

A Stone-Weierstrass approximation theorem for monotone functions

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Abstract

We present an approximation theorem for continuous non-decreasing functions on compact preordered spaces, leading to an algebraic characterization of their corresponding function spaces. As an application, we prove that the family of positive non-decreasing rational functions with non-negative coefficients can uniformly approximate all continuous non-decreasing functions on compact intervals. An explicit approximation formula of this type is provided.

1. Introduction

In this work we establish a Stone-Weierstrass type theorem on the approximation of continuous isotone functions over a compact ordered space.

We recall that a *topological preordered space* $(X, \mathcal{T}, \leq)$ is a triple given by a topological space $(X, \mathcal{T})$ and a *preorder* $\leq$ (reflexive and transitive relation) on it. The preorder is an *order* if it is antisymmetric, in which case the triple is called a *topological ordered space*. A *closed preordered space* is a topological preordered space for which $\leq$ is closed in the product topology. Closed ordered spaces are Hausdorff. A *compact preordered space* is a closed preordered space for which $(X, \mathcal{T})$ is compact. Compact preordered spaces were studied by Nachbin [14, 8].

On a topological preordered space a function $f : X \rightarrow \mathbb{R}$ is said to be *isotone* (non-decreasing) if $x \leq y \Rightarrow f(x) \leq f(y)$.

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On a topological preordered space, given a subset $N \subset X$, we denote $C(X, \leq, \mathbb{R}, N)$ the set of continuous isotone real functions that vanish on N , and we denote $C^+(X, \leq, \mathbb{R}, N)$ the set of non-negative continuous isotone real functions that vanish on N . We omit N if it is empty, so writing $C(X, \leq, \mathbb{R})$ and $C^+(X, \leq, \mathbb{R})$, respectively. We endow $C(X, \mathbb{R})$ with the uniform norm, so if $S \subset C(X, \mathbb{R})$, with $\bar{S}$ we shall mean the closure in the uniform norm.

Given a subset $S \subset C(X, \mathbb{R})$ we denote with N_S the closed subset of X given by

$$N_S = \cap_{f \in S} f^{-1}(0) = \{x \in X : f(x) = 0, \forall f \in S\}.$$

Given a subset $S \subset C(X, \mathbb{R})$ we denote with $\leq_S$ the closed relation given by $x \leq_S y$ if $f(x) \leq f(y)$ for every $f \in S$, that is,

$$\leq_S = \{(x, y) \in X^2 : f(x) \leq f(y), \forall f \in S\}.$$

It is an order if and only if the family S distinguishes (separates) points.

A non-empty subset $F \subset C(X, \mathbb{R})$ is a *semi-algebra* [5] if $f+g, \alpha f, fg \in F$ whenever $f, g \in F$ and $\alpha \geq 0$. It is called a *semi-algebra with identity* (or *unit*) if it contains the unit function 1. According to Bonsall, a semi-algebra is said to be of *type-I* if $f/(1+f) \in F$ whenever $f \in F$.

Let $(X, \mathcal{T}, \leq)$ be a compact preordered space. We are interested on conditions on S which ensure that every element of $C(X, \leq, \mathbb{R})$ or $C^+(X, \leq, \mathbb{R}, N)$ is uniformly approximated by some element of S .

Every compact preordered space is a *normally preordered space* [14, 11] which implies that $\leq = \leq_{C(X, \leq, \mathbb{R})}$ and hence, since every continuous function on X is bounded and can be made non-negative by adding a constant, $\leq = \leq_{C^+(X, \leq, \mathbb{R})}$. A necessary condition for the mentioned approximation property is then $\leq = \leq_S$.

Two approximation theorems of this type have appeared in the literature which establish that this condition, under suitable invariance properties of S , is also sufficient.

The former by Bonsall [5, Thm. 7] can be written as follows

Theorem 1.1. *Let $(X, \mathcal{T}, \leq)$ be a compact preordered space. Let $S \subset C(X, \leq, \mathbb{R})$ be a uniformly closed type-I semi-algebra with identity such that $\leq = \leq_S$, then $S = C^+(X, \leq, \mathbb{R})$.*

The latter by Besnard [2] reads instead

Theorem 1.2. *Let $(X, \mathcal{T}, \leq)$ be a compact ordered space. Let $S \subset C(X, \leq, \mathbb{R})$ be such that $\leq = \leq_S$, S is closed under sum, and it is also closed under any map of the form $f \mapsto h(f)$ where $h : \mathbb{R} \rightarrow \mathbb{R}$ is any continuous non-decreasing piecewise linear function. Then $\bar{S} = C(X, \leq, \mathbb{R})$.*

Taking h equal to a constant shows that S includes all real constants. Also taking $h = ax$, $a \geq 0$, shows that S is a cone, hence convex by closure under sum.

Observe that the Theorem 1.1 characterizes $C^+(X, \leq, \mathbb{R})$ among its subsets, while Theorem 1.2 characterizes $C(X, \leq, \mathbb{R})$ among its subsets.

Another difference is that Theorem 1.1 is more general in dealing with preorders rather than orders.

Bonsall's type-I condition is more elegant, being more algebraic in spirit than Besnard's condition, which requires invariance under a large family of continuous, non-decreasing, piecewise linear functions. These piecewise linear functions lack analyticity due to their corners. On the other hand, Bonsall's theorem imposes a strong semi-algebra condition that is absent in Besnard's result. Additionally, Besnard's proof is simpler and more direct than Bonsall's.

This work presents a theorem that both generalizes and strengthens prior results. Following Besnard's adaptation of the Brosowski–Deutsch proof of the Stone–Weierstrass theorem, our argument incorporates a crucial new element: the identification and careful application of specific fixed-point properties of a subset-preserving function ϕ .

Curiously, Nachbin's theory of topological ordered spaces does not really enter the proof of the following result.

Theorem 1.3. *Let $(X, \mathcal{T}, \leq)$ be a compact preordered space. Let $\phi : [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing continuous function such that $\phi(x) \leq x$ with equality iff $x = 0$ or $x = 1$ (see some examples below). Let $S \subset C^+(X, \leq, \mathbb{R})$ be non-empty and such that*

- (i) *S is a convex cone: $S + S \subset S$ and $\lambda S \subset S$ for every $\lambda > 0$,*
- (ii) *If $f \in S$ then $\phi(f) \in S$,*
- (iii) *S generates $\leq$, that is, $\leq = \leq_S$.*

Then $\bar{S} = C^+(X, \leq, \mathbb{R}, N_S)$. In particular, if additionally

(iv) $N_S = \emptyset$, that is: for every $x \in X$ there is some $f \in S$ such that $f(x) > 0$ (this follows if S contains the identity).

