A Dynamic Competitive Equilibrium Model of Irreversible Capacity Investment with Stochastic Demand and Heterogeneous Producers

Constantinos Kardaras, Alexandros Pavlis and Mihail Zervos

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Abstract

We formulate a continuous-time competitive equilibrium model of irreversible capacity investment in which a continuum of heterogeneous producers supplies a single non-durable good subject to exogenous stochastic demand. Each producer optimally adjusts both output and capacity over time in response to endogenous price signals, while investment decisions are irreversible. Market clearing holds continuously, with prices evolving endogenously to balance aggregate supply and demand through a constant-elasticity demand function driven by a stochastic base component. The model admits a mean-field interpretation, as each producer's decisions both influence and are influenced by the aggregate behaviour of all others. We show that the equilibrium price process can be expressed as a nonlinear functional of the exogenous base demand, leading to a three-dimensional singular stochastic control problem for each producer. We derive an explicit solution to the associated Hamilton-Jacobi-Bellman equation, including a closed-form characterisation of the free-boundary surface separating investment and waiting regions.

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1 Introduction

We consider a continuum of heterogeneous producers of a single non-durable good, subject to exogenous stochastic demand in continuous time. Each producer determines, over time, both the level of output and the amount of investment aimed at improving production efficiency. Producers must commit to their choices, meaning that investments are irreversible. This assumption is particularly plausible in industries with high upfront costs, where the resale of capital may entail substantial losses.

The price of the good is determined in equilibrium as the aggregate outcome of individual producers' actions. In particular, each producer acts as a price taker, meaning that they have no direct influence on the price process. In this competitive setting, the decision problem faced by each producer gives rise to an infinite-horizon singular stochastic control problem.

Continuous time irreversible capacity expansion models featuring a single producer of a good with exogenous prices or demand have attracted considerable attention in the literature. Early references in economics include the pioneering work of Manne [23] (see Van Mieghem [28] for a survey). In the mathematical literature, models of stochastic control with irreversible investment have been studied by Davis, Dempster, Sethi and Vermes [11], Arntzen [4], Øksendal [25], Wang [29], Chiarolla and Haussmann [9], Alvarez [3], Løkka and Zervos [21], Chiarolla and Ferrari [8], Ferrari [16], De Angelis, Federico and Ferrari [12], Federico, Rosestolato and Tacconi [15], Koch and Vargiolu [20], Dammann and Ferrari [10], Han and Yi [18], among others, listed here in rough alphabetical order (see also references therein). Furthermore, capacity expansion models allowing costly reversibility were introduced by Abel and Eberly [1] and were subsequently studied by Guo and Pham [17], Merhi and Zervos [24], Løkka and Zervos [22], De Angelis and Ferrari [13], Federico and Pham [14], and Han, Yi and Zhang [19].

Oligopolistic games of irreversible investment in continuous time have been studied by Back and Paulsen [5] and by Steg [26, 27]. The models in these works consider settings in which identical producers are exposed to an exogenous economic shock modelled as a one-dimensional Itô diffusion. Each individual producer solves a singular stochastic control problem in which the running payoff depends on both the exogenous shock and the capital stocks of all market participants.

Recently, several mean-field investment games have been analysed. These include the finite-horizon irreversible capacity expansion model studied by Campi, De Angelis, Ghio and Livieri [6], in which the drift of the price of the good produced by each firm depends on the average installed capacity across all producers. Cao and Guo [7] study an infinite-horizon partially reversible capacity expansion model in which firms' production levels are modelled as independent copies of a controlled geometric Brownian motion, while the running payoff of each producer depends on the distribution of the long-run production level across all producers. In a similar vein, Aïd, Basei and Ferrari [2] investigate an infinite-horizon irreversible capacity expansion model in which firms' production levels are modelled as independent copies of a controlled geometric Brownian motion with regime switching.

To the best of our knowledge, the model we study is the first in the capacity investment literature to exhibit the key features of a classical dynamic market equilibrium problem. The demand for the good follows a constant-elasticity demand function, where the base demand evolves as a geometric Brownian motion and the prevailing price determines the realised demand level. In response to price signals, producers dynamically adjust both their production capacities and output rates over time. Moreover, market clearing holds at all times: prices evolve endogenously to maintain a continuous balance between aggregate supply and demand.

Alternatively, the model can be interpreted as a mean-field game, in which each producer's decisions both influence and are influenced by the aggregate behaviour of all other producers. Unlike the capacity investment game models discussed above, producers are not statistically identical. In particular, heterogeneity arises from differences in initial production capacities, individual discount rates, and unit costs of capacity expansion as well as production costs.

We determine a Markovian market equilibrium by first solving a nonlinear functional equation that expresses the market-clearing price process of the good as a function of the exogenous base demand process. This solution allows us to represent each producer's objective in terms of a suitable geometric Brownian motion and its running maximum. Building on this result, we reformulate each producer's optimisation problem as a classical three-dimensional singular stochastic control problem.

We solve the singular stochastic control problem that characterises each producer's optimal decisions in equilibrium by first deriving a suitable explicit solution to the associated Hamilton-Jacobi-Bellman (HJB) equation. In particular, we obtain a closed-form expression for the free-boundary surface that separates the so-called "investment" and "wait" regions (see the surface $\mathcal{S}_i^1$, depicted by the purple lines in Figure 1). To the best of our knowledge, this is the first three-dimensional stochastic control problem with a non-trivial explicit solution to have been studied in the literature.

2 Individual optimality and market clearing

Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$ be a filtered probability space supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W, and let $(I, \mathcal{I}, \iota)$ be a σ-finite measure space. We will use W to model the exogenous demand for a certain good in an economy and, ultimately, the good's price process in equilibrium. We use I to index the producers in the sector of the economy producing the given good. We assume that all of the producers are *small*, in the sense that their individual decisions do not impact prices. In other words, we assume that all of the producers are *price takers*. In such a *competitive* setting, we assume that the measure ι on $(I, \mathcal{I})$ is non-atomic. In particular, we assume that $\{i\} \in \mathcal{I}$ and $\iota\big(\{i\}\big) = 0$ for all $i \in I$.

At time $0-$, the individual producer $i \in I$ has installed capability $c_i > 0$. We use the term "capability" to refer to a producer's productive efficiency, which influences its production costs as described below.

We denote by P_t the price at which the good is traded in the economy at time $t \geq 0$. In the competitive setting that we consider, each individual producer takes this price as exogenously given and has no individual influence over it. Indeed, the price process P is the *aggregate outcome* of all producers' actions in equilibrium. We also denote by $C_i(t) > 0$ and $Q_i(t) \geq 0$ the capability and the rate of production that the individual producer $i \in I$ maintains at time $t > 0$. We assume that, over a small time interval $[t, t{+}\Delta t[$, the production cost for producer $i \in I$ is proportional to $C_i^{-a_i}(t) Q_i^{1+b}(t)\, \Delta t$, for some constants $b > 0$ and $a_i \in \,]0, b[$. To simplify the notation in our analysis, we re-parametrise and we assume that the cost of production incurred by producer $i \in I$ during the time interval $[t, t{+}\Delta t[$ is given by

$$\frac{\beta}{\lambda_i^{1/\beta}(1+\beta)^{1+1/\beta}} \frac{Q_i^{1+1/\beta}(t)}{C_i^{\alpha_i/\beta}(t)} \Delta t,$$

for some constants $\beta > 0$, $\alpha_i \in \,]0,1[$ and $\lambda_i > 0$. In this setting, the production cost elasticity $b = 1/\beta$ is the *only* parameter that is common among all heterogeneous producers.

At any time $t \geq 0$, producer $i \in I$ determines both the quantity $Q_i(t)\,\Delta t$ of the good they produce and their expansion $\Delta C_i(t)$ over the time interval $[t, t+\Delta t[$. Over the same time interval, the producer receives the payoff

$$\left(Q_i(t) P_t - \frac{\beta}{\lambda_i^{1/\beta}(1+\beta)^{1+1/\beta}} \frac{Q_i^{1+1/\beta}(t)}{C_i^{\alpha_i/\beta}(t)} \right) \Delta t - \lambda_i k_i \, \Delta C_i(t).$$

Here, the first term is the revenue from sales after deducting production costs, while the constant $\lambda_i k_i > 0$ in the second term is the cost of unit expansion. This expression implies that the optimal for producer $i \in I$ production rate is

$$Q_i^\star(t) = \lambda_i (1+\beta) C_i^{\alpha_i}(t) P_t^\beta, \tag{2.1}$$

which gives rise to the payoff $C_i^{\alpha_i}(t) P_t^{1+\beta} \Delta t$. Accordingly, it is optimal for the individual producer $i \in I$ to adopt an expansion strategy C_i that maximises the performance index

$$\widetilde{J}_{c_i}^{(i)}(C_i \mid P) = \mathsf{E}\left[\int_0^\infty e^{-r_i t} C_i^{\alpha_i}(t) P_t^{1+\beta} \, \mathrm{d}t - k_i \int_{[0,\infty[} e^{-r_i t} \, \mathrm{d}C_i(t) \right], \tag{2.2}$$

where $r_i > 0$ is the producer's discounting rate.

We assume that, over a small time interval $[t, t+\Delta t[$, the demand for the good is given by $D_t P_t^{-\delta} \Delta t$, where $\delta > 0$ is the demand's price elasticity and the process D represents the exogenous base demand corresponding to a unit price. We model D by

$$D_t = D_0 \exp\bigl(\widetilde{\mu} t + \widetilde{\sigma} W_t\bigr),$$

for some constants $D_0 > 0$, $\widetilde{\mu} \in \mathbb{R}$ and $\widetilde{\sigma} > 0$. We determine the price process P by imposing the market-clearing condition, which requires that demand should equal supply. To this end, suppose that all producers have committed on their individual expansion strategies $\{C_i, \ i \in I\}$. In view of (2.1), the market-clearing requirement gives rise to the identity

$$P_t^{-\delta} D_t = (1+\beta) P_t^\beta \int_I \lambda_i C_i^{\alpha_i}(t) \, \iota(\mathrm{d}i),$$

which implies that

$$P_t = X_t^{1/(1+\beta)} \left((1+\beta) \int_I C_i^{\alpha_i}(t) \lambda_i \, \iota(\mathrm{d}i) \right)^{-\gamma/(1+\beta)}, \tag{2.3}$$

where $X = D^\gamma$, with $\gamma = (1+\beta)/(\delta+\beta) \in \,]0,1[$. In particular, X is a geometric Brownian motion, namely,

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t, \quad X_0 = D_0^\gamma, \tag{2.4}$$

for some constants $\mu \in \mathbb{R}$ and $\sigma > 0$.

To ensure that the market-clearing condition (2.3) is well-defined and associated with a real-valued equilibrium price process P, we make the following assumption.

