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Non-vanishing non-linear Static Love Number of a Class of Extremal Reissner-Nordström Black Holes

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Abstract. We compute the tidal Love numbers for a particular axially symmetric configuration of extremal Reissner–Nordström geometry. By exactly solving the non-linear Einstein equations, we investigate the tidal response of extremal Reissner–Nordström black holes in four-dimensional spacetimes under external gravitational fields. We show that, for the specific geometry considered, the static tidal Love number remains finite and non-vanishing to all orders in the external tidal field. By contrast, we verify that the Love number of an isolated extremal Reissner–Nordström black hole remains zero, in agreement with previous expectations. Furthermore, we explicitly calculate the Zerilli–Moncrief master functions and match them with the effective field theory description.

Contents

1 Introduction

Gravitational waves (GWs) and black holes (BHs) represent cornerstone predictions of general relativity, validated by landmark observations such as the detection of GWs originating from BH coalescences by the LIGO and Virgo collaborations [1]. These observations have provided strong empirical support for Einstein's theory of gravity, with no observed deviations from GR to date [2].

During the inspiral of compact binaries—whether involving neutron stars or BHs—tidal forces grow significant as the orbital separation decreases. These interactions influence both the dynamics of the binary system and the characteristics of the emitted GWs. Accurately modeling this interplay is essential for refining waveform templates and testing GR under highly relativistic conditions.

Tidal interactions are quantified by parameters called Love numbers, which describe how an object deforms under the gravitational field of its companion. In particular, the static tidal Love numbers (TLNs) depend on the internal composition and structure of the compact objects subject to deformation [3]. These TLNs first appear at the fifth post-Newtonian order in the GW phase evolution [4], and are especially informative for neutron stars, where their nonzero values can reveal properties of dense nuclear matter.

By contrast, BHs are thought to have zero TLNs due to the absence of material rigidity. This has been demonstrated through linear perturbation theory, which shows that a tidal perturbation proportional to r^ℓ fails to generate an $r^{-\ell-1}$ response, indicating a vanishing static TLN for each multipole order ℓ. Consequently, BHs do not develop a measurable static tidal deformation under linear perturbations [5–15]. This outcome is believed to be connected to certain hidden symmetries in the spacetime structure [16–29].

Recent studies have further confirmed that the static TLNs remain zero even under second-order perturbations in the external tidal field [30, 31]. By using the Ernst formalism [32] and Weyl coordinate framework, the vanishing of the static TLN has been finally proven to persist to all orders in the parity-even tidal deformations in Refs. [33, 34] for the the Schwarzschild BH and in Ref. [35] for the rotating Kerr BHs. Furthermore, it has been understood that the vanishing of the non-linear static TLN is intimately related to the presence of underlying non-linear symmetries.

In this paper we address the question of whether the non-linear static TLN vanishes for Extremal Reissner-Nördstrom (ERN) BHs which generalize Schwarzschild solutions by adding the electric charge as an extra charge besides the mass. We will show that the ERN BHs are indeed responding to the presence of a tidal force and develop a nonvanishing static TLN. This result is obtained by solving exactly Einstein equations by immersing the ERN BH in a external gravitational field and is therefore valid at any order in the external tidal force.

The paper is organized as follows. In section 2, we describe the external spherically symmetric RN solution. In section 3, we analyze a charged axially symmetric extremal RN. For small parameters, it may be regarded as a perturbation of the standard RN. This is analyzed in section 4, where the corresponding Zerilli-Moncrief fields and the corresponding Love numbers are found. In section 5 we match with the EFT and, finally, in section 6 we conclude.

2 The Extremal RN BH

Among the most elegant and instructive exact solutions of Einstein's field equations is the Reissner–Nordström (RN) BH. It represents a non-rotating, spherically symmetric BH that possesses electric charge, and serves as a natural generalization of the Schwarzschild solution. Whereas the Schwarzschild BH is fully characterized by its mass alone, the RN solution is described by two parameters: the mass M and the electric charge Q. This added degree of freedom significantly enriches the geometry and causal structure of the spacetime. QNM of the RN BH have been studied in [36–38].

The RN solution emerges from Einstein's equations coupled to the source-free Maxwell equations. It captures the geometry of spacetime outside a charged, non-rotating, spherically symmetric object. The presence of the electromagnetic field modifies the gravitational dynamics in a subtle but profound way. Most notably, the RN spacetime admits, in general, two distinct horizons: an outer event horizon and an inner Cauchy horizon. These arise provided the charge is smaller than the mass ($|Q| < M$). In the extremal case ($|Q| = M$), the two horizons coincide, leading to a degenerate horizon with special thermodynamic properties. If the charge exceeds the mass ($|Q| > M$), the solution no longer describes a BH, but a naked singularity—raising questions about cosmic censorship.

The RN BH stands out not just for its mathematical elegance, but also for its physical implications. It offers a useful setting for investigating a range of theoretical problems, such as the stability of inner horizons, the behavior of test particles in charged spacetimes, and the nature of singularities. Furthermore, the extremal Reissner–Nordström BH plays a pivotal role in studies of BH thermodynamics and quantum gravity, as it features zero surface gravity and thus vanishing Hawking temperature. These properties make it an important model for understanding extremality and horizon microstructure in various approaches to quantum gravity.

