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Vacuum Structure of the BNT Model of Neutrino Mass Generation

Saiyad Ashanujjaman[a,b] **Siddharth P. Maharathy**[c,d,e]

[a] *Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology, Engesserstraße 7, D-76128 Karlsruhe, Germany*

[b] *Institut für Astroteilchenphysik, Karlsruhe Institute of Technology, Hermann-von-Helmholtz-Platz 1, D-76344 Eggenstein-Leopoldshafen, Germany*

[c] *School of Physics and Institute for Collider Particle Physics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa*

[d] *Indian Institute of Science Education and Research Pune, Dr. Homi Bhabha Road, Pune 411008, India*

[e] *Institute of Physics, Bhubaneswar, Sachivalaya Marg, Sainik School, Bhubaneswar 751005, India*

E-mails: saiyad.ashanujjaman@kit.edu, siddharth.prasad.maharathy@cern.ch

Abstract: We analyze the vacuum structure of the Babu–Nandi–Tavartkiladze (BNT) model of neutrino mass generation, in which the Standard Model is extended by an $SU(2)_L$ scalar quadruplet with hypercharge $Y = 3/2$ and a vector-like $SU(2)_L$ triplet fermion with $Y = 1$, generating neutrino masses via an effective dimension-seven operator. We delineate the theoretical constraints on the model, requiring the scalar potential to be bounded from below in all field directions, ensuring perturbative unitarity of scattering amplitudes, and demanding that the electroweak vacuum corresponds to the global minimum of the potential. We find that the electroweak vacuum is not generically guaranteed to be the global minimum: several charge-breaking stationary points may coexist with—and potentially lie below—it in potential depth. For the electroweak-like vacuum with vanishing quadruplet expectation value, the condition of global stability reduces to two simple mass inequalities involving the doubly- and triply-charged scalars. In contrast, for the general electroweak vacuum with nonzero doublet and quadruplet expectation values—compatible with neutrino-mass generation—no comparably simple analytic condition emerges, and the stability must be assessed for each specific choice of scalar couplings. Nonetheless, by combining the analytic condition governing the relative depth of the two electroweak stationary points with the mass inequalities ensuring the stability of the electroweak-like configuration against charge breaking, we obtain a practical criterion for determining when the general electroweak vacuum is globally stable.

Contents

1 Introduction

Among several observational and theoretical lacunae of the Standard Model (SM), of particular concern is the issue of neutrino mass. A well-founded remedy to this shortcoming is offered by the dimension-five Weinberg-operator-induced seesaw mechanism [1, 2], wherein lepton-number-violating new physics (NP) beyond the SM is invoked at an a priori unknown scale, so that, on integrating out the heavy NP fields, the SM neutrinos are left with observed sub-eV masses after electroweak (EW) symmetry breaking. In canonical type-I [3–7] and type-III [8] seesaw models, neutrino masses consistent with data require either an NP scale $\Lambda_{\rm NP}$ close to the grand unification scale for $\mathcal{O}(1)$ Yukawa couplings, or extremely small Yukawa couplings, $\mathcal{O}(10^{-6})$, if $\Lambda_{\rm NP}$ lies around the TeV scale. In the absence of further suppressions (for example, loop suppression or other additional mechanisms), scenarios aiming at collider-accessible NP ($\Lambda_{\rm NP} \sim$ TeV) with sizable Yukawas typically invoke neutrino mass generation via higher-dimensional operators at tree level [9–15].

A well-known realization is the model proposed by Babu, Nandi, and Tavartkiladze, hereinafter dubbed the *BNT model* [16], where an effective dimension-seven operator induces neutrino masses. The model extends the SM with an $SU(2)_L$ scalar quadruplet of hypercharge $Y = 3/2$ and a vectorlike $SU(2)_L$ triplet fermion with $Y = 1$. Since its proposal in 2009, this model has been explored in various phenomenological contexts, such as lepton flavor violation [17], interpretations of the 750 GeV diphoton excess [18], collider signatures [19–23], and long-lived particle searches [24].

In this work, we investigate the vacuum structure of the BNT model. We delineate the theoretical constraints on its scalar potential, requiring it to be bounded from below in all field directions, perturbative unitarity of scattering amplitudes, and the stability of the electroweak (EW) vacuum as the global minimum. In the BNT framework, neutrino masses arise from the dimension-seven operator generated by the $\lambda_5\,\Delta^\dagger\Phi^3$ interaction. Once the Higgs doublet acquires a VEV, this term induces a tadpole for the neutral component of the scalar quadruplet, implying that $\lambda_5 \neq 0$ necessarily leads to a nonvanishing quadruplet VEV. As a consequence, only the general electroweak vacuum in which both the doublet and quadruplet develop VEVs is compatible with neutrino-mass generation. The electroweak-like configuration with a vanishing quadruplet VEV exists only for $\lambda_5 = 0$ and cannot generate neutrino masses. This observation directly motivates the detailed vacuum-stability analysis presented here and, in particular, the conditions under which the physically relevant electroweak vacuum constitutes the global minimum of the scalar potential.

The rest of the paper is organized as follows. In section 2, we briefly present the scalar sector of the model. The conditions for the potential to be bounded from below and for perturbative unitarity of scattering amplitudes are discussed in section 3. In section 3.3 we present our analysis of the vacuum structure. Finally, we summarize our findings in section 4.

2 The BNT Model

The extended scalar sector of the BNT model is composed of a SM Higgs doublet Φ with $Y = 1/2$ and a Higgs quadruplet Δ with $Y = 3/2$:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} \delta^{+++} \\ \delta^{++} \\ \delta^{+} \\ \delta^{0} \end{pmatrix} . \tag{2.1}$$

The kinetic part of the Higgs sector Lagrangian is given by

$$\mathcal{L}_{\rm kin} = \left|D_\mu\Phi\right|^2 + \left|D_\mu\Delta\right|^2 , \tag{2.2}$$

with $D_\mu\Phi = \partial_\mu\Phi - ig\tau^k W_\mu^k\Phi - i\frac{g'}{2}B_\mu\Phi$ and $D_\mu\Delta = \partial_\mu\Delta - igT^k W_\mu^k\Delta - i\frac{3g'}{2}B_\mu\Delta$, where W_μ^k and B_μ are the $SU(2)_L$ and $U(1)_Y$ gauge fields, and g and g' are the corresponding couplings; τ^k and T^k are, respectively, the generators of 2 and 4 representation of $SU(2)$, where $\tau^k = \sigma^k/2$ with σ^k being the Pauli matrices.[1]

[1]The generators of 4 represenation of $SU(2)$ are not uniquely determined, however, they must satisfy the commutation relations: $[T^k, T^l] = i\epsilon_{klm}T^m$. In a basis with diagonal third generator

The most general renormalizable and gauge-invariant scalar potential can be written as[2]

$$V(\Phi,\Delta) = -\,\mu_\Phi^2\Phi^\dagger\Phi + \mu_\Delta^2\Delta^\dagger\Delta + \lambda_1(\Phi^\dagger\Phi)^2 + \lambda_2(\Delta^\dagger\Delta)^2 + \tilde{\lambda}_2(\Delta^\dagger T^k\Delta)^2 \\ + \lambda_3(\Phi^\dagger\Phi)(\Delta^\dagger\Delta) + \lambda_4(\Phi^\dagger\tau^k\Phi)(\Delta^\dagger T^k\Delta) + (\lambda_5\Delta^\dagger\Phi^3 + \text{h.c.}), \tag{2.4}$$

where μ_Φ^2 and μ_Δ^2 are the mass-squared parameters, and λ_i $(i = 1,...,5)$ and $\tilde{\lambda}_2$ are the independent dimensionless couplings.[3] Hermiticity of the potential allows us to assume that all couplings, except for λ_5, are real. Though λ_5 can pick up a *would-be* CP phase, this phase is unphysical and can always be absorbed through appropriate field redefinitions:

$$\lambda_5 \to \lambda_5 e^{i\omega}, \quad \Phi \to \Phi e^{i\omega'} \quad \text{and} \quad \Delta \to \Delta e^{i(\omega+3\omega')}. \tag{2.5}$$

Therefore, without loss of generality, we also assume λ_5 to be real. Consequently, the potential $V(\Phi,\Delta)$ in Eq. (2.4) is CP conserving. The neutral components of Φ and Δ can be parametrised as

$$\phi^0 = \frac{v_\Phi + \phi_R^0 + i\phi_I^0}{\sqrt{2}} \quad \text{and} \quad \delta^0 = \frac{v_\Delta + \delta_R^0 + i\delta_I^0}{\sqrt{2}}, \tag{2.6}$$

(often referred as *spherical basis*), the corresponding generators are given by

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad T^2 = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \tag{2.3}$$

[2]One can also write Φ and Δ as symmetric rank-1 and rank-3 tensors, respectively, as follows:

$$\Phi^1 = \phi^+, \; \Phi^0 = \phi^0, \; \Phi_i^* = (\Phi^i)^*,$$
$$\Delta^{111} = \Delta^{+++}, \; \Delta^{112} = \frac{\Delta^{++}}{\sqrt{3}}, \; \Delta^{122} = \frac{\Delta^+}{\sqrt{3}}, \; \Delta^{222} = \Delta^0, \; \Delta_{ijk}^* = (\Delta^{ijk})^*.$$

In this tensor notation, various terms in Eq. (2.4) read as

$$\begin{aligned} &\Phi^\dagger\Phi = \Phi_i^*\Phi^i \\ &\Delta^\dagger\Delta = \Delta_{ijk}^*\Delta^{ijk} \\ &4(\Delta^\dagger T^a\Delta)^2 = 18\Delta_{ijk}^*\Delta^{ijp}\Delta_{mnp}^*\Delta^{mnk} - 9(\Delta_{ijk}^*\Delta^{ijk})^2 \\ &4(\Phi^\dagger\tau^a\Phi)(\Delta^\dagger T^a\Delta) = 6\Phi_i^*\Delta^{ijk}\Delta_{jkl}^*\Phi^l - 3\Phi_i^*\Phi^i\Delta_{ijk}^*\Delta^{ijk} \\ &\Delta^\dagger\Phi^3 = \Delta_{ijk}^*\Phi^i\Phi^j\Phi^k \end{aligned}$$

[3]We note in passing that the $\tilde{\lambda}_2$ term was absent in the original formulation of the model [16] and consequently also omitted in several subsequent studies [18, 19, 21–23]. This omission was, however, first pointed out in Ref. [17].

where $v/\sqrt{2}$ and $v_\Delta/\sqrt{2}$, respectively, are the vacuum expectation values (VEVs) acquired by ϕ^0 and δ^0, respectively, after EW symmetry breaking, such that

$$\sqrt{v_\Phi^2 + 3v_\Delta^2} = 246\,\mathrm{GeV}. \tag{2.7}$$

Minimizing the potential $V(\Phi, \Delta)$, we obtain

$$\mu_\Phi^2 = \lambda_1 v_\Phi^2 + \frac{\lambda_3}{2} v_\Delta^2 + \frac{3}{8}\lambda_4 v_\Delta^2 + \frac{3}{2}\lambda_5 v_\Phi v_\Delta, \tag{2.8}$$

$$\mu_\Delta^2 = -(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2) v_\Delta^2 - \frac{\lambda_3}{2} v_\Phi^2 - \frac{3}{8}\lambda_4 v_\Phi^2 - \frac{\lambda_5}{2}\frac{v_\Phi^3}{v_\Delta}. \tag{2.9}$$

It is worth emphasizing a structural feature of the scalar potential: the quartic interaction $\lambda_5 \Delta^\dagger \Phi^3$ induces a linear term in the neutral quadruplet field once the doublet acquires a VEV. Consequently, unless $\lambda_5 = 0$, the minimization condition in Eq. (2.9) necessarily enforces a nonvanishing quadruplet VEV v_Δ. In other words, for generic parameter choices with $\lambda_5 \neq 0$—precisely the regime relevant for neutrino-mass generation (see the discussion at the end of this section)—the scalar potential does not admit stationary configurations with $v_\Delta = 0$.