Then $\bar{S} = C^+(X, \leq, \mathbb{R})$.

As a consequence, if S is uniformly closed then it coincides with $C^+(X, \leq, \mathbb{R}, N_S)$ and so it is invariant under the lattice operations $\vee$ and $\wedge$.

We are not assuming that S contains the (necessarily non-negative) constants.

Remark 1.4. This and the following results can be specialized to the discrete order, $\leq = \Delta = \{(x, x) : x \in X\}$, for which they provide already original statements. The condition $\leq = \leq_S$ reads “for every $x, y \in S$, $x \neq y$, there is $f \in S$, such that $f(x) < f(y)$ ”. The set $C^+(X, \leq, \mathbb{R})$ is the family of continuous non-negative functions.

Remark 1.5. Observe that for any subset N , $C^+(X, \leq, \mathbb{R}, N)$ is uniformly closed and satisfies (i), (ii) and (iii). Actually, (i) holds also for $\lambda = 0$ because $0 \in C^+(X, \leq, \mathbb{R}, N)$. Observe also that $C^+(X, \leq, \mathbb{R})$ is uniformly closed, satisfies (i)-(iv) and all non-negative constants belong to it.

A useful example for function ϕ is $\alpha_1 := 2x^2/(1+x^2) = x \operatorname{sech}(\log x)$. There are other functions with the properties of ϕ including

$$(\alpha) \quad \alpha_a(x) := x \operatorname{sech}(a \log x) = 2x^{1+a}/(1+x^{2a}) \text{ for } 0 < a \leq 1,$$

$$(\beta) \quad \beta(x) := x - \frac{x^2(1-x)^2}{(1+x)^3} = \frac{x(1+2x+5x^2)}{(1+x)^3},$$

$$(\gamma) \quad \gamma(x) := x - \frac{x(1-x)^2}{(1+x)^2} = \left(\frac{2x}{1+x} \right)^2.$$

Of course, it is also easy to construct a piecewise linear function ϕ with the desired properties.

Remark 1.6. If S is stable under squaring $f \rightarrow f^2$, e.g. because it is a semi-algebra, and under the map $f \rightarrow f/(1+f)$ then it is also stable under γ , which shows that Bonsall’s result, Theorem 1.1, follows as a special case of this theorem.

Remark 1.7. Sometimes using β and γ can be convenient as their singularity at -1 together with the cone assumption on S might be used to get that S consists of non-negative functions, not by assumption, but by using the fact that β or γ are well defined on S .

We denote with $\phi^{(n)}$ the composition of ϕ for n times. It can be observed that $\phi^{(n)}$ has fixed points 0 and 1 and converges (non-uniformly) to the characteristic function of $[1, +\infty)$.

We shall need a couple of lemmas. It can be noted that we do not need to assume that $(X, \mathcal{T})$ is Hausdorff. This property would follow from an antisymmetry condition on $\leq$ which we do not impose.

Lemma 1.8. *Let $a, b \in X$ be such that $b \not\leq a$ then for any $\epsilon > 0$ there is $f_{ab} \in S$ such that, $0 \leq f_{ab} \leq 1 + \epsilon$, $f_{ab}(a) \in [0, \epsilon)$ and $f_{ab}(b) = 1$.*

Proof. Since $b \not\leq a$ by (iii) we can find $f \in S$ such that $f(a) < f(b)$. Multiplying f by a positive constant we can further assume that $f(b) = 1$. Since $\phi^{(n)}(f)$ converges pointwise to the characteristic function of $\{f \geq 1\}$, and $\phi^{(n)}(x)$ is decreasing in n for $x > 1$, the maximum of $\phi^{(n)}(f)$ converges to 1. Thus, for large n , $\phi^{(n)}(f) \leq 1 + \epsilon$. Similarly, for sufficiently large n , $\phi^{(n)}(f)(a) < \epsilon$, thus setting $f_{ab} = \phi^{(n)}(f)$ we conclude. $\square$

Lemma 1.9. *Let A, B be two closed (hence compact) subsets of X such that for every $a \in A$ and $b \in B$, $b \not\leq a$. Then for every δ , $0 < \delta < 1$ there is $f_{AB} \in S$ such that $f_{AB}(A) \in [0, \delta)$, $f_{AB}(B) \in [1, 1 + \delta)$, $f_{AB}(X) \in [0, 1 + \delta)$.*

Proof. Let $0 < \epsilon < 1$. For all $a \in A$ and $b \in B$ we can find $f_{ab} \in S$ as in Lemma 1.8. We fix $b \in B$ and let a vary in A . Since f_{ab} is continuous there is an open neighborhood V_a of a such that $f_{ab}(V_a) \subset [0, \epsilon)$. By compactness of A there are $V_1, \dots, V_k$ corresponding to $a_1, \dots, a_k$ such that $A \subset V_1 \cup \dots \cup V_k$. For $k = 1$ let $g_b = f_{a_1 b} \in S$ so that for every $a \in A$, $g_b(a) < 1$, and moreover $g_b(b) = 1$. For $k \geq 2$ let $g_b := \frac{1}{k} \sum_{i=1}^k \phi^{(n)}(f_{a_i b})$, where n is chosen sufficiently large that $\phi^{(n)}([0, 1 + \epsilon)) \subset [0, \frac{k-\epsilon}{k-1})$. Note that $g_b \in S$. Moreover, $g_b(b) = 1$ and for all $a \in A$,

$$g_b(a) < \frac{1}{k}[(k-1)\frac{k-\epsilon}{k-1} + \epsilon] = 1.$$

In both cases, by compactness of A there is $\alpha \in (0, 1)$ such that $g_b(A)$ is a compact subset of $[0, \alpha)$. Let $f_{Ab} := \phi^{(m)}(g_b/\alpha) \in S$ where m can be chosen sufficiently large that $\phi^{(m)}([0, \sup g_b(A)]/\alpha) \subset [0, \delta)$ while $f_{Ab}(b) \in (1, 1 + \delta)$, $f_{Ab}(X) \subset [0, 1 + \delta)$. By continuity of f_{Ab} there is an open neighborhood U_b of b such that $f_{Ab}(U_b) \in (1, 1 + \delta)$. By compactness of B there are $U_1, \dots, U_l$ corresponding to $b_1, \dots, b_l$ such that $B \subset U_1 \cup \dots \cup U_l$. Let us consider $h = \sum_{i=1}^l \phi^{(r)}(f_{Ab_i}) \in S$ where r is chosen sufficiently large that