Assumption 2.1 The functions $I \ni i \mapsto c_i$, $I \ni i \mapsto \alpha_i$, $I \ni i \mapsto \lambda_i$, $I \ni i \mapsto k_i$ and $I \ni i \mapsto r_i$ are all strictly positive and $\mathcal{I}$-measurable. Furthermore,

$$\kappa_0 := (1+\beta) \int_I \lambda_i c_i^{\alpha_i} \,\iota(\mathrm{d}i) < \infty. \tag{2.5}$$

3 Competitive production equilibrium

Before addressing the issue of equilibrium, we note that the general solution to the ODE

$$\mathscr{L}_i u(x) := \frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) - r_i u(x) = 0, \tag{3.1}$$

which is associated with the geometric Brownian motion X defined by (2.4) and the discounting rate $r_i > 0$ of producer i, is given by

$$u(x) = \Delta_1 x^{m_i} + \Delta_2 x^{n_i}, \tag{3.2}$$

for constants $\Delta_1 \in \mathbb{R}$ and $\Delta_2 \in \mathbb{R}$, where $m_i < 0 < n_i$ are the solutions to the quadratic equation

$$\frac{1}{2}\sigma^2 \ell^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\ell - r_i = 0,$$

namely,

$$m_i, n_i = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) \mp \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r_i}}{\sigma^2}.$$

For future reference, we note that the constants $m_i < 0 < n_i$ are such that

$$r_i > \mu \;\Leftrightarrow\; n_i > 1, \quad n_i + m_i - 1 = -\frac{2\mu}{\sigma^2}, \quad n_i m_i = -\frac{2r_i}{\sigma^2} \tag{3.3}$$

$$\text{and} \quad r_i - \mu = \frac{1}{2}\sigma^2 (n_i - 1)(-m_i + 1). \tag{3.4}$$

We now make a further assumption about the data of the optimisation problems (2.2) faced by the individual producers. To this end, we define

$$\overline{\alpha} = \iota\text{-}\operatorname{ess\,sup} \alpha = \inf\bigl\{a \in \,]0,1] \;\big|\; \iota(\alpha > a) = 0\bigr\}. \tag{3.5}$$

Assumption 3.1 The functions $I \ni i \mapsto \alpha_i$ and $I \ni i \mapsto n_i$ are such that[1]

$$\alpha_i \in \left]0, 1\right[\quad \text{and} \quad \varepsilon_i := \frac{(n_i - 1)(1 - \alpha_i)}{\alpha_i} - \frac{1 - \overline{\alpha}}{1 - \overline{\alpha} + \overline{\alpha}\gamma} > 0 \quad \text{for all } i \in I. \tag{3.7}$$

Furthermore,

$$\int_I \lambda_i \left(q \frac{\alpha_i (n_i - 1)}{(r_i - \mu) n_i k_i} \right)^{\alpha_i / (1 - \alpha_i)} \iota(\mathrm{d}i) < \infty \quad \text{for all constants } q > 0.$$

Definition 3.2 The family $\mathcal{A}$ of all admissible production expansion strategies consists of all collections of processes $\{C_i, \ i \in I\}$ such that
(i) the function $(\Omega \times \mathbb{R}_+) \times I \ni \big((\omega, t), i\big) \mapsto C_i(\omega, t)$ is $\mathcal{P} \otimes \mathcal{I}$-measurable, where $\mathcal{P}$ is the progressive σ-algebra on $\Omega \times \mathbb{R}_+$, and
(ii) given any $i \in I$, C_i is an $(\mathcal{F}_t)$-adapted increasing process with càdlàg sample paths such that $C_i(0-) = c_i$ and

$$\mathsf{E}\left[\int_0^\infty e^{-r_i t} C_i^{\alpha_i}(t) X_t \, \mathrm{d}t \right] < \infty. \tag{3.8}$$

Given $i \in I$, the family $\mathcal{A}_i$ of all admissible *individual* expansion strategies consists of all processes C_i that are as in (ii).

The admissibility conditions (3.8) ensure that, in equilibrium, the individual investors' optimisation problems are well-posed. Indeed, the market-clearing condition (2.3) and Assumption 2.1 imply that

$$P_t^{1+\beta} \leq \kappa_0^{-\gamma} X_t \quad \text{for all } t \geq 0.$$

Therefore, $\widetilde{J}_{c_i}^{(i)}(C_i \mid P)$ is well-defined for all $i \in I$.

Definition 3.3 A competitive production equilibrium consists of a continuous $(\mathcal{F}_t)$-adapted price process P and a family of expansion strategies $\{C_i^\star, \, i \in I\} \in \mathcal{A}$ such that the market-clearing condition (2.3) holds and

$$\widetilde{J}_{c_i}^{(i)}(C_i^\star \mid P) = \sup_{C_i \in \mathcal{A}_i} \widetilde{J}_{c_i}^{(i)}(C_i \mid P) \quad \text{for all } i \in I, \tag{3.9}$$

where $\widetilde{J}_{c_i}^{(i)}(C_i \mid P)$ is defined by (2.2).

[1]Note that (3.3) and the conditions in (3.7) imply that

$$n_i > 1 \ \Leftrightarrow \ r_i > \mu \text{ for all } i \in I. \tag{3.6}$$

In the extremal case when $\overline{\alpha} = 1$, the second condition in (3.7) simplifies to (3.6).

4 The main result: a Markovian competitive production equilibrium

In the context of a Markovian equilibrium, intuition suggests that it should be optimal for each producer to increase their capability only when the good's price or demand is at a maximal level, namely, at times $t \geq 0$ when $P_t = \overline{P}_t$ or $X_t = \overline{X}_t$, where

$$\overline{P}_t = \sup_{0 \leq s \leq t} P_s \quad \text{and} \quad \overline{X}_t = \sup_{0 \leq s \leq t} X_s. \tag{4.1}$$

In light of this observation, we restrict attention to expansion strategies of the form

$$C_i(t) = c_i \vee \Psi_i(\overline{P}_t) \quad \text{and} \quad C_i(t) = c_i \vee \Phi_i(\overline{X}_t), \tag{4.2}$$

where $\mathbb{\Psi} = \{\Psi_i,\ i \in I\}$ and $\mathbb{\Phi} = \{\Phi_i,\ i \in I\}$ are in the set $\mathcal{B}$.

Definition 4.1 We define $\mathcal{B}$ to be the set of all families of functions $\mathbb{X} = \{\mathcal{X}_i,\ i \in I\}$ such that[2]

(i) the function $\mathbb{R}_+ \times I \ni (z, i) \mapsto \mathbb{X}(z, i) = \mathcal{X}_i(z)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{I}$-measurable,

(ii) the integrability condition

$$\int_I \lambda_i \mathcal{X}_i^{\alpha_i}(z)\, \iota(\mathrm{d}i) < \infty \tag{4.3}$$

holds true for all $z > 0$, and

(iii) given any $i \in I$, $\mathcal{X}_i$ is the difference of two convex functions[3] such that

$$\lim_{z \downarrow 0} \mathcal{X}_i(z) = 0, \quad \lim_{z \uparrow \infty} \mathcal{X}_i(z) = \infty \quad \text{and} \quad (\mathcal{X}_i)'_-(z) > 0 \text{ for all } z > 0. \tag{4.4}$$

In the context of (4.1) and (4.2), expression (2.3) indicates that, under market-clearing conditions, the good's equilibrium price process P should be characterised by the expressions

$$P^{1+\beta} = X \mathscr{I}[\mathbb{\Psi}](\overline{P}) \quad \text{and} \quad P^{1+\beta} = X \mathscr{I}[\mathbb{\Phi}](\overline{X}),$$

where the operator $\mathscr{I}$ is defined by

$$\mathscr{I}[\mathbb{X}](z) = \left((1+\beta) \int_I \lambda_i \big(c_i^{\alpha_i} \vee \mathcal{X}_i^{\alpha_i}(z) \big)\, \iota(\mathrm{d}i) \right)^{-\gamma}, \quad z \geq 0, \tag{4.5}$$

for $\mathbb{X} = \{\mathcal{X}_i,\ i \in I\} \in \mathcal{B}$.

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[2] Condition (i) in this definition implies that the collections $\{\Psi_i(\overline{P}),\ i \in I\}$ and $\{\Phi_i(\overline{X}),\ i \in I\}$ both satisfy condition (i) in Definition 3.2.

[3] A function $f : \mathbb{R}_+ \to \mathbb{R}$ is the difference of two convex functions if and only if it is absolutely continuous with left-hand derivative that is a function of finite variation. Given such a function, we denote by f'_- its left-hand side first derivative.

Motivated by these observations, we construct a competitive equilibrium starting from a given family $\Psi = \{\Psi_i,\ i \in I\} \in \mathcal{B}$. With the possible exception of the producer indexed by $i \in I$, we assume that every other producer, say $j \in I \setminus \{i\}$, adopts the capacity expansion strategy $C_j(t) = c_j \vee \Psi_j(\overline{P}_t)$. In the competitive setting under consideration, this specification gives rise to the price process P that solves the equation

$$P^{1+\beta} = X\mathscr{I}[\Psi](\overline{P}) =: X\psi(\overline{P}). \tag{4.6}$$

Accordingly, the individual optimality requirement (3.9) reduces to determining an expansion process $C_i^\star$ that maximises the performance criterion

$$\mathsf{E}\left[\int_0^\infty e^{-r_i t} C_i^{\alpha_i}(t)\big(X_t\psi(\overline{P}_t)\big)^{1+\beta}\, \mathrm{d}t - k_i \int_{[0,\infty[} e^{-r_i t}\, \mathrm{d}C_i(t)\right] \tag{4.7}$$

over all admissible C_i.

In Theorem 5.2, we show that the solution to equation (4.6) can be expressed as

$$P^{1+\beta} = X\mathscr{I}[\Phi](\overline{X}) = X\varphi(\overline{X}),$$

where the family $\Phi = \{\Phi_i,\ i \in I\}$ and the function φ are such that

$$\Phi_i(\overline{x}) = \Psi_i\Big(\big(\overline{x}\varphi(\overline{x})\big)^{1/(1+\beta)}\Big).$$

Consequently, maximising (4.7) is equivalent to maximising the performance index

$$\mathsf{E}\left[\int_0^\infty e^{-r_i t} C_i^{\alpha_i}(t)\big(X_t\varphi(\overline{X}_t)\big)^{1+\beta}\, \mathrm{d}t - k_i \int_{[0,\infty[} e^{-r_i t}\, \mathrm{d}C_i(t)\right]. \tag{4.8}$$

We next turn to the individual optimisation problem faced by the singled-out producer $i \in I$. Specifically, we solve the problem of maximising the performance criterion (4.8) over all admissible expansion strategies, as detailed in Sections 7–9. The resulting solution shows that the producer's optimal expansion strategy is given by

$$C_i^\dagger(t) = c_i \vee \Psi_i^\star\big(\overline{P}_t\big) := c_i \vee \left(\frac{\alpha_i(n_i-1)}{(r_i-\mu)n_i k_i}\right)^{1/(1-\alpha_i)} \overline{P}_t^{(1+\beta)/(1-\alpha_i)}. \tag{4.9}$$

Notably, this result *does not* depend on the initial specification of $\psi = \mathscr{I}[\Psi]$ in (4.6). It follows that the solution to the equation

$$P^\star = \Big(X\mathscr{I}[\Psi^\star](\overline{P}^\star)\Big)^{1/(1+\beta)}, \tag{4.10}$$

where $\Psi^\star = \{\Psi_i^\star,\ i \in I\}$ are defined in (4.9), and the production expansion strategies

$$C_i^\star(t) = c_i \vee \Psi_i^\star\big(\overline{P}_t^\star\big) = c_i \vee \left(\frac{\alpha_i(n_i-1)}{(r_i-\mu)n_i k_i}\right)^{1/(1-\alpha_i)} \overline{P}_t^{\star(1+\beta)/(1-\alpha_i)} \tag{4.11}$$

comprise a competitive production equilibrium in *feedback* form. Furthermore, this production equilibrium admits the *closed* form representation given by the expressions

$$P_t^\star = \Big(X_t \, \mathscr{I}[\Phi^\star](\overline{X}_t) \Big)^{1/(1+\beta)} \tag{4.12}$$

$$\text{and} \quad C_i^\star(t) = c_i \vee \Phi_i^\star(\overline{X}_t) := c_i \vee \Psi_i^\star\big(\overline{X}_t \, \varphi(\overline{X}_t)\big)^{1/(1-\alpha_i)}, \tag{4.13}$$

where φ is defined as in (5.4).

We prove the next theorem, which is the main result of the paper, in Section 10.