After EW symmetry breaking, the degrees of freedom with identical electric charges mix, giving rise to several physical Higgs states:

(i) the neutral states ϕ_R^0 and δ_R^0 (ϕ_I^0 and δ_I^0) mix into two CP-even (CP-odd) states h and H (G and A),

(ii) the singly-charged states $\phi^\pm$ and $\delta^\pm$ mix into two mass states $G^\pm$ and $H^\pm$,

(iii) the doubly- and triply-charged states $\delta^{\pm\pm}$ and $\delta^{\pm\pm\pm}$ align with their mass states $H^{\pm\pm}$ and $H^{\pm\pm\pm}$, respectively;

$$\begin{pmatrix} h \\ H \end{pmatrix} = R_\alpha \begin{pmatrix} \phi_R^0 \\ \delta_R^0 \end{pmatrix}, \quad \begin{pmatrix} G \\ A \end{pmatrix} = R_\beta \begin{pmatrix} \phi_I^0 \\ \delta_I^0 \end{pmatrix}, \quad \begin{pmatrix} G^\pm \\ H^\pm \end{pmatrix} = R_\gamma \begin{pmatrix} \phi^\pm \\ \delta^\pm \end{pmatrix} \text{ with } R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and

$$R_\alpha^T m_R^2 R_\alpha = \mathrm{diag}\left(m_h^2, m_H^2\right), \; R_\beta^T m_I^2 R_\beta = \mathrm{diag}\left(0, m_A^2\right), \; R_\gamma^T m_\pm^2 R_\gamma = \mathrm{diag}\left(0, m_{H^\pm}^2\right),$$

where m_R^2, m_I^2 and $m_\pm^2$, respectively, are the mass matrices for CP-even, CP-odd and singly-charged Higgses:

$$m_R^2 = \begin{pmatrix} 2\lambda_1 v_\Phi^2 + \frac{3}{2}\lambda_5 v_\Phi v_\Delta & \lambda_3 v_\Phi v_\Delta + \frac{3}{4}\lambda_4 v_\Phi v_\Delta + \frac{3}{2}\lambda_5 v_\Phi^2 \\ \lambda_3 v_\Phi v_\Delta + \frac{3}{4}\lambda_4 v_\Phi v_\Delta + \frac{3}{2}\lambda_5 v_\Phi^2 & 2(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2) v_\Delta^2 - \frac{\lambda_5 v_\Phi^3}{2 v_\Delta} \end{pmatrix}, \tag{2.10}$$

$$m_I^2 = \frac{1}{2}\lambda_5 v \begin{pmatrix} -9 v_\Delta & 3 v_\Phi \\ 3 v_\Phi & -\frac{v_\Phi^2}{v_\Delta} \end{pmatrix}, \tag{2.11}$$

$$m_{\pm}^2 = (\lambda_4 v_\Delta + 2\lambda_5 v_\Phi)\begin{pmatrix} -\frac{3}{4}v_\Delta & \frac{\sqrt{3}}{4}v_\Phi \\ \frac{\sqrt{3}}{4}v_\Phi & -\frac{v_\Phi^2}{4v_\Delta}\end{pmatrix}, \tag{2.12}$$

and α, β and γ are the respective mixing angles:

$$\tan 2\alpha = \frac{2\lambda_3 v_\Phi v_\Delta + \frac{3}{2}\lambda_4 v_\Phi v_\Delta + 3\lambda_5 v_\Phi^2}{2\lambda_1 v_\Phi^2 - 2(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2)v_\Delta^2 + \frac{3}{2}\lambda_5 v_\Phi v_\Delta \left(1 + \frac{v_\Phi^2}{3v_\Delta^2}\right)} \tag{2.13}$$

$$\tan 2\beta = \frac{6v_\Delta}{v_\Phi}\left(1 - \frac{9v_\Delta^2}{v_\Phi^2}\right)^{-1} \tag{2.14}$$

$$\tan 2\gamma = \frac{2\sqrt{3}v_\Delta}{v_\Phi}\left(1 - \frac{3v_\Delta^2}{v_\Phi^2}\right)^{-1}. \tag{2.15}$$

The mass states G^0 and $G^\pm$ are the so-called *would-be* Nambu-Goldstone bosons eaten by the longitudinal modes of Z and $W^\pm$, and the rest are massive with h being the 125 GeV Higgs observed at the LHC. The physical Higgs masses are given by

$$m_h^2 = \left(2\lambda_1 v_\Phi^2 + \frac{3}{2}\lambda_5 v_\Phi v_\Delta\right) + \left(\lambda_3 v_\Phi v_\Delta + \frac{3}{4}\lambda_4 v_\Phi v_\Delta + \frac{3}{2}\lambda_5 v_\Phi^2\right)\tan\alpha \tag{2.16a}$$

$$= 2(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2)v_\Delta^2 - \frac{\lambda_5 v_\Phi^3}{2v_\Delta} + \left(\lambda_3 v_\Phi v_\Delta + \frac{3}{4}\lambda_4 v_\Phi v_\Delta + \frac{3}{2}\lambda_5 v_\Phi^2\right)\cot\alpha \tag{2.16b}$$

$$m_H^2 = \left(2\lambda_1 v_\Phi^2 + \frac{3}{2}\lambda_5 v_\Phi v_\Delta\right) - \left(\lambda_3 v_\Phi v_\Delta + \frac{3}{4}\lambda_4 v_\Phi v_\Delta + \frac{3}{2}\lambda_5 v_\Phi^2\right)\cot\alpha \tag{2.17a}$$

$$= 2(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2)v_\Delta^2 - \frac{\lambda_5 v_\Phi^3}{2v_\Delta} - \left(\lambda_3 v_\Phi v_\Delta + \frac{3}{4}\lambda_4 v_\Phi v_\Delta + \frac{3}{2}\lambda_5 v_\Phi^2\right)\tan\alpha \tag{2.17b}$$

$$m_A^2 = -\frac{\lambda_5 v_\Phi^3}{2v_\Delta}\left(1 + \frac{9v_\Delta^2}{v_\Phi^2}\right) \tag{2.18}$$

$$m_{H^\pm}^2 = -\left(\frac{\lambda_5 v_\Phi^3}{2v_\Delta} + \frac{\lambda_4 v_\Phi^2}{4}\right)\left(1 + \frac{3v_\Delta^2}{v_\Phi^2}\right) \tag{2.19}$$

$$m_{H^{\pm\pm}}^2 = -\left(\frac{\lambda_5 v_\Phi^3}{2v_\Delta} + \frac{\lambda_4 v_\Phi^2}{2} + 3\tilde{\lambda}_2 v_\Delta^2\right) \tag{2.20}$$

$$m_{H^{\pm\pm\pm}}^2 = -\left(\frac{\lambda_5 v_\Phi^3}{2v_\Delta} + \frac{3\lambda_4 v_\Phi^2}{4} + \frac{9\tilde{\lambda}_2 v_\Delta^2}{2}\right) \tag{2.21}$$

Therefore, we have six free scalar parameters (excluding m_h and v) in our model: v_Δ, α, m_H, m_A, $m_{H^\pm}$ and $m_{H^{\pm\pm}}$ or $m_{H^{\pm\pm\pm}}$. As such, the scalar parameters can be traded with these free parameters as follows:

$$\lambda_1 = \frac{1}{2v_\Phi^2}\left[m_h^2\cos^2\alpha + m_H^2\sin^2\alpha + \frac{3v_\Delta^2}{v_\Phi^2+9v_\Delta^2}m_A^2\right], \tag{2.22}$$

$$\lambda_2 = \frac{1}{2v_\Delta^2}\left[m_h^2\sin^2\alpha + m_H^2\cos^2\alpha + \frac{1}{2}\frac{v_\Phi^2}{v_\Phi^2+9v_\Delta^2}m_A^2 - \frac{3v_\Phi^2}{v_\Phi^2+3v_\Delta^2}m_{H^\pm}^2 + \frac{3}{2}m_{H^{\pm\pm}}^2\right], \tag{2.23}$$

$$\tilde{\lambda}_2 = -\frac{1}{3v_\Delta^2}\left[m_{H^{\pm\pm}}^2 + \frac{v_\Phi^2}{v_\Phi^2+9v_\Delta^2}m_A^2 - \frac{2v_\Phi^2}{v_\Phi^2+3v_\Delta^2}m_{H^\pm}^2\right], \tag{2.24}$$

$$\lambda_3 = \frac{\sin\alpha\cos\alpha}{v_\Phi v_\Delta}(m_h^2 - m_H^2) - \frac{3}{v_\Phi^2+9v_\Delta^2}m_A^2 + \frac{6}{v_\Phi^2+3v_\Delta^2}m_{H^\pm}^2, \tag{2.25}$$

$$\lambda_4 = \frac{4}{v_\Phi^2+9v_\Delta^2}m_A^2 - \frac{4}{v_\Phi^2+3v_\Delta^2}m_{H^\pm}^2, \tag{2.26}$$

$$\lambda_5 = -\frac{2v_\Delta}{v_\Phi}\frac{1}{v_\Phi^2+9v_\Delta^2}m_A^2. \tag{2.27}$$

The remaining two scalar parameters μ_Φ^2 and μ_Δ^2 are then related to the free parameters through the tadpole conditions in Eq. (2.8) and Eq. (2.9).

Both ϕ^0 and δ^0 contribute to the weak gauge boson's masses at tree level:

$$m_W^2 = \frac{1}{4}g^2(v_\Phi^2 + 3v_\Delta^2) \quad \text{and} \quad m_Z^2 = \frac{1}{4}(g^2 + g'^2)(v_\Phi^2 + 9v_\Delta^2), \tag{2.28}$$

that the $\rho = \dfrac{m_W^2}{m_Z^2\cos^2\theta_w}$ parameter takes the form

$$\rho = \frac{v_\Phi^2 + 3v_\Delta^2}{v_\Phi^2 + 9v_\Delta^2}. \tag{2.29}$$

The value of the ρ parameter from the EW precision data $\rho = 1.00031(19)$ [25] leads to an upper bound of $v_\Delta \sim \mathcal{O}(1)$ GeV, i.e. $v_\Delta \ll v_\Phi$. In this limit,

$$\sin\beta \approx \frac{3v_\Delta}{v_\Phi}, \tag{2.30}$$

$$\sin\gamma \approx \frac{\sqrt{3}v_\Delta}{v_\Phi}, \tag{2.31}$$

and

$$m_A^2 \approx -\frac{\lambda_5 v_\Phi^3}{2v_\Delta}, \tag{2.32}$$

$$m_{H^\pm}^2 \approx -\left(\frac{\lambda_5 v_\Phi^3}{2v_\Delta} + \frac{\lambda_4 v_\Phi^2}{4}\right), \tag{2.33}$$

$$m^2_{H^{\pm\pm}} \approx -\left(\frac{\lambda_5 v_\Phi^3}{2v_\Delta} + \frac{\lambda_4 v_\Phi^2}{2}\right), \tag{2.34}$$

$$m^2_{H^{\pm\pm\pm}} \approx -\left(\frac{\lambda_5 v_\Phi^3}{2v_\Delta} + \frac{3\lambda_4 v_\Phi^2}{4}\right), \tag{2.35}$$

and their mass-squared differences become

$$m^2_{H^{\pm\pm\pm}} - m^2_{H^{\pm\pm}} \approx m^2_{H^{\pm\pm}} - m^2_{H^{\pm}} \approx m^2_{H^{\pm}} - m^2_A \approx -\frac{\lambda_4 v_\Phi^2}{4}. \tag{2.36}$$

Further, for small α,

$$m_h^2 \approx 2\lambda_1 v_\Phi^2, \tag{2.37}$$

$$m_H^2 \approx m_A^2 \approx -\frac{\lambda_5 v_\Phi^3}{2v_\Delta}. \tag{2.38}$$

