Equivalence of Synchronization States in the Hybrid Kuramoto Flow

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Abstract

We establish a unified synchronization framework for the all-to-all hybrid Kuramoto model that couples first- and second-order oscillators within a single dynamical system. Although the Kuramoto model has become one of the most widely used paradigms for describing synchronization phenomena-appearing in more than 100,000 scientific studies-the fundamental relationships among distinct synchronization states remain unresolved. In this work, we rigorously prove that full phase-locking, phase-locking, frequency synchronization, and order-parameter synchronization are equivalent for arbitrary hybrid ensembles. The proof combines dissipative energy methods, LaSalle-type compactness arguments, the Poincaré-Bendixson theorem, and Thieme's asymptotically autonomous theory to demonstrate that synchronization equivalence is topological, determined solely by the finite equilibrium structure of the all-to-all network. This result provides a complete mathematical characterization of synchronization in finite oscillator systems and clarifies its geometric invariance across first-, second-, and hybrid-order models.

Keywords: First-, second-, and hybrid-order Kuramoto model; Synchronization; Order parameter; Asymptotically autonomous systems; Poincaré-Bendixson theorem

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Contents

1 Introduction

The Kuramoto model, one of the most representative and widely applied models of synchronization, was proposed nearly five decades ago (Kuramoto [1, 2]). Its synchronization-related applications span physics [3–8], biology [9–11], control theory [12–16], and non-Abelian and operator-valued extensions [7, 17–19], and we refer interested readers to [20] for a comprehensive overview. However, the relationships among different synchronization states—and whether various definitions of synchronization are mathematically equivalent—remain open problems. The objective of this work is to investigate, under the simplest graph topology, namely the all-to-all (complete) coupling with homogeneous interactions, whether distinct notions of synchronization are equivalent for the hybrid Kuramoto model that simultaneously incorporates both inertia and damping effects.

The study of synchronization has a rich and extensive history. Before briefly reviewing its development, let us consider the finite N hybrid Kuramoto model consisting of both first- and second-order oscillators. The dynamics are given by

$$
\begin{aligned}
d_j\dot{\theta}_j &= \omega_j + \frac{\lambda}{N}\sum_{k=1}^{N}\sin(\theta_k-\theta_j), \qquad j\in\{1,\dots,n\},\\
m_j\ddot{\theta}_j + d_j\dot{\theta}_j &= \omega_j + \frac{\lambda}{N}\sum_{k=1}^{N}\sin(\theta_k-\theta_j), \qquad j\in\{n+1,\dots,N\},
\end{aligned}
\tag{1.1}
$$

where the inertia $m_j > 0$, damping $d_j > 0$, natural frequency $\omega_j \in \mathbb{R}$, and coupling strength $\lambda > 0$ are given. The parameter $n \in \{0,\dots,N\}$ determines whether the system (1.1) is purely first order ($n = 0$), purely second order ($n = N$), or genuinely hybrid. The variable $\theta_j = \theta_j(t) \in \mathbb{R}$ denotes the phase of the j-th oscillator. Since the introduction of the order parameter by Kuramoto, measuring synchronization through this quantity has had a profound impact on scientific applications (see also Strogatz [21] and Acebrón et al. [22]). In this work, we refer to this notion as the *order parameter synchronization state* (OPSS). In addition to (OPSS), several alternative mathematical definitions are commonly employed to characterize and analyze synchronization, including the *full phase-locked state* (FPLS), the *phase-locked state* (PLS), and the *frequency synchronization state* (FSS).

Previous studies have primarily focused on determining sufficient or necessary conditions for the emergence of synchronization under various parameter regimes of m_j, d_j, ω_j, and λ. The first-order Kuramoto model introduced in [1, 2] is recovered from (1.1) by taking $n = 0$ and $d_j = 1$. We refer the reader to [21, 23–28] for further developments on this classical setting. For second-order Kuramoto models, the presence of inertia introduces substantially richer and more complex dynamics. Existing works have mainly established necessary or sufficient conditions for phase-locking or frequency synchronization in the inertial regime [25, 29–32]. While these results provide criteria ensuring convergence to synchronized states, they typically address individual synchronization notions rather than the relation.

In this work, we establish a complete synchronization framework for the all-to-all hybrid Kuramoto model:

$$\text{(FPLS)} \Leftrightarrow \text{(PLS)} \Leftrightarrow \text{(FSS)} \Leftrightarrow \text{(OPSS)}.$$

We make some comments here: The implications (FPLS) $\Rightarrow$ (PLS) $\Rightarrow$ (FSS) $\Rightarrow$ (OPSS) are rather natural. The analysis is grounded in classical ODE techniques combining dissipative energy estimates with Barbalat-type convergence arguments to establish asymptotic frequency synchronization. For the reverse implications

$$\text{(FSS)} \Rightarrow \text{(PLS)} \Rightarrow \text{(FPLS)},$$

we observe that the essence of synchronization lies in the *topological structure* of the system. Since the all-to-all network admits only finitely many equilibria (mod 2π), motivated by the LaSalle invariant principle, we construct a nested sequence of compact boxes so that the ω-limit set is contained in the set $\{\dot{\theta}_j = 0\}$. This observation allows the equivalence of synchronization states to be characterized purely by the finiteness of equilibria (see also [33]).

Finally, for the implication (OPSS) $\Rightarrow$ the other synchronization states, we exploit further *symmetry* inherent in the all-to-all topology. The core conceptual step is to reinterpret the Kuramoto flow as an *asymptotically autonomous system* in the sense of Thieme [34], linking the order-parameter dynamics to individual oscillator behavior. On the compactified phase space $\mathbb{R} \times \mathbb{T}$, the pre-compactness of trajectories allows us to apply the Poincaré–Bendixson classification together with the Bendixson–Dulac criterion and Poincaré maps to exclude periodic, homoclinic, and heteroclinic orbits of the limit system. This guarantees that, for the first-, second-, and hybrid-order Kuramoto models, any ω-limit set is contained in that of the limit system—which consists solely of equilibria. Hence, we obtain a complete characterization of the equivalence among all synchronization states.