Theorem 4.2 *Suppose that Assumptions 2.1 and 3.1 hold true. A competitive production equilibrium in the sense of Definition 3.3 is given by (4.10) and (4.11), or equivalently, by (4.12) and (4.13). Furthermore,* $\mathscr{I}[\Psi^\star]$ *and* $\mathscr{I}[\Phi^\star]$, *where* $\mathscr{I}$ *is defined by (4.5), are differences of two convex functions and the stochastic dynamics of the equilibrium price process* P *are given by*

$$\begin{aligned} \mathrm{d}\ln P_t^\star &= \frac{\mu - \frac{1}{2}\sigma^2}{1+\beta}\,\mathrm{d}t + \frac{1}{1+\beta}\,\frac{(\mathscr{I}[\Psi^\star])'_-(\overline{P}_t^\star)}{\mathscr{I}[\Psi^\star](\overline{P}_t)}\,\mathrm{d}\overline{P}_t^\star + \frac{\sigma}{1+\beta}\,\mathrm{d}W_t \\ &= \frac{\mu - \frac{1}{2}\sigma^2}{1+\beta}\,\mathrm{d}t + \frac{1}{1+\beta}\,\frac{(\mathscr{I}[\Phi^\star])'_-(\overline{X}_t)}{\mathscr{I}[\Phi^\star](\overline{X}_t)}\,\mathrm{d}\overline{X}_t + \frac{\sigma}{1+\beta}\,\mathrm{d}W_t, \end{aligned} \tag{4.14}$$

with initial condition

$$P_0^\star = \Big(X_0 \, \mathscr{I}[\Phi^\star](X_0) \Big)^{1/(1+\beta)}. \tag{4.15}$$

Remark 4.3 The optimal expansion strategies $\Psi^\star$ given by (4.11) are *independent* of the original choice of Ψ, in particular, of the specific function $\psi = \mathscr{I}[\Psi]$ used in (4.6). This observation indicates that the form of competitive equilibrium that we derive is rather *robust*. Indeed, even if the prevailing price process P differs from the equilibrium one $P^\star$, the individual producers will still find it optimal to adopt the expansion strategies given by (4.11), provided that P conforms to the structure given in (4.6).

5 The solution to equation (4.6)

Lemma 5.1 *Consider any collection* $\mathbb{X} = \{\mathcal{X}_i, \ i \in I\} \in \mathcal{B}$ *and the operator* $\mathscr{I}$ *that is defined by (4.5). The function* $\chi := \mathscr{I}[\mathbb{X}]$ *is a difference of two convex functions,*

$$\lim_{z \downarrow 0} \chi(z) = \kappa_0^{-\gamma}, \quad \lim_{z \uparrow \infty} \chi(z) = 0, \tag{5.1}$$

$$\chi'_-(z) = -\gamma(1+\beta)\chi^{(1+\gamma)/\gamma}(z) \int_I \lambda_i \alpha_i \mathbf{1}_{\{c_i < \mathcal{X}_i(z)\}} \mathcal{X}_i^{\alpha_i - 1}(z) (\mathcal{X}_i)'_-(z) \, \iota(\mathrm{d}i) \tag{5.2}$$

$$\textit{and} \quad \chi'_-(z) < 0 \textit{ for all } z > \sup\big\{ z \ge 0 \mid \ \chi(z) = \kappa_0^{-\gamma} \big\}, \tag{5.3}$$

where κ_0 *is defined by (2.5).*

Proof. Each of the functions $\mathcal{X}_i^{\alpha_i}$ and $c_i^{\alpha_i} \vee \mathcal{X}_i^{\alpha_i}$ is a difference of two convex functions. Using Fubini's theorem, we can therefore see that

$$\begin{aligned}\frac{1}{(1+\beta)}\big(\chi^{-1/\gamma}(z_2) - \chi^{-1/\gamma}(z_1)\big) &= \int_I \int_{z_1}^{z_2} \lambda_i \big(c_i^{\alpha_i} \vee \mathcal{X}_i^{\alpha_i}\big)'_-(y)\, \mathrm{d}y\, \iota(\mathrm{d}i) \\ &= \int_{z_1}^{z_2} \int_I \lambda_i \big(c_i^{\alpha_i} \vee \mathcal{X}_i^{\alpha_i}\big)'_-(y)\, \iota(\mathrm{d}i)\, \mathrm{d}y \quad \text{for all } z_1 < z_2,\end{aligned}$$

which implies that χ is absolutely continuous with left-hand side derivative given by

$$\chi'_-(z) = -\gamma(1+\beta)\chi^{(1+\gamma)/\gamma}(z) \int_I \lambda_i \big(c_i^{\alpha_i} \vee \mathcal{X}_i^{\alpha_i}\big)'_-(z)\, \iota(\mathrm{d}i),$$

namely, (5.2). The function χ'_- is of finite variation because this is true for all of the functions $\big(c_i^{\alpha_i} \vee \mathcal{X}_i^{\alpha_i}\big)'_-$. Furthermore, (5.1) and (5.3) follow immediately from (4.5), (5.2) and the fact that the functions $\mathcal{X}_i$ are strictly increasing. □

Theorem 5.2 *Consider a family $\Psi = \{\Psi_i,\ i \in I\} \in \mathcal{B}$ and let X be a strictly positive $(\mathcal{F}_t)$-progressive process. Also, define*

$$h(\overline{p}) = \frac{\overline{p}^{1+\beta}}{\mathscr{I}[\Psi](\overline{p})}, \quad \varphi(\overline{x}) = \big(\mathscr{I}[\Psi] \circ h^{\mathrm{inv}}\big)(\overline{x}) \quad \text{and} \quad \Phi_i(\overline{x}) = \Psi_i\Big(\big(\overline{x}\varphi(\overline{x})\big)^{1/(1+\beta)}\Big), \tag{5.4}$$

where h^{inv} denoting the inverse of h. The following statements hold true.

(I) *The family $\Phi = \{\Phi_i,\ i \in I\}$ belongs to $\mathcal{B}$. Furthermore, the function φ is such that $\varphi = \mathscr{I}[\Phi]$,*

$$\big(\overline{x}\varphi(\overline{x})\big)'_- > 0 \quad \text{and} \quad \lim_{\overline{x}\uparrow\infty} \overline{x}\varphi(\overline{x}) = \infty. \tag{5.5}$$

(II) *There exists a unique strictly positive $(\mathcal{F}_t)$-progressive process P such that*

$$P^{1+\beta} = X\mathscr{I}[\Psi](\overline{P}), \tag{5.6}$$

where $\overline{P}_t = \sup_{0\le s\le t} P_s$. This process is given by

$$P = \big(X\mathscr{I}[\Phi](\overline{X})\big)^{1/(1+\beta)}, \tag{5.7}$$

where $\overline{X}_t = \sup_{0\le s\le t} X_s$.

Proof. We first note that

$$\lim_{\overline{p}\downarrow 0} h(\overline{p}) = 0, \quad h'_-(\overline{p}) = \big(\overline{p}^{1+\beta}/\mathscr{I}[\Psi](\overline{p})\big)'_- > 1 \quad \text{and} \quad \lim_{\overline{p}\uparrow\infty} h(\overline{p}) = \infty$$

because $\mathscr{I}[\Psi]$ satisfies (5.1) and (5.3). The claims in part (I) of the theorem follow from these properties of h, the assumption that $\Psi \in \mathcal{B}$ and the equivalences

$$\begin{aligned}\Big(h(\overline{p})\,\big(\mathscr{I}[\Psi] \circ h^{\mathrm{inv}}\big)\big(h(\overline{p})\big)\Big)^{1/(1+\beta)} = \overline{p} \quad &\Leftrightarrow \quad \big(\overline{x}\,\big(\mathscr{I}[\Psi] \circ h^{\mathrm{inv}}\big)(\overline{x})\big)^{1/(1+\beta)} = h^{\mathrm{inv}}(\overline{x}) \\ &\Leftrightarrow \quad h_1(\overline{x}) := \big(\overline{x}\varphi(\overline{x})\big)^{1/(1+\beta)} = h^{\mathrm{inv}}(\overline{x}).\end{aligned}$$

Furthermore, the third identity in these equivalences and the expression

$$\mathscr{I}[\Phi](\overline{x}) = \left((1+\beta) \int_I \lambda_i \Big(c_i^{\alpha_i} \vee \Psi_i^{\alpha_i} \Big(\big(\overline{x} \varphi(\overline{x}) \big)^{1/(1+\beta)} \Big) \Big) \, \iota(\mathrm{d}i) \right)^{-\gamma} = \big(\mathscr{I}[\Psi] \circ h^{\mathrm{inv}} \big)(\overline{x})$$

imply that

$$\mathscr{I}[\Psi] \circ h_1 = \mathscr{I}[\Psi] \circ h^{\mathrm{inv}} = \mathscr{I}[\Phi] = \varphi. \tag{5.8}$$

To establish part (II) of the theorem, let P be the process defined by (5.7). This process is such that $\overline{P} = h_1(\overline{X})$ because h_1 is strictly increasing (see (5.5)). However, this identity and (5.8) imply that

$$\mathscr{I}[\Phi](\overline{X}) = \big(\mathscr{I}[\Psi] \circ h_1 \big)(\overline{X}) = \mathscr{I}[\Psi](\overline{P}),$$

which shows that P is a solution to equation (5.6).

To prove uniqueness, suppose that P is a solution to equation (5.6). Such a process satisfies

$$\overline{P}^{1+\beta} = \overline{X} \mathscr{I}[\Psi](\overline{P}) \quad \Leftrightarrow \quad \overline{P} = h^{\mathrm{inv}}(\overline{X}).$$

However, these identities and (5.8) imply that $\mathscr{I}[\Psi](\overline{P}) = \mathscr{I}[\Phi](\overline{X})$. It follows that every solution P to equation (5.6) of the form given by (5.7). □

6 Asymptotic results

Lemma 6.1 *For any $\mathcal{I}$-measurable functions $\eta : I \to [0, \infty[$, $\zeta : I \to]0, \infty[$ and $\xi : I \to \mathbb{R}$ such that*

$$\int_I \zeta_i \, \iota(\mathrm{d}i) < \infty \quad \text{and} \quad O(y) := \int_I \mathbf{1}_{\{\eta_i < y\}} \zeta_i y^{\xi_i} \, \iota(\mathrm{d}i) < \infty \quad \text{for all } y > 0,$$

it holds that

$$\lim_{y \uparrow \infty} \frac{\ln O(y)}{\ln y} = \iota\text{-}\operatorname{ess\,sup} \xi =: \overline{\xi}. \tag{6.1}$$

Proof. We first note that

$$\begin{aligned} \limsup_{y \uparrow \infty} \frac{\ln O(y)}{\ln y} &\leq \lim_{y \uparrow \infty} \frac{1}{\ln y} \ln \int_I \zeta_i e^{\xi_i \ln y} \, \iota(\mathrm{d}i) = \lim_{y \uparrow \infty} \frac{1}{\ln y} \ln \int_I e^{\xi_i \ln y} \, \widetilde{\iota}(\mathrm{d}i) \\ &= \lim_{y \uparrow \infty} \ln \big\| e^{\xi} \big\|_{L^{\ln y}(\widetilde{\iota})} = \widetilde{\iota}\text{-}\operatorname{ess\,sup} \xi = \iota\text{-}\operatorname{ess\,sup} \xi, \end{aligned} \tag{6.2}$$

where $\widetilde{\iota}$ is the *finite* measure on $(I, \mathcal{I})$ that is equivalent to ι, with Radon-Nikodym derivative given by $\mathrm{d}\widetilde{\iota}/\mathrm{d}\iota = \zeta$.

Next, for $k \in \mathbb{N}$ define $I_k := \{\eta < k\} \in \mathcal{I}$, and note that

$$\begin{aligned} \liminf_{y \uparrow \infty} \frac{\ln O(y)}{\ln y} &\geq \lim_{y \uparrow \infty} \frac{1}{\ln y} \ln \int_I \mathbf{1}_{I_k} e^{\xi_i \ln y} \, \widetilde{\iota}(\mathrm{d}i) \\ &= \lim_{y \uparrow \infty} \ln \big\| \mathbf{1}_{I_k} e^{\xi} \big\|_{L^{\ln y}(\widetilde{\iota})} = \iota\text{-}\operatorname{ess\,sup} \big(\mathbf{1}_{I_k} \xi \big). \end{aligned}$$

Combining this result with the fact that $\bigcup_{k=1}^{\infty} I_k = I$, we obtain

$$\liminf_{y\uparrow\infty} \frac{\ln O(y)}{\ln y} \geq \iota\text{-}\operatorname{ess\,sup} \xi.$$

This last inequality and (6.2) imply (6.1). $\square$

Using this lemma and the equivalence

$$\lim_{y\uparrow\infty} \frac{\ln Q(y)}{\ln y} = -\ell \quad \Leftrightarrow \quad \lim_{y\uparrow\infty} y^{\xi} Q(y) = \begin{cases} 0, & \text{if } \xi < \ell, \\ \infty, & \text{if } \xi > \ell, \end{cases} \tag{6.3}$$

where Q is a strictly positive function, we can establish the following asymptotic result.