Neutrino mass generation For completeness, we briefly recall how neutrino masses arise in the BNT model. Integrating out the heavy vectorlike fermion triplet and the scalar quadruplet generates the effective dimension-seven operator

$$\mathcal{L}_{\text{eff}} \supset \frac{c_{ij}}{\Lambda^3}\left(\overline{L_i^c}\,\tilde{\Phi}^*\right)\left(\tilde{\Phi}^\dagger L_j\right)(\Phi^\dagger\Phi) \;+\; \text{h.c.}, \tag{2.39}$$

where L_i are the lepton doublets, $\tilde{\Phi} = i\sigma_2\Phi^*$, and c_{ij} encodes combinations of the underlying Yukawa couplings and heavy masses. After EW symmetry breaking, this operator yields Majorana masses for the light neutrinos:

$$(m_\nu)_{ij} \;\sim\; c_{ij}\frac{v_\Phi^4}{\Lambda^3}\,. \tag{2.40}$$

In the renormalizable BNT framework, the $\lambda_5\,\Delta^\dagger\Phi^3$ term in the potential in Eq. (2.4) is responsible for the generation of this operator, so that

$$(m_\nu)_{ij} = \lambda_5\left(Y_iY_j' + Y_i'Y_j\right)\frac{v_\Phi^4}{M_\Sigma M_\Delta^2}, \tag{2.41}$$

where Y (Y') denotes the Yukawa coupling of the triplet fermion to Φ (Δ), M_Σ is the bare mass of the vectorlike fermion, and $M_\Delta^2 = \mu_\Delta^2 + (4\lambda_3 + 3\lambda_4)\,v_\Phi^2/8 + \left(4\lambda_2 + 9\tilde{\lambda}_2\right)3v_\Delta^2/4$ is the mass-squared of the neutral quadruplet scalar. Equivalently, using the minimization conditions in Eqs. (2.8)–(2.9), one may express the neutrino masses in terms of the induced quadruplet VEV v_Δ, schematically as

$$(m_\nu)_{ij} \;\propto\; \left(Y_iY_j' + Y_i'Y_j\right)\frac{v_\Phi v_\Delta}{M_\Sigma}, \tag{2.42}$$

making explicit that the same λ_5 coupling that induces v_Δ also controls the overall scale of neutrino masses.

3 Theoretical constraints

3.1 Bounded from below

A necessary condition for the stability of the vacuum comes from requiring that the potential in Eq. (2.4) to be bounded from below for all directions in the field space. Obviously, at large field values, the potential in Eq. (2.4) is dominated by the terms that are quartic in the fields:

$$\begin{aligned} V^{(4)}(\Phi,\Delta) = &\lambda_1(\Phi^\dagger\Phi)^2 + \lambda_2(\Delta^\dagger\Delta)^2 + \tilde{\lambda}_2(\Delta^\dagger T^k\Delta)^2 + \lambda_3(\Phi^\dagger\Phi)(\Delta^\dagger\Delta) \\ &+ \lambda_4(\Phi^\dagger\tau^k\Phi)(\Delta^\dagger T^k\Delta) + (\lambda_5\Delta^\dagger\Phi^3 + \text{h.c.}). \end{aligned} \tag{3.1}$$

Defining $x = (|\Phi|^2 \;\; |\Delta|^2 \;\; |\Phi||\Delta|)^T$ with $|\Phi|^2 = \Phi^\dagger\Phi$, $|\Delta|^2 = \Delta^\dagger\Delta$, and

$$\kappa_1 = \frac{(\Delta^\dagger T^k\Delta)^2}{|\Delta|^4}, \quad \kappa_2 = \frac{(\Phi^\dagger\tau^k\Phi)(\Delta^\dagger T^k\Delta)}{|\Phi|^2|\Delta|^2}, \quad \kappa_3 = \frac{(\Delta^\dagger\Phi^3 + \text{h.c.})}{|\Phi|^3|\Delta|},$$

we can rewrite the potential in the following bi-quadratic form

$$V(\Phi,\Delta)^{(4)} = x^T\Lambda x, \tag{3.2}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & \frac{1}{2}(1-c)(\lambda_3+\kappa_2\lambda_4) & \frac{1}{2}\kappa_3\lambda_5 \\ \frac{1}{2}(1-c)(\lambda_3+\kappa_2\lambda_4) & \lambda_2+\kappa_1\tilde{\lambda}_2 & 0 \\ \frac{1}{2}\kappa_3\lambda_5 & 0 & c(\lambda_3+\kappa_2\lambda_4) \end{pmatrix} \tag{3.3}$$

with $\kappa_1 \in [0, \frac{9}{4}]$, $\kappa_2 \in [-\frac{3}{4}, \frac{3}{4}]$, $\kappa_3 \in [-2, 2]$, and c being arbitrary constants. Now, applying the copositivity conditions [26, 27] to Λ, we find the following conditions:

$$\lambda_1 > 0 \tag{3.4}$$
$$\lambda_2 + \kappa_1\tilde{\lambda}_2 > 0 \tag{3.5}$$
$$c(\lambda_3 + \kappa_2\lambda_4) > 0 \tag{3.6}$$
$$(1-c)(\lambda_3+\kappa_2\lambda_4) + 2\sqrt{\lambda_1(\lambda_2+\kappa_1\tilde{\lambda}_2)} > 0 \tag{3.7}$$
$$\kappa_3\lambda_5 + 2\sqrt{c\lambda_1(\lambda_3+\kappa_2\lambda_4)} > 0 \tag{3.8}$$
$$c(1-c)^2(\lambda_3+\kappa_2\lambda_4)^3 + \kappa_3^2\lambda_5^2(\lambda_2+\kappa_1\tilde{\lambda}_2) - 4c\lambda_1(\lambda_2+\kappa_1\tilde{\lambda}_2)(\lambda_3+\kappa_2\lambda_4) > 0. \tag{3.9}$$

These conditions must be respected for all allowed values of $\kappa_{1,2,3}$ for the potential to be bounded from below. Since κ_1 and κ_2 individually lie in the ranges $[0, \frac{9}{4}]$ and

$[-\frac{3}{4}, \frac{3}{4}]$, respectively, one might expect that the allowed region in the (κ_1, κ_2) plane is the full rectangle spanned by these intervals. This is, however, not the case: κ_1 and κ_2 are correlated and must satisfy

$$|\kappa_2| \leq \frac{1}{2}\sqrt{\kappa_1},$$

so that the true parameter space is confined by the parabola above. For κ_3, the only restriction is the global bound

$$|\kappa_3| \ \leq \ 2,$$

and it can vary essentially independently of κ_1, κ_2. Hence, the full three-dimensional parameter space is obtained by extruding the (κ_1, κ_2) region along the κ_3 axis between -2 and $+2$, forming a "parabolic wedge prism" in $(\kappa_1, \kappa_2, \kappa_3)$ space.

3.2 Perturabtive unitarity

Requiring perturbative unitarity in scattering processes constrains the size of the interactions. The partial-wave decomposition of the scattering amplitude $\mathcal{M}_{i\to f}$ reads as

$$\mathcal{M}_{fi} = iT^{fi} = 16i\pi \sum_j (2j+1) a_j^{fi}(s) P_j(\cos\theta), \tag{3.10}$$

where a_j denotes the j-th partial-wave amplitude, θ is the polar angle between the initial i and final f states, and P_j is the Legendre polynomial of degree j. In the high energy (massless) limit, the dominant contribution comes from the $j = 0$ partial-wave (s-wave) at tree-level

$$a_0^{fi} = -\frac{i}{16\pi}\mathcal{M}_{fi}. \tag{3.11}$$

Unitarity of the S-matrix requires that $|a_0| \leq 1$, $|\mathrm{Re}(a_0)| \leq 1/2$, and $0 \leq \mathrm{Im}(a_0) \leq 1$. In practice, it is sufficient to impose either $|a_0| \leq 1$ or $|\mathrm{Re}(a_0)| \leq 1/2$, which in turn implies that the eigenvalues x_i of the scattering submatrices satisfy $|x_i| \leq \kappa\pi$, with $\kappa = 16$ or 8 depending on the chosen criterion. Following [28], we adopt the former condition.

By virtue of the equivalence theorem [29, 30], unphysical scalar states can be used in place of the longitudinal components of the gauge bosons in the high-energy limit. Compared to $2 \to 2$ scattering, $2 \to 3$ partial-wave amplitudes can be neglected, as they scale inversely with the energy. Moreover, amplitudes involving trilinear vertices are generally suppressed due to intermediate propagators. Therefore, we focus on two-body, tree-level scalar-scalar scattering processes dominated by quartic interactions.

The number of all possible q-charged $2 \to 2$ channels constructed from n neutral, s singly-charged, d doubly-charged, and t triply-charged fields are

- $q = 0 : \frac{n(n+1)}{2} + s^2 + d^2 + t^2$,

- $q = 1 : ns + sd + dt$,
- $q = 2 : nd + st + \frac{s(s+1)}{2}$,
- $q = 3 : nt + sd$,
- $q = 4 : st + \frac{d(d+1)}{2}$,
- $q = 5 : dt$,
- $q = 6 : \frac{t(t+1)}{2}$.

In the present model, $n = 4$, $s = 2$, $d = 1$ and $t = 1$. We present the resulting submatrices $\mathcal{M}_q$ structured in terms of net electric charge q in the initial/final states, with their entries corresponding to the quartic couplings that mediate the scalar-scalar scattering processes in Appendix A. Now, requiring the moduli of the eigenvalues of the submatrices to be $\leq \kappa\pi$, we obtain

$$\begin{aligned}
&\left|4\lambda_2 - 11\tilde{\lambda}_2\right| \leq 2\kappa\pi \\
&\left|4\lambda_2 \pm 9\tilde{\lambda}_2\right| \leq 2\kappa\pi \\
&\left|4\lambda_2 + 3\tilde{\lambda}_2\right| \leq 2\kappa\pi \\
&|4\lambda_3 \pm 3\lambda_4| \leq 4\kappa\pi \\
&|4\lambda_3 - 5\lambda_4| \leq 4\kappa\pi \\
&\left|16\lambda_1 + 4\lambda_3 + 5\lambda_4 \pm 8\sqrt{(16\lambda_1 - 4\lambda_3 - 5\lambda_4)^2 + 3072\lambda_5^2}\right| \leq 8\kappa\pi \\
&\left|12\lambda_2 - \tilde{\lambda}_2 \pm \sqrt{(16\lambda_2^2 - 40\lambda_2\tilde{\lambda}_2 + 793\tilde{\lambda}_2^2}\right| \leq 4\kappa\pi \\
&\left|4\lambda_1 + 4\lambda_2 + 31\tilde{\lambda}_2 \pm \sqrt{(4\lambda_1 - 4\lambda_2 - 31\tilde{\lambda}_2)^2 + 40\lambda_4^2}\right| \leq 4\kappa\pi \\
&\left|8\lambda_1 + 4\lambda_3 + 5\lambda_4 \pm \sqrt{(8\lambda_1 - 4\lambda_3 - 5\lambda_4)^2 + 1536\lambda_5^2}\right| \leq 8\kappa\pi \\
&\left|12\lambda_1 + 20\lambda_2 + 15\tilde{\lambda}_2 \pm \sqrt{(12\lambda_1 - 20\lambda_2 - 15\tilde{\lambda}_2)^2 + 128\lambda_3^2}\right| \leq 4\kappa\pi
\end{aligned}$$

These conditions ensure that perturbative unitarity is respected in all $2 \to 2$ scalar scattering processes and put non-trivial constraints on the parameter space.

3.3 Stability against charge-breaking minima

In Sec. 2, while obtaining the scalar spectrum of the model, we have implicitly assumed that the EW symmetry is spontaneously broken at an electrically neutral configuration in field space, and that this configuration corresponds to the global minimum of the potential. Although the tadpole conditions in Eq. (2.8) and Eq. (2.9) ensure that the chosen vacuum represents an extremum of the potential in Eq. (2.4), it remains necessary to verify that this extremum is indeed stable—i.e., that it corresponds to a true minimum rather than a saddle point or local maximum.

