Martingales, laminates and minimal Korn inequalities

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Abstract

Korn's inequalities show that the L^2-norm of ∇u can be controlled by the L^2-norm of $\mathrm{Sym}(\nabla u)$, which only has $d(d+1)/2$ components. In [J. Math. Pures Appl. 148 (2021), pp. 199-220] Chipot posed the question of *how many scalar measurements are needed to have a Korn-type control on* ∇u when u is in $H_0^1(\Omega)$ and $H^1(\Omega)$, introducing the minimal numbers $N(d,\Omega)$ and $N'(d,\Omega)$ respectively. He proved general bounds and calculated several low-dimensional values of N, N'.

We reframe Chipot's problem in the language of rank-one convexity and quasiconvexity and obtain a purely algebraic characterisation of when such inequalities hold, which yields the sharp bounds

$$\begin{aligned} N(d,\Omega) &= 2d(1-o(1)) \\ N'(d,\Omega) &= 2d-1. \end{aligned}$$

As a consequence, we recover and streamline several of Chipot's results, we obtain a dimension-optimal Korn inequality and several sharp estimates for the best constant for various Korn-type inequalities. Generalisations to the rectangular case and to general L^p estimates are also considered.

The central new ingredient of our approach is a systematic connection between laminates and martingales which produces explicit families of laminates realising these bounds. This method is of independent interest in the calculus of variations: for instance, we use it to obtain a new quick and quantitative proof of Ornstein's non-inequality, valid for all first order homogeneous operators in $\mathbb{R}^{2\times 2}$ and for a large class of operators in general dimensions (including Korn's $\frac{\nabla u+\nabla u^t}{2}$ and $\frac{\nabla u+\nabla u^t}{2} - \mathrm{div}(u)\frac{\mathrm{Id}}{d}$).

1. Introduction

Korn's first inequality, in its arguably most well known form, is the statement that for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, and all $p \in (1,\infty)$ we have

$$\|\nabla u\|_{L^p(\mathbb{R}^d)} \lesssim_{p,d} \|\mathrm{Sym}(\nabla u)\|_{L^p(\mathbb{R}^d)},$$

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where $\mathrm{Sym}(A) := \frac{A+A^t}{2}$ is the symmetric part of A and the norm considered is the Frobenius norm for matrices, i.e. $\|A\|^2 = \mathrm{Tr}(AA^t)$. Korn's second inequality is more general: for $p \in (1, \infty)$ given a (Lipschitz) domain Ω in $\mathbb{R}^d$, for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ we have

$$\|u\|_{W^{1,p}(\Omega)} \lesssim_{p,\Omega} \|u\|_{L^p(\Omega)} + \|\mathrm{Sym}(\nabla u)\|_{L^p(\Omega)}.$$

These two inequalities have many applications (see [28] and references therein), but these are not the inequalities in their strongest form.

For example, if we take $\mathrm{Sym}_0(A) := \mathrm{Sym}\left(A - \mathrm{Tr}(A)\frac{\mathrm{Id}}{d}\right)$ (i.e., the projection of A onto its trace-free symmetric part), we still have[2]

$$\|\nabla u\|_{L^p(\Omega)} \lesssim_{p,\Omega} \|\mathrm{Sym}_0(\nabla u)\|_{L^p(\Omega)}.$$

Chipot analysed this problem more precisely in [22]: namely, he asked what is the smallest number of coordinates that one needs on the right-hand side for an inequality such as Korn's to hold (at least for $p = 2$)? More formally, Chipot studied the quantity $N(d)$, defined as

***Definition* 1.** Let $N(d)$ be the smallest integer k for which there exist $\ell_1, \dots, \ell_k$ linear functionals on the space of $d \times d$ matrices $M_d(\mathbb{R})$ such that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_{i=1}^{k} \|\ell_i(\nabla u)\|^2_{L^2(\mathbb{R}^d)} \tag{1}$$

holds for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ [3].

Similarly, he defined $N'(d)$ as

***Definition* 2.** Let Ω be an open subset of $\mathbb{R}^d$. Let $N'(d, \Omega)$ be the smallest integer k for which there exist $\ell_1, \dots, \ell_k$ linear functionals on $M_d(\mathbb{R})$ such that

$$\|u\|^2_{H^1(\Omega)} \lesssim \|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{k} \|\ell_i(\nabla u)\|^2_{L^2(\Omega)} \tag{2}$$

holds for all $u \in C^\infty(\Omega, \mathbb{R}^d)$ [4].

As we will see, $N'(d, \Omega)$ does not depend on the domain provided the domain is regular enough (say, Lipschitz), so we will continue using $N'(d)$ to denote $N'(d, \Omega)$ for any bounded Lipschitz domain. Henceforth, we will also assume all domains to be bounded and Lipschitz. Chipot proved many important properties of $N(d)$ and $N'(d)$, which we briefly remind the reader:

[2] Note that the second form of this inequality does not hold for $d = 2$

[3] In [22], N is defined for $u \in C_c^\infty(\Omega, \mathbb{R}^d)$, so a priori N depends on the domain. However, one easily sees that N is independent of the domain, so we skip that here.

[4] More care is required in handling Ω in this case. Indeed, the regularity of Ω seems to play a more important role, as it is well known that there exist open sets Ω on which Korn's second inequality fails (see [46]).

Theorem 1.1 ([22]). *For any $d, k \in \mathbb{N}$ we have*

1. *(Proposition 2.1)* $N(d) \leq \frac{d(d-1)}{2} + 1$
2. *(Theorem 2.3)* $N(d) \geq d$ *and the inequality is strict if* d *is odd.*
3. *(Theorem 2.4, Theorem 4.1)* $N(2) = 2, N(3) = 4, N(4) = 4, N(8) \leq 16$
4. *(Theorem 3.3)* $N(kd) \leq k^2 N(d)$
5. *(Theorem 5.1) For any bounded Lipschitz domain* $\Omega \subset \mathbb{R}^2$, $N'(2, \Omega) = 3$. *This in particular implies that* $N'(2)$ *is well defined.*
6. *(Theorem 5.2) For any bounded Lipschitz domain* Ω, $N'(d, \Omega) \geq N(d)$.

In particular, the natural question of the growth of $N(d)$ remained open: does $N(d)$ grow quadratically, or linearly, or at a different rate? It is not hard to see that one can without loss of generality assume the functionals to be orthogonal to each other, so the question is essentially how many coordinates of ∇u are needed to define the norm on $W^{1,p}$. In particular, if $N(d) \approx d$ holds, this implies that, as d grows, only a vanishingly small number of coordinates of ∇u are really needed to determine a norm, i.e. the standard gradient norm is quite redundant. In this regard, $N'(2) \neq N(2)$ is rather surprising, as there seems to be no intrinsic difference between the two spaces.

Our results hinge on the following observation (various forms of this claim have been known in the community of calculus of variations for a long time. See remark 3 for background and history of this result)

Theorem 1.2. *Given a field* $\mathbb{K}$, *let* $n_{\mathbb{K}}(d, k)$ *denote the maximal dimension a subspace* Y *of* $M_d(\mathbb{K})$ *can have without intersecting the set* $\mathcal{R}_k$ *of non-zero matrices of rank at most* k. *Then*

$$N(d) = d^2 - n_{\mathbb{R}}(d, 1) \tag{3}$$

$$N'(d) = d^2 - n_{\mathbb{C}}(d, 1)$$

This connection with algebra allows us to attack the problem from a very different angle and this will lead us to prove several interesting properties of $N(d), N'(d)$, by taking advantage of what is known about $n_{\mathbb{R}}(d, 1)$. We mention here the most important ones. The first, and perhaps most surprising, results we obtain are

Corollary 1.2.1. *For each* $d \in \mathbb{N}$ *larger than one we have*

$$N(d) \leq 2d - 2.$$

This bound is attained if $d = 2^n + 1$ *for some* $n \in \mathbb{N}$. *Moreover, asymptotically we have* $N(d) \sim 2d$, *so the bound is asymptotically sharp.*

Corollary 1.2.2. *For each* $d \in \mathbb{N}$ *we have*

$$N'(d) = 2d - 1.$$

In particular, the inequality $N'(d) \geq N(d)$ *is always strict when* $d > 1$.

We also manage to improve on the best previously known lower bound:

Corollary 1.2.3. *Equality $N(d) = d$ is attained if and only if $d \in \{1, 2, 4, 8\}$. Moreover, we have*

$$N(d) \geq 2^{\lceil \log_2(d) \rceil}.$$

These corollaries by themselves are enough to calculate $N(d)$ for d up to 9; we will later see, using more convoluted methods, how to calculate $N(d)$ for d up to 23 (and we will provide lower and upper bounds for $N(d)$ up to 33). For the sake of the exposition, and as most of the proofs of the required properties of $n_{\mathbb{K}}$ and related functions are rather elementary to prove, we will include their proofs where possible, trying to make them more accessible to analysts (which amounts to keeping the algebraic topology tools to a minimum). Doing so will also allow us to obtain streamlined and shorter proofs of Chipot's original results.

In order to estimate the constants present in these inequalities, we develop a systematic connection between martingales and laminates which allows us to bound the constant from below and (for rank-one convexity) from above by explicitly building laminates that witness these bounds.

As a by-product of our study, we will obtain an explicit and much strengthened form of Korn's first and second inequalities which, to our knowledge, has not been explicitly noted before. Namely, we will prove

$$\|u\|_{H^1(\Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|P_{\mathcal{H}}(\nabla u)\|_{L^2(\Omega)},$$

where $\mathcal{H}$ is the space of Hankel matrices, i.e. matrices that are constant on each skew-diagonal, and $P_{\mathcal{H}}$ denotes the orthogonal projection from $M_d(\mathbb{R})$ onto $\mathcal{H}$. Note that $\mathcal{H} \subset \mathrm{Sym}$ and $\dim(\mathcal{H}) = 2d - 1$, so this is indeed a strengthening of Korn inequalities (and thanks to $N'(d) = 2d - 1$, this is dimensionally optimal). The results mentioned so far are all consequences of the previously mentioned algebraic connection (and will be explored in sections 3, 4 and 5, respectively concerning $N(d), N'(d)$ and the rectangular case); in order to estimate the constants involved, a more refined set of tools is needed. With this goal, we discuss in section 2 a connection between laminates and martingales which we then exploit to calculate the constants. This allows us to prove a bound for the first Korn-Hankel inequality which is sharp up to $\mathcal{O}(\log(d))$, and several new bounds for $p \neq 2$. This does not exhaust the potential applications of the tools we introduce and as an example of this we provide, in section 6, a new quantitative proof of Ornstein's non-inequality result.

1.1. Notation

In this paper, we use the standard Vinogradov notation where, given two positive quantities A, B, $A \lesssim B$ means that there exists a positive constant C such that $A \leq BC$. We write $A \lesssim_x B$ if the implicit constant C is allowed to depend on x. Given two quantities A and B, $A \approx B$ means $A \lesssim B$ and $A \gtrsim B$. We will use $\mathbb{K}$ to denote an unspecified field. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we will always consider $M_{m,d}(\mathbb{K})$ (the ring of $m \times d$ matrices on $\mathbb{K}$) to be equipped with the Frobenius inner product, that is $\langle A, B \rangle =$

$\mathrm{Tr}(AB^*)$ and associated Frobenius norm $\|A\|^2 = \mathrm{Tr}(AA^*)$; when $m = d$, we will write $M_d(\mathbb{K})$. Given a closed subspace X of a Hilbert space H, we will write P_X for the orthogonal projection on X and Q_X for the orthogonal projection on $X^\perp$. We use standard notation for functions and function spaces such as can be found, for instance, in [24].