$\phi^{(r)}([0, \delta)) \subset [0, \delta/l)$. We observe that $h(A) \subset [0, \delta)$ and for every $b \in B$, $b \in U_i$ for some i so

$$h(b) \geq \phi^{(r)}(f_{Ab_i}(b)) > 1.$$

This proves that $h(B) \subset (1, \infty)$ and noting that h is bounded on the compact set X , defining $f := \phi^{(s)}(h) \in S$ we get for sufficiently large s , $f(X) \subset [0, 1 + \delta)$, hence $f(B) \subset (1, 1 + \delta)$ while keeping $f(A) \subset [0, \delta)$. $\square$

Proof of Theorem 1.3. The constant function 0 can be uniformly approximated by elements of S . Indeed, since S is non-empty, take any $f \in S$; then $\frac{1}{n}f \in S$ for all $n \in \mathbb{N}_+$, and $\frac{1}{n}f \rightarrow 0$ uniformly.

Let us prove that if $N_S = \emptyset$ then every other positive constant can be uniformly approximated (if $N_S \neq \emptyset$ the positive constant functions are not to be approximated as they do not belong to $C^+(X, \leq, \mathbb{R}, N_S)$, as they do not vanish on N_S). Indeed, for every $x \in X$ we can find a function $g_x \in S$ such that $g_x(x) > 0$, hence by continuity, there is a neighborhood U_x of x such that $g_x > 0$ over U_x . Let $U_1, \dots, U_k$ be a finite covering of X relative to points $x_1, \dots, x_k$. Let $g = \sum_k g_{x_i} \in S$, then $g > 0$ over X and thus it has a minimum $m > 0$ and a maximum $M \geq m$. Let $\lambda > 0$ be a constant. Let $h = \lambda \phi^{(n)}(g/m)$ then for any ϵ we have for sufficiently large n , $h(X) \subset [\lambda, \lambda + \epsilon)$ which proves that the constant λ is uniformly approximated (actually from above, and this was also the case for the zero function since the elements of S are non-negative).

Let $f \in C^+(X, \leq, \mathbb{R}, N)$ and let m and M be its minimum and maximum over the compact set X . If $m = M$ then either $m = 0$ which has been already shown to be approximable from above, or $m > 0$ which implies that $f > 0$ over X and so $N_S = \emptyset$, which implies, by the previous argument, that any constant, and so the constant function m , is approximable from above.

We can assume that $0 \leq m < M$ and hence, rescaling, we can work with $\tilde{f} = f/(M - m)$ instead of f (since S is a cone, an arbitrary uniform approximation of $\tilde{f}$ leads to a uniform approximation of f). Hence we can assume $f(X) = [m, m + 1]$ without loss of generality. Let $n \in \mathbb{N} \setminus \{0, 1\}$. We want to prove that there is $F \in S$ such that $\|f - F\| < 3/n$.

We set $A_i = f^{-1}([m, m + \frac{i}{n}])$, $B_i = f^{-1}([m + \frac{i+1}{n}, m + 1])$ for each $i = 0, 1, \dots, n - 1$. Since f is continuous and X is compact, the sets A_i, B_i are closed hence compact. Note that if $a \in A_i$ and $b \in B_i$ then $f(b) > f(a)$ thus since f is non-decreasing it cannot be $b \leq a$. For each i by Lemma 1.9 there is $f_i \in S$ such that $f_i(A_i) \subset [0, 1/n)$, $f_i(B_i) \subset [1, 1 + 1/n)$, $f_i(X) \subset [0, 1 + 1/n)$. Let $g \in S$ be a function that approximates the constant m from above,

$m \leq g < m + 1/n$. (this g exists because, for $m = 0$ it was shown previously, while if $m > 0$ then, $f > 0$ over X and so $N_S = \emptyset$ and every constant is approximable from above as observed previously.) Let us consider the function $F = g + \frac{1}{n} \sum_{i=0}^{n-1} f_i$ which belongs to S .

Let $x \in X$. There are two cases to consider, depending on the value of $f(x)$.

Suppose that for some $j \in \{0, 1, \dots, n-1\}$, $m + \frac{j}{n} < f(x) < m + \frac{j+1}{n}$. We have $x \in A_i$ for $j+1 \leq i \leq n-1$ (which implies $f_i(x) \in [0, 1/n)$); we have $x \in B_i$ for $i \leq j-1$ (which implies $f_i(x) \in [1, 1 + 1/n)$), and we have $f_j(x) \in [0, 1 + 1/n)$ (no restriction). Hence with obvious meaning of the notation

$$\begin{aligned} F(x) &= g(x) + \frac{1}{n} \sum_{i=0}^{n-1} f_i(x) \\ &\in [m, m + \frac{1}{n}) + \frac{1}{n} \left\{ (n-1-j)[0, \frac{1}{n}) + [0, 1 + \frac{1}{n}) + j[1, 1 + \frac{1}{n}) \right\} \\ &\subset [m + \frac{j}{n}, m + \frac{1}{n} + \frac{1}{n^2}(n-1-j) + \frac{1}{n}(1 + \frac{1}{n})(1+j)) \\ &\subset [m + \frac{j}{n}, m + \frac{j+1}{n} + \frac{2}{n}) \end{aligned}$$

which, since $f(x)$ belongs to this same interval, proves that $|f(x) - F(x)| < 3/n$.

Suppose that for some $j \in \{0, 1, \dots, n\}$, $f(x) = m + \frac{j}{n}$. We have $x \in A_i$ for $j \leq i \leq n-1$ and $x \in B_i$ for $i \leq j-1$, thus

$$\begin{aligned} F(x) &= g(x) + \frac{1}{n} \sum_{i=0}^{n-1} f_i(x) \\ &\in [m, m + \frac{1}{n}) + \frac{1}{n} \left\{ (n-j)[0, \frac{1}{n}) + j[1, 1 + \frac{1}{n}) \right\} \\ &\subset [m + \frac{j}{n}, m + \frac{1}{n} + \frac{1}{n^2}(n-j) + \frac{1}{n}(1 + \frac{1}{n})j) \\ &\subset [m + \frac{j}{n}, m + \frac{j+1}{n} + \frac{1}{n}) \end{aligned}$$

which, since $f(x)$ belongs to this same interval, proves that $|f(x) - F(x)| < 2/n < 3/n$. $\square$

As a specialization we have the following result that selects some notable functions for ϕ . Moreover, it builds the closed order directly from S , more in the spirit of Bonsall's paper and of algebraic approaches.