The scalar potential in Eq. (2.4) can, in general, admit multiple stationary solutions—both neutral and charge-breaking. In what follows, we identify all stationary points and determine the conditions under which the phenomenologically viable EW vacuum constitutes the global minimum by comparing the potential depths of all extrema.[4] In particular, we compute the potential differences between the desired electroweak vacuum and the remaining extrema. A negative difference implies that the electroweak vacuum is deeper and thus stable against tunnelling into the corresponding configuration, whereas a positive value indicates the presence of a lower, potentially charge-breaking minimum.

Regarding the gauge choice, three real scalar degrees of freedom can always be absorbed through an appropriate gauge fixing. We work in the unitary gauge, where the scalar doublet reduces to a single real neutral component, so its VEV can, without loss of generality, be taken as real and electrically neutral. The only phase-sensitive term in the CP-conserving potential is the quartic interaction $\lambda_5(\Delta^\dagger\Phi^3 + \text{h.c.})$, whose minimization fixes the relevant phase combinations to either 0 or π, so that all stationary configurations—neutral and charge-breaking—can be represented by real VEVs without loss of generality.

The scalar potential of the BNT model admits several stationary configurations, summarized in Table 1. Among these, three correspond to charge-conserving extrema—denoted N_1, N_2, and N_3—referred to as normal minima, where only neutral field components acquire nonzero VEVs. In addition, the potential features fourteen charge-breaking stationary points (denoted $CB1$ to $CB14$), characterized by nonvanishing VEVs of charged fields. Such charge-breaking configurations would spontaneously break electromagnetic gauge invariance and generate a photon mass, rendering them phenomenologically unacceptable.

For clarity, we note that the VEV symbols v, v_0, v_1, v_2, v_3 are reused for notational simplicity across the different stationary configurations. These VEVs should not be interpreted as identical across the various stationary points: their numerical values are, in general, distinct for each configuration. The notation is merely a convenient shorthand.

A brief discussion of some stationary configurations is in order. The stationary point *N1* corresponds to the desired EW vacuum, whose structure has already been presented in Sec. 2. The configuration *N2* also represents an EW-type vacuum

[4]While a metastable vacuum with a lifetime longer than the age of the Universe could, in principle, be acceptable, assessing this would require a dedicated tunnelling analysis, which is beyond the scope of this work. We therefore restrict to the stronger requirement that the EW vacuum lies at the absolute minimum of the potential.

Vacuum	$\langle\sqrt{2}\phi^0\rangle$	$\langle\sqrt{2}\delta^0\rangle$	$\langle\sqrt{2}\delta^+\rangle$	$\langle\sqrt{2}\delta^{++}\rangle$	$\langle\sqrt{2}\delta^{+++}\rangle$
N1	v_Φ	v_Δ	0	0	0
N2	v'_Φ	0	0	0	0
N3	0	v'_Δ	0	0	0
CB1	v	0	v_1	0	0
CB2	v	0	0	v_2	0
CB3	v	0	0	0	v_3
CB4	v	0	v_1	v_2	0
CB5	v	0	v_1	0	v_3
CB6	v	0	0	v_2	v_3
CB7	v	0	v_1	v_2	v_3
CB8	v	v_0	v_1	0	0
CB9	v	v_0	0	v_2	0
CB10	v	v_0	0	0	v_3
CB11	v	v_0	v_1	v_2	0
CB12	v	v_0	v_1	0	v_3
CB13	v	v_0	0	v_2	v_3
CB14	v	v_0	v_1	v_2	v_3

Table 1. Stationary points of the BNT potential: three charge-conserving and fourteen charge-breaking configurations.

and will be treated on the same footing as *N1* in our stability analysis. However, for generic parameters, this is not a true stationary point: the $\lambda_5\,\Delta^\dagger\Phi^3$ interaction induces a tadpole for δ^0 once Φ acquires a VEV, so the condition $v_\Delta = 0$ can be satisfied only if $\lambda_5 = 0$. Thus, *N2* exists as a genuine extremum only if $\lambda_5 = 0$, and otherwise corresponds to the $v_\Delta \to 0$ boundary of *N1*. The stationary point *N3* corresponds to a vacuum structure in which only δ^0 acquires a VEV. Since the absence of a doublet VEV would imply massless fermions, this extremum is incompatible with observations. For this reason, we do not include any additional stationary configurations with $\langle\phi^0\rangle = 0$ in Table 1 and do not consider them further in our analysis. Before proceeding with the comparison of potential depths, it is useful to collect the minimization conditions associated with all stationary configurations.

$$
\begin{aligned}
\mathrm{N1}: \mu_\Phi^2 &= \lambda_1 v_\Phi^2 + \frac{1}{8}\left(4\lambda_3 + 3\lambda_4\right) v_\Delta^2 + \frac{3}{2}\lambda_5 v_\Phi v_\Delta \\
\mu_\Delta^2 &= -(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2) v_\Delta^2 - \frac{1}{8}\left(4\lambda_3 + 3\lambda_4\right) v_\Phi^2 - \frac{\lambda_5}{2}\frac{v_\Phi^3}{v_\Delta} \\
\mathrm{N2}: &\ (\text{exists only for } \lambda_5 = 0)\ \mu_\Phi^2 = \lambda_1 v_\Phi'^2
\end{aligned}
$$

$$\mathrm{CB1}: \mu_\Phi^2 = \lambda_1 v^2 + \frac{1}{8}\left(4\lambda_3 + \lambda_4\right) v_1^2, \quad \mu_\Delta^2 = -\left(\lambda_2 + \frac{1}{4}\tilde{\lambda}_2\right) v_1^2 - \frac{1}{8}\left(4\lambda_3 + \lambda_4\right) v^2$$

$$\mathrm{CB2}: \mu_\Phi^2 = \lambda_1 v^2 + \frac{1}{8}\left(4\lambda_3 - \lambda_4\right) v_2^2, \quad \mu_\Delta^2 = -\left(\lambda_2 + \frac{1}{4}\tilde{\lambda}_2\right) v_2^2 - \frac{1}{8}\left(4\lambda_3 - \lambda_4\right) v^2$$

$$\mathrm{CB3}: \mu_\Phi^2 = \lambda_1 v^2 + \frac{1}{8}\left(4\lambda_3 - 3\lambda_4\right) v_3^2, \quad \mu_\Delta^2 = -\left(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2\right) v_3^2 - \frac{1}{8}\left(4\lambda_3 - 3\lambda_4\right) v^2$$

$$\begin{aligned}
\mathrm{CB4}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{1}{8}\left(4\lambda_3 + \lambda_4\right) v_1^2 + \frac{1}{8}\left(4\lambda_3 - \lambda_4\right) v_2^2 \\
\mu_\Delta^2 &= -\lambda_2(v_1^2 + v_2^2) - \frac{v^2}{6(v_1^2 - v_2^2)}\left\{\left(3\lambda_3 + \lambda_4\right) v_1^2 - \frac{1}{6}\left(3\lambda_3 - \lambda_4\right) v_2^2\right\} \\
6\tilde{\lambda}_2(v_1^2 - v_2^2) &= \lambda_4 v^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{CB5}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{1}{8}\left(4\lambda_3 + \lambda_4\right) v_1^2 + \frac{1}{8}\left(4\lambda_3 - 3\lambda_4\right) v_3^2, \quad \mu_\Delta^2 = -\frac{1}{2}\lambda_3 v^2 - \lambda_2\left(v_1^2 + v_3^2\right) \\
2\tilde{\lambda}_2\left(v_1^2 - 3v_3^2\right) &+ \lambda_4 v^2 = 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{CB6}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{1}{2}\lambda_3\left(v_2^2 + v_3^2\right) - \frac{1}{8}\lambda_4\left(v_2^2 + 3v_3^2\right) \\
\mu_\Delta^2 &= -\lambda_2\left(v_2^2 + v_3^2\right) - \frac{1}{32}\left(16\lambda_3 - 3\lambda_4\right) v^2 - \frac{9v^2 v_3^2}{32 v_2^2}\lambda_4 \\
8\tilde{\lambda}_2 v_2^2 &= \lambda_4 v^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{CB7}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{1}{2}\lambda_3\left(v_1^2 + v_2^2 + v_3^2\right), \quad \mu_\Delta^2 = -\lambda_2\left(v_1^2 + v_2^2 + v_3^2\right) - \frac{\lambda_3}{2} v^2 \\
\tilde{\lambda}_2 &= 0, \quad \lambda_4 = 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{CB8}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{\lambda_3}{2}\left(v_0^2 + v_1^2\right) + \frac{\lambda_4}{8}\left(3v_0^2 + v_1^2\right) + \frac{3}{2}\lambda_5 v v_0 \\
\mu_\Delta^2 &= -\lambda_2\left(v_0^2 + v_1^2\right) - \frac{1}{32}\left(16\lambda_3 + 3\lambda_4\right) v^2 + \frac{9v^2 v_0^2}{32 v_1^2}\lambda_4 + \frac{v^3\left(9v_0^2 + v_1^2\right)}{16 v_0 v_1^2}\lambda_5 \\
8\tilde{\lambda}_2 v_0 v_1^2 &+ \lambda_4 v^2 v_0 + 2\lambda_5 v^3 = 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{CB9}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{\lambda_3}{2}\left(v_0^2 + v_2^2\right) + \frac{\lambda_4}{8}\left(3v_0^2 - v_2^2\right) + \frac{3}{2}\lambda_5 v v_0 \\
\mu_\Delta^2 &= -\lambda_2\left(v_0^2 + v_2^2\right) - \frac{\lambda_3}{2} v^2 - \frac{v^3}{8 v_0}\lambda_5 \\
2\tilde{\lambda}_2 v_0\left(3v_0^2 - v_2^2\right) &+ \lambda_4 v^2 v_0 + \lambda_5 v^3 = 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{CB10}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{\lambda_3}{2}\left(v_0^2 + v_3^2\right) + \frac{3\lambda_4}{8}\left(v_0^2 - v_3^2\right) + \frac{3}{2}\lambda_5 v v_0 \\
\mu_\Delta^2 &= -\lambda_2\left(v_0^2 + v_3^2\right) - \frac{\lambda_3}{2} v^2 - \frac{v^3}{4 v_0}\lambda_5 \\
18\tilde{\lambda}_2 v_0\left(v_0^2 - v_3^2\right) &+ 3\lambda_4 v^2 v_0 + 2\lambda_5 v^3 = 0
\end{aligned}$$

$$\begin{aligned}
\text{CB11}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{\lambda_3}{2}\left(v_0^2+v_1^2+v_2^2\right) \\
&+ \frac{\lambda_4}{8}\left(-13v_0^2+v_1^2-v_2^2+\frac{2v_0\left(30v_0^3v_2-3v_0v_1^2v_2+6\sqrt{3}v_1^2v_2^2-2\sqrt{3}v_0^2(5v_1^2-4v_2^2)\right)}{6v_0^2v_2-3v_2\left(v_1^2-v_2^2\right)-2\sqrt{3}v_0(v_1^2-2v_2^2)}\right) \\
\mu_\Delta^2 &= -\lambda_2\left(v_0^2+v_1^2+v_2^2\right)-\frac{\lambda_3}{2}v^2+\frac{\lambda_4}{8}\frac{3v_0^2v_2+4v_2\left(v_1^2+v_2^2\right)+2\sqrt{3}v_0(v_1^2+2v_2^2)}{6v_0^2v_2-3v_2\left(v_1^2-v_2^2\right)-2\sqrt{3}v_0(v_1^2-2v_2^2)}v^2 \\
&2\tilde{\lambda}_2\left(6v_0^2v_2-3v_2\left(v_1^2-v_2^2\right)-2\sqrt{3}v_0(v_1^2-2v_2^2)\right)+\lambda_4v^2v_2=0 \\
&2\lambda_5\left(6v_0^2v_2-3v_2\left(v_1^2-v_2^2\right)-2\sqrt{3}v_0(v_1^2-2v_2^2)\right)v=\lambda_4\times \\
&\left(-6v_0^3v_2+2\sqrt{3}v_1^2v_2^2+2\sqrt{3}v_0^2(v_1^2-4v_2^2)+v_0(7v_1^2v_2-8v_2^3)\right)
\end{aligned}$$