2 Definitions and Main Results

We begin by introducing a co-rotating frame associated with a weighted average frequency. To avoid an unnecessary proliferation of notation, we shall, whenever convenient, write expressions such as $\omega_j \mapsto \omega_j - d_j(\omega_1 + \ldots + \omega_N)/(d_1 + \ldots + d_N)$, which is to be understood as "ω_j is replaced by $\omega_j - d_j(\omega_1 + \ldots + \omega_N)/(d_1 + \ldots + d_N)$". With this convention, we perform the following change of variables:

$$\omega_j \mapsto \omega_j - \frac{\omega_1 + \cdots + \omega_N}{d_1 + \cdots + d_N} d_j, \quad \theta_j \mapsto \theta_j - \frac{\omega_1 + \cdots + \omega_N}{d_1 + \cdots + d_N} t,$$

which in turn implies

$$\dot{\theta}_j \mapsto \dot{\theta}_j - \frac{\omega_1 + \cdots + \omega_N}{d_1 + \cdots + d_N}, \quad \ddot{\theta}_j \mapsto \ddot{\theta}_j \qquad \text{for all} \quad j \in \{1, \ldots, N\}.$$

Under this change of frame, we may, without loss of generality, impose the normalization

$$\omega_1 + \cdots + \omega_N = 0.$$

which eliminates the uniform drift induced by the collective rotation. We remark that the special case $\omega_1 = \omega_2 = \ldots = \omega_N$ corresponds to identical natural frequencies. With this normalization in place, we now recall the standard definitions of synchronization in the Kuramoto model.

2.1 Definitions of Synchronization State

Definition 2.1 (PSS). *A solution $\theta(t)$ is called a phase synchronization state if for all $j, k \in \{1, \ldots, N\}$,*

$$\lim_{t \to \infty} (\theta_j(t) - \theta_k(t)) = 0.$$

Definition 2.2 (FPLS). *A solution $\theta(t)$ is called a full phase-locked state if for all $j, k \in \{1, \ldots, N\}$,*

$$\lim_{t \to \infty} (\theta_j(t) - \theta_k(t)) = \theta^*_{jk},$$

where θ^*_{jk} are constants for all $j, k \in \{1, \ldots, N\}$. This is equivalent to the statement that $\theta(t)$ converges to a steady state θ^* satisfying the stationary equation:

$$\omega_j + \frac{\lambda}{N} \sum_{k=1}^{N} \sin(\theta^*_{kj}) = 0,$$

for all $j \in \{1, \dots, N\}$.

Definition 2.3 (PLS)**.** *A solution* $\theta(t)$ *is called a phase-locked state if for all* $j, k \in \{1, \dots, N\}$*, there exists a positive constant* L *such that for all* $t > 0$*,*

$$|\theta_j(t) - \theta_k(t)| \leq L.$$

Definition 2.4 (FSS)**.** *A solution* $\theta(t)$ *is called a frequency synchronization state if*

$$\lim_{t\to\infty} \left|\dot{\theta}_j(t)\right| = 0, \quad \textit{for all} \;\; j \in \{1, \dots, N\}.$$

Definition 2.5 (OPSS)**.** *Define order parameter for* $\theta(t)$ *as*

$$Z(t) := R(t)e^{i\Theta(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}, \tag{2.1}$$

where $R(t)$ *and* $\Theta(t)$ *are two real functions. A solution* $\theta(t)$ *is called OP synchronization state if there exist* $Z^* \in \mathbb{C}$ *such that*

$$\lim_{t\to\infty} Z(t) = Z^* \quad \textit{and} \quad |Z^*| \geq \frac{\omega_M}{\lambda}, \tag{2.2}$$

where $\omega_M := \max\limits_{j\in\{1,\dots,N\}} |\omega_j|$. We pause to remark that a necessary condition for the existence of a phase synchronization state (PSS) is that all natural frequencies are identical. In other words, one must assume that

$$\omega_j - \omega_k \equiv 0, \quad \text{for all } j, k \in \{1, \dots, N\}.$$

Moreover, it is evident that if $\theta(t)$ is a full phase-locked state, then it is also a phase-locked state. Additionally, $R : \mathbb{R}^+ \to [0, 1]$ is a continuous function. We also assume that $\Theta(0) \in [0, 2\pi)$ and that $\Theta : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.

2.2 Main Theorems

The main result of this paper is on the equivalence of synchronization states for the hybrid Kuramoto model:

$$\begin{aligned}
&m_j\ddot{\theta}_j + d_j\dot{\theta}_j = \omega_j + \frac{\lambda}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \qquad \sum_{j=1}^{N}\omega_j = 0,\\
&\qquad d_j > 0, \quad \omega_j \in \mathbb{R}, \quad \lambda > 0, \qquad j \in \{1, \dots, N\},\\
&\qquad m_j = 0, \quad j \in \{1, \dots, n\} \quad \text{and} \quad m_j > 0, \quad j \in \{n+1, \dots, N\}.
\end{aligned} \tag{2.3}$$

It will also be convenient to define some notation. For the phase vector $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$ and the natural frequency vector $\omega = (\omega_1, \dots, \omega_N)$, we set

$$\theta_m := \min_{1\le j\le N} \theta_j, \qquad \theta_M := \max_{1\le j\le N} \theta_j, \qquad \omega_M := \max_{1\le j\le N} |\omega_j|\,. \tag{2.4}$$

The main result of this paper is as follows.

