

Consistent Truncations and Generalised Geometry: Scanning through Dimensions and Supersymmetry

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ABSTRACT: We study consistent truncations in the framework of Exceptional Generalised Geometry. We classify the 4-dimensional gauged supergravities that can be obtained as a consistent truncation of 10/11-dimensional supergravity. Any truncation is associated to a (generalised) G_S -structure with singlet intrinsic torsion. We give the full classification for all truncations associated to continuous structure groups and we discuss a few examples with discrete ones. We recover gauged supergravities corresponding to known truncations as well as others for which explicit truncations are still to be constructed. We also summarise similar results obtained in the literature for truncations to $d = 5, 6, 7$ dimensions and we complete them, when needed.

Contents

1	Introduction	2
2	Generalised G-structures and consistent truncations	4
3	Consistent truncations to four dimensions	8
3.1	$\mathcal{N} = 8$ supergravity	11
3.2	$\mathcal{N} = 6$ supergravity	13
3.3	$\mathcal{N} = 5$ supergravity	14
3.4	$\mathcal{N} = 4$ supergravity	15
3.5	$\mathcal{N} = 3$ supergravity	17
3.6	$\mathcal{N} = 2$ supergravity	18
3.6.1	Truncations with only vector multiplets	22
3.6.2	Truncations with only hypermultiplets	25
3.6.3	Truncations with vector and hypermultiplets	27
4	Scanning through supersymmetry and field content	30
4.1	Discrete structures	32
5	Review of consistent truncations to 5, 6 and 7 dimensions	35
5.1	Truncations to 5 dimensions	36
5.1.1	$\mathcal{N} = 8$ supergravity	36
5.1.2	$\mathcal{N} = 6$ supergravity	37
5.1.3	$\mathcal{N} = 4$ supergravity	38
5.1.4	$\mathcal{N} = 2$ supergravity	39
5.2	Truncations to 6 dimensions	43
5.2.1	Maximal supergravity	43
5.2.2	$\mathcal{N} = (2, 1)$ supergravity	45
5.2.3	Half-maximal supergravity	45
5.2.4	Miminal supergravity: $\mathcal{N} = (1, 0)$	48
5.3	Truncations to 7 dimensions	53
5.3.1	Maximal supergravity	54
5.3.2	Half-maximal supergravity	55
6	Conclusions	56
A	Details on $\mathbf{E}_{7(7)}$	59
B	Example: $\mathcal{N} = 4$ truncations	65

1 Introduction

A question that naturally arises in string theory is how to construct low-dimensional effective actions. This is of course crucial if we want to construct string models to be confronted with phenomenological observations, but it has also important applications in other contexts such as understanding the web of supergravity theories.

Supergravity theories have been constructed in any dimension $D = 2, \dots, 11$. While the theories in 11 and 10 dimensions are very constrained and all have an interpretation as low-energy effective theories of M-theory and string theory, respectively, when going down in dimension there is a multitude of supergravities, with or without gauge symmetries, and it is an open question whether they have a string theory origin or not.

A way to address this question is in terms of consistent truncations. 11/10-dimensional supergravity on a background of the form

$$X_{10/11} = X_D \times M, \tag{1.1}$$

with M a compact manifold of dimension $d = 11/10 - D$, can be seen as a D -dimensional theory with an infinite number of fields organised into representations of the symmetry group of the internal manifold. A consistent truncation is a procedure to truncate the theory to a finite set of modes, in such a way that all truncated modes decouple from the lower-dimensional equations of motion and that no dependence on the internal manifold is left. If the truncation is consistent any solution of the truncated theory can be uplifted to a solution of the 11/10-dimensional theory.

The main difficulty in constructing consistent truncations is to find a principle for selecting the modes to be kept in the truncated theory. Starting from a series of explicit constructions (see for instance [1–7]) it has become clear that the formalism of G -structures provides a powerful tool to construct consistent truncations in a systematic way. The notion of G -structure can be extended to the framework of Exceptional Generalised Geometry and Exceptional Field Theory. These are reformulations of 11/10-dimensional supergravity that treat in a geometric way all the symmetries of the theory. Generalised G_S -structures provide a systematic and general derivation of consistent truncations [8]: any generalised G_S -structure on M with constant, singlet intrinsic torsion defines a consistent truncation of 10/11-dimensional supergravity.

This approach allows to give a unified description of truncations in different dimensions and with a different amount of supersymmetry. For instance, all maximally supersymmetric truncations are associated to generalised identity structures and therefore can be seen as generalised Scherk-Schwarz reductions [9–11]. In particular, all maximally supersymmetric truncations on spheres are unified in this class: truncations of 11-dimensional supergravity on S^7 and S^4 , IIB supergravity on S^5 [9, 11] and massive IIA on spheres [12, 13]. Similar classifications can be given for half-maximal and quarter-maximal truncations by considering larger generalised structure groups [8, 14–18].

The generalised G_S -structure fully determines the lower-dimensional truncated theory: field content, bosonic symmetries and supersymmetry.

In [18] these features were exploited to classify the 5-dimensional supergravity theories with $\mathcal{N} = 2$ supersymmetry that can be obtained as consistent truncations of 11/10-dimensional supergravity.

In this paper we will pursue this approach and apply it to truncations of 10/11-dimensional supergravity to 4 dimensions using the framework of $E_{7(7)}$ generalised geometry. The main idea is that this reduces to the study of all possible generalised G_S -structure compatible with a given amount of supersymmetry. We first solve the algebraic problem of finding all possible subgroups G_S of the generalised structure group $E_{7(7)}$. Then, for any G_S , from its embedding in $E_{7(7)}$, we derive the field content and symmetries of the 4-dimensional theory. In particular, the G_S -singlet components of the generalised intrinsic torsion will give in a straightforward way all the components of the embedding tensor of the reduced theory and, hence, all the possible gaugings.

We obtain a classification of 4-dimensional theories with $\mathcal{N} \geq 2$. As in [18], we find that these form a very reduced set of the supergravity theories that can be constructed directly in 4 dimensions.

Some of the theories we present in our classification have already been obtained as explicit truncations on specific manifolds, while some others have not. A priori, nothing guarantees that such truncations exist at all. This is because our algebraic analysis only gives the theories that could be a priori obtained. In this paper, we make hypothesis that the only non-zero components of the intrinsic torsion are G_S -singlets. However, in order to actually construct them, one has to find a compactification manifold with the appropriate G_S -structure and the geometrical features to give a constant, singlet intrinsic torsion.

Consistent truncations have many important applications in the context of the AdS/CFT correspondence, where most of the solutions dual to CFT's or deformations thereof have been first constructed in a lower-dimensional supergravity, which is a consistent truncation of 11/10-dimensional supergravity containing the fields relevant for the solution, and then uplifted to the full theory. The same is true for many black-hole solutions in string or M-theory. Of particular relevance are gauged supergravities in dimensions $4 \leq D \leq 7$. For $D > 4$ a systematic study of consistent truncations in the formalism of Exceptional Generalised Geometry and/or Exceptional field Theory can already be found in the literature (see for instance in [8, 14, 18, 19]). For completeness, we summarise such results and we present them in the language of Exceptional Generalised Geometry. In some cases we fill a few missing points in the classification.

Understanding lower-dimensional supergravity theories as consistent truncations of 11/10-dimensional supergravity has also a more fundamental meaning. Since supergravity theories are non-renormalisable, they make only sense as low-energy effective theories. 11/10-dimensional supergravities are special as their ultraviolet completion is provided by string or M-theory. Thus, understanding whether a lower-dimensional supergravity theory is a consistent truncation of 11/10-dimensional supergravity gives it a proper embedding in a consistent theory of quantum gravity.

The paper is organised as follows. In Section 2 we briefly describe the formalism of $E_{7(7)}$ generalised geometry, which is the relevant one for truncations to 4 dimensions, the

notions of generalised G_S -structure and how this gives the data of the truncated theory. Section 3 contains a summary of our results. We organise them according to the number of supersymmetries of the truncated theory and we also give a brief summary of the main features of the relevant $4d$ supergravities. In Section 4 we discuss in more detail the methods we used to achieve the classification and we illustrate them explicitly in the case of truncations with $\mathcal{N} = 4$ supersymmetry in Appendix B. In Section 5 we summarise the results in the literature about truncations to supergravities in other dimensions, and complete them with a few details, when needed. Our conventions for $E_{7(7)}$ and $SU(8)$ are given in Appendix A.

2 Generalised G-structures and consistent truncations

In this Section we recall the main notions of Exceptional Generalised Geometry (EGG) we will use in the rest of the paper. We follow the conventions of [20] and [18].

Exceptional Generalised Geometry (EGG) is a reformulation of supergravity which gives a unified geometrical description of the bosonic sector of 11/10-dimensional supergravity compactified on a d dimensional manifold M . This is achieved by replacing the tangent bundle TM with a generalised tangent bundle, defined as the extension of TM by appropriate exterior powers of the cotangent bundle, and whose structure group is the exceptional group $E_{d(d)}$.

In this paper we will mainly be interested in compactifications of 11/10-dimensional supergravity to 4 dimensions on backgrounds of the form

$$X_{10/11} = X_4 \times M, \quad (2.1)$$

where the internal manifold M has dimension $d = 7$ or $d = 6$, respectively. In this case the relevant exceptional group is $E_{7(7)} \times \mathbb{R}^+$. The fibres of the generalised tangent bundle transform in the $\mathbf{56}_1$ of $E_{7(7)} \times \mathbb{R}^+$, with the subscript denoting the $\mathbb{R}$ weight, and are called generalised vectors. One can also introduce the dual generalised tangent bundle E^* transforming in the $\mathbf{56}_{-1}$ of $E_{7(7)} \times \mathbb{R}^+$.

The ordinary notions of tensors, Lie derivative and covariant derivatives can be generalised to E [21–23]. Generalised tensors correspond to bundles whose fibres transform in given representations of the exceptional group and can be decomposed as local sums of powers of TM and T^*M .¹ There are four bundles of particular relevance for consistent truncations: the adjoint bundle, $\text{ad}F$, the bundles N and K , and the generalised metric G .

¹Consider, for instance, 11-dimensional supergravity. In this case M has dimension $d = 7$ and the generalised tangent bundle can be written as

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M). \quad (2.2)$$

Its sections $V \in \Gamma(E)$ are locally sums of a vector v , a real two-form ω , a real five-form σ and the (1,7) tensor τ on M : $V \sim v + \omega + \sigma + \tau$. The dual generalised vectors in E^* are obtained by raising indices with the inverse metric on M . Similar decompositions are given for the other tensor bundles.

The adjoint bundle has sections transforming in the adjoint representation, $\mathbf{133}_0 \oplus \mathbf{1}_0$, of $E_{7(7)} \times \mathbb{R}^+$, where $\mathbf{1}_0$ denotes the generator of $\mathbb{R}^+$. They are constructed via the projection $E \times_{\text{ad}} E^*$.

The generalised bundle N is a sub-bundle of the symmetrised product of two copies of the generalised tangent bundle

$$N = \det T^*M \otimes E^* \subset S^2(E), \quad (2.3)$$

and the explicit form of the fibres in terms of $\text{GL}(d)$ tensors can be found in [20, 23]. The bundle K has fibres transforming in the $\mathbf{912}_{-1}$ (see again [23] for more details).

Finally, the generalised metric, G , is defined as the symmetric tensor of rank 2

$$\begin{aligned} G : E \otimes E &\rightarrow \mathbb{R}^+ \\ (V, W) &\rightarrow G(V, W) = G_{MN} V^M W^N, \end{aligned} \quad (2.4)$$

where V, W are generalised vectors. At each point $p \in M$ the generalised metric parameterise the coset

$$G|_p \in \frac{E_{7(7)} \times \mathbb{R}^+}{\text{SU}(8)/\mathbb{Z}_2}, \quad (2.5)$$

where $\text{SU}(8)/\mathbb{Z}_2$ is the maximally compact subgroup of $E_{7(7)}$.

All bosonic transformations of the theory (diffeomorphisms and p -form gauge transformations) are treated in a geometrical way as generalised diffeomorphisms, GDiff , [21, 22]. The infinitesimal GDiff are generated by the Generalised Lie derivative or Dorfman derivative

$$(L_V V')^M = V^N \partial_N V'^M - (\partial \times_{\text{ad}} V)^M{}_N V'^N, \quad (2.6)$$

where $V, V' \in \Gamma(E)$ are two generalised vectors, $\partial_M = (\partial_m, 0, \dots, 0)$ gives the embedding of the ordinary partial derivative ∂_m on M in E^* , and $\times_{\text{ad}}$ denotes again the projection into the adjoint of $E_{7(7)}$.

As in ordinary geometry, one can define a generalised G_S -structure as the reduction of the exceptional structure group to a subgroup $G_S \subset E_{7(7)}$. More precisely, an exceptional G_S -structure defines a G_S -principal subbundle of the $E_{7(7)}$ frame bundle. In all the cases we are interested in, this is equivalent to the existence of globally defined G_S -invariant generalised tensors. As an example, the generalised metric G defines a $\text{SU}(8)$ generalised structure. In what follows we will be interested in generalised structures G_S that are subgroups of $\text{SU}(8)$.² These are defined by generalised vectors, K_I , and/or elements of the adjoint bundle, J_A ,

$$\Xi_i = \{K_I, J_A\}. \quad (2.7)$$

Starting from Ξ_i it is always possible to construct the generalised metric as $G = G(\Xi_i)$ [8].

A generalised G_S -structure is characterised by its intrinsic torsion. Given a G_S -compatible connection $D\Xi_i = 0$, the intrinsic torsion is the part of the torsion of the connection

²Strictly speaking the G_S -structure is a subgroup of $\text{SU}(8)/\mathbb{Z}_2$. However, as discussed below, we are interested in its double cover acting on the fermions of the theory. For simplicity of notation, in the rest of the paper, we will not distinguish between the G_S -structure and its double cover.

D that cannot be eliminated by redefining the connection. It measures the obstruction to finding a torsion-free connection, compatible with the structure [24]. The intrinsic torsion of a G_S -structure is contained in the torsion bundle W

$$W \simeq K \oplus E^* \quad (2.8)$$

which transforms in the $\mathbf{912}_{-1} \oplus \mathbf{56}_{-1}$ of $E_{7(7)} \times \mathbb{R}^+$ and can be decomposed into G_S -representations. The component $\mathbf{56}$ provides the gauging of the scaling symmetry called trombone symmetry. Theories with a trombone symmetry are not lagrangian, but can be studied by looking at the equations of motion. In this paper we will focus only on the $\mathbf{912}$ component, even if our results are trivially extended to include the trombone symmetry.³

Generalised structures are the key ingredient for constructing consistent truncations [8–10]. Consider 11 or 10-dimensional supergravity on a manifold of the form (2.1). Since M is compact, the theory can be seen as an effective 4-dimensional theory with an infinite number of fields organised into representations of

$$GL(4, \mathbb{R}) \times E_{7(7)}. \quad (2.9)$$

The 4-dimensional metric is a singlet of $E_{7(7)}$, the scalars are arranged into the generalised metric, the vectors are sections of the generalised tangent bundle E , while the two-form tensors are sections of the generalised tensor bundle N

$$\begin{aligned} \text{scalars} & \quad G_{MN}(x, y) \in \Gamma(S^2 E^*), \\ \text{vectors} & \quad A_\mu^M(x, y) \in \Gamma(T^* X \otimes E), \\ \text{2-forms} & \quad B_{\mu\nu MN}(x, y) \in \Gamma(\Lambda^2 T^* X \otimes N). \end{aligned} \quad (2.10)$$

A consistent truncation is a procedure to truncate away the infinite towers and construct a 4-dimensional theory with only a finite set of fields. The truncation is called consistent because the modes that have been truncated away decouple from the equation of motion. In doing so, all dependence on the internal coordinates disappears from the 4-dimensional equations of motion and any given solution of the 4-dimensional theory can be uplifted to a full solution of the higher-dimensional one.

If the manifold M admits a generalised G_S -structure with constant singlet intrinsic torsion or zero torsion, then a consistent truncation is guaranteed to exist [8].

The consistent truncation is derived by expanding all bosonic 10/11-dimensional fields in terms of the generalised invariant tensors $\{\Xi_i\}$ defining the G_S -structure. The coefficients in the expansions only depend on the external coordinates x while the dependence on the internal space is in the tensors $\{\Xi_i\}$.

The 4-dimensional scalars are given by the G_S -singlets in the generalised metric G_{MN} . These are singlet deformations of the structure modulo the singlet deformations that do not deform the metric

$$\text{scalars:} \quad h^I(x) \in \mathcal{M} = \frac{C_{E_{7(7)}}(G_S)}{C_{\text{SU}(8)}(G_S)} = \frac{G}{H}, \quad (2.11)$$

³Any connection compatible with $G_S \subset \text{SU}(8)$ does not mix with the trombone, since D is G_S -valued. Adding the trombone simply amounts to taking into account the extra G_S -singlets coming from the $\mathbf{56}$.

where G and H denote the groups that remain in the quotient after the common factors in the numerator and denominator cancel out. The group $G =: G_{\text{iso}}$ gives the isometry group of the scalar manifold.

The vectors are determined by the number of G_S -invariant generalised vectors $\{K_I\}$

$$\text{vectors:} \quad \mathcal{A}_\mu^M(x, y) = A_\mu^I(x) K_I^M \in \Gamma(T^*M \otimes \mathcal{V}), \quad (2.12)$$

where $\mathcal{V} \subset \Gamma(E)$ is the vector space spanned by the $\{K_I\}$. Notice that the singlet generalised vectors determine all the vectors of the reduced theory, coming both from the reduction of the metric and the higher-rank potentials. Thus the vectors K_I generate all symmetries of the reduced theories. This is an important difference with respect to the reductions based on conventional G_S -structures.

The fermionic sector of the truncated theory is constructed in a similar way. The spinors are organised in representations of $SU(8)$. The structure group G_S lifts to its double cover $SU(8)$ and the truncation is obtained by expanding the fermionic fields on the $SU(8)$ singlets in the relevant representations.

The supersymmetry parameters are embedded in the generalised spinor bundle $\mathcal{S}$, which transforms in the $\mathbf{8} \oplus \bar{\mathbf{8}}$ of $SU(8)$. For the truncated theory to have $\mathcal{N}$ supersymmetries the spinor bundle must contain $\mathcal{N}$ G_S -singlets transforming in the fundamental of the relevant R-symmetry group. Thus G_S must be a subgroup of the commutant of the R-symmetry group in $SU(8)$

$$G_S \subseteq C_{SU(8)}(G_R) \quad (2.13)$$

that allows for exactly $\mathcal{N}$ singlets in the spinorial representation of $SU(8)$.

The truncation is consistent thanks to the fact that the intrinsic torsion only consists of constant singlets of the G_S -structure or is zero. Indeed, if there are only singlet representations in the intrinsic torsion, the generalised Levi-Civita connection acts on the invariant generalised tensors Ξ_i as

$$D_M \Xi_i = \Sigma_M \cdot \Xi_i, \quad (2.14)$$

where Σ_M is completely determined in terms of the torsion.⁴ Thus, when plugging the truncated fields in the equations of motion, their derivatives only have expansions in terms of singlets. Since products of singlet representations can never source the non-singlet ones that were truncated away, the truncation is consistent. If the torsion is zero, the invariant tensors are covariantly constant and again no non-singlet terms can be generated in the equations of motion.

As discussed above, the field content and the supersymmetry of the truncated theory are completely determined by the G_S -invariant tensors. As we will now show, the intrinsic torsion of the G_S -structure also encodes the information about the possible gaugings of the truncated theory.

A gauged supergravity is obtained gauging a subgroup of the rigid isometries of the scalar manifold, G_{iso} . The way the gauge group G_{gauge} is embedded in G_{iso} is given by the

⁴ Σ_M is a section of $E^* \otimes \text{adj}(SU(8))$ and $\cdot$ denotes the adjoint action.

embedding tensor (see [25, 26] for a review of this formalism)

$$\Theta : \mathcal{V} \rightarrow \text{Lie}G_{\text{iso}} , \quad (2.15)$$

which is a map from the set of vectors to the Lie algebra of the gauge group G_{gauge} .

In EGG the embedding tensor Θ is identified with the singlet intrinsic torsion. The generalised Lie derivative of the invariant tensors Ξ_i along any invariant generalised vector K_I reduces to

$$L_{K_I}\Xi_i = -T_{\text{int}}(K_I) \cdot \Xi_i , \quad (2.16)$$

where $T_{\text{int}} : \mathcal{V} \rightarrow \text{ad}F$ gives the map from the set of G_S -invariant vectors to the adjoint bundle. Since $T_{\text{int}}(K_I)$ is a G_S -singlet, the map selects the G_S -singlets in the adjoint, namely the elements of the Lie algebra of the commutant group $G = C_{E_{7(7)}}(G_S)$. Since $G = G_{\text{iso}}$ gives the isometries of the scalar manifold, we can identify $-T_{\text{int}}$ with the embedding tensor of the truncated theory. The Leibniz property of the generalised Lie derivative [23, 27] translates into the quadratic condition on the embedding tensor. The generalised Lie derivative (2.16) defines the gauge Lie algebra of the truncated theory with structure constants given by T_{int} . Since the image of the map Θ may not be the whole of $\text{Lie}G_{\text{iso}}$, the gauge group generated by the vectors can be a subgroup of G_{iso}

$$\text{gauge group:} \quad G_{\text{gauge}} \subseteq G_{\text{iso}} . \quad (2.17)$$

Clearly, when the intrinsic torsion is zero, the truncated theory is ungauged.

In summary we see that a G_S -structure completely fixes the data of the truncated theory: the topological properties of the G_S -structure, namely the existence of G_S -invariant non-vanishing generalised tensors, determine the field content and supersymmetry of the theory, while the differential conditions of having only constant, singlet intrinsic torsion (or zero intrinsic torsion), beyond guaranteeing the consistency of the truncation, determine its possible gaugings.

3 Consistent truncations to four dimensions

There is a huge variety of 4-dimensional supergravities with different amount of supersymmetry and different gauge groups. The aim of this paper is to use the formalism of generalised G_S -structures to classify those that can be obtained as consistent truncations of 11/10-dimensional supergravity.

As discussed in the previous section, the algebraic properties of a generalised G_S -structure are enough to fix the field content and supersymmetries of the reduced theory, as well as the truncation ansatz. Then imposing the differential constraint of having singlet, constant intrinsic torsion determines the embedding tensor and the possible gaugings.

We will follow the logic of [18]: we assume that the differential constraints are satisfied and we classify the possible continuous subgroup G_S of $E_{7(7)}$. We also briefly discuss some

cases where the structure group is discrete. Differently from [18] we will not focus on a fixed amount of supersymmetry but we scan through all supersymmetries: $\mathcal{N} = 2, \dots, 8$.⁵

In this section we summarise our results, while the details of the analysis can be found in Section 4 and Appendix B. We organise the presentation by amount of supersymmetry. For any number $\mathcal{N}$ of supercharges we give a brief summary of the corresponding 4-dimensional supergravity and we discuss the main details of the G_S -structure giving rise to the truncation.

For $\mathcal{N} \geq 3$ the generalised structures are always defined only in terms of invariant generalised vectors. In this cases, the G_S -singlets in the adjoint representation are not independent and are obtained as products of the invariant vectors: $(J \sim K_I \times_{\text{ad}} K_J)$.

For $\mathcal{N} = 2$ this is no longer the case. The generalised structure is defined by invariant vectors and invariant adjoint elements, which correspond to the presence of vector and hyper-multiplets in the truncated theory.

For any amount of supersymmetry $\mathcal{N}$ there is a maximal generalised structure, G_S^{max} , corresponding to the largest commutant in $\text{SU}(8)$ of the R-symmetry group that admits exactly $\mathcal{N}$ singlets in the **8** of $\text{SU}(8)$. In the table below we list the R-symmetry groups and the maximal generalised G_S -structure for any amount of supersymmetry. We also give the corresponding invariant generalised tensors.

$\mathcal{N}$	G_R	G_S^{max}	inv. tensors
8	$\text{SU}(8)$	$\mathbb{1}$	$\{K_I\}_{I=1,\dots,56}$
6	$\text{SU}(6) \times \text{U}(1)$	$\text{SU}(2)$	$\{K_I\}_{I=1,\dots,32}$
5	$\text{SU}(5) \times \text{U}(1)$	$\text{SU}(3)$	$\{K_I\}_{I=1,\dots,20}$
4	$\text{SU}(4) \times \text{U}(1)$	$\text{SU}(4)$	$\{K_I\}_{I=1,\dots,12}$
3	$\text{SU}(3) \times \text{U}(1)$	$\text{SU}(5)$	$\{K_I\}_{I=1,\dots,6}$
2	$\text{SU}(2) \times \text{U}(1)$	$\text{SU}(6)$	$\{K, \hat{K}, J_\alpha\}_{\alpha=1,2,3}$

Table 1. R-symmetry groups and the maximal generalised G_S -structure for $\mathcal{N}$ supersymmetries.

As discussed in [8], the truncated theories obtained from maximal structure groups correspond to pure supergravities. For $\mathcal{N} = 5, 6, 8$ this is all one can obtain. For $\mathcal{N} \leq 4$ truncations with extra matter fields can be constructed considering subgroups $G_S \subset G_S^{\text{max}}$ that still give exactly $\mathcal{N}$ singlets in the **8** of $\text{SU}(8)$. In what follows we only list the group G_S that give inequivalent field contents.

Our results provide a complete classification for G_S -structures where G_S is a Lie group, i.e. a continuous group. There could also exist truncations corresponding to G_S -structures defined by discrete groups. We comment on them in Section 4.

The structure of the theories that can be obtained as consistent truncations is very restricted. For instance, the scalar manifolds must necessarily be symmetric and can all be

⁵We also considered truncations with $\mathcal{N} = 1$ and $\mathcal{N} = 0$ supersymmetry. However, the number of truncations is too big to list them in a synthetic and interesting way. We leave their discussion to a future publication.

written as cosets

$$\mathcal{M} = \frac{G}{H}, \quad (3.1)$$

where G is the semisimple non-compact Lie group of global isometries and H is its maximal compact subgroup.

For theories with $\mathcal{N} > 2$ supersymmetry this property is a consequence of supersymmetry. The allowed manifolds are listed in Table 2.

$\mathcal{N}$	8	6	5	4	3
$\mathcal{M}$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{SO^*(12)}{SU(6) \times U(1)}$	$\frac{SU(5,1)}{SU(5) \times U(1)}$	$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(6, n_V)}{SO(6) \times SO(n_V)}$	$\frac{SU(3, n_V)}{S[U(3) \times U(n_V)]}$

Table 2. Homogeneous symmetric manifolds for $\mathcal{N} \geq 3$

For $\mathcal{N} = 2$ the scalar manifold factorises into vector and hypermultiplet spaces, which by supersymmetry, are special Kähler (SK) and quaternionic Kähler (QK) manifolds, respectively. The additional condition of being homogeneous and symmetric is a prediction of EGG. The list of homogeneous, symmetric spaces for theories with $\mathcal{N} = 2$ supersymmetry is given in Table 3.

	SK	n_V	QK	n_H
$\mathcal{M}$	$\frac{SU(1, n_V)}{U(n_V)}$	n_V	$\frac{SU(2, n_H)}{S[U(2) \times U(n_H)]}$	n_H
	$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(2, n_V - 1)}{SO(2) \times SO(n_V - 1)}$	n_V	$\frac{SO(4, n_H)}{SO(4) \times SO(n_H)}$	n_H
	$\frac{SU(1, 1)}{U(1)}$	1	$\frac{G_{2(2)}}{SU(2) \times SU(2)}$	2
	$\frac{Sp(6)}{U(3)}$	6	$\frac{F_{4(4)}}{SU(2) \times USp(6)}$	7
	$\frac{SU(3, 3)}{S[U(3) \times U(3)]}$	9	$\frac{E_{6(2)}}{SU(2) \times SU(6)}$	10
	$\frac{SO^*(12)}{SU(6) \times U(1)}$	15	$\frac{E_{7(-5)}}{SU(2) \times SO(12)}$	16
	$\frac{E_{7(-25)}}{U(1) \times E_{6(6)}}$	27	$\frac{E_{8(-24)}}{SU(2) \times E_{7(7)}}$	28
			$\frac{USp(2, 2n_H)}{USp(2) \times USp(2n_H)}$	n_H

Table 3. Homogeneous symmetric manifolds for $\mathcal{N} = 2$ and the corresponding number of vector (n_V) and hypermultiplets (n_H).

