# GERSTEN CONJECTURE FOR K-THEORY ON HENSELIAN SCHEMES AND $\phi$-MOTIVIC LOCALISATION

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Abstract. A key triviality result for support extension maps for motivic $\mathbb{A}^1$-homotopies of cellular motivic spaces $S$ over a DVR spectrum $B$ is proven. Combining with earlier known results on the K-theory Gersten complex and the K-theory motivic spectrum we achieve a proof of the Gersten Conjecture on essentially smooth local Henselian $B$-schemes. Additionally, we outline generalisations for Cousin complexes associated to motivic $\mathbb{A}^1$- and $\square$-homotopies of cellular $B$-spectra.

The proof is based on two ingredients:

(1) A new "motivic localisation" over $B$, called *$\phi$-motivic*, giving rise to the $\phi$-motivic homotopy category such that the triviality of the support extension maps and the acyclicity of Cousin complexes hold for all objects $S$, not necessarily cellular.

(2) An interpretation of some classes in the motivic $\mathbb{A}^1$-homotopies with support defined with respect to the Morel-Voevodsky motivic homotopy category of smooth $B$-schemes in terms of the construction of $\phi$-motivic homotopy category mentioned in Point (1).

Contents



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# 1. Introdction

## 1.1. **Gersten Conjecture for K-theory.**

1.1.1. *Claim and Results.* The Gersten Conjecture [18], [39, Conjecture 5.10] claims the acyclicity of the complex

$$K_n(U) \to \bigoplus_{u\in U^{(0)}} K_n(u) \to \dots \to \bigoplus_{u\in U^{(c)}} K_{n-c}(u) \to \dots \tag{1.1}$$

for a local regular scheme $U$ [1] [2] [3], where $K_*$ is the algebraic K-theory. The claim is proven

- by Gersten in [18], for DVR spectra $U$ with finite residue fields,
- extended to finite fields algebraic extensions in [42] by Sherman,
- by Quillen in [38], for local schemes $U = \mathrm{Spec}(O_{X,x})$, $X \in \mathrm{Sm}_k$, over a field $k$,
- extended to all local regular equi-characteristic schemes $U$ in [34] by Panin.

Our article proves the claim

- ◦ for essentially smooth local Henselian $B$-schemes $U$ over a DVR spectrum $B$.

*Remark* 1. We remark unpublished preprints by Mochizuki referring to the survey [30]. We do not have a consultation with specialists to comment the status and perspectives.

1.1.2. *Generalisations over $k$ and obstructions over $B$.* The pioneer article [38] gave rise to a list of arguments that recover and generalise the result as the acyclicity of Cousin complexes associated with some classes of cohomology theories over base fields, which includes

- [5] by Bloch and Ogus concerned cohomology theories that got their names,
- [46] by Voevodsky on pretheories with transfers over perfect $k$ [4],
- [31, 32] by Morel for $\mathrm{SH}^{S^1}(k)$-representable coh. th.[5],
- [22] by Panin for coh. th. is sense of [35, Def. 2.1] for any $k$.

Summarising,

> the $\mathbb{A}^1$-invariance and Nisnevich excision of a cohomology theory $A$ on $\mathrm{Sm}_k$ are enough for the acyclicity of the associated Cousin complex over any base field $k$.

*Remark* 2 (Stable Connectivity over $B$). The acyclicity of the Cousin complexes for all $\mathrm{SH}(B)$-representable theories on the schemes $U = X^h_x$, $X \in \mathrm{Sm}_B$, would imply the statement of Connectivity Theorem for $\mathrm{SH}(B)$ by the arguments of [32] for any noetherian separated base scheme $B$ of finite Kull dimension. The latter claim got name Connectivity Conjecture and was disproven in [2] by Ayoub.

*Remark* 3 (Counter-example over $B$). The acyclcity fails for the cohomology theory represented by $\Sigma^\infty_{\mathbb{P}^1}(B - z)$ for any closed point $z$ of positive codimension of any $B \in \mathrm{Sch}$, [7]. So

> the $\mathbb{A}^1$-invariance and Nisnevich excision, and even $\mathrm{SH}(B)$-representability, are not enough for the acyclcity of the Cousin complex over $B$, whenever $\dim B > 0$,

and the proof for (1.1) should use other structures/poperties on/of the K-theory.

1.1.3. *Results for "parts or modifications" of K-theory.* The acyclicity of (1.1) is proven for

- K-theory with finite coefficients $K_*(-, \mathbb{Z}/n)$ [19] by Gillet out of the characteristic, and [17] by Geisser and Levine at the characteristic,
- Milnor K-theory $K^{\mathrm{M}}_*(-)$ [28] by Lüders and Morrow,

*Remark* 4 (Surjectivity Property). For any coh. th. $A$ defined by

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[1] claimed after Gersten's result, and formulated publically in such generality by Quillen.
[2] [27] by Kundu discusses generalisation of the initial morphism injectivity in (1.1) for valuation rings.
[3] The acyclicity holds for $n = 0$, $n = 1$ and for $n = 2$, [36] by Panin.
[4] extended later to K-transfers [26] and framed transfers [15], and extended to arbitrary base fields in [12]
[5] using Gabber's Presentation Lemma [14, 23] for infinite $k$, proven later for finite fields in [21]

+ functors $(X, X-Z) \mapsto A^*(X, X-Z) =: A^*_Z(X)$ for $X \in \mathrm{Sch}_B$, and
+ differentials $\partial\colon A^*(X-Z) \to A^{*+1}_Z(X)$,

and equipped additionally with

+ a module structure over the Milnor-Witt K-theory compatible with the differentials,
+ Gysin isomorphisms $A^*(x) \cong A^*_x(X)$, $x \in X$, compatible with the previous points,

the acycliclty of the Gersten complex for $A$ on any DVR spectrum $U \in \mathrm{Sch}_B$ can be deduced from the surjectivity of the restriction homomorphism

$$A^*(U) \twoheadrightarrow A^*(x), \quad x \in U^{(1)}, \tag{1.2}$$

which holds (0) for $U$ equipped with a retraction to $x$ for any $A$, and (1) for any local $U$ for

- $K^{\mathrm{M}}_*(-)$ by lifting of the generators, and
- $K_n(-)$, $n \leq 2$, because of the isomorphism $K^{\mathrm{M}}_n(x) \to K_n(x)$,

and (2) for Henselian local $U$ for

- $K_*(-, \mathbb{Z}/n)$ by the Rigidity Theorem [13, 44] by Suslin and Gabber, and
- $K_*(-)$ for such $U$ that the residue field at $x$ being an algebraic extension of a finite field because of the computation of K-theory of finite fields [40] by Quillen,

and it would follow for

- $K_3(-)$ for any Henselian local $U$ from the Rigidity Conjecture [29, Conjecture 11.7]. [6]

Nevertheless,

> it is unknown do the surjectivity (1.2) hold for higher groups $K_*(-, \mathbb{Q})$ for Henselian DVR spectra $U$ of a characteristic $(0, p)$, $p > 0$, with an infinite residue field at $x$.

1.1.4. *Reduction to one-dimensional schemes.* The acyclicity of (1.1) for essentially smooth one-dimensional schemes $U$ over $\mathbb{Z}$ implies the acyclicity of (1.1)

- for all schemes of the form $U = X_x$ [20] by Gillet and Levine, and
- for all regular unramified local rings [43] by Skalit.

1.1.5. *Reduction to the injectivity claim.* [4] by Bloch provides a quasi-isomorphism of the complex (1.1) and the 3-term complex

$$K_n(U) \to K_n(U - U \times_B z) \to K_{n-1}(U \times_B z) \tag{1.3}$$

where $z \in B$ is the image of the closed point $x \in U$, for any essentially smooth local $U$ scheme over a DVR.[7] [8] Consequently,

- the complex (1.1) is acyclic except the terms $K_n(U)$ and $\oplus_{x \in U^{(1)}} K_{n-1}(x)$, [9] [10] and
- the acyclicity of (1.1) reduces to the injectivity $K_*(U) \hookrightarrow K_*(U - U \times_B z)$.

1.2. **Proof for essentially smooth local Henselian schemes.** Constructing of Nisnevich squares and $\mathbb{A}^1$-homotopies over netherian separated schemes $B$ of finite Krull dimension allowed in [6, 11, 41] proving of the triviality of the support extension morphisms

$$A_W(U) \to A(U) \tag{1.4}$$

for any closed immersion $W \to U = X^h_x$ of positive relative codimension for all pointed functors $A\colon \mathrm{H}^\bullet_{\mathbb{A}^1,\mathrm{Nis}}(B) \to \mathrm{Set}^\bullet$. A novel instrument allows us to prove the claim for the closed fibre immersion

$$W = U \times_B z \to U = X^h_x, \tag{1.5}$$

where $z \in B$ is the image of $x \in X \in \mathrm{Sm}_B$, for motivic homotopies of cellular motivic $B$-spaces.

[6]The latter two points are mentioned to the author by A. Ananyevsky.

[7]Generalised by [43] by Skalit to geometrically regular $U$ over DVR.

[8]Generalised by [7, 12] to $\mathrm{SH}(B)$-representable coh. th. for any $B \in \mathrm{Sch}$, $\dim B < \infty$.

[9]The bounded acyclicity is proven also in [43] for geometrically regular Henselian $U$ over $B \in \mathrm{Sch}$.

[10]The acyclicity except the mentioned terms is proven also in [36] by Panin for semi-local schemes $U$ of smooth schemes and coh. th. in sense of [35, Def. 2.1] over a DVR.

**Theorem** (Theorems 4.1, 4.3)**.** *Let $B \in \mathrm{Sch}$ be regular of Krull dimension one. Let $U$ be an essentially smooth local henselian scheme over $B$. Let $\rho$ denote a regular function on $B$ that vanishing locus equals a closed point in $B$ as well as its inverse image on $U$.*

*For any non-negative integer $n$, the morphism of the pointed sets*

$$E_\tau\colon \mathrm{Map}_{\mathrm{H}^\bullet_{\mathbb{A}^1,\tau}(B)}(U_+/(U-Z(\rho))_+\wedge S^n, S) \to \mathrm{Map}_{\mathrm{H}^\bullet_{\mathbb{A}^1,\mathrm{Nis}}(B)}(U_+\wedge S^n, S) \tag{1.6}$$

*is trivial for any $S \in \mathrm{H}^\bullet_{\mathbb{A}^1,\tau}(B)$ generated by $\mathbb{P}^{\wedge l}_B$ for all $l \in \mathbb{Z}_{\geq 0}$ via colimits and extensions, and $\tau = \mathrm{Zar}$.*

Though the result holds for the Nisnevich topology, which follows from Remark 23, for simplicity, in Sections 3 and 4.1, we present the proof for

$$\tau = \mathrm{Zar}.$$

This is enough for the triviality of (1.4) for K-theory proven in Section 4.2, because by [1] presheaves of vector bundles on affine schemes are represented by Grassmanians in both $\mathrm{H}^\bullet_{\mathbb{A}^1,\mathrm{Zar}}(B)$ and $\mathrm{H}^\bullet_{\mathbb{A}^1,\mathrm{Nis}}(B)$. Then the acyclicity of the 3-term complex (1.3) follows. Since as mentioned above the complexes (1.1) and (1.3) are quasi-isomorphic [4], and also [7, 12, 43], (1.1) is acyclic.

The method for the triviality of (1.6) in Theorem 4.3 in our article is the lifting of the elements of

$$\mathrm{Map}_{\mathrm{SH}^{S^1}_{\mathbb{A}^1,\tau}(B)}(U_+/(U-Z(\rho))_+\wedge S^n, S)$$

along the closed immersion of open pairs

$$(U, U-Z(\rho)) \xrightarrow{\rho^{\mathrm{Map}}} (\mathbb{A}^N_U, \mathbb{A}^N_U - Z(t_0))^h_{\rho^{\mathrm{Map}}(U)},$$

where $\rho^{\mathrm{Map}}\colon U \to \mathbb{A}^1_U$ is the morphism defined by $\rho$, Theorem 4.1, which reduces the claim on (1.6) to the mentioned above one on (1.4) for $\mathrm{codim}_{U/B} W > 0$, cited as Theorem 4.2.

The proof of the latter lifting in Section 4.1, Theorem 4.1, uses two ingredients:

(1) a new type of motivic equivalences called $\phi$-motivic equivalences defined as morphisms in a certain $\infty$-category denoted $O_B$ in Section 2 based on [8, 9],
(2) an interpretation of the morphisms $U/(U-Z(\rho))\wedge S^l \to \mathbb{P}^{\wedge n}_B$ in $\mathrm{H}^\bullet_{\mathbb{A}^1,\tau}(B)$ in terms of the category of presheaves $\mathrm{Spc}(O_B) = \mathrm{Func}(O_B^{\mathrm{op}}, \mathrm{Spc})$ in Section 3 based on [10].

1.2.1. *$\phi$-motivic localisation and homotopy category.* The localisation of the $\infty$-category $\mathrm{Spc}(O_B)$ with respect to the $\phi$-motivic equivalences gives rise further to the $\phi$-motivic homotopy category over $B$, Definition 5.4. The triviality like for (1.6) holds for any pointed presheaf defined on the $\phi$-motivic homotopy category over $B$, Remark 23, and the acyclicity of the Cousin complexe hold for any $\phi$-motivic prehseaf of spectra.

*Remark* 5. To continue the discussion from Remark 3, we note that following further the arguments of [32] we can deduce Stable Connectivity and Strict Homotopy Invariance Theorems for the $S^1-$ and $\mathbb{P}^1$-stable $\phi$-motivic categories from the acyclcity of the Cousin complexes.