$$\begin{aligned}
\text{CB12}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{\lambda_3}{2}\left(v_0^2+v_1^2+v_3^2\right)+\frac{\lambda_4}{8}\frac{-27v_0^4+v_0^2(51v_1^2-45v_3^2)+2(v_1^2-3v_3^2)^2}{9v_0^2+2v_1^2-6v_3^2} \\
\mu_\Delta^2 &= -\lambda_2\left(v_0^2+v_1^2+v_3^2\right)-\frac{\lambda_3}{2}v^2+\frac{\lambda_4}{8}\frac{9v_0^2}{9v_0^2+2v_1^2-6v_3^2}v^2 \\
&\tilde{\lambda}_2(9v_0^2+2v_1^2-6v_3^2)+\lambda_4v^2=0, \quad 2\lambda_5v(9v_0^2+2v_1^2-6v_3^2)+\lambda_4(9v_0^3-6v_0v_1^2)=0
\end{aligned}$$

$$\begin{aligned}
\text{CB13}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{\lambda_3}{2}\left(v_0^2+v_2^2+v_3^2\right)+\lambda_4\frac{9v_0^4+21v_0^2\left(v_2^2-3v_3^2\right)+4\left(v_2^4+3v_2^2v_3^2\right)}{3v_0^2-4v_2^2} \\
\mu_\Delta^2 &= -\lambda_2\left(v_0^2+v_2^2+v_3^2\right)-\frac{\lambda_3}{2}v^2-\frac{3\lambda_4}{8}\frac{v_2^2-3v_3^2}{3v_0^2-4v_2^2}v^2 \\
&2\tilde{\lambda}_2\left(3v_0^2-4v_2^2\right)+\lambda_4v^2=0, \quad 2\lambda_5v\left(3v_0^2-4v_2^2\right)-3\lambda_4v_0\left(2v_2^2-3v_3^2\right)=0
\end{aligned}$$

$$\begin{aligned}
\text{CB14}: \mu_\Phi^2 &= \lambda_1 v^2 + \frac{\lambda_3}{2}\left(v_0^2+v_1^2+v_2^2+v_3^2\right), \quad \mu_\Delta^2 = -\lambda_2\left(v_0^2+v_1^2+v_2^2+v_3^2\right)-\frac{\lambda_3}{2}v^2 \\
&\tilde{\lambda}_2=0, \quad \lambda_4=0, \quad \lambda_5=0
\end{aligned}$$

Now, to calculate the difference in potential depth between two coexisting minima, we follow the bilinear formalism detailed in Refs. [31–33]; see also Refs. [34–36] for recent works. The potential V in Eq. (2.4) can be decomposed into homogeneous functions of order two, and four in the fields Φ and Δ: $V = V_2 + V_4$. Consequently, the potential can be written as a quadratic polynomial in the real bilinear vector X constructed from independent field bilinears: $x_1 = |\Phi|^2$, $x_2 = |\delta^0|^2$, $x_3 = |\delta^+|^2$, $x_4 = |\delta^{++}|^2$, $x_5 = |\delta^{+++}|^2$, $x_6 = \mathrm{Re}(\phi^{0*}\delta^0)$, $x_7 = \mathrm{Re}(\delta^{0*}\delta^{++})$, $x_8 = \mathrm{Re}(\delta^{+*}\delta^{+++})$

$$V = \underbrace{M^T X}_{V_2} + \underbrace{\frac{1}{2}X^T\Lambda X}_{V_4}, \tag{3.12}$$

where $X = (x_1, ..., x_8)^T$,

$$M = \begin{pmatrix} -\mu_\Phi^2 \\ \mu_\Delta^2 \\ \mu_\Delta^2 \\ \mu_\Delta^2 \\ \mu_\Delta^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 2\lambda_1 & \lambda_3 + \frac{3}{4}\lambda_4 & \lambda_3 + \frac{1}{4}\lambda_4 & \lambda_3 - \frac{1}{4}\lambda_4 & \lambda_3 - \frac{3}{4}\lambda_4 & 2\lambda_5 & 0 & 0 \\ \lambda_3 + \frac{3}{4}\lambda_4 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & 2\lambda_2 - \frac{9}{2}\tilde{\lambda}_2 & 0 & 0 & 0 \\ \lambda_3 + \frac{1}{4}\lambda_4 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{1}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{7}{2}\tilde{\lambda}_2 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & 0 & 4\sqrt{3}\,\tilde{\lambda}_2 & 0 \\ \lambda_3 - \frac{1}{4}\lambda_4 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{7}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{1}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 0 & 0 & 4\sqrt{3}\,\tilde{\lambda}_2 \\ \lambda_3 - \frac{3}{4}\lambda_4 & 2\lambda_2 - \frac{9}{2}\tilde{\lambda}_2 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 0 & 0 & 0 \\ 2\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\sqrt{3}\,\tilde{\lambda}_2 & 0 & 0 & 0 & 0 & 6\tilde{\lambda}_2 \\ 0 & 0 & 0 & 4\sqrt{3}\,\tilde{\lambda}_2 & 0 & 0 & 6\tilde{\lambda}_2 & 0 \end{pmatrix}. \tag{3.13}$$

At any stationary point (denoted by SP), the potential satisfies the minimization conditions $\partial V / \partial \phi_i = 0$, where ϕ_i represents the real scalar components of the model. Multiplying these equations by ϕ_i and summing over all components yields

$$\sum_i \phi_i \frac{\partial V}{\partial \phi_i}\bigg|_{SP} = 0 \quad \Rightarrow 2(V_2)_{SP} + 4(V_4)_{SP} = 0, \tag{3.14}$$

in accordance with Euler's theorem for homogeneous functions. Therefore, the value of the potential at a stationary point can be expressed as

$$V_{SP} = \frac{1}{2}(V_2)_{SP} = -(V_4)_{SP}. \tag{3.15}$$

Next, we define the gradient of V with respect to X as

$$V' = \frac{\partial V}{\partial X^T} = M + \Lambda X. \tag{3.16}$$

The product of V' evaluated at one stationary point ($SP1$) with X evaluated at another ($SP2$) then satisfies

$$X_{SP1}^T V_{SP2}' = 2V_{SP1} + X_{SP1}^T \Lambda X_{SP2}, \tag{3.17}$$

$$X_{SP2}^T V_{SP1}' = 2V_{SP2} + X_{SP2}^T \Lambda X_{SP1} \tag{3.18}$$

Subtracting these two relations and noting that the matrix Λ is symmetric, we obtain the general expression for the difference in potential depth between two stationary points:

$$V_{SP2} - V_{SP1} = \frac{1}{2}\left(X_{SP2}^T V_{SP1}' - X_{SP1}^T V_{SP2}'\right). \tag{3.19}$$

In the following, we analyze the stability of the EW stationary points—i.e. the desired electroweak vacua—against the charge-breaking extrema. Assuming that N_1 or N_2 coexists with one or more CB stationary points, we evaluate the corresponding potential differences using Eq. (3.19).

Stability of $N2$ against charge-breaking extrema The differences in potential depth between coexisting stationary points are then obtained as

$$
\begin{aligned}
V_{CB1} - V_{N2} &= \frac{v_1^2}{32}\left\{8\mu_\Delta^2 + (4\lambda_3 + \lambda_4)\, v_\Phi^2\right\} = \frac{v_1^2}{4}\left(2m^2_{H^{\pm\pm}} - m^2_{H^{\pm\pm\pm}}\right)\\
V_{CB2} - V_{N2} &= \frac{v_2^2}{32}\left\{8\mu_\Delta^2 + (4\lambda_3 - \lambda_4)\, v_\Phi^2\right\} = \frac{v_2^2}{4} m^2_{H^{\pm\pm}}\\
V_{CB3} - V_{N2} &= \frac{v_3^2}{32}\left\{8\mu_\Delta^2 + (4\lambda_3 - 3\lambda_4)\, v_\Phi^2\right\} = \frac{v_3^2}{4} m^2_{H^{\pm\pm\pm}}\\
V_{CB4} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_1^2+v_2^2) + \lambda_4(v_1^2 - v_2^2)\right\} + \frac{\mu_\Delta^2}{4}(v_1^2+v_2^2)\\
&= \frac{1}{4}\left\{v_1^2(2m^2_{H^{\pm\pm}} - m^2_{H^{\pm\pm\pm}}) + v_2^2 m^2_{H^{\pm\pm}}\right\}\\
V_{CB5} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_1^2+v_3^2) + \lambda_4(v_1^2 - 3v_3^2)\right\} + \frac{\mu_\Delta^2}{4}(v_1^2+v_3^2)\\
&= \frac{1}{4}\left\{v_1^2(2m^2_{H^{\pm\pm}} - m^2_{H^{\pm\pm\pm}}) + v_3^2 m^2_{H^{\pm\pm\pm}}\right\}\\
V_{CB6} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_2^2+v_3^2) - \lambda_4(v_2^2 + 3v_3^2)\right\} + \frac{\mu_\Delta^2}{4}(v_2^2+v_3^2) = \frac{1}{4}(v_2^2 m^2_{H^{\pm\pm}} + v_3^2 m^2_{H^{\pm\pm\pm}})\\
V_{CB7} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_1^2+v_2^2+v_3^2) + \lambda_4(v_1^2 - v_2^2 - 3v_3^2)\right\} + \frac{\mu_\Delta^2}{4}(v_1^2+v_2^2+v_3^2)\\
&= \frac{1}{4}\left\{v_1^2(2m^2_{H^{\pm\pm}} - m^2_{H^{\pm\pm\pm}}) + v_2^2 m^2_{H^{\pm\pm}} + v_3^2 m^2_{H^{\pm\pm\pm}}\right\}\\
V_{CB8} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_0^2+v_1^2) + \lambda_4(3v_0^2 + v_1^2) + 12\lambda_5 v v_0\right\} + \frac{\mu_\Delta^2}{4}(v_0^2+v_1^2)\\
&= \frac{1}{4}\left\{v_0^2(3m^2_{H^{\pm\pm}} - 2m^2_{H^{\pm\pm\pm}}) + v_1^2(2m^2_{H^{\pm\pm}} - m^2_{H^{\pm\pm\pm}})\right\} + \frac{3}{8}\lambda_5 v v_0 v_\Phi^2\\
V_{CB9} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_0^2+v_2^2) + \lambda_4(3v_0^2 - v_2^2) + 12\lambda_5 v v_0\right\} + \frac{\mu_\Delta^2}{4}(v_0^2+v_2^2)\\
&= \frac{1}{4}\left\{v_0^2(3m^2_{H^{\pm\pm}} - 2m^2_{H^{\pm\pm\pm}}) + v_2^2 m^2_{H^{\pm\pm}}\right\} + \frac{3}{8}\lambda_5 v v_0 v_\Phi^2\\
V_{CB10} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_0^2+v_3^2) + 3\lambda_4(v_0^2 - v_3^2) + 12\lambda_5 v v_0\right\} + \frac{\mu_\Delta^2}{4}(v_0^2+v_3^2)\\
&= \frac{1}{4}\left\{v_0^2(3m^2_{H^{\pm\pm}} - 2m^2_{H^{\pm\pm\pm}}) + v_3^2 m^2_{H^{\pm\pm}}\right\} + \frac{3}{8}\lambda_5 v v_0 v_\Phi^2\\
V_{CB11} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_0^2+v_1^2+v_2^2) + \lambda_4(3v_0^2 + v_1^2 - v_2^2) + 12\lambda_5 v v_0\right\} + \frac{\mu_\Delta^2}{4}(v_0^2+v_1^2+v_2^2)\\
&= \frac{1}{4}\left\{v_0^2(3m^2_{H^{\pm\pm}} - 2m^2_{H^{\pm\pm\pm}}) + v_1^2(2m^2_{H^{\pm\pm}} - m^2_{H^{\pm\pm\pm}}) + v_2^2 m^2_{H^{\pm\pm}}\right\} + \frac{3}{8}\lambda_5 v v_0 v_\Phi^2\\
V_{CB12} - V_{N2} &= \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_0^2+v_1^2+v_3^2) + \lambda_4(3v_0^2 + v_1^2 - 3v_3^2) + 12\lambda_5 v v_0\right\} + \frac{\mu_\Delta^2}{4}(v_0^2+v_1^2+v_3^2)
\end{aligned}
$$