Let us consider the gravitational field of a charged object with mass M and electric charge Q, assuming a static, spherically symmetric spacetime. The corresponding line element takes the form

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{2.1}$$

where the $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{2.2}$$

The associated electrostatic potential is given by

$$A_\mu = \Phi(r)\,\delta^0_\mu, \qquad \Phi(r) = \frac{Q}{r}. \tag{2.3}$$

There is a singularity at $r = 0$, which is hidden behind the two horizons located at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{2.4}$$

Clearly, the bound

$$M \geq |Q|, \tag{2.5}$$

should be satisfied, since otherwise, there is no horizon and $r = 0$ is a naked singularity, violating the cosmic censorship hypothesis. When the bound (2.5) is saturated, i.e.,

$$M = |Q|, \tag{2.6}$$

the two horizons coincide with $r_\pm = M$. This is the particular case of the RN spacetime, the Extremal RN (ERN) spacetime with metric

$$\mathrm{d}s^2 = -\left(1 - \frac{M}{r}\right)^2 \mathrm{d}t^2 + \frac{1}{\left(1 - \frac{M}{r}\right)^2}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{2.7}$$

The ERN has a single horizon at $r_s = M$, and its near horizon geometry ($r \approx M + \rho$) turns out to be

$$\mathrm{d}s^2 = -\frac{\rho^2}{M^2}\,\mathrm{d}t^2 + \frac{M^2}{\rho^2}\,\mathrm{d}\rho^2 + M^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{2.8}$$

This is the metric of $\mathrm{AdS}_2\times\mathrm{S}^2$, so that the near horizon geometry is a smooth manifold. It is useful to make the change of coordinates

$$\bar{r} = r - M, \tag{2.9}$$

so that the metric (2.7) is written as

$$\mathrm{d}s^2 = -\frac{1}{\left(1 + \frac{M}{\bar{r}}\right)^2}\,\mathrm{d}t^2 + \left(1 + \frac{M}{\bar{r}}\right)^2\left[\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{2.10}$$

whereas, the horizon now is at

$$\bar{r} = 0. \tag{2.11}$$

Similarly, the electrostatic potential turns out to be

$$\Phi(\bar{r}) = \frac{M}{\bar{r} + M}. \tag{2.12}$$

3 The Axisymmetric Extremal RN BH

The metric in Eq. (2.10) can be generalized to the general form [39]

$$\mathrm{d}s^2 = -\frac{1}{(1+\psi)^2}\,\mathrm{d}t^2 + (1+\psi)^2\left[\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{3.1}$$

and

$$\Phi = \frac{\psi}{1+\psi}, \tag{3.2}$$

where $\psi = \psi(\vec{r})$. Einstein equations are satisfied if

$$\frac{\nabla^2\psi}{(1+\psi)^5} = 0. \tag{3.3}$$

In other words, if ψ obeys the Laplace equation

$$\nabla^2\psi = 0, \tag{3.4}$$

on the flat three-dimensional Euclidean space, then Einstein equations are automatically satisfied. In addition, Maxwell equations are written as

$$\vec{\nabla}\left[(1+\psi)^2\vec{\nabla}\left(\frac{\psi}{1+\psi}\right)\right] = 0, \tag{3.5}$$

and are automatically satisfied once Eq. (3.4) holds.

The spherically symmetric ERN corresponds to the spherically symmetric solution

$$\psi(\bar{r}) = \frac{M}{\bar{r}}. \tag{3.6}$$

Clearly, there are more general solutions to Eq. (3.4) which are not spherically symmetric but rather axially symmetric. We will refer to such solutions of (3.4) as Axisymmetric Extremal RN (AERN). As we will see, they describe ERN embedded in external gravitational environments and therefore they are useful to study the static TLNs. Such solutions are of the generic form

$$\psi = \frac{M}{\bar{r}} + \sum_{\ell=1}\left(C_\ell\bar{r}^\ell + \frac{B_\ell}{\bar{r}^{\ell+1}}\right)Y_{\ell m}(\theta,\phi). \tag{3.7}$$

Let us notice however, that the metric of the AERN (3.1) is singular at the points where ψ satisfies

$$1+\psi = 0, \tag{3.8}$$

so that, care should be taken such that ψ given in (3.7) does not violate (3.8). For the $\ell = 2$ case we are interested in, we can take for ψ the following

$$\begin{aligned}\psi =& \frac{M}{\bar{r}} + \frac{Q}{\sqrt{a^2+\bar{r}^2+2\,a\,\bar{r}\cos\theta}} + \frac{Q}{\sqrt{a^2+\bar{r}^2-2\,a\,\bar{r}\cos\theta}} - \frac{2Q}{a}\\ &+ \frac{Q}{\sqrt{b^2+\bar{r}^2+2\,b\,\bar{r}\cos\theta}} + \frac{Q}{\sqrt{b^2+\bar{r}^2-2\,b\,\bar{r}\cos\theta}} - \frac{2Q}{r}.\end{aligned} \tag{3.9}$$

This describes the solution for an extremal RN BHs residing at $\bar{r} = 0$ together with four other extremal RN BHs sitting at $\theta = 0$ and

$$z = a,\ -a,\ b\ \text{ and }\ -b. \tag{3.10}$$

The one at $\bar{r}=0$ is the central BH and the others are accompanying BHs. Clearly, ψ is positive, and hence the metric (3.1) is regular everywhere. Eq. (3.9) is written equivalently as

$$\psi(\bar{r},\theta)=\frac{M}{r}+\frac{Q}{a}\sum_{\ell=1}^{\infty}P_{2\ell}(\cos\theta)\left(\frac{\bar{r}}{a}\right)^{2\ell}+\frac{Q}{\bar{r}}\sum_{\ell=1}^{\infty}P_{2\ell}(\cos\theta)\left(\frac{b}{\bar{r}}\right)^{2\ell}. \tag{3.11}$$