The role played by the category $\mathrm{Sm}_B$ in the Morel-Voevodsky construction is played by the $\infty$-category $O_B$, which definition in Sections 2.2 to 2.4, starts with the category $A_B$ of data

$$(\mathsf{X}, (\rho, (\partial, \varsigma, \beta))), \tag{1.7}$$

where $\mathsf{X} \in \mathrm{Sch}_B$, $\rho$ is some set of sections of line bundles, $\partial, \varsigma$ are $\rho$-monomials, and $\beta$ is some set of sets of $\rho$-monomials. The interpretation is as follows:

- $\mathsf{X}$ is the underlying scheme of a compacitification,
- $\partial$ relates to the "infinitely remote subscheme" called boundary of the compactification,
- $\varsigma$ relates to the support of cohomologies,
- $\beta$ define centres of some blow-ups on $\mathsf{X}$,

- $\rho$ are parameters on $\mathsf{X}$ that control $\partial$, $\varsigma$ and $\beta$.

Then we define $O_B$ as certain full subcategory of the $\infty$-category $\mathrm{Spc}^{\vec{A}_B}$ of copresheaves on the category of arrows $\vec{A}_B = (A_B)^{\Delta^1}$, Definition 2.43. The conditions on the objects

$$\vec{X} = (X \to \underline{X}) \in O_B, \quad X \to \underline{X} \in \mathrm{Spc}^{A_B}, \tag{1.8}$$

in sense of functor $O_B \hookrightarrow \mathrm{Spc}^{\vec{A}_B} \simeq (\mathrm{Spc}^{A_B})^{\Delta^1}$, imply the diagram in the $\infty$-category of copresheaves on $\mathrm{Sch}_B$

$$\prod_{\alpha} \mathbb{P}_B^{N_\alpha} \simeq \mathsf{P} \xleftarrow{b} \mathsf{B} \xleftarrow{e} \mathsf{E} \simeq \mathsf{X}$$

where

- $b$ is a limit of blow-ups that centres are intersections of coordinate divisors and exceptional divisors,
- $e$ is a limit of affine étale morphisms,

The $\phi$-motivic homotopy category is the localisation $\mathrm{Spc}_{\tau,\downarrow,r,\phi,\overline{\underline{\mathbb{A}}},Bl_\rho}(O_B)$ of the $\infty$-category of presheaves $\mathrm{Spc}(O_B)$ with respect to

- $\tau$-equivalences, where $\tau$ is the topology on $A_B$ induced by a topology $\tau$ on $\mathrm{Sch}_B$,
- $\varsigma$-equivalences equivalence $X$ with $\mathrm{cofib}((\mathsf{X}, (\rho, (\partial \cdot \varsigma, 1, \beta))) \to (\mathsf{X}, (\rho, (\partial, 1, \beta))))$
- $\downarrow$-equivalences defined by the counit of the adjunction $\vec{A}_B \simeq (A_B)^{\Delta^1} \leftrightarrows A_B$,
- $r$-equivalences, which allow modifying $\underline{\rho}$ on $\underline{\mathsf{X}}$ in (1.8),
- $Bl_\rho$-equivalences, which are limit morphisms of certain blow-ups controlled by $\rho$, Section 2.5.1,
- $\overline{\underline{\mathbb{A}}^1}$-equivalences, which are like $\square$-equivalences in [3, 24],
- $\phi$-equivalences, which are the main ingredient for the proof of the lifting property of elements in motivic homotopy groups w.r.t. closed immersions of essentially smooth local henselian $B$-schemes.

*Remark* 6 (Illuminating $(\downarrow, r, \varsigma)$-equivalences and the data $\rho$). The $\phi$-motivic homotopy category can be constructed equivalently skipping the data $\rho$ and $\varsigma$ and the arrows category $\vec{A}_B$ in the definitions of $A_B$ and $O_B$, Equation (5.2). Nevertheless, all of this is useful for the proof of the triviality of support extension maps and acyclcity of Cousin complexes discussed in the article.

*Remark* 7 (Continuous presheaves without $\phi$-invariance condition). There is a relation of the subcategory of continuous objects in the category of $(\tau, \downarrow, r, \overline{\underline{\mathbb{A}}}, Bl_\rho)$-invariant objects, i.e. without $\phi$-invariance condition, to the $\square$-homotopy motivic categories form [3, 24]. A difference of the localisation procedures, nevertheless, is the type of blow-ups. Resolution of singularities over $B$ could help to illuminate this difference.

1.2.2. *Interpretation of Morel-Voevodsky $\mathbb{A}^1$-homotopies in the $\infty$-category $O_B$.* The Part (2) interprets elements of

$$\mathrm{Map}_{\mathrm{H}^\bullet_{\mathbb{A}^1,\mathrm{Zar}}(B)}(U_+/(U - Z(\rho))_+ \wedge K, \mathbb{P}_B^{\wedge n}) \tag{1.9}$$

as morphisms in $\mathrm{Spc}_{Bl_\rho,r}(O_B)$ from the object in $O_B$ defined by

$$(U, (\rho, 1, \rho)) \in A_B$$

to the motivic sphere $\mathbb{P}_B^l$ in $A_B$. The construction starts with coding of the element in (1.9) by sets of sections of line bundles on the compactified geometric realisation $|K|_{\overline{\underline{\mathbb{A}}}} \times U$ of $K$ over $U$. The latter sections are replaced by sections on certain blow-up of $|K|_{\overline{\underline{\mathbb{A}}}} \times U$ using Theorem 3.2 from Section 3.3 based on the study of blow-ups of schemes with relation to the data $\rho$, Definition 2.8 applied to blow-ups in the category $O_B$.

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1.4. **Notation.**

(1) $\operatorname{codim}_X(Z)$ is the codimension of $Z$ in $X$. $\operatorname{codim}_X(\emptyset) = \infty$.
(2) $\operatorname{codim}_{U/B} W$ is the relative codimension of $W$ in $U$ over $B$.
(3) $\operatorname{Map}_C(-,-)$, $\operatorname{Map}^{\mathrm{Spc}}_C(-,-)$ and $\operatorname{Map}^{\mathrm{Spt}}_C(-,-)$ denote respectively the mapping set or the mapping space, or the mapping spectrum in a category or an $\infty$-category $C$, or a stable $\infty$-category $C$.
(4) $S^1_{\mathbb{A}^1} \simeq \Delta^1_{\mathbb{A}^1}/\partial\Delta^1_{\mathbb{A}^1}$, $S^l_{\mathbb{A}^1} \simeq (S^1_{\mathbb{A}^1})^{\wedge l}$
(5) Let $\operatorname{ShifColim}_C(-)$ denote the subcategory of $C$ generated via shifts and colimits.
(6) $\operatorname{Subsets}(S)$ is the set of subsets of a set $S$.
(7) Let $X$ be a scheme.
   $X^{(c)}$ is the set of points on codimension $c$.
   $X_x$ and $X^h_x$ are the local and Henselian local schemes of a scheme $X$ at $x \in X$.
   Given set of sections of line bundles $I$ on $X$, $Z(I) = Z_X(I)$ is the vanishing locus.
   Given closed subscheme $Z$ in $X$, $I(Z)$ is the sheaf of vanishing ideals.
   Given divisor $D$ on $X$, $I(D) = I_X(D)$ is the sheaf of ideals defined by $D$. Let
$$Z_X(D) := Z_X(I_X(D)).$$
(8) $A \cdot B = \{a \cdot b \in Mon | a \in A, b \in B\} \in \operatorname{Subsets}(Mon)$ for some $A, B \in \operatorname{Subsets}(Mon)$.
(9) $2^S$ is a set of subsets of a given set $S$.
(10) $\operatorname{Mon}(S) = S^{\mathrm{Mon}}$ is the monoid of monomials of a given set $S$.

(11) We use the identification of $\operatorname{Mon}(2^S) = (2^S)^{\mathrm{Mon}}$ and the submonoid is the monoid of subsets of $S^{\mathrm{Mon}}$ generated by subsets of $S$, Notation 8.
(12) A $\cup$-closed set is a set $\beta$ such that for each $\beta_1, \beta_2 \in \beta$, $\beta_1 \cup \beta_2 \in \beta$. $\beta^{\cup}$ is the minimal $\cup$-closed subset containing given subset $\beta$.
(13) $\mathrm{Spc}^{\bullet}$ denotes the $\infty$-category of pointed spaces.
(14) $S^C = \operatorname{Func}(C, S)$, $S(C) = \operatorname{Func}(C^{\mathrm{op}}, S)$.
(15) Sch and $\mathrm{Sch}_B$ are the categories of Noetherian separated schemes and $B$-schemes.
(16) Spc, Spt are $\infty$-categories of spaces and spetra.
(17) $\mathrm{Spc}_{\mathbb{A}^1,\tau}(\mathrm{Sch}_B)$ and $\mathrm{Spt}_{\mathbb{A}^1,\tau}(\mathrm{Sch}_B)$ are the subcategories of $\mathbb{A}^1$-invariant Nisnevich excisive presheaves of spaces and spectra on $\mathrm{Sch}_B$, and similarly for $\mathrm{Sm}_B$. $\mathrm{Spt}_{\mathbb{A}^1,\tau}(B) = \mathrm{Spt}_{\mathbb{A}^1,\tau}(\mathrm{Sch}_B)$.
(18) Given $\infty$-category $C$, a presheaf $F \in \mathrm{Spc}(\mathrm{Spc}^C)$ is continuous, if for any $X \simeq \varprojlim X_\alpha$ in $C$, the canonical morphism $F(X) \leftarrow \varinjlim_\alpha F(X_\alpha)$ is an equivalence.
(19) For any presheaf $F \in \mathrm{Spc}(C)$ on an $\infty$-category $C$, the same symbol $F$ denotes the associated continuous presheaf on the $\infty$-category of copresheaves $C^{\mathrm{Spc}}$, Notation def:continuous.
(20) $\Delta^{\bullet}$ denotes the category of finite linearly ordered sets with objects $\{0, \dots n\} = [n] = \Delta^n$.
   $\Delta^{\bullet \leq l}$ is the subcategory spanned by $\Delta^n$, $n \leq l$.
   $\Delta^{\times\bullet} = (\Delta^{\bullet\leq 1})^{\mathbb{Z}_{\geq 0}}$.
(21) A cubical set$S$ is a functor
$$S\colon (\Delta^{\times\bullet})^{\mathrm{op}} \to \mathrm{Set}.$$
(22) Let $\rho$ be a subset in a monoid $N$, $\operatorname{Mon}(\rho)$ denotes the submonoid generated by $\rho$.
(23) Sch is the category of noetherian separates schemes.
   $\mathrm{Sch}_B$ is the category of $B$-schemes of such type.
   $\mathrm{Sm}_B$ is the category of smooth $B$-schemes.

(24) Let $c\colon X \to Y$, $f\colon F \to G$ be morphisms of an $\infty$-category. The morphism $c$ satisfies LLP w.r.t. the morphism $f$ or $f$ satisfies RLP w.r.t. $c$ if for any morphisms $X \to F$

and from $Y \to G$ such that the outer square in the diagram is commutative

$$\begin{tikzcd} X \arrow[r] \arrow[d] & F \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & G \end{tikzcd}$$

the space of dashed arrows that fit into the commutative diagram is non empty.

(25) Let $c\colon X \to Y$, $f\colon F \to G$ be morphisms of an $\infty$-category. The morphism $c$ satisfies uLLP w.r.t. the morphism $f$ or $f$ satisfies uRLP w.r.t. $c$ if for any morphisms $X \to F$ and from $Y \to G$ such that the outer square in the diagram is commutative

$$\begin{tikzcd} X \arrow[r] \arrow[d] & F \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & G \end{tikzcd}$$

the space of dashed arrows that fit into the commutative diagram is contractible, i.e. the morphism

$$\mathrm{Map}(Y,F) \to \mathrm{Map}(X,F) \times_{\mathrm{Map}(X,G)} \mathrm{Map}(Y,G).$$

(26) Given a morphism of schemes $X \to X'$, and closed subscheme $Z'$ in $X'$, $X^h_{Z'} := X^h_{Z' \times_{X'} X}$.

(27) $\mathcal{S}_{\setminus\{0\}}(X) = \mathcal{S}(X) \setminus \{0\} = \mathcal{S}(X) - 0$.

(28) Given $f\colon X \to X'$ in Sch, $f^*\colon (\mathcal{S} - \{0\})(X') \dashrightarrow (\mathcal{S} - \{0\})(X)$ is a partly defined map induced by $f^*\colon \mathcal{S}(X') \to \mathcal{S}(X)$ and immersion $\mathcal{S}_{\setminus\{0\}} \to \mathcal{S}$.

## 2. $\phi$-MOTIVIC LOCALISATION

### 2.1. Schemes category preparations.

2.1.1. *Line bundles and sections.*

**Definition 2.1.** Define the morphism of presheaves on Sch

$$\mathcal{S} \to \mathcal{L}; \quad (s, L) \mapsto L, \tag{2.1}$$

where $\mathcal{L}$ denotes the presheaf of line bundles on Sch, and $\mathcal{S}$ is the presheaf of line bundles sections, i.e.

$$\mathcal{S}(X) = \{(s \in L(X), L \in \mathcal{L}(X))\}.$$

*Remark* 8. The presheaf of sets $\mathcal{S}(X)$ has the monoid structure with respect to the tensor products of line bundles and products of sections.

*Remark* 9. Each submomoid of $\mathcal{S}(X)$ contains all invertible sections, i.e. elements $(s, L)$ such that $Z_X(s) = \emptyset$, because the set $\mathcal{S}(X)$ is the set of isomorphicm classes of the pairs $(s, L)$.

**Definition 2.2.** Given $X \in$ Sch, an element $s \in \mathcal{S}(X)$ is *irreducible* if the scheme $Z_X(s)$ is irreducible.

2.1.2. *Subpresheaves of $\mathcal{S}^{\times l}$.*

**Definition 2.3.** We use notation $\mathcal{S}^{\times l}_{\mathcal{L}}$ for powers of $\mathcal{S}$ over $\mathcal{L}$ w.r.t. the morphism (2.1), and

$$\mathcal{S}^{\times \mathfrak{p}}_{\mathcal{L}} := \prod_{l \in \mathfrak{L}} \mathcal{S}^{\times \mathfrak{S} \times_{\mathfrak{L}} \{l\}}_{\mathcal{L}}$$

for a given morphism of sets $\mathfrak{p}\colon \mathfrak{S} \to \mathfrak{L}$.