2. Preliminary results in calculus of variations

We briefly recall some concepts from calculus of variations: a function $f\colon M_{m,d}(\mathbb{R}) \to \mathbb{R}$ is said to be rank-one convex if it is convex on each rank-one line, i.e. on each segment connecting two matrices A, B such that $\mathrm{rank}(A - B) = 1$. A rank-one convex function is locally Lipschitz ([61, lemma 5.6]). Similarly, a continuous function f is said to be quasiconvex if, for all $u \in W_0^{1,\infty}(B_{\mathbb{R}^d}(0,1), \mathbb{R}^m)$ we have

$$\int_{B_{\mathbb{R}^d}(0,1)} f(A + \nabla u) \geq |B_{\mathbb{R}^d}(0,1)| f(A),$$

where $|B_{\mathbb{R}^d}(0,1)|$ denotes the d-dimensional Lebesgue measure of the unit ball $B_{\mathbb{R}^d}(0,1)$. It is well known (see [61, Proposition 5.3] and [56, Lemma 4.3]) that quasiconvex functions are rank-one convex, and the opposite implication (known as Morrey's problem) is known to be false for $m \geq 3, d \geq 2$, as proved by Šverák ([65], see also [37]). In dimension $m = 2, d \geq 2$, it is not known whether rank-one convexity and quasiconvexity are equivalent in general, though several positive results are known for particular functions and particular domains (see [40], [34], [55]). We define the rank-one convex envelope f^{rc} and the quasiconvex envelope f^{qc} of a function $f\colon M_{m,d} \to \mathbb{R}$ respectively as

$$f^{rc}(A) = \sup\{g(A)\colon g \leq f \text{ and } g \text{ rank} - \text{one convex}\}$$

and

$$f^{qc}(A) := \sup\{g(A)\colon g \leq f \text{ and } g \text{ quasiconvex}\}.$$

The envelopes admit an alternate description by duality. For f^{rc} we have (see [25, Theorem 6.10] for the formula for the envelope, [59] and [61, Section 9.1] for definitions and generalities on laminates):

$$f^{rc}(A) = \inf\left\{\langle f, \mu\rangle \colon \mu \in \mathcal{M}_A\right\}, \tag{4}$$

where $\mathcal{M}_A$ represents the set of laminate measures (of finite order) having A as their barycentre. Similarly, for the quasiconvex envelope we have Dacorogna's formula ([25, Theorem 6.9], [49, Appendix])

$$f^{qc}(A) = \inf_{u \in W_0^{1,\infty}(B_{\mathbb{R}^d}(0,1), \mathbb{R}^m)} \frac{1}{|B_{\mathbb{R}^d}(0,1)|} \int_{B_{\mathbb{R}^d}(0,1)} f(A + \nabla u(x)) \mathrm{d}x \tag{5}$$

It is not hard to prove that, for a continuous function f, either $f^{qc} \equiv -\infty$ or $f^{qc} > -\infty$ and f^{qc} is quasiconvex, and similarly for rank-one convexity. It follows that for homogeneous functions f, the existence of a lower quasiconvex bound is equivalent to $f^{\mathrm{qc}}(0) = 0$, a necessary condition for which is that $f^{\mathrm{rc}}(0) = 0$.

In this section, we focus on determining when the rank-one convex envelope of certain functions is real valued (in other words, whether f has a lower rank-one convex envelope). The functions of interest have the form

$$f_{X,p,C}(A) = C^p \|P_X(A)\|^p - \|Q_X(A)\|^p, \tag{6}$$

where X is a subspace of $M_d(\mathbb{R})$, $\|\cdot\|$ is the Frobenius norm and P_X, Q_X are the associated orthogonal projections onto X and $X^\perp$ respectively. In the following we will denote a function of the class above as f or f_p when X and C are clear from the context.

The reasons for studying this class of functions are manifold. In this paper, we are interested in their relations with differential inequalities. In this regard, we note that f^{qc} is real valued iff $f^{qc}(0) = 0$, and that this is equivalent, thanks to (5), to the inequality

$$\|Q_X(\nabla u)\|_{L^p(\mathbb{R}^d)} \leq C \|P_X(\nabla u)\|_{L^p(\mathbb{R}^d)} \tag{7}$$

holding for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Since rank-one convexity is a necessary condition for quasi-convexity, such a study is naturally interesting.

For $p = 2$, the function above becomes a quadratic form, namely

$$f(A) = C\langle P_X(A), A\rangle - \langle Q_X(A), A\rangle.$$

For quadratic functions, it is known (and easy to prove, see [25, Theorem 5.25(i)]) that quasiconvexity is equivalent to rank-one convexity which in turn is equivalent to $f(a \otimes b) \geq 0$ for all $a, b \in \mathbb{R}^d$. Hence, $f^{qc}(0) = 0$ is equivalent to

$$C\langle P_X(A), A\rangle - \langle Q_X(A), A\rangle = f(A) \geq 0$$

for all A such that $\mathrm{rank}(A) = 1$. In other words,

$$\inf_{A\,:\ \mathrm{rank}(A)=1} \frac{\|P_X(A)\|^2}{\|Q_X(A)\|^2} \geq \frac{1}{C}.$$

In this section, we want to extend this result to $p \in (1, \infty)$ and provide a sharpness result there as well. In particular, we will prove the following:

Theorem 2.1. *Let f_C be a function of the form (6). Then there exists a C such that f_C^{rc} is real-valued if and only if $X^\perp$ does not contain rank-one matrices. In that case, we can choose $C = (p^* - 1)B$, where*

$$B^{-1} = \inf_{A\colon \mathrm{rank}(A)=1} \frac{\|P_X(A)\|}{\|Q_X(A)\|}$$

and $p^* = \max\left(p, \frac{p}{p-1}\right)$.

In some particular cases, we can also obtain a sharp estimate on C:

Theorem 2.2. *Let X be a subspace of $M_d(\mathbb{R})$ such that*

1. *For all matrices $R \in M_d(\mathbb{R})$ having rank-one,*
$$\|P_X(R)\| \geq c\|Q_X(R)\|.$$
2. *There exist $A \in X, B \in X^\perp$ such that $\|A\| = \|B\| = 1$, $\operatorname{rank}(A \pm cB) = 1$.*

Then the function
$$f(A) = C^p\|P_X(A)\|^p - \|Q_X(A)\|^p$$
has real-valued rank-one convex envelope if and only if $C \geq c(p^* - 1)$.

To prove this result, we will need to introduce a connection between laminates and martingales. Before we do so formally, let us describe the connection heuristically: a finite order laminate (with barycentre in 0) is constructed by applying a series of splits to the original Dirac measures δ_0, that is to say operations where one passes from the measure δ_A to $t\delta_B + (1-t)\delta_C$ (for some appropriate t, B, C) which preserve the barycentre of the measure. This can be seen as a martingale on the dyadic tree, with each split represented by a branching in the tree. Let us set some notation concerning dyadic trees:

***Definition* 3** (Dyadic trees)**.** Let us denote the dyadic tree by $\mathcal{T}$, i.e. the set of finite sequences $(\varepsilon_0, \dots, \varepsilon_n)$ with elements $\varepsilon_i \in \{0, 1\}$. Given such a sequence σ, $|\sigma|$ will denote its length $|\sigma| = n + 1$. We will write $T_m := \{\sigma \in \mathcal{T} \colon |\sigma| = m\}, \mathcal{T}_m := \{\sigma \in \mathcal{T} \colon |\sigma| \leq m\}$. Given a sequence $\sigma = (\epsilon_0, \epsilon_1, \dots, \epsilon_m)$ and a number $\varepsilon \in \{0, 1\}$, we will write $\sigma \frown \varepsilon$ for the sequence obtained by adding ε in final position, i.e.
$$\sigma \frown \varepsilon = (\epsilon_0, \dots, \epsilon_m, \varepsilon).$$

***Definition* 4** (Associated martingale)**.** Let μ be a laminate of order n.

Then there exists a function $M \colon \mathcal{T}_n \to M_d(\mathbb{R})$ such that for all $x \in \mathcal{T}_{n-1}$
$$t(x)M(x \frown 0) + (1 - t(x))M(x \frown 1) = M(x), \tag{8}$$
and
$$\operatorname{rank}\big(M(x \frown 0) - M(x \frown 1)\big) \leq 1 \tag{9}$$
Let $(X_i)_{i \in \mathbb{N} \cup \{0\}}$ denote the random walk on $\mathcal{T}$ such that for every $x \in \mathcal{T}_n$, whose length we denote as m, we have
$$\mathbb{P}(X_{m+1} = x \frown 0 | X_m = x) = t(x)$$
and
$$\mathbb{P}(X_{m+1} = x \frown 1 | X_m = x) = 1 - t(x).$$
We define the associated dyadic martingale as $M(X_{i \wedge n})$.

Condition (9) can be reformulated in a more natural manner: if we denote by $\mathrm{d}M_i$ the increments of the martingale (i.e. $\mathrm{d}M_i = M_{i+1} - M_i$), then assuming $X_{i+1} = x \frown 0$ we have

$$\begin{aligned}\mathrm{d}M_i &= M(x \frown 0) - M(x)\\ &= M(x \frown 0) - t(x)M(x \frown 0) - (1-t(x))M(x \frown 1)\\ &= (1-t(x))(M(x \frown 0) - M(x \frown 1))\end{aligned}$$

and similarly if we instead assume $X_{i+1} = x \frown 1$

$$\mathrm{d}M_i = -t(x)(M(x \frown 0) - M(x \frown 1)),$$

so condition (9) is satisfied if and only if $\mathrm{rank}(\mathrm{d}M_i) \leq 1$ almost everywhere.

Remark 1. Many other results connecting certain classes of martingales with convexity properties are known. For example, Aumann and Hart in [7] found a similar result for bi-convexity and bi-martingales.