It is easy then to see that in the limit

$$a\to\infty,\qquad b\to 0, \tag{3.12}$$

such that

$$\begin{aligned}\frac{Q}{a^3}&=C_2=\text{finite},\\ Q\,b^2&=B_2=\text{finite},\end{aligned} \tag{3.13}$$

we get that ψ reduces to

$$\psi=\frac{M}{\bar{r}}+\left(C_2\bar{r}^2+\frac{B_2}{\bar{r}^3}\right)P_2(\cos\theta). \tag{3.14}$$

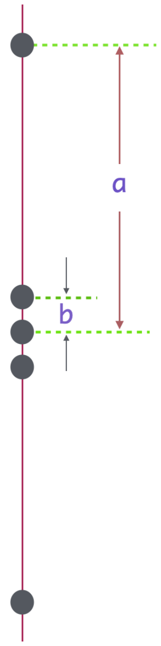

Figure 1. The BH configuration of Eq. (3.9).

The situation is completely analogous of a conducting sphere in an external constant electric field, where the later is generated by similarly by an appropriate limit of two charges at a distance, as in Eqs. (3.12) and (3.13). In what follows we regard Eq. (3.14) as a regulated limit of a smooth geometry that realizes the configuration (3.14), so no singularity issue arises. Furthermore, (3.14) should not be interpreted as an isolated RN black hole. It is the appropriate scaling limit of a system consisting of two widely separated black holes together with a central cluster of black holes.

4 The AERN as tidal perturbation of ERN

For a small enough C_ℓ, B_ℓ so that $C_\ell \ll r/(r-M)^{\ell+1}$ and $B_\ell \ll r(r-M)^\ell$, the metric (3.1) becomes

$$\begin{aligned}\mathrm{d}s^2=&-\left(1-\frac{M}{r}\right)^2\left[1-2\left(\frac{B_\ell}{r^{\ell+1}\left(1-\frac{M}{r}\right)^\ell}+C_\ell r^\ell\left(1-\frac{M}{r}\right)^{\ell+1}\right)\mathrm{P}_\ell(\cos\theta)\right]\mathrm{d}t^2\\ &+\left[1+2\left(\frac{B_\ell}{r^{\ell+1}\left(1-\frac{M}{r}\right)^\ell}+C_\ell r^\ell\left(1-\frac{M}{r}\right)^{\ell+1}\right)\mathrm{P}_\ell(\cos\theta)\right]\left[\frac{\mathrm{d}r^2}{\left(1-\frac{M}{r}\right)^2}+r^2\mathrm{d}\Omega^2\right]\end{aligned} \tag{4.1}$$

Similarly, the electrostatic potential becomes:

$$\Phi=\frac{\psi}{1+\psi}=\underbrace{\frac{\psi_0}{1+\psi_0}}_{\Phi_0}+\frac{\delta\psi}{(1+\psi_0)^2}=\Phi_0+\left(\frac{B_\ell}{r^{\ell+1}\left(1-\frac{M}{r}\right)^{\ell-2}}+C_\ell r^\ell\left(1-\frac{M}{r}\right)^{\ell+3}\right)\mathrm{P}_\ell(\cos\theta). \tag{4.2}$$

It can be interpreted as a tidal perturbation of the Papapetrou metric. Notice that the tidal field formally diverges at large distances, where perturbation theory breaks down. In realistic situations, the external tidal field behaves as an increasing source only within a limited spatial region, and it naturally vanishes as one moves toward spatial infinity. Therefore, assuming that the tidal field proportional to C_ℓ is a small perturbation to the dominant M/r term, there is no singularity issue. This means that for $C_\ell = \mathcal{C}_\ell M^{-\ell}$, we have $r \ll M/\mathcal{C}_\ell^{1/\ell}$ and the ψ can be regarded as an approximation of an exact solution as we have already noticed.

However, one should not look simply at the metric but instead at the Zerilli-Moncrief function, which we will construct in a moment. Employing now the usual parametrization of $\delta g_{\mu\nu}$ as

$$\delta g_{\mu\nu} = \left(f\, H_0, \frac{H_2}{f}, r^2 K, r^2 \sin^2\theta K \right) P_\ell(\cos\theta), \tag{4.3}$$

we can read off the metric perturbations of the ERN from Eq. (4.1) as

$$H_0 = H_2 = K = 2\left(\frac{B_\ell}{r^{\ell+1}\left(1-\frac{M}{r}\right)^\ell} + C_\ell r^\ell \left(1-\frac{M}{r}\right)^{\ell+1} \right). \tag{4.4}$$

Similarly, the perturbation of the electrostatic potential $\delta\Phi$ turns out to be

$$\delta\Phi = \left[C_\ell\, r^\ell \left(1-\frac{M}{r}\right)^{\ell+2} + \frac{B_\ell}{r^{\ell+1}\left(1-\frac{M}{r}\right)^{\ell-1}} \right] P_\ell(\cos\theta). \tag{4.5}$$

The electric field is

$$E^i = -\sqrt{-g}\, F^{i0}, \tag{4.6}$$

so that the perturbation of the electrostatic potential $\delta\Phi$ gives rise to a corresponding perturbation of the radial electric field

$$\delta E^r = E \sin\theta P_\ell(\cos\theta), \tag{4.7}$$

where E is given by

$$E = C_\ell\, \ell (r-M)^{\ell+1} - \frac{\ell(\ell+1)B_\ell}{(r-M)^\ell}. \tag{4.8}$$