**Definition 2.4.** The category $sL^{\mathrm{div}}$ is a full subcategory in Set(Sch) that objects are presheaves $F$ of the form
$$F(X) = \{s \in \mathcal{S}_{\mathcal{L}}^{\times\mathfrak{p}} | \cap_{\gamma\in\Gamma_\alpha} Z(s_\gamma) \subset Z(\delta_\alpha), \alpha \in A\} \tag{2.2}$$
for a morphism of finite sets $\mathfrak{p}\colon \mathfrak{S} \to \mathfrak{L}$ and a set $A$ with subsets $\Gamma_\alpha \subset \mathfrak{S}$ and sections $\delta_\alpha \in \mathcal{S}(X)$ for each $\alpha \in A$.

**Definition 2.5.** The category $sL^{\emptyset}$ is a full subcategory in Set(Sch) that objects are presheaves $F$ of the form
$$F(X) = \{s \in \mathcal{S}_{\mathcal{L}}^{\times\mathfrak{p}} | \cap_{\gamma\in\Gamma_\alpha} Z(s_\gamma) \subset \emptyset, \alpha \in A\} \tag{2.3}$$
for a morphism of finite sets $\mathfrak{p}\colon \mathfrak{S} \to \mathfrak{L}$ and a set $A$ with subsets $\Gamma_\alpha \subset \mathfrak{S}$ for each $\alpha \in A$.

2.1.3. *Retract closed immersions.*

**Definition 2.6.** An morphism $c\colon X \to \overline{X}$ in $\mathcal{A}_B$ is a retract-closed immersion if it is a closed immersion and there is a morphism $\overline{X} \to X$ such that the composite
$$X \xrightarrow{c} \overline{X} \to X$$
equals $\mathrm{id}_X$.

2.1.4. *Equational morphisms.*

**Definition 2.7.** A morphism $X \to Y$ in Sch is *equational* if it is isomorphic to the composite
$$X \simeq Z_{\mathbb{A}^N_Y}(V) - Z_{\mathbb{A}^N_Y}(C) \to \mathbb{A}^N_Y \to Y$$
for some sets of regular functions $V$ and $C$ on $\mathbb{A}^N_Y$.

2.2. **$\infty$-category $\vec{\mathcal{A}}_B$.**

2.2.1. *Category $A_B$.*

**Definition 2.8.** An $A$-object over $B$ is the data
$$X = (\mathsf{X}, (\rho, (\partial, \varsigma, \beta))), \tag{2.4}$$
where

(o1) $\mathsf{X} \in \mathrm{Sch}_B$,
(o2) $\rho \in \mathrm{Subsets}(\mathcal{S}(X)_{\setminus\{0\}})$, Notation 28, such that for each finite subset $\rho^{\mathrm{Subset}} \subset \rho$,
  - $\mathrm{codim}_{\mathsf{X}}(Z(\rho^{\mathrm{Subset}})) \geq \#\rho^{\mathrm{Subset}}$,
  - connected components of the scheme $Z_X(\rho^{\mathrm{Subset}})$ are irreducible and reduced, and moreover,
  - the scheme $Z_X(\rho^{\mathrm{Subset}})$ is irreducible if $\#\rho^{\mathrm{Subset}} = 1$, i.e. for each $\rho^{\mathrm{el}} \in \rho$, the scheme $Z_X(\rho^{\mathrm{el}})$ is irreducible,

(o3) $\partial, \varsigma \in \rho^{\mathrm{Mon}}$, where $\rho^{\mathrm{Mon}} = \mathrm{Mon}(\rho)$ is the submonoid of $\mathcal{S}(X)$ generated by $\rho$, Remark 8,
(o4) $\beta$ is a $\cup$-closed set of finite subsets of $\rho^{\mathrm{Mon}}$, Notation 12, 11.

**Notation 1.** Denote by $\beta^{\mathrm{Mon}}$ the submonoid in the monoid of subsets of the monoid of monomials $\rho^{\mathrm{Mon}}$ generated by $\beta$.

**Definition 2.9.** An $A$-morphism over $B$
$$x\colon X = (\mathsf{X}, (\rho, (\partial, \varsigma, \beta))) \to \underline{X} = (\underline{\mathsf{X}}, (\underline{\rho}, (\underline{\partial}, \underline{\varsigma}, \underline{\beta}))) \tag{2.5}$$
is defined by

(m1) a morphism of $B$-schemes
$$\mathsf{x}\colon \mathsf{X} \to \underline{\mathsf{X}} \tag{2.6}$$
such that
(m2) $\rho^{\mathrm{Mon}} \supset \mathsf{x}^*(\underline{\rho}^{\mathrm{Mon}})$,
(m3) $\partial \leq \mathsf{x}^*(\underline{\partial})$, $\varsigma \leq \mathsf{x}^*(\underline{\varsigma})$,

(m4) $\beta^{\mathrm{Mon}} \supset \mathsf{x}^*(\underline{\beta}^{\mathrm{Mon}})$,

*Remark* 10. For any morphism (2.5) in $A_B$, the set of subschemes $Z_X(\rho)$ for all $\rho \in \mathcal{S}(\mathsf{X})$ contains the set of irreducible components of the subschemes $Z(\mathsf{x}^*(\underline{\rho}))$ for all $\underline{\rho} \in \mathcal{S}(\underline{\mathsf{X}})$.

**Definition 2.10.** $A_B$ is the category that objects and morphisms are $A$-objects and $A$-morphisms over $B$, Definitions 2.8 and 2.9, and the composition of morphisms of the form $x$ (2.5) is induced by the composition of the respective morphisms $\mathsf{x}$ (2.6) in Sch.

2.2.2. *Category $\vec{A}_B$.*

**Definition 2.11.** $\vec{A}_B$ is the category of arrows of $A_B$, i.e. the morphisms of the category $A_B$, i.e.

$$\vec{A}_B := (A_B)^{\Delta^1}.$$

**Notation 2.** We use notation

$$\vec{X} \colon X = (\mathsf{X}, (\rho, (\partial, \varsigma, \beta))) \to \underline{X} = (\underline{\mathsf{X}}, (\underline{\rho}, (\underline{\partial}, \underline{\varsigma}, \underline{\beta}))) \tag{2.7}$$

for objects of $\vec{A}_B$.

**Definition 2.12.** Let $\mathbf{a}$ be a class of morphisms in $A_B$ closed with respect to base changes along arbitrary morphisms in $A_B$. We define the class of morphisms $\mathbf{a}$ in $\vec{A}_B$ as the class generated by the classes $\mathbf{a}_r$ and $\mathbf{a}^r$ via the composition. Here $\mathbf{a}_r$ and $\mathbf{a}^r$ denote the classes of morphisms in $\vec{A}_B$ defined respectively by the commutative squares in $A_B$ of the form

$$\begin{tikzcd} \breve{X} \arrow[r, "\mathbf{a}"] \arrow[d] \arrow[dr, phantom, "\mathbf{a}_r"] & X \arrow[d] & \breve{X} \arrow[r, "\mathbf{a}"] \arrow[d] \arrow[dr, phantom, "\mathbf{a}^r"] & X \arrow[d] \\ \underline{\breve{X}} \arrow[r, "\simeq", "\mathbf{a}"'] & \underline{X}, & \underline{\breve{X}} \arrow[r, "\mathbf{a}"'] & \underline{X}, \end{tikzcd} \tag{2.8}$$

where the vertical arrows define the objects in $\vec{A}_B$ and the horizontal ones define the morphisms,

- the left bottom arrow is an isomorphism, and the right square is a pullback, and
- all the horizontal arrows morphisms belong to $\mathbf{a}$.

2.2.3. *Monoidal structure.*

**Definition 2.13.** Given $X_1, X_2 \in A_B$,

$$X_1 \times X_2 = (\mathsf{X}_1, \rho_1, \partial_1, \varsigma_1, \beta_1) \times (\mathsf{X}_2, \rho_2, \partial_2, \varsigma_2, \beta_2) :=$$
$$(\mathsf{X}_1 \times \mathsf{X}_2, \rho_1 \cup \rho_2, \partial_1 + \partial_2, \varsigma_1 + \varsigma_2, \beta_1 \cup \beta_2),$$

where $\rho_\alpha$, $\partial_\alpha$, $\varsigma_\alpha$, $\beta_\alpha$, $\alpha = 1, 2$, at the right side stand for the inverse images along the projections $\mathsf{X}_1 \times \mathsf{X}_2 \to \mathsf{X}_\alpha$.

2.2.4. *Functors to schemes.*

**Definition 2.14.** $\mathrm{Sch}^U_B$ is the functor

$$\begin{array}{rcrcl} \mathrm{Sch}^U_B & : & O^c_B & \to & \mathrm{Sch}_B \\ & & (x \colon X \to \underline{X}) & \mapsto & \mathsf{X}, \end{array} \tag{2.9}$$

where $X$, $\mathsf{X}$, and $\partial$ are from (2.5).

**Notation 3.** For any $F \in \mathrm{Spc}(\mathrm{Sch}_B)$, the same symbol denotes the inverse image of $F$ along the functor (2.9).

**Definition 2.15.** $\mathrm{Sch}^V$ is the functor

$$\begin{array}{rcrcl} \mathrm{Sch}^V & : & O^c_B & \to & \mathrm{Sch}_B \\ & & (x \colon X \to \underline{X}) & \mapsto & \mathsf{X} - Z(\partial), \end{array} \tag{2.10}$$

where $X$, $\mathsf{X}$, and $\partial$ are from (2.5).

**Definition 2.16.** $\mathrm{Sch}^Q$ is the functor

$$\begin{array}{lclcl} \mathrm{Sch}^Q & : & O^c_B & \to & \mathrm{Spc}(\mathrm{Sch}_B) \\ & & (x\colon X \to \underline{X}) & \mapsto & (\mathsf{X} - Z(\partial))/(\mathsf{X} - Z(\partial) - Z(\varsigma)), \end{array} \tag{2.11}$$

where $X$, $\mathsf{X}$, and $\partial$ are from (2.5).

The natural morphism in $\mathrm{Spc}(\mathrm{Sch}_B)$

$$(\mathsf{X} - Z(\partial)) \to (\mathsf{X} - Z(\partial))/(\mathsf{X} - Z(\partial) - Z(\varsigma))$$

defines the morphism of functors

$$\mathrm{Sch}^V \to \mathrm{Sch}^Q. \tag{2.12}$$

2.2.5. *$\infty$-category $\vec{\mathcal{A}}_B$.*

**Definition 2.17.** The $\infty$-category $\vec{\mathcal{A}}_B$ is the $\infty$-category $(\vec{A}_B)^{\mathrm{Spc}}$ of co-presheaves on the category of $\vec{A}_B$.

**Notation 4.** For any $F \in \mathrm{Spc}(\mathrm{Spc}^{\mathrm{Sch}_B})$, the same symbol denotes both the inverse image along the functor

$$(\mathrm{Sch}^U_B)^{\mathrm{Spc}}\colon (A_B)^{\mathrm{Spc}} \to (\mathrm{Sch}_B)^{\mathrm{Spc}}$$

induced by (2.9) and the inverse image along the composite with the functor

$$(\mathrm{Sch}^U_B)^{\mathrm{Spc}}\colon (\vec{A}_B)^{\mathrm{Spc}} \to (A_B)^{\mathrm{Spc}}$$

induced by the functor

$$\begin{array}{lcl} \vec{A}_B & \to & A_B; \\ (x\colon X \to X') & \mapsto & X. \end{array} \tag{2.13}$$

*Remark* 11. In view of the Grothendieck correspondence objects in $\vec{\mathcal{A}}_B$ are defined by diagrams in the category $\vec{A}_B$, which are diagrams in $A_B$ of the form

$$D \times \Delta^1 \to A_B, \tag{2.14}$$

where $D$ is an $\infty$-category indexing the diagram in $\vec{A}_B$.

**Definition 2.18.** Given class of morphisms $\mathbf{a}$ in $\vec{A}_B$, the same symbol $\mathbf{a}$ denotes the class of morphisms in $\vec{\mathcal{A}}_B$ that are defined by term-wise $\mathbf{a}$-morphisms of diagrams in $\vec{A}_B$ according to Remark 11.

2.3. **Types of morphisms in $\vec{\mathcal{A}}_B$.**

2.3.1. *Types of morphisms with respect to $\rho$.*

**Definition 2.19.** A morphism $x\colon X \to \underline{X}$ given by (2.5) is called

- *$\nu$-morphism* if $\mathsf{x}\colon \mathsf{X} \to \underline{\mathsf{X}}$ induces a well defined map
$$\underline{\rho} \to \rho,$$
- *$\eta$-morphism* if
$$\partial = \mathsf{x}^*(\underline{\partial}), \quad \varsigma = \mathsf{x}^*(\underline{\varsigma}),$$
- *$\mu$-morphism* if for any $\rho^{\mathrm{el}} \in \rho$ there is $\underline{\rho}^{\mathrm{el}} \in \underline{\rho}$ such that $Z_{\mathsf{X}}(\rho^{\mathrm{el}}) \subset Z_{\mathsf{X}}(\mathsf{x}^*(\underline{\rho}^{\mathrm{el}}))$, and $\partial = \mathsf{x}^*(\underline{\partial})$, $\varsigma = \mathsf{x}^*(\underline{\varsigma})$, $\beta = \mathsf{x}^*(\beta')$.
- *$\pi$-morphism* if there is a map $r\colon \rho \to \underline{\rho}$ such that
$$\mathsf{x}^*(r(\rho^{\mathrm{el}})) = \rho^{\mathrm{el}},\ \forall \rho^{\mathrm{el}} \in \rho,$$
$$\underline{\partial} = r(\partial), \quad \underline{\varsigma} = r(\varsigma),$$
$$\underline{\beta}^{\mathrm{Mon}} = \{r(\beta^{\mathrm{el}}) | \beta^{\mathrm{el}} \in \beta^{\mathrm{Mon}}\},$$

- $\tau$*-morphism* if it is a $\pi$-morphism such that $\mathsf{x}\colon \mathsf{X} \to \underline{\mathsf{X}}$ induces a bijection

$$\underline{\rho} \xrightarrow{\simeq} \rho.$$

**Lemma 2.1.** *For any $X \in A_B$ given by* (2.4) *and morphism $\mathsf{x}\colon \mathsf{X}' \to \mathsf{X}$ in $\mathrm{Sch}_B$, the set of $\mu$-morphisms $x\colon X' \to X$ in $A_B$ given by* (2.5) *that image along the functor* (2.10) *equals $\mathsf{x}$ is either empty or consists of unique element.*

*Proof.* The inverse image along the morphism $\mathsf{x}$ uniquely defines the data $(\rho, \partial, supp, \beta)$. If the latter data satisfies conditions form Definition 2.8 it defines the unique morphism $x\colon X' \to X$, otherwise the set is empty. □

*Remark* 12. Any $\pi$-morphism is a $\mu$-morphism.