In our classification, for $\mathcal{N} \geq 3$ we recover, as expected, all the classes of manifolds listed in Table 2, while for $\mathcal{N} = 2$, some manifolds are missing (see Section 3.6).

For theories with $\mathcal{N} \leq 4$ supersymmetry we find that the possible matter content is very constrained: there is a maximum number of vector and/or hypermultiplets and only a few possibilities are allowed.

For any amount of supersymmetry, the embedding tensor of the truncated theory is given by the G_S -singlets components of the intrinsic torsion. From its analysis one can derive all the gaugings of the global isometries of the theory. In this article we are not interested in performing a detailed study of the gaugings. However, for completeness, for any amount of supersymmetry we discuss in which representations of the global isometry group the singlet intrinsic torsion transforms.

Finally, let us recall that the ungauged supergravity in 4 dimensions is invariant under electromagnetic duality, which is realised on-shell. Together, electric and dual magnetic vectors form a linear representation of the global symmetry group G_{iso} . In EGG they are all associated to the G_S -invariant generalised vectors in the fundamental of $E_{7(7)}$.

3.1 $\mathcal{N} = 8$ supergravity

The field content of $\mathcal{N} = 8$ supergravity in four dimensions [28] consists of the graviton, 28 electric vectors, 70 scalars, 8 gravitini and 56 gaugini, organised into a single (gravity) multiplet. The R-symmetry is $SU(8)$ and the scalars parameterise the manifold

$$\mathcal{M} = \frac{E_{7(7)} \times \mathbb{R}^+}{SU(8)/\mathbb{Z}_2}, \quad (3.2)$$

with $G_{\text{iso}} = E_{7(7)} \times \mathbb{R}^+$ the rigid isometry group. The subgroups of G_{iso} that can be gauged are determined by the embedding tensor Θ_I^α , with $I = 1, \dots, 56$ and $\alpha = 1, \dots, 133$, which transforms in the **912**₋₁ of $E_{7(7)}$ [29, 30]

$$D_\mu = \nabla_\mu - g A_\mu^I \Theta_I^\alpha t_\alpha \quad (3.3)$$

where t_α are the $E_{7(7)}$ generators.

In generalised geometry maximally supersymmetric truncations correspond to a generalised identity structure $G_S = \mathbb{1}$. The structure is defined by 56 generalised vectors K_I , $I = 1, \dots, 56$, that give a Leibniz parallelisation of the generalised tangent bundle [9, 10, 31, 32]

$$L_{K_I} K_J = X_{IJ}^K K_K, \quad (3.4)$$

where X_{IJ}^K are constant and $G(K_I, K_J) = \delta_{IJ}$ with G the generalised metric.

The generalised vectors K_I transform in the **28**_c of the $SU(8)$ R-symmetry, and in the **28** and **28'** of $SL(8, \mathbb{R})$ (see Appendix A)

$$K_I = \{K_{ij}, K^{ij}\} \quad I = 1, \dots, 56 \quad i, j = 1, \dots, 8, \quad (3.5)$$

where K_{ij} are the 28 electric vectors of the truncated theory and K^{ij} their magnetic duals. The scalar manifold (3.2) is trivially obtained from (2.11).

The tensor $X_{IJ}{}^K$ gives the intrinsic torsion of the identity structure: it transforms in the **912**₋₁ and is related to the embedding tensor of the truncated theory

$$X_{IJ}{}^K = \Theta_I{}^\alpha (t_\alpha)_J{}^K, \quad (3.6)$$

where t_α are the generators of the scalar isometry group $G_{\text{iso}} = E_{7(7)}$. The Leibniz property of the generalised Lie derivative [23, 27]

$$[X_I, X_J] = -X_{IJ}{}^K X_K, \quad (3.7)$$

with $(X_I)_J{}^K = X_{IJ}{}^K$ a matrix, translates into the quadratic constraint on the embedding tensor.

The theory with maximal gauge group $SO(8)$ was constructed in [33] and was shown to be the consistent truncation of 11-dimensional supergravity on S^7 in [34]. It was re-interpreted as a generalised Scherck-Schwarz reduction in [9]. As shown in [9], the intrinsic torsion only belongs to the component **36** in the decomposition

$$\mathbf{912} = \mathbf{36} \oplus \mathbf{36}' \oplus \mathbf{420} \oplus \mathbf{420}' \quad (3.8)$$

under $SL(8, \mathbb{R}) \subset E_{7(7)}$. The generalised Lie derivative (3.4) among the generalised vectors (3.5) gives the $SO(8)$ algebra

$$X_{[ii'] [jj']}^{[kk']} = -X_{[ii'] [jj']}^{[kk']} = R^{-1} (\delta_{ij} \delta_{i'j'}^{kk'} - \delta_{i'j} \delta_{ij'}^{kk'} - \delta_{ij} \delta_{i'j'}^{kk'} + \delta_{i'j} \delta_{ij'}^{kk'}), \quad (3.9)$$

where $[i, i']$, with $i, i' = 1, \dots, 8$, are antisymmetrised $SL(8, \mathbb{R})$ indices, $\delta_{ij}^{kk'} = \delta_i^{[k} \delta_j^{k']}$, and R the radius of S^7 . The tensor (3.9) reproduces the 4-dimensional embedding tensor for the $SO(8)$ electric gauging.

In [35] $\mathcal{N} = 8$ supergravity with a dyonic $ISO(7)$ gauging has been obtained as a consistent truncation of massive type IIA supergravity on S^6 . In this case it is natural to branch the various representations under $SL(7, \mathbb{R}) \subset SL(8, \mathbb{R}) \subset E_{7(7)}$ so that the globally defined vectors arrange into

$$\begin{aligned} \mathbf{56} &= \mathbf{28} \oplus \mathbf{28}' = \mathbf{21} \oplus \mathbf{7} \oplus \mathbf{21}' \oplus \mathbf{7}' \\ K_I &= (K_{ij}, K^{ij}) = (K_{ab}, K_{a8}, K^{ab}, K^{a8}), \end{aligned} \quad (3.10)$$

with $i, j = 1, 8$ and $a, b = 1, \dots, 7$. The non-zero components of $X_{IJ}{}^K$ in (3.4) have a simple expression in terms of $SL(8, \mathbb{R})$ indices [13]

$$\begin{aligned} X_{[ii'] [jj']}^{[kk']} &= -X_{[ii'] [jj']}^{[kk']} = 8\delta_{[i}^{[k} \theta_{i']}^{j]} \delta_{j'}^{k']}, \\ X_{[ii'] [kk']}^{[jj']} &= -X_{[ii'] [kk']}^{[jj']} = 8\delta_{[j}^{[i} \xi_{i']}^{k]} \delta_{j'}^{k']}, \end{aligned} \quad (3.11)$$

where the tensors

$$\theta_{ij} = \frac{1}{2R} \begin{pmatrix} \mathbb{1}_7 & \\ & 0 \end{pmatrix} \quad \text{and} \quad \xi^{ij} = \frac{m}{2} \begin{pmatrix} 0_7 & \\ & 1 \end{pmatrix} \quad (3.12)$$

give the components of the embedding tensor corresponding to an element of the **28** and the singlet in the decomposition of the **36** of $\text{SL}(8, \mathbb{R})$ under $\text{SL}(7, \mathbb{R})$ [13, 35]. They correspond to the gauging of the $\text{ISO}(7)$ group as it can be seen from the covariant derivative in 4 dimensions

$$D_\mu = \nabla_\mu - g A_\mu^{ab} t_{[a}^c \theta_{b]c} + g(\theta_{ab} A_\mu^{a8} - \frac{m}{2} A_{\mu a 8}) t_8^b \quad (3.13)$$

where t_a^c and t_8^b are $\text{SL}(8, \mathbb{R})$ generators, and $t_{ab} = t_{[a}^c \theta_{b]c}$ gives the embedding of the $\text{SO}(7)$ generators. From (3.13) one easily sees that, while the $\text{SO}(7)$ gauging is purely electric, the $\mathbb{R}^+$ is dyonic for non-zero Roman mass m .

3.2 $\mathcal{N} = 6$ supergravity

We find one truncation with $\mathcal{N} = 6$ supersymmetry corresponding to the generalised structure $G_S = \text{SU}(2)$.⁶ From the commutant of G_S in $\text{SU}(8)$ we recover the R-symmetry of $\mathcal{N} = 6$ supersymmetry

$$\text{C}_{\text{SU}(8)}(\text{SU}(2)_S) = \text{SU}(6)_R \times \text{U}(1)_R. \quad (3.14)$$

The structure group embeds in $\text{E}_{7(7)}$ as

$$\text{E}_{7(7)} \rightarrow \text{SU}(2)_S \times \text{SO}^*(12), \quad (3.15)$$

where $\text{G}_{\text{iso}} = \text{SO}^*(12)$ is the global isometry group. The G_S -structure is determined by invariant generalised vectors only. From the decomposition of the generalised tangent bundle under (3.15)

$$\mathbf{56} \rightarrow (\mathbf{2}, \mathbf{12}) \oplus (\mathbf{1}, \mathbf{32}') \quad (3.16)$$

we see that there are 32 generalised vectors K_I transforming in the $\mathbf{32}'$ of the isometry group $\text{SO}^*(12)$. The vectors K_I are normalised to $G(K_I, K_J) = \delta_{IJ}$, where again G is the generalised metric.

By further decomposing under the R-symmetry $\text{SO}^*(12) \supset \text{SU}(6) \times \text{U}(1)$, the invariant vectors split into the singlet and **15** representations (and their conjugates) $\text{SU}(6)$

$$K^0 \in \mathbf{1}_{-6} \quad K^{[ij]} \in \mathbf{15}_2 \quad K'^0 \in \mathbf{1}_6 \quad K'^{[ij]} \in \overline{\mathbf{15}}_{-2}. \quad (3.17)$$

They give the 16 vectors of the truncated theory and their magnetic duals

$$A_\mu^I = (A_\mu^0, A_\mu^{[ij]}, A_{\mu 0}, A_{\mu [ij]}). \quad (3.18)$$

The scalar manifold is obtained from (2.11) with $G = \text{SO}^*(12)$ and $H = \text{SU}(6) \times \text{U}(1)$, the R-symmetry

$$\mathcal{M} = \frac{\text{SO}^*(12)}{\text{SU}(6) \times \text{U}(1)}. \quad (3.19)$$

From representation theory one can check that this truncation can be obtained as a consistent truncation of the $\mathcal{N} = 8$ theory where only the $\text{SU}(2)_S$ -singlets are kept.

⁶We also find $\mathcal{N} = 6$ truncations corresponding to $G_S = \text{U}(1)$ and $G_S = \mathbb{Z}_2$. They have the same field content as the $G_S = \text{SU}(2)$ one.

The data above reproduce the field content of 4d $\mathcal{N} = 6$ supergravity, namely 1 graviton, 6 gravitini, 16 vectors, 26 Majorana spin 1/2 fields and 30 scalars, all organised in a single multiplet of the $SU(6) \times U(1)$ R-symmetry group. The scalars parametrise the manifold (3.19).

The gauging of the scalar isometries are given by the embedding tensor Θ_I^α

$$D_\mu = \nabla_\mu - g A_\mu^I \Theta_I^\alpha t_\alpha, \quad (3.20)$$

where t_α ($\alpha = 1, \dots, 66$) are the $SO^*(12)$ generators and A_μ^I are the 16 electric vectors and their magnetic duals ($I = 1, \dots, 32$). The linear and quadratic constraints imply that Θ_I^α transforms in the **352** of $SO^*(12)$. This is exactly the representation of singlet intrinsic torsion in the decomposition of the **912** under (3.15). From the structure of the vector fields, we see that the largest subgroup of the scalar isometries that can be gauged is an electric (magnetic) $SO(6) \times SO(2)$.

$\mathcal{N} = 6$ gauged supergravity with $SO(6)$ gauge group was obtained as the consistent truncation of type II supergravity on $AdS_4 \times \mathbb{CP}^3$ in [36], where the relation to the maximally supersymmetric truncation of 11-dimensional supergravity was also discussed.

3.3 $\mathcal{N} = 5$ supergravity

Also for $\mathcal{N} = 5$ supersymmetry we find only one truncation, which reproduces the field content and embedding tensor of $\mathcal{N} = 5$ supergravity. The R-symmetry is $SU(5) \times U(1)$ and the fields are arranged into a single gravity multiplet containing 1 graviton, 5 gravitini, 10 vectors, 11 Majorana spin 1/2 fields and 10 scalars parameterising the manifold

$$\mathcal{M} = \frac{SU(1, 5)}{SU(5) \times U(1)}. \quad (3.21)$$

The truncation corresponds to the generalised structure $G_S = SU(3)$. From the embedding in $SU(8)$ and $E_{7(7)}$

$$\begin{aligned} SU(8) &\supset SU(3)_S \times SU(5)_R \times U(1)_R \\ E_{7(7)} &\supset SU(3)_S \times SU(1, 5), \end{aligned} \quad (3.22)$$

we find that the R-symmetry is $SU(5) \times U(1)$, as expected, and the scalar isometries are $G_{\text{iso}} = SU(1, 5)$. Combining (3.22) with (2.11) we recover the scalar manifold (3.21).

From the decomposition of the fundamental of $E_{7(7)}$ under (3.22)

$$\begin{aligned} \mathbf{56} &= (\mathbf{3}, \mathbf{6}) \oplus (\bar{\mathbf{3}}, \bar{\mathbf{6}}) \oplus (\mathbf{1}, \mathbf{20}) \\ &= [(\mathbf{3}, \mathbf{5})_1 \oplus (\mathbf{3}, \mathbf{1})_{-5} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{5}})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{1})_5] \oplus [(\mathbf{1}, \mathbf{10})_{-3} \oplus (\mathbf{1}, \bar{\mathbf{10}})_3], \end{aligned} \quad (3.23)$$

it follows that the G_S -structure is defined by 20 invariant generalised vectors

$$\{K_I\} = \{K_{[mnp]}\} \quad m, n, p = 0, \dots, 5 \quad (3.24)$$

transforming in the **20** of the global isometry group and satisfying the normalisation condition $G(K_I, K_J) = \delta_{IJ}$, with G the generalised metric. Splitting the $SU(1, 5)$ indices into

0 and $i, j = 1, \dots, 5$, the vectors splits into $K_{[ij]0}$ and $K_{[ijk]}$, giving the 10 vectors of the truncated theory in the $\mathbf{10}_{-3}$ of the R-symmetry group and their magnetic duals

$$A_\mu^I = (A_\mu^{[ij]} A_{\mu[ij]}) \quad i, j = 1, \dots, 5. \quad (3.25)$$

The intrinsic torsion contains two SU(3) singlets in the $\mathbf{70}$ and $\overline{\mathbf{70}}$ of SU(5, 1)

$$W_{\text{int}} = \mathbf{70} \oplus \overline{\mathbf{70}}, \quad (3.26)$$

corresponding to the components of the embedding tensor

$$\Theta_I^\alpha = (\theta_{[mm],p}, \theta^{[mn],p}) \quad m, n, p = 0, \dots, 5, \quad (3.27)$$

with $\theta_{[mn,p]} = \theta^{[mn,p]} = 0$. From the representations of the generalised vectors it follows that the largest compact gauging is an electric (magnetic) subgroup SO(5) of $G_{\text{iso}} = \text{SU}(5, 1)$.

$\mathcal{N} = 5$ supergravity can also be obtained as a truncation of $\mathcal{N} = 8$ gauged supergravity where only the G_S -singlets are kept. An example with gauge group SO(5), the largest possible compact gauging, was constructed directly in four dimensions in [37]. No explicit truncation with just $\mathcal{N} = 5$ is known to us.

3.4 $\mathcal{N} = 4$ supergravity

For $\mathcal{N} = 4$ supergravity two kinds of multiplet are possible: the graviton and vector multiplets. The former consists of 1 graviton, 4 gravitini, 6 vectors, 4 Majorana spin 1/2 fields and 2 scalars, while the latter are formed by 1 vector, 4 Majorana spin 1/2 fields and 6 scalars, all transforming in representation of the $\text{SU}(4)_R \sim \text{SO}(6)_R$ R-symmetry. In a theory with n_V vector multiplets the scalar manifold is given by

$$\mathcal{M} = \frac{\text{SO}(6, n_V)}{\text{SO}(6)_R \times \text{SO}(n_V)} \times \frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}, \quad (3.28)$$

where the first factor is parametrised by scalars in the vector multiplets and the second by those in the gravity multiplet. The global symmetry group is $G_{\text{iso}} = \text{SO}(6, n_V) \times \text{SL}(2, \mathbb{R})$.

The gaugings of the theory

$$D_\mu = \nabla_\mu - g A_\mu^{Ia} f_{aI}{}^{JK} t_{JK} + g A_\mu^{I(a} \epsilon^{b)c} \xi_{cI} t_{ab}, \quad (3.29)$$

with ∇_μ the spin-connection and t_{IJ} and t_{ab} the generators of $\text{SO}(6, n_V)$ and $\text{SL}(2, \mathbb{R})$, respectively, are determined by the embedding tensor [38]

$$\Theta_I^\alpha = (\xi_{aI}, f_{aIJK}) \quad a = 1, 2 \quad I = 1, \dots, 6 + n_V, \quad (3.30)$$

whose components transform as doublets of $\text{SL}(2, \mathbb{R})$ and as the fundamental and the three-index anti-symmetric representations of $\text{SO}(6, n_V)$, respectively.

We find six inequivalent truncations with $\mathcal{N} = 4$ supersymmetry associated to the structure groups

$$G_S = \text{Spin}(6 - n_V), \quad n_V = 0, \dots, 6, \quad (3.31)$$

with $\text{Spin}(1) = \text{Spin}(0) = \mathbb{Z}_2$ (see Appendix B for more details on the continuous structures). In what follows we will define the structure in terms of the corresponding orthogonal groups, since these are the ones acting on the invariant generalised vectors. The G_S -structures correspond to the embeddings⁷

$$\begin{aligned} E_{7(7)} &\supset \text{SO}(6 - n_V)_S \times \text{SO}(6, n_V) \times \text{SL}(2, \mathbb{R}) \\ \text{SU}(8) &\supset \text{SO}(6 - n_V)_S \times \text{SO}(6) \times \text{SO}(n_V) \times \text{U}(1). \end{aligned} \quad (3.32)$$

The generalised $\text{SO}(6 - n_V)$ -structure is defined by an $\text{SL}(2, \mathbb{R})$ doublet of $6 + n_V$ generalised vectors in the decomposition of the generalised tangent bundle

$$\mathbf{56} = [(\mathbf{6} - n_V, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{6} + n_V, \mathbf{2})] \oplus [(\mathbf{S}_{\text{SO}(6-n_V)}, \mathbf{S}_{\text{SO}(6, n_V)}, \mathbf{1}) \oplus c.c.], \quad (3.33)$$

where $\mathbf{S}_{\text{SO}(6-n_V)}$ and $\mathbf{S}_{\text{SO}(6, n_V)}$ denote the spinorial representations of $\text{SO}(6 - n_V)$ and $\text{SO}(6, n_V)$ respectively. The K_{Ia} singlets satisfy the compatibility conditions

$$s(K_{Ia}, K_{Jb}) = \kappa^2 \delta_{IJ} \epsilon_{ab} \quad \forall I, J = 1, \dots, 6 + n_V \quad \forall a, b = \pm, \quad (3.34)$$

where $s(V, V')$ is the $E_{7(7)}$ symplectic invariant, ϵ_{ab} is the $\text{SL}(2, \mathbb{R})$ invariant antisymmetric tensor and $\kappa = (\det T^*M)^{1/2}$ (see also [14]). The generalised vectors give $6 + n_V$ electric vectors and $6 + n_V$ magnetic vectors in the truncated theory

$$A_\mu^{I\alpha} = (A_\mu^{I+}, A_\mu^{I-}) \quad I = 1, \dots, 6 + n_V, \quad (3.35)$$

where $\pm$ denote the charges under $\text{SO}(2) \subset \text{SL}(2, \mathbb{R})$. Moreover, under $\text{SO}(6, n_V) \supset \text{SO}(6) \times \text{SO}(n_V)$, the singlet vectors decompose as $\mathbf{6} + n_V = (\mathbf{6}, 1) \oplus (\mathbf{1}, n_V)$

$$\begin{aligned} A_\mu^{I+} &= (A_\mu^{i+}, A_{\mu a}^+) \quad i = 1, \dots, 6 \\ A_\mu^{I-} &= (A_\mu^{i-}, A_{\mu a}^-) \quad a = 1, \dots, n_V \end{aligned} \quad (3.36)$$

where the vectors in the $\mathbf{6}$ belong to the gravity multiplet and singlets to the vector multiplets.

The scalar isometries are $G_{\text{iso}} = \text{SO}(6, n_V) \times \text{SL}(2, \mathbb{R})$, and, from (3.32) and (2.11) we reproduce the scalar manifold (3.28).

The G_S -singlets in the intrinsic torsion reproduce the embedding tensor (3.30)

$$W_{\text{int}} = (\mathbf{6} + \mathbf{n}_V, \mathbf{2}) \oplus (\mathbf{X}_{[IJK]}, \mathbf{2}), \quad (3.37)$$

where the first term transforms in the fundamental of $\text{SO}(6 + n_V)$ and the second in three-index anti-symmetric representation. Both are doublets of $\text{SL}(2, \mathbb{R})$. From the number of vectors we see that the maximal compact gauging is the electric (magnetic) subgroup $\text{SO}(4) \times \text{U}(1)$ of the R-symmetry.

Four-dimensional $\mathcal{N} = 4$ gauged supergravities without and with vector multiplets were constructed since the seventies [39–44, 44, 45] and the general reformulation in terms

⁷For $n_V = 0, 2, 4$ the truncations correspond to regular branching while for $n_V = 1, 3$ they arise from non-regular ones.

of the embedding tensor can be found in [38, 46]. Some examples of the truncations discussed above can be found in the literature. Pure supergravity ($n_V = 0$) with $\text{SO}(4)$ gauging is obtained truncating 11-dimensional supergravity on S^7 [47] or as a reduction of IIB supergravity on $S^1 \times S^5$ [48]. $\mathcal{N} = 4$ supergravity with $n_V = 3$ vector multiplets (and gauge group $\text{SO}(3) \ltimes_{\mathbf{3} \oplus \mathbf{3}} \text{nil}_{(6,3)}$) was derived in [6] as the universal truncation of 11-dimensional supergravity on tri-Sasakian manifolds. The derivation in the context of EGG can be found in [49]. In [6] an $\mathcal{N} = 4$ supergravity with $n_V = 4$ was found when the tri-Sasakian manifold is taken to be N^{010} . The extra vector multiplet corresponds to a Betti-multiplet arising from the non-trivial cohomology of N^{010} .

3.5 $\mathcal{N} = 3$ supergravity

The field content of $\mathcal{N} = 3$ supergravity [50] is again given by one gravity multiplet, consisting of 1 graviton, 3 gravitini, 3 vectors and 1 Majorana spin 1/2 field, and n_V vector multiplets containing 1 vector, 4 Majorana spin 1/2 fields and 6 scalars. The R-symmetry is $\text{SU}(3) \times \text{U}(1)$ and the scalars parameterise the coset

$$\mathcal{M} = \frac{\text{SU}(3, n_V)}{\text{S}(\text{U}(3) \times \text{U}(n_V))}, \quad (3.38)$$

where the denominator is locally isomorphic to $\text{SU}(3) \times \text{U}(n_V)$.

The gaugings of the scalar isometry group $\text{SU}(3, n_V)$ are given by the embedding tensor

$$\Theta_I^\alpha = (\theta_{IJ}^K, \theta^{IJ}_K), \quad (3.39)$$

where $I, J, K = 1, \dots, 3 + n_V$ and $\theta_{IJ}^K = \theta_{[IJ]}^K$.

$\mathcal{N} = 3$ truncations correspond to generalised structures $G_S \subseteq \text{SU}(5)$, where $\text{SU}(5)$ is the commutant of the R-symmetry in $\text{SU}(8)$

$$\text{SU}(8) \supset \text{SU}(5)_S \times \text{SU}(3) \times \text{U}(1). \quad (3.40)$$

The largest structure, $G_S = \text{SU}(5)$, corresponds to a truncation to pure supergravity. It is defined by six invariant generalised vectors

$$\{K_I\} = \{K_i, K^i\}, \quad i = 1, 2, 3, \quad (3.41)$$

satisfying

$$s(K_I, K_J) = \kappa^2 \delta_{IJ}. \quad (3.42)$$

Decomposing the fundamental of $\text{E}_{7(7)}$ under

$$\text{E}_{7(7)} \supset \text{SU}(5)_S \times \text{U}(1) \times \text{SU}(3), \quad (3.43)$$

the six singlet vectors arrange into two triplets of the R-symmetry group

$$\mathbf{56} = (\mathbf{5}, \mathbf{3})_1 \oplus (\mathbf{1}, \mathbf{3})_{-5} \oplus (\mathbf{10}, \mathbf{1})_{-3} \oplus c.c \quad (3.44)$$

corresponding to the three vectors in the gravity multiplet and their duals. Combining (2.11) with (3.43) and (3.40), it is easy to verify that there are no scalars. Thus we recover the field content of pure supergravity.

For $G_S \subset \text{SU}(5)$ extra vector multiplets are possible. Since the vectors in the vector multiplets are singlets of the R-symmetry, they correspond to G_S -singlets in $(\mathbf{10}, \mathbf{1})_{-3}$ and $(\overline{\mathbf{10}}, \mathbf{1})_3$ of (3.44). Together with those in the gravity multiplets they transform in the fundamental and antifundamental of the global $\text{SU}(3, n_V)$ symmetry group

$$(\mathbf{3} + n_V) \oplus (\overline{\mathbf{3} + n_V}). \quad (3.45)$$

Under $\text{SU}(3) \times \text{SU}(n_V) \times \text{U}(1)$ the $\mathbf{3} + n_V$ vectors split as $(\mathbf{3}, \mathbf{1})_{-1}$ and $(\mathbf{1}, n_V)_{3/n_V}$ (plus their conjugates) and give the electric and magnetic vectors of the truncated theory

$$\{A_\mu^I\} = \{A_\mu^{ij}, A_\mu^a, A_{\mu ij}, A_{\mu a}\} \quad i = 1, 2, 3, \quad a = 1, \dots, n_V. \quad (3.46)$$

Decomposing the intrinsic torsion under $G_S \times \text{SU}(3, n_V)$ we find that the G_S -singlets transform in the

$$\mathbf{N} \oplus \frac{\mathbf{N}(\mathbf{N} - 2)(\mathbf{N} + 1)!}{2\mathbf{N}!} \quad (\mathbf{N} = \mathbf{3} + n_V) \quad (3.47)$$

of the global symmetry group $G_{\text{iso}} = \text{SU}(3, n_V)$. These representations correspond to the $\text{SU}(3, n_V)$ irreducible representations of (3.39).

The list of allowed truncation is given in the table below

n_V	G_S	$\mathcal{M}$
0	$\text{SU}(5)$	$\mathbf{1}$
1	$\text{SU}(3) \times \text{SU}(2)$	$\frac{\text{SU}(3,1)}{\text{SU}(3) \times \text{U}(1)}$
2	$\text{U}(1)^2$	$\frac{\text{SU}(3,2)}{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}$
3	$\text{U}(1)$	$\frac{\text{SU}(3,3)}{\text{SU}(3) \times \text{SU}(3) \times \text{U}(1)}$
4	$\mathbb{Z}_6$	$\frac{\text{SU}(3,4)}{\text{SU}(3) \times \text{SU}(4) \times \text{U}(1)}$

Table 4. $\mathcal{N} = 3$ truncations and associated G_S -structures

Four-dimensional gauged $\mathcal{N} = 3$ supergravity coupled to vector multiplets was constructed in [50], and explicit gaugings inside $\text{SO}(3, n_V)$ were studied in [51]. We are not aware of explicit truncations to $\mathcal{N} = 3$ supergravity that are not subtruncations of a more supersymmetric theory.