Lemma 6.2 *Suppose that Assumption 2.1 holds true and let $\Psi = \{\Psi_i,\ i \in I\} \in \mathcal{B}$ be such that*

$$\Psi_i(\overline{p}) = K_i\, \overline{p}^{(1+\beta)/(1-\alpha_i)},$$

for some constant $\beta > 0$ and some $\mathcal{I}$-measurable functions $K > 0$ and α such that $\alpha_i \in]0,1[$ for all $i \in I$. Also, let h, φ, $\Phi = \{\Phi_i,\ i \in I\}$ be as in Theorem 5.2 and define $\psi = \mathscr{I}[\Psi]$. The functions φ and ψ are such that

$$\lim_{\overline{p}\uparrow\infty} \frac{\ln \psi(\overline{p})}{\ln \overline{p}} = -\frac{\overline{\alpha}(1+\beta)\gamma}{1-\overline{\alpha}}, \quad \lim_{\overline{p}\uparrow\infty} \frac{\ln\big(-\psi'_-(\overline{p})\big)}{\ln \overline{p}} = -\frac{\overline{\alpha}(1+\beta)\gamma}{1-\overline{\alpha}} - 1, \tag{6.4}$$

$$\lim_{\overline{x}\uparrow\infty} \frac{\ln \varphi(\overline{x})}{\ln \overline{x}} = -\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}+\overline{\alpha}\gamma}, \quad \lim_{\overline{x}\uparrow\infty} \frac{\ln\big(-\varphi'_-(\overline{x})\big)}{\ln \overline{x}} = -\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}+\overline{\alpha}\gamma} - 1, \tag{6.5}$$

$$\lim_{\overline{x}\uparrow\infty} \frac{\ln\big(\overline{x}\varphi(\overline{x})\big)}{\ln \overline{x}} = \frac{1-\overline{\alpha}}{1-\overline{\alpha}+\overline{\alpha}\gamma} \quad \text{and} \quad \lim_{\overline{x}\uparrow\infty} \frac{\ln\big(x\varphi(\overline{x})\big)'_-}{\ln \overline{x}} = -\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}+\overline{\alpha}\gamma}. \tag{6.6}$$

where $\overline{\alpha}$ is defined by (3.5).

Proof. In view of the definition of $\mathscr{I}[\Psi]$, we can see that

$$(1+\beta)\int_I \lambda_i \Psi_i^{\alpha_i}(\overline{p})\, \iota(\mathrm{d}i) \leq \psi^{-1/\gamma}(\overline{p}) \leq \kappa_0 + (1+\beta)\int_I \lambda_i \Psi_i^{\alpha_i}(\overline{p})\, \iota(\mathrm{d}i),$$

where κ_0 is defined by (2.5). These inequalities, Lemma 6.1 and the fact that the function $]0,1[\ \ni a \mapsto a/(1-a)$ is increasing imply the first limit in (6.4). On the other hand, combining the expression

$$\begin{aligned} \ln\big(-\psi'_-(\overline{p})\big) = {} & \ln\big(\gamma(1+\beta)\big) + \frac{\gamma+1}{\gamma}\ln\psi(\overline{p}) \\ & + \ln \int_I \mathbf{1}_{\left\{(c_i/K_i)^{(1-\alpha_i)/(1+\beta)} < \overline{p}\right\}} \frac{\alpha_i(1+\beta)\lambda_i K_i^{\alpha_i}}{1-\alpha_i}\, \overline{p}^{\alpha_i(1+\beta)/(1-\alpha_i)-1}\, \iota(\mathrm{d}i), \end{aligned}$$

which follows from (5.2), with Lemma 6.1 and the first limit in (6.4), we obtain the second limit in (6.4).

Using the definitions of h and φ as well as (6.4), we can see that

$$\lim_{\overline{p}\uparrow\infty}\frac{\ln h(\overline{p})}{\ln\overline{p}}=1+\beta-\lim_{\overline{p}\uparrow\infty}\frac{\ln\psi(\overline{p})}{\ln\overline{p}}=(1+\beta)\left(1+\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}}\right),$$

$$\lim_{\overline{x}\uparrow\infty}\frac{\ln h^{\mathrm{inv}}(\overline{x})}{\ln\overline{x}}=\lim_{\overline{x}\uparrow\infty}\frac{\ln h^{\mathrm{inv}}(\overline{x})}{\ln h\big(h^{\mathrm{inv}}(\overline{x})\big)}=(1+\beta)^{-1}\left(1+\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}}\right)^{-1}$$

$$\text{and}\quad\lim_{\overline{x}\uparrow\infty}\frac{\ln\varphi(\overline{x})}{\ln\overline{x}}=\lim_{\overline{x}\uparrow\infty}\frac{\ln\psi\big(h^{\mathrm{inv}}(\overline{x})\big)}{\ln h^{\mathrm{inv}}(\overline{x})}\frac{\ln h^{\mathrm{inv}}(\overline{x})}{\ln\overline{x}}=-\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}+\overline{\alpha}\gamma}.$$

Furthermore,

$$\lim_{\overline{p}\uparrow\infty}\frac{\ln\big((\beta+1)\overline{p}^{\beta}\psi^{-1}(\overline{p})\big)}{\ln\overline{p}}=\lim_{\overline{p}\uparrow\infty}\frac{\ln\big(\overline{p}^{\beta+1}(-\psi)'_-(\overline{p})\psi^{-2}(\overline{p})\big)}{\ln\overline{p}}=\beta+\frac{\overline{\alpha}(1+\beta)\gamma}{1-\overline{\alpha}}.$$

The last two of these limits and the equivalence in (6.3) imply that

$$\begin{aligned}\lim_{\overline{p}\uparrow\infty}\overline{p}^{\xi}h'_-(\overline{p})&=\lim_{\overline{p}\uparrow\infty}\overline{p}^{\xi}\left(\frac{(\beta+1)\overline{p}^{\beta}}{\psi(\overline{p})}+\frac{\overline{p}^{\beta+1}(-\psi)'_-(\overline{p})}{\psi^2(\overline{p})}\right)\\&=\begin{cases}0, & \text{if } \xi<-(1+\beta)\big(1+\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}}\big)+1,\\ \infty, & \text{if } \xi>-(1+\beta)\big(1+\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}}\big)+1.\end{cases}\end{aligned}$$

Combining this observation with (6.3), we obtain

$$\lim_{\overline{p}\uparrow\infty}\frac{\ln h'_-(\overline{p})}{\ln\overline{p}}=(1+\beta)\left(1+\frac{\overline{\alpha}\gamma}{1-\overline{\alpha}}\right)-1.$$

In view of the limits that we have derived thus far and the calculation

$$\lim_{\overline{x}\uparrow\infty}\frac{\ln\big(-\varphi'_-(\overline{x})\big)}{\ln\overline{x}}=\lim_{\overline{x}\uparrow\infty}\left(\frac{\ln(-\psi)'_-\big(h^{\mathrm{inv}}(\overline{x})\big)}{\ln h^{\mathrm{inv}}(\overline{x})}-\frac{\ln h'_-\big(h^{\mathrm{inv}}(\overline{x})\big)}{\ln h^{\mathrm{inv}}(\overline{x})}\right)\frac{\ln h^{\mathrm{inv}}(\overline{x})}{\ln\overline{x}},$$

we can see that the second limit in (6.5) also holds true.

Finally, the limits in (6.6) can be seen using similar arguments. □

7 A singular stochastic control problem and its HJB equation

In view of the analysis in Sections 4 and 5, we now consider the problem of maximising the performance index defined by (4.8) over all admissible C_i. In particular, we determine $C_i^\dagger\in\mathcal{A}_i$ that maximises the performance index

$$J^{(i)}_{c_i,x,\overline{x}}(C)=\mathsf{E}\left[\int_0^\infty e^{-r_it}C^{\alpha_i}(t)X_t\varphi(\widetilde{X}_t)\,\mathrm{d}t-k_i\int_{[0,\infty[}e^{-r_it}\,\mathrm{d}C(t)\right],\tag{7.1}$$

for $x = \overline{x}$, where X is given by (2.4) and

$$\widetilde{X}_t = \overline{x} \vee \overline{X}_t = \overline{x} \vee \sup_{0 \le s \le t} X_s. \tag{7.2}$$

Here, we allow for any initial condition $\overline{x} \ge x$ because we will solve the problem using dynamic programming. Additionally, φ is *any* function satisfying the conditions of following assumption.

Assumption 7.1 The function φ can be expressed as a difference of two convex functions and has the properties listed in (5.5). Moreover, for every $\varepsilon > 0$, there exist constants $\widetilde{K}(\varepsilon)$ and $\widetilde{x}(\varepsilon) > 1$ such that

$$\overline{x}\varphi(\overline{x}) \le \widetilde{K}(\varepsilon)\, \overline{x}^{(1-\overline{\alpha})/(1-\overline{\alpha}+\overline{\alpha}\gamma)+\varepsilon} \quad \text{for all } \overline{x} > \widetilde{x}(\varepsilon). \tag{7.3}$$

We are thus faced with the singular stochastic control problem whose value function is defined by

$$v^{(i)}(c_i, x, \overline{x}) = \sup_{C \in \mathcal{A}_i} J^{(i)}_{c_i, x, \overline{x}}(C), \tag{7.4}$$

where the class of admissible controls $\mathcal{A}_i$ is specified by Definition 3.2. The HJB equation of this control problem is given by

$$\max\bigl\{\mathscr{L}_i w(c, x, \overline{x}) + c^{\alpha_i} x \varphi(\overline{x}),\ w_c(c, x, \overline{x}) - k_i\bigr\} = 0, \tag{7.5}$$

with boundary condition

$$w_{\overline{x}}(c, x, \overline{x}) = 0, \tag{7.6}$$

where the operator $\mathscr{L}_i$ is defined by (3.1).

We will show that the the value function $v^{(i)}$ identifies with the solution $w = w^{(i)}$ to the HJB equation (7.5) with boundary condition (7.6) that is of the form presented by Problem 7.2 below. This involves a function $G_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that $G_i(\cdot, \overline{x})$ is C^1,

$$(G_i)_c(c, \overline{x}) > 0, \quad \lim_{c \downarrow 0} G_i(c, \overline{x}) = 0 \quad \text{and} \quad \lim_{c \uparrow 0} G_i(c, \overline{x}) = \infty \quad \text{for all } \overline{x} > 0. \tag{7.7}$$

It also involves the unique solutions $\Phi_i(\overline{x})$ to the equations

$$G_i\bigl(\Phi_i(\overline{x}), \overline{x}\bigr) = \overline{x}, \quad \overline{x} > 0, \tag{7.8}$$

which give rise to a function Φ_i that is required to be the difference of two convex functions and such that

$$(\Phi_i)'_-(\overline{x}) > 0 \text{ for all } \overline{x} > 0, \quad \lim_{\overline{x} \downarrow 0} \Phi_i(\overline{x}) = 0 \quad \text{and} \quad \lim_{\overline{x} \uparrow \infty} \Phi_i(\overline{x}) = \infty. \tag{7.9}$$

The functions G and Φ define the free-boundary surfaces

$$\mathcal{S}_i^1 = \bigl\{(c, x, \overline{x}) \in \mathbb{R}_+ \mid\ 0 < x \le \overline{x} \text{ and } x = G_i(c, \overline{x})\bigr\} \tag{7.10}$$

$$\text{and} \quad \mathcal{S}_i^2 = \bigl\{(c, x, \overline{x}) \in \mathbb{R}_+ \mid\ 0 < x \le \overline{x} \text{ and } c = \Phi_i(\overline{x})\bigr\} \tag{7.11}$$

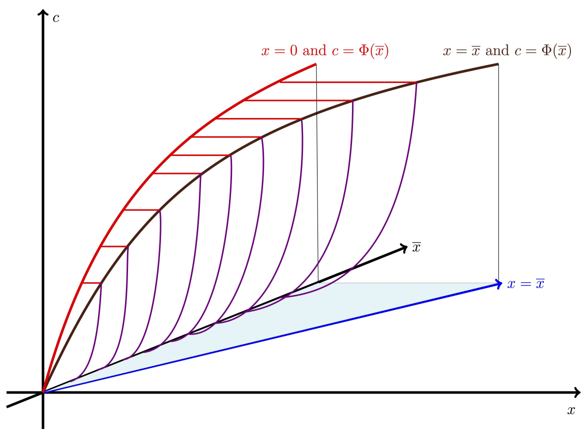

Figure 1: Graph illustrating the surfaces $\mathcal{S}_i^1$ (purple lines) and $\mathcal{S}_i^2$ (red lines).