We will also need the following result concerning martingales:

Theorem 2.3 (Burkholder [18], see [58] for an exposition)**.** *Let $\mathcal{H}$ be a Hilbert space and let X_i, Y_i be two $\mathcal{H}$-valued martingales (with respect to the same filtration). If (X_i) is differentially subordinated to (Y_i), i.e. $\|\mathrm{d}X_i\| \leq \|\mathrm{d}Y_i\|$ for all $i \in \mathbb{N}$ and $\|X_0\| \leq \|Y_0\|$, then for all $i \in \mathbb{N}$*

$$\|X_i\|_{L^p(\mathbb{P})} \leq (p^*-1)\|Y_i\|_{L^p(\mathbb{P})}, \tag{10}$$

where $p^* = \max\left(p, \frac{p}{p-1}\right)$. Moreover, the inequality is sharp [5] *among martingales defined on dyadic trees such that $\|\mathrm{d}X_i\| = \|\mathrm{d}Y_i\|$.*

Proof of Theorem 2.1. It suffices to prove that $f^{rc}_{X,p,C_p}(0) = 0$. To do so, let us recall that

$$f^{rc}(x) = \inf_{\mu\colon [\mu]=x, \mu \text{ is a laminate}} \langle f, \mu\rangle.$$

So it suffices to prove that $\langle f_{X,p,C_p}, \mu\rangle \geq 0$ for laminates with barycentre 0 if $C \gtrsim (p^*-1)$. Let μ denote such a laminate, and let n be its order. By hypothesis, we have

$$\sup_{\mathrm{rank}(A)=1} \frac{\|Q_X(A)\|}{\|P_X(A)\|} = k < \infty.$$

We claim that $C \geq k(p^*-1)$ suffices. For simplicity, we assume $k=1$, the proof in the general case is the same mutatis mutandis. We now associate to μ an $M_d(\mathbb{R})$-valued martingale as in the definition above, which we call M_i. It is easy to see that $A_i := P_X(M_i), B_i := Q_X(M_i)$ are martingales and that $A_0 = B_0 = M_0 = 0$. Moreover, since $\mathrm{rank}(M_{i+1} - M_i) \leq 1$ and $\|P_X(R)\| \geq \|Q_X(R)\|$ for all rank-one matrices, it follows

[5]That is to say, no smaller constant works

that B_i is differentially subordinated to A_i. Using Burkholder's inequality (10), it follows that if $c_p \geq (p^* - 1)$, we have

$$c_p \|A_i\|_{L^p(\mathbb{P})} \geq \|B_i\|_{L^p(\mathbb{P})}$$

for all $i \in \{0, \ldots, n\}$ and taking $i = n$ we get that

$$\langle f, \nu \rangle = \int f \mathrm{d}L_{M_n} = c_p^p \|A_n\|_{L^p(\mathrm{d}\mathbb{P})}^p - \|B_n\|_{L^p(\mathrm{d}\mathbb{P})}^p \geq 0,$$

where L_{M_n} is the law of the random variable. On the other hand, assume that Y contains a rank-one matrix R, which we assume without loss of generality to have norm 1. Then

$$f_{X,p,C}^{rc}(0) \leq \inf_{t>0} \frac{f_{X,p,C}(-tR) + f_{X,p,C}(tR)}{2} = \inf_{t>0} -t^p = -\infty.$$

□

Proof of Theorem 2.2. To prove the other direction, i.e. $C < (p^* - 1)$ implies $f_{p,c_p}^{rc}(0) = -\infty$, we focus on a particular class of martingales: namely, let f_n be any dyadic martingale and g_n a ± 1-transform of f_n, i.e. a martingale such that $dg_n = \varepsilon_n df_n$ with ε_n a predictable sequence of signs. Then, we define

$$M_n = Af_n + Bg_n.$$

This matrix-valued martingale satisfies $\operatorname{rank}(dM_n) \leq 1$ a.e., so it is associated to a laminate with barycentre at 0. From the sharpness statement in Theorem 2.3 the result follows. □

Remark 2. The problem of determining conditions for the rank-one convexity of a function (in particular as a necessary condition for quasiconvexity) is a very active area of research with a long history; see [8], [9], [45], [4] and references therein for a more detailed history of this approach in its relation with the Iwaniec-Martin conjecture. The technique of using laminates has also been in use, in various forms, for quite some time, see [49]. We particularly highlight Faraco's results with the use of the so-called staircase laminates (see [32]) and [12], where a connection is noted between laminates and the Burkholder function in the context of studying the rank-one convexity of some functions related to the Iwaniec-Martin conjecture, which is in a similar spirit to ours. Theorem 2.2 was originally developed in a rougher form in [20], where we applied it to a particular function. We believe that the results we have here do not exhaust the applicability of the technique, which will be broadened in future work.

3. Korn's first inequality and the quantity $N(d)$

The goal of this section is to prove the first part of Theorem 1.2, which we restate here for the reader's convenience:

Theorem 1.2 (Part 1).

$$d^2 - n_{\mathbb{R}}(d,1) = N(d). \tag{11}$$

From this, we will deduce a sharp lower bound, an asymptotically sharp upper bound, we will calculate all $N(d)$ up to 19 and provide ranges for $N(d)$ up to $d = 33$ (though these results could quite likely be extended further, we refrain from doing that here). Before proving the theorem, we note that the definition can be restated in a way that is slightly more amenable to our process:

Lemma 3.1. *Let $M(d)$ be defined as*

$$M(d) := \inf \left\{ \dim(X) \colon \forall u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d) \ \|\nabla u\|_{L^2(\mathbb{R}^d)} \lesssim \|P_X(\nabla u)\|_{L^2(\mathbb{R}^d)} \right\}.$$

Then $M(d) = N(d)$.

Proof. First, let X be a subspace of $M_d(\mathbb{R})$ such that $\|\nabla u\|_{L^2(\mathbb{R}^d)} \lesssim \|P_X(\nabla u)\|_{L^2(\mathbb{R}^d)}$ for all smooth compactly supported functions. Let $e_1, \ldots, e_{\dim(X)}$ be an orthogonal basis of X. Then let $l_i(\cdot) := \langle e_i, \cdot \rangle$. By construction we have

$$\|\nabla u\|^2_{L^2(\mathbb{R}^d)} \lesssim \|P_X(\nabla u)\|^2_{L^2(\mathbb{R}^d)} = \sum \|l_i(\nabla u)\|^2_{L^2(\mathbb{R}^d)}.$$

It follows that $N(d) \leq M(d)$. To prove the other direction, let $l_1, \ldots, l_m$ be a set of linear functionals witnessing $N(d) = m$. Select for each of them a vector $x_i \in M_d(\mathbb{R})$ such that $l_i(\cdot) = \langle x_i, \cdot \rangle$. Define $X := \operatorname{span}(x_1, \ldots, x_m)$. By minimality of the set $\{l_i\}$, $\{x_1, \ldots, x_m\}$ is a basis of X. By finite dimensionality of X it then follows that

$$\|P_X(y)\|^2 \approx_{\dim(X)} \sum |\langle x_i, y \rangle|^2,$$

hence

$$\|\nabla u\|^2_{L^2(\mathbb{R}^d)} \leq \sum \|l_i(\nabla u)\|^2_{L^2(\mathbb{R}^d)} = \sum \|\langle x_i, \nabla u \rangle\|^2_{L^2(\mathbb{R}^d)} \approx_{\dim(X)} \|P_X(\nabla u)\|^2_{L^2(\mathbb{R}^d)},$$

proving the other direction $N(d) \geq M(d)$. □

Proof of Part 1 of Theorem 1.2. The definition can be equivalently stated as: let $\mathcal{X}$ be the class of subspaces X of $M_d(\mathbb{R})$ for which there exists a constant $C = C_X$ such that

$$C\|P_X(\nabla u)\|^2_{L^2(\mathbb{R}^d)} \geq \|Q_X(\nabla u)\|^2_{L^2(\mathbb{R}^d)}$$

holds for all admissible u. In these terms we have $N(d) = \min\{\dim(X) \colon X \in \mathcal{X}\}$. Let us define $f_X \colon M_d(\mathbb{R}) \to \mathbb{R}$ as

$$f_{C,X}(A) := C\|P_X(A)\|^2 - \|Q_X(A)\|^2.$$

Hence X is admissible if and only if there exists a C such that

$$f^{qc}_{C,X}(0) = 0.$$

Since $f_{C,X}$ is a quadratic form, this is equivalent to

$$f^{rc}_{C,X}(0) = 0.$$

By applying Theorem 2.1, one obtains that such C exists if and only if $X^\perp$ avoids rank-one matrices, proving the result. □

Remark 3. We are of course not the first to observe a connection between rank-one properties and inequalities such as Korn's. Indeed, results of this kind have been known for a long time, see for instance [21], [56, Lemma 2.7], [23], [33], [50]. Similar results also hold in other spaces: Boman [10] investigated a problem with a method that is reminiscent of our approach (or, to be more precise, vice versa): in particular, his condition $\hat{A}$ is equivalent to $\operatorname{span}(\hat{A})$ being, in our vocabulary, admissible. In this light, our result can be seen as a $W^{1,p}$ version of his Theorem 1.

It seems however that this algebraic characterisation has never been taken full advantage of as we do here.

Remark 4. An alternative proof can easily be obtained by using ellipticity of the operator $T_X(u) := P_X(\nabla u)$ (and indeed it is not hard to see that Theorem 3.2 in [22] is essentially the ellipticity condition for T_X). Our proof, however, has the advantage of allowing one to control the constants associated with the inequalities, whereas using ellipticity only ensures the existence of such constants.

The number $n(d, 1)$ can also be interpreted in a different way, which will prove itself to be more amenable to our study.

***Definition* 5.** Let $d, k \in \mathbb{N}$, and let $\mathcal{B}(\mathbb{K}^d \times \mathbb{K}^d, \mathbb{K}^k)$ denote the set of bilinear maps $f\colon \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^k$. We say that such a map is nonsingular if $f(a, b) = 0$ implies that $a = 0$ or $b = 0$. If $\mathcal{B}_{ns}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^k)$ denotes the set of nonsingular bilinear mappings, we define $g(d)$ as

$$g_{\mathbb{K}}(d) := \min \left\{ k\colon \mathcal{B}_{ns}(\mathbb{K}^d \times \mathbb{K}^d, \mathbb{K}^k) \neq \emptyset \right\}.$$

Proposition 3.1. *For each $d \in \mathbb{N}$ we have*

$$d^2 - n_{\mathbb{K}}(d, 1) = g_{\mathbb{K}}(d).$$

Proof (folklore). Note that $f\colon \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}^k$ with k being minimal implies f is surjective. Moreover, note that any such map f bilinear and nonsingular naturally induces $L\colon \mathbb{K}^d \otimes \mathbb{K}^d \to \mathbb{K}^k$, so taking $L^{-1}(0)$ provides a subspace avoiding rank-one matrices, and

$$\dim(M_d(\mathbb{K})) - \dim(L^{-1}(0)) = \dim(\mathbb{K}^k) = g_{\mathbb{K}}(d),$$

so

$$d^2 - n_{\mathbb{K}}(d, 1) \leq g_{\mathbb{K}}(d).$$

On the other hand given such a dimension-minimal subspace X, one can define L as the natural quotient map

$$L\colon \mathbb{K}^d \otimes \mathbb{K}^d \simeq M_d(\mathbb{K}) \twoheadrightarrow M_d(\mathbb{K})/X \simeq \mathbb{K}^{d^2 - n_{\mathbb{K}}(d,1)}$$

and this implies

$$g_{\mathbb{K}}(d) \le d^2 - n_{\mathbb{K}}(d,1).$$

□

As we will mostly focus on $g_{\mathbb{R}}(d)$, we will from now on write $g(d)$ for $g_{\mathbb{R}}(d)$.

3.1. First consequences

The function $g(d)$ has been studied in depth since the 1960s. In particular, as we will see, there exist asymptotically sharp lower and upper bounds for it and we can now transfer those to $N(d)$. Since some of the results for $g(d)$ are rather complex and usually derived with techniques from algebraic topology, we first focus on using analytical techniques. To start, let us show how one can use this result to obtain short proofs of some of the results obtained in [22].