$$= \frac{1}{4}\left\{v_0^2(3m_{H^{\pm\pm}}^2 - 2m_{H^{\pm\pm\pm}}^2) + v_1^2(2m_{H^{\pm\pm}}^2 - m_{H^{\pm\pm\pm}}^2) + v_3^2 m_{H^{\pm\pm\pm}}^2\right\} + \frac{3}{8}\lambda_5 v v_0 v_\Phi^2$$

$$V_{CB13} - V_{N2} = \frac{v_\Phi^2}{32}\left\{4\lambda_3(v_0^2 + v_2^2 + v_3^2) + \lambda_4(3v_0^2 - v_2^2 - 3v_3^2) + 12\lambda_5 v v_0\right\} + \frac{\mu_\Delta^2}{4}(v_0^2 + v_1^2 + v_3^2)$$

$$= \frac{1}{4}\left\{v_0^2(3m_{H^{\pm\pm}}^2 - 2m_{H^{\pm\pm\pm}}^2) + v_2^2 m_{H^{\pm\pm}}^2 + v_3^2 m_{H^{\pm\pm\pm}}^2\right\} + \frac{3}{8}\lambda_5 v v_0 v_\Phi^2$$

$$V_{CB14} - V_{N2} = \frac{1}{8}\left(v_0^2 + v_1^2 + v_2^2 + v_3^2\right)\left(\lambda_3 v_\phi^2 + 2\mu_\Delta^2\right) = \frac{1}{8}\left(v_0^2 + v_1^2 + v_2^2 + v_3^2\right)\left(3m_{H^{\pm\pm}}^2 - m_{H^{\pm\pm\pm}}^2\right)$$

Here, $m_{H^{\pm\pm}}$, $m_{H^{\pm\pm\pm}}$ denote the physical scalar masses evaluated at the stationary point N_2.

If $N2$ is a true minimum, all mass-squared eigenvalues are positive. Moreover, as argued above, $N2$ exists as a stationry point only if $\lambda_5 = 0$. Consequently, several of the potential differences are manifestly positive, namely $V_{CB2} - V_{N2} > 0$, $V_{CB3} - V_{N2} > 0$, and $V_{CB6} - V_{N2} > 0$. Thus, $N2$ is necessarily stable against charge-breaking extrema of types $CB2$, $CB3$ and $CB6$. For the remaining configurations, however, $N2$ is not guaranteed to be the global minimum, as some charge-breaking stationary points may coexist with—and potentially lie below—it in potential depth. Ensuring that $N2$ is stable against all charge-breaking extrema then requires the mass inequalities

$$2m_{H^{\pm\pm}}^2 - m_{H^{\pm\pm\pm}}^2 > 0, \quad 3m_{H^{\pm\pm}}^2 - 2m_{H^{\pm\pm\pm}}^2 > 0. \tag{3.20}$$

Stability of $N1$ vs $N2$ The potential difference between the two electroweak stationary points $N1$ and $N2$ is

$$V_{N2} - V_{N1} = \frac{1}{4}\left\{\lambda_1 v_\Phi^2(v_\Phi'^2 - v_\Phi^2) + v_\Delta^4\left(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2\right) - \lambda_5 v_\Phi^3 v_\Delta\right\},$$

Since $N2$ exists only for $\lambda_5 = 0$, the last term vanishes. Noting the electroweak symmetry–breaking conditions $\sqrt{v_\Phi^2 + 3v_\Delta^2} \approx 246$ GeV (Eq. (2.7)) and $v_\Phi' \approx 246$ GeV, we have $v_\Delta' > v_\Delta$, so the first term is negative definite. The second term is positive definite as a consequence of the bounded-from-below conditions, which require $\lambda_1 > 0$ and $\lambda_2 + \frac{9}{4}\tilde{\lambda}_2 > 0$ (Eqs. (3.4)–(3.5)). Together, these observations imply that the relative depth of $N1$ and $N2$ depends on the balance between the negative first term and the positive second term, so neither stationary point is guaranteed to be globally deeper. The condition for $N1$ to be deeper than $N2$ is

$$\lambda_1 v_\Phi^2(v_\Phi'^2 - v_\Phi^2) + v_\Delta^4\left(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2\right) > 0, \tag{3.21}$$

which, using $v_\Phi^2 + 3v_\Delta^2 = v_\Phi'^2$, simplifies to

$$3\lambda_1 v_\Phi^2 + v_\Delta^2 \left(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2\right) > 0. \tag{3.22}$$

Stability of $N1$ against charge-breaking extrema The potential differences between $N1$ and the various charge-breaking configurations are given by

$$\begin{aligned}
V_{CB1} - V_{N1} &= -\frac{\tilde{\lambda}_2}{2} v_1^2 v_\Delta^2 - \frac{\lambda_4}{16}\left(v^2 v_\Delta^2 + v_1^2 v_\Phi^2\right) - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left(3v^2 v_\Delta^2 + v_1^2 v_\Phi^2\right) \\
V_{CB2} - V_{N1} &= -\frac{\tilde{\lambda}_2}{2} v_2^2 v_\Delta^2 - \frac{\lambda_4}{8}\left(v^2 v_\Delta^2 + v_2^2 v_\Phi^2\right) - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left(3v^2 v_\Delta^2 + v_2^2 v_\Phi^2\right) \\
V_{CB3} - V_{N1} &= -\frac{3\lambda_4}{16}\left(v^2 v_\Delta^2 + v_3^2 v_\Phi^2\right) - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left(3v^2 v_\Delta^2 + v_3^2 v_\Phi^2\right) \\
V_{CB4} - V_{N1} &= -\frac{3\tilde{\lambda}_2}{4} v_2^2 v_\Delta^2 + \lambda_4 \frac{v^2 v_\Delta^2(-7v_1^2 + 8v_2^2) - 3v_\Phi^2(v_1^4 + v_1^2 v_2^2 - 2v_2^4)}{48(v_1^2 - v_2^2)} \\
&\quad - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left\{3v^2 v_\Delta^2 + (v_1^2 + v_2^2)v_\Phi^2\right\} \\
V_{CB5} - V_{N1} &= -\frac{9\tilde{\lambda}_2}{8} v_3^2 v_\Delta^2 + \frac{\lambda_4}{16}\left\{\frac{3v^2 v_1^2 v_\Delta^2}{v_1^2 - 3v_3^2} - v_\Phi^2(v_1^2 + 3v_3^2)\right\} - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\{3v^2 v_\Delta^2 + (v_1^2 + v_3^2)v_\Phi^2\} \\
V_{CB6} - V_{N1} &= -\frac{3\tilde{\lambda}_2}{8}(2v_2^2 + 3v_3^2)v_\Delta^2 + \frac{\lambda_4}{16}\left\{\frac{3v^2 v_\Delta^2}{4v_2^2}(-2v_2^2 + 3v_3^2) - v_\Phi^2(2v_2^2 + 3v_3^2)\right\} \\
&\quad - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left\{3v^2 v_\Delta^2 + (v_2^2 + v_3^2)v_\Phi^2\right\} \\
V_{CB7} - V_{N1} &= -\frac{3\tilde{\lambda}_2}{8} v_\Delta^2(2v_2^2 + 3v_3^2) - \frac{\lambda_4}{16}(v_2^2 + 2v_2^2 + 3v_3^2)v_\Phi^2 \\
&\quad - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left\{3v^2 v_\Delta^2 + (v_1^2 + v_2^2 + v_3^2)v_\Phi^2\right\} \\
V_{CB8} - V_{N1} &= -\frac{\lambda_4}{16} v_1^2 v_\Phi^2 + \frac{\lambda_5}{8v_0 v_\Delta}\left\{v^3 v_\Delta^3 - 3v^2 v_0 v_\Delta^2 v_\Phi + 3v v_0^2 v_\Delta v_\Phi^2 - v_0(v_0^2 + v_1^2)v_\Phi^3\right\} \\
V_{CB9} - V_{N1} &= -\frac{3\tilde{\lambda}_2}{4} v_2^2 v_\Delta^2 - \frac{\lambda_4}{8} v_2^2 v_\Phi^2 + \frac{\lambda_5}{8v_0 v_\Delta}\left\{v^3 v_\Delta^3 - 3v^2 v_0 v_\Delta^2 v_\Phi + 3v v_0^2 v_\Delta v_\Phi^2 - v_0(v_0^2 + v_2^2)v_\Phi^3\right\} \\
V_{CB10} - V_{N1} &= -\frac{9\tilde{\lambda}_2}{8} v_3^2 v_\Delta^2 - \frac{3\lambda_4}{16} v_3^2 v_\Phi^2 + \frac{\lambda_5}{8v_0 v_\Delta}\left\{v^3 v_\Delta^3 - 3v^2 v_0 v_\Delta^2 v_\Phi + 3v v_0^2 v_\Delta v_\Phi^2 - v_0(v_0^2 + v_3^2)v_\Phi^3\right\} \\
V_{CB14} - V_{N1} &= -\frac{3\tilde{\lambda}_2}{8}(2v_2^2 + 3v_3^2)v_\Delta^2 - \frac{\lambda_4}{16}(v_1^2 + 2v_2^2 + 3v_3^2)v_\Phi^2 \\
&\quad - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left\{v^2 v_\Delta^2 - 2v v_0 v_\Delta v_\Phi + v_\Phi^2(v_0^2 + v_1^2 + v_2^2 + v_3^2)\right\} \\
V_{CB11} - V_{N1} &= -\frac{3\tilde{\lambda}_2}{4} v_2^2 v_\Delta^2 - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left\{v^2 v_\Delta^2 - 2v v_0 v_\Delta v_\Phi + (v_0^2 + v_1^2 + v_2^2)v_\Phi^2\right\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{f_\Delta v_\Delta^2 + f_{\Delta\Phi} v_\Delta v_\Phi + f_\Phi v_\Phi^2}{16\left\{6v_0^2 v_2 - 3v_1^2 v_2 + 3v_2^3 - 2\sqrt{3}v_0(v_1^2 - 2v_2^2)\right\}} \\
V_{CB12} - V_{N1} =& -\frac{9\tilde{\lambda}_2}{8} v_3^2 v_\Delta^2 - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left\{v^2 v_\Delta^2 - 2vv_0 v_\Delta v_\Phi + (v_0^2 + v_1^2 + v_3^2)v_\Phi^2\right\} \\
&- \frac{\lambda_4}{16}\frac{3vv_\Delta(3v_0^2 - 2v_1^2)(vv_\Delta - 2v_0 v_\Phi) + \left\{(9v_0^4 + 3v_0^2(v_1^2 + 9v_3^2) + 2(v_1^4 - 9v_3^4)\right\} v_\Phi^2}{9v_0^2 + 2v_1^2 - 6v_3^2} \\
V_{CB13} - V_{N1} =& -\frac{3\tilde{\lambda}_2}{8}(2v_2^2 + 3v_3^2)v_\Delta^2 - \frac{\lambda_5}{8}\frac{v_\Phi}{v_\Delta}\left\{v^2 v_\Delta^2 - 2vv_0 v_\Delta v_\Phi + (v_0^2 + v_2^2 + v_3^2)v_\Phi^2\right\} \\
&+ \frac{\lambda_4}{16}\frac{3vv_\Delta(2v_2^2 - 3v_3^2)(vv_\Delta - 2v_0 v_\Phi) + 2(4v_2^4 - 9v_0^2 v_3^2 + 6v_2^2 v_3^2)v_\Phi^2}{3v_0^2 - 4v_2^2},
\end{aligned}
$$

where

$$
\begin{aligned}
f_\Delta =& v^2\left\{-6v_0^2 v_2 + 7v_1^2 v_2 - 8v_2^3 + 2\sqrt{3}v_0(v_1^2 - 4v_2^2)\right\} \\
f_{\Delta\Phi} =& -2v\left\{-6v_0^3 v_2 + 2\sqrt{3}v_1^2 v_2^2 + 2\sqrt{3}v_0^2(v_1^2 - 4v_2^2) + v_0(7v_1^2 v_2 - 8v_2^3)\right\} \\
f_\Phi =& -6v_0^4 v_2 + v_0^2 v_2(v_1^2 - 20v_2^2) + 2\sqrt{3}v_0^3(v_1^2 - 4v_2^2) + 2\sqrt{3}v_0(v_1^4 + v_1^2 v_2^2 - 4v_2^4) \\
&+ 3v_2(v_1^4 + v_1^2 v_2^2 - 2v_2^4)
\end{aligned}
$$

Some of these expressions can be further simplified by using the *additional* minimization conditions that apply to the CB configurations—that is, the CB-specific tadpole equations beyond the μ_Φ^2 and μ_Δ^2 relations already imposed at the $N1$ stationary point. However, even after incorporating these extra conditions, the resulting expressions do not, in general, reduce to particularly useful analytic forms.