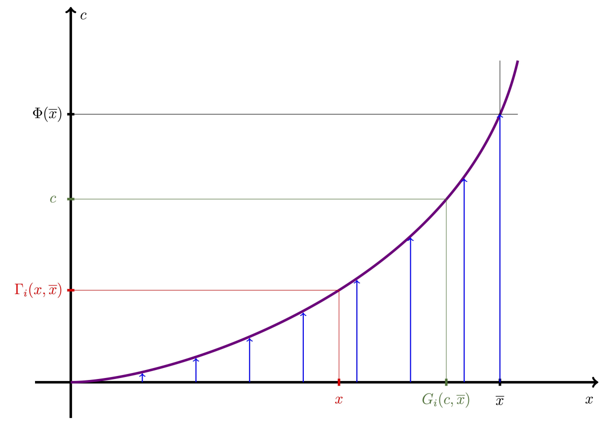

Figure 2: Graph illustrating the functions G_i and Γ_i.

that can be illustrated by Figures 1 and 2. These surfaces partition the control problem's state space

$$\mathcal{C} = \left\{ (c, x, \overline{x}) \in \mathbb{R}_+^3 \mid \ 0 < x \leq \overline{x} \right\} \tag{7.12}$$

into the sets

$$\mathcal{C}_i^1 = \left\{ (c, x, \overline{x}) \in \mathbb{R}_+ \mid \ 0 < x \leq \overline{x} \text{ and } c \leq \Gamma_i(x, \overline{x}) \right\}, \tag{7.13}$$

$$\mathcal{C}_i^2 = \left\{ (c, x, \overline{x}) \in \mathbb{R}_+ \mid \ 0 < x \leq \overline{x} \text{ and } \Gamma_i(x, \overline{x}) < c < \Phi_i(\overline{x}) \right\} \tag{7.14}$$

$$\text{and} \quad \mathcal{C}_i^3 = \left\{ (c, x, \overline{x}) \in \mathbb{R}_+ \mid \ 0 < x \leq \overline{x} \text{ and } \Phi_i(\overline{x}) \leq c \right\}, \tag{7.15}$$

(7.16)

where

$$\Gamma_i(\cdot, \overline{x}) = G_i^{\text{inv}}(\cdot, \overline{x}), \tag{7.17}$$

in which expression, $G_i^{\text{inv}}(\cdot, \overline{x})$ the inverse of the function $G_i^{\text{inv}}(\cdot, \overline{x})$ (see also Figure 2).

We postulate that the optimal expansion strategy can be described informally as follows. At time 0, if the initial state $(c, x, \overline{x})$ is originally inside the region $\mathcal{C}_i^1$, then it is optimal to exert minimal action so that the state process is repositioned on the boundary surface $\mathcal{S}_i^2$. In view of standard singular stochastic control theory, such an action is associated with the requirement that candidate w for the value function $v^{(i)}$ should satisfy (7.21) as well as the inequality in (7.22). Beyond such a possible jump at time 0, it is optimal to exert minimal action so that the state process is prevented from entering the interior of $\mathcal{C}_i^1$. On the other hand, it is optimal to exert no control effort while the state process takes values in the interior of the set $\mathcal{C}_i^2 \cup \mathcal{C}_i^3$, which is associated with the inequalities (7.19) and (7.20), as well as the equalities in (7.22). The significance of the surface $\mathcal{S}_i^2$ arises from the fact that, eventually, it is optimal to exert minimal effort so as to prevent the state process falling below the curve defined by $x = \overline{x}$ and $c = \Phi_i(\overline{x})$, which is the intersection of $\mathcal{S}_i^2$ with the boundary of the state space defined by $x = \overline{x}$. In particular, it is optimal that minimal control effort should be exercised so that the state process takes values on the surface $\mathcal{S}_i^2$ at all times after this surface has been reached.

Problem 7.2 *Determine a function* $G_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ *and a function* $w = w^{(i)} : \mathcal{C} \to \mathbb{R}$, *satisfying the following conditions.*

(I) *The function* $G_i(\cdot, \overline{x})$ *is* C^1, *while the functions* $G_i(c, \cdot)$ *and* Φ_i *are differences of two convex functions. Furthermore,* G_i *and* Φ_i *satisfy (7.7)–(7.9).*

(II) *The function* $w(\cdot, \cdot, \overline{x})$ *is* $C^{1,2}$ *in the interior of* $\mathcal{C}$ *for all* $\overline{x} > 0$.

(III) *Given any* $c \geq 0$ *and* $x > 0$, *the function* $w\big(c, x, \cdot\big) :]x, \infty[\to \mathbb{R}$ *is the difference of two convex functions. Furthermore,*

$$w_{\overline{x}}\big(c, \overline{x}, \overline{x}\big) := \lim_{x \uparrow \overline{x}} w_{\overline{x}-}\big(c, x, \overline{x}\big) = 0 \quad \textit{for all } 0 < \Phi_i(\overline{x}) \leq c. \tag{7.18}$$

(IV) *The function* w *is such that*

$$w_c(c, x, \overline{x}) < k_i \quad \textit{for all } 0 < x < \overline{x} \textit{ and } \Phi_i(\overline{x}) \leq c, \tag{7.19}$$

$$w_c(c, x, \overline{x}) < k_i \quad \textit{for all } 0 < x \leq \overline{x} \textit{ and } \Gamma_i(x, \overline{x}) < c < \Phi_i(\overline{x}), \tag{7.20}$$

$$w_c(c, x, \overline{x}) = k_i \quad \textit{for all } 0 < x \leq \overline{x} \textit{ and } c \leq \Gamma_i(x, \overline{x}), \tag{7.21}$$

and

$$\mathscr{L}_i w(c, x, \overline{x}) + c^{\alpha_i} x \varphi(\overline{x}) \begin{cases} = 0 & \textit{for all } 0 < x < \overline{x} \textit{ and } 0 < \Phi_i(\overline{x}) \leq c, \\ = 0 & \textit{for all } 0 < x \leq \overline{x} \textit{ and } \Gamma_i(x, \overline{x}) < c < \Phi_i(\overline{x}), \\ < 0 & \textit{for all } 0 < x < \overline{x} \textit{ and } c \leq \Gamma_i(x, \overline{x}), \end{cases} \tag{7.22}$$

where $\mathscr{L}_i$ *is defined by (3.1).*

8 The solution to the control problem's HJB equation

In this section, we construct a solution $w = w^{(i)}$ to the HJB equation (7.5) with boundary condition (7.6) that is as in Problem 7.2 and identifies with the value function $v^{(i)}$ defined by (7.4). To this end, we consider the function

$$w(c, x, \overline{x}) = \begin{cases} c^{\alpha_i} \big(A_i(\overline{x}) + \int_{\Phi_i(\overline{x})}^{c} f_i(y)\, \mathrm{d}y\big) x^{n_i} + \frac{1}{r_i - \mu} c^{\alpha_i} x \varphi(\overline{x}), & \text{if } (c, x, \overline{x}) \in \mathcal{C}_i^3, \\ B_i(c, \overline{x}) x^{n_i} + \frac{1}{r_i - \mu} c^{\alpha_i} x \varphi(\overline{x}), & \text{if } (c, x, \overline{x}) \in \mathcal{C}_i^2 \cup \mathcal{S}_i^1, \\ w\big(\Gamma_i(x, \overline{x}), x, \overline{x}\big) - k_i \big(\Gamma_i(x, \overline{x}) - c\big), & \text{if } (c, x, \overline{x}) \in \mathcal{C}_i^1 \setminus \mathcal{S}_i^1, \end{cases} \tag{8.1}$$

where $\mathcal{S}_i^2$, $\mathcal{C}_i^1$, $\mathcal{C}_i^2$, $\mathcal{C}_i^3$ and Γ_i are given by (7.11), (7.13)–(7.15) and (7.17). In view of the general solution (3.2) to the ODE $\mathscr{L}_i u(x) = 0$, we have taken the coefficient of x^{m_i} to be 0 here because, otherwise, the function w would not satisfy the transversality condition that is required for its identification with the control problem's value function.

To determine the function B_i and the free-boundary function G_i, we appeal to the so called "smooth-pasting condition" of singular stochastic control. In particular, we require that $w(\cdot, x, \overline{x})$ should be C^2 along the free-boundary point $G_i(c, \overline{x})$, which suggests the equations

$$\lim_{x \uparrow G_i(c,\overline{x})} w_c(c, x, \overline{x}) = (B_i)_c(c, \overline{x}) G_i^{n_i}(c, \overline{x}) + \frac{\alpha_i}{r_i - \mu} c^{\alpha_i - 1} \varphi(\overline{x}) G_i(c, \overline{x}) = k_i \tag{8.2}$$

$$\text{and} \quad \lim_{x \uparrow G_i(c,\overline{x})} w_{cx}(c, x, \overline{x}) = n_i (B_i)_c(c, \overline{x}) G_i^{n_i - 1}(c, \overline{x}) + \frac{\alpha_i}{r_i - \mu} c^{\alpha_i - 1} \varphi(\overline{x}) = 0. \tag{8.3}$$

The solution to this system of equations is given by

$$G_i(c, \overline{x}) = \frac{(r_i - \mu) n_i k_i}{\alpha_i (n_i - 1)} \frac{c^{1-\alpha_i}}{\varphi(\overline{x})} \quad \text{and} \quad (B_i)_c(c, \overline{x}) = -\frac{k_i}{n_i - 1} G_i^{-n_i}(c, \overline{x}). \tag{8.4}$$

For G_i given by the first of these expressions, we can see that function Γ_i defined by (7.17) and the unique solution to equation (7.8) are given by

$$\Gamma_i(x, \overline{x}) = \left(\frac{\alpha_i (n_i - 1)}{(r_i - \mu) n_i k_i} \right)^{1/(1-\alpha_i)} \big(x \varphi(\overline{x}) \big)^{1/(1-\alpha_i)} \tag{8.5}$$

$$\text{and} \quad \Phi_i(\overline{x}) = \left(\frac{\alpha_i (n_i - 1)}{(r_i - \mu) n_i k_i} \right)^{1/(1-\alpha_i)} \big(\overline{x} \varphi(\overline{x}) \big)^{1/(1-\alpha_i)}. \tag{8.6}$$

In particular, we note that Φ_i is as in (4.13), while

$$0 < \Gamma_i(x, \overline{x}) \leq \Gamma_i(\overline{x}, \overline{x}) = \Phi_i(\overline{x}) \quad \text{for all } 0 < x < \overline{x}. \tag{8.7}$$

Furthermore, Assumptions 3.1 and 7.1 imply that $\{\Phi_i, \ i \in I\} \in \mathcal{B}$ and, given any $\varepsilon \in \,]0, \varepsilon_i[$,

$$0 < \Phi_i(\overline{x}) \leq \left(\widetilde{K}(\varepsilon) \frac{\alpha_i (n_i - 1)}{(r_i - \mu) n_i k_i} \right)^{1/(1-\alpha_i)} \overline{x}^{\frac{1-\overline{\alpha}}{(1-\overline{\alpha}+\overline{\alpha}\gamma)(1-\alpha_i)} + \frac{\varepsilon}{1-\alpha_i}} \quad \text{for all } \overline{x} > \widetilde{x}(\varepsilon) > 1.$$

