g_{hH^\pm H^\mp} g_{Vff}^L g_{H^\pm H^\mp V}}{(s - m_Z^2) (s_{34} - m_{H^\pm}^2)} + \frac{g_{hH^\pm H^\mp} g_{hff}^L g_{hff}^R}{(t - m_\mu^2) (3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{8\pi\alpha g_{hH^\pm H^\mp}}{s (s_{34} - m_{H^\pm}^2)} + \frac{g_{HH^\pm H^\mp} g_{hff}^L g_{Hff}^R}{(t - m_\mu^2) (m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{g_{hVV} g_{Vff}^L g_{H^\pm H^\mp V}}{(s - m_Z^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{m_\mu g_{hff}^R g_{H^\pm H^\mp V} \cdot g_{Vff}^{R-L}}{(t - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{2\sqrt{2}\pi\alpha m_\mu c_{\beta-\alpha} g_{H^\pm ff}}{s_W^2 t_{14} [2(m_h^2 + m_{H^\pm}^2 + m_\mu^2) - m_W^2 - s_{34} - t - u]} - \frac{m_\mu g_{hff}^R g_{H^\pm ff}^2}{t_{14} (u - m_\mu^2)} \\
& - \frac{g_{hH^\pm H^\mp} g_{hff}^L g_{hff}^R}{(u - m_\mu^2) (3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& - \frac{g_{HH^\pm H^\mp} g_{hff}^R g_{Hff}^L}{(u - m_\mu^2) (m_h^2 - m_H^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{m_\mu g_{hff}^R g_{H^\pm H^\mp V} \cdot g_{Vff}^{R-L}}{(u - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})}, \tag{11}
\end{aligned}$$

$$\begin{aligned}
F_6 = & - \frac{2g_{hH^\pm H^\mp} g_{Vff}^R g_{H^\pm H^\mp V}}{(s - m_Z^2) (s_{34} - m_{H^\pm}^2)} + \frac{8\pi\alpha m_\mu \cdot g_{hff}^{L+R}}{(t - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{8\pi\alpha g_{hH^\pm H^\mp}}{s (s_{34} - m_{H^\pm}^2)} - \frac{2g_{hVV} g_{Vff}^R g_{H^\pm H^\mp V}}{(s - m_Z^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& - \frac{2m_\mu g_{Vff}^R g_{H^\pm H^\mp V} \cdot g_{hff}^{L+R}}{(t - m_\mu^2) (m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})} \\
& + \frac{8\pi\alpha g_{hH^\pm H^\mp}}{s (2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u)} + \frac{g_{hH^\pm H^\mp} g_{H^\pm ff}^2}{t_{14} (2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u)} \\
& - \frac{2g_{hH^\pm H^\mp} g_{Vff}^R g_{H^\pm H^\mp V}}{(s - m_Z^2) (2m_h^2 + m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t - u)} \\
& - C_6 \frac{\sqrt{2}\pi\alpha m_\mu c_{\beta-\alpha} g_{H^\pm ff}}{s_W^2 t_{14} [2(m_h^2 + m_{H^\pm}^2 + m_\mu^2) - m_W^2 - s_{34} - t - u]} \\
& + \frac{g_{hH^\pm H^\mp} g_{H^\pm ff}^2}{(s_{34} - m_{H^\pm}^2) (m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)} \\
& - C_6 \frac{\sqrt{2}\pi\alpha m_\mu c_{\beta-\alpha} g_{H^\pm ff}}{s_W^2 (s_{34} - m_W^2) (m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{m_\mu g_{H^\pm ff}^2 \cdot g_{hff}^{L+R}}{(t - m_\mu^2)(m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)} + \frac{m_\mu g_{H^\pm ff}^2}{t_{14}(u - m_\mu^2)} \cdot g_{hff}^{L+R} \\
& + \frac{8\pi\alpha m_\mu \cdot g_{hff}^{L+R}}{(u - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24})} \\
& - \frac{2m_\mu g_{Vff}^R g_{H^\pm H^\mp V} \cdot g_{hff}^{L+R}}{(u - m_\mu^2)(m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24})}, \tag{12}
\end{aligned}$$

$$\begin{aligned}
F_7 = & \frac{8\pi\alpha g_{hff}^L}{m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24}} \left[ \frac{1}{t - m_\mu^2} + \frac{1}{u - m_\mu^2} \right] \\
& - \frac{2g_{hff}^L g_{H^\pm H^\mp V}}{m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24}} \left[ \frac{g_{Vff}^R}{t - m_\mu^2} + \frac{g_{Vff}^L}{u - m_\mu^2} \right] \\
& - \frac{2\sqrt{2}\pi\alpha c_{\beta-\alpha} g_{H^\pm ff}}{s_W^2(s_{34} - m_W^2)(m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)} \\
& + \frac{g_{hff}^L g_{H^\pm ff}^2}{(t - m_\mu^2)(m_h^2 + 2m_{H^\pm}^2 + 3m_\mu^2 - s - t_{24} - u)}, \tag{13}
\end{aligned}$$