*Remark* 13. Any $\tau$-morphism is a $\nu$-morphism.

2.3.2. *Closed immersions.*

**Definition 2.20.** A morphism $c\colon X \to \overline{X}$ in $A_B$ is a $\rho$*-ci-closed immersion* if

- the morphism $\mathrm{Sch}^U_B(c)$ is a closed immersion in Sch,
- there is a subset $e$ of $\rho^{\mathrm{Mon}}(\overline{X})$ such that

$$\mathsf{X} = Z_{\overline{\mathsf{X}}}(e).$$

**Lemma 2.2.** *Let $c\colon X \to \overline{X}$ given by* (2.5) *be a $\pi$-$\rho$-ci-closed immersion in $A_B$. Then*

$$\underline{\rho} = r(\rho) \amalg e,$$

*where $r$ and $e$ are form Definitions 2.19 and 2.20.*

*Proof.* Lct $\underline{\rho}^{\mathrm{el}} \in \underline{\rho}$, then $c^*(\underline{\rho}^{\mathrm{el}}) \in \mathcal{S}(\mathsf{X})$ is either 0 or in $\rho$. Define

$$\underline{\rho}^{\mathrm{el}\prime} = \begin{cases} \emptyset, & c^*(\underline{\rho}^{\mathrm{el}}) = 0 \in \mathcal{S}(\mathsf{X}), \\ r(c^*(\underline{\rho}^{\mathrm{el}})), & c^*(\underline{\rho}^{\mathrm{el}})\rho \end{cases}.$$

Since

$$\#((e \cup \underline{\rho}^{\mathrm{el}\prime})) = \mathrm{codim}_{\overline{\mathsf{X}}}((e \cup \underline{\rho}^{\mathrm{el}\prime})) = \mathrm{codim}_{\overline{\mathsf{X}}}(\{\underline{\rho}^{\mathrm{el}}\} \cup (e \cup \underline{\rho}^{\mathrm{el}\prime})) = \#(\{\underline{\rho}^{\mathrm{el}}\} \cup (e \cup \underline{\rho}^{\mathrm{el}\prime})),$$

$\underline{\rho}^{\mathrm{el}} \in (e \cup \underline{\rho}^{\mathrm{el}\prime})$. □

*Remark* 14. *$\rho$-ci-closed immersions in $\vec{A}_B$ and in $\vec{A}_B$* are defined by Definition 2.21 and Definitions 2.12 and 2.18.

**Definition 2.21.** A morphism $c\colon X \to \overline{X}$ in $A_B$ is a *retract-closed immersion* if $\mathrm{Sch}^U_B(c)$ from (2.9) is a retract-closed immersion in Sch, Definition 2.6.

*Remark* 15. Definition 2.21 and Definition 2.12. define *retract-closed immersions in $\vec{A}_B$*. Then Definition 2.18 defines *retract closed immersions in $\vec{A}_B$*.

2.3.3. *Blow-ups.*

**Definition 2.22** (Blow-up in $A_B$)**.** Given $X \in A_B$ (2.4) and a closed subscheme $Z$ of the scheme $\mathsf{X}$, a blow-up of $X$ w.r.t. $Z$ is an $\eta$-morphism

$$Bl_Z(X) \to X \tag{2.15}$$

that image along the functor (2.9) is isomorphic to the morphism $Bl_Z(\mathsf{X}) \to \mathsf{X}$ in Sch.

**Definition 2.23** (Blow-up in $\vec{A}_B$)**.** Given $x \in \vec{A}_B$ (2.5),

(1) A blow-up of $x$ w.r.t. a closed subscheme $Z$ in $\mathsf{X}$ and
(2) a blow-up of $x$ w.r.t. a closed subscheme $Z'$ in $\mathsf{X}'$

are morphisms defined by the commutative squares in $A_B$

$$\begin{tikzcd} Bl_Z(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & X \arrow[d] & Bl_{\mathsf{X}\times_{\mathsf{X}'}Z'}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & X \arrow[d] \\ \underline{X} \arrow[r, "\mathrm{id}_{\underline{X}}"'] & \underline{X}, & Bl_{Z'}(\underline{X}) \arrow[r] & \underline{X}, \end{tikzcd} \tag{2.16}$$

where the vertical arrows define objects in $\vec{A}_B$ and the horizontal arrows define morphisms.

**Definition 2.24** ($\rho$-blow-up)**.** An *elementary $\rho$-blow-up-morphism* in $A_B$ is a morphism equivalent to a bow-up morphism (2.15) for some $X \in A_B$ given by (2.4), and

$$Z = Z(\rho^{\mathrm{Subset}}), \quad \rho^{\mathrm{Subset}} \subset \rho.$$

A *$\rho$-blow-up-morphism* in $A_B$ is a composite of a sequence of elementary $\rho$-blow-up-morphisms.

**Definition 2.25.** Given $X \in A_B$ (2.4), denote by $\rho^{\mathrm{F}}$ the set of subsets $\rho^{\mathrm{subset}}$ of $\rho^{\mathrm{Mon}}$ such that $Bl_{\rho^{\mathrm{subset}}}(\mathsf{X}) \to \mathsf{X}$ is a $\rho$-blow-up.

**Definition 2.26.** A closed subscheme $C$ of a scheme $X$ is good w.r.t a closed subset $Z$ of $X$, if either $C \subset Z$ or $\operatorname{codim}_X(C \cap Z) \geq \operatorname{codim}_X C + \operatorname{codim}_X Z$.

A morphism of schemes $\widetilde{X} \to X$ is a good blow-up w.r.t. a closed subset $Z$ of $X$, if it is isomorphic to the blow-up $Bl_C(X) \to X$ for closed subscheme $Z$ of $X$ goodw.r.t. $C$.

**Definition 2.27** ($\beta$-blow-up)**.** An *elementary $\beta$-blow-up-morphism* in $A_B$ is a morphism equivalent to a bow-up morphism (2.15) for some $X \in A_B$ given by (2.4), and

$$Z = Z(\beta^{\mathrm{el}}), \quad \beta^{\mathrm{el}} \in \beta$$

that is a good blow-up w.r.t. the vanishing locus of each element of $\beta$, Definition 2.26, and a $\mu$-morphism, Definition 2.19.
A *$\beta$-blow-up-morphism* in $A_B$ is a composite of a sequence of elementary $\beta$-blow-up-morphisms.

*Remark* 16. *$\rho$-blow-up-morphisms* and *$\beta$-blow-up-morphisms* in $\vec{A}_B$ and $\mathrm{Spc}^{\vec{A}_B}$ are defined by Definitions 2.12 and 2.18 w.r.t Definition 2.24 and Definition 2.27.

**Definition 2.28** ($\mathscr{F}$-object)**.** An $\mathscr{F}$-object in $A_B$ is an object $X$ (2.4) such that $\beta \subset \rho^{\mathrm{F}}$.

**Lemma 2.3.** *A $\beta$-blow-up of an $\mathscr{F}$-object is an $\mathscr{F}$-object.*

*Proof.* $bl^*(\beta^{\mathrm{el}}) = bl^*_{\mathrm{proper}}(\beta^{\mathrm{el}}) \cdot e^p$ ☐

**Lemma 2.4.** *Given $X \in A_B$, for any finite set of finite subsets $\rho^{SC}$ in $\rho^{\mathrm{Mon}}$ and any set of finite subsets $\rho^{SG}$ in $\rho^{\mathrm{Mon}}$, there is a finite sequence of elementary $\rho$-blow-ups*

$$X = X_0 \leftarrow \cdots = X_j \leftarrow Bl_{Z(\beta_j^{\mathrm{el}})}(X_j) = X_{j+1} \leftarrow \cdots = X_l \tag{2.17}$$

*such that each $Z(\beta_j^{\mathrm{el}})$ is good w.r.t inverse images of $Z(\rho^G)$ for all $\rho^G \in \rho^{SG}$ and for each $\rho^C \in \rho^{SC}$, the blow-up $Bl_{Z(bl^*(\rho^C))}(X_l) \to X_l$ admits a section, where $bl$ is the composite of* (2.17).

*Proof.* The proof is by a cascade induction with respect to parameters

(P1) $\#\rho^{SC}$,
(P2) $(\sum_{\rho^C \in \rho^{SC}} \#Z(\rho^C), \sum_{\rho^C \in \rho^{SC}} \dim Z(\rho^C))$,
(P3) $\sum_{\rho^C \in \rho^{SC}} \sum_{\rho^{\text{Mon-el}} \in \rho^C} \deg \rho^{\text{Mon-el}}$.

(P1 Base) Suppose $\#\rho^{SC} = 0$. Then the claim holds.
(P1 Step) Suppose the claim holds for any $\#\rho^{SC}$ less then the given one.
(P2 Base) Suppose $Z(\rho^C) = \emptyset$ for each $\rho^C \in \rho^{SC}$. Then the claim holds.
(P2 Step) Suppose the claim holds if (P2) is less than for the given $\rho^{SC}$.

(P3 Base) Suppose $\rho^{SC}$ is a set of subsets of $\rho$. Let $\rho_f^C \in \rho^{SC}$.

By induction assumption on (P2) there is a sequences of elementary $\rho$-blow-ups that trivialise blow-ups with centres of the forms

$$Z(\rho_f^C \cup \rho^G), \quad \rho^C \in \rho^{SC}, \rho^G \in \rho^{SG},$$

and

$$Z(\rho_f^C \cup \rho^C), \quad \rho^C \in \rho^{SC}$$

that are not equal to $Z(\rho_f^C)$, and are goodw.r.t. each element of $\rho^{SC}$ and $\rho^{SG}$. Composing with the blow-up with respect to $\rho_f^C$ reduces the claim to the claim for a smaller (P1).

(P3 Step) Suppose the claim would hold if (P3) would be less than the given one. Let $\rho_f^C \in \rho^{SC}$.

$$\rho^{SC} = (\rho_f^C, \hat{\rho}^{SC}),$$

and suppose

$$\rho_f^C = (\hat{\rho}_f^C, a \cdot b), \quad a, b \in \rho^{\mathrm{Mon}} - \{1\}.$$

By the induction assumption on (P3) there is a sequence of elementary $\rho$-blow-ups $bl$ that trivialises blow-up with the centre $Z(\hat{\rho}_f^C, a)$ and is good w.r.t. $\rho^{SG}$ and $(\hat{\rho}_f^C, a)$. Then the claim reduces to the claim for the data

$$bl^*(\rho^{SC}, \rho^{SG}) = (bl^*(\rho_f^C), bl^*(\hat{\rho}^{SC})), bl^*(\rho^{SG}))$$

which is equivalent to the one with

$$bl^*(\rho_f^C) = (bl^*(\hat{\rho}_f^C), bl^*(a)bl^*(b))$$

substituted by

$$(bl^*_{\mathrm{proper}}(\hat{\rho}_f^C), bl^*_{\mathrm{proper}}(a)bl^*(b)), \tag{2.18}$$

because $bl$ is good w.r.t. $(\hat{\rho}_f^C, a)$. The claim for (2.18) is equivalent to the one for

$$(bl^*_{\mathrm{proper}}(\hat{\rho}_f^C), bl^*(b)), \tag{2.19}$$

because $Z(bl^*_{\mathrm{proper}}(\hat{\rho}_f^C), bl^*_{\mathrm{proper}}(a)) = \emptyset$, since $b$ trivialises blow-up with the centre $Z(\hat{\rho}_f^C, a)$. The claim for (2.19) holds by the induction assumption on (P3). □

**Lemma 2.5.** *Given $X \in A_B$, for any finite subset $\dot{\beta}$ in $\beta$ let*

$$\beta^{\lim} = \prod_{\dot{\beta}^{\mathrm{el}} \in \dot{\beta}^{\cup}} \dot{\beta}^{\mathrm{el}} \tag{2.20}$$

*there is a finite sequence of elementary $\beta$-blow-ups*

$$X = X_0 \leftarrow \cdots = X_j \leftarrow Bl_{Z(\beta_j^{\mathrm{el}})}(X_j) = X_{j+1} \leftarrow \cdots = X_l$$

*such that the morphism $\mathsf{X}_l \leftarrow Bl_{Z(\beta^{\mathrm{el}})}(\mathsf{X}) \times_{\mathsf{X}} \mathsf{X}_l$ adimts a section.*

*Proof.* Denote by $\beta_n$ the set of subsets of $\rho$ formed by unions of $\#\beta - n$ different elements in $\beta$

$$\beta_n = \{\bigcup \beta^{\mathrm{Subset}} | \#\beta^{\mathrm{Subset}} = \beta - n\}.$$

Let

$$\beta_n^{\mathrm{el}} = \prod_{\dot{\beta}^{\mathrm{el}} \in \beta_n} \dot{\beta}^{\mathrm{el}}$$

Then inverse image of the ideal of $\beta_n^{\mathrm{el}}$ along the blow-up

$$Bl_{\beta_{n-1}^{\mathrm{el}}}(\mathsf{X}) \to \mathsf{X}$$

equals the product of a product of exceptional divisors ideals and the product of the vanishing ideals of the inverse images of $Z_{\mathsf{X}}(\beta^{\mathrm{Subset}})$ for all $\beta^{\mathrm{Subset}}$ such that $\#\beta^{\mathrm{Subset}} = \beta - n$ and the latter inverse images have empty intersection in $Bl_{\beta_{n-1}^{\mathrm{el}}}(\mathsf{X})$. So the blow-up of $Bl_{\beta_{n-1}^{\mathrm{el}}}(\mathsf{X})$ defined by the inverse image of $\beta_n^{\mathrm{el}}$ is a $\beta$-blow-up. □

2.3.4. *Étale morphisms and infinitesimal neighbourhoods.*

**Definition 2.29** (Étale in $\vec{A}_B$)**.** A morphism $e$ in $A_B$ is called *étale* if $\mathrm{Sch}^U_B(e)$ is étale, where $\mathrm{Sch}^U_B$ is from (2.9). The class of étale morphisms is denoted Et.