Corollary 1.10. *Let $(X, \mathcal{T})$ be a compact space. Let S be a non-empty subset of the set $C(X, \mathbb{R})$. Suppose*

(a) S is a convex cone: $S + S \subset S$, $\lambda S \subset S$ for every $\lambda > 0$,

(b) One of the following non-excluding possibilities:

1. S consists of non-negative functions and if $f \in S$ then $\chi(f) := \frac{2f^2}{1+f^2}$ belongs to S ,
2. If $f \in S$ then $\gamma(f) := \left(\frac{2f}{1+f}\right)^2$ is well defined and belongs to S .
3. S consists of positive functions, its contains the positive constants and if $f \in S$ then both f^2 and $-1/f$ belong to S .

Then $\bar{S} = C^+(X, \leq_S, \mathbb{R}, N_S)$. In particular, if additionally

(c) $N_S = \emptyset$, that is, for every $x \in X$ there is some $f \in S$ such that $f(x) > 0$ (this follows if S contains the identity, e.g. in case b3),

then $\bar{S} = C^+(X, \leq_S, \mathbb{R})$.

Note that (b)3 implies $N_S = \emptyset$ because $1 \in S$.

Proof. It is clear that $\leq_S$ is a closed preorder so $(X, \mathcal{T}, \leq_S)$ is a compact preordered space. As mentioned in Remark 1.7 if there where $f \in S$ such that for some x , $f(x) < 0$ then for some positive constant $\lambda > 0$, $f' = \lambda f$, would be such that $f'(x) = -1$, but then $\gamma(f)$ would not be well defined. We conclude that in case 2 the set S consists of non-negative functions. We just need to prove $3 \Rightarrow 1$. Since $\frac{f}{1+f} = 1 - (1+f)^{-1}$ and $1+f \in S$, we have $-(1+f)^{-1} \in S$ and so $\frac{f}{1+f} \in S$. Due to the invariance under squaring $\chi(f) \in S$. The functions in S are positive thus they are non-negative and so 1 is proved. $\square$

We have already observed (Remark 1.6) that Theorem 1.3 implies Bon-sall's. Let us prove that Besnard's theorem also follows.

Proposition 1.11. *Theorem 1.3 implies Besnard's Theorem 1.2.*

Proof. Under the assumption of Besnard's theorem, consider the projections $p_+(x) = \max\{x, 0\}$ and $p_-(x) = \min\{x, 0\}$, and define $S^+ = p_+(S)$, $S^- = p_-(S)$. The functions $p_+, p_- : \mathbb{R} \rightarrow \mathbb{R}$ are piecewise linear non-decreasing functions, thus $p_+(f), p_-(f) \in S$ for $f \in S$. The set S^+ is closed under sum, because of $0 \leq p_+(f) + p_+(g) \in S$, thus $p_+(f) + p_+(g) = p_+(p_+(f) + p_+(g)) \in S_+$. Moreover, for $a \geq 0$, $p_+(af) = ap_+(f)$ thus S_+ is a convex cone. But we

can find a piecewise linear function $\phi : [0, \infty) \rightarrow [0, \infty)$ which respects the conditions of Theorem 1.3, and extending it to a function $\check{\phi}$ equal to zero for $x < 0$, we have that $\check{\phi}$ is piecewise linear and so for $f \in S$, $\phi(p_+(f)) = \check{\phi}(f) = p_+(\check{\phi}(f)) \in S^+$ which means that S^+ is invariant under ϕ . Thus, by Theorem 1.3 $\overline{S^+} = C^+(X, \leq, \mathbb{R})$. Now consider S^- , then $-S^- \subset C^+(X, \geq, \mathbb{R})$ where $\geq$ is the transposed order. Arguing as above we get $\overline{-S^-} = C^+(X, \geq, \mathbb{R})$ and so $\overline{S^-} = -\overline{-S^-} = -C^+(X, \geq, \mathbb{R}) = C^-(X, \leq, \mathbb{R})$. Finally,

$$\overline{S} \supset \overline{S^+} + \overline{S^-} = C^+(X, \leq, \mathbb{R}) + C^-(X, \leq, \mathbb{R}) = C(X, \leq, \mathbb{R}),$$

The other inclusion being clear. $\square$

The following result does not follow from Bonsall's theorem since he requires that S is uniformly closed from the outset.

Theorem 1.12. *Let $\mathbb{R}^n$ be endowed with the Cartesian product order: $x \leq y$ iff $x_k \leq y_k$ for $k = 1, \dots, n$. Let $X \subset \mathbb{R}^n$ be a compact subset endowed with the induced topology and order (denoted in the same way). Let S be the family of positive isotone rational functions $f := p(x)/q(x)$ on X such that (i) the coefficients of p and q are non-negative, (ii) $\deg(p) = \deg(q)$. Then $\overline{S} = C^+(X, \leq, \mathbb{R})$.*

Conditions (ii) can be dropped but its presence makes the theorem somewhat stronger.

Remark 1.13. For $n = 1$, the theorem states that every non-negative continuous isotone function on $[a, b] \subset \mathbb{R}$ can be uniformly approximated on $[a, b]$ by rational functions $p(x)/q(x)$ that are positive, isotone and with non-negative coefficients.

Proof. To start with $\leq \subset \leq_S$ because all the elements of S are isotone. Let $x \leq_S y$, we want to prove that $x \leq y$ and hence $\leq = \leq_S$ due to the arbitrariness of x and y . If not $y_k < x_k$ for some k . Now consider the function $g : X \rightarrow \mathbb{R}$, $z \mapsto g(z) = \frac{z_k + c}{z_k + c + 1}$, $c > 0$. Since z_k is continuous over X it reaches a minimum and so $c > 0$ can be chosen so that $z_k + c$ is positive on X . With this choice $g \in S$ thus $x \leq_S y$ implies $g(x) \leq g(y)$ which reads $x_k \leq y_k$, a contradiction. We proved $\leq = \leq_S$.