Corollary 3.1.1. *$N(d)$ is non-decreasing.*

Proof. The corollary follows from composing the nonsingular bilinear map $f\colon \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}^{N(d+1)}$ with the standard inclusion $\iota\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ defined as $\iota(x,y) = ((x,0),(y,0))$. □

Corollary 3.1.2. *For each $d \in \mathbb{N}$ we have $N(d) \ge d$*

Proof. It suffices to notice that given a nonsingular bilinear map $f\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^k$, $f(x,\cdot)$ is injective, so $\operatorname{rank}(f(x,\cdot)) \ge d$. □

Similarly, we can obtain a somewhat more enlightening proof that $N(4) = 4$:

Corollary 3.1.3.

$$N(4) = 4,$$
$$N(8) = 8.$$

Proof. Thanks to the lower bound on $N(d)$, it suffices to prove $N(4) \le 4$, $N(8) \le 8$. $N(4) \le 4$ follows by considering the map $f(x,y) = x \times_H y$, where $\times_{\mathbb{H}}$ denotes the quaternion multiplication. $N(8) = 8$ follows by considering the map $h(x,y) = x \times_{\mathbb{O}} y$, i.e. the octonion multiplication. □

The lower bound can actually be strengthened:

Theorem 3.2. *We have*

$$N(d) > d \tag{12}$$

unless $d \in \{1,2,4,8\}$, in which case equality holds. We also have the general lower bound

$$N(d) \ge 2^{\lceil \log_2(d) \rceil}$$

for all d.

Proof. Assume $N(d) = d$ and let f be a nonsingular bilinear map that witnesses it. Then $(\mathbb{R}^d, +, f)$ is a real finite dimensional division algebra. The only such algebras are ([14, corollary 1]) $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and the first result follows. The second result is proved as follows: such a nonsingular bilinear map f induces a map between real projective spaces

$$\tilde{f}\colon \mathbb{RP}^{d-1} \times \mathbb{RP}^{d-1} \to \mathbb{RP}^{k-1},$$

so we have the following morphism between cohomologies:

$$\tilde{f}^*\colon H(\mathbb{RP}^{k-1}, \mathbb{Z}_2) \to H(\mathbb{RP}^{d-1} \times \mathbb{RP}^{d-1}, \mathbb{Z}_2).$$

We recall that thanks to Künneth's theorem (see [42, Theorem 3.15]), we have

$$H(\mathbb{RP}^{d-1} \times \mathbb{RP}^{d-1}, \mathbb{Z}_2) \simeq H(\mathbb{RP}^{d-1}, \mathbb{Z}_2) \otimes H(\mathbb{RP}^{d-1}, \mathbb{Z}_2)$$

Using the fact that $H(\mathbb{RP}^d, \mathbb{Z}_2) \simeq \mathbb{Z}_2[X]/(X^{d+1})$ ([42, Theorem 3.19]), we can equivalently write

$$\tilde{f}^*\colon \mathbb{Z}_2[Z]/(Z^k) \to \mathbb{Z}_2[X]/(X^d) \otimes \mathbb{Z}_2[Y]/(Y^d) \simeq \mathbb{Z}_2[X, Y]/(X^d, Y^d),$$

where (X^d, Y^d) is the ideal generated by X^d, Y^d. By degree preservation it follows that $\tilde{f}^*(Z) = aX + bY$ and by bilinearity $a = b = 1$. We hence obtain that

$$(X + Y)^k \in (X^d, Y^d),$$

where (X^d, Y^d) is the ideal generated by X^d, Y^d. By the binomial theorem, it follows that this happens if and only if

$$k \geq \min \left\{ t\colon \binom{t}{j} \equiv_2 0 \text{ for all } t - d < j < d \right\}$$

The minimal such k (which coincides with the right hand side) is denoted $d \circ d$ and as one easily checks the binomial characterisation implies $d \circ d \geq 2^{\lceil \log_2(d) \rceil}$. □

Remark 5. A more detailed proof of $g(d) \geq d \circ d$ is presented in [42, Theorem 3.21] [6]. For more details on this estimate, its extension to other fields etc, we refer the reader to [63, Chapter 12], and to [31] for more information on the Hopf-Stiefel function $m \circ d$.

Remark 6. The lower bound $g(d) \geq d \circ d$ is by no means sharp. Indeed, if we define $\tilde{g}(d)$ as the minimal dimension such that a continuous nonsingular map $f\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{\tilde{g}(d)}$ satisfying $f(x, -y) = -f(x, y)$ and $f(-x, y) = -f(x, y)$ exists[7] (without requiring linearity), the same bound holds by the same proof.

This result allows us to fully calculate $N(d)$ for $d \leq 8$, as shown in table 3.1.

We turn to the upper bound on $N(d)$:

[6] The theorem is only stated for $d = 2^n$ but the method is easily generalised

[7] Such a map is called bi-skew

Table 1: First 8 values of $N(d)$

d	$N(d)$
1	1
2	2
3	4
4	4
5	8
6	8
7	8
8	8

Theorem 3.3. *For $d, k \in \mathbb{N}$ we have*

$$N(d) \leq 2d - 1$$

and indeed more generally

$$N(kd) \leq (2k-1)N(d).$$

Proof. It suffices to prove that $g_{\mathbb{K}}(d) \leq 2d-1$. To do so, we identify $\mathbb{K}^d \simeq \mathbb{K}_{d-1}[X]$, the set of polynomials of degree $\leq d-1$ over $\mathbb{K}$, and then define f as polynomial multiplication $\mathbb{K}_{d-1}[X] \times \mathbb{K}_{d-1}[X] \to \mathbb{K}_{2d-2}[X] \simeq \mathbb{K}^{2d-1}$. To prove the upper bound for $N(kd)$, namely $N(kd) \leq (2k-1)N(d)$, it suffices to extend the previous polynomial multiplication trick to $(\mathbb{R}^d, f)$, where $f\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{g(d)}$ is a nonsingular bilinear map. More precisely, let us first identify $\mathbb{R}^{kd} \simeq \ell_2^k(\mathbb{R}^d)$, the set of sequences $(x_1, \ldots, x_k)$ with each $x_i \in \mathbb{R}^d$) and define the map $h\colon \ell_2^k(\mathbb{R}^d) \times \ell_2^k(\mathbb{R}^d) \to \ell_2^{2k-1}(\mathbb{R}^{g(d)})$ as

$$h((x_1, \ldots, x_k), (y_1, \ldots, y_k))_l = \sum_{i+j=l} f(x_i, y_j).$$

To obtain the explicit inequality we claimed, it suffices to notice that in our proof of equality of $N(d)$ and $g_{\mathbb{R}}(d)$, we can take $X = \ker(\tilde{f})$, where $\tilde{f}$ is the map induced by f on $M_d(\mathbb{R}) \simeq \mathbb{R}^d \otimes \mathbb{R}^d$. Making this explicit in our case proves the inequality. □

As we will see later, the estimate $g_{\mathbb{R}}(d) \leq 2d-1$ is not optimal. On the other hand, the bound $g_{\mathbb{K}}(d) \leq 2d-1$ cannot be improved in general:

Proposition 3.2. *Let $\mathbb{K}$ be an algebraically closed field. Then*

$$g_{\mathbb{K}}(d) = 2d - 1$$

Proof (Folklore). To see this, it suffices to argue that given a subspace $V \subset M_d(\mathbb{K})$ that avoids $\mathcal{R}_1$, then $\mathbb{P}(V) \subset \mathbb{P}(M_d(\mathbb{K}))$ avoids $\mathbb{P}(\mathcal{R}_1)$. Since $\dim(\mathbb{P}(M_d(\mathbb{K}))) = d^2 - 1$ and $\mathbb{P}(\mathcal{R}_1)$ has dimension $2d-2$, the result follows from the fact that the intersection of two projective varieties X, Y in $\mathbb{P}^d$ cannot be empty if $\dim(X) + \dim(Y) \geq d$ (see [41, Theorem 7.2]). □

The above upper bound $N(d) \leq 2d-1$ can be turned into an explicit inequality:

Theorem 3.4. *Let $\mathcal{H}(d)$ be the set of Hankel matrices in $M_d(\mathbb{R})$, i.e. the set of matrices that are constant on skew-diagonals. Then for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$*

$$\|Q_{\mathcal{H}(d)}(\nabla u)\|_{L^2(\mathbb{R}^d)} \leq C(d)\|P_{\mathcal{H}(d)}(\nabla u)\|_{L^2(\mathbb{R}^d)}. \tag{13}$$

3.2. Improved estimates on $g(d)$

We now return to the problem of describing the behaviour of $N(d)$, that is, that of $g(d)$. As we have mentioned before, we have the bounds $2^{\lceil \log_2(d) \rceil} \leq g(d) \leq 2d-1$ (for $d>1$, of course). First, we mention the following improved upper bound on $g(d)$, essentially due to [52].

Proposition 3.3. *Let $d \geq 2$. Then $g(d) \leq 2d-2$.*

Proof. We can apply Theorem 5.5 of [52] (with $h = d-1, k = 2d-2$) to obtain the existence of a nonsingular bilinear map

$$f \colon \mathbb{R}^d \times \mathbb{R}^{d-1+\tau(2d-2,d-1)} \to \mathbb{R}^{2d-2},$$

where $\tau(2d-2, d-1)$ is defined as follows:

$$\tau(2d-2,d-1) = \operatorname{Card}\left\{i \colon (d-1)_i = 0 \text{ and } (2d-2)_i \neq (d-1)_i\right\},$$

with $(a)_i$ denoting the i-th binary digit of a. Since $\tau(2d-2, d-1) \neq 0$, the result follows. □

Remark 7. For d even, the proof can be simplified substantially: identify $\mathbb{R}^d \simeq \mathbb{C}^{\frac{d}{2}}$ and then use complex polynomial multiplication.

While the upper bound seems quite sharp (being attained infinitely many times), the lower bound leaves much to be desired. This is not surprising, as our proof only required very basic tools from algebraic topology. To improve it, one needs some more sophisticated topological obstructions. Let us briefly sketch the idea behind the lower bounds. The main ingredient one needs is to make use of the connection of the problem with algebraic topology: namely, that the existence of a bilinear nonsingular map $f \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^k$ implies the existence of an immersion of $\mathbb{RP}^{d-1}$ in $\mathbb{R}^k$ (see [60]). Hence $g(d) \geq \operatorname{Imm}(\mathbb{RP}^{d-1}) + 1$, where $\operatorname{Imm}(M)$ is the minimal k such that M immerses in $\mathbb{R}^k$. We can now leverage the many results concerning $\operatorname{Imm}(\mathbb{RP}^d)$ [8] to obtain the remaining results:

Theorem 3.5. *The following bounds hold: (see [6, Theorem 1.4] and [54, Theorem 4.8] respectively)*

$$\operatorname{Imm}(\mathbb{RP}^d) = 2d - \mathcal{O}(\log(d))$$
$$\operatorname{Imm}(\mathbb{RP}^{2^n}) = 2^{n+1} - 1$$

[8] The following is by no means an exhaustive list, and we refer the interested reader to [27] for a complete list of the bounds known

It follows that

$$N(2^n+1) \geq \mathrm{Imm}(\mathbb{RP}^{2^n}) + 1 = 2^{n+1}$$

and

$$N(d) \sim 2d,$$

thus proving the remaining claims.

Remark 8. It is not known whether in general $g(d) = \mathrm{Imm}(\mathbb{RP}^{d-1}) + 1$. Conditional on this equation being true, the inequality $g_{\mathbb{R}}(d) \leq 2d-2$ is attained if and only if $d = 2^n+1$.