3.6 $\mathcal{N} = 2$ supergravity

In $\mathcal{N} = 2$ supergravity the fields are arranged into a graviton multiplet, n_V vector multiplets and n_H hyper-multiplets all carrying a representation of the $\text{SU}(2)_R \times \text{U}(1)_R$ R-symmetry. The graviton multiplet contains the metric, the graviphoton, and an $\text{SU}(2)_R$ doublet of

gravitini. The vector multiplets consist of a vector, 2 spin 1/2 and 1 complex scalar, while the hypermultiplets contain 2 spin 1/2 fermions and 4 real scalar fields.

The scalars in the vector multiplets parametrise a special Kähler manifold $\mathcal{M}_V$ of complex dimension n_V and those in the hypermultiplets parametrise a quaternionic Kähler manifold $\mathcal{M}_H$ of real dimension $4n_H$. Together, the scalar manifold is given by the product

$$\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H. \quad (3.48)$$

The isometry group of the scalar manifolds also splits into two factors $G_{\text{iso}} = G_V \times G_H$ acting on the scalars in the vector and in the hypermultiplets, respectively.

The gauging of the scalar isometries can be expressed in terms of the embedding tensor, which consists of two parts

$$(\Theta_{\tilde{I}}^a, \Theta_{\tilde{I}}^A) \quad (3.49)$$

corresponding to symmetries of the vector and hyper-multiplet scalars. In the above equation $\tilde{I} = 0, \hat{0}, \dots, 2n_V$ runs over the number of electric and magnetic vectors in the theory, while $a = 1, \dots, \dim G_V$ and $A = 1, \dots, \dim G_H$ span the generators of G_V and G_H . The embedding tensor determines the combination of Killing vectors on $\mathcal{M}_V$ and $\mathcal{M}_H$ that are gauged

$$k_{\tilde{I}}^i = \Theta_{\tilde{I}}^a k_a^i(\phi) \quad k_{\tilde{I}}^x = \Theta_{\tilde{I}}^A k_A^x(q), \quad (3.50)$$

where ϕ^i , $i = 1, \dots, n_V$, are the scalars in the vector multiplets and q^x , $x = 1, \dots, 4n_H$ those in the hypermultiplets. On the scalars the gaugings are defined via the covariant derivatives

$$\begin{aligned} \mathcal{D}_\mu \phi^i &= \partial_\mu \phi^i + ig k_{\tilde{I}}^i A_\mu^{\tilde{I}}, \\ \mathcal{D}_\mu q^x &= \partial_\mu q^x + ig k_{\tilde{I}}^x A_\mu^{\tilde{I}}. \end{aligned} \quad (3.51)$$

In generalised geometry the largest structure group that is compatible with $\mathcal{N} = 2$ supersymmetry in four dimensions is $G_S = \text{SU}(6)$, the commutant in $\text{SU}(8)$ of the R-symmetry. The decomposition of the spinorial representation of $\text{SU}(8)$ under the embedding $\text{SU}(8) \supset \text{SU}(6) \times \text{SU}(2) \times \text{U}(1)$

$$\mathbf{8} \rightarrow (\mathbf{6}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{2})_{-3}, \quad (3.52)$$

contains the two $\text{SU}(6)$ -singlets corresponding to the supersymmetry parameters of the truncated theory.

The $\text{SU}(6)$ -structure is defined by two generalised vectors $K, \hat{K} \in \Gamma(E)$ and a triplet of weighted adjoint elements $J_\alpha \in \Gamma((\det T^*M)^{1/2} \text{ad} F)$, with $\alpha = 1, 2, 3$, defining a highest root $\mathfrak{su}(2)$ subalgebra of $\mathfrak{e}_{7(7)}$ [20]. Together they satisfy

$$J_\alpha \cdot K = J_\alpha \cdot \hat{K} = 0 \quad \text{Tr}(J_\alpha, J_\beta) = -2\sqrt{q(K)}\delta_{\alpha\beta}, \quad (3.53)$$

with $q(K)$ the quartic invariant of $E_{7(7)}$. The vectors K and $\hat{K}$ are the only singlets in the decomposition under $E_{7(7)} \supset \text{SU}(6) \times \text{SU}(2) \times \text{U}(1)$ of the fundamental of $E_{7(7)}$

$$\mathbf{56} = (\mathbf{6}, \mathbf{2})_{-2} \oplus (\mathbf{15}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_{-6} \oplus c.c., \quad (3.54)$$

and give the graviphoton A_μ^0 and its magnetic dual. The singlets in the adjoint

$$\begin{aligned} \mathbf{133} = & (\mathbf{35}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \\ & \oplus [(\mathbf{6}, \mathbf{2})_4 \oplus (\mathbf{15}, \mathbf{1})_{-4} \oplus c.c] \oplus (\mathbf{20}, \mathbf{2})_0 \end{aligned} \quad (3.55)$$

correspond to the generators of $SU(6) \times SU(2) \times U(1)$, where the $SU(2)$ R-symmetry is generated by the invariant tensors J_α .

Moreover, since $C_{E_{7(7)}}(SU(6)) = C_{SU(8)}(SU(6)) = SU(2) \times U(1)$, from (2.11) it follows that the scalar manifold is trivial. As expected, this corresponds to the truncation to pure supergravity.

In order to obtain truncations with vector and/or hypermultiplets, the generalised structure must be a subgroup of $SU(6)$. Generically the G_S structure is defined by a set of generalised G_S -invariant vectors and adjoint elements

$$\{K_{\tilde{I}}, J_A\} \quad \tilde{I} = 0, \hat{0}, \dots, 2n_V \quad A = 1, \dots, \dim G_H \quad (3.56)$$

satisfying

$$J_A \cdot K_{\tilde{I}} = 0 \quad (3.57)$$

for any $\tilde{I} = 0, \hat{0}, \dots, 2n_V$ and $A = 1 \dots \dim G_H$. The generalised vectors $K_{\tilde{I}}$ transform as a vector of the $Sp(2 + 2n_V, \mathbb{R})$ symplectic group and satisfy

$$s(K_{\tilde{I}}, K_{\tilde{J}}) = \kappa^2 \Omega_{\tilde{I}\tilde{J}} \quad (3.58)$$

where $s(\cdot, \cdot)$ is the $E_{7(7)}$ symplectic invariant (see for instance [20]) and $\Omega_{\tilde{I}\tilde{J}}$ is the $Sp(2 + 2n_V, \mathbb{R})$ invariant matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_{n_V+1} \\ -\mathbb{1}_{n_V+1} & 0 \end{pmatrix}. \quad (3.59)$$

Condition (3.57) implies that the extra singlet generalised vectors must be invariant under the $SU(2)$ R-symmetry and therefore must come from the G_S -singlets in the $(\mathbf{15}, \mathbf{1})_2$ and its conjugate in (3.54)

$$\{K_{\tilde{I}}\} = \{K, \hat{K}, K_i, \hat{K}_i\} \quad i = 1, \dots, n_V. \quad (3.60)$$

They give the vectors in the vector multiplets and their magnetic duals

$$A^{\tilde{I}} = \{A_\mu^0, A_{0\mu}, A_\mu^i, A_{i\mu}\} \quad i = 1, \dots, n_V. \quad (3.61)$$

The invariant adjoint elements generate the group G_H

$$[J_A, J_B] = \kappa f_{AB}^C J_C, \quad (3.62)$$

of the isometries of the hypermultiplet scalar manifold and can be normalised as

$$\text{tr}(J_A J_B) = \kappa^2 \eta_{AB}, \quad (3.63)$$

where η_{AB} is a diagonal matrix with -1 and $+1$ entries in correspondence with compact and non-compact generators of G_H , respectively. The group G_H always contains the $SU(2)$

R-symmetry, as it can be seen from (3.55), with the other invariant adjoint elements coming from the G_S -singlets in the $(\mathbf{20}, \mathbf{2})_0$.

The scalar manifold is again given by (2.11). For G_S -structures that give strict $\mathcal{N} = 2$ supersymmetry, one can show that the coset manifold (2.11) factorises [18]. Indeed, the commutant $C_{E_{7(7)}}(G_S)$ cannot contain elements that change the structure while leaving the generalised metric invariant. This means that the elements of $C_{E_{7(7)}}(G_S)$ must split into two groups

$$C_{E_{7(7)}}(G_S) = C_{G_{J_A}}(G_S) \times C_{G_{K_I}}(G_S) \quad (3.64)$$

where $C_{G_{K_I}}(G_S)$ is the subgroup of $E_{7(7)}$ that leaves invariant all generalised vectors K_I while $C_{G_{J_A}}(G_S)$ is the one leaving fixed the adjoint elements J_A .⁸ Then the scalar manifold factorises as expected

$$\begin{aligned} \mathcal{M} &= \frac{C_{G_{J_A}}(G_S)}{C_{H_{J_A}}(G_S)} \times \frac{C_{G_{K_I}}(G_S)}{C_{H_{K_I}}(G_S)} \\ &= \frac{G_V}{H_V} \times \frac{G_H}{H_H} = \mathcal{M}_V \times \mathcal{M}_H. \end{aligned} \quad (3.65)$$

In the above expressions G_V and G_H (H_V and H_H) are the groups that remain after cancellation of possible common factors between the numerators and denominators, and correspond to the isometries of the vectors and hypermultiplet scalars. From (3.65) it follows that the number of non-compact invariant adjoint singlets determine the number of hypermultiplets of the truncated theory.

The two components of the embedding tensor in (3.49) are reflected in the torsion of the G_S -structure, which also has two components

$$\begin{aligned} L_{K_{\tilde{I}}} K_{\tilde{J}} &= -T_{\text{int}}(K_{\tilde{I}}) \cdot K_{\tilde{J}} = t_{\tilde{I}\tilde{J}}^{\tilde{K}} K_{\tilde{K}}, \\ L_{K_I} J_A &= -T_{\text{int}}(K_I) \cdot J_A = p_{IA}^B J_B, \end{aligned} \quad (3.66)$$

where the matrices $(t_{\tilde{I}})_{\tilde{J}}^{\tilde{K}}$ and $(p_{\tilde{I}})_A^B$ are constant and give the elements of Lie algebras of G_V and G_H respectively.

For pure $\mathcal{N} = 2$ supergravity, the intrinsic torsion of the $SU(6)$ structure contains two singlet representations

$$W_{\text{int}} \supset (\mathbf{1}, \mathbf{3})_{-3} \oplus (\mathbf{1}, \mathbf{3})_3 \quad (3.67)$$

transforming in the adjoint of $SU(2)_R$. The only non-zero component in (3.66) is $p_{\tilde{I}\alpha}^{\beta}$ with $\tilde{I} = (0, \hat{0})$ labelling the graviphoton and its dual. It corresponds to the FI-terms for the gauging of the R-symmetry action on the fermions. For $G_S \subset SU(6)$ extra singlets appear in the intrinsic torsion giving a large variety of possible gaugings. We comment further on this in the rest of the section.

We find truncations containing only vectors multiplets or hypermultiplets, and some truncations with both.

⁸In the same way, we denote by $C_{H_{K_I}}(G_S)$ and $C_{H_{J_A}}(G_S)$ denote subgroups of $SU(8)$ that leave invariant all generalised vectors K_I and all adjoint elements J_A , respectively.

3.6.1 Truncations with only vector multiplets

Truncations with no hypermultiplets correspond to G_S -structures with only a triplet of singlet invariant adjoint elements J_α , $\alpha = 1, 2, 3$, which generate the $SU(2)$ R-symmetry of the truncated theory. This means that the relevant G_S -structures embed as

$$\begin{aligned} G_S &\subset SO^*(12) \sim Spin^*(12) \subset E_{7(7)}, \\ G_S &\subset SU(6) \subset SU(8), \end{aligned} \quad (3.68)$$

where $SO^*(12)$ is the stabiliser of the triplet of J_α .

The group $SO^*(12)$ embeds in $E_{7(7)}$ as

$$SO^*(12) \times SU(2)_R \subset E_{7(7)}. \quad (3.69)$$

The compatibility condition (3.53) implies that the generalised vectors $K_{\tilde{I}}$ are invariant under the $SU(2)_R$ generated by J_α . It follows that in the decomposition of the fundamental of $E_{7(7)}$ under (3.69)

$$\mathbf{56} = (\mathbf{12}, \mathbf{2}) \oplus (\mathbf{32}', \mathbf{1}) \quad (3.70)$$

the extra G_S -invariant vectors can only come from the $(\mathbf{32}', \mathbf{1})$ component. Thus there is an upper bound of $n_V \leq 15$ vector multiplets. Decomposing the $(\mathbf{32}', \mathbf{1})$ under $SU(6) \times U(1)$ we recover the two components $(\mathbf{15}, \mathbf{1})_2 \oplus c.c$ of (3.54).

The torsion in the $\mathbf{912}$ decomposes under (3.69) as

$$\mathbf{912} = (\mathbf{351}', \mathbf{1}) \oplus (\mathbf{32}', \mathbf{3}) \oplus (\mathbf{220}, \mathbf{2}) \oplus (\mathbf{12}, \mathbf{2}). \quad (3.71)$$

For truncations with only vector multiplets, one can check that the intrinsic torsion is contained in the components $(\mathbf{351}', \mathbf{1}) \oplus (\mathbf{32}', \mathbf{3})$, where the former provides the gauging of the vector multiplet isometries and the latter of the $SU(2)_R$ symmetry. We will not discuss the intrinsic torsion and the possible gaugings in more details.

We find truncations corresponding to all scalar manifolds listed in Table 3 but the last one. This is expected since the groups at the numerator and denominator cannot be contained in $E_{7(7)}$ and $SU(8)$, respectively.

1. We find a class of truncations with scalar manifold

$$\mathcal{M}_V = \frac{SU(1, n_V)}{SU(n_V) \times U(1)} \quad (3.72)$$

with $n_V = 1, \dots, 4$, vector multiplets. Together with the graviphoton and their magnetic duals, the vectors transform in the $(\mathbf{1} + n_V) \oplus (\mathbf{1} + n_V)'$ representations of the global isometry group.

The structure group for $n_V = 1$ is

$$G_S = SU(4) \times SU(2), \quad (3.73)$$

while for $n_V = 2, 3, 4$ it reduces to

$$G_S = SU(5 - n_V) \times U(1), \quad (3.74)$$

with $SU(5 - n_V) \subset SU(4)$.

2. We also find the family of truncations with scalar manifolds

$$\mathcal{M}_V = \frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)} \times \frac{\text{SO}(2, n_V - 1)}{\text{SO}(2) \times \text{SO}(n_V - 1)} \quad (3.75)$$

and $n_V = 1, \dots, 5$ vector multiplets, corresponding to the generalised structure groups

$$G_S = \text{Spin}(7 - n_V) \times \text{SU}(2). \quad (3.76)$$

The n_V vectors coming from the $(\mathbf{15}, \mathbf{1})_2$ and $(\overline{\mathbf{15}}, \mathbf{1})_2$ in (3.54) combine with the two singlets giving the graviphoton and its magnetic dual into the $(\mathbf{2}, \mathbf{1} + n_V)$ representation of the $\text{SL}(2, \mathbb{R}) \times \text{SO}(2, n_V - 1)$ global isometry.

3. There is a truncation with $n_V = 1$ vector multiplet and the scalar manifold

$$\mathcal{M}_V = \frac{\text{SU}(1, 1)}{\text{U}(1)}. \quad (3.77)$$

It is associated to the structure group

$$G_S = \text{USp}(6) \subset \text{SU}(6). \quad (3.78)$$

Under this embedding the $(\mathbf{15}, \mathbf{1})_2$ splits into $(\mathbf{14}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_2$, giving, together with its conjugate, an extra vector and its magnetic dual. Together with the graviphoton and its magnetic dual they transform in the $\mathbf{4}$ of the global symmetry group $\text{SU}(1, 1)$. It corresponds to the third line of Table 3.

4. We also find a truncation with $n_V = 6$ vector multiplets and scalar manifold

$$\mathcal{M}_V = \frac{\text{USp}(6)}{\text{U}(3)}. \quad (3.79)$$

It is associated to $G_S = \text{SU}(2)$, which embeds in $\text{SU}(6)$ as $\text{SU}(2)_S \times \text{SU}(3) \subset \text{SU}(6)$. The decomposition $(\mathbf{15}, \mathbf{1})_2 = (\mathbf{6}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{3})$ gives the six extra vectors. Together with the graviphoton and their magnetic duals, they transform in the $\mathbf{14}'$ of the global isometry group $\text{USp}(6)$.

5. The truncation with scalar manifold

$$\mathcal{M}_V = \frac{\text{SU}(3, 3)}{\mathcal{S}[\text{SU}(3) \times \text{SU}(3)]} \quad (3.80)$$

is obtained from a $G_S = \text{U}(1)$ structure, embedded in $\text{SU}(6) \supset \text{SU}(3) \times \text{SU}(3) \times \text{U}(1)$. The theory has 9 vector multiplets, whose vectors come from the decomposition of the $(\mathbf{15}, \mathbf{1})_2 \ni (\mathbf{3}, \mathbf{3})_{0,2}$ under this breaking. The 9 vectors, the graviphoton and their magnetic dual arrange in the $\mathbf{20}$ of the $\text{SU}(3, 3)$ global isometry group.

6. Finally, we also find a truncation with $n_V = 15$ vector multiplets with scalar manifold

$$\mathcal{M}_V = \frac{\text{SO}^*(12)}{\text{SU}(6) \times \text{U}(1)}. \quad (3.81)$$

It corresponds to a $\mathbb{Z}_2$ structure, where the $\mathbb{Z}_2$ is in the centre of $\text{SO}^*(12)$, since it must commute with $\text{SU}(2)_R$

$$C_{E_{7(7)}}(\text{SU}(2)_R \times \text{SO}^*(12)) = C(\text{SO}^*(12)). \quad (3.82)$$

When embedded in $\text{SU}(6)$, the $\mathbb{Z}_2$ structure acts as a reflection on the fundamental of $\text{SU}(6)$. The vectors in $(\mathbf{15}, \mathbf{2})_2$ of (3.54) can be seen as six-dimensional 2-forms and therefore are left invariant by the $\mathbb{Z}_2$ action. Together with the graviphoton and their magnetic duals, they transform in the $\mathbf{32}'$ of the global isometry group (see also Section 4.1).

In Table 5 below we summarise the truncations we find according to the number of vector multiplets. The symbol $\star$ denote the G_S -structures coming from special branchings (see Section 4). For any truncation, we list the largest G_S -structure that allows for it.

n_V	G_S	$\mathcal{M}_v$
1	$\text{SU}(4) \times \text{SU}(2)$	$\frac{\text{SU}(1,1)}{\text{U}(1)}$
	$\text{USp}(6)^\star$	$\frac{\text{SU}(1,1)}{\text{U}(1)}$
2	$\text{Spin}(5) \times \text{SU}(2)^\star$	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^2$
	$\text{SU}(3) \times \text{U}(1)$	$\frac{\text{SU}(2,1)}{\text{SU}(2) \times \text{U}(1)}$
3	$\text{SU}(2) \times \text{U}(1)$	$\frac{\text{SU}(3,1)}{\text{SU}(3) \times \text{U}(1)}$
	$\text{Spin}(4) \times \text{SU}(2)$	$\frac{\text{SO}(2,2)}{\text{SO}(2) \times \text{SO}(2)} \times \frac{\text{SU}(1,1)}{\text{U}(1)}$
4	$\text{U}(1)$	$\frac{\text{SU}(4,1)}{\text{SU}(4) \times \text{U}(1)}$
	$\text{Spin}(3) \times \text{SU}(2)^\star$	$\frac{\text{SO}(3,2)}{\text{SO}(3) \times \text{SO}(2)} \times \frac{\text{SU}(1,1)}{\text{U}(1)}$
5	$\text{Spin}(2) \times \text{SU}(2)$	$\frac{\text{SO}(4,2)}{\text{SO}(4) \times \text{SO}(2)} \times \frac{\text{SU}(1,1)}{\text{U}(1)}$
6	$\text{SU}(2)^\star$	$\frac{\text{Sp}(6)}{\text{U}(3)}$
9	$\text{U}(1)$	$\frac{\text{SU}(3,3)}{\text{SU}(3) \times \text{SU}(3) \times \text{U}(1)}$
15	$\mathbb{Z}_2$	$\frac{\text{SO}^*(12)}{\text{SU}(6) \times \text{U}(1)}$

Table 5. $\mathcal{N} = 2$ truncations with $n_H = 0$.

The only examples of truncation with only vector multiplets are provided by the STU models. These can be obtained truncating 4-dimensional $\mathcal{N} = 8$ supergravity by imposing the invariance under the $\text{U}(1)^4$ Cartan generators of the $\text{SO}(8)$ gauge group [52]. The 4-dimensional theory is gravity coupled to three vector multiplets, where the scalar parameterise $\mathcal{M}_V = \left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^3$. The STU model has various realisations both in 11-dimensional and type II supergravity (see for instance [53–55]). In our classification it corresponds to

the truncation in item 2) with $n_V = 3$ and can also be obtained as a subtruncation of the theory in item 5) with $G_S = \text{U}(1)$.

3.6.2 Truncations with only hypermultiplets

In order to have truncations with only hypermultiplets, the G_S -structures must only admit the invariant generalised vectors K and $\hat{K}$. This corresponds to the embeddings

$$\begin{aligned} G_S &\subset \text{E}_{6(2)} \times \text{U}(1) \subset \text{E}_{7(7)}, \\ G_S &\subset \text{SU}(6) \times \text{SU}(2) \times \text{U}(1) \subset \text{SU}(8), \end{aligned} \quad (3.83)$$

where $\text{E}_{6(2)}$ is the stabiliser of the vector $X = K + i\hat{K}$ in the **56**

$$\mathbf{56} = \mathbf{1}_3 \oplus \mathbf{27}_{-1} \oplus c.c. \quad (3.84)$$

The hypersmultiplet scalars are given by the non-compact generators of $\text{E}_{7(7)}$ that are singlet of both G_S and the $\text{U}(1)$ R-symmetry and transform non-trivially under the $\text{SU}(2)$ R-symmetry. This means that, in the decomposition of the adjoint of $\text{E}_{7(7)}$ under $\text{SU}(6) \times \text{SU}(2) \times \text{U}(1) \subset \text{E}_{6(2)} \times \text{U}(1) \subset \text{E}_{7(7)}$, the extra singlets must be in the

$$(\mathbf{20}, \mathbf{2})_0 \in \mathbf{78}_0, \quad (3.85)$$

giving a maximum number of $n_H = 10$ hypermultiplets.

For truncations with only hypermultiplets the intrinsic torsion takes a very simple form. Since the only vectors are the graviphoton and its magnetic dual, only abelian gauging of the hypermultiplet isometries are possible. It is straightforward to check that for all the truncations listed below the only components of the intrinsic torsion in (3.66) are $p_{0A}{}^B$ and $p_{\hat{0}A}{}^B$ transforming in the adjoint representation of the hyperscalar isometry group.

As in the previous section, we present the truncations we find according to the geometry of the scalar manifolds as given in Table 3.

1. A first family of truncations has scalar manifold

$$\mathcal{M}_H = \frac{\text{SU}(2, n_H)}{S[\text{U}(2) \times \text{U}(n_H)]} \quad n_H = 1, 2, 3. \quad (3.86)$$

For $n_H = 2, 3$ the corresponding G_S -structure is

$$G_S = \text{SU}(2)^{4-n_H} \times \text{U}(1), \quad (3.87)$$

and, for $n_H = 1$, it enhances to $G_S = \text{SU}(3) \times \text{SU}(3)$.

2. A second family of truncations has scalar manifold

$$\mathcal{M}_H = \frac{\text{SO}(4, n_H)}{\text{SO}(4) \times \text{SO}(n_H)} \quad (3.88)$$

for $n_H = 2, \dots, 6$. The corresponding G_S -structures are listed below. For $n_H = 5$ the truncation automatically enhances to $n_H = 6$ (see Section 4.1).

n_H	2	3	4	6
G_S	$SU(2) \times SU(2) \times U(1)$	$SU(2) \times U(1)$	$U(1) \times U(1)$	$U(1)$

3. The scalar manifold

$$\mathcal{M}_H = \frac{G_{2(2)}}{SU(2) \times SU(2)} \quad (3.89)$$

corresponds to a truncation with $n_H = 2$ hypermultiplets. The truncation comes from a $SU(3)$ -structure, which embeds as $SU(3)_S \times G_{2(2)} \subset E_{6(2)}$ and $SU(3)_S \times SU(2) \subset SU(6)$.

4. The truncation with scalar manifold

$$\mathcal{M}_H = \frac{E_{6(2)}}{SU(2) \times SU(6)} \quad (3.90)$$

is obtained with a $G_S = \mathbb{Z}_3$ structure (see Section 4.1 for an explicit expression). The only $\mathbb{Z}_3$ -invariant terms in the fundamental of $E_{7(7)}$ are the $E_{6(2)}$ -singlets, so that there are no vector multiplets. On the contrary, $\mathbb{Z}_3$ acts trivially on the whole $(\mathbf{20}, \mathbf{2})$ in the adjoint, thus giving $n_H = 10$ hypermultiplets. It is easy to verify that the $\mathbb{Z}_3$ singlets in the $(\mathbf{20}, \mathbf{2})$ span the coset (3.90).

We do not find some of the quaternionic scalar manifolds listed in Table 3. Clearly the scalar manifolds

$$\mathcal{M}_H = \frac{E_{7(-5)}}{SU(2) \times SO(12)} \quad \text{and} \quad \mathcal{M}_H = \frac{E_{8(-24)}}{SU(2) \times E_{7(7)}} \quad (3.91)$$

are too big to be realised in truncations of $E_{7(7)}$ generalised geometry. On the contrary, the manifolds

$$\mathcal{M}_H = \frac{F_{4(4)}}{SU(2) \times USp(6)} \quad n_H = 7 \quad (3.92)$$

and

$$\mathcal{M}_H = \frac{USp(2, 2n_H)}{USp(2) \times USp(2n_H)} \quad n_H \leq 3 \quad (3.93)$$

are in principle allowed. However, they cannot be obtained in a consistent truncation since the G_S -structures that are compatible with them always contain extra singlets and hence give larger truncations.

Consider first the manifold (3.92). To obtain it, G_S must be discrete since (2.11) implies

$$G_S = C_{SU(8)}(USp(6) \times SU(2) \times U(1)). \quad (3.94)$$

Moreover, from the embedding of $\mathrm{USp}(6)$ in $\mathrm{SU}(6)$ and Schur's lemma it follows that G_S can only be of the form $(a\mathbb{1}_6, b\mathbb{1}_2)$, and so it also commutes with $\mathrm{SU}(6)$, with a number of adjoint singlets allowing to reconstruct the full coset (3.90) and not simply (3.92).

The same reasoning holds for the scalar manifolds (3.93). Shur's lemma guarantees that $\mathrm{USp}(2n)$ always enhances to $\mathrm{SU}(2n)$. So the scalar manifolds corresponding to $n_H = 3$ and $n_H = 2$, are both enhanced to the truncation (3.90), while that with $n_H = 1$ enhances to the truncation in (3.86) with $n_H = 2$.

n_H	G_S	$\mathcal{M}_h$
1	$\mathrm{SU}(3) \times \mathrm{SU}(3)$	$\frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)}$
2	$\mathrm{SU}(3)^*$	$\frac{\mathrm{G}_2}{\mathrm{SO}(4)}$
	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$\frac{\mathrm{SU}(2,2)}{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)}$
3	$\mathrm{SU}(2) \times \mathrm{U}(1)^*$	$\frac{\mathrm{SO}(4,3)}{\mathrm{SO}(4) \times \mathrm{SO}(3)}$
	$\mathrm{SU}(2) \times \mathrm{U}(1)$	$\frac{\mathrm{SU}(2,3)}{\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{U}(1)}$
4	$\mathrm{U}(1)^2$	$\frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)}$
6	$\mathrm{U}(1)$	$\frac{\mathrm{SO}(4,6)}{\mathrm{SO}(4) \times \mathrm{SO}(6)}$
10	$\mathbb{Z}_3$	$\frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(6) \times \mathrm{SU}(2)}$

Table 6. $\mathcal{N} = 2$ truncations with $n_V = 0$. For any truncation, we list the largest G_S -structure that allows for it.

3.6.3 Truncations with vector and hypermultiplets

We also find a number of truncations with both vector and hypermultiplets.