Combining this estimate with the identity

$$\frac{(1-\overline{\alpha})\alpha_i}{(1-\overline{\alpha}+\overline{\alpha}\gamma)(1-\alpha_i)} + \frac{\alpha_i \varepsilon_i}{1-\alpha_i} = n_i - 1$$

and the inequality

$$\frac{1-\overline{\alpha}}{(1-\overline{\alpha}+\overline{\alpha}\gamma)(1-\alpha_i)} + \frac{\alpha_i \varepsilon_i}{1-\alpha_i} < n_i,$$

which follow from Assumption 3.1, we can see that there exists $\vartheta_i > 0$ such that

$$x \Phi_i^{\alpha_i}(\overline{x}) \leq \left(\widetilde{K}(\vartheta_i) \frac{\alpha_i (n_i - 1)}{(r_i - \mu) n_i k_i} \right)^{\alpha_i/(1-\alpha_i)} \overline{x}^{n_i - \vartheta_i} \quad \text{for all } \overline{x} > \widetilde{x}(\vartheta_i) \tag{8.8}$$

$$\text{and} \quad \Phi_i(\overline{x}) \leq \left(\widetilde{K}(\vartheta_i) \frac{\alpha_i (n_i - 1)}{(r_i - \mu) n_i k_i} \right)^{1/(1-\alpha_i)} \overline{x}^{n_i - \vartheta_i} \quad \text{for all } \overline{x} > \widetilde{x}(\vartheta_i). \tag{8.9}$$

On the other hand, the solution to the ODE in (8.4) is given by

$$B_i(c,\overline{x}) = \widetilde{f}_i(\overline{x}) + \frac{k_i}{(n_i-1)\big(n_i(1-\alpha_i)-1\big)} c^{-n_i(1-\alpha_i)+1}\overline{x}^{\,-n_i}\Phi_i^{n_i(1-\alpha_i)}(\overline{x}), \tag{8.10}$$

where $\widetilde{f}_i$ is a function that we will determine later.

The requirement that w should be continuous is reflected by the identity

$$\lim_{c\uparrow\Phi(\overline{x})} w(c,x,\overline{x}) = \lim_{c\downarrow\Phi(\overline{x})} w(c,x,\overline{x}),$$

which gives rise to the expressions

$$\Phi_i^{\alpha_i}(\overline{x})A_i(\overline{x}) = B_i\big(\Phi_i(\overline{x}),\overline{x}\big) = \widetilde{f}_i(\overline{x}) + \frac{k_i}{(n_i-1)\big((1-\alpha_i)n_i-1\big)}\overline{x}^{\,-n_i}\Phi_i(\overline{x}). \tag{8.11}$$

To determine the functions A_i and f_i, we first recall that, once the state process takes values on the surface $\mathcal{S}_i^1$, it should be optimal to be kept there at all times by means of minimal control effort exerted when the state process is on the boundary line defined by $x=\overline{x}$ and $\overline{x}=\Phi_i(\overline{x})$. This observation and the "smooth-pasting condition" of singular stochastic control suggest that w should satisfy

$$\lim_{c\downarrow\Phi_i(\overline{x}),\,x\uparrow\overline{x}} w_c(c,x,\overline{x}) = k_i \quad\text{and}\quad \lim_{c\downarrow\Phi_i(\overline{x}),\,x\uparrow\overline{x}} w_{cx}(c,x,\overline{x}) = 0,$$

which give rise to the system of equations

$$\Phi_i^{\alpha_i-1}(\overline{x})\Big(\alpha_i A_i(\overline{x}) + \Phi_i(\overline{x})f_i\big(\Phi_i(\overline{x})\big)\Big)\overline{x}^{n_i} + \frac{\alpha_i}{r_i-\mu}\Phi_i^{\alpha_i-1}(\overline{x})\overline{x}\varphi(\overline{x}) = k_i \tag{8.12}$$

$$\text{and}\quad n_i\Phi_i^{\alpha_i-1}(\overline{x})\Big(\alpha_i A_i(\overline{x}) + \Phi_i(\overline{x})f_i\big(\Phi_i(\overline{x})\big)\Big)\overline{x}^{n_i} + \frac{\alpha_i}{r_i-\mu}\Phi_i^{\alpha_i-1}(\overline{x})\overline{x}\varphi(\overline{x}) = 0. \tag{8.13}$$

The solution to this system is given by $\Phi_i(\overline{x})$ as in (8.6) and by

$$f_i\big(\Phi_i(\overline{x})\big) = -\frac{k_i}{n_i-1}\overline{x}^{\,-n_i}\Phi_i^{-\alpha_i}(\overline{x}) - \alpha_i\Phi_i^{-1}(\overline{x})A_i(\overline{x}), \tag{8.14}$$

which is equivalent to

$$f_i(y) = -\frac{k_i}{n_i-1}y^{-\alpha_i}\big(\Phi_i^{\mathrm{inv}}\big)^{-n_i}(y) - \alpha_i y^{-1}A_i\big(\Phi_i^{\mathrm{inv}}(y)\big), \tag{8.15}$$

where Φ_i^{inv} is the inverse function of Φ_i.

To complete this part of the analysis, we note that the requirement that w should satisfy (7.18) gives rise to the expression

$$\lim_{x\uparrow\overline{x}} w_{\overline{x}}(c,x,\overline{x}) = c^{\alpha_i}\bigg(\Big((A_i)'_-(\overline{x}) - f_i\big(\Phi_i(\overline{x})\big)(\Phi_i)'_-(\overline{x})\Big)\overline{x}^{n_i} + \frac{1}{r_i-\mu}\overline{x}\varphi'_-(\overline{x})\bigg) = 0. \tag{8.16}$$

Substituting the expression given by (8.14) for f_i in this equation and using the identities

$$\frac{1}{r_i - \mu} \overline{x} \varphi(\overline{x}) \Phi_i^{\alpha_i}(\overline{x}) = \frac{n_i k_i}{\alpha_i (n_i - 1)} \Phi_i(\overline{x}) \tag{8.17}$$
$$\text{and} \quad \frac{1}{r_i - \mu} \big(\overline{x} \varphi(\overline{x})\big)'_- \Phi_i^{\alpha_i}(\overline{x}) = \frac{(1-\alpha_i) n_i k_i}{\alpha_i (n_i - 1)} (\Phi_i)'_-(\overline{x}),$$

which follow from (8.6), we obtain

$$\big(\Phi_i^{\alpha_i} A_i\big)'_-(\overline{x}) = -\frac{(1-\alpha_i) k_i}{\alpha_i} \overline{x}^{\,-n_i} (\Phi_i)'_-(\overline{x}) - \frac{k_i}{\alpha_i (n_i - 1)} \big(\overline{x}^{\,-n_i} \Phi_i(\overline{x})\big)'_-.$$

In view of (8.9) and the integration by parts formula, we can see that both of the integrals in the identity

$$\overline{x}^{\,-n_i} \Phi_i(\overline{x}) = n \int_{\overline{x}}^\infty y^{\,-n_i - 1} \Phi_i(y) \, \mathrm{d}y - \int_{\overline{x}}^\infty y^{\,-n_i} (\Phi_i)'_-(y) \, \mathrm{d}y$$

converge. It follows that

$$\begin{aligned} A_i(\overline{x}) &= -\frac{\big((1-\alpha_i)(n_i - 1) + 1\big) k_i}{\alpha_i (n_i - 1)} \overline{x}^{\,-n_i} \Phi_i^{1-\alpha_i}(\overline{x}) + \frac{(1-\alpha_i) n_i k_i}{\alpha_i} \Phi_i^{-\alpha_i}(\overline{x}) \int_{\overline{x}}^\infty y^{-n_i - 1} \Phi_i(y) \, \mathrm{d}y \\ &= -\frac{k_i}{\alpha_i (n_i - 1)} \overline{x}^{\,-n_i} \Phi_i^{1-\alpha_i}(\overline{x}) + \frac{(1-\alpha_i) k_i}{\alpha_i} \Phi_i^{-\alpha_i}(\overline{x}) \int_{\overline{x}}^\infty y^{-n_i} (\Phi_i)'_-(y) \, \mathrm{d}y. \end{aligned} \tag{8.18}$$

Lemma 8.1 *Suppose that Assumptions 2.1 and 3.1 hold true. Also, let φ be any function satisfying Assumption 7.1 and consider the free-boundary functions G_i and Φ_i given by (8.4) and (8.6), as well as the functions A_i, f_i and B_i given by (8.10), (8.11), (8.15) and (8.18). The function w given by (8.1) is a solution to the HJB equation (7.5) with boundary condition (7.6). Furthermore, there exist constants $K > 0$ and $\vartheta_i > 0$ such that*

$$0 < x \Gamma_i^{\alpha_i}(x, \overline{x}) \le K(1 + \overline{x}^{n_i - \vartheta_i}), \quad 0 < x \Phi_i^{\alpha_i}(\overline{x}) \le K(1 + \overline{x}^{n_i - \vartheta_i}), \tag{8.19}$$

and

$$0 < w(c, x, \overline{x}) \le K \Big(1 + \overline{x}^{n_i - \vartheta_i} + (1 + \overline{x}^{n_i - \vartheta_i}) \big(c \Phi_i^{-1}(\overline{x})\big)^{\alpha_i} + \overline{x}^{n_i - \vartheta_i} \big(c^{-1} \Phi_i(\overline{x})\big)^{n_i (1-\alpha_i) - 1}\Big) \tag{8.20}$$

for all $(c, x, \overline{x}) \in \mathcal{C}$.

Proof. We first note that the functions Φ and G satisfy the requirements of Problem 7.2 because $1 - \alpha_i > 0$, $\varphi \in \mathcal{V}$ and $\{\Phi_i, \ i \in I\} \in \mathcal{B}$ (see also the discussion after (8.9)). In what follows, we use $K > 0$ as a generic constant. The estimates in (8.19) follow immediately from the expressions of Γ_i and Φ_i in (8.5) and (8.6), the estimate (8.8) and the facts that Φ_i is continuous and $\lim_{\overline{x} \downarrow 0} \Phi_i(\overline{x}) = 0$. The estimate (8.9) and the first expression for A_i in (8.18) imply that

$$\Phi_i^{\alpha_i}(\overline{x}) \big| A_i(\overline{x}) \big| \le K(\overline{x}^{\,-n_i} + \overline{x}^{\,-\vartheta_i}) \quad \text{and} \quad \overline{x} \varphi(\overline{x}) \le K \Phi_i^{1-\alpha_i}(\overline{x}) \le \Phi_i^{-\alpha_i}(\overline{x}) \overline{x}^{n_i - \vartheta_i}.$$

On the other hand, (8.11) implies that

$$\big|\widetilde{f}_i(\overline{x})\big| \le K\Big(\Phi_i^{\alpha_i}(\overline{x})\big|A_i(\overline{x})\big| + \overline{x}^{\,-n_i}\Phi_i(\overline{x})\Big) \le K(\overline{x}^{\,-n_i} + \overline{x}^{\,-\vartheta_i}),$$

while (8.10) yields

$$\big|B_i(c,\overline{x})\big| \le K\Big(\overline{x}^{\,-n_i} + \overline{x}^{\,-\vartheta_i} + c\overline{x}^{\,-n_i}\big(c^{-1}\Phi_i(\overline{x})\big)^{n_i(1-\alpha_i)}\Big).$$

Using (8.14) and the second expression for A_i in (8.18), we obtain

$$f_i\big(\Phi_i(\overline{x})\big) = -(1-\alpha_i)k_i\Phi_i^{-1-\alpha_i}(\overline{x})\int_{\overline{x}}^{\infty} y^{-n_i}(\Phi_i)'_-(y)\,\mathrm{d}y < 0.$$

The upper bound in (8.20) follows from these estimates and the definition (8.1) of w.