It seems clear that the problem of determining additional values of $N(d)$ (and more ambitiously that of determining a closed form for $N(d)$) is fundamentally topological rather than analytical, and one that will require new tools.

For specific values, more can be said. Namely, one can get (see [1], [29] for proofs) the following larger table [9]:

Table 2: Values of $N(d)$

d	$N(d)$	d	$N(d)$	d	$N(d)$
1	1	12	17	23	39
2	2	13	19	24	39
3	4	14	23	*25	[40,47]
4	4	15	23	*26	[42,48]
5	8	16	23	*27	[46,49]
6	8	17	32	*28	[46,50]
7	8	18	32	*29	[47,51]
8	8	19	33	*30	[47,54]
9	16	*20	[33,35]	*31	[47,54]
10	16	21	35	32	54
11	17	22	39	33	64

3.3. The optimal constant $C(X)$

While in the standard versions of Korn's inequality ($Y = \mathrm{Sym}$ and Sym_0, for example), one can usually calculate C explicitly and prove it independent of the dimension d, this is not the case for Korn-Hankel, and indeed it is almost never the case for inequalities induced by an admissible space X. To see why, we need the following auxiliary lemma:

Lemma 3.6. *Let Z be a subspace of $M_d(\mathbb{R})$. Let u, v be two independent random vectors that are uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$ and let $R = u \otimes v$. Then*

$$\mathbb{E}(\|P_Z(R)\|^2) = \frac{\dim(Z)}{d^2}.$$

[9] For the numbers marked with $*$, the precise value is not known and the table reports the best known bounds.

Proof. To see this, first notice that since $\|P_Z(A)\|^2 = \sum_i \|P_{\mathrm{span}(x_i)}(A)\|^2$ where the x_i form an orthonormal basis of Z, it suffices to prove the claim for a one-dimensional space Z. In that case, let B be a unit vector in Z and notice that

$$\begin{aligned}
\mathbb{E}(\|P_Z(R)\|^2) &= \mathbb{E}\left(\mathrm{Tr}(vu^tB)^2\right) \\
&= \mathbb{E}(\mathrm{Tr}(u^tBv)^2) \\
&= \mathbb{E}(\mathrm{Tr}(B^tuu^tBvv^t)) \\
&= \mathrm{Tr}(B^t\mathbb{E}(uu^t)B\mathbb{E}(vv^t)) \\
&= \frac{1}{d^2}\mathrm{Tr}(B^tB) \\
&= \frac{1}{d^2}.
\end{aligned}$$

□

This lemma allows us to investigate the dimension-dependence of the constants in inequalities of Korn type as follows:

Proposition 3.4. *Let X be a subspace of $M_d(\mathbb{R})$ and assume that, for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ we have*

$$\|Q_X(\nabla u)\|_{L^2(\mathbb{R}^d)} \le C(X)\|P_X(\nabla u)\|_{L^2(\mathbb{R}^d)}.$$

Then we must have

$$C(X)^2 \ge \frac{d^2}{\dim(\mathrm{X})} - 1$$

Proof. We have already seen that the Korn-type inequality holds with constant $C(X)$ if and only if for all matrices R of rank-one we have

$$C(X)^2\|P_X(R)\|^2 \ge \|Q_X(R)\|^2.$$

If this holds for all matrices of rank-one, then it must follow that

$$\mathbb{E}\bigg(C(X)^2\|P_X(R)\|^2 - \|Q_X(R)\|^2\bigg) \ge 0,$$

where R is the same random variable as in the lemma. Applying the lemma, the result follows. □

The previous proposition leaves open the question of the sharpness of this bound. By applying concentration inequalities, we prove that the bound is almost sharp for most randomly chosen X, provided we stay away from the critical range $\dim(X) \approx d$.

Theorem 3.7. *Let $k, d \in \mathbb{N}$ with $k \le d^2$. We define*

$$K(k,d) := \inf\bigg\{C(X)\colon X \subset M_d(\mathbb{R}), \dim(X) = k\bigg\}.$$

Then for any $\alpha > 1$ if $k \gtrsim d^\alpha$ we have

$$K(k,d) = \sqrt{\frac{d^2}{k} - 1} + \mathcal{O}\left(\frac{1}{d^\beta}\right),$$

where $\beta \in (0,1)$ depends on α.

Proof. Let $\mathrm{Gr_k}(M_d(\mathbb{R}))$ denote the Grassmannian of order k on $M_d(\mathbb{R})$, i.e. the set of k-dimensional subspaces of $M_d(\mathbb{R})$. As is well known (see [30, Proposition 7.2]) we can equip this space with a Haar measure. Let X be a random variable which is uniformly distributed on $\mathrm{Gr}_k(M_d(\mathbb{R}))$ with respect to the aforementioned Haar measure, and let A be a matrix of norm 1 in $M_d(\mathbb{R})$. It is not difficult to calculate that

$$\mathbb{E}(\|P_X(A)\|^2) = \frac{k}{d^2}.$$

One way to see this is to notice first that, since $P_X = \sum P_{x_i}$ where $\{x_i\}$ is an orthonormal basis of X, we can without loss of generality assume $X = \langle x \rangle$, and then rotational invariance proves the result. By an elementary concentration inequality (see [26]) we have that

$$\mathbb{P}\left(\left|\|P_X(A)\|^2 - \frac{k}{d^2}\right| > \varepsilon\frac{k}{d^2}\right) \lesssim \exp\left(\frac{k}{2}(\ln(1+\varepsilon) - \varepsilon)\right),$$

Moreover, since the map $(u,v) \to \|P_X(u \otimes v)\|^2$ is Lipschitz as a map $\mathbb{S}^{d-1} \times_{\ell_2} \mathbb{S}^{d-1} \to \mathbb{R}$ (and its Lipschitz constant is easily bound by 2 again), it follows that once we find a suitable $\frac{k}{d^2}\frac{\varepsilon}{100}$ net of $\mathbb{S}^d \times \mathbb{S}^d$, we obtain that for all matrices R of rank and norm 1 we have by the union bound

$$\mathbb{P}\left(\left|\|P_X(R)\|^2 - \frac{k}{d^2}\right| \geq \varepsilon\frac{k}{d^2}\right) \lesssim \mathcal{N}\left(\mathbb{S}^{d-1} \times_{\ell_2} \mathbb{S}^{d-1}, \frac{\varepsilon}{1000}\frac{k}{d^2}\right)\exp\left(-\frac{k\varepsilon^2}{4}\right).$$

Since $\mathcal{N}\left(\mathbb{S}^d \times \mathbb{S}^d, \varepsilon\right) \lesssim (1 + \frac{2}{\varepsilon})^{2d}$, it follows that

$$\mathbb{P}\left(\left|\|P_X(R)\|^2 - \frac{k}{d^2}\right| \geq \varepsilon\frac{k}{d^2}\right) \lesssim \exp(C_\varepsilon d \ln d)\exp\left(-\frac{k\varepsilon^2}{4}\right).$$

Hence if k grows at least as fast as d^α, the probability goes to 0 as $d \to \infty$, so there exist subspaces $X \subset M_d(\mathbb{R})$ of dimension k such that

$$\inf_R \|P_X(R)\|^2 \geq \frac{k}{d^2}(1-\varepsilon)$$
$$\sup_R \|P_X(R)\|^2 \leq (1+\varepsilon)\frac{k}{d^2}$$

The claim follows. □

Remark 9. Similarly, we can prove that

$$\left\lceil \frac{d^2}{2} \right\rceil \leq h(d) \leq \left\lceil \frac{d^2}{2} \right\rceil + 1,$$

where $h(d) := \min\{\dim(X)\colon C(X) = 1\}$. For even d, it is not hard to construct an explicit subspace that attains $C(X) = 1$ of dimension $\frac{d^2}{2}$, thus providing us with an optimal version of Korn's inequality with constant one:

$$X = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A, B \in M_{\frac{d}{2}}(\mathbb{R}) \right\}.$$

This space is not a subspace of $\mathrm{Sym}(d)$, and hence it is not a direct generalisation of the standard form of Korn's inequality, unless $d = 2$. This, however, should not come as a surprise: it is not hard to prove that the only subspace of $\mathrm{Sym}(d)$ (other than itself) for which $C(X) = 1$ is Sym_0.

It is not hard to see that our argument actually applies to a slightly larger region, i.e. $k \gtrsim_\varepsilon d \log(d)$. On the other hand, as we will see later, there are clear topological obstructions in the "small k" region $k \lesssim d$ (namely, $X^\perp \cap \mathcal{R}_1 \neq \emptyset$ becomes inevitable for k slightly below $2d$), so the best we can hope for is for our argument to be extended to $k \gtrsim d$. It turns out that, up to a dependence on the constant on ε, this can be attained.

To see this, notice that our argument introduced the $d \log(d)$ factor when we applied our naive ϵ-net argument. If, instead of considering each individual rank-one matrix and then finding a net, one considers all of them at the same time, this can be avoided, but the argument becomes more technical. More precisely, let us define $\gamma_2(T)$ for a set $T \subset S^{d-1}$ to be

$$\gamma_2(T) := \mathbb{E}\left(\sup_{x \in T} \langle g, x \rangle \right),$$

where g is a gaussian vector, i.e. $g \sim \mathcal{N}(0, \mathrm{I}_d)$. This quantity, known as Gaussian width (see [67, Chapter 9] and references therein), is the exact right measure of size that is needed for our probabilistic argument to go through. Indeed we have that, for any $k \gtrsim \frac{\gamma_2^2(T)}{\varepsilon^2}$ and letting P_X denote the projection onto a random element of $\mathrm{Gr}_k(\mathbb{R}^d)$ we have (see [51, Section 4])

$$\mathbb{P}\left(\text{for all } x \in T,\ (1 - \varepsilon) \leq \|P_X(x)\| \frac{d}{\sqrt{k}} \leq (1 + \varepsilon) \right) > \frac{1}{2}.$$

To apply this to $T := \mathcal{R}_1 \cap S_{M_d(\mathbb{R})}$, we wish to calculate $\gamma_2(T)$. This is not hard to do:

$$\gamma_2(T) = \mathbb{E}\left(\sup_{\|x\| = \|y\| = 1} \langle g, x \otimes y \rangle \right) = \mathbb{E}(\|g\|_{\mathrm{Op}}) \approx \sqrt{d}.$$

It follows that the argument in the proof of Theorem 3.7 can be extended to the region

$$k \gtrsim \frac{\gamma_2^2(T)}{\varepsilon^2} = \frac{d}{\varepsilon^2}.$$

In other words, on the region $d = o(k)$ we have, as d goes to infinity,

$$K(k,d) = \sqrt{\frac{d^2}{k}} - 1 + o(1)$$

Remark 10. In case one is interested, more broadly, in linear operators $T(\nabla u)$ such that $\|T(\nabla u)\|_{L^2(\mathbb{R}^d)} \gtrsim \|\nabla u\|_{L^2(\mathbb{R}^d)}$, while maintaining[10] $\|T\|_F \gtrsim d$ then it is possible to obtain such an operator with a much smaller range than in the projection case. In particular, a nonsingular bilinear map is said to be normed if $\|f(x,y)\| = \|x\|\|y\|$. In that case, (let F denote the extension of f to $M_d(\mathbb{R})$) the same rank-one convexity argument as in section 2 proves that

$$\|F(\nabla u)\|_{L^2(\mathbb{R}^d)} \geq \|\nabla u\|_{L^2(\mathbb{R}^d)},$$

and it is not hard to prove that $\|F\| = 1$. Not much is known about the minimal dimension $\tilde{g}(d)$ such that a normed bilinear nonsingular map $f\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{\tilde{g}(d)}$ exists: to the author's knowledge only the following values are known: for $d \leq 10$, $g(d) = \tilde{g}(d)$ and $\tilde{g}(16) \in [29, 32]$. Asymptotically, one can prove that $\tilde{g}(d) = \mathcal{O}\left(\frac{d^2}{\log(d)}\right)$, but it is not known if this is optimal. For other rings and fields, more is known: for $\mathbb{K} = \mathbb{C}$ (and for all fields $\mathbb{K}$ with positive characteristic) we have $\tilde{g}_{\mathbb{C}}(d) = \mathcal{O}(d^{1.62})$; lower-bound-wise we have $\tilde{g}_{\mathbb{Z}}(d) = \Omega(d^{\frac{6}{5}})$.