*Remark* 17 (Étale in $\vec{\mathcal{A}}_B$). Then the classes Et called étale morphisms in $\vec{A}_B$ and $\vec{\mathcal{A}}_B$ are defined by Definitions 2.12 and 2.18 w.r.t Definition 2.29.

**Definition 2.30** (Infinitesimal étale neighbourhoods)**.** The class of morphisms

$$\mathrm{InfEtNeigh} = \mathrm{uLLP}(\mathrm{Et}),$$

Notation 25, is called *infinitesimal étale neighbourhoods in* $\vec{\mathcal{A}}_B$.

**Definition 2.31** (Étale in $\mathrm{Spc}(\vec{\mathcal{A}}_B)$)**.** The class of *étale morphisms in* $\mathrm{Spc}(\vec{\mathcal{A}}_B)$ is the class

$$\mathrm{uRLP}(\mathrm{InfEtNeigh}),$$

Notation 25.

2.3.5. *Étale-equational-morphisms.*

**Definition 2.32** (EtEq-morphisms in $A_B$)**.** A morphism in $A_B$ is called *étale-equational* if the image along the functor $\mathrm{Sch}^U_B\colon A_B \to \mathrm{Sch}$ from (2.9) is equational and étale, Definition 2.7.

**Definition 2.33** ($\mathrm{EtEq}_\rho$-morphisms in $A_B$)**.** $\mathrm{EtEq}_\rho$-morphisms in $A_B$ are étale-equational $\tau$-morphisms, Definition 2.19.

*Remark* 18 ($\mathrm{EtEq}_\rho$-morphisms in $\vec{\mathcal{A}}_B$). The class of morphisms $\mathrm{EtEq}_\rho$ in $\vec{\mathcal{A}}_B$ is defined by Definitions 2.12 and 2.18 with respect to Definition 2.33.

2.3.6. *Morphisms $\mathscr{H}$.*

**Definition 2.34.** $\mathscr{H}$ denotes the class of morphisms in $\vec{A}_B$ that is the image along the functor

$$A_B \to \vec{A}_B; X \mapsto \mathrm{id}_X,$$

of the class of $\rho$-ci-closed immersions in $A_B$ of the form

$$c\colon X \to \overline{X}$$

such that $U(c)$ is a closed immersion of essentially smooth local henselian $B$-schemes.

2.4. **$\infty$-category $O_B$.**

2.4.1. *Regularity.*

**Definition 2.35.** $A_{\mathrm{rs},B}$ is the subcategory of $A_B$ spanned by objects of the form (2.4) such that strata of the filtration formed by schemes $Z(\rho^{\mathrm{Subset}})$ for all subsets $\rho^{\mathrm{Subset}} \subset \rho$ are regular.

**Definition 2.36.** $\vec{A}_{\mathrm{rs},B}$ and $\vec{\mathcal{A}}_{\mathrm{rs},B}$ are the subcategories of $\vec{A}_B$ and $\vec{\mathcal{A}}_B$ formed by the objects defined by diagrams in $A_B$ that belong to $A_{\mathrm{rs},B}$ of the respective type, Remark 11.

2.4.2. *$\mathscr{P}$-objects.*

**Definition 2.37.** An object $X \in A_B$ is a *$\mathscr{P}$-object* if there is an element in $\rho^{\mathrm{Mon}}(X)$ that is a section of a very ample line bundle on $\mathsf{X}$.

**Definition 2.38.** *$\mathscr{P}$-objects* in $\vec{A}_B$ and $\vec{\mathcal{A}}_B$ are such objects that are defined by diagrams of $\mathscr{P}$-objects in $A_B$ of the respective type, Remark 11.

2.4.3. *$\beta$-objects.*

**Definition 2.39.** An object $X \in A_B$ is a *$\beta$-object* if for any $\beta^{\mathrm{Subset}} \subset \beta$ there is a filtration

$$\emptyset = c_0 \subset \cdots \subset c_i \subset \cdots \subset c_{n-1} \subset \cup(\beta^{\mathrm{Subset}})$$

where $n = \# \cup (\beta^{\mathrm{Subset}})$, such that $\# c_i = i$ for each $i = 0, \ldots, n-1$, and there are retractions

$$r_i \colon Z_{\mathsf{X}}(c_{i-1}) \to Z_{\mathsf{X}}(c_i), \quad \forall i = 1, \ldots, n-1.$$

**Definition 2.40.** A *$\beta$-object* in $\vec{\mathcal{A}}_B$ is an object given by a diagram of $\beta$-objects in $A_B$, Remark 11.

2.4.4. *$\mathbb{P}$-object.*

**Definition 2.41.** A *$\mathbb{P}$-morphism in $A_B$* is a morphism $X \to \underline{X}$ that image along the functor $\mathrm{Sch}_B^U$ is isomorphic to

$$\mathsf{X} \simeq \prod_{\alpha \in A} (\mathbb{P}^{N_\alpha}) \times \underline{\mathsf{X}} \xrightarrow{p} \underline{\mathsf{X}}$$

for some set $A$ and integers $N_\alpha$, $\alpha \in A$, and $\rho$ is the union of $p^*(\underline{\rho})$ and a subset

$$\rho_P \subset \bigcup_{\alpha \in A} \{t_{1,i}, \ldots, t_{N_\alpha,i}, t_{\infty,i}\} \subset \mathcal{S}(\mathsf{X})$$

of the set of inverse images of the coordinate sections $t_{1,i}, \ldots, t_{N_\alpha,i}, t_{\infty,i} \in \mathcal{S}(\mathbb{P}^{N_\alpha})$ on the multiplicands $\mathbb{P}^{N_\alpha}$.

**Definition 2.42.** A *$\mathbb{P}$-object in $\vec{\mathcal{A}}_B$* is an object defined by a diagram, Remark 11, in subcategory of the comma-category $A_B/(B, (\emptyset, , , \emptyset))$ that objects are $\mathbb{P}$-morphisms.

2.4.5. *$\infty$-category $O_B$.*

**Definition 2.43** ($\infty$-category $O_B$)**.** The $\infty$-category $O_B$ is the full subcategory of $\vec{\mathcal{A}}_B$ formed by objects of the form $\tilde{X}$ such that there is a diagram in $\vec{\mathcal{A}}_{\mathrm{rs},B}$, Definition 2.36,

$$X \xleftarrow{b} \mathrm{Bl}_\rho(X) \xleftarrow{b} B \xleftarrow{e} E \simeq \tilde{X}$$

such that

(O1) $X$ is a $\mathscr{P}$-$\mathscr{F}$-$\beta$-$\mathbb{P}$-object in $\vec{\mathcal{A}}_B$, Definitions 2.28, 2.38, 2.40 and 2.42.
(O2) $b$ is a composite of a finite set of elementary $\rho$-blow-up morphisms, $b$ is a $\mu$-$\rho$-blow-up morphism, Remark 16,
(O3) $e$ is an $\mathrm{EtEq}_\rho$-morphism, Remark 18.

## 2.5. Localisations and endo-functors.

2.5.1. *$Bl_\beta$-localisation and $Bl_\rho$–localisation.* In this subsection, we define two types of localisations on $\mathrm{Spc}(O_B)$ called $Bl_\beta$-localisation and $Bl_\rho$-localisation. The definitions and results for the first type, $Bl_\beta$, are written below. The definitions and results for $Bl_\rho$ are the same where the role of the set $\beta$ is played by the set of subsets of $\rho$.

**Definition 2.44.** An elementary *$\beta$-blow-up-equivalence* in $\vec{A}_B$ is a $\beta$-blow-up-morphism given by (2.16) such that

$$Z \subset Z_{\mathsf{X}}(\partial), \quad Z' \subset Z_{\mathsf{X}'}(\partial').$$

Let $Bl_\beta$ denote the classes of morphisms in $\vec{A}_B$ and $O_B$ generated by elementary $\beta$-blow-up-equivalences and called $\beta$-blow-up-equivalences. Let $\mathrm{Spc}_{Bl_\beta}(O_B)$ denote the subcategory of $Bl_\beta$-invariant objects in $\mathrm{Spc}(O_B)$. Let

$$L_{Bl_\beta} \colon \mathrm{Spc}(O_B) \to \mathrm{Spc}_{Bl_\beta}(O_B)$$

denote the localisation functor.

**Definition 2.45.** For any $X \in O_B$, $D_{\rho\text{-bl-eq}}(X)$ is the subcategory of the comma $\infty$-category $O_B/X$ formed by $\rho$-blow-up-equivalences, and $D_{\beta\text{-bl-eq}}(X)$ is the subcategory formed by $\rho$-blow-up-equivalences that are defined by elements of $\beta^{\mathrm{Mon}}$.

**Lemma 2.6.** *(1) The subcategory of the comma $\infty$-category $O_B/X$ formed by $\rho$-blow-up-equivalences is cofinal in $D_{\rho\text{-bl-eq}}(X)$. (2) The subcategory of the comma $\infty$-category $O_B/X$ formed by $\beta$-blow-up-equivalences is cofinal in $D_{\beta\text{-bl-eq}}(X)$.*

*Proof.* The claim for $\beta$-blow-ups follows from Lemma 2.5. The claim for $\rho$-blow-ups follows from Lemma 2.4. ☐

**Lemma 2.7.** *For any $F \in \mathrm{Spc}(O_B)$, and any $X \in O_B$, there is a natural equivalence*

$$L_\beta(F)(X) \simeq \varinjlim_{(\tilde{X}\to X)\in D_{\beta\text{-bl-eq}}(X)} F(\tilde{X}).$$

*Proof.* Because of Lemma 2.6,

$$\varinjlim_{(\tilde{X}\to X)\in D_{\beta\text{-bl-eq}}(X)} F(\tilde{X}) \simeq \varinjlim_{(\tilde{X}\to X)\in D_{\beta\text{-bl-eq}}(X)} F(\tilde{X})$$

and for any $\beta$-blow-up-equivalence $b'\colon \tilde{X}' \to X'$ in $A_B$ and morphism $X \to X'$ there s a $\beta$-blow-up-equivalence $b\colon \tilde{X} \to X$ equipped with a commutative square

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "b"] & \tilde{X}' \arrow[d, "b'"] \\ X \arrow[r] & X'. \end{tikzcd}$$

Then it follows that the class of $\beta$-blow-up-equivalences in $O_B$ satisfies similar property. Then the claim follows. ☐

The functor

$$\mathrm{limBl}_\beta\colon A_B \to \mathrm{Spc}^{A_B}$$

that takes $X$ to

$$\mathrm{limBl}_\beta(X) = \lim D_{\beta\text{-bl-eq}}(X) \tag{2.21}$$

induces the functor

$$\mathrm{limBl}_\beta\colon \vec{A}_B \to \mathrm{Spc}^{\vec{A}_B} \simeq \vec{\mathcal{A}}_B$$

and moreover, the functor $\mathrm{limBl}_\beta\colon \vec{\mathcal{A}}_B \to \vec{\mathcal{A}}_B$. The latter functor restricts to the functor

$$\mathrm{limBl}_\beta\colon O_B \to O_B. \tag{2.22}$$

*Remark* 19. By the construction the functor $Bl_\beta$ from (2.22) preserves limits.

**Lemma 2.8.** *The arrow $\mathrm{limBl}_\beta(X) \to X$ is the initial object of the $\infty$-category of pro-objects of the $\infty$-category $D_{\beta\text{-bl-eq}}(X)$ for any $X \in O_B$.*

*Proof.* The claim holds for any $X \in \vec{A}_B$ by the construction of $\mathrm{limBl}_\beta$ from (2.21). Then it follows for any $X \in \vec{\mathcal{A}}_B$. ☐

**Lemma 2.9.** *For any continuous $F \in \mathrm{Spc}(O_B)$, and any $X \in O_B$, there is a natural equivalence*

$$L_\beta(F)(X) \simeq F(\mathrm{limBl}_\beta(X)).$$

*Proof.* The claim follows from Lemmas 2.7 and 2.8. ☐

**Lemma 2.10.** *For any $X \in O_B$ such that $\beta$ is finite,* $\mathrm{limBl}_\beta(X))$ *is equivalent to $Bl_{\beta^{\mathrm{lim}}}(X)$ form* (2.20).

*Proof.* The claim follow because blow-up on $Bl_{\beta^{\mathrm{lim}}}(X)$ defined by the inverse image of each element $\beta^{\mathrm{el}} \in \beta$ is identity because (2.20) includes such multiplicant. ☐

*Remark* 20. It follows from the construction that the functor $\mathrm{limBl}_\beta$ form (2.22) preserves limits.

2.5.2. $(\overline{\underline{\mathbb{A}}}, Bl_\beta)$*-localisation.*

**Definition 2.46.**

$$\begin{array}{rcll} \Delta^1_{\overline{\underline{\mathbb{A}}}} & := & \left((\mathbb{P}^1_B, ((t_0,t_\infty),(t_\infty,1,\{\{t_\infty\}\}))) \to (B,\emptyset,,,\emptyset)\right) & \in \vec{A}_B, \\ \Delta^1_{\overline{\underline{\mathbb{A}}},\mathrm{all}} & := & \mathrm{id}_{(\mathbb{P}^1_B,((t_0,t_\infty),(t_\infty,1,\emptyset)))} & \in \vec{A}_B, \end{array}$$

where the morphism in the second row is the identity morphism.