In view of the expression (8.1) of w the function $w_{\overline{x}}\big(c,x,\cdot\big) :]x,\infty[\to \mathbb{R}$ is the difference of two convex functions for all $c \ge 0$ and $x > 0$. By construction, the function w_c is continuous along the surface $\mathcal{S}_i^1$. The continuity of w_x and w_{xx} across the surface $\mathcal{S}_i^1$ follows from the observations that, given any $(c,x,\overline{x}) \in \mathcal{C}_i \setminus \mathcal{S}_i^1$,

$$\begin{aligned} w_x(c,x,\overline{x}) &= \lim_{c\downarrow\Gamma_i(x,\overline{x})} w_x(c,x,\overline{x}) + \Big(w_c\big(\Gamma_i(x,\overline{x}),x,\overline{x}\big) - k_i\Big)(\Gamma_i)_x(x,\overline{x}) \\ &= \lim_{c\downarrow\Gamma_i(x,\overline{x})} w_x(c,x,\overline{x}) = w_x\big(\Gamma_i(x,\overline{x}),x,\overline{x}\big) \qquad (8.21) \\ \text{and}\quad w_{xx}(c,x,\overline{x}) &= \lim_{c\downarrow\Gamma_i(x,\overline{x})} w_{xx}(c,x,\overline{x}) + w_{cx}\big(\Gamma_i(x,\overline{x}),x,\overline{x}\big)(\Gamma_i)_x(x,\overline{x}) \\ &= \lim_{c\downarrow\Gamma_i(x,\overline{x})} w_{xx}(c,x,\overline{x}) = w_{xx}\big(\Gamma_i(x,\overline{x}),x,\overline{x}\big), \qquad (8.22) \end{aligned}$$

where we have used the definition (8.1) of w as well as (8.2) and (8.3). The definition (8.1) of w and the expression (8.11) imply that w_x and w_{xx} are continuous across the surface $\mathcal{S}_i^2$. On the other hand, we use equation (7.8) as well as the expressions of $(B_i)_c$ and f in (8.4) and (8.14) to obtain

$$\begin{aligned} \lim_{c\downarrow\Phi_i(\overline{x})} w_c(c,x,\overline{x}) &= -\frac{k_i}{n_i-1}\overline{x}^{\,-n_i}x^{n_i} + \frac{\alpha_i}{r_i-\mu}\Phi_i^{\alpha_i-1}(\overline{x})x\varphi(\overline{x}) \\ &= -\frac{k_i}{n_i-1}G_i^{-n_i}\big(\Phi_i(\overline{x}),\overline{x}\big)x^{n_i} + \frac{\alpha_i}{r_i-\mu}\Phi_i^{\alpha_i-1}(\overline{x})x\varphi(\overline{x}) \\ &= \lim_{c\uparrow\Phi_i(\overline{x})} w_c(c,x,\overline{x}). \end{aligned}$$

It follows that the function $w(\cdot,\cdot,\overline{x})$ is $C^{1,2}$ in the interior of $\mathcal{C}$ for all $\overline{x} > 0$.

By construction, we will prove that w provides a solution to Problem 7.2 if we show that

$$\begin{aligned} w_c(c,x,\overline{x}) &< k_i \quad \text{for all } 0 < x < \overline{x} \text{ and } \Phi_i(\overline{x}) < c, \qquad (8.23) \\ w_c(c,x,\overline{x}) &< k_i \quad \text{for all } 0 < x \le \overline{x} \text{ and } \Gamma_i(x,\overline{x}) < c \le \Phi_i(\overline{x}), \qquad (8.24) \\ \text{and}\quad \mathscr{L}_i w(c,x,\overline{x}) + c^{\alpha_i}x\varphi(\overline{x}) &\le 0 \quad \text{for all } 0 < x < \overline{x} \text{ and } c \le \Gamma_i(x,\overline{x}). \qquad (8.25) \end{aligned}$$

To this end, consider first any $(c,x,\overline{x})$ such that $0<x\leq\overline{x}$ and $\Gamma_i(x,\overline{x})<c\leq\Phi_i(\overline{x})$. Using (8.3), we calculate

$$\begin{aligned}
w_{cx}(c,x,\overline{x}) &= n_i(B_i)_c(c,\overline{x})x^{n_i-1}+\frac{\alpha_i}{r_i-\mu}c^{\alpha_i-1}\varphi(\overline{x})\\
&= \frac{\alpha_i}{r_i-\mu}c^{\alpha_i-1}\left(1-\left(\frac{x}{G_i(c,\overline{x})}\right)^{n_i-1}\right)\varphi(\overline{x})>0 \quad \text{for all } x\in\big]0,G_i(c,\overline{x})\big[.
\end{aligned}$$

Combining this inequality with (8.2), we obtain (8.24).

Next, we consider any $(c,x,\overline{x})$ such that $0<x\leq\overline{x}$ and $\Phi_i(\overline{x})<c$ and we use (8.16) to derive

$$w_{c\overline{x}}(c,x,\overline{x})=\frac{\alpha_i}{r_i-\mu}c^{\alpha_i-1}x^{n_i}\varphi'_-(\overline{x})\big(x^{-n_i+1}-\overline{x}^{\,-n_i+1}\big)<0.$$

In view of this inequality and (8.24), we can see that

$$w_c(c,x,\overline{x})=w_c\big(c,x,\Phi_i^{\text{inv}}(c)\big)-\int_{\overline{x}}^{\Phi_i^{\text{inv}}(c)}w_{c\overline{x}}(c,x,y)\,\mathrm{d}y<k_i.$$

To establish (8.25), we use the definition (8.1) of w, as well as (8.21) and (8.22), to obtain

$$\begin{aligned}
&\mathscr{L}_iw(c,x,\overline{x})+c^{\alpha_i}x\varphi(\overline{x})\\
&\qquad=\mathscr{L}_iw\big(\Gamma_i(x,\overline{x}),z,x\big)+r_ik_i\big(\Gamma_i(x,\overline{x})-c\big)+c^{\alpha_i}x\varphi(\overline{x})\\
&\qquad=-\Gamma_i^{\alpha_i}(x,\overline{x})x\varphi(\overline{x})+r_ik_i\big(\Gamma_i(x,\overline{x})-c\big)+c^{\alpha_i}x\varphi(\overline{x})\\
&\qquad=-\int_c^{\Gamma_i(x,\overline{x})}\big(\alpha_iu^{\alpha_i-1}x\varphi(\overline{x})-r_ik_i\big)\,\mathrm{d}u \quad \text{for all } c<\Gamma_i(x,\overline{x}).
\end{aligned} \tag{8.26}$$

In view of (3.3) and (3.4), we can see that the free-boundary point $G_i(c,\overline{x})$ given by (8.4) is the unique solution to the equation

$$\int_0^{G_i(c,\overline{x})}y^{-m_i-1}\big(\alpha_ic^{\alpha_i-1}y\varphi(\overline{x})-r_ik_i\big)\,\mathrm{d}y=0.$$

Therefore, $\alpha_ic^{\alpha_i-1}x\varphi(\overline{x})-r_ik_i>0$ for all $x\geq G_i(c,\overline{x})$, which implies that

$$\alpha_ic^{\alpha_i-1}x\varphi(\overline{x})-r_ik_i>0 \quad \text{for all } c\leq\Gamma_i(x,\overline{x})$$

because $\Gamma_i(\cdot,\overline{x})$ is the inverse of the strictly increasing function $G_i(\cdot,\overline{x})$. However, this conclusion and (8.26) imply that w satisfies (8.25).

To prove the positivity of w and complete the proof, we first use (8.16) to calculate

$$\begin{aligned}
\int_{\Phi_i(\overline{x})}^c f(y)\,\mathrm{d}y &= \int_{\overline{x}}^{\Phi_i^{\text{inv}}(c)}f\big(\Phi_i(y)\big)(\Phi_i)'_-(y)\,\mathrm{d}y\\
&= A_i\big(\Phi_i^{\text{inv}}(c)\big)-A_i(\overline{x})+\frac{1}{r_i-\mu}\int_{\overline{x}}^{\Phi_i^{\text{inv}}(c)}y^{-n_i+1}\varphi'_-(y)\,\mathrm{d}y.
\end{aligned}$$

We next use integration by parts to obtain

$$\int_{\overline{x}}^{\Phi_i^{\rm inv}(c)} y^{-n_i+1}\varphi'_-(y)\,\mathrm{d}y = \big(\Phi_i^{\rm inv}(c)\big)^{-n_i+1}\varphi\big(\Phi_i^{\rm inv}(c)\big) - \overline{x}^{\,-n_i+1}\varphi(\overline{x}) + (n_i-1)\int_{\overline{x}}^{\Phi_i^{\rm inv}(c)} y^{-n_i}\varphi(y)\,\mathrm{d}y.$$

On the other hand, we use (8.17) and the second expression for A_i in (8.18) to obtain

$$A_i(\overline{x}) + \frac{1}{r_i-\mu}\overline{x}^{\,-n_i+1}\varphi(\overline{x}) = \frac{k_i}{\alpha_i}\overline{x}^{\,-n_i}\Phi_i^{1-\alpha_i}(\overline{x}) + \frac{(1-\alpha_i)k_i}{\alpha_i}\Phi_i^{-\alpha_i}(\overline{x})\int_{\overline{x}}^{\infty} y^{-n_i}(\Phi_i)'_-(y)\,\mathrm{d}y > 0,$$

which implies that

$$A_i\big(\Phi_i^{\rm inv}(c)\big) + \frac{1}{r_i-\mu}\big(\Phi_i^{\rm inv}(c)\big)^{-n_i+1}\varphi\big(\Phi_i^{\rm inv}(c)\big) > 0.$$

In view of these observations, we can see that, given any $(c,x,\overline{x}) \in \mathcal{C}_i^3$,

$$\begin{aligned} &w(c,x,\overline{x}) \\ &\quad = c^{\alpha_i}\left(A_i\big(\Phi_i^{\rm inv}(c)\big) + \frac{1}{r_i-\mu}\big(\Phi_i^{\rm inv}(c)\big)^{-n_i+1}\varphi\big(\Phi_i^{\rm inv}(c)\big) + \frac{n_i-1}{r_i-\mu}\int_{\overline{x}}^{\Phi_i^{\rm inv}(c)} y^{-n}\varphi(y)\,\mathrm{d}y\right)x^{n_i} \\ &\quad\quad + \frac{1}{r_i-\mu}c^{\alpha_i}x^{n_i}\big(x^{-n_i+1} - \overline{x}^{\,-n_i+1}\big)\varphi(\overline{x}) > 0. \end{aligned}$$

Given any $(c,x,\overline{x}) \in \mathcal{C}_i^2$, the fact that $(B_i)_c(c,\overline{x}) < 0$ (see (8.4)) implies that

$$\begin{aligned} w(c,x,\overline{x}) &\geq B_i\big(\Phi_i(\overline{x}),\overline{x}\big)x^{n_i} + \frac{1}{r_i-\mu}c^{\alpha_i}x\varphi(\overline{x}) \\ &= \Phi_i^{\alpha_i}(\overline{x})A_i(\overline{x}) + \frac{1}{r_i-\mu}c^{\alpha_i}x\varphi(\overline{x}) = w\big(\Phi_i(\overline{x}),x,\overline{x}\big) > 0. \end{aligned} \tag{8.27}$$

Finally, combining the observations that

$$0 < w(c,x,\overline{x}), \quad 0 < w_x(c,x,\overline{x}) \quad \text{and} \quad \lim_{\substack{(c,x,\overline{x})\in\mathcal{C}_i^2 \\ (c,x,\overline{x})\to(0,0,0)}} w(c,x,\overline{x}) = 0$$

with (8.21), we can see that w is strictly positive in $\mathcal{C}_i^1$ as well. □

9 The solution to the control problem studied in Sections 7 and 8

Theorem 9.1 *Consider the singular stochastic control problem formulated in Section 7 and suppose that Assumptions 2.1 and 3.1 hold true. Also, let φ be any function satisfying Assumption 7.1. The value function $v^{(i)}$ of the optimisation problem that the individual producer $i \in I$ faces, which is defined by (7.4), identifies with the function $w = w^{(i)}$ in Lemma 8.1, namely,*

$$w(c_i, x, \overline{x}) = \sup_{C \in \mathcal{A}_i} J^{(i)}_{c_i,x,\overline{x}}(C) \quad \textit{for all } (c_i, x, \overline{x}) \in \mathcal{C},$$

where $J^{(i)}_{c_i,x,\overline{x}}$ is defined by (7.1). Furthermore, the expansion process

$$C_i^\dagger(t) = c_i \vee \Big(\Gamma_i(\overline{X}_t, \overline{x}) \mathbf{1}_{\{\overline{X}_t < \overline{x}\}} + \Phi_i\big(\overline{X}_t\big) \mathbf{1}_{\{\overline{X}_t \geq \overline{x}\}} \Big), \tag{9.1}$$

where $\overline{X}$ is defined by (4.1), while Γ_i and Φ_i are given by (8.5) and (8.6), is admissible as well as optimal for the individual producer $i \in I$.