We can now define the quantities [40–42]

$$\begin{aligned} Z_1 =& \frac{Q_1}{\Lambda}\sqrt{(l-1)(l+2)}, \\ Z_2 =& -2E - \frac{2M}{r}\frac{Q_1}{\Lambda}, \end{aligned} \tag{4.9}$$

where

$$Q_1 = 2re^{-2\lambda}\left[H_2 - (1 + r\partial_r\lambda)K - r\partial_r K\right] + \ell(\ell+1)\, r\, K, \qquad e^\lambda = \left(1-\frac{M}{r}\right)^{-1}, \tag{4.10}$$

and

$$\Lambda = (\ell-1)(\ell+2) + \frac{6M}{r} - \frac{4M^2}{r^2}. \tag{4.11}$$

Then, the Zerilli-Moncrief master functions [40, 43] are

$$\begin{aligned} \Psi_1 =& \sqrt{\frac{\ell-1}{2\ell+1}}Z_1 - \sqrt{\frac{\ell+2}{2\ell+1}}Z_2, \\ \Psi_2 =& \sqrt{\frac{\ell+2}{2\ell+1}}Z_1 + \sqrt{\frac{\ell-1}{2\ell+1}}Z_2. \end{aligned} \tag{4.12}$$

After calculations, these turn out to be

$$\Psi_1 = \frac{2C_\ell\,\ell\sqrt{(2\ell+1)(\ell+2)}}{(\ell+2)r-2M}\; r^{\ell+2}\left(1-\frac{M}{r}\right)^{\ell+1}, \tag{4.13}$$

$$\Psi_2 = \frac{2B_\ell\,(\ell+1)\sqrt{(2\ell+1)(\ell-1)}}{(\ell-1)r+2M}\; r^{-\ell+1}\left(1-\frac{M}{r}\right)^{-\ell}. \tag{4.14}$$

They should satisfy the (time-independent) Zerilli-Moncrief equation [40]

$$-\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}\Psi_i + V_i\Psi_i = 0, \qquad i = 1,2, \tag{4.15}$$

where, for the extreme RN,

$$\frac{\mathrm{d}r}{\mathrm{d}r_*} = \left(1-\frac{M}{r}\right)^2, \tag{4.16}$$

and

$$V_1 = \frac{(r-M)^2}{r^6\Big[(\ell+2)r-2M\Big]^2}\Big[\ell(\ell+1)(\ell+2)^2r^4 - 2(\ell-1)(\ell+2)^2Mr^3 \\ - 12(\ell+2)M^2r^2 + 8(\ell+3)M^3r - 8M^4\Big], \tag{4.17}$$

$$V_2 = \frac{(r-M)^2}{r^6\Big[(\ell-1)r+2M\Big]^2}\Big[-2\left(l^3-3l+2\right)Mr^3 + 8(l-2)M^3r \\ - 12(l-1)M^2r^2 - (l-1)^2l(l+1)r^4 + 8M^4\Big]. \tag{4.18}$$

An equivalent expression for the Zerilli-Moncrief equation is

$$-\left(1-\frac{M}{r}\right)^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\left(1-\frac{M}{r}\right)^2\frac{\mathrm{d}\Psi_{\mathrm{i}}}{\mathrm{d}r}\right) + V_i\Psi_i = 0, \qquad i = 1,2, \tag{4.19}$$

Defining now the static TLNs k^ℓ as the coefficients in the expansion for $r \gg M$ of the linear combination

$$a_\ell\Psi_1 + b_\ell\Psi_2 \sim r^{\ell+1}\left[1+\cdots+k^\ell\left(\frac{1}{r}\right)^{2\ell+1}+\cdots\right], \tag{4.20}$$

for some (to be specified) coefficients a_ℓ, b_ℓ, we obtain

$$a_\ell\Psi_1 + b_\ell\Psi_2 = 2a_\ell C_\ell\ell\sqrt{\frac{2\ell+1}{\ell+2}}r^{\ell+1}\left[1+\frac{b_\ell}{a_\ell}\frac{B_\ell}{C_\ell}\left(\frac{\ell+1}{\ell}\right)\sqrt{\frac{\ell+2}{\ell-1}}r^{-2\ell-1}\right] \tag{4.21}$$

and thus the TLN is

$$k^\ell = \frac{b_\ell}{a_\ell}\frac{B_\ell}{C_\ell}\left(\frac{\ell+1}{\ell}\right)\sqrt{\frac{\ell+2}{\ell-1}}. \tag{4.22}$$

Let us stress at this point that for an isolated RN black hole in an external gravitational field we have $C_\ell \neq 0$ and $B_\ell = 0$. Indeed, for the $\ell = 2$ tidal perturbation described by ψ of

Eq. (3.14), we have that $Qb^2 \to 0$ as $b \to 0$, so that the second line in Eq. (3.14) is missing. In this case, the charge Q sources an external tidal gravitational field proportional to C_2 in the particular limit we considered above. However, because the RN black hole is isolated now, we also have $B_2 = Qb^2 = 0$. Thus, we once again confirm that the static tidal Love number of an isolated extremal RN black hole vanishes, i.e., $k^\ell = 0$.