$$\begin{aligned}
F_8 = & \frac{8\pi\alpha g_{hff}^R}{m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - s_{34} - t_{14} - t_{24}} \left[ \frac{1}{t - m_\mu^2} + \frac{1}{u - m_\mu^2} \right] \\
& - \frac{2g_{hff}^R g_{H^\pm H^\mp V}}{m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 - m_Z^2 - s_{34} - t_{14} - t_{24}} \left[ \frac{g_{Vff}^L}{t - m_\mu^2} + \frac{g_{Vff}^R}{u - m_\mu^2} \right] \\
& - \frac{2\sqrt{2}\pi\alpha c_{\beta-\alpha} g_{H^\pm ff}}{s_W^2 t_{14} [2(m_h^2 + m_{H^\pm}^2 + m_\mu^2) - m_W^2 - s_{34} - t - u]} + \frac{g_{hff}^R g_{H^\pm ff}^2}{t_{14}(u - m_\mu^2)}, \tag{14}
\end{aligned}$$

The operator  $P_k$  are chosen as folllow:  $P_1 = P_R$ ,  $P_2 = P_L$ ,  $P_3 = P_R \not{k}_3$ ,  $P_4 = P_L \not{k}_4$ ,  $P_5 = P_L \not{k}_3$ ,  $P_6 = P_R \not{k}_4$ ,  $P_7 = P_L(\not{k}_3 \not{k}_4 - \not{k}_4 \not{k}_3)1$ , and  $P_8 = P_R(\not{k}_3 \not{k}_4 - \not{k}_4 \not{k}_3)$ . In addition, the coefficient  $C_i$  are of the form:

$$C_1 = \frac{m_h^2 - m_{H^\pm}^2}{m_W^2} - 1, \tag{15}$$

$$C_2 = m_\mu^2 g_{hff}^{L+R} g_{Vff}^{R-L} + g_{hff}^R g_{Vff}^L (m_h^2 + m_{H^\pm}^2 + m_\mu^2 - s_{34} - t), \tag{16}$$

$$C_3 = m_\mu^2 g_{hff}^{L+R} g_{Vff}^{L-R} + g_{hff}^L g_{Vff}^R (m_h^2 + m_{H^\pm}^2 + m_\mu^2 - s_{34} - t), \tag{17}$$

$$C_4 = m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 + \frac{(m_h^2 - m_{H^\pm}^2)^2}{m_W^2} - 2s_{34} - t_{14} - t_{24}, \tag{18}$$

$$C_5 = m_h^2 - m_{H^\pm}^2 - \frac{(m_h^2 - m_{H^\pm}^2)^2}{m_W^2} - t + t_{14} + t_{24} - u, \tag{19}$$

$$C_6 = \frac{m_h^2 - m_{H^\pm}^2}{m_W^2} + 1, \tag{20}$$

$$C_7 = m_\mu^2 g_{hff}^{L+R} g_{Vff}^{R-L} + g_{hff}^L g_{Vff}^L (m_h^2 + m_{H^\pm}^2 + m_\mu^2 - s_{34} - u), \tag{21}$$

$$C_8 = m_h^2 - m_{H^\pm}^2 - \frac{(m_h^2 - m_{H^\pm}^2)(m_H^2 - m_{H^\pm}^2)}{m_W^2} - t + t_{14} + t_{24} - u, \tag{22}$$

$$C_9 = -2(m_h^2 + m_{H^\pm}^2 + m_\mu^2) + 2s_{34} + t + u, \tag{23}$$

$$C_{10} = g_{hVV} g_{Vff}^{R-L} + C_9 \frac{\pi\alpha}{c_W s_W^3}, \tag{24}$$

$$C_{11} = m_h^2 + 3m_{H^\pm}^2 + 2m_\mu^2 + \frac{(m_h^2 - m_{H^\pm}^2)(m_H^2 - m_{H^\pm}^2)}{m_W^2} - 2s_{34} - t_{14} - t_{24}, \quad (25)$$

$$C_{12} = \frac{m_\mu(m_h^2 - m_{H^\pm}^2)}{m_W^2} + t - t_{14}, \quad (26)$$

$$C_{13} = m_h^2 + m_{H^\pm}^2 - s_{34} - 2(m_h^2 + m_{H^\pm}^2 + m_\mu^2 - s_{34} - t), \quad (27)$$

$$C_{14} = m_\mu^2 g_{hff}^{L+R} g_{Vff}^{L-R} + g_{hff}^R g_{Vff}^R (m_h^2 + m_{H^\pm}^2 + m_\mu^2 - s_{34} - u). \quad (28)$$

Where  $g_{hff}^{L+R} = (g_{hff}^L + g_{hff}^R)$ ,  $g_{Vff}^{L-R} = (g_{Vff}^L - g_{Vff}^R)$ . The amplitude for the process  $\mu^+\mu^- \rightarrow H^\pm H^\mp H$  is obtained by substituting all general couplings associated with the external  $h$  field above with the corresponding couplings of  $H$  to the relevant particles.

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