The truncations associated to continuous G_S -structures are quite limited and are listed in Table 9. Several such truncations already appeared in the literature. For instance, we recover all $\mathcal{N} = 2$ consistent truncations of 11-dimensional supergravity on coset manifolds derived in [7]. The truncation with $n_V = n_H = 1$ and $G_S = \mathrm{SU}(3)$ corresponds to the universal truncation on a SE_7 manifold originally derived in [3]. A particular example is the truncation on $S^7 = \frac{\mathrm{SU}(4)}{\mathrm{SU}(3)}$.⁹ The truncation on $V_{5,2}$ with $n_V = 1$ and $n_H = 2$ is obtained from a $G_S = \mathrm{SO}(3)$ structure, while that on M^{110} with $n_V = 2$ and $n_H = 1$ corresponds to $G_S = \mathrm{SU}(2) \times \mathrm{U}(1)$. The truncations with $n_V = 3$ and $n_H = 1$ on Q^{111} and $N(k, l)$ are associated to a $G_S = \mathrm{U}(1)^2$ and the truncation on $N(1, -1)$ with $n_V = 5$ and $n_H = 1$ is given by a $\mathrm{U}(1)$ -structure.

The $\mathrm{SO}(3)$ -structure with $n_V = 1$ vector and $n_H = 2$ hypermultiplets is also associated to the consistent truncations derived in [58] of 11-dimensional supergravity on $\Sigma_3 \times S^4$, where $\Sigma_3 = H_3/\Gamma, S^3/\Gamma$ or $\mathbb{R}^3/\Gamma$, with Γ a discrete group of isometries.

⁹This truncations was extended in [56] to include a skew-wiffing mode. In the context EGG both truncations have been reproduced in [57].

We also recover the consistent type IIA truncations on coset manifolds discussed in [5]. The coset manifolds are $\frac{G_2}{\text{SU}(3)}$, $\frac{\text{Sp}(2)}{\text{SU}(2) \times \text{U}(1)}$ and $\frac{\text{SU}(3)}{\text{U}(1) \times \text{U}(1)}$, and the truncated theories contain $n_H = 1$ hypermultiplets and $n_V = 1$, $n_V = 2$ and $n_V = 3$ vector multiplets, respectively. They correspond to the G_S -structure $\text{SU}(3)$, $\text{SU}(2) \times \text{U}(1)$ and $\text{U}(1)^2$ in Table 9.

The set of possible truncations enlarges if we consider discrete structure groups or direct products of a continuous and a discrete factor. For truncations of this kind we just give some examples so that, differently from the case of continuous structure groups, our list is not exhaustive.

Discrete structures appear when, starting from a truncation with only vector (or hyper) multiplets, we add extra hyper (vectors) to have a maximum matter content (see Section 4.1). In this way, we find the two truncations in Table 7, with purely discrete structure groups.

G_S	n_V	n_H	$\mathcal{M}_V \times \mathcal{M}_H$
$\mathbb{Z}_4$	1	6	$\frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SO}(4,6)}{\text{SO}(4) \times \text{SO}(6)}$
$\mathbb{Z}_3$	9	1	$\frac{\text{SU}(3,3)}{\text{SU}(3) \times \text{SU}(3) \times \text{U}(1)} \times \frac{\text{SU}(2,1)}{\text{SU}(2) \times \text{U}(1)}$

Table 7. Discrete groups with $\mathcal{N} = 2$.

Another family of truncations associated to a discrete structure is obtained starting from $\mathcal{N} = 4$ truncations and imposing a G_S -structure that is the (direct) product of the original G_S -structure times a discrete factor. A non-exhaustive list of truncations of this type is provided in Table 8.

G_S	n_V	n_H	$\mathcal{M}_V \times \mathcal{M}_H$
$\text{SU}(2) \times \mathbb{Z}_2$	1	3	$\frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SO}(4,3)}{\text{SO}(4) \times \text{SO}(3)}$
$\text{U}(1) \times \mathbb{Z}_2$	1	4	$\frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SO}(4,4)}{\text{SO}(4) \times \text{SO}(4)}$
$\text{U}(1) \times \mathbb{Z}_2$	2	3	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^2 \times \frac{\text{SO}(4,3)}{\text{SO}(4) \times \text{SO}(3)}$
$\text{SU}(2) \times \mathbb{Z}_2$	2	2	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^2 \times \frac{\text{SO}(4,2)}{\text{SO}(4) \times \text{SO}(2)}$
$\text{U}(1) \times \mathbb{Z}_2$	3	2	$\left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)^3 \times \frac{\text{SO}(4,3)}{\text{SO}(4) \times \text{SO}(3)}$

Table 8. Discrete groups with $\mathcal{N} = 2$ from $\mathcal{N} = 4$.

The truncations with $n_V = 2$ and $n_H = 2$ on $\frac{\text{Sp}(2)}{\text{Sp}(1)}$ and $n_V = 3$ and $n_H = 2$ on $N(1,1)$ mentioned in [7] are in this class. They correspond to the last two lines of Table 8. The $\frac{\text{Sp}(2)}{\text{Sp}(1)}$ theory is obtained imposing a $\mathbb{Z}_2$ invariance on the $\mathcal{N} = 4$ theory with $n_V = 3$ vector multiplets of Section 3.4. Similarly, the $N(1,1)$ is given by a $\mathbb{Z}_2 \times \text{U}(1)$ structure and comes from the $\mathcal{N} = 4$ theory with $n_V = 4$ vector multiplets.

In all these cases the embedding tensor is easily derived decomposing the **912** under $G_S \times G_{\text{iso}}$. We will not discuss it in this paper.

$n_H \backslash n_V$	1	2	3
1	$G_S = \text{SU}(3)^*$	$\mathcal{M}_V = \frac{\text{SU}(1,1)}{\text{U}(1)}$ $\mathcal{M}_H = \frac{\text{G}_2}{\text{SO}(4)}$	$G_S = \text{U}(1)$ $\mathcal{M}_V = \frac{\text{SU}(1,1)}{\text{U}(1)}$ $\mathcal{M}_H = \frac{\text{SU}(2,3)}{\text{SU}(2) \times \text{SU}(3) \times \text{U}(1)}$
	$G_S = \text{SU}(2) \times \text{SU}(2) \times \text{U}(1)$	$\mathcal{M}_V = \frac{\text{SU}(1,1)}{\text{U}(1)}$ $\mathcal{M}_H = \frac{\text{SU}(2,2)}{\text{SU}(2) \times \text{SU}(2) \times \text{U}(1)}$	
	$G_S = \text{U}(1)$	$G_S = \text{U}(1)$	
2	$G_S = \text{SU}(2) \times \text{U}(1)^*$	$\mathcal{M}_V = \left(\frac{\text{SU}(1,1)}{\text{U}(1)} \right)^2$ $\mathcal{M}_H = \frac{\text{SU}(2,1)}{\text{SU}(2) \times \text{U}(1)}$	–
	$G_S = \text{SU}(2) \times \text{U}(1)$	$\mathcal{M}_V = \frac{\text{SU}(2,1)}{\text{SU}(2) \times \text{U}(1)}$ $\mathcal{M}_H = \frac{\text{SU}(2,1)}{\text{SU}(2) \times \text{U}(1)}$	
3	$G_S = \text{U}(1)^2$	–	–
5	$G_S = \text{U}(1)$	–	–

Table 9. $\mathcal{N} = 2$ truncations with n_V and n_H

4 Scanning through supersymmetry and field content

In this section we provide some details about the derivation of the results discussed in Section 3. As already mentioned in the previous sections, the classification consists in determining all generalised G_S -structures with singlet intrinsic torsion that give rise to inequivalent truncated theories with different amount of supersymmetry.

Since we are interested in supersymmetric truncations, the allowed G_S -structures must be subgroups of $SU(8)$, the double cover of the maximally compact subgroup of $E_{7(7)}$, under which the spinors of the theory transform. We first consider continuous G_S -structures and then, at the end of the section, we comment on discrete ones.

We first focus on truncations to pure supergravity with $2 \leq \mathcal{N} < 8$. For these cases, for any amount of supersymmetry $\mathcal{N}$, the corresponding structure G_S^{max} is the largest generalised structure compatible with the given amount of supersymmetry, and it is determined by

$$G_S^{max} = C_{SU(8)}(G_R), \quad (4.1)$$

where G_R is the R-symmetry group. Both G_R and the corresponding G_S^{max} are listed in Table 1. The idea is to find the explicit embedding of G_S^{max} in the $E_{7(7)}$ and $SU(8)$ generators given in Appendix A, for any fixed $\mathcal{N}$.

An economic way to do so is to use the fact that all the branchings in Table 1

$$SU(8) \supset G_S^{max} \times G_R, \quad (4.2)$$

correspond to maximal regular subalgebrae $\mathfrak{h}$ of $\mathfrak{su}(8)$.¹⁰ Then we find it useful to derive the allowed G_S^{max} -structures just by looking at their Cartan subalgebrae.

The idea is to construct a generic $U(1)$ -structure as a linear combination of the Cartan subalgebra of $SU(8)$

$$\mu_{\vec{\lambda}} = \sum_{i=1}^7 \lambda_i H_i \quad \lambda_i \in \mathbb{N}, \quad (4.3)$$

where H_i are the Cartan generators of $SU(8)$.¹¹ We let the number λ_i run over $n_i = 0, \dots, N$ with $N \in \mathbb{N}$. A priori the coefficient λ_i could be real numbers. However, since the generators H_i only have rational entries and multiplication by a global factor will not change the $U(1)$ -structure, we can restrict our study to $\lambda_1, \dots, \lambda_7 \in \mathbb{Z}$. Then, using the freedom in the tracelessness condition, it is possible to consider only positive λ_i . Any set $\vec{\lambda} = \{\lambda_1, \dots, \lambda_7\}$ defines a different $U(1)$ -structure, $\mu_{\vec{\lambda}}$.

¹⁰In this paper we need to distinguish between regular and special maximal subalgebrae of a Lie group G [59, 60]. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a maximal regular subalgebra of $\mathfrak{g}$ if it has the same rank as $\mathfrak{g}$, if not, $\mathfrak{h}$ is called special. We are interested in the embedding of the algebra $\mathfrak{g}_S$ of the G_S -structure group into $\mathfrak{h}$. We call the embedding $\mathfrak{g}_S \subset \mathfrak{h}$ a regular branching or a special branching depending on whether $\mathfrak{h}$ is a regular or special (maximal) subalgebra.

¹¹All Cartan algebrae are isomorphic, so it is enough to study just one of them. We take $(H_i)^\alpha{}_\beta = i\delta_i^\alpha \delta_{i\beta} - \frac{i}{8}\delta_\beta^\alpha$ (see Appendix A for our conventions).

Now we look for singlets under $\mu_{\vec{\lambda}}$ in the spinorial and adjoint of $SU(8)$, and in the fundamental and adjoint representations of $E_{7(7)}$

$$\begin{aligned}\mu_{\vec{\lambda}} \cdot \epsilon &= 0, \\ \mu_{\vec{\lambda}} \cdot V &= 0, \\ \mu_{\vec{\lambda}} \cdot R &= 0, \\ \mu_{\vec{\lambda}} \cdot R_{SU(8)} &= 0,\end{aligned}\tag{4.4}$$

where $\epsilon \in \mathbf{8}$ and $R_{SU(8)} \in \mathbf{63}$ of $SU(8)$, while $V \in \mathbf{56}$ and $R \in \mathbf{133}$ of $E_{7(7)}$. The explicit expressions for the various representations are given in Appendix A.

The first equation (4.4) determines the number $\mathcal{N}$ of supersymmetries preseved by the theory, whereas the remaining three determine the bosonic field content. More precisely the second equation gives the number $2n_V$ of invariant generalised vectors, namely the vectors of the truncated theory, together with their magnetic duals. The third equation determines the number n_{adF} of $E_{7(7)}$ generators that commute with the structure, while the fourth gives the number n_c of compact ones. The difference $n_{\text{adF}} - n_c$ between the $E_{7(7)}$ and the $SU(8)$ -singlets gives the number of scalars in the truncated theory.

With the help of Mathematica, we then look for solutions of (4.4) and classify them according the number of singlets in the various representations

$$\mu_{\vec{\lambda}} \leftrightarrow \{\mathcal{N}, 2n_V, n_{\text{adF}}, n_c\}.\tag{4.5}$$

If two $U(1)$ -structures $\mu_{\vec{\lambda}}$ give the same singlet content, they are considered as equivalent. Notice that from the knowledge of the $U(1)$ generator we can reconstruct the full structure group by looking at the commutation relations among the $U(1)$ singlets in the adjoint of $SU(8)$.

For $\mathcal{N} \geq 5$ we find only one solution for each amount of supersymmetry, characterised by the singlets

$\mathcal{N}$	8	6	5
n_V	28	16	10
$n_{\text{adF}} - n_c$	70	30	10

The $\mathcal{N} = 8$ truncation is trivially realised as it corresponds to the identity structure (4.3), while for $\mathcal{N} = 6$ and $\mathcal{N} = 5$ we recover $G_S^{\text{max}} = SU(2)$ and $G_S^{\text{max}} = SU(3)$. As expected, the solutions are unique since the theories only contain the gravity multiplet. The singlets in the table above reproduce the field content of the gravity multiplet and the dimension of the associated scalar manifolds.

For $2 \leq \mathcal{N} \leq 4$ we find several independent solutions $\mu_{\vec{\lambda}}$ for each $\mathcal{N}$. The solutions corresponding to G_S^{max} are singled out by the values

$\mathcal{N}$	4	3	2
n_V	6	3	1
$n_{\text{adF}} - n_c$	2	0	0

As in the previous cases we can reconstruct the full structures $G_S^{max} = \text{SU}(8 - \mathcal{N})$.

For any given $\mathcal{N}$, the other solutions $\mu_{\vec{\lambda}}$ correspond to truncations with extra matter fields. The values n_V , n_{adF} and n_c allow to derive their field content, while the $G_S \subset G_S^{max}$ can be reconstructed from the corresponding $\text{U}(1)$ generator.

However, these solutions do not exhaust all possible truncations with $2 \leq \mathcal{N} \leq 4$ supersymmetry and continuous G_S -structures. This is because the trick of looking only at the Cartan subalgebrae holds only for subgroups coming from regular subalgebrae of $\text{SU}(8)$.¹² For G_S -structures coming from special subalgebrae, we have not been able to find a similar algorithmic procedure.

For this reason, we proceeded to a systematic scan of the different subgroup $G_S \subset G_S^{max}$ for fixed $\mathcal{N}$ and looked for solutions of equations (4.4) where now $\mu_{\vec{\lambda}}$ is replaced by a generic element of the algebra G_S , embedded in $E_{7(7)}$ and $\text{SU}(8)$ via G_S^{max} . This analysis gives the results discussed in Section 3. As an example, the explicit derivation for $\mathcal{N} = 4$ is given in Appendix B.

4.1 Discrete structures

The knowledge of the explicit embedding of G_S^{max} allow us to study also some discrete structures.

Discrete structures are subtler and classifying them all is out of the scope of this paper. However, some cases can be easily studied. The analysis is the same as for continuous groups, with the difference that now we have to solve

$$\begin{aligned} g \cdot \epsilon &= \epsilon, \\ g \cdot R_{\text{SU}(8)} &= R_{\text{SU}(8)}, \\ g \cdot V &= V, \\ g \cdot R &= R, \end{aligned} \tag{4.6}$$

where g is any element of the discrete group.

A first instance where discrete G_S -structures appear is in truncations with $2 \leq \mathcal{N} \leq 4$ supersymmetry and the largest number of matter multiplets.

Consider first the truncation to $\mathcal{N} = 2$ supergravity with $n_V = 15$ vector multiplets and no hypermultiplets (see Section 3.6.1). Recall that truncations with only vector multiplets are given by G_S -structures

$$G_S \subset \text{SU}(6) \subset \text{SO}^*(12), \tag{4.7}$$

where $\text{SO}^*(12)$ is the stabiliser of the triplet of adjoint singlets generating the $\text{SU}(2)$ R-symmetry. Moreover, the compatibility condition (3.53) implies that the singlets generalised vectors can only belong the representation $(\mathbf{32}', \mathbf{1})$ in the decomposition (3.70) of

¹²For any regular structure $G_S \subset G_S^{max}$, the roots of G_S are a subset of those of G_S^{max} . Thus, as it happens with the roots, the weights of G_S are a subset of those of G_S^{max} as well. The $\text{U}(1)$ representing G_S is regular in both G_S and G_S^{max} and, therefore, it preserves information about the weights. This allows to reproduce the G_S -structure field content with a $\text{U}(1)$ -structure. For special cases roots and weights can not be found as subsets of the ones of G_S^{max} , so one is forced to study them independently.

the fundamental of $E_{7(7)}$ under $SO^*(12) \times SU(2)_R$, which in turn splits as

$$(\mathbf{32}', \mathbf{1}) = (\mathbf{1}, \mathbf{1})_{-6} \oplus (\mathbf{1}, \mathbf{1})_6 \oplus (\mathbf{15}, \mathbf{1})_2 \oplus (\overline{\mathbf{15}}, \mathbf{1})_{-2} \quad (4.8)$$

under $E_{7(7)} \supset SO^*(12) \times SU(2)_R \supset SU(6) \times U(1)_R \times SU(2)_R$.

In order to preserve all 15 vectors, the group G_S must commute with the whole $SU(6)$. Since we want G_S to be also a subgroup of $SU(6)$, we expect it to be an element of the center of $SU(6)$. It is easy to check that $G_S = \mathbb{Z}_2$

$$g_{\mathbb{Z}_2} = \begin{pmatrix} -\mathbb{1}_6 & \\ & \mathbb{1}_2 \end{pmatrix} \in SU(8). \quad (4.9)$$

Consider now a generalised vector in the notation of (A.41)

$$V^{\alpha\beta} = (V^{mn}, V^{mi}, V^{ij}), \quad (4.10)$$

where $\alpha = (m, i)$, $i = 1, 2 \in SU(2)$ and $m = 1, \dots, 6 \in SU(6)$. The action of $g_{\mathbb{Z}_2}$ on V leaves invariant the components V^{ij} and V^{mn} , corresponding to the generalised vectors K and $\hat{K}$ and the 15 vector multiplets, respectively.

Using the expressions in Appendix A for the generators of $SO^*(12)$ in terms of those of $SU(6)$, one can check that $G_S = \mathbb{Z}_2$ belongs to the center of $SO^*(12)$.¹³ Thus from (2.11), we recover, as expected, the scalar manifold

$$\mathcal{M}_V = \frac{SO^*(12)}{SU(6) \times U(1)}. \quad (4.11)$$

The truncation to $\mathcal{N} = 2$ supergravity with no vector multiplets and $n_H = 10$ hypermultiplets is obtained along the same lines. In this case we look for a subgroup

$$G_S \subset SU(6) \subset E_{6(2)}, \quad (4.12)$$

with $E_{6(2)}$ the stabiliser of the generalised vectors K and $\hat{K}$, and

$$E_{7(7)} \supset E_{6(2)} \times U(1)_R. \quad (4.13)$$

The extra hypermultiplets come from the component $(\mathbf{20}, \mathbf{2})_0$ in the decomposition of the $\mathbf{78}_0$ under $SU(6) \times SU(2)_R$

$$\mathbf{78}_0 = (\mathbf{35}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{20}, \mathbf{2})_0. \quad (4.14)$$

The structure that leaves invariant the whole $(\mathbf{20}, \mathbf{2})_0$ is now $G_S = \mathbb{Z}_3$, which is again in the centre of $SU(6)$ and $E_{6(2)}$, and embeds in $SU(8)$ as

$$g_{\mathbb{Z}_3} = e^{\frac{2}{3}\pi i \mu_{U(1)_R}} = \begin{pmatrix} e^{\frac{2\pi}{3}i} \mathbb{1}_6 & \\ & \mathbb{1}_2 \end{pmatrix}, \quad (4.15)$$

¹³It is interesting to note that the same $\mathbb{Z}_2$ is contained in all the G_S of Table 5 and, in all cases it belongs to the center.

where $\mu_{U(1)_R}$ is given in (A.40).¹⁴ The action of g_{Z_3} on a generalised vector (4.10) leaves invariant only the component V^{ij} , namely the generalised vectors K and $\hat{K}$. Using again (2.11), the scalar coset manifold is

$$\mathcal{M}_H = \frac{E_{6(2)}}{SU(6) \times SU(2)}. \quad (4.16)$$

Discrete structures can also be used to construct $\mathcal{N} = 2$ truncations with a maximal amount of vector and hypermultiplets. The idea is to start with a truncation with only vectors (hypers) and to enhance it to a truncation preserving the same number of vectors (hypers) and a maximum number of hypers (vectors) with a discrete G'_S .

To see how it works consider the truncation of Section 3.6.1 with only one vector multiplet and scalar manifold

$$\mathcal{M}_V = \frac{SU(1,1)}{U(1)}. \quad (4.17)$$

The corresponding structure is $G_S = SU(4) \times SU(2)$ and it embeds in $SU(8)$ as

$$SU(8) \supset SU(4) \times SU(2) \times SU(2)_R \times U(1)_R. \quad (4.18)$$

In order to leave invariant the same amount of vectors, the new structure must be a subgroup of $SU(6)$ that also commutes with $SU(4) \times SU(2)$. These requirements lead to a discrete structure of the form

$$g_{G'_S} = \begin{pmatrix} e^{\theta i} \mathbb{1}_4 & & \\ & e^{-2\theta i} \mathbb{1}_2 & \\ & & \mathbb{1}_2 \end{pmatrix}. \quad (4.19)$$

By construction

$$C_{SU(8)}(G'_S) = SU(4) \times SU(2) \times SU(2)_R \times U(1)_R, \quad (4.20)$$

which now becomes the denominator of the full scalar manifold. The $U(1)_R$ is the denominator of (4.17), while

$$SU(4) \times SU(2) \times SU(2)_R \sim SO(6) \times SO(4) \quad (4.21)$$

is the denominator of the quaternionic Kähler manifold

$$\mathcal{M}_H = \frac{SO(4,6)}{SO(4) \times SO(6)}. \quad (4.22)$$

By embedding (4.19) in $E_{7(7)}$ it is straightforward to verify that, for $\theta = \frac{\pi}{2}$, we have

$$C_{E_{7(7)}}(G'_S) = SU(1,1) \times SO(4,6). \quad (4.23)$$

This is the truncation in the first line of Table 7. The one in the second line is obtained similarly.

¹⁴As it happens for truncations with only vector multiplets, the same Z_3 is contained in all the G_S of Table 6 and, in all cases, it belongs to the centre.

The same approach could in principle be applied to any truncation with only vector or hypermultiplets by taking higher-order $\mathbb{Z}_p$ -structures. However it is beyond the scope of this paper to perform a complete analysis of such truncations.

We simply want to mention that there are cases where we know from the start that truncations of this kind cannot be constructed. The obstruction comes again from Schur's lemma.

Consider, as an example, the truncation with $n_V = 2$ and $n_H = 0$, associated to $G_S = \text{USp}(4) \times \text{SU}(2)$. We could try to extend the truncation by adding $n_H = 5$ hypermultiplets to obtain the scalar manifold

$$\mathcal{M}_V \times \mathcal{M}_H = \left(\frac{\text{SU}(1,1)}{\text{U}(1)} \right)^2 \times \frac{\text{SO}(4,5)}{\text{SO}(4) \times \text{SO}(5)}, \quad (4.24)$$

with $\text{SO}(5) \sim \text{USp}(4)$ and $\text{SO}(4) \sim \text{SU}(2) \times \text{SU}(2)_R$. However, since the embedding of $\text{USp}(4)$ in $\text{SU}(4)$ is such that the **4** of $\text{SU}(4)$ goes into the **4** of $\text{USp}(4)$, Schur's lemma implies that any group element commuting with $G_S = \text{USp}(4) \times \text{SU}(2)$ commutes with $\text{SU}(4) \times \text{SU}(2)$ as well. Thus, instead of the truncation in (4.24) we end up again with the truncation of the previous example.

We checked that this behaviour is true whenever the original G_S -structure is a $\text{USp}(n)$ group. In addition, this behaviour holds in all cases where the original G_S -structure corresponds to a special branching of $\text{SU}(6)$ where the fundamental of $\text{SU}(6)$ decomposes into just one irreducible representation of G_S . As an example, consider the branching

$$\text{SU}(2) \times \text{SU}(3) \subset \text{SU}(6), \quad (4.25)$$

in which the fundamental of $\text{SU}(6)$ breaks as $\mathbf{6} = (\mathbf{2}, \mathbf{3})$. This branching leads to either the truncation with structure $G_S = \text{SU}(2)$, $n_V = 6$ vector multiplets and coset manifold $\frac{\text{Sp}(6)}{\text{U}(3)}$ in (3.79) or the truncation with $G_S = \text{SU}(3)$, $n_H = 2$ hypermultiplets and coset manifold $\frac{\text{G}_2}{\text{SO}(4)}$ in (3.89). However, it is not possible to find a discrete structure leading to a truncation with vector and hypermultiplets with scalar manifold $\mathcal{M}_V \times \mathcal{M}_H = \frac{\text{Sp}(6)}{\text{U}(3)} \times \frac{\text{G}_2}{\text{SO}(4)}$. Indeed Schur's lemma implies that any discrete structure commuting with $\text{SU}(2) \times \text{SU}(3)$ will commute with $\text{SU}(6)$ as well, leaving the two possibilities in (3.81) or (3.90).

Let us stress that this is just an observation derived from examples, which could provide a hint at how some discrete structures can be found. It is not meant to be a formal proof.

5 Review of consistent truncations to 5, 6 and 7 dimensions

The classification of the supergravity theories that can be obtained as consistent truncations of 11/10-dimensional supergravity can be performed along the same lines at least for truncations to $D \geq 4$ dimensions. Of particular interest are the truncations to 5, 6 and 7 dimensions since they provide valuable tools in many instances of the gauge/gravity duality.

The difference with respect to the analysis of Section 3 is in the exceptional group $E_{d(d)}$ determining the theory, which changes depending on the dimension of the compactification

manifold. The generalised structure groups, the (double cover of) the maximal compact subgroups and the relevant tensor bundles for truncations to 5, 6 and 7 dimensions are listed in Table 10 below

D	$E_{d(d)}$	E	$\text{ad}F$	N	W	$\tilde{H}_d$	$\mathcal{S}$
5	$E_{6(6)}$	$\mathbf{27}_1$	$\mathbf{78}_0$	$\mathbf{27}'_2$	$\mathbf{351}'_{-1}$	$\text{USp}(8)$	$\mathbf{8}$
6	$\text{Spin}(5, 5)$	$\mathbf{16}^c_{-1}$	$\mathbf{45}_0$	$\mathbf{10}_2$	$\mathbf{144}^c_{-1}$	$\text{USp}(4) \times \text{USp}(4)$	$(\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4})$
7	$\text{SL}(5, \mathbb{R})$	$\mathbf{10}_1$	$\mathbf{24}_0$	$\mathbf{5}_2$	$\mathbf{40}_{-1} \oplus \mathbf{15}'_{-1}$	$\text{USp}(4)$	$\mathbf{4}$

Table 10. Exceptional geometries for truncations to 5,6 and 7 dimensions

Truncations to $D = 5, 6, 7$ within generalised geometry have already been studied in the literature (see for instance [8–19]). In this section, for completeness, we summarise the main results, present them all in the language of Exceptional Generalised Geometry and complete them when needed.

5.1 Truncations to 5 dimensions

Consistent truncations to 5 dimensions are associated to generalised structures in $E_{6(6)}$ generalised geometry and have been studied in [9, 10, 14, 17, 18].

In 5 dimensions supergravity theories exist with $2 \leq \mathcal{N} \leq 8$ supercharges. The supersymmetry parameters ϵ_i , with $i = 1, \dots, \mathcal{N}$, transform in the fundamental of the $\text{USp}(\mathcal{N})$ R-symmetry group and are symplectic (pseudo) Majorana spinors: $\epsilon^i = \Omega^{ij} \epsilon_j^c$, where Ω^{ij} is the $\text{USp}(\mathcal{N})$ symplectic invariant. Thus only even numbers of supercharges are allowed.

Truncations to $\mathcal{N} \geq 4$ supergravity are associated to generalised G_S -structures defined only by globally invariant vectors, while for $\mathcal{N} = 2$ supersymmetry invariant adjoint elements are also needed.