5 Matching with the EFT

Any object, observed from sufficiently large distances, appears as a point source. Therefore, we may make use of the point-particle effective field theory (EFT) in order to describe the properties of a finite-size object from the point of view of an observed located far away from it. In the EFT approach the tidal Love numbers are defined without ambiguities, which in the case of General Relativity are caused by non-linearities and gauge invariance. More specifically, if we consider the point-particle EFT of a massive, charged, self-gravitating object, the EFT action can be written as [29]

$$S_{EFT} = S_{\text{bulk}} + S_{\text{pp}} + S_{\text{finite-size}} \tag{5.1}$$

where

$$S_{\text{bulk}} = \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu}) \tag{5.2}$$

is the Einstein-Maxwell action describing gravitation and electromagnetism in the bulk,

$$S_{\text{pp}} = \int \mathrm{d}\tau \left(-M \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} + QA_\mu \frac{dx^\mu}{d\tau} \right) \tag{5.3}$$

describes a charged point-particle, and

$$\begin{aligned} S_{\text{finite}-\text{size}} = \sum_{\ell=1}^{\infty} \int \mathrm{d}\tau \Bigg[& \frac{\lambda_\ell^{(E)}}{2\ell!} \left(\partial_{(a_1} \cdots \partial_{a_{\ell-1}} E_{a_\ell)_T}\right)^2 + \frac{\lambda_\ell^{(B)}}{4\ell!} \left(\partial_{(a_1} \cdots \partial_{a_{\ell-1}} B_{a_\ell)_T b}\right)^2 \\ & + \frac{\lambda_\ell^{(C_E)}}{2\ell!} \left(\partial_{(a_1} \cdots \partial_{a_{\ell-2}} E^{(2)}_{a_{\ell-1} a_\ell)_T}\right)^2 + \frac{\lambda_\ell^{(C_B)}}{4\ell!} \left(\partial_{(a_1} \cdots \partial_{a_{\ell-2}} B^{(2)}_{a_{\ell-1} a_\ell)_T|b}\right)^2 \\ & + \frac{\eta_\ell^{(E)}}{\ell!} \left(\partial_{(a_1} \cdots \partial_{a_{\ell-1}} E_{a_\ell)_T}\right) \left(\partial_{(a_1} \cdots \partial_{a_{\ell-2}} E^{(2)}_{a_{\ell-1} a_\ell)_T}\right) \\ & + \frac{\eta_\ell^{(B)}}{2\ell!} \left(\partial_{(a_1} \cdots \partial_{a_{\ell-1}} B_{a_\ell)_T b}\right) \left(\partial_{(a_1} \cdots \partial_{a_{\ell-2}} B^{(2)}_{a_{\ell-1} a_\ell)_T|b}\right) \Bigg] . \end{aligned} \tag{5.4}$$

Here, $(\cdots)_T$ denotes the symmetrized traceless component of the enclosed indices, E_a and B_a are the electric and magnetic field

$$E_a \equiv F_{0a} = \dot{A}_a - \partial_a A_0 \, , \qquad B_{ab} \equiv F_{ab} = \partial_a A_b - \partial_b A_a \, , \tag{5.5}$$

and $E^{(2)}_{ab}, B^{(2)}_{ab}$ are the electric and magnetic components of the Weyl tensor,

$$E^{(2)}_{ab} \equiv C_{0a0b} = -\frac{1}{2} \partial_a \partial_b h_{00} \, , \qquad B^{(2)}_{ab|c} \equiv C_{0abc} = \frac{1}{2} \left(\partial_a \partial_b h_{c0} - \partial_a \partial_c h_{b0}\right) = \partial_a \partial_{[b} h_{c]0} \, . \tag{5.6}$$

The tidal Love numbers are therefore defined to be the coefficients of the effective field theory couplings. There are pure electromagnetic ($\lambda_\ell^{(E)}, \lambda_\ell^{(B)}$), pure gravitational ($\lambda_\ell^{(C_E)}, \lambda_\ell^{(C_B)}$) and mixed gravitational-electromagnetic ($\eta_\ell^{(E)}, \eta_\ell^{(B)}$) Love numbers. Focusing in our case, where the magnetic parts are absent, calculations in the EFT model give the following expressions for the radial electric field and the linearized Weyl tensor:

$$
\begin{aligned}
E_r \propto r^{\ell-1} P_\ell(\cos\theta) \Bigg[\bar{d}^{(E)} \left(1 - (-1)^\ell \frac{2^\ell \sqrt{\pi}(\ell+1)}{\ell\,\Gamma(\frac{1}{2}-\ell)} \lambda_\ell^{(E)} r^{-2\ell-1} \right) \\
- \bar{c}^{(E)} (-1)^\ell \frac{2^{\ell-1}\sqrt{\pi}(\ell+1)}{\ell\,\Gamma(\frac{1}{2}-\ell)} \eta_\ell^{(E)} r^{-2\ell-1} \Bigg], \qquad (5.7)
\end{aligned}
$$

$$
\begin{aligned}
C_{0r0r} \propto r^{\ell-2} P_\ell(\cos\theta) \Bigg[\bar{c}^{(E)} \left(1 + (-1)^\ell \frac{(\ell+1)(\ell+2)}{\ell(\ell-1)} \frac{2^\ell \sqrt{\pi}}{\Gamma(\frac{1}{2}-\ell)} \lambda_\ell^{(C_E)} r^{-2\ell-1} \right) \\
+ \bar{d}^{(E)} (-1)^\ell \frac{(\ell+1)(\ell+2)}{\ell(\ell-1)} \frac{2^{\ell+1}\sqrt{\pi}}{\Gamma(\frac{1}{2}-\ell)} \eta_\ell^{(E)} r^{-2\ell-1} \Bigg]. \qquad (5.8)
\end{aligned}
$$