For the configuration $CB7$, the minimization conditions imply $\tilde{\lambda}_2 = 0$, $\lambda_4 = 0$, and the potential difference takes the following form when expressed in terms of the physical masses evaluated at the stationary point $N1$

$$
V_{CB7} - V_{N1} = \frac{3v^2 v_\Delta^2 + (v_1^2 + v_2^2 + v_3^2)\, v_\Phi^2}{4\,(v_\Phi^2 + 9v_\Delta^2)} m_A^2.
$$

If $N1$ is a true minimum, all mass-squared eigenvalues are positive. Therefore the expression above is strictly positive: $V_{CB7} - V_{N1} > 0$, and thus $N1$ is necessarily stable against charge-breaking extrema of type $CB7$.

For CB14, the minimization conditions set $\tilde{\lambda}_2 = 0$, $\lambda_4 = 0$ and $\lambda_5 = 0$, so that the potential difference vanishes identically: $V_{CB14} - V_{N1} = 0$. The origin of this degeneracy is straightforward: in this limit, the scalar potential depends only on the invariants $\Phi^\dagger\Phi$ and $\Delta^\dagger\Delta$, and is therefore insensitive to the orientation of the triplet in $SU(2)$ space. Consequently, the apparent charge-breaking VEVs of the triplet can be rotated away, continuously mapping the would-be $CB14$ configuration into the $N1$ one. In other words, $CB14$ ceases to be a genuinely charge-breaking vacuum in this limit. For generic scalar-potential parameters—i.e. whenever at least one of $\tilde{\lambda}_2, \lambda_4$ and λ_5 is nonzero—this degeneracy is lifted, and no $CB14$ stationary point coexists with $N1$.

At this point, it becomes clear that $N1$ is invariably stable against the configurations $CB7$ and $CB14$. For the remaining charge-breaking extrema, while it is evident that $N1$ is not generically guaranteed to be the global minimum—since several charge-breaking stationary points may coexist with, and potentially lie below, it in potential depth—the corresponding expressions involve several competing terms and cannot be reduced to simple positivity conditions on masses or couplings. As a result, no useful analytic criterion can be extracted for most cases. Therefore, for these configurations, the stability of $N1$ must be assessed for each specific choice of scalar couplings.

A more practical route to establishing the globality of $N1$ exploits the special role of $N2$. Equation (3.20) ensures that $N2$ is stable against all charge-breaking extrema, while Eq. (3.21) (or Eq. (3.22)) provides the analytic condition under which $N1$ lies deeper than $N2$. When both conditions are satisfied simultaneously, $N1$ is necessarily deeper than every other stationary configuration and thus constitutes the global minimum of the scalar potential.

Finally, it is worth emphasizing the phenomenological relevance of the two electroweak stationary points. Neutrino masses in the BNT model arise from the dimension-seven operator induced by the $\lambda_5\, \Delta^\dagger \Phi^3$ interaction, which necessarily requires $\lambda_5 \neq 0$ and hence a nonvanishing quadruplet VEV. As a consequence, only the general electroweak vacuum $N1$ is compatible with neutrino-mass generation, whereas the electroweak-like configuration $N2$—which exists solely for $\lambda_5 = 0$—cannot generate neutrino masses and is therefore phenomenologically irrelevant. Although from a purely theoretical standpoint it would suffice to ensure that $N1$ is stable against charge-breaking extrema, the requirement of neutrino-mass generation imposes an additional constraint: $N2$ must not lie below $N1$. The analytic condition derived in Eq. (3.21) (or Eq. (3.22)), together with the mass inequalities of Eq. (3.20), thus provides a practical and phenomenologically necessary criterion ensuring that $N1$ is the global minimum of the scalar potential.

4 Summary and outlook

In this work we have undertaken a systematic study of the vacuum structure of the Babu–Nandi–Tavartkiladze (BNT) model of neutrino mass generation, imposing from the outset the general theoretical requirements that the scalar potential be bounded from below in all field directions and that the quartic couplings satisfy perturbative unitarity constraints obtained from scalar–scalar scattering amplitudes. We, then, classified the rich set of stationary configurations admitted by the scalar potential, derived the corresponding minimization conditions, and evaluated the potential differences between all coexisting extrema. This provides a complete basis for comparing the relative depths of the various stationary points and determining the conditions under which the electroweak vacuum constitutes the global minimum.

A first important result is that the electroweak-like vacuum $N2$, characterized by a vanishing quadruplet VEV, is stable against all charge-breaking extrema whenever the two simple mass inequalities collected in Eq. (3.20) are satisfied. For the general electroweak vacuum $N1$, in which both the doublet and quadruplet acquire VEVs, no comparably simple analytic condition emerges: most charge-breaking configurations lead to potential differences containing several competing terms, preventing the extraction of generic positivity constraints on masses and/or couplings. Consequently, the stability of $N1$ must be assessed for each specific choice of scalar couplings.

We have shown, however, that the interplay between the two EW stationary points yields a practical and complete set of analytic criteria. The condition in Eq. (3.21) (or Eq. (3.22)) determines when $N1$ lies deeper than $N2$, while the inequalities Eq. (3.20) ensure that $N2$ is deeper than all CB stationary points. When these conditions are simultaneously satisfied, $N1$ is necessarily deeper than every other stationary configuration and therefore constitutes the global minimum of the scalar potential.

Finally, we emphasize the phenomenological significance of these results. Neutrino masses in the BNT model arise through the dimension-seven operator generated by the $\lambda_5\, \Delta^\dagger \Phi^3$ interaction, which requires $\lambda_5 \neq 0$ and hence a nonvanishing quadruplet VEV. This singles out $N1$ as the only electroweak vacuum compatible with neutrino-mass generation, while $N2$—which exists only for $\lambda_5 = 0$—cannot produce neutrino masses and is therefore phenomenologically disfavored. The analytic conditions derived here thus provide the necessary and sufficient criteria for establishing the global stability of the physically relevant electroweak vacuum of the BNT model.

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A $\quad 2 \to 2$ scalar-scalar scattering submatrices

- $\underline{q = 6}$: $\left(\frac{1}{\sqrt{2}}\delta^{+++}\delta^{+++}\right)$

$$\mathcal{M}_6 = 2\left(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2\right)$$

- $\underline{q=5}$: $(\delta^{+++}\delta^{++})$

$$\mathcal{M}_5 = 2\left(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2\right)$$

- $\underline{q=4}$: $\left(\delta^{+++}\phi^{+}, \delta^{+++}\delta^{+}, \frac{1}{\sqrt{2}}\delta^{++}\delta^{++}\right)$

$$\mathcal{M}_4 = \begin{pmatrix} \lambda_3 + \frac{3}{4}\lambda_4 & 0 & 0 \\ 0 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & 2\sqrt{6}\tilde{\lambda}_2 \\ 0 & 2\sqrt{6}\tilde{\lambda}_2 & 2\lambda_2 + \frac{1}{2}\tilde{\lambda}_2 \end{pmatrix}$$

- $\underline{q=3}$: $(\delta^{++}\phi^{+}, \delta^{+++}\phi^0_R, \delta^{+++}\phi^0_I)$

$$\mathcal{M}_3^a = \begin{pmatrix} \lambda_3 + \frac{1}{4}\lambda_4 & \frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & i\frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 \\ \frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & \lambda_3 - \frac{3}{4}\lambda_4 & 0 \\ i\frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & 0 & \lambda_3 - \frac{3}{4}\lambda_4 \end{pmatrix}$$

- $\underline{q=3}$: $(\delta^{++}\delta^{+}, \delta^{+++}\delta^0_R, \delta^{+++}\delta^0_I)$

$$\mathcal{M}_3^b = \begin{pmatrix} 2\lambda_2 + \frac{7}{2}\tilde{\lambda}_2 & \frac{3}{\sqrt{2}}\tilde{\lambda}_2 & i\frac{3}{\sqrt{2}}\tilde{\lambda}_2 \\ \frac{3}{\sqrt{2}}\tilde{\lambda}_2 & 2\lambda_2 - \frac{9}{2}\tilde{\lambda}_2 & 0 \\ -i\frac{3}{\sqrt{2}}\tilde{\lambda}_2 & 0 & 2\lambda_2 - \frac{9}{2}\tilde{\lambda}_2 \end{pmatrix}$$

- $\underline{q=2}$: $(\delta^{+}\phi^{+}, \delta^{++}\phi^0_R, \delta^{++}\phi^0_I, \delta^{+++}\phi^{-}, \phi^{+}\phi^{+})$

$$\mathcal{M}_2^a = \begin{pmatrix} \lambda_3 - \frac{1}{4}\lambda_4 & \frac{1}{\sqrt{2}}\lambda_4 & i\frac{1}{\sqrt{2}}\lambda_4 & 0 & 0 \\ \frac{1}{\sqrt{2}}\lambda_4 & \lambda_3 - \frac{1}{4}\lambda_4 & 0 & \frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & \sqrt{6}\lambda_5 \\ -i\frac{1}{\sqrt{2}}\lambda_4 & 0 & \lambda_3 - \frac{1}{4}\lambda_4 & i\frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & i\sqrt{6}\lambda_5 \\ 0 & \frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & -i\frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & \lambda_3 + \frac{3}{4}\lambda_4 & 6\lambda_5 \\ 0 & \sqrt{6}\lambda_5 & -i\sqrt{6}\lambda_5 & 6\lambda_5 & 4\lambda_1 \end{pmatrix}$$

- $\underline{q=2}$: $(\delta^{+}\delta^{+}, \delta^{++}\delta^0_R, \delta^{++}\delta^0_I, \delta^{+++}\delta^{-})$

$$\mathcal{M}_2^b = \begin{pmatrix} 4\lambda_2 + \tilde{\lambda}_2 & 2\sqrt{6}\tilde{\lambda}_2 & i2\sqrt{6}\tilde{\lambda}_2 & 0 \\ 2\sqrt{6}\tilde{\lambda}_2 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & 0 & \frac{3}{\sqrt{2}}\tilde{\lambda}_2 \\ -i2\sqrt{6}\tilde{\lambda}_2 & 0 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & i\frac{3}{\sqrt{2}}\tilde{\lambda}_2 \\ 0 & \frac{3}{\sqrt{2}}\tilde{\lambda}_2 & -i\frac{3}{\sqrt{2}}\tilde{\lambda}_2 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 \end{pmatrix}$$