Theorem 2.1 (Equivalence of Synchronization)**.** *Let $\theta(t)$ be a solution of the hybrid Kuramoto model* (2.3)*. Then the following statements are equivalent:*

1. *The solution $\theta(t)$ is a full phase-locked state* (FPLS)*.*
2. *The solution $\theta(t)$ is a phase-locked state* (PLS)*.*
3. *The solution $\theta(t)$ is a frequency synchronization state* (FSS)*.*
4. *The solution $\theta(t)$ is an order parameter synchronization state* (OPSS)*.*

Consequently, any of the above equivalent properties may be used as the definition of a synchronization state. Moreover, in the special case $\omega_j = 0$, the previous synchronization notions are equivalent to (PSS) *as well.*

Regarding Theorem 2.1, our proof strategy is divided into two steps. First, we establish the equivalence among (FPLS), (PLS), and (FSS). Then, we show that (OPSS) is also equivalent to the previous three notions. The most challenging part of the first step is to prove that (FSS) implies (PLS), and subsequently that (PLS) implies (FPLS). Before giving the full proof, we observe that, by definition, (FPLS) automatically implies (PLS). Next, using a Lyapunov-like auxiliary function, we demonstrate that (PLS) implies (FSS). For the reader's convenience, we provide a direct proof here. A similar argument first appeared in Hemmen–Wreszinski [35].

Lemma 2.2. *Let $\theta(t)$ be a solution of the hybrid Kuramoto model* (2.3)*. If $\theta(t)$ is a* (PLS)*, then it is also a* (FSS)*.*

Proof. Multiplying $\dot\theta_j$ in (2.3), summing over $j = 1, \dots, N$ and integrating the term over $(0, t)$, we obtain

$$\begin{aligned}\sum_{j=1}^N \frac{m_j}{2}\dot\theta_j^2(t) + \int_0^t \sum_{j=1}^N d_j \dot\theta_j^2(s)\mathrm{d}s &= \sum_{j=1}^N \frac{m_j}{2}\dot\theta_j^2(0) + \sum_{j=1}^N \omega_j\left(\theta_j(t) - \theta_j(0)\right) \\ &\quad + \int_0^t \frac{\lambda}{N}\sum_{j=1}^N\sum_{k=1}^N \sin(\theta_k(s) - \theta_j(s))\dot\theta_j(s)\mathrm{d}s.\end{aligned}$$

Since $\theta(t)$ is a (PLS), all phase differences $\theta_j - \theta_k$ remain uniformly bounded. Hence

$$\sum_{j=1}^N \omega_j\left(\theta_j(t) - \theta_j(0)\right) = \sum_{j=2}^N \omega_j(\theta_j(t) - \theta_1(t)) - \sum_{j=1}^N \omega_j\theta_j(0)$$

is uniformly bounded in t. Also, one may observe

$$\int_0^t \sum_{j=1}^N \sum_{k=1}^N \sin(\theta_k(s) - \theta_j(s))\dot{\theta}_j(s)\mathrm{d}s = \sum_{j<k} \cos(\theta_k(s) - \theta_j(s))\Big|_0^t,$$

which is also uniformly bounded in t. Therefore,

$$\lim_{t\to\infty} \int_0^t \sum_{j=1}^N d_j \dot{\theta}_j^2(s)\mathrm{d}s \quad \text{exists and bounded.}$$

On the other hand, from (2.3) we see that both $\dot{\theta}_j(t)$ and $\ddot{\theta}_j(t)$ are bounded, which implies the uniform continuity of $\sum_j d_j \dot{\theta}_j^2(t)$. Applying Barbalat's lemma [36, Theorem 1], we obtain

$$\lim_{t\to\infty} \sum_{j=1}^N d_j \dot{\theta}_j^2(t) = 0,$$

hence $\dot{\theta}_j(t) \to 0$ for all j. Therefore, $\theta(t)$ is a (FSS). $\square$

We proceed to show first that (FSS) $\Rightarrow$ (PLS) and then (PLS) $\Rightarrow$ (FPLS); afterwards, we verify that (OPSS) is in fact equivalent to these three synchronization states. To streamline the argument, we begin with the following key lemma.

Lemma 2.3. *Let $\psi = (\psi_1, \dots, \psi_N) \in \mathbb{T}^N := (\mathbb{R}/(2\pi\mathbb{Z}))^N$. Consider the system*

$$g_j(\psi) := \omega_j + \frac{\lambda}{N} \sum_{k=1}^N \sin(\psi_k - \psi_j) = 0, \qquad j = 1, \dots, N, \tag{2.5}$$

with parameters satisfying $\sum_{j=1}^N \omega_j = 0$. Then the system either has no solution or admits only finitely many phase-locked solutions up to a common phase shift, i.e., the set of solutions ψ is finite modulo the transformation $\psi \mapsto \psi + s\mathbf{1}$, $s \in \mathbb{R}$.

Proof. Let us rewrite the system as follows:

$$g_j(\psi) = \omega_j + \lambda r \sin(\Psi - \psi_j) = 0, \quad re^{i\Psi} = \frac{1}{N}\sum_{j=1}^N e^{i\psi_j}, \quad j = 1, \dots, N. \tag{2.6}$$

Therefore, it is clear that if $\lambda < \omega_M$, then the system has no solution since $r \le 1$ and $|\sin(\cdot)| \le 1$. Hence we assume $\lambda > \omega_M$. From (2.6) we obtain

$$\sin(\Psi - \psi_j) = -\frac{\omega_j}{\lambda r} := s_j, \quad \text{with} \quad r \in \left[\frac{\omega_M}{\lambda}, 1\right].$$

Thus each $\cos(\Psi - \psi_j)$ must satisfy

$$\cos(\Psi - \psi_j) = \sigma_j \sqrt{1 - \left(\frac{\omega_j}{\lambda r}\right)^2} := c_j, \qquad \sigma_j \in \{\pm 1\}.$$

The order-parameter identity in (2.6) yields the self-consistency condition

$$rN = \sum_{j=1}^{N} \cos(\Psi - \psi_j) = \sum_{j=1}^{N} \sigma_j \sqrt{1 - \left(\frac{\omega_j}{\lambda r}\right)^2}, \text{ with } \quad r \in \left[\frac{\omega_M}{\lambda}, 1\right].$$

Fix a sign vector $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{\pm 1\}^N$ and set $x = \lambda r \in [\omega_M, \lambda]$. One may define

$$H_\sigma(x) := x - \frac{\lambda}{N} \sum_{j=1}^{N} \sigma_j \sqrt{1 - \left(\frac{\omega_j}{x}\right)^2}.$$

Since H_σ is a nontrivial real-analytic function on (ω_M, ∞), its zeros are isolated. Hence $H_\sigma(x) = 0$ has at most finitely many solutions within the compact interval $[\omega_M, \lambda]$. Because there are at most 2^N choices of σ, it follows that the total number of admissible values of r is finite. For each fixed pair (r, σ), the angle

$$\Delta_j := \Psi - \psi_j = \operatorname{arctan2}(s_j, c_j) \in (-\pi, \pi]$$

is uniquely determined. Thus the relative phase configuration $(\psi_1 - \Psi, \ldots, \psi_N - \Psi)$ is uniquely fixed.