5.1.1 $\mathcal{N} = 8$ supergravity

Five-dimensional ungauged supergravity with maximal symmetry was constructed in [61] and its gauging have been studied in several papers (see for instance [62–64]). The field content consists of the graviton, 8 gravitini, 27 vectors, 28 gravitini and 42 scalars. The fermions transform in the $\mathbf{8}$ and $\mathbf{48}$ of $\text{USp}(8)$, while the vector fields transform in the fundamental of $E_{6(6)}$. The scalar parameterise the manifold

$$\mathcal{M} = \frac{E_{6(6)}}{\text{USp}(8)}. \quad (5.1)$$

The truncation corresponds to a generalised Scherk-Schwarz reduction, where the $\mathbf{27}$ globally generalised vectors K_I , $I = 1, \dots, 27$, define a Leibniz parallelisation [9, 10]

$$L_{K_I} K_J = X_{IJ}{}^K K_K, \quad (5.2)$$

with X_{IJ}^K constant and again $[X_I, X_J] = -X_{IJ}^K X_K$. The vectors are normalised to $G(K_I, K_J) = \delta_{IJ}$, with G the generalised metric. They give the 27 vectors of the truncated theory, while from (2.11) one recovers the scalar manifold (5.1).

The intrinsic torsion X_{IJ}^K transforms in the **351** and gives the embedding tensor of the truncated theory.

$\mathcal{N} = 8$ supergravity with gauge group $\text{SO}(6)$ [62–64] is obtained as the truncation of type IIB on $AdS_5 \times S^5$, whose consistency was proven only thanks to the results of [9, 10]. In this case, decomposing the singlet intrinsic torsion under $E_{6(6)} \supset \text{SL}(6, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$

$$\mathbf{351} = (\mathbf{21}, \mathbf{1}) + (\mathbf{15}, \mathbf{3}) + (\mathbf{84}, \mathbf{2}) + (\mathbf{6}, \mathbf{2}) + (\mathbf{105}, \mathbf{1}), \quad (5.3)$$

one can show [9] that X_{IJ}^K belongs to the component $(\mathbf{21}, \mathbf{1})$ and takes the form

$$\begin{aligned} X_{[ii'][jj']}^{[kk']} &= R^{-1}(\delta_{ij}\delta_{i'j'}^{kk'} - \delta_{i'j}\delta_{ij'}^{kk'} - \delta_{ij'}\delta_{i'j}^{kk'} + \delta_{i'j'}\delta_{ij}^{kk'}), \\ X_{[ii']\beta k}^{\gamma j} &= R^{-1}(\delta_{ik}\delta_{i'}^j - \delta_{i'k}\delta_i^j)\delta_\beta^\gamma, \end{aligned} \quad (5.4)$$

where $i, j \dots = 1, \dots, 6$ are $\text{SL}(6, \mathbb{R})$ indices and R is the radius of S^5 . The tensors above give the embedding tensor for the $\text{SO}(6)$ electric gauging. The consistency of other compact and non-compact gaugings can also be obtained in this way [11].

5.1.2 $\mathcal{N} = 6$ supergravity

$\mathcal{N} = 6$ pure supergravity with gauge group $\text{SU}(3) \times \text{U}(1)$ was constructed in [63] as a consistent truncation of the $\mathcal{N} = 8$ theory. The fields are arranged in the graviton multiplet, which consists of the graviton, 6 gravitini, 15 vectors, 20 spin 1/2 and 14 scalars.

The $\mathcal{N} = 6$ theory corresponds to a truncation with a generalised $G_S = \text{SU}(2)$ structure, defined by 15 invariant vectors K_I , as it can be seen from the embedding

$$\begin{aligned} E_{6(6)} &\supset \text{SU}^*(6) \times \text{SU}(2) \\ \mathbf{27} &= (\overline{\mathbf{15}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}). \end{aligned} \quad (5.5)$$

The generalised vectors are normalised to $G(K_I, K_J) = \delta_{IJ}$, with G the generalised metric, and give the 15 vectors in the graviton multiplet. Using again (2.11) with $C_{\text{USp}(8)}(\text{SU}(2)) = \text{USp}(6)$, we recover the scalar manifold

$$\mathcal{M} = \frac{\text{SU}^*(6)}{\text{USp}(6)}. \quad (5.6)$$

The possible gaugings are encoded in the singlet intrinsic torsion transforming in

$$W_{\text{int}} = \overline{\mathbf{105}} \oplus \mathbf{21} \quad (5.7)$$

of the $\text{SU}^*(6)$ global symmetry group.

The $\mathcal{N} = 6$ theory of [63] is believed to describe a subsector of the chiral primary operators of $\mathcal{N} = 4$ Super Yang-Mills [65], which is identified using the $\text{SL}(2, \mathbb{R})$ symmetry. It is still unclear whether a supergravity truncation can be constructed.

5.1.3 $\mathcal{N} = 4$ supergravity

For $\mathcal{N} = 4$ supersymmetry there are two kinds of multiplet: the graviton multiplet, containing the metric, 4 gravitini, 6 vectors, 4 spin 1/2 fermions and 1 real scalar, and vector multiplets, consisting of 1 vector, 4 spin 1/2 fermions and 5 real scalars, each. The scalar manifold is the coset

$$\mathcal{M} = \text{SO}(1, 1) \times \frac{\text{SO}(5, n_V)}{\text{SO}(5) \times \text{SO}(n_V)}, \quad (5.8)$$

where $\text{SO}(1, 1)$ is parameterised by the scalar in the graviton multiplet and the other factor by those in the vector multiplets.

The gauging of the global isometry group $G_{\text{iso}} = \text{SO}(1, 1) \times \text{SO}(5, n_V)$

$$D_\mu = \nabla_\mu - g A_\mu^I (f_I^{JK} t_{JK} + \xi^J t_{JK} + \xi_I t_0) - g A_\mu^0 \xi^{IJ} t_{IJ}, \quad (5.9)$$

where t_{IJ} and t_0 are the generators of $\text{SO}(5, n)$ and $\text{SO}(1, 1)$, are determined by the embedding tensor [38], with components

$$\Theta_I^\alpha = (\xi_I, \xi_{[IJ]}, f_{[IJK]}) \quad I, J, K = 1, \dots, 5 + n_V, \quad (5.10)$$

where the bracket denotes full antisymmetrisation.

In the context of Exceptional Geometry and Exceptional Field Theory, consistent truncations to $\mathcal{N} = 4$ supergravity theories have been studied in [8, 14].

There exists only one family¹⁵ of truncations corresponding to

$$G_S = \text{Spin}(5 - n_V) \subseteq \text{USp}(4) \subset \text{USp}(8), \quad (5.11)$$

where $n_V = 0, \dots, 5$ is the number of vector multiplets, and $\text{Spin}(1) = \text{Spin}(0) = \mathbb{Z}_2$.

The G_S -structure is defined by $6 + n_V$ invariant generalised vectors according to

$$\begin{aligned} E_{6(6)} \supset \text{SO}(1, 1) \times \text{SO}(5, n) \times \text{SO}(5 - n) \\ \mathbf{27} \ni (\mathbf{1}, \mathbf{1})_{-4} \oplus (\mathbf{5} + n, \mathbf{1})_2 = \{K_0, K_I\} \quad I = 1, \dots, 5 + n_V, \end{aligned} \quad (5.12)$$

satisfying the compatibility conditions (see [8] for more details)

$$\begin{aligned} c(K_0, K_0, V) &= 0, \\ c(K_0, K_I, K_J) &= \eta_{IJ} \kappa^2 \quad \forall V \in \Gamma(E) \\ c(K_I, K_J, K_K) &= 0 \quad \forall I, J, K = 1, \dots, 5 + n_V \end{aligned} \quad (5.13)$$

where $c(V, V', V'')$ is the $E_{6(6)}$ cubic invariant, $\eta_{IJ} = \text{diag}(-\mathbb{1}_5, \mathbb{1}_{n_V})$ is the flat $\text{SO}(5, n_V)$ metric.

The generalised vectors $\{K_{\tilde{I}}\}$ with $\tilde{I} = 0, \dots, 5$ and $\tilde{I} = 6 \dots 5 + n_V$ determine the vectors in the gravity multiplets and those in the vector multiplets, respectively. From (2.11) one recovers the scalar manifold (5.8).

¹⁵As discussed in [8] the $\text{SO}(5)$ subgroups $G_S = \text{SU}(2) \times \text{U}(1)$ and $G_S = \text{U}(1)^2$ also lead to an $\mathcal{N} = 4$ truncation. They are both subgroups of $G_S = \text{SO}(4)$ and one can show that they give the same truncated theory as the $\text{SO}(4)$ case.

The singlet intrinsic torsion has components

$$W_{\text{int}} = (\mathbf{5} \oplus n_V)_{-4} \oplus (\mathbf{X}_{IJ})_2 \oplus (\mathbf{X}_{IJK})_2 \quad (5.14)$$

where $\mathbf{X}_{IJ}$ and $\mathbf{X}_{IJK}$ denote the antisymmetric two and three-tensors representation of $\text{SO}(5 + n_V)$ and the subscripts give the $\text{SO}(1, 1)$ charge. The generalised Lie derivative among the singlet generalised vectors

$$L_{K_{\tilde{I}}} K_{\tilde{J}} = \Theta_{\tilde{I}} \cdot K_{\tilde{J}} = \Theta_{\tilde{I}}^{\hat{\alpha}} (t_{\hat{\alpha}})_{\tilde{J}}^{\tilde{K}} K_{\tilde{K}} = X_{\tilde{I}\tilde{J}}^{\tilde{K}} K_{\tilde{K}}, \quad (5.15)$$

with $(X_{\tilde{I}})_{\tilde{J}}^{\tilde{K}} = X_{\tilde{I}\tilde{J}}^{\tilde{K}}$ and $[X_{\tilde{I}}, X_{\tilde{J}}] = -X_{\tilde{I}\tilde{J}}^{\tilde{K}} X_{\tilde{K}}$, reproduces the embedding tensor of the truncated theory

$$X_{IJ}{}^K = -f_{IJ}{}^K - \frac{1}{2} \eta_{IJ} \xi^K + \delta_{[I}^J \xi_{K]} \quad X_{I0}{}^0 = \xi_I \quad X_{0I}{}^J = -\xi_I{}^J. \quad (5.16)$$

There are several examples of half-maximal truncations to 5 dimensions. For instance, the truncation [1, 2] of type IIB supergravity on squashed Sasaki-Einstein manifolds to gravity coupled to two vector multiplets and gauge group $\text{Heis}_3 \times U(1)$ is reproduced with a generalised $G_S = \text{SO}(3)$ structure [8], which corresponds to the ordinary $\text{SU}(2)$ structure of the original truncation.

A generalised $\text{SU}(2)$ structure can also be used to derive a consistent truncation on β -deformed Sasaki-Einstein manifolds [8] to give a continuous family of $\mathcal{N} = 4$ theories with two vector multiplets and $U(1) \times \text{Heis}_3$ gauging. This family contains the truncation to pure gauged supergravity of [66].

A generalised $U(1)$ structure leads to the most general truncation of 11-dimensional supergravity around the Maldacena-Nun ez solution with $\mathcal{N} = 4$ supersymmetry [67]. The truncated theory [8] consists of half-maximal supergravity coupled to three vector multiplets and with $U(1) \times \text{ISO}(3)$ gauge group, which reproduces the reduction of 7-dimensional gauged supergravity of [68]. On the other hand, if one includes the trombone symmetry a larger truncation around the Maldacena-Nun ez solution to maximal supergravity can be obtained [69, 70]. In this case there is no Lagrangian for the truncated theory.

There are also examples of truncations to ungauged supergravity. For instance, a $G_S = \text{SO}(3)$ on $K3 \times T^2$ gives a consistent truncation of 11-dimensional supergravity to ungauged supergravity with two vector multiplets [14].

5.1.4 $\mathcal{N} = 2$ supergravity

In 5 dimensions $\mathcal{N} = 2$ supergravity coupled to matter multiplets contains the gravity multiplet, vector, tensor and hypermultiplets [71–73]. The gravity multiplet consists of the graviton, 2 gravitini transforming as a doublet of the R-symmetry group $\text{SU}(2)_R$ and the graviphoton. A vector multiplet consists of a vector, 2 spin-1/2 fermions in the fundamental of $\text{SU}(2)_R$ and a complex scalar, while in a tensor multiplet the vector is replaced by a two-form, which is dual to a vector. Thus vector and tensor multiplets have the same number of degrees of freedom. The scalars of the vector and tensor multiplets are grouped together and parametrise a very special real manifold $\mathcal{M}_{\text{VT}}$. Finally, a hypermultiplet

consists of 4 real scalars and an R-symmetry doublet of spin-1/2 fermions. The scalars of the hypermultiplets parameterise a quaternionic Kähler manifold $\mathcal{M}_H$. The isometry group splits into the isometries of the vector/tensor and hypermultiplet scalar manifolds

$$G_{\text{iso}} = G_{\text{VT}} \times G_H. \quad (5.17)$$

The Killing vectors determining the gaugings are the combinations

$$k_{\tilde{I}}^i(\phi) = \Theta_{\tilde{I}}^a k_a^i(\phi) \quad k_{\tilde{I}}^x(q) = \Theta_{\tilde{I}}^A k_A^x(q) \quad (5.18)$$

of the vectors generating G_{iso} via the embedding tensor

$$\Theta_{\tilde{I}}^\alpha = (\Theta_{\tilde{I}}^a, \Theta_{\tilde{I}}^A), \quad (5.19)$$

where the indices a and A run over the dimensions of G_{VT} and G_H , respectively. In (5.18) ϕ^i with $i = 1, \dots, n_{\text{VT}}$ and q^x with $x = 1, \dots, 4n_H$ denote the scalars in the vector and hypermultiplets, respectively.

The classification of $\mathcal{N} = 2$ supergravity theories in 5 dimensions that have an 11/10-dimensional origin can be found in [18]. Here we summarise the main results.

In order to have $\mathcal{N} = 2$ supersymmetry the generalised structures must be

$$G_S \subseteq \text{USp}(6) = \text{C}_{\text{USp}(8)}(\text{SU}(2)_R), \quad (5.20)$$

where $\text{USp}(8)$ is the double cover of the maximal compact subgroup of $E_{6(6)}$.

$G_S = \text{USp}(6)$ gives the truncation to pure supergravity and it is defined by a singlet generalised vector $K \in \mathbf{27}$ of positive norm with respect to the $E_{6(6)}$ cubic invariant

$$c(K, K, K) = 6\kappa^2, \quad \kappa^2 \in \Gamma(\det T^*M), \quad (5.21)$$

and a triplet of singlet weighted adjoint elements J_α defining a highest weight $\mathfrak{su}(2)$ subalgebra of $\mathfrak{e}_{6(6)}$. The pair (K, J_α) satisfies

$$\begin{aligned} J_\alpha \cdot K &= 0, \\ \text{tr}(J_\alpha J_\beta) &= -c(K, K, K)\delta_{\alpha\beta}, \end{aligned} \quad (5.22)$$

with $\alpha = 1, 2, 3$. The singlet generalised vector gives the graviphoton of the truncated theory, while the singlets J_α are the generators of the $\text{SU}(2)$ R-symmetry. From (2.11) it follows that the scalar manifold is trivial, as expected.

Also in this case, there are truncations with only vector/tensor multiplets and only hypermultiplets, and truncations with both. The generalised structures are defined by a set of singlet generalised vectors $K_{\tilde{I}}$ with $\tilde{I} = 0, \dots, n_{\text{VT}}$ and a set of singlet adjoint tensors J_A , with $A = 1, \dots, \dim G_H$, satisfying

$$\begin{aligned} J_A \cdot K_{\tilde{I}} &= 0, \\ c(K_{\tilde{I}}, K_{\tilde{J}}, K_{\tilde{K}}) &= 6\kappa^2 C_{\tilde{I}, \tilde{J}, \tilde{K}}, \quad \forall \tilde{I}, \tilde{J}, \tilde{K} = 0, \dots, n_{\text{VT}} \\ [J_A, J_B] &= \kappa f_{AB}^C J_C, \quad \forall A, B = 1, \dots, \dim G_H \\ \text{tr}(J_A, J_B) &= \kappa^2 \eta_{AB}, \end{aligned} \quad (5.23)$$

where $C_{\tilde{I}\tilde{J}\tilde{K}}$ is a symmetric, constant tensor (it gives the tensor of the same name in the truncated theory), f_{AB}^C are the structure constants of $\mathfrak{g}_H$, and η_{AB} is a diagonal matrix with entries -1 and +1 for the compact and non-compact generators of G_H , respectively.

The scalar manifold is given by (2.11) and it factorises in

$$\mathcal{M} = \mathcal{M}_{\text{VT}} \times \mathcal{M}_H = \frac{G_{\text{VT}}}{H_{\text{VT}}} \times \frac{G_H}{H_H}. \quad (5.24)$$

By construction, all scalar manifolds are homogeneous and symmetric spaces.

The intrinsic torsion transforms in **351** of $E_{6(6)}$ and the G_S -singlet components are determined by

$$\begin{aligned} L_{K_{\tilde{I}}} K_{\tilde{J}} &= -T_{\text{int}}(K_{\tilde{I}}) \cdot K_{\tilde{J}} = t_{\tilde{I}\tilde{J}}^{\tilde{K}} K_{\tilde{K}} \\ L_{K_{\tilde{I}}} J_A &= -T_{\text{int}}(K_{\tilde{I}}) \cdot J_A = p_{\tilde{I}A}^B J_B, \end{aligned} \quad (5.25)$$

with $(t_{\tilde{I}})_{\tilde{J}}^{\tilde{K}}$ and $(p_{\tilde{I}})_A^B$ constant matrices giving the two components of the embedding tensor. They also determine the elements of Lie algebras of G_{VT} and G_H respectively. A detailed analysis of the intrinsic torsion and the possible gaugings can be found in [18].

Truncations with only vector/tensor multiplets are associated to generalised structures that only admit as adjoint singlets the triplet J_α defining the $SU(2)$ R-symmetry. This means that

$$G_S \subset SU^*(6), \quad (5.26)$$

where $SU^*(6)$ is the stabiliser of the J_α 's. The G_S -structures are defined by the set of singlets

$$(K_{\tilde{I}}, J_\alpha) \quad \alpha = 1, 2, 3 \quad \tilde{I} = 0, \dots, n_{\text{VT}}, \quad (5.27)$$

where, by the compatibility condition (5.23), the generalised vectors $K_{\tilde{I}}$ belong to the component **(15, 1)** in the breaking of the **27** under $E_{6(6)} \supset SU^*(6) \times SU(2)_R$. In the truncated theory, the vector $K_0 = K$ gives the graviphoton while the other K_I , with $I = 1 \dots n_{\text{VT}}$, give the vectors in the vector/tensor multiplets.

The truncations obtained in [18] are listed in Table 11.

n_{VT}	G_S	$\mathcal{M}_{\text{VT}}$
$1 \dots 6$	$\text{Spin}(6 - n_{\text{VT}})$	$\mathbb{R}^+ \times \frac{\text{SO}(n_{\text{VT}}-1, 1)}{\text{SO}(n_{\text{VT}}-1)}$
5	$SU(2)$	$\frac{\text{SL}(3, \mathbb{R})}{\text{SO}(3)}$
8	$U(1)$	$\frac{\text{SL}(3, \mathbb{C})}{\text{SU}(3)}$
14	$\mathbb{Z}_2$	$\frac{\text{SU}^*(6)}{\text{USp}(6)}$

Table 11. $\mathcal{N} = 2$ truncations with $n_H = 0$

For truncations with only hypermultiplets the only singlet in E must be the vector K . Thus the associated to generalised structures must be

$$G_S \subset F_{4(4)}, \quad (5.28)$$

where now $F_{4(4)}$ is the stabiliser of the generalised vector K . The G_S -structures are defined by the singlet K and a set of adjoint tensors J_A , with $A = 1, \dots, \dim G_H$.

There are only two possible truncations

n_H	G_S	$\mathcal{M}_H$
1	SU(3)	$\frac{SU(2,1)}{SU(2) \times U(1)}$
2	SO(3)	$\frac{G_{2(2)}}{SO(4)}$

Table 12. $\mathcal{N} = 2$ truncations with $n_V = 0$.

As mentioned in [18], there are two other symmetric spaces¹⁶

$$\begin{aligned}\mathcal{M} &= \frac{SO_0(4, n_V)}{SO(4) \times SO(n_V)} \\ \mathcal{M} &= \frac{F_{4(4)}}{USp(6) \times SU(2)}\end{aligned}\tag{5.29}$$

which could, in principle, correspond to a consistent truncation with hypermultiplets. They should correspond to subtruncations of half-maximal and maximal supergravity with a discrete G_S -structure. However, the requirement that $G_S \subset USp(8)$ and Schur lemma are enough to show that such truncations are not allowed.

Finally, the truncations with both vector/tensor and hypermultiplets, are simply¹⁷

n_{VT}	n_H	G_S	$\mathcal{M}_{VT}$	$\mathcal{M}_H$
1	1	$SU(2) \times U(1)$	$\mathbb{R}^+$	$\frac{SU(2,1)}{SU(2) \times U(1)}$
2	1	$U(1)$	$\mathbb{R}^+ \times \frac{SO(1,1)}{SO(3)}$	$\frac{SU(2,1)}{SU(2) \times U(1)}$
4	1	$U(1)$	$\mathbb{R}^+ \times \frac{SO(3,1)}{SO(3)}$	$\frac{SU(2,1)}{SU(2) \times U(1)}$

Table 13. $\mathcal{N} = 2$ truncations with vector/tensor and hypermultiplets

To our knowledge there are no examples of truncations with only vector multiplets nor only hypermultiplets.

On the other hand, all cases in Table (13) correspond known truncations of 11-dimensional supergravity around AdS_5 solutions. The $G_S = U(1)$ structure in the third line gives the truncation of 11-dimensional supergravity around the Maldacena-Nuñez solution [67] with $\mathcal{N} = 2$ supersymmetry [17]. The resulting 5-dimensional theory contains 4 vector and 1 hypermultiplet and has gauge group $SO(3) \times U(1)$. The first line should

¹⁶ $SO_0(4, n_V)$ denotes the connected component of $SO(4, n_V)$.

¹⁷The truncation with $n_V = 3$ automatically enhances to $n_V = 4$.

correspond to the subtruncation of this theory to one vector and one hypermultiplet and gauging $U(1) \times \mathbb{R}^+$ derived in [74]. Finally, the other $G_S = U(1)$ structure gives the truncation of 11-dimensional supergravity around the BBBW solutions [75]. This is an infinite family of the $\mathcal{N} = 2$ solutions generalising the Maldacena-Nuñez one. The truncated theory contains two vectors and one hypermultiplet, with gauge group $U(1) \times \mathbb{R}^+$ [17]. This truncation extends the one of [76].

5.2 Truncations to 6 dimensions

In 6 dimensions the spinorial representation is reducible and the supersymmetry parameters are symplectic Majorana-Weyl spinors. The amount of allowed supersymmetry is $(\mathcal{N}_-, \mathcal{N}_+)$ with the supersymmetry parameters transforming in the fundamental of the $USp(\mathcal{N}_+) \times USp(\mathcal{N}_-)$ R-symmetry group. Depending on the values of $\mathcal{N}_-$ and $\mathcal{N}_+$ one can construct chiral and non chiral theories.

For truncations to 6 dimensions the relevant exceptional group is $E_{5(5)} = \text{Spin}(5, 5) \sim \text{SO}(5, 5)$ [13, 15, 16, 19], and the G_S -structures leading to minimal supergravities with $(\mathcal{N}_-, \mathcal{N}_+)$ supersymmetry are [8]

$(\mathcal{N}_-, \mathcal{N}_+)$	G_S	$G_R = C_{USp(4) \times USp(4)}(G_S)$
(2, 2)	$\mathbb{1}$	$USp(4) \times USp(4)$
(2, 1)	$SU(2)$	$USp(4) \times SU(2)$
(2, 0)	$USp(4)$	$USp(4)$
(1, 1)	$SU(2) \times SU(2)$	$SU(2) \times SU(2)$
(1, 0)	$SU(2) \times USp(4)$	$SU(2)$

Table 14. Structure and R-symmetry groups for six-dimensional truncations

5.2.1 Maximal supergravity

There are three possible maximal supergravity algebras in six dimensions: the non-chiral $\mathcal{N} = (2, 2)$ algebra and two chiral ones, $\mathcal{N} = (4, 0)$ and $\mathcal{N} = (3, 1)$.

The full non-linear $\mathcal{N} = (2, 2)$ theory has been constructed in [77]. It contains the graviton, 5 two-forms, 8 gravitini, 16 vectors, 40 spin 1/2 fields and 25 scalars which parameterise the coset

$$\mathcal{M} = \frac{SO(5, 5)}{SO(5) \times SO(5)}, \quad (5.30)$$

where $SO(5, 5)$ is the global isometry group and $SO(5) \times SO(5) \sim USp(4)_l \times USp(4)_r$ ¹⁸ its maximal compact subgroup. The fermions are left- and right-handed symplectic Majorana-Weyl. In particular the supersymmetry parameters transform in the $(\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4})$ of $USp(4) \times USp(4)$.

¹⁸The subscript l and r denote left and right chirality.

The gauging of the theory are given in terms of the embedding tensor

$$D_\mu = \nabla_\mu - g A_\mu^I \Theta_I^{AB} t_{AB} \quad (5.31)$$

where $t_{AB} = t_{[AB]}$ with $A, B = 1, \dots, 10$ are the $\text{SO}(5, 5)$ generators. The embedding tensor Θ can be written in terms of a tensor θ_{JA} transforming in the $\mathbf{144}^c$ of $\text{SO}(5, 5)$

$$\Theta_I^{AB} = -\theta^{L[A} \gamma^{B]}_{LI} \quad (5.32)$$

where γ_{AIJ} are the $\text{SO}(5, 5)$ gamma matrices and the tensor θ^{IA} satisfies $\gamma_{AIJ} \theta^{JA} = 0$ [78]. The theory with $\text{SO}(5)$ gauge group was obtained in [79] as a circle reduction of $\text{SO}(5)$ maximal supergravity in 7 dimensions. The classification of all possible gaugings can be found in [78].

The chiral theories $\mathcal{N} = (4, 0)$ and $\mathcal{N} = (3, 1)$ are more exotic since the graviton is replaced by tensor fields with mixed symmetries and self-duality conditions. For both theories only the linearised actions are known (see [80–83] for more details).

Only the non-chiral theory $\mathcal{N} = (2, 2)$ can be obtained as a consistent truncation. Recall that the supersymmetry parameters of the truncated theory are given by G_S -singlets in the generalised spinor bundle $\mathcal{S}$ and that the R-symmetry is $\text{C}_{\tilde{H}_d}(G_S)$. Since maximally supersymmetric truncations are associated to a $G_S = 1$, the supersymmetry parameters are given by the full generalised spinor bundle and transform as

$$(\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}) \quad (5.33)$$

of the $\text{USp}(4) \times \text{USp}(4)$ R-symmetry group (see Table 10). This structure is only compatible with $\mathcal{N} = (2, 2)$ supersymmetry.

The $G_S = 1$ is associated to a generalised Leibniz parallelisation defined by 16 generalised vectors K_I , $I = 1, \dots, 16$, transforming in the spinorial representation of $\text{SO}(5, 5)$. The K_I are normalised with the generalised metric $G(K_I, K_J) = \eta_{IJ}$, with η_{IJ} the $\text{SO}(5, 5)$ invariant metric. They give the 16 vectors of the truncated theory.

The projection on the bundle N of the symmetric product of the singlet generalised vectors defines 10 singlet tensors Q_M , which, in the truncated theory, give the 5 two-forms and their duals. The scalar manifold is given by (2.11) and reproduces (5.30).

The vectors realise a Leibniz algebra

$$L_{K_I} K_J = X_{IJ}^K K_K, \quad (5.34)$$

where the constant intrinsic torsion

$$X_{IJ}^K = -\frac{1}{4} \Theta_I^{AB} \gamma_{AB}^K{}_J \quad (5.35)$$

transforms in the $\mathbf{144}^c$ of $\text{SO}(5, 5)$ and gives the embedding tensor (5.32).