It follows that the only way the constant can be uniform in dimension is if $\dim(X) \approx d^2$, and that $C(\mathcal{H}(d)) \gtrsim \sqrt{d}$. Somewhat surprisingly (compared to its parent case $X = \mathrm{Sym}$, which is quite close to being optimal), $C(\mathcal{H}(d))$ grows much faster than polynomially. Indeed we can, with a more refined analysis, pinpoint $C(\mathcal{H}(d))$ quite precisely:

Proposition 3.5.

$$\log(C(\mathcal{H}(d))) = \frac{2G}{\pi} d + \mathcal{O}(\log(d)).$$

where G is Catalan's constant.

Proof. To see this, notice that, given two vectors u and v in $\mathbb{R}^d$, $P_{\mathcal{H}(d)}(u \otimes v) = u \star v$, where $u \star v$ is the polynomial coefficient convolution operation induced by $\mathbb{R}^d \simeq \mathbb{R}_{\leq d-1}[X]$ and $u * v$ denotes the standard convolution operation. Hence the problem of determining $C(\mathcal{H}(d))$ is equivalent to that of determining

$$C(\mathcal{H}(d))^{-1} \approx \inf_{u,v \in \mathbb{R}^d \setminus \{0\}} \frac{\|u \star v\|}{\|u\|\|v\|}.$$

Notice moreover that $\|u * v\| \leq \|u \star v\| \leq \sqrt{d}\|u * v\|$. Using this and selecting the coefficients properly, it is not hard to obtain an exponential bound[11]; this can be improved

[10] Such a normalisation is necessary, for otherwise it would suffice to consider P_X/k for k large enough.

[11] One can just take $u = \sum_i \binom{d}{i} e_i$ and $v = \sum_i \binom{d}{i}(-1)^i e_i$; then $u * v = \sum_k (-1)^k \binom{d}{k} e_{2k}$ and it follows that $C(\mathcal{H}(d))^{-1} \leq \frac{1}{\sqrt{\binom{2d}{d}}}$.

by noticing that, by Parseval, we can rephrase the problem as one about the $L^2(\mathbb{T})$ norm of polynomials with real coefficients:

$$\inf_{u,v\in\mathbb{R}^d\setminus\{0\}} \frac{\|u*v\|}{\|u\|\|v\|} = \inf_{\deg(p),\deg(q)\le d-1} \frac{\|p(z)q(z)\|_{L^2(\mathbb{T})}}{\|p(z)\|_{L^2(\mathbb{T})}\|q(z)\|_{L^2(\mathbb{T})}}.$$

Since, for a polynomial in $\mathbb{R}_{d-1}[X]$, we have that the $L^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$ norms are equivalent up to a polynomial factor, i.e.

$$\frac{1}{\sqrt{d}}\|p\|_{L^\infty(\mathbb{T})} \le \|p\|_{L^2(\mathbb{T})} \le \|p\|_{L^\infty(\mathbb{T})},$$

we have

$$\log\left(C(\mathcal{H}(d))\right) = \sup_{\deg(p),\deg(q)\le d-1} \log\left(\frac{\|p\|_{L^\infty(\mathbb{T})}\|q\|_{L^\infty(\mathbb{T})}}{\|pq\|_{L^\infty(\mathbb{T})}}\right) + \mathcal{O}(\log(d)).$$

The value of the right hand side is already known, see [15, Theorem 2, Remark 2] and [62, corollary 2.8, Theorem 2.9, Remark 2.10]:

$$\sup_{\deg(p),\deg(q)\le d-1} \log\left(\frac{\|p\|_{L^\infty(\mathbb{T})}\|q\|_{L^\infty(\mathbb{T})}}{\|pq\|_{L^\infty(\mathbb{T})}}\right) = d\log(\delta) + \mathcal{O}(\log(d)),$$

where $\delta = e^{2G/\pi}$, proving the result. □

This leaves open the question whether, in the regime $\dim(X) \approx 2d$, an optimality result such as Theorem 3.7 can be attained. While the optimal regime remains outside of our reach, if one is content with only near-optimality, then the result can be substantially improved:

Proposition 3.6. *For $k \ge (4+\delta)d$, with $\delta > 0$, there exists a subspace X of $M_d(\mathbb{R})$ such that $C(X) \approx_\delta \sqrt{d}$.*

Proof. Let G be a Gaussian matrix that maps onto X. Then by Gordon's min-max inequality (see [36, Theorem A, corollary 1.2]) we have

$$\mathbb{E}\left(\min_{A\in\mathcal{R}_1\cap S}\|GA\|\right) \gtrsim \sqrt{k} - \gamma_2(\mathcal{R}_1\cap S) \approx \sqrt{k} - 2\sqrt{d}.$$

It is well known that with high probability

$$d - \sqrt{k} \lesssim \sigma_1(G) \lesssim \sqrt{k} + d,$$

so in our regime $\sigma_1(G) \approx d$. It follows that

$$\mathbb{E}\left(\min_{A\in\mathcal{R}_1\cap S}\|P_X(A)\|\right) \gtrsim \frac{1}{d}\mathbb{E}\left(\min_{A\in\mathcal{R}_1\cap S}\|G(A)\|\right) \gtrsim \frac{\sqrt{k}-2\sqrt{d}}{d}.$$

This, together with proposition 3.4 implies that there exists a subspace X such that

$$C(X) \approx \sqrt{d}.$$

Indeed, by applying standard concentration arguments it follows that this happens with high probability. □

Remark 11. Gordon's theorem cannot be pushed any further than $k > (4+\delta)d$, as the estimate above becomes negative and hence useless. This is no coincidence: the statistical dimension of the convex cone generated by $\mathcal{R}_1$, i.e. the unit ball with respect to the nuclear norm, has statistical dimension $4d$, so from the viewpoint of these Gaussian arguments, it is not possible to guarantee that $X^\perp$ does not intersect $\mathcal{R}_1$.

3.4. Extending to $p \neq 2$: $N_p(d)$

We have so far been interested only in the dimensionality problem in $W^{1,2}$, but the results above extend to $W^{1,p}$ for $p \in (1,\infty)$. More precisely we can define $N_p(d)$ in the same way as $N(d)$, just with p instead of 2 in (1):

***Definition* 6.** Let $p \in (1,\infty)$. We define $N_p(d)$ to be the smallest integer k for which there exist $\ell_1,\dots,\ell_k$ functionals on $M_d(\mathbb{R})$ such that

$$\|\nabla u\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{i=1}^k \|\ell_i(\nabla u)\|_{L^p(\mathbb{R}^d)} \tag{14}$$

holds for all $u \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$.

In the same way as for $N_2(d)$ we have:

Lemma 3.8. *Let $M_p(d)$ be defined as*

$$M_p(d) := \inf\left\{\dim(X)\colon \forall u \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)\ \|\nabla u\|_{L^p(\mathbb{R}^d)} \lesssim \|P_X(\nabla u)\|_{L^p(\mathbb{R}^d)}\right\}.$$

Then $M_p(d) = N_p(d)$.

Mutatis mutandis, we can follow the proof of Theorem 1.2 to obtain $N_p(d) \geq g(d)$, but the opposite direction of the proof does not go through any more (since being rank-one convex is only necessary for quasiconvexity). However, since we do not care about the precise constant, a different argument can be employed to prove the upper bound as well.

Theorem 3.9. *For $p \in (1,\infty)$ we have*

$$N_p(d) = N_2(d).$$

Proof. We already have $N_p(d) \geq g(d)$, so it suffices to prove $N_2(d) \geq N_p(d)$. To prove it, let X be an admissible subspace for $N_2(d)$. Then the operator $T(u) := P_X(\nabla u)$ is an elliptic operator for $p=2$, so it is elliptic in the injective sense (see [66, Theorems 5.1, 5.3], hence we have the bound

$$\|\nabla u\|_{L^p(\mathbb{R}^d)} \lesssim_{T,d,p} \|Tu\|_{L^p(\mathbb{R}^d)},$$

which proves the claim. □

Remark 12. The same questions that we posed for $p = 2$ (the best decay that can be obtained with $\dim(\mathrm{X}) = N_p(d)$ and the best $\dim(X)$ that can be attained while maintaining a constant ≥ 1) can be asked for $p \neq 2$. In this case, we remark that the author [20] has obtained dimension-free bounds for various standard forms of Korn's inequality.

While for $p = 2$, the issue of determining the precise value of the constant $C(p, X)$ such that

$$\|Q_X(u)\|_{L^p(\mathbb{R}^d)} \leq C(p, X)\|P_X(u)\|_{L^p(\mathbb{R}^d)}$$

holds can be explicitly answered (at least a priori) using

$$C(X, 2)^{-1} = \inf_{A \in \mathcal{R}_1} \frac{\|P_X(A)\|}{\|Q_X(A)\|},$$

the situation is much more complicated for $p \neq 2$. Indeed, the question of determining the L^p constant is still open even for the standard version of Korn's inequality (see [20] for more on this). However, we can employ our method to obtain bounds on a similar constant, namely C^{rc}, which is defined as

$$C^{\mathrm{rc}}(X, p) := \inf \left\{ c \colon f^{rc}_{p,c,X}(0) = 0 \right\}.$$

Indeed, Theorem 2.1 allows us to prove

Proposition 3.7.

$$C^{\mathrm{rc}}(X, p) \leq C(2, d)(p^* - 1).$$

Focusing in particular on the Hankel-Korn inequality, we can take advantage of the sharp part of 2.2. Indeed, we can embed the scalar martingales by defining $M_n = \frac{f_n}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{g_n}{2}(e_1 \otimes e_2 - e_2 \otimes e_1)$ and the theorem then allows us to show

$$C^{\mathrm{rc}}(\mathcal{H}(d), p) \geq (p^* - 1).$$

It is then natural to conjecture

Conjecture.

$$C(\mathcal{H}(d), p) = C^{\mathrm{rc}}(\mathcal{H}(d), p) = C(\mathcal{H}(d), 2)(p^* - 1). \tag{15}$$

Let us point out that, for $d = 2$, we have $C^{\mathrm{rc}}(\mathcal{H}(2), p) = C(\mathcal{H}(2), 2)(p^* - 1)$, so the conjecture becomes one of quasiconvexity at 0 of $f^{rc}_{\mathcal{H}(2),C^{\mathrm{rc}},p}$ and that in this case the conjecture is similar to (and indeed, implied by) the Iwaniec-Martin conjecture (see [44]). Indeed, Theorem 2.2 can be used to obtain a new proof of the lower bound in the Iwaniec-Martin conjecture. This will be further explored in future work.