The vector ψ itself is not unique: a simultaneous shift

$$\Psi \mapsto \Psi + s, \qquad \psi_j \mapsto \psi_j + s,$$

leaves all differences $\Psi - \psi_j$ invariant. Therefore the set of solutions corresponding to (r, σ) forms exactly a one-dimensional orbit generated by the global phase-shift symmetry. $\square$

We pause to remark that to the best of the authors' knowledge, the finiteness of roots for such a system was first conjectured in [37], where the case $N = 3$ was rigorously proved and an intuitive argument for the finiteness when $N > 3$ was also provided. For further studies on the finiteness of equilibria, we refer the reader to the numerical and formally analytical results in [38], [39], and [40], which confirm the finiteness of equilibria for various classes of coupling configurations and network topologies.

Lemma 2.4 (Landau-Kolmogorov inequality)**.** *Let $I \subset \mathbb{R}^+$ be an open interval and $f \in C^2(I)$ satisfy $\|f\|_{L^\infty(I)} < \infty$ and $\|\ddot{f}\|_{L^\infty(I)} < \infty$. Then*

$$\|\dot{f}\|_{L^\infty(I)} \le 2 \, \|f\|_{L^\infty(I)}^{\frac{1}{2}} \, \|\ddot{f}\|_{L^\infty(I)}^{\frac{1}{2}}.$$

Proof. Fix $x \in I$ and then choose small enough $\delta > 0$ such that $x \pm \delta \in I$. By Taylor's theorem, there exist $\xi_- \in (x-\delta, x)$ and $\xi_+ \in (0, x+\delta)$ such that

$$f(x+\delta) = f(x) + \delta \dot{f}(x) + \frac{\delta^2}{2}\ddot{f}(\xi_+), \qquad f(x-\delta) = f(x) - \delta \dot{f}(x) + \frac{\delta^2}{2}\ddot{f}(\xi_-).$$

Subtracting these two expressions gives

$$2\delta \left|\dot{f}(x)\right| \le |f(x+\delta) - f(x-\delta)| + \frac{\delta^2}{2}\left|\ddot{f}(\xi_+) - \ddot{f}(\xi_-)\right| \le 2\|f\|_{L^\infty(I)} + \delta^2 \|\ddot{f}\|_{L^\infty(I)}.$$

Hence

$$\left|\dot{f}(x)\right| \le \frac{\|f\|_{L^\infty(I)}}{\delta} + \frac{\delta}{2}\|\ddot{f}\|_{L^\infty(I)}.$$

Optimizing in δ by taking $\delta = \sqrt{2\|f\|_{L^\infty(I)}/\|\ddot{f}\|_{L^\infty(I)}}$ yields

$$\left|\dot{f}(x)\right| \le 2\,\|f\|^{\frac{1}{2}}_{L^\infty(I)}\,\|\ddot{f}\|^{\frac{1}{2}}_{L^\infty(I)}.$$

Taking the supremum over x completes the proof. □

We note in passing that the constant 2 is sharp on $I = \mathbb{R}^+$ [41]. For a more general form of the Landau–Kolmogorov inequality, see [42].

Finally, we provide a fundamental a priori estimate for $\dot{\theta}(t)$.

Lemma 2.5. *The solution $\theta(t)$ of (2.3) satisfies*

$$\begin{aligned}
&\left|\dot{\theta}_j(t)\right| \le \frac{|\omega_j|}{d_j} + \frac{\lambda}{d_j}, && j \in \{1, \dots, n\},\\
&\left|\dot{\theta}_j(t)\right| \le \left|\dot{\theta}_j(0)\right| e^{-\frac{d_j}{m_j}t} + \left(\frac{|\omega_j|}{m_j} + \frac{\lambda}{m_j}\right)\left(1 - e^{-\frac{d_j}{m_j}t}\right), && j \in \{n+1, \dots, N\},
\end{aligned}$$

for all $t \ge 0$.

3 Proof of Theorem 2.1

We are now in a position to prove Theorem 2.1.

3.1 Proof of (FPLS) ⇔ (PLS) ⇔ (FSS)

Lemma 3.1. *Let $\theta(t)$ be a solution of the hybrid Kuramoto model (2.3). Then the following implications hold:*

$$(\mathrm{FSS}) \Rightarrow (\mathrm{PLS}) \quad \textit{and} \quad (\mathrm{PLS}) \Rightarrow (\mathrm{FPLS}).$$

Proof. (FSS) $\Rightarrow$ (PLS): Recall (2.3). We know that both $|\ddot{\theta}|$ and $|\dddot{\theta}|$ are uniformly bounded in $t > 0$, since we assume (FSS). By applying Lemma 2.4, we obtain that $\ddot{\theta}_j$ also converges to zero for each $j \in \{1, \dots, N\}$.

Next, we observe that if we sum (2.3) over j from 1 to N, and then integrate over the interval $[0, t]$, we obtain

$$\lim_{t\to\infty} \sum_j d_j \theta_j = \sum_j d_j \theta_j(0) + \sum_j m_j \dot{\theta}_j(0) := C_0.$$

We define the following algebraic equation:

$$g_0(\theta) := d_1\theta_1 + \ldots + d_N\theta_N = C_0. \tag{3.1}$$

Recalling (2.5), it is clear that g_j is periodic and satisfies

$$g_j(\theta_1, \dots, \theta_k, \dots, \theta_N) = g_j(\theta_1, \dots, \theta_k + 2\pi, \dots, \theta_N),$$

for all $j, k \in \{1, \dots, N\}$.