Given a metric perturbation of the form (4.3) and a electrostatic potential perturbation $\delta\Phi$, the expressions for the radial electric and linearized Weyl tensor are given by

$$
E_r = -\partial_r(\delta\Phi), \qquad (5.9)
$$

$$
C_{0r0r} = -\frac{1}{2}\partial_r^2 \left(\frac{\Delta}{r^2} H_0(r) \right) + \frac{2M}{r^2} \partial_r(\delta\Phi), \qquad (5.10)
$$

where $\Delta = (r-M)^2$ for the extremal RN black hole. If we calculate these, we find that

$$
E_r = r^{-\ell-3} \left(1 - \frac{M}{r} \right)^{-\ell} \left[((\ell+1)r - 2M) B_\ell - r^{2\ell+1} \left(1 - \frac{M}{r} \right)^{2\ell+1} (\ell r + 2M) C_\ell \right], \qquad (5.11)
$$

$$
\begin{aligned}
C_{0r0r} = - r^{-\ell-5} \left(1 - \frac{M}{r} \right)^{-\ell} \Bigg[\left((\ell+1)(\ell+2)r^2 - 2(2\ell+5)Mr + 8M^2 \right) B_\ell \\
+ r^{2\ell+1} \left(1 - \frac{M}{r} \right)^{2\ell+1} (\ell(\ell-1)r^2 + 2(2\ell-3)Mr + 8M^2) C_\ell \Bigg]. \qquad (5.12)
\end{aligned}
$$

After a few computations, we can find that in the limit $M \ll r$ we have

$$
\begin{aligned}
E_r = r^{\ell-1} P_\ell(\cos\theta) \Bigg[(\ell+1) B_\ell \sum_{n=0}^{\infty} \binom{\ell+n-1}{n} \left(\frac{M}{r} \right)^n r^{-2\ell-1} \\
- \ell C_\ell \sum_{k=0}^{\ell+1} \binom{\ell+1}{k} (-1)^k \left(\frac{M}{r} \right)^k \Bigg] \qquad (5.13)
\end{aligned}
$$

Since $M \ll r$ we can approximately keep only the $n = k = 0$ terms and get

$$
E_r \simeq -\ell C_\ell r^{\ell-1} P_\ell(\cos\theta) \left[1 - \frac{(\ell+1)}{l} \frac{B_\ell}{C_\ell} r^{-2\ell-1} \right]. \qquad (5.14)
$$

A similar process for C_{0r0r} yields

$$C_{0r0r} \simeq -(\ell+1)\ell C_\ell r^{\ell-2} P_\ell(\cos\theta)\left[1+\frac{(\ell+2)}{\ell}\frac{B_\ell}{C_\ell}r^{-2\ell-1}\right]. \tag{5.15}$$

These expressions should be compared with eqs. (5.7) and (5.8) in order to extract the TLNs. A quick glance makes it obvious that, while we have four matching conditions, we have five unknowns: $\bar{c}^{(E)}, \bar{d}^{(E)}, \lambda_\ell^{(E)}, \lambda_\ell^{(C_E)}, \eta_\ell^{(E)}$. However, after some algebra, we can get

$$E_r \propto \bar{d}^{(E)} r^{\ell-1} P_\ell(\cos\theta)\left[1+\left(\lambda_\ell^{(E)}+\frac{1}{2}\frac{\bar{c}^{(E)}}{\bar{d}^{(E)}}\eta_\ell^{(E)}\right) f_\ell\, r^{-2\ell-1}\right], \tag{5.16}$$

$$C_{0r0r} \propto \bar{c}^{(E)} r^{\ell-2} P_\ell(\cos\theta)\left[1+\left(\lambda^{(C_E)}+2\frac{\bar{d}^{(E)}}{\bar{c}^{(E)}}\eta_\ell^{(E)}\right) q_\ell\, r^{-2\ell-1}\right], \tag{5.17}$$

where

$$f_\ell = (-1)^{\ell+1}\frac{(\ell+1)2^\ell\sqrt{\pi}}{\ell\Gamma\left(\frac{1}{2}-\ell\right)}, \tag{5.18}$$

$$q_\ell = (-1)^\ell\left(\frac{\ell+2}{\ell-1}\right)\frac{(\ell+1)2^\ell\sqrt{\pi}}{\ell\Gamma\left(\frac{1}{2}-\ell\right)} = \left(\frac{\ell+2}{\ell-1}\right) f_\ell. \tag{5.19}$$

If we now define

$$\bar{\lambda}_\ell^{(E)} = \left(\lambda_\ell^{(E)}+\frac{1}{2}\frac{\bar{c}^{(E)}}{\bar{d}^{(E)}}\eta_\ell^{(E)}\right) f_\ell, \tag{5.20}$$

$$\bar{\lambda}_\ell^{(C_E)} = \left(\lambda_\ell^{(C_E)}+2\frac{\bar{d}^{(E)}}{\bar{c}^{(E)}}\eta_\ell^{(E)}\right) q_\ell, \tag{5.21}$$

we obtain

$$E_r \propto \bar{d}^{(E)} r^{\ell-1} P_\ell(\cos\theta)\left[1+\bar{\lambda}_\ell^{(E)} r^{-2\ell-1}\right], \tag{5.22}$$

$$C_{0r0r} \propto \bar{c}^{(E)} r^{\ell-2} P_\ell(\cos\theta)\left[1+\bar{\lambda}_\ell^{(C_E)} r^{-2\ell-1}\right]. \tag{5.23}$$