Remark 13. The study of the quasiconvexity of these rank-one convex envelopes is an active area of research, particularly for $X = \mathcal{C}(2)$ (i.e. the set of conformal 2×2 matrices) thanks to its relation with the Iwaniec-Martin conjecture (see [9], [45], [4]) for more on this connection). In this case, it turns out that the envelope can be explicitly described in terms of the Burkholder function, whose conjectured quasiconvexity has been studied intensively, see [39], [2], [38], [5] and [3].

4. Korn's second inequality: $N'(d)$

We now deal with $N'(d)$, i.e. the minimal dimension of $X \subset M_d(\mathbb{R})$ such that for all $u \in H^1(\Omega)$ we have

$$\|u\|_{H^1(\Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|P_X(\nabla u)\|_{L^2(\Omega)}.$$

First, let us note that the correct definition should read $N'(d, \Omega)$, as it can a priori depend on the domain. We will use the term $N'(d)$ to refer to $N'(d, \Omega)$ for any open bounded set Ω with Lipschitz boundary. [12] In [22] the following was proved:

Theorem 4.1. 1. $N'(2) = 3$

2. $N'(d) \geq N(d)$
3. $N'(d) \geq d$ *and the inequality is strict if* d *is odd and* $d > 1$.

We prove a closed form for $N'(d)$. Namely

$$N'(d) = g_{\mathbb{C}}(d) = 2d - 1.$$

This implies the previous theorem in a strengthened form, since both inequalities stated above have to be strict (provided $d > 1$).

Proof of $N'(d) \geq g_{\mathbb{C}}(d)$. Let X be admissible for $N'(d)$, i.e. let us assume that

$$\|u\|_{H^1(\Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|P_X(\nabla u)\|_{L^2(\Omega)}$$

holds. Assume, for the sake of contradiction, that $\dim(X) < g_{\mathbb{C}}(d)$. It follows that $X^{\mathbb{C}\perp}$ (i.e. the orthogonal complement of the complexification of X) contains matrices of rank 1, hence there exist $a, b \in \mathbb{C}^d$ such that

$$P_{X^{\mathbb{C}}}(a \otimes b) = 0.$$

Let us define the set $S := \{u \in W^{1,2}(\Omega) \colon P_X(\nabla u) = 0\}$. We claim S is infinite-dimensional. To see this, take $g \colon \mathbb{C}^d \to \mathbb{C}$ entire, and define

$$u_g(x) = \mathrm{Re}(g(bx)a).$$

It follows that

$$\nabla u_g = \mathrm{Re}(g'(bx)a \otimes b),$$

hence $u_g \in S$, so S is infinite-dimensional. Since X is admissible, it follows that for all $u \in S$ we have

$$\|u\|_{W^{1,2}(\Omega)} \lesssim \|u\|_{L^2(\Omega)},$$

so the two norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{L^2}$ are equivalent on S. Since the embedding $H^1(\Omega) \to L^2(\Omega)$ is compact, it follows that the unit ball of S is totally bounded, which implies S is finite-dimensional, a contradiction. □

[12] One can extend this to, say, John domains, but for simplicity we stick with this simpler case. Let us note that the geometry of Ω does play a role: namely, if $\partial\Omega$ is too irregular, Korn's second inequality fails.

Proof of $N'(d) \le g_{\mathbb{C}}(d)$. We prove the result for domains that are star-shaped with respect to a ball. Standard partition of unity and reflection arguments extend the result to Lipschitz domains. Let X be a subspace of $M_d(\mathbb{C})$ such that X avoids all rank one matrices. Let T_X denote the differential operator $T_X(u) = P_X(\nabla u)$. Thanks to the rank-one condition, we have that the family $T_X(\cdot)_{i,j}$ of scalar differential operators satisfies condition (C) of [48], hence thanks to [48, Theorem 4] we obtain that, for each $u \in C^\infty(\Omega)$ we have

$$u(x) = \mathcal{P}(u)(x) + \int_\Omega K(x,y) T_X(u)(y) \mathrm{d}y,$$

where K is a vector-valued Calderón-Zygmund kernel and $\mathcal{P}(u)(x)$ is defined as

$$\mathcal{P}(u)(x) = \int_\Omega \sum_{|\alpha| \le l} \partial_y^\alpha \left(\frac{(x-y)^\alpha}{\alpha!} \omega(y) \right) u(y) \mathrm{d}y,$$

where ω is a mollifier supported in $B(0,1)$ and $l \in \mathbb{N}$ depends on Ω, T and ω but not on u. It follows that

$$\|u\|_{H^1(\Omega)} \lesssim_{d,\Omega,X} \|T_X(u)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

and the result follows by density. □

One can define $N'_p(d,\Omega)$ in the same manner as N_p was defined from N. The same argument as above then proves that $N'_p(d,\Omega) = N'_2(d,\Omega) = 2d-1$.

Remark 14. The above results can also be interpreted (and proved) in terms of $\mathbb{C}$-ellipticity, a strengthening of the concept of ellipticity. Namely, a differential operator of order k of the form

$$T(u) = \sum_{|\alpha|=k} T_\alpha \partial_\alpha u$$

having symbol $T(\xi) = \sum T_\alpha \xi^\alpha$ is said to be $\mathbb{C}$-elliptic if $\ker(T(\xi)) = 0$ for all $\xi \in \mathbb{C}^n \setminus \{0\}$, and if it is then the inequality $\|u\|_{W^{k,p}(\Omega)} \lesssim \|u\|_{L^p(\Omega)} + \|T(u)\|_{L^p(\Omega)}$ holds for suitably well behaved domains. It is not hard to see that, in our context, T_X being $\mathbb{C}$-elliptic is equivalent to $X^{\perp^{\mathbb{C}}}$ not containing any rank-one matrices. The concept of $\mathbb{C}$-ellipticity has been independently discovered many times: Smith [64] first obtained a result of this kind, followed by Boman [11], Kałamajska [48] and more recently by Breit, Diening, and Gmeineder [16]. Boman's proof, in particular, implies a strengthening of our theorem to Boman domains, which as is known ([17]) coincide with John domains. Since Korn's second inequality cannot in general be extended to domains that are not John ([46]), this is in a sense the largest class possible.

In particular, we have

Theorem 4.2 (Second Korn-Hankel inequality). *Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Then for all $p \in (1,\infty)$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ we have*

$$\|u\|_{W^{1,p}(\Omega)} \le C'(\Omega,p) \left(\|u\|_{L^p(\Omega)} + \|P_{\mathcal{H}(d)}(\nabla u)\|_{L^p(\Omega)} \right)$$

Remark 15. The issue of determining the constants associated with the inequality induced by an admissible subspace X is much more subtle in this case than it was for $N(d)$: the geometry of the domain plays a very important role, as shown in [53]. More precisely, it is known that for the standard form of Korn's second inequality, the constant is not uniformly bounded with respect to (Lipschitz) domains. This is not surprising, as one expects for physical reasons that the constant should blow up for very thin strips.

It is not clear what geometric quantities one should expect $C'(\Omega, X, p)$ to depend on.

5. The rectangular case: $N(m, d)$

The Korn-type inequalities we have considered so far have one restriction: namely, they only apply to vector fields $u\colon \mathbb{R}^d \to \mathbb{R}^d$. The goal of this section is to extend our results to vector fields $u\colon \mathbb{R}^m \to \mathbb{R}^d$, or in other words to the spaces $H^1_{(0)}(\Omega \subset \mathbb{R}^m; \mathbb{R}^d)$.

Definition 7 ($N(m, d)$)**.** We define $N(m, d)$ to be the smallest integer k for which there exist linear $\ell_1, \ldots, \ell_k$ functionals on $M_{m,d}(\mathbb{R})$ such that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^m)} \lesssim \sum_{i=1}^{k} \|\ell_i(\nabla u)\|^2_{L^2(\mathbb{R}^m)} \tag{16}$$

holds for all $u \in C_c^\infty(\mathbb{R}^m, \mathbb{R}^d)$.

One similarly defines $N'(m, d)$, with the usual caveats on Ω being a regular-enough domain, and $n_{\mathbb{K}}(m, d, k), g_{\mathbb{K}}(m, d)$. The same method we used in the previous sections allows us to prove the following theorem

Theorem 5.1. *Let $m, d \in \mathbb{N}$. Then*

$$N(m, d) = md - n_{\mathbb{R}}(m, d, 1) = g_{\mathbb{R}}(m, d)$$

and

$$N'(m, d) = md - n_{\mathbb{C}}(m, d, 1) = g_{\mathbb{C}}(m, d).$$

Remark 16. This also provides an easy proof that both $N(m, d)$ and $N'(m, d)$ are symmetric with respect to m and d (since $g_{\mathbb{K}}$ is) which, while not surprising (especially in light of the easily provable fact that $N(d, 1) = N(1, d) = d$), has proved rather elusive to prove directly.

The same argument as in Proposition 3.2 proves

$$N'(m, d) = m + d - 1.$$

Moreover, we have the following bounds:

Theorem 5.2. *Let $m, d \in \mathbb{N}$. Then*

$$\max\{d, m\} \leq m \circ d \leq N(m, d) \leq N'(m, d).$$

Sketch. It is clear by the immersion $\mathbb{R} \hookrightarrow \mathbb{C}$ that $g_\mathbb{R} \leq g_\mathbb{C}$, so the rightmost upper bound follows. To prove that $N(m,d) \geq m \circ d$, we adapt the proof of Theorem 3.2: a nonsingular bilinear map $f\colon \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^{g(m,d)}$ implies the existence of a morphism $\tilde{f}$

$$\tilde{f}\colon \mathbb{Z}_2[X]/(X^m) \otimes \mathbb{Z}_2[Y]/(Y^d) \to \mathbb{Z}_2[Z]/(Z^{g(m,d)})$$

satisfying $\tilde{f}(X,Y) = X+Y$, hence

$$(X+Y)^{g(m,d)} \in (X^m, Y^d),$$

i.e. $g(m,d) \geq m \circ d$. The left-most lower bound follows by standard algebraic considerations. □

Following the proof of Proposition 3.3, we can also explicitly characterise the values of m, d for which the equality $N(m,d) = N'(m,d)$ holds:

Proposition 5.1. *Let $m, d \in \mathbb{N}$. Then $N(m,d) = N'(m,d) = m+d-1$ if and only if $m-1, d-1$ are dyadically disjoint, i.e. $\{i\colon (m-1)_i = 1\} \cap \{i\colon (d-1)_i = 1\} = \emptyset$.*

Proof. As in Proposition 3.3, we employ Theorem 5.5 of [52] to obtain the existence of a nonsingular bilinear map

$$f\colon \mathbb{R}^m \times \mathbb{R}^{d-1+\tau(m+d-2,m-1)} \to \mathbb{R}^{m+d-2},$$

where

$$\tau(m+d-2, m-1) = \operatorname{Card}\{i\colon (m-1)_i = 0 \text{ and } (m+d-2)_i \neq (d-1)_i\}.$$

It follows that, if $m-1, d-1$ are not dyadically disjoint then $\tau > 0$. On the other hand, if they are dyadically disjoint then $m \circ d = m+d-1$, and since $m \circ d \leq g(m,d) \leq m+d-1$, the result follows. □

As in section 3.2, we can use more involved topological arguments to prove stronger lower bounds, e.g.[13]

$$g_\mathbb{R}(m,d) \geq \min\Big(\operatorname{Imm}(\mathbb{RP}^{m-1}), \operatorname{Imm}(\mathbb{RP}^{d-1})\Big) + 1.$$

For more on bounds on $g_\mathbb{R}(m,d)$ (and many explicit values of it), we refer the reader to [29] and references therein.