Partition $\mathbb{R}^N$ into a grid of boxes $\{U_n\}_{n=1}^{\infty}$, each being a closed hypercube with side length 2π. For a fixed initial condition, i.e., a fixed value of C_0, we apply Lemma 2.3. Together with the constraint $g_0 = C_0$ (cf. (3.1)), it follows that the system of equations

$$g_0(\theta) = C_0, \quad g_1(\theta) = 0, \dots, g_N(\theta) = 0$$

has only finitely many roots in each box U_n.

Using these roots $\{r_n^1, r_n^2, \dots, r_n^m\} \subset U_n$ as centers, we construct disjoint open balls

$$\{B^\epsilon_{n1}, B^\epsilon_{n2}, \dots, B^\epsilon_{nm}\} \subset \mathbb{R}^N. \tag{3.2}$$

Then there exists $\delta > 0$ such that

$$|g_1(\theta_1, \dots, \theta_N)| > \delta \quad \text{and} \quad |g_0 - C_0| > \delta \tag{3.3}$$

simultaneously on the closed and bounded set

$$U_n \setminus \bigcup_{j=1}^{m} B^\epsilon_{nj}. \tag{3.4}$$

By periodicity, the same argument applies to all boxes $\{U_n\}_{n=1}^{\infty}$, yielding the same roots, open balls, and the same uniform bound δ, so that

$$|g_1(\psi_1, \dots, \psi_N| > \delta \quad \text{and} \quad |g_0 - C_0| > \delta \tag{3.5}$$

hold simultaneously on $U_n \setminus \bigcup_{j=1}^m B^\epsilon_{nj}$ for all $n \in \mathbb{N}$.

We now argue by contradiction. Suppose that θ is not a (PLS). Then there exists at least one index j such that θ_j is unbounded; without loss of generality, take $j = 1$. The trajectory $(\theta_1(t), \ldots, \theta_N(t))$ lies in $\mathbb{R}^N$, and since θ_1 is unbounded, the trajectory must pass through infinitely many distinct boxes. Consequently, the inequalities $|g_1| > \delta$ and $|g_0 - C_0| > \delta$ are satisfied infinitely often. In other words, for every $\tilde{n} \in \mathbb{N}$, there exists a time $t_{\tilde{n}}$ such that

$$|g_1(\theta_1(t_{\tilde{n}}), \ldots, \theta_N(t_{\tilde{n}}))| > \delta \quad \text{and} \quad |g_0(\theta_1(t_{\tilde{n}}), \ldots, \theta_N(t_{\tilde{n}})) - C_0| > \delta. \tag{3.6}$$

This contradicts the assumption that $\theta(t)$ is a (FSS), since the left-hand side of (2.3) tends to zero as $t \to \infty$. Therefore, $\theta(t)$ must be a (PLS).

(PLS) $\Rightarrow$ (FPLS): We follow the previous argument. Given a sufficiently small $\epsilon > 0$, we construct disjoint open balls B_{nl}^{ϵ} centered at r_n^l, for all $n \in \mathbb{N}$ and $l \in \{1, \ldots, m\}$. From the preceding proof, we already know that $\theta(t)$ is a (PLS), or equivalently, a (FSS) (cf. Lemma 2.2). By a similar reasoning, we deduce that

$$|g_j(\theta_1, \ldots, \theta_N)| + |g_0(\theta_1, \ldots, \theta_N) - C_0| \tag{3.7}$$

is bounded below by a positive constant for all $j \in \{1, \ldots, N\}$ and for all

$$(\theta_1, \ldots, \theta_N) \in \mathbb{R}^N \setminus \bigcup_{n=1}^{\infty} \bigcup_{l=1}^{m} B_{nl}^{\epsilon}. \tag{3.8}$$

Therefore, there exists a sufficiently large $T > 0$ such that the trajectory eventually enters one of these balls B_{nl}^{ϵ} for some $n \in \mathbb{N}$ and $l \in \{1, \ldots, m\}$; that is,

$$(\theta_1(t), \ldots, \theta_N(t)) \in B_{nl}^{\epsilon}, \quad \text{for all } t > T. \tag{3.9}$$

By applying the same argument iteratively, we can find a smaller radius $\epsilon_1 < \epsilon$ and show that the trajectory will ultimately be confined within the smaller ball $B_{nl}^{\epsilon_1}$. Repeating this process indefinitely, we conclude that the trajectory converges to a specific point r_n^j, which corresponds to an equilibrium.

Hence, if $\theta(t)$ is a (PLS), then $\theta(t)$ is also a (FPLS). This completes the proof of Lemma 3.1. $\square$

Combining Lemma 2.2 with Lemma 3.1, we obtain the equivalence among (FPLS), (PLS), and (FSS). From now on, we refer to these equivalent notions collectively as the synchronization state (SS).

3.2 Proof of (SS) $\Leftrightarrow$ (OPSS)

Let us consider the Lemma as follows:
Lemma 3.2. *Let $\theta(t)$ be a solution of the hybrid Kuramoto model (2.3). Then the following implication holds:*

$$\text{(SS)} \Rightarrow \text{(OPSS)}.$$

Proof. Suppose that $\theta(t)$ exhibits (SS). Then there exist constants $\theta_j^* \in \mathbb{R}$, depending on the initial data (cf. (3.1)), such that

$$\lim_{t\to\infty} \theta_j(t) = \theta_j^*, \quad \text{for all } j \in \{1, \dots, N\}.$$

Consequently, the order parameter $Z(t)$ converges to a constant as $t \to \infty$. In other words, there exist $R^* \in (0, 1]$ and $\Theta^* \in \mathbb{R}$ such that

$$\lim_{t\to\infty} R(t) = R^*, \quad \text{and} \quad \lim_{t\to\infty} \Theta(t) = \Theta^*.$$

If condition (2.2) does not hold, then there exists some k such that

$$|\omega_k| > |Z(t)|\,|\lambda| = R(t)|\lambda|, \quad \text{for } t \text{ sufficiently large}.$$

In this case, $|\dot{\theta}_k(t)|$ remains bounded below by a positive constant, implying that $\theta(t)$ cannot be an (FSS). This contradiction completes the proof. □

We now turn to proving the converse of Lemma 3.2. Our argument is based on Thieme's theory of asymptotically autonomous convergence. The key result, stated as Theorem 3.3, is due to Thieme [34]. A detailed exposition of this theorem, together with its proof and various generalizations, can be found in Thieme [34].