Therefore, while it is impossible to determine $\lambda_\ell^{(E)}, \lambda_\ell^{(C_E)}, \eta_\ell^{(E)}$ by matching, it is possible to determine $\bar{\lambda}_\ell^{(E)}$ and $\bar{\lambda}_\ell^{(C_E)}$, i.e. the "shifted" TLNs by the gravitational-electromagnetic mixing. Matching with our expressions (5.14) & (5.15) yields

$$\bar{\lambda}_\ell^{(E)} = -\frac{(\ell+1)}{\ell}\frac{B_\ell}{C_\ell}, \tag{5.24}$$

$$\bar{\lambda}_\ell^{(C_E)} = \frac{(\ell+2)}{\ell}\frac{B_\ell}{C_\ell}. \tag{5.25}$$

Now, all that is left is to equate the two expressions (4.22) & (5.25) (i.e. $k^\ell = \bar{\lambda}_\ell^{(C_E)}$) for the TLN in order to find the appropriate linear combination of Ψ_1, Ψ_2. We find that

$$b_\ell = \frac{\sqrt{(\ell-1)(\ell+2)}}{\ell+1} a_\ell. \tag{5.26}$$

Therefore, the appropriate linear combination is

$$\Psi = \Psi_1 + \frac{\sqrt{(\ell-1)(\ell+2)}}{\ell+1}\Psi_2. \tag{5.27}$$

6 Conclusions

Our construction isolates two irreducible pieces of the static, axisymmetric response of an ERN BH embedded in a distant, tidal environment. The coefficients C_ℓ and B_ℓ play conjugate roles: C_ℓ multiplies the growing, far–zone solution that represents the applied tidal field of the remote companions, while B_ℓ multiplies the decaying, near–zone solution that encodes the induced multipole. In gauge–invariant language this separation is realized by the Zerilli–Moncrief master fields, with $\Psi_1 \propto C_\ell$ and $\Psi_2 \propto B_\ell$. Horizon regularity together with asymptotic matching selects a unique observable combination (5.27) whose far–zone behavior determines the measurable response. The entire static polarizability is then governed by one dimensionless ratio:

$$k^\ell \; = \; \bar{\lambda}_\ell^{(C_E)} \; = \; \frac{\ell+2}{\ell}\,\frac{B_\ell}{C_\ell}, \qquad (r \gg M)\,, \tag{6.1}$$

in exact agreement between the wave–mechanics (Zerilli–Moncrief) and point–particle EFT descriptions derived in the main text.

In addition, within a transparent perturbative regime (small B_ℓ, C_ℓ and $r \ll M/\mathcal{C}_\ell^{1/\ell}$), we established a one–parameter description of the static polarizability of an ERN black hole under a geometrically realized, multi–source tide. Isolated black holes retain vanishing static TLNs. Tidally coupled, multi–source configurations behave as polarizable objects with an effective response encoded in B_ℓ/C_ℓ and captured identically by the Zerilli–EFT matching.

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References

[1] LIGO Scientific, VIRGO, KAGRA collaboration, R. Abbott et al., *Tests of General Relativity with GWTC-3*, 2112.06861.

[2] A. Kehagias and A. Riotto, *Can We Detect Deviations from Einstein's Gravity in Black Hole Ringdowns?*, 2411.12428.

[3] E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*. Cambridge University Press, 2014, 10.1017/CBO9781139507486.

[4] E. E. Flanagan and T. Hinderer, *Constraining neutron star tidal Love numbers with gravitational wave detectors*, *Phys. Rev. D* **77** (2008) 021502, [0709.1915].

[5] T. Binnington and E. Poisson, *Relativistic theory of tidal Love numbers*, *Phys. Rev. D* **80** (2009) 084018, [0906.1366].

[6] T. Damour and A. Nagar, *Relativistic tidal properties of neutron stars*, *Phys. Rev. D* **80** (2009) 084035, [0906.0096].

[7] T. Damour and O. M. Lecian, *On the gravitational polarizability of black holes*, *Phys. Rev. D* **80** (2009) 044017, [0906.3003].

[8] P. Pani, L. Gualtieri, A. Maselli and V. Ferrari, *Tidal deformations of a spinning compact object*, *Phys. Rev. D* **92** (2015) 024010, [1503.07365].

[9] P. Pani, L. Gualtieri and V. Ferrari, *Tidal Love numbers of a slowly spinning neutron star*, *Phys. Rev. D* **92** (2015) 124003, [1509.02171].

[10] R. A. Porto, *The Tune of Love and the Nature(ness) of Spacetime*, *Fortsch. Phys.* **64** (2016) 723–729, [1606.08895].

[11] A. Le Tiec and M. Casals, *Spinning Black Holes Fall in Love*, *Phys. Rev. Lett.* **126** (2021) 131102, [2007.00214].

[12] H. S. Chia, *Tidal deformation and dissipation of rotating black holes*, *Phys. Rev. D* **104** (2021) 024013, [2010.07300].

[13] A. Le Tiec, M. Casals and E. Franzin, *Tidal Love Numbers of Kerr Black Holes*, *Phys. Rev. D* **103** (2021) 084021, [2010.15795].

[14] E. Poisson, *Tidally induced multipole moments of a nonrotating black hole vanish to all post-Newtonian orders*, *Phys. Rev. D* **104** (2021) 104062, [2108.07328].

[15] A. Kehagias, D. Perrone and A. Riotto, *A Short Note on the Love Number of Extremal Reissner-Nordstrom and Kerr-Newman Black Holes*, 2406.19262.

[16] L. Hui, A. Joyce, R. Penco, L. Santoni and A. R. Solomon, *Static response and Love numbers of Schwarzschild black holes*, *JCAP* **04** (2021) 052, [2010.00593].