- $\underline{q=1}$: $(\delta^{+++}\delta^{--}, \delta^{++}\delta^{-}, \delta^{+}\delta^{0}_{R}, \delta^{+}\delta^{0}_{I}, \phi^{+}\phi^{0}_{R}, \phi^{+}\phi^{0}_{I}, \delta^{++}\phi^{-}, \delta^{+}\phi^{0}_{R}, \delta^{+}\phi^{0}_{I}, \phi^{+}\delta^{0}_{R}, \phi^{+}\delta^{0}_{I})$

$$\mathcal{M}_1 = \begin{pmatrix} A' & B' \\ B'^{\dagger} & C' \end{pmatrix},$$

with

$$A' = \begin{pmatrix}
2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 4\sqrt{3}\tilde{\lambda}_2 & \frac{3}{\sqrt{2}}\tilde{\lambda}_2 & -i\frac{3}{\sqrt{2}}\tilde{\lambda}_2 & \frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & -i\frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 \\
4\sqrt{3}\tilde{\lambda}_2 & 2\lambda_2 + \frac{7}{2}\tilde{\lambda}_2 & 2\sqrt{6}\tilde{\lambda}_2 & -i2\sqrt{6}\tilde{\lambda}_2 & \frac{1}{\sqrt{2}}\lambda_4 & -i\frac{1}{\sqrt{2}}\lambda_4 \\
\frac{3}{\sqrt{2}}\tilde{\lambda}_2 & 2\sqrt{6}\tilde{\lambda}_2 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 0 & \frac{\sqrt{3}}{4}\lambda_4 & -i\frac{\sqrt{3}}{4}\lambda_4 \\
i\frac{3}{\sqrt{2}}\tilde{\lambda}_2 & i2\sqrt{6}\tilde{\lambda}_2 & 0 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & i\frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{4}\lambda_4 \\
\frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & \frac{1}{\sqrt{2}}\lambda_4 & \frac{\sqrt{3}}{4}\lambda_4 & -i\frac{\sqrt{3}}{4}\lambda_4 & 2\lambda_1 & 0 \\
i\frac{\sqrt{3}}{2\sqrt{2}}\lambda_4 & i\frac{1}{\sqrt{2}}\lambda_4 & i\frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{4}\lambda_4 & 0 & 2\lambda_1
\end{pmatrix}$$

$$B' = \begin{pmatrix}
\multicolumn{4}{c}{0_{4\times 5}} \\
\sqrt{6}\lambda_5 & \sqrt{3}\lambda_5 & -i\sqrt{3}\lambda_5 & \\
-i\sqrt{6}\lambda_5 & -i\sqrt{3}\lambda_5 & -\sqrt{3}\lambda_5 & \raisebox{1ex}{$0_{2\times 2}$}
\end{pmatrix}$$

$$C' = \begin{pmatrix}
\lambda_3 + \frac{1}{4}\lambda_4 & \frac{1}{\sqrt{2}}\lambda_4 & -i\frac{1}{\sqrt{2}}\lambda_4 & 0 & 0 \\
\frac{1}{\sqrt{2}}\lambda_4 & \lambda_3 + \frac{1}{4}\lambda_4 & 0 & \frac{\sqrt{3}}{4}\lambda_4 & i\frac{\sqrt{3}}{4}\lambda_4 \\
i\frac{1}{\sqrt{2}}\lambda_4 & 0 & \lambda_3 + \frac{1}{4}\lambda_4 & -i\frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{4}\lambda_4 \\
0 & \frac{\sqrt{3}}{4}\lambda_4 & i\frac{\sqrt{3}}{4}\lambda_4 & \lambda_3 - \frac{3}{4}\lambda_4 & 0 \\
0 & -i\frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{4}\lambda_4 & 0 & \lambda_3 - \frac{3}{4}\lambda_4
\end{pmatrix}$$

- $\underline{q=0}$: $\left(\delta^{3+}\delta^{3-}, \delta^{2+}\delta^{2-}, \delta^{+}\delta^{-}, \delta^{+}\phi^{-}, \phi^{+}\delta^{-}, \phi^{+}\phi^{-}, \frac{\delta^0_R\delta^0_R}{\sqrt{2}}, \delta^0_R\delta^0_I, \frac{\delta^0_I\delta^0_I}{\sqrt{2}}, \delta^0_R\phi^0_R, \delta^0_R\phi^0_I, \delta^0_I\phi^0_R, \delta^0_I\phi^0_I, \frac{\phi^0_R\phi^0_R}{\sqrt{2}}, \phi^0_R\phi^0_I, \frac{\phi^0_I\phi^0_I}{\sqrt{2}}\right)$

$$\mathcal{M}_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with

$$A = \begin{pmatrix}
4\lambda_2 + 9\tilde{\lambda}_2 & 2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & 0 & 0 & \lambda_3 + \frac{3\lambda_4}{4} & \frac{1}{\sqrt{2}}(2\lambda_2 - \frac{9}{2}\tilde{\lambda}_2) \\
2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2 & 4\lambda_2 + \tilde{\lambda}_2 & 2\lambda_2 + \frac{7}{2}\tilde{\lambda}_2 & 0 & 0 & \lambda_3 + \frac{1}{4}\lambda_4 & \frac{1}{\sqrt{2}}(2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2) \\
2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2 & 2\lambda_2 + \frac{7}{2}\tilde{\lambda}_2 & 4\lambda_2 + \tilde{\lambda}_2 & 0 & 0 & \lambda_3 - \frac{1}{4}\lambda_4 & \frac{1}{\sqrt{2}}(2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2) \\
0 & 0 & 0 & \lambda_3 - \frac{\lambda_4}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_3 - \frac{\lambda_4}{4} & 0 & 0 \\
\lambda_3 + \frac{3}{4}\lambda_4 & \lambda_3 + \frac{1}{4}\lambda_4 & \lambda_3 - \frac{1}{4}\lambda_4 & 0 & 0 & 4\lambda_1 & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{3}{4}\lambda_4) \\
\frac{1}{\sqrt{2}}(2\lambda_2 - \frac{9}{2}\tilde{\lambda}_2) & \frac{1}{\sqrt{2}}(2\lambda_2 - \frac{3}{2}\tilde{\lambda}_2) & \frac{1}{\sqrt{2}}(2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2) & 0 & 0 & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{3}{4}\lambda_4) & 3(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2)
\end{pmatrix}$$

$$B = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}(2\lambda_2 - \frac{9\tilde{\lambda}_2}{2}) & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{3}{4}\lambda_4) & 0 & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{3}{4}\lambda_4) \\ 0 & \frac{1}{\sqrt{2}}(2\lambda_2 - \frac{3\tilde{\lambda}_2}{2}) & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{1}{4}\lambda_4) & 0 & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{1}{4}\lambda_4) \\ 0 & \frac{1}{\sqrt{2}}(2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2) & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}(\lambda_3 + \frac{1}{4}\lambda_4) & 0 & \frac{1}{\sqrt{2}}(\lambda_3 + \frac{1}{4}\lambda_4) \\ 0 & 0 & \frac{\sqrt{3}}{4}\lambda_4 & -i\frac{\sqrt{3}}{4}\lambda_4 & i\frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{\sqrt{2}}\lambda_5 & i\sqrt{3}\lambda_5 & -\frac{\sqrt{3}}{\sqrt{2}}\lambda_5 \\ 0 & 0 & \frac{\sqrt{3}\lambda_4}{4} & i\frac{\sqrt{3}\lambda_4}{4} & -i\frac{\sqrt{3}\lambda_4}{4} & \frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{\sqrt{2}}\lambda_5 & -i\sqrt{3}\lambda_5 & -\frac{\sqrt{3}}{\sqrt{2}}\lambda_5 \\ 0 & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{3}{4}\lambda_4) & 0 & 0 & 0 & 0 & \sqrt{2}\lambda_1 & 0 & \sqrt{2}\lambda_1 \\ 0 & \lambda_2 + \frac{9}{4}\tilde{\lambda}_2 & 0 & 0 & 0 & 0 & \frac{1}{2}(\lambda_3 + \frac{3}{4}\lambda_4) & 0 & \frac{1}{2}(\lambda_3 + \frac{3}{4}\lambda_4) \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}(2\lambda_2 - \frac{9\tilde{\lambda}_2}{2}) & \frac{1}{\sqrt{2}}(2\lambda_2 - \frac{3\tilde{\lambda}_2}{2}) & \frac{1}{\sqrt{2}}(2\lambda_2 + \frac{9}{2}\tilde{\lambda}_2) & 0 & 0 & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{3}{4}\lambda_4) & \lambda_2 + \frac{9}{4}\tilde{\lambda}_2 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{4}\lambda_4 & 0 & 0 \\ 0 & 0 & 0 & i\frac{\sqrt{3}}{4}\lambda_4 & -i\frac{\sqrt{3}}{4}\lambda_4 & 0 & 0 \\ 0 & 0 & 0 & -i\frac{\sqrt{3}}{4}\lambda_4 & i\frac{\sqrt{3}}{4}\lambda_4 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{4}\lambda_4 & \frac{\sqrt{3}}{4}\lambda_4 & 0 & 0 \\ \frac{1}{\sqrt{2}}(\lambda_3 - \frac{3}{4}\lambda_4) & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{1}{4}\lambda_4) & \frac{1}{\sqrt{2}}(\lambda_3 + \frac{1}{4}\lambda_4) & \frac{\sqrt{3}}{\sqrt{2}}\lambda_5 & \frac{\sqrt{3}}{\sqrt{2}}\lambda_5 & \sqrt{2}\lambda_1 & \frac{1}{2}(\lambda_3 + \frac{3}{4}\lambda_4) \\ 0 & 0 & 0 & -i\sqrt{3}\lambda_5 & i\sqrt{3}\lambda_5 & 0 & 0 \\ \frac{1}{\sqrt{2}}(\lambda_3 - \frac{3}{4}\lambda_4) & \frac{1}{\sqrt{2}}(\lambda_3 - \frac{1}{4}\lambda_4) & \frac{1}{\sqrt{2}}(\lambda_3 + \frac{1}{4}\lambda_4) & -\frac{\sqrt{3}}{\sqrt{2}}\lambda_5 & -\frac{\sqrt{3}}{\sqrt{2}}\lambda_5 & \sqrt{2}\lambda_1 & \frac{1}{2}(\lambda_3 + \frac{3}{4}\lambda_4) \end{pmatrix}$$

$$D = \begin{pmatrix} 2\lambda_2 + \frac{9\tilde{\lambda}_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3(\lambda_2 + \frac{9}{4}\tilde{\lambda}_2) & 0 & 0 & 0 & 0 & \frac{1}{2}(\lambda_3 + \frac{3}{4}\lambda_4) & 0 & \frac{1}{2}(\lambda_3 + \frac{3}{4}\lambda_4) \\ 0 & 0 & \lambda_3 + \frac{3\lambda_4}{4} & 0 & 0 & 0 & \frac{3}{\sqrt{2}}\lambda_5 & 0 & -\frac{3}{\sqrt{2}}\lambda_5 \\ 0 & 0 & 0 & \lambda_3 + \frac{3\lambda_4}{4} & 0 & 0 & 0 & -3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 + \frac{3}{4}\lambda_4 & 0 & 0 & 3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 + \frac{3}{4}\lambda_4 & \frac{3}{\sqrt{2}}\lambda_5 & 0 & -\frac{3}{\sqrt{2}}\lambda_5 \\ 0 & \frac{1}{2}(\lambda_3 + \frac{3}{4}\lambda_4) & \frac{3}{\sqrt{2}}\lambda_5 & 0 & 0 & \frac{3}{\sqrt{2}}\lambda_5 & 3\lambda_1 & 0 & \lambda_1 \\ 0 & 0 & 0 & -3\lambda_5 & 3\lambda_5 & 0 & 0 & 2\lambda_1 & 0 \\ 0 & \frac{1}{2}(\lambda_3 + \frac{3}{4}\lambda_4) & -\frac{3}{\sqrt{2}}\lambda_5 & 0 & 0 & -\frac{3}{\sqrt{2}}\lambda_5 & \lambda_1 & 0 & 3\lambda_1 \end{pmatrix}$$

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