Theorem 3.3. *Let (X, d) be a metric space and consider the nonautonomous ODE (the original system)*

$$\dot{x} = F(t, x), \qquad x \in X,$$

and its limiting autonomous ODE (the limit system)

$$\dot{y} = f(y), \qquad y \in X,$$

with associated semiflows φ and ϕ, respectively. Assume that $F(t, x) \to f(x)$ locally uniformly in x as $t \to \infty$ and the equilibria of ϕ are isolated compact ϕ-invariant subsets of X. Further, the ω-ϕ-limit set of any pre-compact ϕ-orbit contains a ϕ-equilibrium. Then, let the point (s, x), with $s \geq t_0$ and $x \in X$, have a pre-compact φ-orbit. Then the following alternative holds:

- *$\varphi(t, s; x) \to e$ as $t \to \infty$, for some ϕ-equilibrium e;*
- *the ω-φ-limit set of (s, x) contains finitely many ϕ-equilibria which are chained to each other in a cyclic way.*

We rewrite equation (2.3) (see also (1.1)) in terms of the order parameter (cf. (2.1)) as

$$\begin{aligned} & m_j\ddot{\theta}_j + d_j\dot{\theta}_j = \omega_j + \lambda R(t)\sin(\Theta(t) - \theta_j), \qquad \sum_{j=1}^N \omega_j = 0, \\ & \quad d_j > 0, \quad \omega_j \in \mathbb{R}, \quad \lambda > 0, \qquad j \in \{1, \dots, N\}, \\ & \quad m_j = 0, \quad j \in \{1, \dots, n\} \quad \text{and} \quad m_j > 0, \quad j \in \{n+1, \dots, N\}. \end{aligned} \tag{3.10}$$

where $\theta_j \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ for all $j \in \{1, \ldots, N\}$. Assume that $\theta(t)$ represents an order-parameter synchronization state (OPSS). On the other hand, we obtain its limiting autonomous counterpart:

$$
\begin{aligned}
& m_j\ddot{\theta}_j + d_j\dot{\theta}_j = \omega_j + \lambda R^* \sin(\Theta^* - \theta_j), \qquad \sum_{j=1}^{N} \omega_j = 0, \\
& \quad d_j > 0, \quad \omega_j \in \mathbb{R}, \quad \lambda > 0, \qquad j \in \{1, \ldots, N\}, \\
& \quad m_j = 0, \quad j \in \{1, \ldots, n\} \quad \text{and} \quad m_j > 0, \quad j \in \{n+1, \ldots, N\}.
\end{aligned} \tag{3.11}
$$

where $\theta_j \in \mathbb{T}$ for all $j \in \{1, \ldots, N\}$ and the order parameter satisfies

$$
R(t) \to R^*, \quad \Theta(t) \to \Theta^* \quad \text{as} \quad t \to \infty.
$$

For each $j \in \{1, \ldots, N\}$, we regard (3.10) as the original system and (3.11) as the limiting system. Our goal is to exclude periodic solutions of the limit system (3.11). Using only the average-controlled asymptotic behavior (OPSS) to control the individual asymptotic behavior (SS) is generally impossible. We need to exploit the symmetry of the all-to-all coupling topology. When we consider (3.10) and its limiting system (3.11), we observe that for any fixed $j \in \{1, \ldots, N\}$, the j-th component can be viewed as a decoupled single equation. Therefore, our goal is to characterize the omega-limit set of the j-th equation of (3.11) for any pre-compact orbit. In the following, we analyze only the more intricate second-order case $m_j > 0$; the case $m_j = 0$ is omitted since it can be treated analogously.

For any fixed $j \in \{n+1, \ldots, N\}$, let $v_j = \dot{\theta}_j$. Then the original system can be rewritten as follows, and the corresponding limiting system takes the form:

$$
\begin{aligned}
\dot{v}_j &= -\frac{d_j}{m_j} v_j + \frac{\omega_j}{m_j} + \frac{\lambda}{m_j} R(t) \sin(\Theta(t) - \theta_j) := F_1^j(v_j, \theta_j, t), \\
\dot{\theta}_j &= v_j := F_2^j(v_j),
\end{aligned} \tag{3.12}
$$

where $(v_j, \theta_j) \in \mathbb{R} \times \mathbb{T}$ for each $j \in \{1, \ldots, N\}$. Also, the limit equation of (3.12) is given by

$$
\begin{aligned}
\dot{v}_j &= -\frac{d_j}{m_j} v_j + \frac{\omega_j}{m_j} + \frac{\lambda}{m_j} R^* \sin(\Theta^* - \theta_j) := f_1^j(v_j, \theta_j), \\
\dot{\theta}_j &= v_j := f_2^j(v_j),
\end{aligned} \tag{3.13}
$$

where $(v_j, \theta_j) \in \mathbb{R} \times \mathbb{T}$ for each $j \in \{1, \ldots, N\}$ and the order parameter satisfies

$$
R(t) \to R^*, \quad \Theta(t) \to \Theta^* \quad \text{as} \quad t \to \infty.
$$

We note that since the system is 2π-periodic, the vector field is invariant under the transformation $\theta \mapsto \theta + 2\pi$. Therefore, we first consider the dynamics projected onto the torus $\mathbb{T}^N$ and subsequently lift the trajectories back to the original space $\mathbb{R}^N$.

Lemma 3.4. *Let $\theta(t)$ be a solution of the hybrid Kuramoto model* (2.3)*. Then the following implication holds:*

$$(\text{OPSS}) \Rightarrow (\text{SS}).$$

Proof. If $\omega = 0$, we can directly apply an argument similar to that in Lemma 2.2 to conclude that $\dot{\theta} \to 0$. Hence, the proof is complete in this case. Therefore, we now consider the case $\omega \neq 0$.