**Notation 5.** $\Delta^{\times l}_{\overline{\underline{\mathbb{A}}}} := (\Delta^1_{\overline{\underline{\mathbb{A}}}})^{\times l} = \prod_{\{1,\dots,l\}} (\Delta^1_{\overline{\underline{\mathbb{A}}}})$, and $\Delta^0_{\overline{\underline{\mathbb{A}}}} := (\Delta^1_{\overline{\underline{\mathbb{A}}}})^{\times 0} = (\Delta^1_{\overline{\underline{\mathbb{A}}}})^{\times 0}$.

**Definition 2.47.** $\Delta^{\bullet\leq 1}_{\overline{\underline{\mathbb{A}}}}, \Delta^{\times\bullet}_{\overline{\underline{\mathbb{A}}}} \in \mathcal{A}_B$ are the objects

$$\Delta^{\bullet\leq 1}_{\overline{\underline{\mathbb{A}}}} := \varprojlim \left( \Delta^0_{\overline{\underline{\mathbb{A}}}} \rightrightarrows \Delta^1_{\overline{\underline{\mathbb{A}}}} \right).$$

$$\Delta^{\times\bullet}_{\overline{\underline{\mathbb{A}}}} := \prod_{\mathbb{Z}_{\geq 1}} (\Delta^{\bullet\leq 1}_{\overline{\underline{\mathbb{A}}}}) \simeq \lim_{l\in\mathbb{Z}_{\geq 1}} ((\Delta^{\bullet\leq 1}_{\overline{\underline{\mathbb{A}}}})^{\times l}).$$

*Remark* 21. The image of $\Delta^{\bullet\leq 1}_{\overline{\underline{\mathbb{A}}}}, \Delta^{\times\bullet}_{\overline{\underline{\mathbb{A}}}}$ is $\vec{\mathcal{A}}_B$ is contained in $O_B$.

**Definition 2.48.** The $\infty$-category $\mathrm{Spc}_{\overline{\underline{\mathbb{A}}},\beta}(O_B)$ is the subcategory in $\mathrm{Spc}(O_B)$ spanned by objects invariant with respect to morphisms of the form $X \leftarrow X \times \Delta^1_{\overline{\underline{\mathbb{A}}}}$, and $\beta$-blow-up-equivalences.

$$L_{\overline{\underline{\mathbb{A}}},\beta} \colon \mathrm{Spc}(O_B) \to \mathrm{Spc}_{\overline{\underline{\mathbb{A}}},Bl_\beta}(O_B)$$

is the localisation functor. An $(\overline{\underline{\mathbb{A}}},\beta)$-equivalence is a morphism that image along $L_{\overline{\underline{\mathbb{A}}},\beta}$ is an equivalence.

**Lemma 2.11.** *For any $F \in \mathrm{Spc}(O_B)$ and all $X \in O_B$, there is a natural equivalence*

$$(L_{\overline{\underline{\mathbb{A}}},\beta}F)(X) \simeq \varinjlim_{\Delta^{\times\bullet}} (L_\beta(F))(X \times \Delta^{\times l}_{\overline{\underline{\mathbb{A}}}}).$$

*Proof.* $\Delta^1_{\overline{\underline{\mathbb{A}}},\mathrm{all}}$ and $\Delta^1_{\overline{\underline{\mathbb{A}}}}$ are intervals in $O_B[w^{-1}_{Bl_\beta}]$, where $w_{Bl_\beta}$ is the class of $Bl_\beta$-equivalences. □

**Corollary 2.12.** *For any continuous $F \in \mathrm{Spc}(O_B)$ and all $X \in O_B$, there is a natural equivalence*

$$(L_{\overline{\underline{\mathbb{A}}},\beta}F)(X) \simeq F(\mathrm{limBl}_\beta(X \times \Delta^{\times\bullet}_{\overline{\underline{\mathbb{A}}}})).$$

2.5.3. *Endo-functor $F \mapsto F^{Bl_r}$.*

**Definition 2.49.** Let $\chi$ denote the class of morphisms in $A_B$ that image along the functor $\mathrm{Sch}^U_B \colon A_B \to \mathrm{Sch}$ from (2.9) is an isomorphism. Denote by $\chi^r$ the class of arrows in $\vec{A}_B$ defined by Definition 2.12. Denote by the same symbol $\chi^r$ the class of arrows in $\vec{\mathcal{A}}_B$ defined by Definition 2.18. Denote by the same symbol $\chi$ the class of arrows in $O_B$ that belong to the class $\chi$ in $\vec{\mathcal{A}}_B$.

**Lemma 2.13.** *Let $X \in O_B$. For any $\chi^r$-morphism*

$$f \colon X^r \to X$$

*in $\vec{\mathcal{A}}_B$ there is a $\chi^r$-morphism*

$$f' \colon X^{r'} \to X$$

*in $O_B$ equipped with a $\chi^r$-morphism $X^{r'} \to X^r$ that the composite with $f$ is equivalent to $f'$.*

**Definition 2.50.** For any $X \in O_B$ we consider the category $D(X)$ that objects are $\chi^r$-morphisms of the form

$$X^r \to X$$

and morphisms given by commutative triangles of $\chi^r$ in $O_B$.

For any $F \in \mathrm{Spc}(O_B)$, the assertion

$$F^{Bl}(X) := \varinjlim_{X^r \in D(X)} L_{Bl_\rho} F(X^r) \tag{2.23}$$

defines the object $F^{Bl_r} \in \mathrm{Spc}(\vec{\mathcal{A}}_B)$. Moreover, the assertion (2.23) defines the endo-functor on $\mathrm{Spc}(O_B)$ equipped with the natural transformation form the identity one

$$F \mapsto F^{Bl_r}, \quad F \to F^{Bl_r}.$$

2.5.4. *Endo-functor $F \mapsto F^{|K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}}}$.*

**Definition 2.51.** Let $K$ be a pointed cubical set, Notation 21. Then $|K|_{\underline{\bar{\mathbb{A}}}}$ is the image of $K$ along the left Kan extension of the functor

$$\begin{array}{ccc} (\Delta^{\bullet \leq 1})^{\mathbb{Z}_{\geq 0}} & \to & \mathrm{Spc}(O_B) \\ \prod_{i\in\mathbb{Z}_{\geq 0}} \Delta^{p_i} & \mapsto & \prod_{i\in\mathbb{Z}_{\geq 0}} \Delta^{p_i}_{\underline{\bar{\mathbb{A}}}} \end{array}$$

Similarly for $|K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}}$ and $\Delta^{p_i}_{\underline{\bar{\mathbb{A}}},\mathrm{all}}$.

For any cubical set $K$, there is the endo-functor $F \mapsto F^{|K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}}}$ on $\mathrm{Spc}(O_B)$ such that there is a natural equivalence

$$(F^{|K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}}})(X) \simeq \mathrm{Map}^{\mathrm{Spc}}_{\mathrm{Spc}^\bullet(O_B)}(X \wedge |K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}}), F) \tag{2.24}$$

2.5.5. *Properties of $L_{\underline{\bar{\mathbb{A}}},\beta}$: continuous presheaves and endo-functors.*

**Lemma 2.14.** *Let $F \in \mathrm{Spc}(O_B)$ be continuous, then $L_{\underline{\bar{\mathbb{A}}},\beta}(F) \in \mathrm{Spc}(O_B)$ is continuous, Item 18 and definition 2.48.*

*Proof.* Since $F$ is continuous, by Lemma 2.9

$$(L_{\underline{\bar{\mathbb{A}}},\beta}F)(X) \simeq \varinjlim_{\Delta^l \in \Delta^{\times\bullet}} F(\mathrm{limBl}_\beta(X \times \Delta^{\times l}_{\underline{\bar{\mathbb{A}}}}))$$

for all $X \in O_B$. By Remark 20 and since $F$ is continuous,

$$F(\mathrm{limBl}_\beta(X \times \Delta^{\times l}_{\underline{\bar{\mathbb{A}}}})) \simeq \varinjlim_\alpha F(\mathrm{limBl}_\beta(X_\alpha \times \Delta^{\times l}_{\underline{\bar{\mathbb{A}}}})).$$

Then since colimits commute with colimits,

$$\varinjlim_{\Delta^l \in \Delta^\bullet} F(\mathrm{limBl}_\beta(X \times \Delta^{\times l}_{\underline{\bar{\mathbb{A}}}})) \simeq \varinjlim_\alpha \varinjlim_{\Delta^l \in \Delta^\bullet} F(\mathrm{limBl}_\beta(X_\alpha \times \Delta^{\times\bullet}_{\underline{\bar{\mathbb{A}}}})).$$

$\square$

**Lemma 2.15.** *Let $F \in \mathrm{Spc}(O_B)$ be continuous, then $F^{|K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}}} \in \mathrm{Spc}(O_B)$ is continuous, Item 18 and definition 2.51.*

*Proof.* By (2.24) and since

$$(\varprojlim_\alpha X_\alpha) \times |K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}} \simeq \varprojlim_\alpha (X_\alpha \times |K|_{\underline{\bar{\mathbb{A}}}}),$$

it follows that

$$F^{|K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}}}(\varprojlim_\alpha X_\alpha) \simeq \varinjlim_\alpha F^{|K|_{\underline{\bar{\mathbb{A}}},\mathrm{all}}}(X_\alpha).$$

$\square$

**Lemma 2.16.** *Let $F \in \mathrm{Spc}(O_B)$. Then $L_{\underline{\bar{\mathbb{A}}},\beta}(F^{Bl_r}) \simeq L_{\underline{\bar{\mathbb{A}}},\beta}(F)^{Bl_r}$.*

*Proof.* It follows form Definition 2.46 that for any $X \in \vec{A}_B$, the subset
$$\rho(X \times \Delta^{\times l}_{\underline{\mathbb{A}}}) \subset \mathcal{S}(\mathsf{X} \times (\mathbb{P}^1)^{\times l})$$
equals the inverse image of $\rho(X)$ along the projection $\mathsf{X} \times \Delta^{\times l}_{\underline{\mathbb{A}}} \to \mathsf{X}$, the natural morphism is an equivalence
$$(F^{\Delta^{\times l}_{\underline{\mathbb{A}}}})^{Bl_r} \xrightarrow{\simeq} (F^{Bl_r})^{\Delta^{\times l}_{\underline{\mathbb{A}}}}.$$
So the claim follows because $(-)^{Bl_r}$ preserve colimits in $\mathrm{Spc}(O_B)$. □

**Lemma 2.17.** *Let $F \in \mathrm{Spc}(O_B)$. Then $L_{\overline{\underline{\mathbb{A}}},\beta}(F^{|K|_{\underline{\mathbb{A}},\mathrm{all}}}) \simeq (L_{\overline{\underline{\mathbb{A}}},\beta}(F))^{|K|_{\underline{\mathbb{A}},\mathrm{all}}}$.*

*Proof.* The claim follows because
$$(F^{|K|_{\underline{\mathbb{A}},\mathrm{all}}})^{\Delta^{\times l}_{\underline{\mathbb{A}}}} \simeq (F^{\Delta^{\times l}_{\underline{\mathbb{A}}}})^{|K|_{\underline{\mathbb{A}},\mathrm{all}}},$$
and $(-)^{|K|_{\underline{\mathbb{A}},\mathrm{all}}}$ preserve colimits in $\mathrm{Spc}(O_B)$. □

2.5.6. *$\phi$-equivalences.*

**Definition 2.52.** A morphism $\xi$ in $O_B$ is an $\phi$-equivalence if

- $\xi \in \mathrm{uLLP}(\mathcal{S} \to \mathcal{L}, \mathcal{L} \to *)$, where $*$ is the terminal object of $\mathrm{Spc}(O_B)$ and $\mathcal{S} \to \mathcal{L}$ is from Definition 2.1, and
- $\xi$ is an infinitesimal étale neighbourhood in $\vec{\mathcal{A}}_B$, Definition 2.30.

2.6. $(\phi, \overline{\underline{\mathbb{A}}}, \beta)$**-invariant objects.** We recollect some results and consequences on $\phi$-invariant objects from [8, 9] used in the next section.

**Theorem 2.1** ([9])**.** *For any $F \in sL^{\emptyset}$, $L_{\overline{\underline{\mathbb{A}}},\beta}(F)$ is $\phi$-invariant.*

**Theorem 2.2** ([9])**.** *Let $F$ be continuous and $(\phi, \overline{\underline{\mathbb{A}}}, \beta)$-invariant. Then $\pi_0(F)$ satisfies RLP w.r.t. $\mathscr{H}$-morphisms, Definition 2.34.*

**Theorem 2.3** ([8])**.** *Let $F \in \mathrm{Spc}(O_B)$ be invariant w.r.t. $(\phi, \overline{\underline{\mathbb{A}}}, \beta)$-equivalences. Then $F^{Bl}$ is invariant w.r.t. $(\phi, \overline{\underline{\mathbb{A}}}, \beta)$-equivalences.*

**Theorem 2.4** ([8])**.** *Let $F \in \mathrm{Spc}(O_B)$ be invariant w.r.t. $(\phi, \overline{\underline{\mathbb{A}}}, \beta)$-equivalences. Then $F^{|S^1|_{\overline{\underline{\mathbb{A}}}^1_{\mathrm{all}}}}$ is invariant w.r.t. $(\phi, \overline{\underline{\mathbb{A}}}, \beta)$-equivalences. where $S^1 = \Delta^1/\partial\Delta^1$.*

## 3. Presentation of morphisms in $\mathrm{H}^{\bullet}_{\mathbb{A}^1,\tau}(\mathrm{Sm}_B)$.

Throughout the section $B \in \mathrm{Sch}$, and $\tau = \mathrm{Zar}$ or $\tau = \mathrm{Nis}$ is the Zariski or Nisnevich topology on $\mathrm{Sch}_B$ or $\mathrm{Sm}_B$.