The $\text{SO}(5)$ gauged supergravity of [79] is obtained as a truncation of massless type IIA on S^4 [79, 84] and has been rederived using generalised geometry in [13]. It is convenient

to consider the embedding of S^4 in $\mathbb{R}^5$ and then decompose $\text{SO}(5, 5)$ under $\text{SL}(5, \mathbb{R})$. Then non-zero component of the intrinsic torsion are

$$\begin{aligned} X_{[ii'][jj']}^{[kk']} &\sim R^{-1}(\delta_{ij}\delta_{j'i'}^{kk'} - \delta_{ij'}\delta_{ji}^{kk'} - \delta_{i'j}\delta_{j'i}^{kk'} + \delta_{i'j'}\delta_{ji}^{kk'}), \\ X_{[ii']j}^k &\sim -R^{-1}(\delta_{ji'}\delta_i^k - \delta_{ji}\delta_{i'}^k), \end{aligned} \quad (5.36)$$

with R the radius of S^4 and $\delta_{ij} \sim \theta_{ij}$ are the non trivial components of the tensor θ^{IA} in (5.32). The tensors $X_{[ii'][jj']}^{[kk']}$ and $X_{[ii']j}^k$ reproduce the $\text{SO}(5)$ gauge algebra.

5.2.2 $\mathcal{N} = (2, 1)$ supergravity

There also exists an $\mathcal{N} = (2, 1)$ supergravity [85–87]. The R-symmetry is $\text{USp}(4) \times \text{USp}(2)$ and the fields are arranged in the graviton multiplet consisting of the graviton, 1 self-dual and 5 anti-self dual two forms in **5** of $\text{USp}(4)$, a $\text{USp}(2)$ doublet of positive chirality gravitini and 4 doublets of negative chirality gravitini, 8 vectors in $(\mathbf{4}, \mathbf{2})$ of the R-symmetry, 10 positive chirality and 4 negative chirality spin1/2 spinors in the $(\mathbf{5}, \mathbf{2})$ and $(\mathbf{4}, \mathbf{1})$ of $\text{USp}(4) \times \text{USp}(2)$ respectively, and 5 scalars, which are neutral under $\text{USp}(2)$. The theory is anomalous.

The field content of the $\mathcal{N} = (2, 1)$ theory is easily deduced from Exceptional Generalised Geometry. The theory corresponds to a $G_S = \text{USp}(2)$ structure embedded in $\text{SO}(5, 5)$ as

$$\text{SO}(5, 5) \supset \text{SO}(5, 1) \times \text{USp}(2)_R \times \text{USp}(2)_S. \quad (5.37)$$

It gives 8 singlet generalised vectors K_I and 6 singlet tensors Q_M in the bundle N

$$\begin{aligned} \mathbf{16} &\rightarrow (\mathbf{4}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{4}, \mathbf{2}, \mathbf{1}), \\ \mathbf{10} &\rightarrow (\mathbf{6}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}). \end{aligned} \quad (5.38)$$

corresponding to the vectors and tensors in the graviton multiplet of the truncated theory. The scalar manifold is given by (2.11)

$$\mathcal{M} = \frac{\text{SO}(5, 1)}{\text{SO}(5)}. \quad (5.39)$$

5.2.3 Half-maximal supergravity

In six dimensions half-maximal supergravity can be chiral and non chiral.

The non chiral theory has $\mathcal{N} = (1, 1)$ supersymmetry with the supersymmetry parameters transforming as left- and right-handed doublets of the $\text{SO}(4)_R \sim \text{SU}(2) \times \text{SU}(2)$ R-symmetry. There are two types of multiplets, the graviton and vector multiplets. The graviton multiplet contains the graviton, 4 gravitini, 4 vectors, 1 two-form, 4 spin 1/2 fermions and 1 scalar, while a vector multiplet consists of 1 vector, 4 1/2 fermions and 4 scalars. All fermions are symplectic Majorana-Weyl and transform in $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ of the R-symmetry group.¹⁹

¹⁹For Anti de Sitter backgrounds, which are relevant for gauged supergravity, the description in terms of chiral fermions does not hold: each pair of chiral spinors is arranged into an 8-dimensional pseudo-Majorana spinor. At the same time the R-symmetry is broken to the diagonal $\text{SU}(2)$ in $\text{SU}(2) \times \text{SU}(2)$.

For n_V vector multiplets, the dilaton and the scalars in the vector multiplets parameterise the coset

$$\mathcal{M} = \frac{\mathrm{SO}(4, n_V)}{\mathrm{SO}(4) \times \mathrm{SO}(n_V)} \times \mathbb{R}^+. \quad (5.40)$$

The most general form of the gauged theory was constructed in [88]. It has gauge group $F(4)$ and generalises [89, 90] (see also [91] for more recent results).

Even if a fully $\mathrm{SO}(4, n_V) \times \mathbb{R}^+$ covariant formulation of six-dimensional gauged supergravity has not been constructed, the components of the embedding tensors have been identified [92] in the Kac-Moody approach to supergravity

$$\Theta_I^{JK} = f_I^{JK} + \delta_I^{[J} \xi^{K]}, \quad \Theta_I^0 = \xi_I, \quad (5.41)$$

where $I, J, K = 1, \dots, 4 + n_V$ and Θ_I^{JK} and Θ_I^0 give the gaugings of $G \subset \mathrm{SO}(4, n_V)$ and $\mathbb{R}^+$, respectively.

The chiral theory [93, 94] has supersymmetry $\mathcal{N} = (2, 0)$. The supersymmetry parameters are left-handed symplectic Majorana-Weyl transforming in the $\mathbf{4}$ of the $\mathrm{USp}(4)_R$ R-symmetry group. The field consists of the gravity multiplet (the graviton, 4 left-handed gravitini and 5 two-forms) coupled to n_T tensor multiplets (1 anti self-dual two-form, 4 right-handed symplectic Majorana spinors and 5 scalars). The scalars parameterise the manifold

$$\mathcal{M} = \frac{\mathrm{SO}(5, n_T)}{\mathrm{SO}(5) \times \mathrm{SO}(n_T)} \quad (5.42)$$

The supersymmetry transformations and field equations for the graviton multiplet coupled to n_T tensor multiplets are given in [94, 95]. Being chiral, the theory is anomalous. The only anomaly free theory has $n_T = 21$ tensor multiplets and it can be obtained as a reduction of type IIB on $K3$ [93].

In the context of generalised geometry, truncations to half-maximal supergravity have been studied in [14–16]. We review them in the conventions of Exceptional Generalised Geometry.

The truncation to minimal $\mathcal{N} = (1, 1)$ supergravity is associated to the structure

$$G_S = \mathrm{SO}(4) \sim \mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \quad (5.43)$$

where each of the two $\mathrm{SU}(2)$ factors embeds in one $\mathrm{USp}(4)$ as $\mathrm{SU}(2)_S \times \mathrm{SU}(2)_R \subset \mathrm{USp}(4)$ (see Table 14). From the embedding

$$\begin{aligned} \mathrm{SO}(5, 5) &\supset (\mathrm{SU}(2)_S \times \mathrm{SU}(2)_R) \times (\mathrm{SU}(2)_S \times \mathrm{SU}(2)_R) \times \mathbb{R}^+ \\ \mathbf{16} &\rightarrow (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})_1 \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})_{-1} \oplus (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})_{-1}, \\ \mathbf{10} &\rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_{-2} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})_0 \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})_0, \end{aligned} \quad (5.44)$$

it follows that the structure is defined by 2 singlets Q and $\hat{Q}$ in the bundle N and 4 singlet generalised vectors K_I , $I = 1, \dots, 4$, transforming in the $\mathbf{1}$ and $\mathbf{4}$ of the $\mathrm{SO}(4)_R \sim \mathrm{SU}(2)_R \times \mathrm{SU}(2)_R$ R-symmetry group.

The singlets K_I give the 4 vectors in the gravity multiplets, while Q and $\hat{Q}$ give the two-form in the same multiplet and its dual. From $C_{\text{Spin}(5,5)}(G_S) = \mathbb{R}^+$ and (2.11), one recovers the scalar manifold of minimal supergravity

$$\mathcal{M} = \mathbb{R}^+. \quad (5.45)$$

To have extra vector multiplets the structure is further reduced to the diagonal $\text{SU}(2)$ in $G_S = \text{SO}(4)$ and to subgroups thereof. The allowed truncation are listed in Table 15. Notice that the $n_V = 3$ case is missing as it enhances to $n_V = 4$.

n_V	1	2	4
G_S	$\text{SU}(2)$	$\text{U}(1)$	$\mathbb{Z}_2$

Table 15. Generalised structure groups for truncation with $n_V \neq 0$

The structures are defined by the pair of singlet Q and $\hat{Q}$ in the bundle N and by $4 + n_V$ generalised vectors K_I transforming in $(\mathbf{4} + \mathbf{n}_V)_{-1}$ of the global isometry group $G_{\text{iso}} = \text{SO}(4, n_V) \times \mathbb{R}^+$.

The tensor Q , $\hat{Q}$ and K_I defining the generalised structure satisfy the algebraic conditions [14]

$$\begin{aligned} \eta(Q, Q) &= \eta(\hat{Q}, \hat{Q}) = 0, \\ \eta(Q, \hat{Q}) &= \kappa^2, \quad I = 1, \dots, 4 + n_V, \\ K_I \times_N K_J &= \eta_{IJ} \hat{Q}, \end{aligned} \quad (5.46)$$

where $\eta(\cdot, \cdot)$ is the $\text{SO}(5, 5)$ invariant (see [20]) while η_{IJ} is the $\text{SO}(4, n_V)$ invariant metric, as well as the differential ones [14]

$$\begin{aligned} L_{K_I} K_J &= -X_{IJ}{}^K K_K, \\ L_{K_I} Q &= X_I Q, \\ \partial \times_E Q &= \tilde{X}^I K_I, \end{aligned} \quad (5.47)$$

with $X_{IJK} = X_{IJ}{}^L \eta_{LK} = X_{[IJK]}$. The singlet components of the intrinsic torsion, $(X_{IJK}, X_I, \tilde{X}_I)$, transform in the

$$W_{\text{int}} = \mathbf{X}_{IJK, -1} \oplus \mathbf{N}_{-1} \oplus \mathbf{N}_3 \quad (5.48)$$

of the global isometry group $G_{\text{iso}} = \text{SO}(4, n_V) \times \mathbb{R}^+$ and give the components of the embedding tensor. With respect to (5.41), there are two independent vector components.²⁰

Minimal supergravity theory with $F(4)$ gauge group was obtained as a consistent truncation of massive type IIA supergravity on (the upper hemisphere of) S^4 in [96], and also as a consistent truncation of type IIB on a S^2 warped over a Riemann surface in [97]. The latter truncation extends those in [98].

²⁰We thank G. Bossard for confirming this results. See also Section 5.3.2.

In the context of Exceptional Field Theory the truncation of type IIB on AdS_6 times the warped product of S^2 and a Riemann surface to minimal $F(4)$ supergravity was recovered in [15]. In [16] the truncation was extended to extra vector multiplets. The truncations with $n_V = 1$ and $n_V = 2$ vector multiplets correspond to the $SU(2)$ and $U(1)$ structures of Table 15. On the other hand, the truncations in [16] with 3 vector multiplets transforming as triplets of the $SU(2)_R$ symmetry and with 3 vector multiplets in the $\mathbf{3}$ and one in the $\mathbf{1}$ of $SU(2)_R$ cannot be obtained in our analysis since all the extra vectors are singlets of the $SU(2)$ R-symmetry. It would be interesting to understand what is the origin of this mismatch.

Truncations to minimal ungauged supergravities are obtained from 11-dimensional supergravity on $K3 \times S^2$ or type IIA on $K3$ [14].

The chiral $\mathcal{N} = (2, 0)$ theories correspond to structures that embed uniquely in one of the two $USp(4)$ factors. The allowed G_S -structures are listed in the table below

n_T	0	1	2	3	5
G_S	$USp(4)$	$SU(2) \times SU(2)$	$SU(2)_{\text{diag}}$	$U(1)$	$\mathbb{Z}_2$

Table 16. Generalised structure groups for truncation to chiral $\mathcal{N} = (2, 0)$ supergravity

Decomposing the bundles E and N under

$$SO(5, 5) \supset G_S \times SO(5, n_T) \quad (5.49)$$

with $G_{\text{iso}} = SO(5, n_T)$ the global isometry group, one sees that there are no singlets generalised vectors and $5 + n_T$ singlets Q_i , $i = 1, \dots, 5 + n_T$, in N . They transform in the fundamental of G_{iso} and satisfy

$$\eta(Q_i, Q_j) = \delta_{ij} \quad (5.50)$$

with $\eta(\cdot, \cdot)$ the $SO(5, 5)$ invariant.

The tensors Q_i with $i = 1, \dots, 5$ give the 5 tensors in the gravity multiplet, while each of those with $i = 6, \dots, 5 + n_T$ give the tensor in the tensor multiplets. From (2.11), the scalar manifold is

$$\mathcal{M} = \frac{SO(5, n_T)}{SO(5) \times SO(n_T)}. \quad (5.51)$$

It is easy to see that for any $n_T = 0, \dots, 5$, there are no G_S -singlets in the intrinsic torsion [14]. This implies that only truncations to ungauged supergravity are possible, thus giving only Minkowski vacua. We are not aware of explicit examples of such truncations.

5.2.4 Minimal supergravity: $\mathcal{N} = (1, 0)$

The generic $\mathcal{N} = (1, 0)$ chiral supergravity contains, beside the gravity multiplet, n_T tensor, n_V vector and n_H hypermultiplets. The gravity multiplet consists of the graviton, a $USp(2)_R$ doublet of left-handed gravitini and an anti-self-dual two-form. Each tensor

multiplet is formed by a self-dual two-form, an $\text{USp}(2)_R$ doublet of right-handed spin 1/2 fermions and a scalar. The scalar in the n_T tensor multiplets parameterise the coset space

$$\mathcal{M}_T = \frac{\text{SO}(1, n_T)}{\text{SO}(n_T)}. \quad (5.52)$$

Each vector multiplet contains a vector and a $\text{USp}(2)_R$ doublet of left-handed spin 1/2 fermions. A hypermultiplet consists of a right-handed spin 1/2 fermion and 4 real scalars. The scalars in the n_H hypermultiplets parameterise a quaternionic Kähler manifold of negative curvature. The full list of allowed manifolds can be found in [99, 100]. In consistent truncation we are only interested homogeneous symmetric spaces ²¹

$$\begin{aligned} \mathcal{M} &= \frac{\text{USp}(2n_H, 2)}{\text{USp}(2n_H) \times \text{USp}(2)}, \\ \mathcal{M} &= \frac{\text{SU}(n_H, 2)}{\text{SU}(n_H) \times \text{SU}(2) \times \text{U}(1)}, \\ \mathcal{M} &= \frac{\text{SO}(n_H, 4)}{\text{SO}(n_H) \times \text{SO}(4)}. \end{aligned} \quad (5.53)$$

For theories with $n_T > 1$ tensor multiplets, the (anti-)self-duality condition of the tensors does not allow for a Lagrangian formulation, but the equations of motion, supersymmetry variations and pseudo-Lagrangian were derived in [94, 103–105]. A standard Lagrangian formulation is possible for $n_T = 1$ since the self-dual two-form in the tensor multiplet combines with the anti-self-dual two-form in the gravity multiplet to give a two-form with no self-duality property.

While the gauging of the isometries of the hypermultiplet sector takes the ordinary form

$$\mathcal{D}_\mu \phi^x = \partial_\mu \phi^x + ig k_I^x A_\mu^I. \quad (5.54)$$

where ϕ^x are the scalars in the hypermultiplets and k_I^x are Killing vector fields, the self-duality of the two-forms in the tensor multiplets makes the gauging of the isometries of the tensor scalar manifold more involved.

The full gaugings were studied in [106] in the case of magical supergravities²² in terms of an embedding tensor that takes the form²³

$$\Theta_I^\alpha = (\Theta_I^{ij}, \Theta_I^{AB}) \quad (5.55)$$

where $\Theta_I^{ij} = -\gamma_{IJ}^{[i} \theta^{j]J}$ and $\gamma_{IJ}^i \theta_i^J = 0$, with γ_{IJ}^i the $\text{SO}(5, 5)$ gamma metrics. The first term in (5.55) determines the gaugings of the tensor multiplet isometry group and the second one those of the hypermultiplet scalars.

²¹The other homogeneous spaces in in [99–102] are clearly too big to be obtained from $E_{5(5)}$ geometry.

²²Magical supergravities are a class of theories characterised by a fixed number of tensor and vector multiplets and arbitrary number of hypermultiplets. In six dimensions magical supergravities exist for $(n_T, n_V) = (2, 2), (3, 4), (5, 8), (9, 16)$.

²³The embedding tensor could a priori contain an extra term gauging the symmetries of the theory that only act on the vectors. However consistency of the gauging sets it to zero. See [106] for more details

The construction via the embedding tensor shows that the non-abelian gauge algebra only closes for the values $n_T = 2, 3, 5, 9$ [106]. In all other cases the non-abelian gauging of the tensors isometry groups are not known.

For a generic number of tensor, vector and hypermultiplet the $\mathcal{N} = (1, 0)$ suffers of gravitational, gauge and mixed anomalies [93] (see [107–110] for anomaly cancellation in gauged supergravity and [111–115] for more recent references). The Green-Schwarz mechanism for anomaly cancellation constrains the number of multiplet to satisfy [107]

$$n_H - n_V + 29n_T = 273. \quad (5.56)$$

It is easy to see that none of the theories that can be obtained as consistent truncations satisfy the anomaly cancellation condition. However, for completeness, we still present the results of our classification.

In Exceptional Geometry, minimal $\mathcal{N} = (1, 0)$ supergravity corresponds to the generalised structures

$$G_S \subseteq \mathrm{USp}(2) \times \mathrm{USp}(4). \quad (5.57)$$

The structures are defined by the a set of singlets in the bundles N , E , $(\det T^*M)^{1/2}\mathrm{ad}F$, respectively

$$\{Q_i, K_I, J_A\}. \quad (5.58)$$

In the truncated theory the singlets Q_i , with $i = 0, \dots, n_T$, determine the two-forms in the gravity and the tensor multiplets, K_I the vectors ($I = 1, \dots, n_V$), while the scalars in the hypermultiplets are associated to J_A , with $A = 1, \dots, \dim G_H$, where G_H is the isometry group of the hyperscalars. As we will show below, the embedding of the G_S -structures in $\mathrm{SO}(5, 5)$ puts bounds on the allowed number of multiplets.

Truncations to the gravity multiplet only are associated to $G_S = \mathrm{USp}(2) \times \mathrm{USp}(4)$. As discussed in [20], the structure is defined by a single tensor Q and a triplet J_α , $\alpha = 1, 2, 3$, defining a highest weight $\mathfrak{su}(2)$ subalgebra of $\mathrm{SO}(5, 5)$. The tensors Q and J_α satisfy

$$\begin{aligned} \eta(Q, Q) &= \kappa^2, \\ J_\alpha \cdot Q &= 0, \\ \mathrm{tr}(J_\alpha, J_\beta) &= -\eta(Q, Q)\delta_{\alpha\beta}, \end{aligned} \quad (5.59)$$

where $\eta(\cdot, \cdot)$ is the $\mathrm{SO}(5, 5)$ invariant and $\cdot$ denotes the adjoint action. The triplet J_α are the generators of the $\mathrm{USp}(2)$ R-symmetry. Decomposing the intrinsic torsion under $\mathrm{SU}(2)_R \times \mathrm{USp}(2) \times \mathrm{USp}(4)$ [20]

$$W_{\mathrm{int}} = (\mathbf{1}, \mathbf{2}, \mathbf{4}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{4}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{4}) \oplus (\mathbf{3}, \mathbf{2}, \mathbf{4}) \quad (5.60)$$

one sees that there are no G_S -singlets. This implies

$$\partial \times_E Q = 0, \quad (5.61)$$

so that only truncations to ungauged supergravity are possible.

For truncations with tensor, vector and hypermultiplets the embedding of the structures in $\text{SO}(5, 5)$ puts bounds on the allowed number of multiplets. In order to have truncations with only tensor multiplets, the G_S -structure must be a subgroup of $\text{SU}(2) \times \text{Spin}(1, 5)$, the stabiliser of the triplet J_α . Decomposing the bundles N under $\text{SO}(5, 5) \supset \text{SU}(2)_R \times \text{SU}(2) \times \text{Spin}(1, 5)$

$$\mathbf{10} = (\mathbf{1}, \mathbf{1}, \mathbf{6}) \oplus (\mathbf{2}, \mathbf{2}, \mathbf{1}), \quad (5.62)$$

and using the fact that the singlet tensors must be invariant under the R-symmetry, we see that the maximum number of allowed singlets is 6, which gives $n_T = 5$ tensor multiplets.

Similarly, for truncations with only hypermultiplets the structures must be subgroups of the stabiliser of the singlet tensor Q , namely $G_S \subset \text{Spin}(4, 5)$. Recall that the hypermultiplets scalars are associated to the G_S -singlets among the non-compact element of $\text{SO}(5, 5)$ which transform non-trivially under the R-symmetry. Thus in the decomposition of the adjoint bundle under $\text{SO}(5, 5) \supset \text{SO}(5) \times \text{SO}(4) \subset \text{Spin}(4, 5)$ the relevant elements must be in the representation

$$\mathbf{45} \supset (\mathbf{5}, \mathbf{2}, \mathbf{2}), \quad (5.63)$$

which sets the bound $n_H \leq 5$ for the number of hypermultiplets. The number of singlets among the generalised vectors follows from the specific G_S group, but cannot clearly exceed n_V , the dimension of the generalised tangent bundle.

As a consequence, none of the theories we find as consistent truncations satisfy the anomaly cancellation condition.

We find only a few examples of truncations to gauged supergravity. The theories we find contain tensors and vector multiplets, but no hypermultiplets.

The truncations and corresponding G_S -structures are listed in Table 17.

G_S	n_T	n_V	G_{sym}	$\mathcal{R}_T$	$\mathcal{R}_V$
$\text{SU}(2) \times \text{SU}(2)_{\text{diag}}$	1	1	$\text{SO}(1, 1_T)$	2	-1
$\text{SU}(2) \times \text{U}(1)$	1	2	$\text{SO}(1, 1_T) \times \text{U}(1)$	2	$2(-1)$
$\text{SU}(2)$	2	2	$\text{SO}(1, 2_T)$	3	2 _{real}
$\text{U}(1)$	3	4	$\text{SO}(1, 3_T) \times \text{U}(1)$	$(\mathbf{2}, \mathbf{2})_0$	$(\mathbf{2}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{2})_{-2}$
$\mathbb{Z}_2$	5	8	$\text{SO}(1, 5_T) \times \text{SU}(2)$	$(\mathbf{6}, \mathbf{1})$	$(\bar{\mathbf{4}}, \mathbf{2})$

Table 17. Truncations with tensor and vector multiplets. $\mathcal{R}_T$ and $\mathcal{R}_V$ denote the representation of G_{sym} in which tensors and vectors transform.

The case $n_T = n_V = 1$ reproduces the field content of the Salam-Sezgin model [116].

The last three entries in Table 17 correspond to six-dimensional magical supergravities [106, 117, 118] with no hypermultiplets.²⁴ In these theories the vectors carry spinorial

²⁴The magical supergravity with $n_T = 9$ and $n_V = 16$ cannot be obtained as a truncation since its symmetry group $\text{SO}(1, 9)$ does not embeds in $\text{SO}(5, 5)$.

representations of the tensor global isometry group and, when extra symmetry groups not acting on the tensors are present, they are also charged under these latters.

The property of having vectors fields with "spinorial" charges under the isometry group of the tensors, is also common to the theories in the first two lines in Table 17.

The G_S -structures of Table 17 are defined by the singlets $\{Q_i, K_I, J_\alpha\}$ satisfying a generalisation of the algebraic conditions (5.59) for any $i, j = 0 \dots, n_T$, $I = 1, \dots, n_V$ and $\alpha = 1, 2, 3$. The singlets J_α in the adjoint generate the $SU(2)_R$ symmetry

$$[J_\alpha, J_\beta] = \epsilon_{\alpha\beta}{}^\gamma J_\gamma \quad (5.64)$$

and are normalised to $\text{tr}(J_\alpha, J_\beta) = \delta_{\alpha\beta}$. Altogether the singlets $\{Q_i, K_I, J_\alpha\}$ satisfy the compatibility conditions

$$\begin{aligned} \eta(Q_i, Q_j) &= \eta_{ij} \kappa^2 \\ (Q_i \times_{\text{ad}} Q_j) \cdot K_I &= -\kappa^2 \frac{1}{4} (\gamma_{ij})^J{}_I K_J \\ J_A \cdot Q_i &= 0 \end{aligned} \quad (5.65)$$

where η_{ij} is the invariant $SO(1, n_T)$ metric, $\frac{1}{4}(\gamma_{ij})^J{}_I$ are the generators of $SO(1, n_T)$, as well as the differential conditions

$$\begin{aligned} \partial \times_E Q_i &= X_i^I K_I, & L_{K_I} J_\alpha &= p_{I\alpha}{}^\beta J_\beta, \\ L_{K_I} Q_i &= X_{Ii}{}^j Q_j & L_{K_I} K_J &= X_{IJ}{}^K K_K, \end{aligned} \quad (5.66)$$

with $X_{Ii}{}^j \propto X_i^J (\gamma^j)_{IJ} - X^{jJ} (\gamma_i)_{IJ}$. The tensors X_J^i , $p_{I\alpha}{}^\beta$ and $X_{IJ}{}^K$ are the component of the intrinsic torsion.

In all the examples in Table 17 the component $X_{IJ}{}^K$ vanishes. Moreover, since the vectors are in spinorial representations of the tensor isometry group, X_i^I transforms in a tensorial representation of $SO(1, n_T)$ of dimension $(1 + n_T)n_V$, which has two irreducible components, a trace part and a traceless one

$$X_I^{(tr)} = (\gamma^i)_{IJ} X_i^J \quad X_i^{(0)I} = X_i^I + \frac{1}{4} \gamma^{IJ} X_J. \quad (5.67)$$

This is easily seen by looking at the G_S -singlets in the intrinsic torsion in the table below, where, in each line, the first element corresponds to $X_i^{(0)I}$, the second to $X_I^{(tr)}$ and the last one to $p_{I\alpha}{}^\beta$.

The requirement that the differential conditions (5.66) give a Leibniz algebra sets to zero the trace component of the singlet intrinsic torsion.

Recall that the intrinsic torsion determines the embedding tensor of the truncated theory, where X_i^I , $X_{Ii}{}^j$ and $p_{I\alpha}{}^\beta$ provide the gauging of the tensor isometries and R-symmetry, while $X_{IJ}{}^K$ determine the structure constants of the tensor scalar isometry group and of possible extra symmetries that do not act on the scalars. Since the representation in which $X_{IJ}{}^K$ transforms never appears in the intrinsic torsion of Table 18, we find that only abelian gaugings of the tensor isometries and of the R-symmetry are possible. Thus we reproduce exactly the embedding tensor derived in [106] for magical supergravities.

n_T	n_V	G_{sym}	W_{int}
1	1	$\text{SO}(1, 1_T)$	$\mathbf{1}_3 \oplus \mathbf{1}_{-1} \oplus \mathbf{3}_{-1}$
1	2	$\text{SO}(1, 1_T) \times \text{U}(1)$	$\mathbf{1}_{3,\pm 2} \oplus \mathbf{1}_{-1,\pm 2} \oplus \mathbf{3}_{-1,\pm 2}$
2	2	$\text{SO}(1, 2_T)$	$(\mathbf{4}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{3})$
3	4	$\text{SO}(1, 3_T) \times \text{U}(1)$	$((\mathbf{3}, \mathbf{2})_2 \oplus (\mathbf{2}, \mathbf{3})_{-2}, \mathbf{1}) \oplus ((\mathbf{1}, \mathbf{2})_{-2} \oplus (\mathbf{2}, \mathbf{1})_2, \mathbf{1})$ $\oplus ((\mathbf{1}, \mathbf{2})_2 \oplus (\mathbf{2}, \mathbf{1})_{-2}, \mathbf{3})$
5	8	$\text{SO}(1, 5_T) \times \text{SU}(2)$	$(\mathbf{20}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{3})$

Table 18. Singlet intrinsic torsion in representations of $G_{\text{sym}} \times \text{SU}(2)_R$.

The other truncations we find give ungauged supergravity.