Proof. Fix any initial condition $(c_i, x, \overline{x}) \in \mathcal{C}$ and consider any expansion process $C \in \mathcal{A}_i$. Using Itô's formula and the fact that the process $\widetilde{X} = \overline{x} \vee \overline{X}$ is continuous and increases on the set $\{X_t = \widetilde{X}_t\}$, we obtain

$$\begin{aligned}
&e^{-r_i T} w\big(C(T-), X_T, \widetilde{X}_T\big) \\
&\quad = w(c_i, x, \overline{x}) + \int_0^T e^{-r_i t} \mathscr{L}_i w\big(C(t), X_t, \widetilde{X}_t\big) \, \mathrm{d}t + \int_0^T e^{-r_i t} w_c\big(C(t), X_t, \widetilde{X}_t\big) \, \mathrm{d}C_t^{\mathrm{c}} \\
&\qquad + \sum_{0 \leq t < T} e^{-r_i t} \Big(w\big(C(t), X_t, \overline{X}_t\big) - w\big(C(t-), X_t, \widetilde{X}_t\big) \Big) \\
&\qquad + \int_0^T e^{-r_i t} w_{\overline{x}}\big(C(t), \widetilde{X}_t, \widetilde{X}_t\big) \, \mathrm{d}\widetilde{X}_t + M_T,
\end{aligned}$$

where

$$M_T = \int_0^T e^{-r_i t} \sigma X_t w_x\big(C(t), X_t, \widetilde{X}_t\big) \, \mathrm{d}W_t. \tag{9.2}$$

In view of this identity, the positivity of w and the fact that w satisfies the HJB equation

(7.5) with boundary condition (7.6), we can see that

$$\begin{aligned}
\int_0^T e^{-r_i t} C^{\alpha_i}(t) X_t \varphi(\widetilde{X}_t)\,\mathrm{d}t - k_i \int_{[0,T[} e^{-r_i t}\,\mathrm{d}C(t) \\
&= w(c_i, x, \overline{x}) - e^{-r_i T} w\big(C(T-), X_T, \widetilde{X}_T\big) \\
&\quad + \int_0^T e^{-r_i t}\Big(\mathscr{L}_i w\big(C(t), X_t, \widetilde{X}_t\big) + C^{\alpha_i}(t) X_t \varphi(\widetilde{X}_t)\Big)\,\mathrm{d}t \\
&\quad + \int_0^T e^{-r_i t}\Big(w_c\big(C(t), X_t, \widetilde{X}_t\big) - k_i\Big)\,\mathrm{d}C^{\mathrm{c}}(t) \\
&\quad + \sum_{0 \le t < T} e^{-r_i t} \int_0^{\Delta C(t)} \Big(w_c\big(C(t-) + y, X_t, \widetilde{X}_t\big) - k_i\Big)\,\mathrm{d}y + M_T \\
&\le w(c_i, x, \overline{x}) - e^{-r_i T} w\big(C(T-), X_T, \widetilde{X}_T\big) + M_T.
\end{aligned}$$

Taking expectations, we are faced with the inequality

$$\begin{aligned}
\mathsf{E}\bigg[\int_0^{T\wedge\tau_n} e^{-r_i t} C^{\alpha_i}(t) X_t \varphi(\widetilde{X}_t)\,\mathrm{d}t - k_i \int_{[0,T\wedge\tau_n[} e^{-r_i t}\,\mathrm{d}C(t)\bigg] \\
\le w(c_i, x, \overline{x}) - \mathsf{E}\Big[e^{-r_i(T\wedge\tau_n)} w\big(C((T\wedge\tau_n)-), X_{T\wedge\tau_n}, \widetilde{X}_{T\wedge\tau_n}\big)\Big],
\end{aligned} \tag{9.3}$$

where (τ_n) is a localising sequence for the local martingale M. Recalling the admissibility condition (3.8), the fact that φ is bounded and the positivity of w, we can pass to the limit as $n, T \to \infty$ using the monotone convergence theorem to obtain $J^{(i)}_{c_i,x,\overline{x}}(C) \le w(c_i, x, \overline{x})$. It follows that $v^{(i)}(c_i, x, \overline{x}) \le w(c_i, x, \overline{x})$ because C has been an arbitrary admissible expansion process.

To establish the reverse inequality, we first note that the expansion process given by (9.1) is such that

$$C_i^{\dagger\,\alpha_i}(t) \le c_i^{\alpha_i} + \Phi_i^{\alpha_i}(\overline{x}) + \Phi_i^{\alpha_i}\big(\overline{X}_t\big).$$

Combining this inequality with the estimates in (8.19), the assumption that $n_i > 1$ and Lemma 1 in [24], we can see that

$$\mathsf{E}\bigg[\int_0^\infty e^{-r_i t} C_i^{\dagger\,\alpha_i}(t) X_t\,\mathrm{d}t\bigg] < \infty.$$

Therefore, $C^\dagger$ is admissible because it satisfies the admissibility condition (3.8). This capacity schedule is such that the (9.3) holds with equality, namely,

$$\begin{aligned}
\mathsf{E}\bigg[\int_0^{T\wedge\tau_n} e^{-r_i t} C_i^{\dagger\,\alpha_i}(t) X_t \varphi(\widetilde{X}_t)\,\mathrm{d}t - k_i \int_{[0,T\wedge\tau_n[} e^{-r_i t}\,\mathrm{d}C_i^\dagger(t)\bigg] \\
= w(c_i, x, \overline{x}) - \mathsf{E}\Big[e^{-r_i(T\wedge\tau_n)} w\big(C_i^\dagger(T\wedge\tau_n)-), X_{T\wedge\tau_n}, \widetilde{X}_{T\wedge\tau_n}\big)\Big].
\end{aligned} \tag{9.4}$$

Using the expressions (8.5) and (8.6) of Γ_i and Φ_i, as well as the estimates in (8.20), we can see that

$$
\begin{aligned}
0 < e^{-r_i T} w\big(C_i^\dagger(T), X_T, \widetilde{X}_T\big) &\leq K e^{-r_i T}\Big(1 + w\big(\Phi_i(\overline{X}_t), X_T, \overline{X}_T\big)\Big) \\
&\leq K e^{-r_i T}\big(1 + \overline{X}_T^{n_i - \vartheta_i}\big),
\end{aligned} \tag{9.5}
$$

where $K > 0$ is a generic constant. Therefore, we can pass to the limit as $n \to \infty$ in (9.4) using the monotone and the dominated convergence theorems to obtain

$$
\begin{aligned}
&\mathsf{E}\bigg[\int_0^T e^{-r_i t} C_i^{\dagger\,\alpha_i}(t) X_t \varphi(\widetilde{X}_t)\,\mathrm{d}t - k_i \int_{[0,T[} e^{-r_i t}\,\mathrm{d}C_i^\dagger(t)\bigg] \\
&\qquad\qquad = w(c_i, x, \overline{x}) - \mathsf{E}\Big[e^{-r_i T} w\big(C_i^\dagger(T-), X_T, \widetilde{X}_T\big)\Big].
\end{aligned}
$$

Furthermore, we can use the admissibility of the expansion process $C_i^\dagger$, (9.5), Lemma 1 in [24] and the monotone and dominated convergence theorems to pass to the limit as $T \uparrow \infty$ and obtain $J^{(i)}_{c_i,x,\overline{x}}\big(C_i^\dagger\big) = w(c_i, x, \overline{x})$, which implies that $v^{(i)}(c_i, x, \overline{x}) \geq w(c_i, x, \overline{x})$ as well as the optimality of $C_i^\dagger$. □

10 Proof of Theorem 4.2

The following observations prove that the expressions (4.10) and (4.11), or equivalently, (4.12) and (4.13), provide a production equilibrium in the sense of Definition 3.3.

1. Consider any family $\Phi = \{\Phi_i,\ i \in I\} \in \mathcal{B}$, where $\mathcal{B}$ is introduced by Definition 4.1, and suppose that the function $\varphi = \mathscr{I}[\Phi]$ satisfies the requirements of Assumption 7.1.

2. With the possible exception of the producer indexed by $i \in I$, suppose that every other producer, say $j \in I \setminus \{i\}$, adopts the capacity expansion strategy $C_j(t) = c_j \vee \Phi_j(\overline{X}_t)$. In the competitive setting that we have considered, this assumption and the clearing condition (2.3) give rise to the price process

$$
P = \big(X\varphi(\overline{X})\big)^{1/(1+\beta)}. \tag{10.1}
$$

3. Producer i, who has been singled out, is faced with the task of maximising the performance index

$$
\widetilde{J}^{(i)}_{c_i}(C_i \mid P) = J^{(i)}_{c_i,\overline{x},\overline{x}}(C_i)
$$

over all $C_i \in \mathcal{A}$, where $\widetilde{J}^{(i)}_{c_i}(C_i \mid P)$ and $J^{(i)}_{c_i,x,\overline{x}}(C_i)$ are the performance criteria defined by (2.2) and (7.1).

4. Theorem 9.1 implies that the expansion strategy

$$C_i^\dagger(t) = \left(\frac{\alpha_i(n_i-1)}{(r_i-\mu)n_i k_i}\right)^{1/(1-\alpha_i)} \big(\overline{x}\varphi(\overline{x})\big)^{1/(1-\alpha_i)}$$
$$= c_i \vee \Psi_i^\star\Big(\big(\overline{x}\varphi(\overline{x})\big)^{1/(1+\beta)}\Big) \overset{(10.1)}{=} c_i \vee \Psi_i^\star(\overline{P}_t),$$

where

$$\Psi_i^\star(\overline{p}) = \left(\frac{\alpha_i(n_i-1)}{(r_i-\mu)n_i k_i}\right)^{1/(1-\alpha_i)} \overline{p}^{(1+\beta)/(1-\alpha_i)}, \tag{10.2}$$

is such that (3.9) holds true. In particular, this expansion strategy is optimal for producer i.

5. The family $\Psi^\star = \{\Psi_i^\star,\ i \in I\}$ defined by (10.2) belongs to $\mathcal{B}$. By Theorem 5.2, the solution to the equation $P^{1+\beta} = X\mathscr{I}[\Psi^\star](\overline{P})$ is given by $P^{1+\beta} = X\mathscr{I}[\Phi^\star](\overline{X})$, where

$$\Phi_i^\star(\overline{x}) = \Psi_i^\star\Big(\big(\overline{x}\varphi^\star(\overline{x})\big)^{1/(1+\beta)}\Big)$$

and the function $\varphi^\star$ satisfies $\varphi^\star = \mathscr{I}[\Phi^\star]$ and possesses the properties listed in (5.5).

6. The function $\varphi^\star$ all of the requirements of Assumption 7.1, by virtue of Lemma 6.2 and (10.2).

7. In light of the observations above, the price process

$$P^\star = \big(X\varphi^\star(\overline{X})\big)^{1/(1+\beta)} = \big(X\psi^\star(\overline{P})\big)^{1/(1+\beta)}, \tag{10.3}$$

where $\psi^\star = \mathscr{I}[\Psi^\star]$, and the expansion strategies

$$C_i^\star(t) = c_i \vee \Psi_i^\star\big(\overline{X}_t\,\varphi(\overline{X}_t)\big)^{1/(1-\alpha_i)} = c_i \vee \Psi_i^\star(\overline{P}_t^\star)$$

comprise a competitive production equilibrium.

Finally, (4.14) and (4.15) follow immediately from (2.4), (10.3) and an application of Itô's formula. □

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