Let us prove that S satisfies (a), (b)1, (c) of Cor. 1.10. Condition (a) is satisfied because the product or sum of two polynomials with non-negative coefficients gives a polynomial with non-negative coefficients. The condition

on the degrees is easily proved noting that $\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + q_1 p_2}{q_1 q_2}$ and $\deg(p_1) = \deg(q_1)$, $\deg(p_2) = \deg(q_2)$ so both terms in the numerator have the same degree of the denominator. The degree cannot get reduced by the sum on the numerator because, even if the highest degree terms of two addenda $p_1 q_2$, $q_1 p_2$ involve the same variables, the sum cannot vanish as the coefficients of the polynomials are non-negative.

Condition (b)1 follows from the fact that the composition of isotone functions is isotone so if $f \in S$, $\chi(f) = 2f^2/(1+f^2)$ is also isotone. Note that since f is positive $\chi(f)$ is positive, and also if $f = p/q$ since $\chi(f) = \frac{2p^2}{p^2+q^2}$ it is also rational with non-negative coefficients. Moreover since $\deg(p) = \deg(q)$, $\deg(p^2) = \deg(p^2 + q^2)$, by the same argument on the non-negativity of the coefficients used above. In conclusion, $\chi(f) \in S$. Condition (c), $N_S = \emptyset$, follows because $1 \in S$.

We conclude from Cor. 1.10 that $\bar{S} = C^+(X, \leq, \mathbb{R})$. $\square$

In the literature the subject of approximation of continuous isotone functions by isotone polynomials is well explored [9] [7, Chap. 10, Thm. 3.3(i)-(ii)]. In those studies the polynomial can have negative coefficients while in our version we use rational functions but restrict the coefficients to the non-negative real numbers.

Theorem 1.14. *Let $X \subset \mathbb{R}^n$ be a compact subset endowed with the induced topology. Let S be the family of positive rational functions $f := p(x)/q(x)$ on X such that the coefficients of p and q are non-negative. Then $\bar{S} = C^+(X, \mathbb{R})$.*

Proof. Let $\leq$ be the discrete order on X . It is closed because the induced topology on X is Hausdorff. Every function is isotone with respect to the discrete order, thus $\leq \subset \leq_S$. Let $x \leq_S y$, we want to prove that $x = y$ and hence $\leq = \leq_S$ due to the arbitrariness of x and y . If not $x_k \neq y_k$ for some k . Now, consider the functions $g, h : X \rightarrow \mathbb{R}$, $z \mapsto g(z) = \frac{z_k + c}{z_k + c + 1}$, $c > 0$ and $z \mapsto h(z) = \frac{1}{z_k + c}$. Since z_k is continuous over X it reaches a minimum and so $c > 0$ can be chosen so that $z_k + c$ is positive on X . With this choice $g, h \in S$ thus $x \leq_S y$ implies $g(x) \leq g(y)$ and $h(x) \leq h(y)$ which read $x_k \leq y_k$ and $y_k \leq x_k$, respectively, hence $x_k = y_k$, a contradiction. We proved $\leq = \leq_S$.

Let us prove that S satisfies (a), (b)1, (c) of Cor. 1.10. Condition (a) is satisfied because the product or sum of two polynomials with non-negative coefficients gives a polynomial with non-negative coefficients.

Condition (b)1 follows since if f is positive $\chi(f)$ is positive, and also if $f = p/q$ since $\chi(f) = \frac{2p^2}{p^2+q^2}$ it is also rational with non-negative coefficients. In conclusion, $\chi(f) \in S$. Condition (c), $N_S = \emptyset$, follows because $1 \in S$.

We conclude from Cor. 1.10 that $\bar{S} = C^+(X, \mathbb{R})$. $\square$

2. Algebraic characterization of $C^+(X, \leq, \mathbb{R})$

On a compact preordered space $(X, \mathcal{T}, \leq)$ the family of differences of continuous isotone functions

$$C_{BV}(X, \leq, \mathbb{R}) := \{f - g, f, g \in C(X, \leq, \mathbb{R})\} = C(X, \leq, \mathbb{R}) - C(X, \leq, \mathbb{R})$$

coincides with the family of differences of non-negative continuous isotone functions $C_{BV}(X, \leq, \mathbb{R}) = C^+(X, \leq, \mathbb{R}) - C^+(X, \leq, \mathbb{R})$, indeed it is sufficient to add to both f and g the same sufficiently large constant, to make them non-negative.

Proposition 2.1. *Let $(X, \mathcal{T}, \leq)$ be a compact ordered space. The set $C_{BV}(X, \leq, \mathbb{R})$ is an algebra. Moreover, we have the identity $\overline{C_{BV}(X, \leq, \mathbb{R})} = C(X, \mathbb{R})$, namely every continuous function can be uniformly approximated by differences of continuous isotone functions.*

The proof follows from the fact that the product of two non-negative continuous isotone functions is again non-negative continuous and isotone,

$$C^+(X, \leq, \mathbb{R}) C^+(X, \leq, \mathbb{R}) \subset C^+(X, \leq, \mathbb{R}).$$

Proof. The set $C_{BV}(x, \leq, \mathbb{R})$ is an algebra, indeed if $f_1, f_2, g_1, g_2 \in C^+(X, \leq, \mathbb{R})$

$$(f_1 - g_1)(f_2 - g_2) = (f_2 f_2 + g_2 g_2) - (g_1 f_2 + f_1 g_2)$$

where the functions in parenthesis on the right-hand side belong to $C^+(X, \leq, \mathbb{R})$. Since $C(X, \leq, \mathbb{R})$ includes the constants and separates points (by complete order regularity [14]), so does $C_{BV}(x, \leq, \mathbb{R})$. The claim now follows from Stone Theorem [15] [16]. $\square$

The following result establishes an algebraic characterization of $C^+(X, \leq, \mathbb{R})$. A result of the same type but demanding stability under the whole family of non-decreasing continuous functions and focused on $C(X, \leq, \mathbb{R})$ can be found in Besnard [3, Thm. 4] [4, Thm. 1] (improving a previous

approach by the same author [1] that also demanded for stability under lattice operations). The advantage of working with $C^+(X, \leq, \mathbb{R})$ instead of $C(X, \leq, \mathbb{R})$ is that the former set constitutes a semi-algebra.

An involution on an algebra A over $\mathbb{C}$ is a conjugate linear map $f \rightarrow f^*$, that is $(f + ag)^* = f^* + \bar{a}g^*$ for $a \in \mathbb{C}$, such that $f^{**} = f$ and $(fg)^* = g^*f^*$. The pair $(A, *)$ is called a $*$ -algebra. An element is *self-adjoint* or *hermitian* if $f = f^*$. For each $f \in A$ there exist unique hermitian elements $g, h \in A$ such that $f = g + ih$.