We find a family of truncations with only tensor multiplets, associated to the structures

n_T	1	2	3	4	5
G_S	$\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)$	$\text{SU}(2)_{\text{diag}} \times \text{SU}(2)$	$\text{SU}(2) \times \text{U}(1)_{\text{diag}}$	$\text{U}(1)$	$\mathbb{Z}_2$

Table 19. Generalised structure groups for truncation with only tensor multiplets

The commutant of G_S in $\text{SO}(5, 5)$ gives the global isometry group of the tensor multiplet scalars

$$G_{\text{iso}} = \text{SO}(1, n_T) = \text{C}_{\text{SO}(5,5)}(G_S), \quad (5.68)$$

and, from (2.11), we recover the scalar manifold (5.52).

Since for any value of $n_T \leq 5$ there are no G_S -singlets in intrinsic torsion, all the truncations in this family give ungauged supergravity. The only differential constraint is again $\partial \times_E Q_i = 0$, with $i = 0, \dots, n_T$.

Finally we find two truncations with $n_T = 3$ tensor multiplets and $n_H = 1, 2$ hypermultiplets, corresponding to the $G_S = \text{U}(1)$ and $G_S = \mathbb{Z}_4$, respectively. The scalar of the truncated theories are

$$\mathcal{M} = \frac{\text{SO}(1, 3)}{\text{SO}(3)} \times \frac{\text{SU}(n_H, 2)}{\text{SU}(n_H) \times \text{SU}(2) \times \text{U}(1)} \quad n_H = 1, 2 \quad (5.69)$$

and, again, there are no singlets in the intrinsic torsion.

5.3 Truncations to 7 dimensions

Consistent truncations to 7 dimensional supergravity have been discussed in the context of Exceptional Generalised Geometry or Exceptional Field theory in [9, 13, 15, 16, 19]. The relevant generalised geometry is $\text{E}_{4(4)} = \text{SL}(5, \mathbb{R})$. The generalised tangent bundle

transforms in the **10** of $\text{SL}(5, \mathbb{R})$, while the supersymmetry parameters transform as spinors of $\text{USp}(4)$, the maximally compact subgroup of $\text{SL}(5, \mathbb{R})$ (see Table 10). The allowed amount of supersymmetry gives maximal and half-maximal supergravities.

5.3.1 Maximal supergravity

Maximal supergravity in 7 dimensions has $\mathcal{N} = 4$ supercharges, transforming in the fundamental of the $\text{USp}(4)$ R-symmetry group. The fields are organised in the gravity multiplet, consisting of the graviton, 4 gravitini, 10 vectors, 5 two-forms, 16 spin 1/2 fermions and 14 scalars. The bosonic fields carry non trivial representations of the global symmetry group $\text{SL}(5, \mathbb{R})$ group, while the fermions are symplectic Majorana and transform under $\text{USp}(4)$ [119, 120]. The 14 scalars parameterise the coset

$$\mathcal{M} = \frac{\text{SL}(5, \mathbb{R})}{\text{SO}(5)}. \quad (5.70)$$

The gaugings of the global $\text{SL}(5, \mathbb{R})$ symmetry are given by

$$D_\mu = \nabla_\mu - g A_\mu^{[ij]} \Theta_{[ij],k}^l t^k_l, \quad (5.71)$$

where $i, j, k, l = 1, \dots, 5$ are $\text{SL}(5, \mathbb{R})$ indices, $A_\mu^{[ij]}$ are the 10 vectors and t^k_l (with $t^k_k = 0$) are the $\text{SL}(5, \mathbb{R})$ generators. The embedding tensor has two components [120]

$$\Theta_I^\alpha = (Y_{(ij)}, Z^{[ij],k}) \quad Z^{[ij],k} = 0 \quad (5.72)$$

transforming in the **15** and **40'** of $\text{SL}(5, \mathbb{R})$.

In generalised geometry, truncations to maximal supergravity correspond to a $G_S = \mathbb{1}$ structure defined by 10 generalised vectors $\{K_I\} = \{K_{[ij]}\}$ transforming in the **10** of $\text{SL}(5, \mathbb{R})$ ($I = 1, \dots, 10$ and $i, j = 1, \dots, 5$). They realise a Leibniz parallelisation

$$L_{K_I} K_J = X_{IJ}^K K_K \quad I, J, K = 1, \dots, 10 \quad (5.73)$$

and are normalised to $G(K_I, K_J) = \delta_{IJ}$, with G the generalised metric. They give the 10 vectors of the truncated theory.

As in the six-dimensional case, the generalised singlet vectors also define a parallelisation of the bundle N via the projection $E \times_N E$. The 5 tensors in N give the 5 two-forms of the truncated theory. The scalars are again given by (2.11).

The generalised Lie derivative among the vectors K_I (5.73) determines the intrinsic torsion X_{IJ}^K , which transforms as the

$$W_{\text{int}} = \mathbf{40}' \oplus \mathbf{15} \quad (5.74)$$

of the $\text{SL}(5, \mathbb{R})$ global isometry group. X_{IJ}^K reproduces the embedding tensor (5.72) and determines the gaugings of the truncated theory.

Examples of truncations to maximal supergravity are the truncation of 11-dimensional supergravity on S^4 with gauge group $\text{SO}(5)$ [121, 122] and of massless IIA theory on S^3 with

gauge group ISO(4) [84]. In generalised geometry the truncations have been reproduced in [13]. In both cases the intrinsic torsion is only in the **15** component [9]

$$X_{[ii'],[jj']}^{[kk']} \sim -R^{-1}(Y_{ij}\delta_{i'j'}^{[kk']} - Y_{ij'}\delta_{i'j}^{[kk']} - Y_{i'j}\delta_{ij'}^{[kk']} + Y_{i'j'}\delta_{ij}^{[kk']}), \quad (5.75)$$

where R is the radius of the internal manifold and $i, j = 1, \dots, 5$ are again $\text{SL}(5, \mathbb{R})$ indices. For the M-theory truncation on S^4 (5.73) reproduces the algebra of the $\text{SO}(5)$ with

$$Y_{ii'} \sim \text{diag}(1, 1, 1, 1, 1), \quad (5.76)$$

while for the truncation of type IIA on S^3 it gives the algebra of the $\text{ISO}(4)$

$$Y_{ii'} \sim \text{diag}(1, 1, 1, 1, 0). \quad (5.77)$$

5.3.2 Half-maximal supergravity

Half maximal supergravity has $\mathcal{N} = 2$ supersymmetry with $\text{SU}(2)$ R-symmetry. Its field content consists of the gravity multiplet (the graviton, 2 gravitini, 3 vectors, 2 spin 1/2 fermions, a two-form and a scalar) and n_V vector multiplets, each containing a vector, 2 spin 1/2 fermions and 3 scalars [123]. The fermions are all symplectic Majorana and the scalars in the vector multiplets parameterise the coset

$$\mathcal{M} = \frac{\text{SO}(3, n_V)}{\text{SO}(3) \times \text{SO}(n_V)} \times \mathbb{R}^+, \quad (5.78)$$

where the $\mathbb{R}^+$ factor is parameterised by the scalar in the gravity multiplet.

The full embedding tensor formalism for half-maximal supergravity in 7 dimensions has not been worked out yet. Using the results from Kac-Moody analysis [92], two components of the embedding tensor

$$\Theta_I^\alpha = (f_I^{JK} + \delta_I^{[J} \xi^{K]}, \xi_I). \quad (5.79)$$

have been studied in [124]. The tensors ξ^I and f_I^{JK} transform as the fundamental and the three-index anti-symmetric representations of $\text{SO}(3, n_V)$. They give the gaugings

$$D_\mu = \nabla_\mu - A_\mu^I (f_I^{JK} t_{JK} + \xi^J t_{IJ} + \xi_I t_0), \quad (5.80)$$

where t_{IJ} and t_0 are the generators of $\text{SO}(3, n_V)$ and of the $\mathbb{R}^+$ shifts.

In generalised geometry, truncations to half-maximal supegravity were classified in [15, 16, 19]. They correspond to the generalised structures

$$G_S = \text{SO}(3 - n_V) \subset \text{USp}(4) \quad n_V = 0, 1, 2, 3 \quad (5.81)$$

with $\text{SO}(0) = \mathbb{Z}_2$. The truncation with $n_V = 2$ vector multiplets automatically enhances to the one with $n_V = 3$.

The G_S -structures are defined by $3 + n_V$ singlet generalised vectors K_I and a singlet element Q of the bundle N (see Table 10 again) satisfying the compatibility conditions

$$\begin{aligned} K_I \times_N K_J - \frac{1}{4} \eta_{IJ} K_K \times_N K^K &= 0, \\ \epsilon(K_I, K_J, Q) &= \eta_{IJ} \kappa^2, \end{aligned} \quad I, J, K = 1, \dots, 3 + n_V, \quad (5.82)$$

where $\epsilon(\cdot, \cdot, \cdot)$ is the $\text{SL}(5)$ invariant tensor and η_{IJ} is the $\text{SO}(3, n_V)$ invariant metric. The generalised vectors K_I with $I = 1, 2, 3$ and the tensor Q give the 3 vectors and the one-form of the gravity multiplets, while the remaining vectors are associated to the vector multiplets. From (2.11) one reproduces the scalar manifold (5.78).

The differential conditions

$$\begin{aligned} L_{K_I} K_J &= X_{IJ}{}^K K_K, \\ L_{K_I} Q &= \xi_I Q, \\ dQ &= \xi_0 (J_I \times_N J^I), \end{aligned} \tag{5.83}$$

with $X_{IJK} = X_{[IJK]}$ give the singlet components of the intrinsic torsion

$$W_{\text{int}} = \mathbf{X}_{IJK} \oplus \mathbf{N} \oplus \mathbf{1} \tag{5.84}$$

transforming in the three-index antisymmetric, fundamental and singlet representation of the global isometry group $\text{SO}(3, n_V)$. With respect to (5.79) we find an extra singlet component, whose presence is also confirmed by a Kac-Moody analysis.²⁵ The components of W_{int} should then give the full content of the embedding tensor of the truncated theory.

Minimal half-maximal supergravity with gauge group $\text{SU}(2)$ [125] was obtained in [126] by truncating 11-dimensional supergravity on S^4 . The same theory can be obtained as a truncation of massive IIA on a deformation of S^3 [127].

A complete classification of half-maximal truncations of massless and massive type IIA on $AdS_7 \times M_3$, where M_3 is an S^2 fibration over a segment were studied in [15, 16]. For massive IIA only the truncation to pure supergravity with $G_S = \text{SO}(3)$ is allowed and it corresponds to the truncation in [127]. In massless IIA, there exists a truncation with $\text{U}(1)$ structure on S^3 giving gauged supergravity with one-vector multiplet and gauge group $\text{SU}(2) \times \text{U}(1)$. The same truncation can be obtained by dimensionally reducing and keeping only the $\text{U}(1)$ invariant modes of the maximally supersymmetric consistent truncation of 11-dimensional supergravity on S^4 .

The truncation with $n_V = 3$ and $G_S = \mathbb{Z}_2$ corresponds to the theory in [128] and it can be obtained by further imposing a $\mathbb{Z}_2$ structure to the truncation of 11-dimensional on S^4 to maximal supergravity.

6 Conclusions

In this article we pursued the programme of classifying lower dimensional supergravities that can be obtained as consistent truncations of 11/10-dimensional supergravity, using the formalism of Exceptional Geometry. A consistent truncation is determined by its field content and gauge symmetries. In Exceptional Geometry these properties are captured by an exceptional G_S -structure with singlet, constant intrinsic torsion. The field content of the reduced theory, as well as its supersymmetry and bosonic symmetries are given by globally defined G_S -invariant generalised tensors on the compactification manifold M .

²⁵We thank G. Bossard for confirming this results.

Our main result is the classification of the truncations to 4 dimensional supergravity. In this case, the exceptional structure group is $E_{7(7)}$, and, since we want supersymmetric truncations, the possible G_S -structures are subgroups of $SU(8)$, the double cover of the maximal compact subgroup of $E_{7(7)}$.

In the formalism of G_S -structures the derivation of a consistent truncation consists of an algebraic problem and a differential one. In this paper we focused on the algebraic question, namely the classification of the subgroups $G_S \subset SU(8)$ and the derivation of the field content and symmetries of the truncated theory in terms of G_S singlets. We do not address the algebraic problem of checking that the intrinsic torsion of the G_S -structure consists of G_S -singlets only (or is zero). We simply assume that this is the case.

We first scan through supersymmetry and determine the largest $G_S \subset SU(8)$ compatible with the fixed the number $2 \leq \mathcal{N} \leq 8$ of supercharges. For $\mathcal{N} \geq 5$ the 4-dimensional theory is unique, due to the large amount of supersymmetry. For $2 \leq \mathcal{N} \leq 4$ there is a $G_S^{max} \subset SU(8)$ structure group, corresponding to the truncation to minimal supergravity. Then for any fixed $2 \leq \mathcal{N} \leq 4$, we scanned for all continuous $G_S \subset G_S^{max}$ leading to inequivalent truncations allowing for extra matter multiplets. We find a limited number of possible truncations, which are listed in (3.28) and Tables 4 - 9. For any $2 \leq \mathcal{N} \leq 4$ there is a truncation with maximal number of matter multiplets, which corresponds to a discrete structure group. In Tables 7 and 8 we list other examples of truncations associated to discrete G_S -structure groups. However, in this case our analysis is far from being complete and what we give are just few examples.

Even if our analysis is performed looking explicitly at the various embedding of the structure groups G_S into $SU(8)$ and $E_{7(7)}$, there are some general features that emerge. In particular we can use group theory to exclude some of the a priori allowed G_S -structures. There are two types of embeddings of G_S into $SU(8)$, what we call regular embeddings, where the number of Cartan generators is preserved, and special ones, when the number is not preserved. We find that Schur's lemma excludes a good deal of the special branchings. It would be interesting to see if more rigorous group theoretical arguments can be used to justify this result.

A general prediction of generalised geometry is that the scalar manifolds of the truncated theories must all be homogeneous and symmetric. In [129] it was shown that coset manifolds $\frac{G}{H}$, where H does not contain non-trivial G -invariant subgroups, have H as structure group. Our analysis suggest that this result extends to generalised geometry: all coset manifold compactifications $\frac{G}{H}$ correspond to a generalised H -structure.

The same algebraic approach can be applied to truncations to any external dimension $D \geq 4$. These supergravity theories play an important role in the gauge/gravity duality.

For truncations to 5, 6 and 7 dimensions most of the results have already been obtained in the literature (see for instance [8–19]) in Exceptional Generalised Geometry and/or Exceptional Field Theory. Section 5 is devoted to a review of these results, with the goal of presenting them in a uniform language. Along the way, we completed them with some missing details. In particular we derived all components of the embedding tensors for half-maximal supergravities in 6 and 7 dimensions and checked that the extra terms we

find with respect to the literature are indeed predicted by the Kac-Moody analysis of the reduced theory. We also complete the analysis of the allowed truncations to 6 dimensions with different supersymmetry and matter content. For the chiral theories $\mathcal{N} = (2, 1)$ and $\mathcal{N} = (1, 0)$ we only find anomalous theories, since the limited number of allowed matter fields does not fulfill the anomaly cancellation conditions.

With this paper we complete the classification of all supergravity theories that have an 11/10-dimensional origin for any amount of supersymmetry in dimension larger than 4, and with $\mathcal{N} \geq 2$ in dimension 4. It is straightforward to perform the same analysis for $\mathcal{N} = 1$ truncations in 4 dimensions. However, in this case we find a large variety of allowed theories and we did not find a nice and interesting way of presenting them. We leave this to a further publication. The large structure groups of the generalised tangent bundle makes it possible to construct also truncations with no supersymmetry. This is an interesting direction to explore in the future.

A similar classification for truncations to 3 dimensions requires $E_{8(8)}$ generalised geometry. Its structure is more involved than those described in this article as it requires additional covariantly-constrained fields to close the algebra of the generalised Lie derivative, which makes it harder to introduce the notion of G_S -structures. These issues were resolved in [130] where a classification of consistent truncations with $\mathcal{N} \geq 4$ was given. It would be interesting to extend this analysis to consistent truncations with less supersymmetry, which might be relevant for the study of truncations around GK geometries [131], for instance.

Our classifications only provides a list of 4-dimensional theories that can a priori be obtained as consistent truncations. In some cases the explicit truncations have been worked out in the literature (we mention the examples known to us in the various sections), for others an explicit higher dimensional realisation is still missing. This would imply finding manifolds with the right differential properties to give the required singlet intrinsic torsion. For maximal supersymmetry, it is possible to provide necessary and sufficient conditions that an embedding tensor has to fulfil to give rise to a generalised Leibniz parallelisation and hence to be associated to a consistent truncations [32, 132, 133]. It would be interesting to see whether similar conditions can be found also for less supersymmetric truncations. These would allow to identify lower-dimensional theories with interesting features for the construction of black-holes or solutions relevant for the AdS/CFT correspondence, and then uplift them to full 11/10-dimensional solutions.

Finally, it would be interesting to explore the space of vacua, *AdS* in particular, of the theories we find in our analysis. In 6 and 7 dimensions a complete classification of the possible *AdS* vacua is known [134, 135]. *AdS* vacua for the $\mathcal{N} = 2$ 5-dimensional supergravities found in [18] were studied in [136] based on the analysis of the embedding tensor and the supersymmetry equation. A similar approach can be extended to the other theories discussed in this article.

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A Details on $E_{7(7)}$

The classification of the $G_S \subset E_{7(7)}$ -structures is determined by determining the number of G_S -singlets in the generalised spinor, vector and adjoint bundles. This amounts at solving the equations

$$\begin{aligned} g \cdot \epsilon &= \epsilon \\ g \cdot R_{\text{SU}(8)} &= R_{\text{SU}(8)} \\ g \cdot V &= V \\ g \cdot R &= R, \end{aligned} \tag{A.1}$$

for all $g \in G_S$, where ϵ and $R_{\text{SU}(8)}$ are in the spinorial and adjoint representation of $\text{SU}(8)$ and V and R are in **56** and **133** of $E_{7(7)}$.

Since the action of the groups G_S on the vector and tensors spaces we are interested is linear, for continuous G_S , i.e Lie groups, we can use the fact that any element connected to the identity is the exponentiation of the Lie algebra elements, $g = e^{\lambda t}$, where t is a generator in the Lie algebra and λ a parameter, to replace (4.6) with its Lie algebra analog

$$\begin{aligned} t \cdot \epsilon &= 0 \\ t \cdot R_{\text{SU}(8)} &= 0 \\ t \cdot V &= 0 \\ t \cdot R &= 0, \end{aligned} \tag{A.2}$$

for all $\forall t \in \mathfrak{g}_S$, where $\mathfrak{g}_S$ denotes the Lie algebra of G_S .

In this appendix we give our conventions for the action of $E_{7(7)}$ and $\text{SU}(8)/\mathbb{Z}_2$, its maximally compact subgroup. The main references for this appendix are [22, 26].

We are mostly interested in the fundamental and adjoint representations, **56** and **133** of $E_{7(7)}$. We will denote $E_{7(7)}$ fundamental indices by $M, N, P = 1, \dots, 56$ and adjoint indices by $A, B, C = 1, \dots, 133$.

There are two relevant decompositions of $E_{7(7)}$, one according to $\text{SL}(8)$ and the other according to $\text{SU}(8)$. We will relate the two via $\text{SO}(8)$ representations, since

$$\text{SO}(8) = \text{SL}(8) \cap \text{SU}(8). \tag{A.3}$$

The group $\text{SO}(8)$ has three inequivalent representations of dimension 8: $\mathbf{8}_v$, $\mathbf{8}_c$ and $\mathbf{8}_s$. The $\mathbf{8}_v$ is identified with the $\mathbf{8}$ of $\text{SL}(8)$ under the branching $\text{SO}(8) \subset \text{SL}(8)$, whereas the $\mathbf{8}_s$ is identified with the $\mathbf{8}$ of $\text{SU}(8)$ under the branching $\text{SO}(8) \subset \text{SU}(8)$. We denote $\mathbf{8}_v$ indices by a, b , $\mathbf{8}_c$ indices by $\dot{\alpha}, \dot{\beta}$ and $\mathbf{8}_s$ indices by α, β , all of them running from 1 to 8. These three representations are connected by triality. $\text{SO}(8)$ indices are raised and lowered with the three invariant tensors: δ^{ab} , $C^{\alpha\beta}$ and $C^{\dot{\alpha}\dot{\beta}}$. In our conventions $C^{\alpha\beta} = \delta^{\alpha\beta}$ and $C^{\dot{\alpha}\dot{\beta}} = \delta^{\dot{\alpha}\dot{\beta}}$.

We are interested in the $\text{SO}(8)$ generators in the $\mathbf{8}_s$ representation, which we will use as intertwiners between the $\text{SL}(8)$ and $\text{SU}(8)$ representations,

$$(\gamma_{ab})^{\alpha\beta} = (\gamma_{[a}\gamma_{b]})^{\alpha\beta} = (\gamma_{[a})^{\alpha}{}_{\dot{\gamma}}(\gamma_{b]})^{\dot{\beta}}{}_{\delta} C^{\dot{\gamma}\delta}, \quad (\text{A.4})$$

where the gamma matrices are

$$\begin{aligned} \gamma_1 &= \sigma_2 \otimes \mathbb{1}_2 \otimes \sigma_1 & \gamma_5 &= \sigma_3 \otimes \sigma_2 \otimes \sigma_1 \\ \gamma_2 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 & \gamma_6 &= -\mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_3 \\ \gamma_3 &= -\sigma_2 \otimes \sigma_1 \otimes \sigma_3 & \gamma_7 &= -\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_2 \\ \gamma_4 &= -\sigma_1 \otimes \sigma_2 \otimes \sigma_1 & \gamma_8 &= -i\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \end{aligned} \quad (\text{A.5})$$

with Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.6})$$

We will also need the four-gamma antisymmetric product

$$\gamma^{abcd}_{\alpha\beta} = (\gamma^{[a}\gamma^b\gamma^c\gamma^{d]})_{\alpha\beta} = \gamma^{[a}{}_{\alpha\dot{\alpha}}\gamma^b{}_{\dot{\beta}}\gamma^c{}_{\delta\dot{\gamma}}\gamma^{d]}{}_{\beta\dot{\delta}} C^{\dot{\alpha}\dot{\beta}} C^{\dot{\gamma}\dot{\delta}}. \quad (\text{A.7})$$

A.1 $\text{SL}(8)$ decomposition

Under $\text{SL}(8) \subset \text{E}_{7(7)}$ the fundamental representations of $\text{E}_{7(7)}$ decompose as

$$\begin{aligned} \mathbf{56} &= \mathbf{28} \oplus \mathbf{28}' \\ V &= (V^{ab}, \tilde{V}_{ab}) \end{aligned} \quad (\text{A.8})$$

where $V^{ab}, \tilde{V}_{ab}$ are two index antisymmetric tensors $V^{ab} = -V^{ba}$, while the adjoint gives

$$\begin{aligned} \mathbf{133} &= \mathbf{63} \oplus \mathbf{70} \\ \mu &= (\mu^a{}_b, \mu^{abcd}) \end{aligned} \quad (\text{A.9})$$

with $\mu^a{}_a = 0$ and $\mu^{abcd} = \mu^{[abcd]}$.

Contractions between two vectors are given by

$$V^M W_M = \frac{1}{2} V^{ab} W_{ab} + \frac{1}{2} V_{ab} W^{ab}. \quad (\text{A.10})$$

The action of $\mathbf{133}$ on the $\mathbf{56}$ becomes

$$\begin{aligned} (\mu \cdot V)^{ab} &= \mu^a{}_c V^{cb} + \mu^b{}_c V^{ac} + \frac{1}{2} \mu^{abcd} \tilde{V}_{cd} \\ (\mu \cdot \tilde{V})_{ab} &= -\mu^c{}_a \tilde{V}_{cb} - \mu^c{}_b \tilde{V}_{ac} + \frac{1}{2} (*\mu)_{abcd} V^{cd}, \end{aligned} \quad (\text{A.11})$$

where $(*\mu)_{abcd} = \frac{1}{4!}\epsilon_{abcdefgh}\mu^{efgh}$, and the commutator of two adjoints reads

$$\begin{aligned}(\mu \cdot \mu')^a_b &= \mu^a_c \mu'^c_b - \mu'^a_c \mu^c_b + \frac{1}{3!}\mu^{acde}(*\mu')_{bcde} \\ (\mu \cdot \mu')^{abcd} &= -4(\mu^{[a}_e \mu'^{bcd]e} - \mu'^{[a}_e \mu^{bcd]e}).\end{aligned}\tag{A.12}$$

For computational reasons, it is more convenient to treat the elements of the **56** and the **133** as 56-dimensional vectors and 56×56 dimensional matrices. In this way the action of the $E_{7(7)}$ adjoint on the **56** becomes a matrix multiplication

$$V^M \rightarrow \mu^M_N V^N.\tag{A.13}$$

The idea is to flatten the antisymmetric tensors V^{ab} and $\tilde{V}_{ab}$ into two 28-component vectors whose elements are ordered as $(V^{12}, V^{13}, \dots, V^{78})$, and then to construct a 56-dimensional vector

$$V^M = (V^{ab}, \tilde{V}_{ab}) \quad a < b.\tag{A.14}$$

Similarly, a generic $E_{7(7)}$ Lie algebra element can be written as a 56×56 matrix in terms of a basis of generators:

$$\mu^M_N = \mu^a_b (t_a^b)^M_N + \frac{1}{4!}\mu^{abcd}(t_{abcd})^M_N,\tag{A.15}$$

where²⁶

$$\begin{aligned}(t_a^b)^M_N &= \begin{pmatrix} (t_a^b)^{cd}_{ef} & 0 \\ 0 & (t_a^b)_{cd}^{ef} \end{pmatrix} = \begin{pmatrix} 4\delta^{[c}_{[e}(t_a^b)^{d]}_{f]} & 0 \\ 0 & -4\delta^{[e}_{[c}(t_a^b)^{f]}_{d]} \end{pmatrix} \\ (t_{abcd})^M_N &= \begin{pmatrix} 0 & (t_{abcd})^{efgh} \\ (t_{abcd})_{efgh} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4!\delta_{abcd}^{efgh} \\ \varepsilon_{abcdefgh} & 0 \end{pmatrix},\end{aligned}\tag{A.17}$$

and where the generators $(t_a^b)^c_d$ of $SL(8)$ in the fundamental are 8×8 matrices defined as

$$(t_a^b)^c_d = \delta_a^c \delta_d^b - \frac{1}{8}\delta_a^b \delta_d^c.\tag{A.18}$$

Thus, the action of a generic $E_{7(7)}$ generator on the fundamental is given by:

$$\mu^M_N V^N = \begin{pmatrix} \mu^a_{a'} V^{a'b} + \mu^b_{b'} V^{ab'} + \frac{1}{2}\mu^{abcd} V_{cd} \\ -\mu^{a'}_a V_{a'b} - \mu^{b'}_b V_{ab'} + \frac{1}{2}(*\mu)_{abcd} V^{cd} \end{pmatrix}.\tag{A.19}$$

Note how a $\frac{1}{2}$ factor has been introduced due to our contraction conventions, to avoid over-counting.

Finally, denoting by t_A the full set of $E_{7(7)}$ generators (A.17), the adjoint representation of $E_{7(7)}$ can be obtained from the commutator as:

$$t_A \cdot t_B = [t_A, t_B] = f_{AB}^C t_C.\tag{A.20}$$

²⁶When flattening matrix indices we must take into account that each contribution in the $SL(8)$ basis appears twice, due to the antisymmetry in the ab indices. More explicitly, if we fix the first index of an adjoint element μ^{ab}_N , its action is given by

$$\mu^{ab}_N V^N = \frac{1}{2} \left(\sum_{c,d} \mu^{ab}_{cd} V^{cd} + \sum_{c,d} \mu^{abcd} V_{cd} \right) = \sum_{c < d} \mu^{ab}_{cd} V^{cd} + \sum_{d < c} \mu^{ab}_{cd} V^{cd}.\tag{A.16}$$

A.2 SU(8)/ $\mathbb{Z}_2$ decomposition

Under SU(8)/ $\mathbb{Z}_2$ the fundamental of $E_{7(7)}$ decomposes as

$$\begin{aligned} \mathbf{56} &= \mathbf{28} \oplus \overline{\mathbf{28}} \\ V &= (V^{\alpha\beta}, \bar{V}_{\alpha\beta}) \end{aligned} \quad (\text{A.21})$$

where $\bar{V}^{\alpha\beta} = V_{\alpha\beta}^*$, and the adjoint decomposes as

$$\begin{aligned} \mathbf{133} &= \mathbf{63} \oplus \mathbf{70} \\ \mu &= (\mu^\alpha_\beta, \mu^{\alpha\beta\gamma\delta}) \end{aligned} \quad (\text{A.22})$$

with $\mu^\alpha_\alpha = 0$ and $\mu^{\alpha\beta\gamma\delta} = \mu^{[\alpha\beta\gamma\delta]}$.