Finally, we mention that the Korn-Hankel inequalities hold in this context as well, with virtually no modifications necessary to either the argument nor the formulation of the inequalities. Moreover, by using the same trick of reframing the matter as one concerning the norm of polynomials, one can prove

$$C(\mathcal{H}(m,d), 2) = \frac{1}{\mu^\mathbb{R}_{m+d,d}} + \mathcal{O}(\log(dm)),$$

[13] We make no claim of sharpness for this lower bound, which we present only as an example of the bounds that can be attained by using more topological tools.

where $\mu^{\mathbb{R}}_{m+d,d}$ is defined as

$$\mu^{\mathbb{R}}_{m+d,d} = \inf \left\{ \frac{\|pq\|_{L^\infty(\mathbb{T})}}{\|p\|_{L^\infty(\mathbb{T})}\|q\|_{L^\infty(\mathbb{T})}} : p \in \mathbb{R}_{m-1}[X], q \in \mathbb{R}_{d-1}[X] \right\}.$$

Bounds similar (albeit more complicated to state) to the ones obtained in the case $d = m$ can then be proved by taking advantage of the known properties of μ; we refer the interested reader to [19] for details.

6. A quantitative proof of Ornstein's non-inequality

In [57], Ornstein proved the following result:

Theorem 6.1. *Let L be a homogeneous differential operator of order 1 with constant coefficients, i.e. $L(u) = T(\nabla u)$ for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$, with $T \in \mathrm{Lin}(M_{d,m}(\mathbb{R}), \mathbb{R}^k)$. Then there exists a constant C such that for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$*

$$\|\nabla u\|_{L^1(\mathbb{R}^d)} \leq C\|L(u)\|_{L^1(\mathbb{R}^d)}$$

if and only if T is injective.

To be precise, Ornstein proved this result for higher orders as well, but here we focus on operators of order 1. This theorem has a clear connection with calculus of variations, for it can be equivalently stated in terms of quasiconvexity: namely, let $f_{L,C}(A) := C\|L(A)\| - \|A\|$. Then there exists C large enough such that $f^{qc}_{C,L}(0) = 0$ if and only if L is invertible (and a similar framing can be obtained for the higher-order version of the theorem as well). This connection has been taken advantage of several times to obtain proofs of the theorem via calculus of variations: in particular Kirchheim and Kristensen ([50, Theorem 1.3]) obtained a far reaching generalisation of Ornstein's result, and Faraco and Guerra ([33]) used a similar method to obtain a very efficient proof of Ornstein's result (for operators of order 1 and 2) in $\mathbb{R}^{2\times 2}$. Both proofs, however, are qualitative, in that they do not provide witnesses to the failure of the inequality. In that direction, Conti, Faraco and Maggi constructed in [23] an explicit family of laminates which proves the failure of the inequality for Korn's operator P_{Sym}. In this section, we extend their result by using our method to construct such a family for any operator L which contains a matrix of rank 2 in its kernel; moreover, we provide a quantitative bound on the failure of the inequality, showing that the necessary constant blows up like $p^* - 1$. In a different quantitative direction, we mention the work [47], where Riesz products are used to construct functions that witness the failure of the L^1 inequality.

To prove our result, we notice that the method applied in Theorem 2.2 of embedding scalar martingales can be used in a much more general setting:

Theorem 6.2. *Let L be a homogeneous differential operator of order 1 with constant coefficients, i.e. $L(u) = T(\nabla u)$ for all $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$, with $T \in \mathrm{Lin}(M_{d,m}(\mathbb{R}), \mathbb{R}^k)$. If* *there exists a non-zero matrix of rank at most* 2 *in* $\ker(T)$*, then there is no constant* C *such that for all* $u \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$

$$\|\nabla u\|_{L^1(\mathbb{R}^d)} \leq C\|Lu\|_{L^1(\mathbb{R}^d)}.$$

Moreover, the minimal constant C_p *such that*

$$f_p(A) := C_p^p\|T(A)\|^p - \|A\|^p$$

is rank-one convex (which is finite if and only if $\ker(T)$ *does not contain rank-one matrices) grows at least like* $p^* - 1$ *as* $p \to 1$ *or* $p \to \infty$*.*

Remark 17. If $d = 2$, the theorem applies to all operators $T(\nabla u)$ where T is not injective, proving the full Ornstein result. It also applies to $L = P_{\mathrm{Sym}}, P_{\mathrm{Sym}_0}$. It is not difficult to see that the same method can be applied to the more general case of $\|T(\nabla u)\|_{L^1(\mathbb{R}^d)} \gtrsim \|S(\nabla u)\|_{L^1(\mathbb{R}^d)}$.

Proof. If $\ker(T)$ contains an element of rank 1, the result is trivially true, so let us assume that there exists $A \in \ker(T)$ having rank 2 and that $\ker(T) \cap \mathcal{R}_1 = \emptyset$. We can write $A = u \otimes v + w \otimes z$ for some vectors u, v, w, z. Define $B := u \otimes v - w \otimes z$. It is clear that $\mathrm{rank}(A - B) = \mathrm{rank}(A + B) = 1$ and that $B \notin \ker(T)$, for if it did then $A - B$ would belong to $\ker(T)$, a contradiction. We can now apply the same martingale construction as in the proof of Theorem 2.2: namely, let us construct first a (not necessarily orthogonal) projection P onto $\ker(T)$ such that $P(B) = 0$, and define the function

$$f(A) = c_p^p\|T(A)\|^p - \|P(A)\|^p.$$

By considering the laminate associated with the martingale $M_n = g_nA + f_nB$, where g_n is a ± 1 transform of f_n and both start at 0, the result follows. □

Remark 18. This proof can be seen as an extension of the laminate approach that Conti, Faraco and Maggi developed in [23]. Indeed, specialising the proof to $L = \mathrm{Sym}$ leads to a family of laminates with the same support: the difference between the two constructions is then essentially one of choosing f_n.

If one is only interested in proving the failure of the inequality, then it is not necessary to utilise Burkholder's result or its sharpness: it suffices to construct two dyadic martingales f_n, g_n such that $\mathrm{d}g_n = \varepsilon_n \mathrm{d}f_n$ and $\sup \|f_n\|_{L^1(\mathbb{P})} < \infty$ while $\|g_n\|_{L^1(\mathbb{P})} \to \infty$. Such an example can be found in [58, Section 3.2.2]. We construct here an alternative example: let $\Omega = [0, 1]$, $\mathcal{F}_n$ denote the dyadic filtration and construct f_n via $f_0 = 0$, $\mathrm{d}f_n = 2^{n-1}(2\chi_{I_n} - \chi_{I_{n-1}})$, where $I_n = [0, 2^{-n}]$. For g_n, we simply impose $\mathrm{d}g_n = (-1)^{n-1}\mathrm{d}f_n$. While the L^1 norm of f_n remains bounded (indeed one easily sees that $\|f_n\|_{L^1(\mathbb{P})} \leq 2$), $\|g_n\|_{L^1}$ grows linearly: to see this let $T_k = I_k^c \cap I_{k-1}$

$$g_{n|T_k} = \sum_{i=1}^{k-1}(-1)^{i-1}2^{i-1} + (-1)^k 2^{k-1} = (-1)^k \frac{2^{k+1} + (-1)^k}{3},$$

hence, since all the T_k are disjoint, it follows that

$$\|g_n\|_{L^1} \geq \sum_{k \leq n} \frac{2^{k+1} + (-1)^k}{3} 2^{-k} = \frac{n}{3} + \mathcal{O}(1).$$

Remark 19. The following probabilistic interpretation of the above is possible: consider a game played, at each turn, by tossing a coin. The martingale f_n denotes the payoff obtained by playing n turns with the following strategy: betting each turn that the toss will result in heads, doubling the bet each turn, until the first tail is tossed, at which point one stops playing. g_n is then the payoff obtained by a similar strategy, where one doubles the bet at each turn and stops once tails appears for the first time, but instead of betting on heads each turn, alternates: first heads, then tails, then heads etc.

In [35] it was proved that for any homogeneous even multiplier T_m of order zero, one has

$$\|T_m\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \gtrsim p^* - 1.$$

A similar result, under some additional hypotheses, was proved for matrix-valued multipliers in [13, Theorem 14]. These results, however, concern themselves with the norm of T^{-1} or, in other words, with $C^{\mathrm{qc}}(L,p)$, proving $C^{\mathrm{qc}}(L,p) \gtrsim p^* - 1$. Our result, on the other hand, deals with $p^* - 1 \lesssim C^{\mathrm{rc}}(L,p) \leq C^{\mathrm{qc}}(L,p)$.

Finally, we remark that it is possible to avoid the use of the projection P (instead considering M_n in its entirety directly), and this approach would probably allow for a (slightly) more accurate estimate of $C^{\mathrm{rc}}(L,p)$. However, the complexity of the Burkholder side of the problem increases very significantly, as shown in [43].

7. Conclusion

There are several interesting questions that we have not managed to answer. The most natural one, finding a closed form for $g_{\mathbb{R}}(d)$, seems to be mainly topological in nature. We point out that even the upper bounds are still getting improved, see [29]. Similarly, determining the constants associated to the second form of Korn-type inequalities is mostly a geometrical question, and its answer is unknown even for the standard form of Korn's inequality. Some questions that seem more approachable are:

1. Is there an X_d of dimension $= N(d) + \mathcal{O}(1)$ such that $C(X_d) \approx \sqrt{d}$? If not, what is the best possible rate of growth in terms of d? Is it exponential, as suggested by the example $C(\mathcal{H}(d))$?
2. What more can be said about $C(X,p)$? For example, under which conditions is it true that $C(X,p) = C(X,2)(p^* - 1)$? As we mentioned before, this problem is connected to determining whether certain rank-one convex functions are quasiconvex (and as such, it is an interesting question in and of itself, even ignoring the applications that obtaining a sharp constant for the inequalities might have).
3. As we have seen, there exist subspaces X of dimension less than or equal to $\lceil \frac{d^2}{2} \rceil + 1$ for which $C(X,2) = 1$. A somewhat explicit description of these spaces would be interesting, since at the moment the smallest subspace satisfying $C(X,2) = 1$ that can be explicitly described is, to our knowledge, Sym_0.

Let us finally mention a natural direction to explore: extending the previous results to problems concerning higher derivatives; we will return to this question in future work.

Acknowledgements

The author wishes to thank Prof. Petrović for his comments and insights on the topic of bilinear nonsingular maps, and Prof. Kristensen for the many insightful conversations and comments, and for pointing out Boman's papers ([10] in particular) to us and the fact that Theorem 1.2 was already known in the calculus of variations literature.

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