We consider the case $m_j > 0$; the case $m_j = 0$ can be treated by a similar argument. We equip $(v_j, \theta_j) \in \mathbb{R} \times \mathbb{T}$ with the *product metric*

$$d\big((v^1, \theta^1), (v^2, \theta^2)\big) := |v^1 - v^2| + \mathrm{dist}_{\mathbb{T}}(\theta^1, \theta^2),$$

where $\mathrm{dist}_{\mathbb{T}}$ denotes the standard geodesic distance on the torus $\mathbb{T}$. Assuming that $\theta(t)$ is an (OPSS), we may consider the corresponding *limit system* (3.13) associated with (3.12). Our goal is to verify the assumptions of Theorem 3.3. It is easy to check that $F(\tilde{v}, \tilde{\theta}, t) \to f(\tilde{v}, \tilde{\theta})$ uniformly in $(\tilde{v}, \tilde{\theta})$ as $t \to \infty$. Also, we already know that every trajectory of the limit system (3.13) is precompact since Lemma 2.5 and the compactness of $\mathbb{T}$, and moreover, the equilibrium points of (3.13) can be explicitly solved and there are at most two equilibria. Using the Poincaré–Bendixson theorem, the ω-limit set of any orbit of the limit system (3.13) is either a single equilibrium, a periodic orbit, or a connected set composed of finitely many equilibria together with their homoclinic and heteroclinic connections. Again, the system (3.13) possesses exactly two equilibria $e_1, e_2 \in \mathbb{R} \times \mathbb{T}$ when $\lambda R^* > |\omega_j|$, and only one equilibrium e_1 when $\lambda R^* = |\omega_j|$. The ω-limit set of the limiting system (3.13) can only be one of the following: a contractible periodic orbit, a contractible homoclinic orbit, a single heteroclinic orbit connecting the two equilibria, two heteroclinic orbits connecting the two equilibria, a non-contractible periodic orbit, a non-contractible homoclinic orbit, or a set of equilibria.

First, by the Bendixson–Dulac theorem, any contractible closed orbit in $\mathbb{R} \times \mathbb{T}$ cannot occur, since the divergence satisfies

$$\mathrm{div}(F_1^j, F_2^j) = \frac{\partial}{\partial v_j} F_1^j + \frac{\partial}{\partial \theta_j} F_2^j = -\frac{d_j}{m_j} < 0,$$

whose sign is strictly negative and never vanishes. Therefore, both a contractible periodic orbit and a contractible homoclinic orbit are impossible. Next, if there exist two heteroclinic orbits connecting the two equilibria, they can be further classified into two topologically distinct cases: a pair of contractible heteroclinic orbits, which together bound a simply connected region on $\mathbb{R} \times \mathbb{T}$, or a pair of non-contractible heteroclinic orbits, which wind once around the cylindrical direction of $\mathbb{R} \times \mathbb{T}$ and therefore cannot be continuously deformed into each other within the phase space. Let us get rid of the first case. By lifting this pair of contractible heteroclinic orbits back to $\mathbb{R} \times \mathbb{R}$

and consider the Lyapunov-type function

$$L(v_j(t), \theta_j(t)) := \frac{1}{2} m_j v_j^2 - \omega_j \theta_j - \lambda R^* \cos(\Theta^* - \theta_j). \tag{3.14}$$

Let $(v_j, \theta_j) \in \mathbb{R}^2$ satisfy (3.13). A straightforward calculation yields

$$\frac{d}{dt} L(v_j(t), \theta_j(t)) = -d_j v_j^2 < 0 \quad \text{when} \quad v_j \neq 0. \tag{3.15}$$

Therefore, if there exist two heteroclinic orbits connecting the two equilibria e_1 and e_2, the Lyapunov function L must satisfy $L(e_1^+) > L(e_2^+)$ along one orbit and $L(e_1^+) < L(e_2^+)$ along the other, which is impossible.

Next, we exclude the more delicate cases of a non-contractible periodic orbit, a non-contractible homoclinic orbit, and a pair of non-contractible heteroclinic orbits. Lifting the dynamics from $\mathbb{R} \times \mathbb{T}$ to $\mathbb{R} \times \mathbb{R}$, all such trajectories become unbounded in the lifted space, regardless of whether they wind with positive or negative orientation on the cylinder. Without loss of generality, we may assume that the angular component satisfies $\limsup_{t\to\infty} \theta_j(t) \to +\infty$. The other case can be argued by a similar method.

Since $\lambda R^* \geq |\omega_j| = \omega_j$, there exists at least one $\theta_0 \pmod{2\pi}$ satisfying

$$\omega_j + \lambda R^* \sin(\Theta^* - \theta_0) = 0.$$

Without loss of generality, assume $\theta_0 \in [0, 2\pi)$. Consider the cross section

$$\Sigma^+ := \{\, (v_j, \theta_j) : \theta_j = \theta_0 + 2\pi,\ v_j > 0 \,\}.$$

Fix $\theta_j(0) = \theta_0$ and regard the initial speed $v_0 > 0$ as a free parameter. Define a Poincaré-like map P so that $P(v_0) \in \Sigma^+$; namely, starting from the initial condition $(v_j(0), \theta_j(0)) = (v_0, \theta_0)$, the value $P(v_0)$ represents the velocity component of the solution upon its first return to the cross section Σ^+. This is indeed inevitable. Since the lifted trajectory is unbounded with $\theta_j(t) \to +\infty$, let

$$\tau(v_0) := \inf\{\, t > 0 : \ \theta_j(t) = \theta_0 + 2\pi \,\}$$

be the first hitting time of $\theta_j = \theta_0 + 2\pi$. By continuity of $\theta_j(t)$ and monotonic escape in the angular direction, $\tau(v_0) < \infty$. At $t = \tau(v_0)$ we cannot have $v_j(\tau(v_0)) = 0$, because then, by uniqueness of solutions, the trajectory would stick to the equilibrium and be trivial, contradicting unboundedness. Nor can $v_j(\tau(v_0)) < 0$, since this would violate the choice of positive orientation toward $\theta_j \to +\infty$ at the first crossing. Therefore the Poincaré return velocity satisfies $P(v_0) := v_j(\tau(v_0)) > 0$.