The element $f \in A$ is *normal* if $ff^* = f^*f$ in which case it generates a commutative $*$ -algebra. If A is unital $1 = 1^*$ because $1^* = 1(1^*) = (1(1^*))^* = ((1^*))^* = 1$. A homomorphism of $*$ -algebras $\varphi : A \rightarrow B$, that preserves adjoints $\varphi(f^*) = \varphi(f)^*$ is a $*$ -homomorphism. If it is bijective, it is a $*$ -isomorphism.

A Banach $*$ -algebra B is a $*$ -algebra with a complete submultiplicative norm such that $\|f^*\| = \|f\|$. If it has unit such that $\|e\| = 1$ we call it a unital Banach $*$ -algebra B . A C^* -algebra is a Banach $*$ -algebra B such that

$$\|f^*f\| = \|f\|^2.$$

A bijective $*$ -homomorphism $\varphi : A \rightarrow B$ such that $\|\varphi(f)\| = \|f\|$ is a $*$ -isometry.

Theorem 2.2. *Let B be a commutative unital complex C^* -algebra. Let H be the subset of hermitian elements, and assume there is a subset $C \subset B$ with the following properties:*

- (i) $C + C \subset C$, and $aC \subset C$ for every $a > 0$,
- (ii) C is closed,
- (iii) $\overline{C - C} = H$,
- (iv) We have one of the following
 1. C consists of non-negative elements (non-negative spectrum) and if $f \in C$ then $\chi(f) := \frac{2f^2}{1+f^2}$ belongs to C ,
 2. If $f \in C$ then $\gamma(f) := \left(\frac{2f}{1+f}\right)^2$ is well defined and belongs to C .

Let X be the set of characters. Endow X with a the weak*-topology $\mathcal{T}$ and with the preorder $x \leq y$ if $x(f) \leq y(f)$ for every $f \in C$. Then $(X, \mathcal{T}, \leq)$

is a compact ordered space. Moreover, the Gelfand map $\mathcal{G} : B \rightarrow C(X, \mathbb{C})$, $f \mapsto \hat{f}$, $\hat{f}(x) := x(f)$, is a bijective unit-preserving $*$ -isometry which maps bijectively C to $C^+(X, \leq, \mathbb{R})$.

Observe that properties (iv)1 and (iv)2 are satisfied if the maps $f \mapsto f^2$ and $f \mapsto \frac{1}{1+f}$ send C to itself, where the former condition $C^2 \subset C$ would follow if C were a semi-algebra. Note also that we do not assume that the identity is in C , that C is a semi-algebra, or that it is stable under lattice operations. However, these properties are recovered from the proved isometry and the properties of $C^+(X, \leq, \mathbb{R})$.

Remark 2.3. Of course, a more general version of the theorem holds with (iv) involving stability under a function ϕ with the properties of Theorem 1.3. Here we just wanted to give a more concrete realization of the result.

Proof. From Banach-Alaoglu the topology $\mathcal{T}$ of X is Hausdorff and compact. Since for $f \in C$ the function $\hat{f} : X \rightarrow \mathbb{R}$ is real [13, Thm. 2.1.9] and continuous the preorder $\leq$ is closed. Suppose $x \leq y$ and $y \leq x$, then for every $f \in C$, $x(f) \leq y(f)$ and $y(f) \leq x(f)$, that is $(y - x)|_C = 0$. But since $y - x : B \rightarrow \mathbb{C}$ is real-linear $(y - x)|_{C-C} = 0$ and since it is continuous, by (iii), $(y - x)|_H = 0$. Again by the complex-linearity $y - x = 0$. This proves that $\leq$ is an order and $(X, \mathcal{T}, \leq)$ is a compact ordered space.

By the Gelfand-Naimark theorem there is a bijective $*$ -isometry $\mathcal{G} : B \rightarrow C(X, \mathbb{C})$, thus we need only to show that $\mathcal{G}(C)$ coincides with $C^+(X, \leq, \mathbb{R})$. Since $\mathcal{G}$ is linear if $\hat{f}, \hat{g} \in \mathcal{G}(C)$, $a > 0$, by (i) $a\hat{f} \in \mathcal{G}(C)$ and $\hat{f} + \hat{g} \in \mathcal{G}(C)$.

Since $\mathcal{G}$ is an isometry, if $\hat{f}_k \rightarrow f$ uniformly in $C(X, \mathbb{R})$, then regarding $\hat{f}_k, f$ as functions in $C(X, \mathbb{C})$ we have $\hat{f}_k \rightarrow f = \hat{h}$ uniformly, where f belonging to $C(X, \mathbb{C})$ is really the image of some $h \in B$. Since $\mathcal{G}$ is an isometry $\hat{f}_k \rightarrow \hat{h}$, and since by (ii) C is closed, $h \in C$, which implies $f \in \mathcal{G}(C)$, that is $\mathcal{G}(C)$ is closed.

Let $x \in X$ and suppose that $\hat{f}(x) = 0$ for every $\hat{f} \in \mathcal{G}(C)$, then $x(f) = 0$ for every $f \in C$, hence $x(g) = 0$ for every $g \in C - C$ and finally, since x is continuous, $x|_H = 0$ by (iii), $H = \overline{C - C}$, and finally $x = 0$ by the fact that every element of B can be written in the form $b = h_1 + ih_2$ for $h_1, h_2 \in H$. This is a contradiction since characters are non-zero by definition, thus we conclude that $N_{\mathcal{G}(C)} = \emptyset$.

Condition (iv)2 implies that the spectrum of any $f \in C$ is non-negative, for -1 does not belong to the spectrum of $a^{-1}f$ for $a > 0$ and so $-a$ does not belong to the spectrum of f .

But Gelfand calculus shows that for $f \in \mathcal{G}(C)$, $\mathcal{G}(\chi(f)) = \chi(\hat{f})$ in case 1, and $\mathcal{G}(\gamma(f)) = \gamma(\hat{f})$ in case 2, thus $\mathcal{G}(C)$ is in the domain of the operation and is preserved by it in each case. The assumptions of Thm. 1.3 are satisfied, and using the closure of $\mathcal{G}(C)$ we get $\mathcal{G}(C) = C^+(X, \leq, \mathbb{R})$. $\square$

Remark 2.4. Properties (i) – (iv) do not really require that B is commutative thanks to functional calculus on C^* -algebras. They can be used to define a special type of cone on B . The theorem then implies that the category of commutative unital complex C^* -algebra endowed with such a cone is equivalent to that of compact ordered spaces (see Thm. 2.5).