The action of the adjoint of $E_{7(7)}$ on the $\mathbf{56}$ decomposes as

$$\begin{aligned} (\mu \cdot V)^{\alpha\beta} &= \mu^\alpha_\gamma V^{\gamma\beta} + \mu^\beta_\gamma V^{\alpha\gamma} + \frac{1}{2} \mu^{\alpha\beta\gamma\delta} \bar{V}_{\gamma\delta} \\ (\mu \cdot \bar{V})_{\alpha\beta} &= -\mu^\gamma_\alpha \bar{V}_{\gamma\beta} - \mu^\gamma_\beta \bar{V}_{\alpha\gamma} + \frac{1}{2} (*\mu)_{\alpha\beta\gamma\delta} V^{\gamma\delta}, \end{aligned} \quad (\text{A.23})$$

and the commutator of two adjoints reads

$$\begin{aligned} (\mu \cdot \mu')^\alpha_\beta &= \mu^\alpha_\gamma \mu'^\gamma_\beta - \mu'^\alpha_\gamma \mu^\gamma_\beta + \frac{1}{3!} \mu^{\alpha\gamma\delta\delta'} (*\mu')_{\beta\gamma\delta\delta'} \\ (\mu \cdot \mu')^{\alpha\beta\gamma\delta} &= -4(\mu^{[\alpha}_{\delta'} \mu'^{\beta\gamma\delta]\delta'} - \mu'^{[\alpha}_{\delta'} \mu^{\beta\gamma\delta]\delta'}). \end{aligned} \quad (\text{A.24})$$

A.3 Relation between SL(8) and SU(8)/ $\mathbb{Z}_2$ decomposition

The idea is to express the $E_{7(7)}$ generators in terms of SU(8) generators. This can be done by connecting the SL(8) and SU(8)/ $\mathbb{Z}_2$ basis.

The relation between a vector V in the SL(8) and SU(8)/ $\mathbb{Z}_2$ basis is obtained using the SO(8) generators in (A.4)

$$\begin{aligned} V^{\alpha\beta} &= \frac{1}{4\sqrt{2}} (V^{ab} \gamma_{ab} + i \tilde{V}_{ab} \gamma^{ab})^{\alpha\beta} \\ \bar{V}_{\alpha\beta} &= \frac{1}{4\sqrt{2}} (V^{ab} \gamma_{ab} - i \tilde{V}_{ab} \gamma^{ab})_{\alpha\beta}. \end{aligned} \quad (\text{A.25})$$

In matrix notation this is performed in terms of the unitary matrix

$$S^M_{\underline{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} \gamma^{ab}_{\alpha\beta} & \frac{1}{2} \gamma^{ab\alpha\beta} \\ -\frac{i}{2} \gamma_{ab\alpha\beta} & \frac{i}{2} \gamma_{ab}^{\alpha\beta} \end{pmatrix}. \quad (\text{A.26})$$

Explicitly, the change of basis in 56 flattened indices is obtained as

$$\begin{pmatrix} V^{\alpha\beta} \\ V_{\alpha\beta} \end{pmatrix} = V^{\underline{M}} = (S^\dagger)^{\underline{M}}_M V^M = \frac{1}{2\sqrt{2}} \begin{pmatrix} \frac{1}{2} \gamma_{ab}^{\alpha\beta} V^{ab} + \frac{i}{2} \gamma^{ab\alpha\beta} V_{ab} \\ \frac{1}{2} \gamma_{ab\alpha\beta} V^{ab} - \frac{i}{2} \gamma^{ab}_{\alpha\beta} V_{ab} \end{pmatrix}. \quad (\text{A.27})$$

To connect the elements of the $\mathbf{133}$ of $E_{7(7)}$ in the SL(8) basis (A.8) to those in the SU(8) basis (A.21), it is convenient to recall how those two split under the common SO(8) factor.

Under $\text{SL}(8) \supset \text{SO}(8)$, the elements of the **63** decompose as

$$\begin{aligned} \mathbf{63} &= \mathbf{28} \oplus \mathbf{35}_v \\ \mu^a{}_b &= \mu^{(a)a}{}_b + \mu^{(s)s}{}_b, \end{aligned} \quad (\text{A.28})$$

where $\mu_{ab}^{(a)} = \mu_{[ab]}$ and $\mu_{ab}^{(s)} = \mu_{(ab)}$ are the antisymmetric and symmetric components. Similarly the elements of the **70** decompose as

$$\begin{aligned} \mathbf{70} &= \mathbf{35}_c \oplus \mathbf{35}_s \\ \mu^{abcd} &= \mu^{[abcd]}_+ + \mu^{[abcd]}_-, \end{aligned} \quad (\text{A.29})$$

where $\mu^{[abcd]\pm}$ correspond to self-dual and anti-self dual totally antisymmetric rank four tensors

$$\mu^{[abcd]\pm} = \pm(*\mu)^{[abcd]\pm}. \quad (\text{A.30})$$

On the other hand, under $\text{SU}(8) \supset \text{SO}(8)$ we have the following splitting

$$\begin{aligned} \mathbf{63} &= \mathbf{28} \oplus \mathbf{35}_s \\ \mu^\alpha{}_\beta &= \mu^{(a)\alpha}{}_\beta + i\mu^{(s)s}{}_\beta, \end{aligned} \quad (\text{A.31})$$

where $\mu_{\alpha\beta}^{(a)} = \mu_{[\alpha\beta]}$ and $\mu_{\alpha\beta}^{(s)} = \mu_{(\alpha\beta)}$ are again the (real) antisymmetric and symmetric components, and

$$\begin{aligned} \mathbf{70} &= \mathbf{35}_v \oplus \mathbf{35}_c \\ \mu^{\alpha\beta\gamma\delta} &= \mu^{[\alpha\beta\gamma\delta]}_+ + i\mu^{[\alpha\beta\gamma\delta]}_- \end{aligned} \quad (\text{A.32})$$

with $\mu^{[\alpha\beta\gamma\delta]\pm} = \pm(*\mu)^{[\alpha\beta\gamma\delta]\pm}$ (real) self-dual and anti-self dual totally antisymmetric rank four tensors.

We can use the $\text{SO}(8)$ triality to connect the representations of $\text{SU}(8)$, in $\mathbf{8}_s$ indices, to those of $\text{SL}(8)$ in $\text{SO}(8)$ vector indices

$$\begin{aligned} \mathbf{28} : \quad & \mu^{(a)a}{}_b = \frac{1}{4}(\gamma^a{}_b)_{\alpha\beta} \mu^{(a)\alpha\beta} \\ \mathbf{35}_v : \quad & \mu^{(s)a}{}_b = \frac{1}{4}(\gamma^{ac}\gamma_{bc})_{[\alpha\beta\gamma\delta]}_+ \mu^{[\alpha\beta\gamma\delta]}_+ \\ \mathbf{35}_c : \quad & \mu^{[abcd]}_+ = \frac{3}{2} \frac{1}{4!} (\gamma^{[ab}\gamma^{cd]})_{[\alpha\beta\gamma\delta]}_- \mu^{[\alpha\beta\gamma\delta]}_- \\ \mathbf{35}_s : \quad & \mu^{[abcd]}_- = \frac{1}{4} \frac{1}{4!} (\gamma^{[abcd]}_-)_{\alpha\beta} \mu^{(s)\alpha\beta}. \end{aligned} \quad (\text{A.33})$$

By plugging (A.33) into (A.15) we can express an $\text{E}_{7(7)}$ adjoint element acting on the **56** as²⁷

$$\mu^{\underline{M}}{}_{\underline{N}} = \mu^{\alpha\beta}(t_{\alpha\beta})^{\underline{M}}{}_{\underline{N}} + \frac{1}{4!} \mu^{\alpha\beta\gamma\delta}(t_{\alpha\beta\gamma\delta})^{\underline{M}}{}_{\underline{N}}, \quad (\text{A.35})$$

²⁷Note that the generators can be expressed either in the $\text{SL}(8)$ basis (A.17) or the $\text{SU}(8)$ basis, for which we have to rotate them according to (A.27), this is:

$$(t_A)^{\underline{M}}{}_{\underline{N}} = (S^\dagger)^{\underline{M}}{}_M (t_A)^M{}_N S^N{}_{\underline{N}}. \quad (\text{A.34})$$

where $\mu^{\alpha\beta} \in \mathbf{63}$ and $\mu^{\alpha\beta\gamma\delta} \in \mathbf{70}$ of $SU(8)$ and, omitting fundamental $E_{7(7)}$ indices for the sake of simplicity,

$$\begin{aligned} t_{\alpha\beta} &= \frac{1}{4}\gamma^a{}_{b\alpha\beta} t_a{}^b - i\frac{1}{4}\frac{1}{4!}\gamma^{[abcd]}{}_{\alpha\beta} t_{abcd} \\ t_{\alpha\beta\gamma\delta} &= \frac{1}{4}\gamma^{ac}{}_{[\alpha\beta]}\gamma_{bc}{}_{[\gamma\delta]} t_a{}^b - i\frac{3}{2}\frac{1}{4!}\gamma^{[ab}{}_{[\alpha\beta}\gamma^{cd]}{}_{\gamma\delta]} t_{abcd}. \end{aligned} \quad (\text{A.36})$$

The generators $t_a{}^b$ and t_{abcd} are given in (A.17). It is straightforward, albeit tedious, to check that the action in the $SU(8)$ basis matches (A.23):

$$\left(\mu^{\gamma\delta} t_{\gamma\delta} + \frac{1}{4!} \mu^{\gamma\delta\gamma'\delta'} t_{\gamma\delta\gamma'\delta'} \right) \cdot V = \begin{pmatrix} \mu^{\alpha}{}_{\alpha'} V^{\alpha'\beta} + \mu^{\beta}{}_{\beta'} V^{\alpha\beta'} + \frac{1}{2} \mu^{\alpha\beta\alpha'\beta'} V_{\alpha'\beta'} \\ -\mu^{\alpha'}{}_{\alpha} V_{\alpha'\beta} - \mu^{\beta'}{}_{\beta} V_{\alpha\beta'} + \frac{1}{2} (*\mu)_{\alpha\beta\alpha'\beta'} V^{\alpha'\beta'} \end{pmatrix}. \quad (\text{A.37})$$

A.4 Embeddings for $\mathcal{N} = 2$ truncations

We conclude the appendix with some details about some of the embeddings relevant for the classifications of $\mathcal{N} = 2$ truncations.

Since $\mathcal{N} = 2$ truncations are associated to G_S -structures that are subgroups of $SU(6)$, we are interested in the branching

$$\begin{aligned} SU(8) &\supset SU(6) \times SU(2)_R \times U(1)_R \\ \mathbf{63} &= (\mathbf{35}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{2}, \mathbf{2})_{-4} \oplus (\bar{\mathbf{6}}, \mathbf{2})_4 \end{aligned} \quad (\text{A.38})$$

The explicit embedding of the $SU(6)$ and $SU(2)_R$ generators is given by

$$\mu_{SU(6) \times SU(2)_R} = \begin{pmatrix} \mu_{SU(6)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mu_{SU(2)_R} \end{pmatrix}, \quad (\text{A.39})$$

while $U(1)_R$ embeds as

$$\mu_{U(1)_R} = \begin{pmatrix} i\mathbb{1}_6 & \\ & -3i\mathbb{1}_2 \end{pmatrix}. \quad (\text{A.40})$$

Accordingly, the $SU(8)$ fundamental indices split into $\alpha = (m, i)$, where $i = 1, 2 \in SU(2)_R$ and $m = 1, \dots, 6 \in SU(6)$. With this choice a generalised vector (A.21) splits under (A.38) as

$$V^{\alpha\beta} = (V^{mn}, V^{mi}, V^{ij}), \quad (\text{A.41})$$

and similarly for its conjugate.

In studying truncations with only vector multiplets we need to consider G_S -structures

$$G_S \subset SU(6) \subset SO^*(12) \quad (\text{A.42})$$

where $SO^*(12)$ is the stabiliser of the triplet of adjoint singlets generating the $SU(2)_R$ R-symmetry and

$$E_{7(7)} \supset SO^*(12) \times SU(2)_R \supset SU(6) \times U(1)_R \times SU(2)_R. \quad (\text{A.43})$$

The $SO^*(12)$ generators are given, in terms of the $SU(6) \times SU(2)_R$ indices (α, i) , as:

$$\begin{aligned} SU(6) &= \mathbf{35}_0 : (\mu_{SU(6)})^{\alpha\beta} (t_{\alpha\beta})^M{}_N \\ U(1)_R &= \mathbf{1}_0 : (\mu_{U(1)_R})^{\alpha\beta} (t_{\alpha\beta})^M{}_N \\ \bar{\mathbf{15}}_{-4} &: \frac{1}{4} \mu^{mnij} (t_{mni j})^M{}_N \\ \mathbf{15}_4 &: \frac{1}{2} \mu^{mnpq} (t_{mnpq})^M{}_N. \end{aligned} \quad (\text{A.44})$$

Similarly, in studying truncations with only hypermultiplets we need to consider G_S -structures

$$G_S \subset \mathrm{SU}(6) \subset \mathrm{E}_{6(2)} \quad (\text{A.45})$$

where $\mathrm{E}_{6(2)}$ is the stabiliser of the generalised vectors K and $\hat{K}$, and

$$\mathrm{E}_{7(7)} \supset \mathrm{E}_{6(2)} \times \mathrm{U}(1)_R. \quad (\text{A.46})$$

The $\mathrm{E}_{6(2)}$ generators are given, in terms of the $\mathrm{SU}(6) \times \mathrm{SU}(2)_R$ indices (α, i) , as:

$$\begin{aligned} \mathrm{SU}(6) &= (\mathbf{35}, \mathbf{1}) : (\mu_{\mathrm{SU}(6)})^{\alpha\beta} (t_{\alpha\beta})^M{}_N \\ \mathrm{SU}(2)_R &= (\mathbf{1}, \mathbf{3}) : (\mu_{\mathrm{SU}(2)_R})^{\alpha\beta} (t_{\alpha\beta})^M{}_N \\ (\mathbf{20}, \mathbf{2}) &: \mu^{imnp} (t_{imnp})^M{}_N. \end{aligned} \quad (\text{A.47})$$

B Example: $\mathcal{N} = 4$ truncations

In this appendix we give the details of the derivation of the truncations with $\mathcal{N} = 4$ supersymmetry of Section 3.4.

To classify the possible $\mathcal{N} = 4$ truncations we have to look at G_S -structures that are subgroups

$$G_S \subseteq \mathrm{SU}(4), \quad (\text{B.1})$$

where $\mathrm{SU}(4)$ is the commutant in $\mathrm{SU}(8)$ of the $\mathrm{SU}(4)_R$ symmetry

$$\mathrm{SU}(8) \supset \mathrm{SU}(4) \times \mathrm{SU}(4)_R, \quad (\text{B.2})$$

and only preserve 4 singlets in the **8** of $\mathrm{SU}(8)$.

The idea is to proceed from the largest to the smallest subgroup $G_S \subset \mathrm{SU}(8)$, with the property (B.1). The largest subgroup of $\mathrm{SU}(8)$ leading to a $\mathcal{N} = 4$ truncation is $\mathrm{SU}(4)_S \cong \mathrm{Spin}(6)$ [8]. The $\mathrm{SU}(4)_S$ generators are embedded in $\mathrm{SU}(8)$ as anti-hermitean matrices of the form

$$\mu_{\mathrm{SU}(4)_S} \sim \begin{pmatrix} \mathfrak{su}(4)_4 & \\ & 0_4 \end{pmatrix}. \quad (\text{B.3})$$

This defines the embedding of the **4** of $\mathrm{SU}(4)_S$ in the **8** of $\mathrm{SU}(8)$ according to (B.2). Using the expressions of Appendix A we also build the embedding of the $\mathrm{SU}(4)_S$ generators in all relevant representations of $\mathrm{E}_{7(7)}$: **56** and **133**.

Then the singlets in the **8** and **63** of $\mathrm{SU}(8)$ and the **56** and **133** of $\mathrm{E}_{7(7)}$ are given by the solutions to the equations

$$\begin{aligned} \mu_{\mathrm{SU}(4)_S} \cdot \epsilon &= 0 \\ \mu_{\mathrm{SU}(4)_S} \cdot V &= 0 \\ \mu_{\mathrm{SU}(4)_S} \cdot R &= 0 \\ \mu_{\mathrm{SU}(4)_S} \cdot R_{\mathrm{SU}(8)} &= 0. \end{aligned} \quad (\text{B.4})$$

We find 12 singlets in the **56**, 18 in the **133** and 16 in the **63**, in agreement with the branchings:

$$\begin{aligned}
E_{7(7)} &\rightarrow \mathrm{SU}(4)_S \times \mathrm{SU}(4) \times \mathrm{U}(1) \\
\mathbf{56} &\rightarrow (\mathbf{1}, \mathbf{6})_{-2} \oplus (\mathbf{6}, \mathbf{1})_2 \oplus (\mathbf{4}, \mathbf{4})_0 \oplus \text{c.c.} \\
\mathbf{133} &\rightarrow (\mathbf{1}, \mathbf{1})_4 \oplus (\mathbf{1}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{15})_0 \oplus (\mathbf{15}, \mathbf{1})_0 \oplus (\mathbf{4}, \bar{\mathbf{4}})_2 \oplus (\bar{\mathbf{4}}, \mathbf{4})_{-2} \\
&\quad \oplus (\mathbf{4}, \bar{\mathbf{4}})_{-2} \oplus (\bar{\mathbf{4}}, \mathbf{4})_2 \oplus (\mathbf{6}, \mathbf{6})_0,
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
\mathrm{SU}(8) &\rightarrow \mathrm{SU}(4)_S \times \mathrm{SU}(4) \times \mathrm{U}(1) \\
\mathbf{8} &\rightarrow (\mathbf{4}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{4})_{-1} \\
\mathbf{63} &\rightarrow (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{15})_0 \oplus (\mathbf{15}, \mathbf{1})_0 \oplus (\mathbf{4}, \bar{\mathbf{4}})_2 \oplus (\bar{\mathbf{4}}, \mathbf{4})_{-2}.
\end{aligned} \tag{B.6}$$

The 12 singlets in the **56** correspond to 6 vectors plus their magnetic duals, whereas the 2 singlets in $E_{7(7)}/\mathrm{SU}(8)$ correspond to 2 scalars parametrizing the manifold $\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)}$. Together with the spin 2 degrees of freedom, these form the bosonic content of the gravity multiplet in a duality covariant form. The truncation associated to this structure is pure supergravity.

In order to obtain an $\mathcal{N} = 4$ theory with matter coupled to gravity, the largest subgroup of $\mathrm{SU}(4)_S$ is $\mathrm{USp}(4)_S \cong \mathrm{Spin}(5)_S$, whose generators are the subset of $\mathrm{SU}(4)_S$ generators in (B.3) preserving the symplectic form

$$\omega_{\alpha\beta} = \begin{pmatrix} & \mathbb{1}_2 \\ -\mathbb{1}_2 & \\ & 0_4 \end{pmatrix} \tag{B.7}$$

As for the $\mathrm{SU}(4)_S$ -structure, we can embed the $\mathrm{USp}(4)_S$ generators in the relevant representations and determine the singlets in the **56** and **133** of $E_{7(7)}$ and in the **63** of $\mathrm{SU}(8)$ imposing (B.4) with $\mu_{\mathrm{USp}(4)_S}$.

Beside the singlets corresponding to the gravity multiplet, we get one extra vector (with its magnetic dual) and 6 scalars parameterising the manifold $\mathrm{SO}(6, 1)/\mathrm{SO}(6)_R$. All together they constitute the bosonic field content of an $\mathcal{N} = 4$ vector multiplet.

These results are in agreement with the further breaking $\mathrm{SU}(4) \supset \mathrm{USp}(4)$ in (B.5)

$$\begin{aligned}
\mathrm{SU}(8) &\rightarrow \mathrm{USp}(4)_S \times \mathrm{SU}(4) \times \mathrm{U}(1) \\
\mathbf{8} &\rightarrow (\mathbf{4}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{4})_{-1} \\
\mathbf{63} &\rightarrow (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{15})_0 \oplus (\mathbf{5}, \mathbf{1})_0 \oplus (\mathbf{10}, \mathbf{1})_0 \oplus (\mathbf{4}, \bar{\mathbf{4}})_2 \oplus (\mathbf{4}, \mathbf{4})_{-2},
\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}
E_{7(7)} &\rightarrow \mathrm{USp}(4)_S \times \mathrm{SU}(4) \times \mathrm{U}(1) \\
\mathbf{56} &\rightarrow (\mathbf{1}, \mathbf{6})_{-2} \oplus (\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{5}, \mathbf{1})_2 \oplus (\mathbf{4}, \mathbf{4})_0 \oplus \text{c.c.} \\
\mathbf{133} &\rightarrow (\mathbf{1}, \mathbf{1})_4 \oplus (\mathbf{1}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{6})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{15})_0 \oplus (\mathbf{5}, \mathbf{1})_0 \oplus (\mathbf{10}, \mathbf{1})_0 \\
&\quad \oplus (\mathbf{4}, \bar{\mathbf{4}})_2 \oplus (\mathbf{4}, \mathbf{4})_{-2} \oplus (\mathbf{4}, \bar{\mathbf{4}})_{-2} \oplus (\mathbf{4}, \mathbf{4})_2 \oplus (\mathbf{5}, \mathbf{6})_0.
\end{aligned} \tag{B.9}$$

The next $G_S \subset \text{SU}(4)$ preserving $\mathcal{N} = 4$ supersymmetry and providing extra matter is $\text{Spin}(4)_S \cong \text{SU}(2)_S \times \text{SU}(2)_S \subset \text{USp}(4)_S$. Schematically, its Lie algebra reads²⁸

$$\mu_{\text{Spin}(4)_S} \sim \begin{pmatrix} \mathfrak{su}(2)_2 & & \\ & \mathfrak{su}(2)_2 & \\ & & 0_4 \end{pmatrix}. \quad (\text{B.10})$$

This corresponds to the branching:

$$\begin{aligned} \text{SU}(8) &\rightarrow \text{SU}(2)_S \times \text{SU}(2)_S \times \text{SU}(4) \times \text{U}(1) \\ \mathbf{8} &\rightarrow (\mathbf{2}, \mathbf{1}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{4})_{-1} \\ \mathbf{63} &\rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{15})_0 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus \dots, \end{aligned} \quad (\text{B.11})$$

and

$$\begin{aligned} \text{E}_{7(7)} &\rightarrow \text{SU}(2)_S \times \text{SU}(2)_S \times \text{SU}(4) \times \text{U}(1) \\ \mathbf{56} &\rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{6})_{-2} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_2 \oplus \dots \oplus \text{c.c.} \\ \mathbf{133} &\rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1})_4 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{6})_0 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{6})_0 \\ &\quad \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{15})_0 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \oplus \dots, \end{aligned} \quad (\text{B.12})$$

where $\dots$ denote non-singlets representations of $\text{Spin}(4)_S$, which we have omitted for simplicity of notation. The numbers of vector and scalar singlets corresponds to a truncation with two vector multiplets, whose 12 scalars parametrize the manifold $\frac{\text{SO}(6,2)}{\text{SO}(6)_R \times \text{SO}(2)}$.

From (B.10), one immediately realises that the diagonal combination, $\text{SU}(2)_{S,\text{diag}}$, of the two $\text{SU}(2)$'s also preserves the same amount of supersymmetries. This diagonal $\text{SU}(2)_{S,\text{diag}}$ corresponds to the branching

$$\begin{aligned} \text{SU}(8) &\rightarrow \text{SU}(2)_{S,\text{diag}} \times \text{SU}(4) \times \text{U}(1) \\ \mathbf{8} &\rightarrow (\mathbf{2}, \mathbf{1})_1 \oplus (\mathbf{2}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{4})_{-1} \\ \mathbf{63} &\rightarrow (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{15})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus \dots, \end{aligned} \quad (\text{B.13})$$

and

$$\begin{aligned} \text{E}_{7(7)} &\rightarrow \text{SU}(2)_{S,\text{diag}} \times \text{SU}(4) \times \text{U}(1) \\ \mathbf{56} &\rightarrow (\mathbf{1}, \mathbf{6})_{-2} \oplus (\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_2 \oplus \dots \oplus \text{c.c.} \\ \mathbf{133} &\rightarrow (\mathbf{1}, \mathbf{1})_4 \oplus (\mathbf{1}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{6})_0 \oplus (\mathbf{1}, \mathbf{6})_0 \oplus (\mathbf{1}, \mathbf{6})_0 \\ &\quad \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{15})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus \dots. \end{aligned} \quad (\text{B.14})$$

and gives three vector multiplets, whose 18 scalars parametrize the coset $\frac{\text{SO}(6,3)}{\text{SO}(6)_R \times \text{SO}(3)}$.

Finally, the last $G_S \subset \text{SU}(4)_S$ group preserving $\mathcal{N} = 4$ consists of the $\text{U}(1)_S \subset \text{SU}(2)_{S,\text{diag}}$. In spinor indices, it is given by²⁹:

$$\mu_{\text{U}(1)_S} = \begin{pmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \\ & & & & 0_4 \end{pmatrix}, \quad (\text{B.15})$$

²⁸In our explicit realization, this $\text{SU}(2) \times \text{SU}(2)$ arises upon a permutation of the spinor coordinates $\epsilon_2 \leftrightarrow \epsilon_3$. As far as the structure is concerned, this permutation is not needed. We include it for readability.

²⁹Here we keep the permutation $\epsilon_2 \leftrightarrow \epsilon_3$ introduced before.

and it corresponds to the branching

$$\begin{aligned}
\text{SU}(8) &\rightarrow \text{U}(1)_S \times \text{SU}(4) \times \text{U}(1) \\
\mathbf{8} &\rightarrow 2 \times [(\mathbf{1}, \mathbf{1})_{1,1} \oplus (\mathbf{1}, \mathbf{1})_{-1,1}] \oplus (\mathbf{1}, \mathbf{4})_{0,-1} \\
\mathbf{63} &\rightarrow \mathbf{1}_{(0,0)} \oplus \mathbf{15}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \\
&\quad \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \dots,
\end{aligned} \tag{B.16}$$

and

$$\begin{aligned}
\text{E}_{7(7)} &\rightarrow \text{U}(1)_S \times \text{SU}(4) \times \text{U}(1) \\
\mathbf{56} &\rightarrow \mathbf{6}_{(0,-2)} \oplus \mathbf{1}_{(0,2)} \oplus \mathbf{1}_{(0,2)} \oplus \mathbf{1}_{(0,2)} \oplus \dots \oplus \text{c.c.} \\
\mathbf{133} &\rightarrow \mathbf{1}_{(0,4)} \oplus \mathbf{1}_{(0,-4)} \oplus \mathbf{6}_{(0,0)} \oplus \mathbf{6}_{(0,0)} \oplus \mathbf{6}_{(0,0)} \oplus \mathbf{6}_{(0,0)} \\
&\quad \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{15}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \\
&\quad \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \mathbf{1}_{(0,0)} \oplus \dots
\end{aligned} \tag{B.17}$$

From the singlets in the above decomposition, one sees that the $\text{U}(1)_S$ structure preserves 4 vector multiplets on top of the gravity multiplet. The 24 scalars in the vector multiplets parametrize the manifold $\frac{\text{SO}(6,4)}{\text{SO}(6)_R \times \text{SO}(4)}$.

These are all the unequivalent continuous G_S structures one can find with $\mathcal{N} = 4$ supersymmetries. Since the field content of each truncation is fixed by the singlets of the corresponding structure, every structure can be systematically studied by solving (A.2). The number of independent solutions to this equation provides the number of singlets in each representation and, therefore, the field content for each truncation.

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