Lemma 3.5. *Let $(v(t; v_0), \theta(t; v_0))$ solve (3.13) with $(v(0; v_0), \theta(0; v_0)) = (v_0, \theta_0)$, and let $\tau(v_0) > 0$ be the first time $\theta(t; v_0) = \theta_0 + 2\pi$, assuming $v(\tau(v_0); v_0) > 0$. Define $P(v_0) := v(\tau(v_0); v_0)$. Then τ and P are of class C^1 in a neighborhood of v_0.*

Proof. Let $G(t, v_0) = \theta(t; v_0) - (\theta_0 + 2\pi)$. Since $G(\tau(v_0), v_0) = 0$ and $\partial_t G = v(\tau(v_0); v_0) \neq 0$, the implicit function theorem gives $\tau \in C^1$ and the first identity. The second follows by the chain rule applied to $P(v_0) = v(\tau(v_0); v_0)$. □

We now exclude the possibility of periodic orbits. We notice that if P admits a fixed point, then a periodic orbit exists. By (3.14) and (3.15), we have

$$|P(v_0)|^2 - |v_0|^2 = \frac{4\pi\omega_j}{m_j} - \frac{2d_j}{m_j}\int_0^{\tau(v_0)} v_j^2(t)\,dt. \tag{3.16}$$

Denoting the derivative with respect to v_0 by a prime, differentiating (3.16) gives

$$P(v_0)P'(v_0) - v_0 = -\frac{d_j}{m_j}|v_j(\tau(v_0))|^2\tau'(v_0) = -\frac{d_j}{m_j}P^2(v_0)\tau'(v_0). \tag{3.17}$$

On the other hand, integrating (3.13) over $(0, t)$ yields

$$\dot\theta_j(t) = \dot\theta_j(0)e^{-\frac{d_j}{m_j}t} + e^{-\frac{d_j}{m_j}t}\int_0^t \left(\frac{\omega_j}{m_j} + \frac{\lambda R^*}{m_j}\sin(\Theta^* - \theta_j(s))\right)e^{\frac{d_j}{m_j}s}\,ds.$$

Substituting $t = \tau(v_0)$, we obtain

$$P(v_0) = v_0 e^{-\frac{d_j}{m_j}\tau} + e^{-\frac{d_j}{m_j}\tau}\int_0^\tau \left(\frac{\omega_j}{m_j} + \frac{\lambda R^*}{m_j}\sin(\Theta^* - \theta_j(s))\right)e^{\frac{d_j}{m_j}s}\,ds. \tag{3.18}$$

Differentiating (3.18) with respect to v_0, we obtain

$$P'(v_0) = e^{-\frac{d_j}{m_j}\tau} - \frac{d_j}{m_j}\tau'(v_0)P(v_0). \tag{3.19}$$

Combining (3.17) and (3.19), we conclude that

$$P(v_0) = e^{\frac{d_j}{m_j}\tau(v_0)}v_0 > v_0. \tag{3.20}$$

Hence, no positive fixed point can exist, and consequently no periodic orbit $(v_j, \theta_j) \in \mathbb{R} \times \mathbb{T}$ arises.

We exclude the cases a non-contractible homoclinic orbit, and a pair of non-contractible heteroclinic orbits. If such orbits existed, by an argument analogous to the previous one, when lifted to $\mathbb{R} \times \mathbb{R}$, we obtain *homoclinic/heteroclinic ladder of equilibria* that advances monotonically in the θ_j-direction:

$$(0, \theta_0) \longrightarrow (0, \theta_0 + 2\pi) \longrightarrow (0, \theta_0 + 4\pi) \longrightarrow \cdots \longrightarrow (0, \theta_0 + 2k\pi) \longrightarrow \cdots.$$

We know that the orbit satisfies: there exists a time sequence $\{t_k\}$ such that

$$(v_j(t_k), \theta_j(t_k)) = (v_j(t_k), \theta_0 + 2k\pi).$$

Because the trajectory repeatedly approaches the equilibria (homoclinic or heteroclinic ladder of equilibria), there exists a subsequence $\{t_{k_\ell}\}$ for which $v_j(t_{k_\ell}) \to 0$ as $\ell \to \infty$. However, by applying the previous argument $P(v_j(t_{k_l})) > v_j(t_{k_l})$ (cf. (3.20)), we conclude that such a situation cannot occur.

So far, by applying Theorem 3.3, we can only guarantee that for any γ-orbit (which is automatically pre-compact) of (3.12), the trajectory $\gamma(t, s; v_j, \theta_j)$ either converges to an equilibrium point of (3.13), or the ω-γ-limit sets of (s, v_j, θ_j) contain a single heteroclinic orbit connecting the two equilibria of (3.13). If only the first case occurs, the proof is complete. Also, the latter case is "mild," because when we lift the system back to $\mathbb{R} \times \mathbb{R}$, the cyclic chain formed by these contractible homoclinic or heteroclinic orbits remains bounded. If the ω-γ-limit sets of (s, v_j, θ_j) contain finitely many equilibria of (3.13) connected by such bounded homoclinic or heteroclinic orbits, then we know that γ remains bounded (even after lifting to $\mathbb{R} \times \mathbb{R}$). Therefore, we conclude that such trajectories must eventually become (SS). We complete the proof. $\square$

Finally, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By definition, we have (FPLS) $\Rightarrow$ (PLS). Combining Lemma 2.2 and Lemma 3.1, we obtain

$$\text{(FPLS)} \Leftrightarrow \text{(PLS)} \Leftrightarrow \text{(FSS)},$$

which are denoted by (SS). Finally, Lemma 3.2 and Lemma 3.4 together yield

$$\text{(SS)} \Leftrightarrow \text{(OPSS)}.$$

This completes the proof. $\square$

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Declarations

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Data Availability Statement

This study does not involve any data.

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