Theorem 2.5. *There is a one-to-one correspondence (a duality) between:*

1. *Compact ordered spaces $(X, \mathcal{T}, \leq)$, and*
2. *Pairs (B, C) , where B is a commutative unital complex C^* -algebra and C is a subset of B satisfying the conditions (i)-(iv) of Theorem 2.2.*

This correspondence is implemented by the following mutually inverse structure-preserving maps:

Map I (Algebra $\rightarrow$ Space): Given (B, C) , construct the space $(X, \mathcal{T}, \leq)$ where X is the Gelfand spectrum (set of characters) of B , $\mathcal{T}$ is the weak*-topology, and the order $\leq$ is defined by $x \leq y \iff x(f) \leq y(f)$ for all $f \in C$.

Map II (Space $\rightarrow$ Algebra): Given $(X, \mathcal{T}, \leq)$, construct the pair (B, C) where $B = C(X, \mathbb{C})$ (the continuous complex functions) and $C = C^+(X, \leq, \mathbb{R})$ (the non-negative continuous isotone real functions).

Proof. We must show that Map I and Map II are well-defined and that they are mutually inverse (up to canonical isomorphism).

1. Map I is Well-Defined.

Given a pair (B, C) , Theorem 2.2 proved that the constructed triple $(X, \mathcal{T}, \leq)$ is indeed a compact ordered space. Furthermore, the Gelfand map $\mathcal{G}$ is a unit-preserving *-isometry that establishes a canonical isomorphism:

$$\mathcal{G} : (B, C) \rightarrow (C(X, \mathbb{C}), C^+(X, \leq, \mathbb{R})).$$

2. Map II is Well-Defined.

Given a compact ordered space $(X, \mathcal{T}, \leq)$, we must show that the pair (B, C) , where $B = C(X, \mathbb{C})$ and $C = C^+(X, \leq, \mathbb{R})$, satisfies the required conditions (i)-(iv) of Theorem 2.2.

- (i) Convex Cone: If $f, g \in C$, then $f + g \in C$ and $\lambda f \in C$ for $\lambda > 0$, by the definition of non-negative isotone functions.
- (ii) Closed: $C \subset C(X, \mathbb{R})$ is obtained imposing to its elements the condition $f \geq 0$ and for each $x, y \in X$, $x \leq y$, the conditions $f(x) \leq f(y)$. These conditions are uniformly closed so is their intersection C . Since C is a uniformly closed subset of $C(X, \mathbb{R})$, it is also a uniformly closed subset of $C(X, \mathbb{C})$.
- (iii) $\overline{C - C} = H$: The set of hermitian elements H in $B = C(X, \mathbb{C})$ is $C(X, \mathbb{R})$. By Prop. 2.1 the uniform closure of the set of differences of continuous isotone functions equals $C(X, \mathbb{R})$. Thus, $\overline{C - C} = C(X, \mathbb{R}) = H$.
- (iv) ϕ -Invariance: For any continuous non-decreasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ (satisfying the properties of Theorem 1.3), if $f \in C$, then $\phi(f)$ is also continuous, non-negative, and isotone. Since conditions (iv)1 and (iv)2 rely on specific functions χ and γ which are of this type, C is closed under these operations.

Thus, Map II is well-defined.

3. Mutually Inverse Property.

We verify the composition of the maps.

Map II $\circ$ Map I: Starting with (B, C) and applying Map I followed by Map II results in the structure $(C(X, \mathbb{C}), C^+(X, \leq, \mathbb{R}))$. By the result of Theorem 2.2, this resulting structure is canonically $*$ -isomorphic to the starting structure (B, C) .

Map I $\circ$ Map II: Starting with $(X, \mathcal{T}, \leq)$, Map II yields

$$(B', C') = (C(X, \mathbb{C}), C^+(X, \leq, \mathbb{R})).$$

Map I takes the Gelfand spectrum of B' . By the Gelfand-Naimark Theorem, the spectrum is canonically homeomorphic to X . Furthermore, the order generated by C' is precisely the original order $\leq$ of the compact ordered space X (by complete order regularity of the latter). Thus, the resulting structure is identical to the starting structure: $(X, \mathcal{T}, \leq)$.

Since the mappings are mutually inverse (up to canonical isomorphism), the correspondence is established. $\square$

3. Bernstein's type realization of the rational approximation

In this section we provide an explicit rational approximation of continuous isotone functions over $[0, b] \subset \mathbb{R}$ whose existence was established in Theorem 1.12.

Theorem 3.1. *Let $b > 0$ and let $f : [0, b] \rightarrow \mathbb{R}$ be a non-negative, non-decreasing continuous function. Extend f to $[0, \infty)$ by setting $f(x) = f(b)$ for $x > b$. For each integer $n \geq 1$, define the rational function*

$$R_n(f; x) = \sum_{k=0}^n f\left(\frac{bk}{n+1-k}\right) \binom{n}{k} \frac{b^{n-k} x^k}{(b+x)^n}, \quad x \in [0, b].$$

Then:

1. $R_n(f; x)$ is continuous and the ratio of polynomials with non-negative coefficients.
2. $R_n(f; x)$ is non-decreasing on $[0, b]$.
3. As $n \rightarrow \infty$, $R_n(f; x)$ converges uniformly to $f(x)$ on $[0, b]$.

Proof. Part 1: Form of the function

We write the expression with a common denominator:

$$R_n(f; x) = \frac{1}{(b+x)^n} \sum_{k=0}^n f\left(\frac{bk}{n+1-k}\right) \binom{n}{k} b^{n-k} x^k.$$

Let $P_n(x)$ denote the summation in the numerator. Since f is non-negative, $b > 0$, and the binomial coefficients are positive, every coefficient in the polynomial $P_n(x)$ is non-negative. The denominator is $(b+x)^n = \sum_{j=0}^n \binom{n}{j} b^{n-j} x^j$, which also has strictly positive coefficients. Thus, $R_n(f; x)$ is a rational function with non-negative coefficients.

Part 2: Monotonicity

Let $p = \frac{x}{b+x}$. Since $x \in [0, b]$, we have $p \in [0, 1/2]$. The function $x \mapsto p(x)$ is strictly increasing. Let X be a binomial random variable with parameters n and p , i.e., $X \sim \text{Bin}(n, p)$. We can rewrite the sum as an expectation:

$$R_n(f; x) = \mathbb{E}[f(g_n(X))], \quad \text{where } g_n(k) = \frac{bk}{n+1-k}.$$

First, observe that $g_n(k)$ is a strictly increasing function of k for $0 \leq k \leq n$. Since f is non-decreasing, the composite function $h(k) = f(g_n(k))$ is non-decreasing in k .