3.1. **Notation $H^K_\tau$.** The fully faithful functor $\mathrm{Sm}_B \to \mathrm{Sch}_B$ induces the fully faithful functors
$$E^{\mathrm{Spc}}\colon \mathrm{Spc}(\mathrm{Sm}_B) \to \mathrm{Spc}(\mathrm{Sch}_B), \quad E^{\mathrm{H}}\colon \mathrm{Spc}_{\mathbb{A}^1,\tau}(\mathrm{Sm}_B) \to \mathrm{Spc}_{\mathbb{A}^1,\tau}(\mathrm{Sch}_B).$$

**Definition-Notation 1.** Given $S \in \mathrm{Spc}_{\mathbb{A}^1,\tau}(\mathrm{Sch}_B)$ and pointed cubical set $K$,
$$H^K_\tau(S)\colon \mathrm{Spc}(\mathrm{Sch}_B) \to \mathrm{Spt}; W \mapsto \mathrm{Map}^{\mathrm{Spc}}_{\mathrm{Spc}_{\mathbb{A}^1,\tau}(\mathrm{Sch}_B)}(W \wedge K, S) \tag{3.1}$$

**Notation 6.** $H_\tau(S) := H^K_\tau(S)$ for $K = \Delta^0_+$. So $H_\tau(S)(W) \simeq \mathrm{Map}^{\mathrm{Spc}}_{\mathrm{Spc}_{\mathbb{A}^1,\tau}(\mathrm{Sch}_B)}(W, S)$.

**Notation 7.** We skip $S$ or $\tau$ for $H^K_\tau(S)$ from (3.1), when the context defines any of them.

**Notation 8.** We use notation $H^K$ for the presheaf on $A_B$ and $\vec{A}_B$ that is the inverse image long the functor $\vec{A}_B \to A_B \to \mathrm{Spc}(\mathrm{Sch}_B)$, and for the associated continuous presheaf on $\vec{\mathcal{A}}_B$, and its restriction on $O_B$.

3.2. **Intersections of line bundle sections and blow-ups.**

**Definition 3.1.** The category $sL^{\subset}$ is a full subcategory in $\mathrm{Set}(O_B)$ that objects are presheaves of sets $F$ on the $\infty$-category $O_B$ defined by presheaves on $A_B$ of the form

$$F(X) = \{s \in \mathcal{S}_{\mathcal{L}}^{\times\mathfrak{p}} | \bigcap_{\gamma\in\Gamma_\alpha} Z(s_\gamma) \subset Z(e_\alpha), \bigcap_{\gamma\in\Gamma_\epsilon} Z(s_\gamma) = \emptyset\} \tag{3.2}$$

for a morphism of finite sets $\mathfrak{p}\colon \mathfrak{S} \to \mathfrak{L}$, a set $A$ with subsets and elements

$$\Gamma_\alpha \subset \mathfrak{S}, \quad e_\alpha \in (s \cup \rho^{\mathrm{Mon}})$$

for each $\alpha \in A$.

**Theorem 3.1.** *For any $F \in sL^{\subset}$, there is a morphism in* $\mathrm{Set}(A_B)$

$$F' \to F$$

*that is an equivalence on $A_{\mathrm{rs},B}$ for some $F' \in \mathrm{Set}(A_B)$ of the form*

$$F' = (F'')_r^{\mathrm{limBl}}, \quad F'' \in sL^{\mathrm{div}},$$

*Definition 2.4.*

*Proof.* Let $X \in A_B$.

Let $F$ be given by (3.2). Let $F' \in sL^{\mathrm{div}}$ be the subpresheaf of $\mathcal{S}_{\mathcal{L}}^{\times\mathfrak{p}}$ defined by (2.2) with conditions like in (3.2) with the same $\alpha$ and $\gamma_\alpha$ as in (2.2) for $F$.

The element $f$ is defined by a set of sections of line bundles $s$ on $\mathsf{X}$. Then inverse images of $s$ on the limit of blow-ups $\tilde{X}$ of $X$ with centres $Z(s_\alpha)$ define an element of $F'$. By [10] there is a finite sequence of morphisms

$$\tilde{X}_n \xrightarrow{b_n} X_n \xrightarrow{r_n} \dots \xrightarrow{b_l} X_l \xrightarrow{r_l} \tilde{X}_{l-1} \xrightarrow{b_{l-1}} \dots \xrightarrow{r_1} \tilde{X}_0 \xrightarrow{\simeq} X_0 = X$$

in $A_B$ such that

- each $b_l$ is a $\rho$-blow-up,
- each $r_l$ is a $\pi$-morphism such that $\mathrm{Sch}_B^U(r_n)$ is an isomorphism in $\mathrm{Sch}_B$ and
- the composite decomposes as

$$\tilde{X}_n \to \tilde{X} \to X.$$

Hence the inverse images of $s$ defines an element in $F''(\tilde{X}_n)$. The latter element defines an element in $F'(X)$. ☐

**Lemma 3.1.** *For any $F \in sL^{\mathrm{div}}$, there is a morphism in* $\mathrm{Set}(A_B)$

$$F' \to F$$

*that induces an equivalence on $A_{\mathrm{rs},B}$ for some $F' \in \mathrm{Set}(A_B)$ such that $F' \in sL^{\emptyset}$, Definition 2.5.*

*Proof.* Any element $f \in F(X)$ for any $X \in A_{\mathrm{rs},B}$ is defined by the set of irreducible multiplicands of the sections defining the element $f$, and conversely, the set of irreducible multiplicands is defined by $f$. The latter sets of irreducible multiplicands form elements $f' \in F'_\alpha(X)$ for presheaves $F'_\alpha \in sL^{\emptyset}$ with the required morphism. ☐

**Corollary 3.2.** *For any $F \in sL^{\subset}$, there is a morphism in* $\mathrm{Set}(A_B)$

$$F' \to F$$

*that is an equivalence on $A_{\mathrm{rs},B}$ for some $F' \in \mathrm{Set}(A_B)$ such that*

$$F' = (F'')_r^{\mathrm{limBl}}, \quad F'' \in sL^{\emptyset},$$

*Definition 2.5.*

*Proof.* The claim follows by the combination of Theorem 3.1 and lemma 3.1. ☐

### 3.3. Presentation.

**Definition 3.2.** Let $(\bigcup_n ((sL^{\subset})^{(\Delta^1_{\mathbb{A}})^{\times n\alpha}}))^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$ denote the subcategory of presheaves generated by objects of the form

$$(F')^{(\Delta^1_{\mathbb{A}})^{\times n\alpha}}, \quad F' \in sL^{\subset}$$

via colimits.

**Definition 3.3.** A $\overset{\circ}{\mathrm{Z}}$ar-covering of $X \in A_B$ is a subset $c$ in $\mathcal{S}(X)$ such that there is the equality $\bigcup_{s\in c}(\overset{\circ}{X} - Z(s)) = \overset{\circ}{X}$ of open subschemes in $\overset{\circ}{X} = \mathrm{Sch}^V(X) \in \mathrm{Sch}$.

**Lemma 3.2.** *Any* $\mathrm{Sch}^{V*}(\mathrm{Zar})$*-coverings of* $X \in A_B$ *admits a refinement by a* $\overset{\circ}{\mathrm{Z}}$ar*-covering.*

**Definition 3.4.** $\mathrm{Cov}(X)$ is the diagram of $\overset{\circ}{\mathrm{Z}}$ar-coverings of $X \in A_B$ with morphisms given by refinements. $\mathrm{Cov}_l(X)$ is the set of $\overset{\circ}{\mathrm{Z}}$ar-coverings with $\#c = l$, Definition 3.3.

**Definition 3.5.** We define the endo-functor $L_{\mathrm{Zar}}$ on $\mathrm{Spt}(B)$ such that

$$L^{(1)}_{\mathrm{Zar}}(F) \simeq \varinjlim_{e\in\mathrm{Cov}(X)} \check{C}_e(F) \tag{3.3}$$

Let $\check{C}_e(F)$ denote $U \mapsto \varinjlim_l F(\tilde{X}^{\times l} \times_X U)$ and $\check{C}^{\mathbb{A}^1}_e(F)$ be the $\Delta^{\bullet}_{\mathbb{A}^1}$-geometric realisation in $\mathrm{Spc}(\mathrm{Sch}_X)$. $\check{C}_{e,X}(F) = \check{C}_e(F)(X)$, $\check{C}^{\mathbb{A}^1}_{e,X}(F) = \check{C}^{\mathbb{A}^1}_e(F)(X)$.

**Lemma 3.3.** *There is an equivalence*

$$L^{(1),l}_{\mathrm{Zar}}(F)(X) \simeq \varinjlim_{l} L^{(1),l}_{\mathrm{Zar}}(F)(X)$$

*where*

$$L^{(1),l}_{\mathrm{Zar}}(F)(X) = \coprod_{e\in\mathrm{Cov}_l(X)} \check{C}_{e,X}(F).$$

*Proof.* The claim is provided by the definition of $L_{\mathrm{Zar}}$ (3.3), and because (1) the set $\mathrm{Cov}(X)$ is the union of sets $\mathrm{Cov}_l(X)$ (2) for each $l$, the subdiagram of $\mathrm{Cov}(X)$ formed by $\mathrm{Cov}_l(X)$ is discrete. $\square$

**Lemma 3.4.** *There is a natural equivalence* $L^{(1),l}_{\mathrm{Zar}} L_{\mathbb{A}^1}(F)(X) \simeq \coprod_{e\in\mathrm{Cov}_l(X)} \check{C}^{\mathbb{A}^1}_{e,X}(F)$.

**Lemma 3.5.** *For any* $F \in (sL^{\subset})^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$, $L^{(1)}_{\mathrm{Zar}} L_{\mathbb{A}^1}(F)(X)$ *is in* $(\bigcup_n ((sL^{\subset})^{(\Delta^1_{\mathbb{A}})^{\times n\alpha}}))^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$.

*Proof.* It from Lemma 3.4 and Lemma 3.3 that there are natural equivalences

$$L^{(1)}_{\mathrm{Zar}} L_{\mathbb{A}^1}(F)(X) \simeq \varinjlim_{l} L^{(1),l}_{\mathrm{Zar}} L_{\mathbb{A}^1}(F)(X) \simeq \varinjlim_{l} \coprod_{e\in\mathrm{Cov}_l(X)} \check{C}^{\mathbb{A}^1}_{e,X}(F).$$

Then we note that $(X \mapsto \coprod_{e\in\mathrm{Cov}_l(X)} \check{C}^{\mathbb{A}^1}_{e,X}(F)) \in sL^{\subset}$. $\square$

**Lemma 3.6.** *For any* $F \in (sL^{\subset})^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$, $L_{\mathrm{Zar},\mathbb{A}^1}(X)$ *is in* $(\bigcup_n ((sL^{\subset})^{(\Delta^1_{\mathbb{A}})^{\times n\alpha}}))^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$.

*Proof.* Since there is an equivalence

$$L_{\mathrm{Zar},\mathbb{A}^1}(F) \simeq \varinjlim_{p} (L^{(1)}_{\mathrm{Zar}} L_{\mathbb{A}^1})^{\circ p}(F),$$

and $HS$ is in Hence the claim for the endo-functor $L_{\mathrm{Zar},\mathbb{A}^1}$ reduces to the claim for the endo-functors $L^{(1)}_{\mathrm{Zar}} L_{\mathbb{A}^1}$ provided by Lemma 3.5. $\square$

**Theorem 3.3.** *Let $\tau = \mathrm{Zar}$. Let $S = \mathbb{P}^l_B/(\mathbb{P}^l_B - 0_B)$. Then there is a morphism in $\mathrm{Spc}(A_B)$*

$$F \to \mathrm{Sch}^{Q^*}(H),$$

*that is an equivalence and such that $F$ is in $(\bigcup_n((sL^{\subset})^{(\Delta^1_{\underline{\mathbb{A}}})^{\times n_\alpha}}))^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$.*

*Proof.* For any $X \in A_B$, $H(X) = L_{\mathrm{Zar},\mathbb{A}^1}(HS)((\mathsf{X} - Z(\partial))//(\mathsf{X} - Z(\partial) - Z(\varsigma)))$, where $HS = \mathrm{Map}(-, \mathbb{P}^l_B/(\mathbb{P}^l_B - 0_B))$, and $(\mathsf{X} - Z(\partial))//(\mathsf{X} - Z(\partial) - Z(\varsigma)) = \mathsf{X}\amalg_{\mathsf{X}-Z(\partial)}(\mathsf{X} - Z(\partial)) \times (\Delta^1_{\mathbb{A}^1}/\Delta^0_{\mathbb{A}^1})$. Since $HS$ is in $sL^{\subset}$, $L_{\mathrm{Zar},\mathbb{A}^1}(HS)$ is in $(\bigcup_n((sL^{\subset})^{(\Delta^1_{\underline{\mathbb{A}}})^{\times n_\alpha}}))^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$ by Lemma 3.6. Then $H$ is in $(\bigcup_n((sL^{\subset})^{(\Delta^1_{\underline{\mathbb{A}}})^{\times n_\alpha}}))^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$ because sections of line bundles on $(\mathsf{X} - Z(\partial))//(\mathsf{X} - Z(\partial) - Z(\varsigma))$ are defined by sections of line bundles on $\mathsf{X} \times \Delta^1_{\mathbb{A}^1}$. □

**Theorem 3.4.** *Let $\tau = \mathrm{Zar}$. Let $S = \mathbb{P}^l_B/(\mathbb{P}^l_B - 0_B)$, $K$ be a pointed cubical set. Then there is a morphism in $\mathrm{Spc}(A_B)$*

$$F \to \mathrm{Sch}^{Q^*}(H(S)).$$

*that is an equivalence on $A_{\mathrm{rs},B}$ and such that $F$ is generated by objects of the form*

$$((F')^{\mathrm{limBl}}_r)^{(\Delta^1_{\underline{\mathbb{A}}})^{\times n_\alpha}}, \quad F' \in sL^{\emptyset}$$