[17] P. Charalambous, S. Dubovsky and M. M. Ivanov, *On the Vanishing of Love Numbers for Kerr Black Holes*, *JHEP* **05** (2021) 038, [2102.08917].

[18] P. Charalambous, S. Dubovsky and M. M. Ivanov, *Hidden Symmetry of Vanishing Love Numbers*, *Phys. Rev. Lett.* **127** (2021) 101101, [2103.01234].

[19] L. Hui, A. Joyce, R. Penco, L. Santoni and A. R. Solomon, *Ladder symmetries of black holes. Implications for love numbers and no-hair theorems*, *JCAP* **01** (2022) 032, [2105.01069].

[20] L. Hui, A. Joyce, R. Penco, L. Santoni and A. R. Solomon, *Near-zone symmetries of Kerr black holes*, *JHEP* **09** (2022) 049, [2203.08832].

[21] P. Charalambous, S. Dubovsky and M. M. Ivanov, *Love symmetry*, *JHEP* **10** (2022) 175, [2209.02091].

[22] M. M. Ivanov and Z. Zhou, *Vanishing of Black Hole Tidal Love Numbers from Scattering Amplitudes*, *Phys. Rev. Lett.* **130** (2023) 091403, [2209.14324].

[23] T. Katagiri, M. Kimura, H. Nakano and K. Omukai, *Vanishing Love numbers of black holes in general relativity: From spacetime conformal symmetry of a two-dimensional reduced geometry*, *Phys. Rev. D* **107** (2023) 124030, [2209.10469].

[24] G. Bonelli, C. Iossa, D. P. Lichtig and A. Tanzini, *Exact solution of Kerr black hole perturbations via CFT2 and instanton counting: Greybody factor, quasinormal modes, and Love numbers*, *Phys. Rev. D* **105** (2022) 044047, [2105.04483].

[25] A. Kehagias, D. Perrone and A. Riotto, *Quasinormal modes and Love numbers of Kerr black holes from AdS_2 black holes*, *JCAP* **01** (2023) 035, [2211.02384].

[26] J. Ben Achour, E. R. Livine, S. Mukohyama and J.-P. Uzan, *Hidden symmetry of the static response of black holes: applications to Love numbers*, *JHEP* **07** (2022) 112, [2202.12828].

[27] R. Berens, L. Hui and Z. Sun, *Ladder symmetries of black holes and de Sitter space: love numbers and quasinormal modes*, *JCAP* **06** (2023) 056, [2212.09367].

[28] V. De Luca, J. Khoury and S. S. C. Wong, *Nonlinearities in the tidal Love numbers of black holes*, *Phys. Rev. D* **108** (2023) 024048, [2305.14444].

[29] M. Rai and L. Santoni, *Ladder Symmetries and Love Numbers of Reissner–Nordström Black Holes*, 2404.06544.

[30] M. M. Riva, L. Santoni, N. Savić and F. Vernizzi, *Vanishing of nonlinear tidal Love numbers of Schwarzschild black holes*, *Phys. Lett. B* **854** (2024) 138710, [2312.05065].

[31] M. M. Riva, L. Santoni, N. Savić and F. Vernizzi, *Vanishing of Quadratic Love Numbers of Schwarzschild Black Holes*, 2410.03542.

[32] F. J. Ernst, *New formulation of the axially symmetric gravitational field problem*, *Phys. Rev.* **167** (1968) 1175–1179.

[33] A. Kehagias and A. Riotto, *Black Holes in a Gravitational Field: The Non-linear Static Love Number of Schwarzschild Black Holes Vanishes*, 2410.11014.

[34] O. Combaluzier-Szteinsznaider, L. Hui, L. Santoni, A. R. Solomon and S. S. C. Wong, *Symmetries of Vanishing Nonlinear Love Numbers of Schwarzschild Black Holes*, 2410.10952.

[35] L. R. Gounis, A. Kehagias and A. Riotto, *The vanishing of the non-linear static love number of Kerr black holes and the role of symmetries*, *JCAP* **03** (2025) 002, [2412.08249].

[36] K. D. Kokkotas and B. F. Schutz, *Black Hole Normal Modes: A WKB Approach. 3. The Reissner-Nordstrom Black Hole*, *Phys. Rev. D* **37** (1988) 3378–3387.

[37] E. Berti and K. D. Kokkotas, *Quasinormal modes of Reissner-Nordström-anti-de Sitter black holes: Scalar, electromagnetic and gravitational perturbations*, *Phys. Rev. D* **67** (2003) 064020, [gr-qc/0301052].

[38] E. Berti and K. D. Kokkotas, *Asymptotic quasinormal modes of Reissner-Nordstrom and Kerr black holes*, *Phys. Rev. D* **68** (2003) 044027, [hep-th/0303029].

[39] A. Papapetrou, *A Static Solution of the Equations of the Gravitational Field for an Arbitary ChargeDistribution*, *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences* , **51** (1945 - 1948) 191–204.

[40] V. Moncrief, *Stability of Reissner-Nordstrom black holes*, *Phys. Rev. D* **10** (1974) 1057–1059.

[41] S. Chandrasekhar, *On the equations governing the perturbations of the Reissner-Nordström Black Hole*, *Proc. Roy. Soc. Lond. A* **365** (1979) 453–465.

[42] B. C. Xanthopoulos and S. Chandrasekhar, *Metric and electromagnetic perturbations of the Reissner-Nordström black hole,*, *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences,* **378** (1981) 73–88.

[43] F. J. Zerilli, *Effective potential for even parity Regge-Wheeler gravitational perturbation equations*, *Phys. Rev. Lett.* **24** (1970) 737–738.