Second, the family of binomial distributions is stochastically increasing in the parameter p . That is, if $p_1 < p_2$, then X_{p_1} is stochastically smaller than X_{p_2} , implying $\mathbb{E}[\phi(X_{p_1})] \leq \mathbb{E}[\phi(X_{p_2})]$ for any non-decreasing function ϕ .

Since x increases p , and increasing p increases the expectation of the non-decreasing function $f(g_n(X))$, it follows that $R_n(f; x)$ is non-decreasing on $[0, b]$.

Part 3: Uniform Convergence

Since f is continuous on the compact interval $[0, b]$ (and constant thereafter), it is uniformly continuous on $[0, \infty)$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon/2$ whenever $|u - v| < \delta$. Let $M = \sup |f(x)| = f(b)$.

Let $\hat{p} = X/n$ be the sample proportion. Note that $\mathbb{E}[\hat{p}] = p$ and $\text{Var}(\hat{p}) = \frac{p(1-p)}{n}$. We analyze the argument of f inside the expectation. Let $\delta_n = 1/n$. The argument is:

$$Y_n = \frac{bk}{n+1-k} = \frac{b(k/n)}{1 - (k/n) + (1/n)} = \frac{b\hat{p}}{1 - \hat{p} + \delta_n}.$$

Define the mapping $\psi_n(t) = \frac{bt}{1 - t + \delta_n}$. We compare $Y_n = \psi_n(\hat{p})$ to x . Recall that $x = \frac{bp}{1-p}$. We define the intermediate value $x_n = \psi_n(p) = \frac{bp}{1-p + \delta_n}$.

Step 3a: Deterministic error bound. First, we bound $|x_n - x|$:

$$|x_n - x| = \left| \frac{bp}{1-p + \delta_n} - \frac{bp}{1-p} \right| = \frac{bp\delta_n}{(1-p)(1-p + \delta_n)}.$$

Since $x \in [0, b]$, we have $p \in [0, 1/2]$. Thus $1-p \geq 1/2$ and $1-p + \delta_n \geq 1/2$.

$$|x_n - x| \leq \frac{2b}{n}.$$

For sufficiently large n , $|x_n - x| < \delta/2$.

Step 3b: Stochastic error bound. We now bound $|\psi_n(\hat{p}) - \psi_n(p)|$. The derivative of ψ_n is:

$$\psi'_n(t) = \frac{b(1 + \delta_n)}{(1 - t + \delta_n)^2}.$$

This derivative blows up near $t = 1$, so we cannot use a global Lipschitz constant. However, we only care about $p \in [0, 1/2]$. Consider the interval $S = [0, 3/4]$. If $t \in S$, then $1 - t + \delta_n \geq 1/4$.

$$\sup_{t \in S} |\psi'_n(t)| \leq \frac{b(1+1)}{(1/4)^2} = 32b.$$

Let $K = 32b$. By the Mean Value Theorem, if $\hat{p} \in S$ and $p \in [0, 1/2]$, then:

$$|\psi_n(\hat{p}) - \psi_n(p)| \leq K|\hat{p} - p|.$$

Step 3c: Splitting the expectation. We want to bound $|R_n(f; x) - f(x)| \leq \mathbb{E}[|f(Y_n) - f(x)|]$. Define the "good" set $A = \{|\hat{p} - p| < \eta\}$ where $\eta = \min(\frac{1}{4}, \frac{\delta}{2K})$. Note that if we are in set A , since $p \leq 1/2$, we have $\hat{p} < 3/4$, so we are in the region S where the Lipschitz bound holds.

Split the expectation:

$$\mathbb{E}[|f(Y_n) - f(x)|] = \mathbb{E}[|f(Y_n) - f(x)|\mathbb{I}_A] + \mathbb{E}[|f(Y_n) - f(x)|\mathbb{I}_{A^c}].$$

Case 1 (In A):

$$|Y_n - x| \leq |Y_n - x_n| + |x_n - x| \leq K|\hat{p} - p| + \frac{2b}{n}.$$

Inside A , $K|\hat{p} - p| < K\eta \leq \delta/2$. Choose n large enough so $2b/n < \delta/2$. Then $|Y_n - x| < \delta$, which implies $|f(Y_n) - f(x)| < \varepsilon/2$.

Case 2 (In A^c): The integrand is bounded by $2M$. By Chebyshev's inequality:

$$\mathbb{P}(A^c) = \mathbb{P}(|\hat{p} - p| \geq \eta) \leq \frac{p(1-p)}{n\eta^2} \leq \frac{1}{4n\eta^2}.$$

This converges to 0 uniformly for all $p \in [0, 1/2]$.

Combining the terms:

$$|R_n(f; x) - f(x)| \leq \frac{\varepsilon}{2} + 2M \left(\frac{1}{4n\eta^2} \right).$$

For sufficiently large n , the second term is less than $\varepsilon/2$. Thus $|R_n(f; x) - f(x)| < \varepsilon$ for all $x \in [0, b]$. $\square$

4. Conclusions

We have established a general approximation theorem for isotone functions on a compact preordered space (Thm. 1.3). The theorem employs a family of functions S , where the core requirement is its invariance under a single transformation ϕ with specific fixed-point properties. This minimalist invariance condition provides considerable flexibility, allowing ϕ to be chosen as an algebraic expression.

As a concrete application, we obtained the uniform approximation on compact intervals of continuous isotone functions by isotone rational functions with non-negative coefficients (Thm. 1.12), complete with an explicit realization of Bernstein type (Sec. 3).

Ultimately, this framework offers a new means to characterize function spaces that can be interpreted as families of continuous isotone functions over a compact set X endowed with a closed order (Sec. 2).

Our primary motivation comes from spacetime geometry as we intend to apply this algebraic characterization to recover the causal structure of spacetime. In fact, we showed in previous work that every stably causal spacetime (a time-oriented Lorentzian manifold) is quasi-uniformizable and can thus be densely embedded into a compact ordered space via Nachbin's compactification [10, Thm. 4.15] [12]. This connection opens a potential pathway toward unifying General Relativity with Quantum Mechanics, a direction of investigation we plan to explore in future work.

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