*via colimits.*

*Proof.* The claim follows form Theorem 3.3, Theorem 3.2. □

**Theorem 3.5.** *Let $\tau = \mathrm{Zar}$. Let $S = \mathbb{P}^l_B/(\mathbb{P}^l_B - 0_B)$, $K$ be a pointed cubical set. Then there is a morphism in $\mathrm{Spc}(A_B)$*

$$F \to \mathrm{Sch}^{Q^*}(H^K(S)).$$

*that is an equivalence on $A_{\mathrm{rs},B}$ and such that $F$ is in $(\bigcup_n((sL^{\subset})^{(\Delta^1_{\underline{\mathbb{A}}})^{\times n_\alpha}}))^{\mathrm{Colim}}_{\mathrm{Spc}(A_B)}$, in other words $F$ is generated by objects of the form*

$$(((F')^{\mathrm{limBl}}_r)^{(\Delta^1_{\underline{\mathbb{A}}})^{\times n_\alpha}})^{|K|_{\underline{\mathbb{A}},\mathrm{all}}}, \quad F' \in sL^{\emptyset}$$

*via colimits.*

*Proof.* The claim follows form Theorem 3.4, Equation (3.1), because the endo-functor $(-)^{|K|_{\underline{\mathbb{A}},\mathrm{all}}}$ on $\mathrm{Spt}(A_B)$ preserves colimits. □

## 4. Injectivity Theorem

### 4.1. Lifting propery and Injectivity.

**Theorem 4.1.** *Let $\tau = \mathrm{Zar}$, and $B \in \mathrm{Sch}$ be a regular scheme of a finite Krull dimension. Let $S \in \mathrm{Spc}_{\mathbb{A}^1,\tau}(B)$ belong to the subcategory generated by $\mathbb{P}^{\wedge l}_B$ for all $l \in \mathbb{Z}_{\geq 0}$ via colimits and extensions. Let $n \in \mathbb{Z}_{\geq 0}$, and $H^{S^n} = H^{S^n}(S)$, Notation 7.*

*Then*

$$\mathrm{Sch}^{Q^*}(H^{S^n}) \in \mathrm{Spc}(O_B)$$

*is $(\phi, \underline{\overline{\mathbb{A}^1}}, Bl_\beta)$-invariant, and the preshaef*

$$\pi_0(\mathrm{Sch}^{Q^*}(H^{S^n})) \in \mathrm{Set}(O_B)$$

*satisfies RLP w.r.t. $\mathscr{H}$-morphisms, Notation 24, Definition 2.34, and Definition 2.36.*

*Proof.* By Theorems 3.2 and 3.5 there is an equivalence in $\mathrm{Spc}(O_B)$

$$F' \to \mathrm{Sch}^{Q^*}(H^{S^n}) \tag{4.1}$$

such that $F$ is generated by objects of the form

$$F' = (((F'')^{\mathrm{limBl}}_r)^{(\Delta^1_{\underline{\mathbb{A}}})^{\times N}})^{|S^n|_{\underline{\mathbb{A}},\mathrm{all}}}, \quad F'' \in \Sigma^\infty(sL^{\emptyset})$$

via colimits and extensions. By Lemmas 2.16 and 2.17

$$L_{\overline{\underline{\mathbb{A}}},\beta}(F') \simeq (((L_{\overline{\underline{\mathbb{A}}},\beta}(F''))_r^{\mathrm{limBl}})^{(\Delta^1_{\overline{\underline{\mathbb{A}}}})^{\times N}})^{|S^n|_{\overline{\underline{\mathbb{A}}},\mathrm{all}}}). \tag{4.2}$$

By Theorem 2.1 $L_{\overline{\underline{\mathbb{A}}},\beta}(F'')$ is $\phi$-invariant. By Theorem 2.3 and Theorem 2.4 the right side presheaf of (4.2) is $\phi$-invariant. Thus $L_{\overline{\underline{\mathbb{A}}},\beta}(F')$ is $(\phi, \overline{\underline{\mathbb{A}^1}}, Bl_\beta)$-invariant. Then $\pi_0 L_{\overline{\underline{\mathbb{A}}},\beta}(F')$ satisfies RLP w.r.t. $\mathscr{H}$-morphisms by Theorem 2.2. □

**Theorem 4.2** ([6, 11, 32, 41])**.** *Let $B \in \mathrm{Sch}$ be a regular scheme of a finite Krull dimension. Let $U$ be a local henselian essentially smooth scheme over $B$, and $Z$ be a closed subscheme of positive relative codimension. Let $S \in \mathrm{H}^{\bullet}_{\mathbb{A}^1,\mathrm{Nis}}(B)$, and $K$ be a pointed cubical set, Notation 21. Then the morphism of the pointed sets*

$$E_{\mathrm{Nis}}\colon \mathrm{Map}_{\mathrm{H}^{\bullet}_{\mathbb{A}^1,\mathrm{Nis}}(B)}(U_+/(U-Z)_+ \wedge K, S) \to \mathrm{Map}_{\mathrm{H}^{\bullet}_{\mathbb{A}^1,\mathrm{Nis}}(B)}(U_+ \wedge K, S)$$

*is trivial.*

**Theorem 4.3.** *Let $B \in \mathrm{Sch}$ be regular of Krull dimension one. Let $U$ be an essentially smooth local henselian scheme over $B$. Let $\rho$ denote a regular function on $B$ that vanishing locus equals a closed point in $B$ as well as its inverse image on $U$.*

*For any non-negative integer $n$, the morphism of the pointed sets*

$$E_\tau\colon \mathrm{Map}_{\mathrm{H}^{\bullet}_{\mathbb{A}^1,\tau}(B)}(U_+/(U-Z(\rho))_+ \wedge S^n, S) \to \mathrm{Map}_{\mathrm{H}^{\bullet}_{\mathbb{A}^1,\mathrm{Nis}}(B)}(U_+ \wedge S^n, S) \tag{4.3}$$

*is trivial for for any $S \in \mathrm{H}^{\bullet}_{\mathbb{A}^1,\tau}(B)$ such that $\pi_0\mathrm{Sch}^{Q^*}(H^{S^n}(S)) \in \mathrm{Set}(O_B)$ satisfies RLP w.r.t. $\mathscr{H}$-morphisms, and $\tau = \mathrm{Zar}$.*

*Proof.* The object

$$\mathcal{U} = \mathrm{id}_{(U,(\{\rho\},(1,\rho,\emptyset))))} \in O_B$$

in $\vec{A}_{\mathrm{rs},B}$, Definition 2.36, goes by applying the natural transformation of the functors $\mathrm{Sch}^V \to \mathrm{Sch}^Q$ from (2.12) to the morphism

$$U/(U-Z) \to U \in \mathrm{Spc}(\mathrm{Sch}_B).$$

Then the morphism (4.3) is equivalent to to the morphisms

$$E\colon \mathrm{Sch}^{Q^*}(H^{S^n})(\mathcal{U}) \to \mathrm{Sch}^{V^*}(H^{S^n}_{\mathrm{Nis}})(\mathcal{U})$$

induced by the natural transformation $\mathrm{Sch}^V \to \mathrm{Sch}^Q$ and morphism $H^{S^n} \to H^{S^n}_{\mathrm{Nis}}$ defined by (3.1).

Let $\rho^{\mathrm{Map}}\colon U \to \mathbb{A}^N_U$ be a morphism such that $Z(\rho^{\mathrm{Map}*}(t_0)) = Z(\rho)$. Let

$$\tilde{\mathcal{U}} = (\mathrm{id}_{(\mathbb{A}^N_U,(\{t_0\},(1,t_0,\emptyset))))})^h_{\rho^{\mathrm{Map}}(U)},$$

where $(-)^h$ stands for the Henselisation defined as the limit of étale neighbourhoods. The morphism

$$\tilde{c}\colon \mathcal{U} \to \tilde{\mathcal{U}}$$

is in $\mathscr{H}$, Definition 2.34. By assumption $\pi_0\mathrm{Sch}^{Q^*}(H^{S^n})$ satisfies RLP w.r.t. $\mathscr{H}$. Then for any $e \in \mathrm{Sch}^{Q^*}(H^{S^n})(\mathcal{U})$, there is a lift $\tilde{e}$ of $e$, i.e. there an element

$$\tilde{e} \in \mathrm{Sch}^{Q^*}(H^{S^n})(\tilde{\mathcal{U}}), \quad \tilde{c}^*(\tilde{e}) = e. \tag{4.4}$$

Since by Theorem 4.2, the morphism

$$\mathrm{Sch}^{Q^*}(H^{S^n}_{\mathrm{Nis}})(\tilde{\mathcal{U}}) \to \mathrm{Sch}^{V^*}(H^{S^n}_{\mathrm{Nis}})(\tilde{\mathcal{U}})$$

is trivial, for any $\tilde{e}$ as above, there is the equality $\tilde{E}(\tilde{e}) = *$, where

$$\tilde{E}\colon \mathrm{Sch}^{Q^*}(H^{S^n})(\tilde{\mathcal{U}}) \to \mathrm{Sch}^{V^*}(H^{S^n}_{\mathrm{Nis}})(\tilde{\mathcal{U}}).$$

Hence by (4.4) $E(e) = *$ for any $e \in \mathrm{Sch}^{Q^*}(H^{S^n})(\mathcal{U})$. □

4.2. **Acyclicity.** Theorem 4.3 and [7] imply the following consequence.

**Corollary 4.4.** *Let $B$ be a DVR spectrum.*

*Let $S \in \mathrm{SH}_{\mathbb{A}^1,\mathrm{Zar}}(\mathrm{Sm}_B)$. Suppose that*

*(1) $S$ satisfies descent with respect to Nisnevich squares.*
*(2) $S$ belongs to the subcategory of $\mathrm{SH}_{\mathbb{A}^1,\mathrm{Zar}}(\mathrm{Sm}_B)$ generated by suspension spectra of $\mathbb{P}^l_B/(\mathbb{P}^l_B - 0_B)$ via shifts and colimits.*

*Then for any essentially smooth local henselian $U$ over $B$ the Cousin complex*

$$\mathrm{S}^n(U) \to \bigoplus_{u\in U^{(0)}} \mathrm{S}^n_u(U) \to \cdots \to \bigoplus_{u\in U^{(c)}} \mathrm{S}^{c+n}_u(U) \to \dots, \tag{4.5}$$

*where*

$$\mathrm{S}^l_W(U) = \pi_0\mathrm{Map}_{\mathrm{SH}_{\mathbb{A}^1,\mathrm{Nis}}(B)}(U/(U-W), L_{\mathbb{A}^1,\mathrm{Nis}}(S)[l]),$$

*is acyclic on $U$.*

*Proof.* By Theorem 4.3 the complex

$$\mathrm{S}^n(U) \to \mathrm{S}^n_{\eta\times_B U}(U) \to \mathrm{S}^{1+n}_{z\times_B U}(U) \to \dots,$$

where $z \in B$ is the image of the closed point of $U$, and $\eta \in B$ be the image of the generic point, is acyclic. The morphism from the Cousin complex (4.5) to the complex

$$\mathrm{S}^n(U) \to \mathrm{S}^n_{\eta\times_B U}(U) \to \mathrm{S}^{1+n}_{z\times_B U}(U) \to \dots,$$

to the Cosuin complex is a quasi-isomorphism by [7] for $U = X^h_x$ for any $X \in \mathrm{Sm}_B$, $x \in X$. Consequently, the morphism is a quasi-isomorphism for any essentially smooth local Henselian $U$ because the Cousin complex is compatible with affine transition morphisms of schemes. Hence the (4.5) is acyclic. □

Further, using [1, 33, 45], we deduce the claim of the Gersten conjecture for $K$-theory for local henselian essentially smooth schemes $U$ over a one-dimensional regular base scheme $B$.

**Theorem 4.5.** *Let $B$ be a DVR spectrum. Let $U$ be an essentially smooth local henselian scheme over $B$. The complex* (1.1)*is acyclic.*

*Proof.* The K-theory presheaf is isomorphic to the presheaf of homotopies of $S^1$-spectra presheaf K on $\mathrm{Sm}_B$ associated by Secal's machine to the $\infty$-commutative monoid formed by nerves of the groupoids of vector bundles $\mathrm{Vect} = \bigoplus_L \mathrm{Vect}_L$,

The presheaves $\mathrm{Vect}_L \in \mathrm{Spc}(\mathrm{Sm}_B)$ satisfy Nisnevich descent by [1, Proof of Th 5.2.3]. Hence $\mathrm{K} \in \mathrm{Spc}(\mathrm{Sm}_B)$ satsfies Nisnevich descent. Since presheaves $K_n$ are $\mathbb{A}^1$-invariant on smooth affine $B$-schemes for all $n \geq 0$ by [39, Prop 4.1]and Poincar´e Duality [39, Section 7.1], the presheaf K is $\mathbb{A}^1$-invariant. Then it follows form by [1, Th 5.1.3] that $L_{\mathbb{A}^1,\mathrm{Zar}}(\mathrm{K}) \simeq L_{\mathbb{A}^1,\mathrm{Nis}}(\mathrm{K})$ in $\mathrm{Spt}(\mathrm{Sm}_B)$ and

$$K_n(X) \simeq \pi_n(L_{\mathbb{A}^1,\mathrm{Zar}}(\mathrm{K})) \simeq \pi_n(L_{\mathbb{A}^1,\mathrm{Nis}}(\mathrm{K})). \tag{4.6}$$

Because of Bott periodicity, [45, Th 6.8], and by (4.6), there is a $\mathbb{P}^1$-spectum

$$(L_{\mathbb{A}^1,\mathrm{Zar}}(\mathrm{K}), L_{\mathbb{A}^1,\mathrm{Zar}}(\mathrm{K}), \dots) \in \mathrm{Spc}(\mathrm{Sm}_B)[\mathbb{P}^{\wedge -1}_B] \tag{4.7}$$

that represents K-theory. Since by [33, Prop 3.13, 4.14]

$$K_n(X) \simeq \mathrm{Map}_{\mathrm{H}_{\mathbb{A}^1,\mathrm{Nis}}}(X \wedge S^n, \mathrm{Gr}_{\infty,\infty} \times B),$$

the schemes $\mathrm{Gr}_{N,L,B}$ are cellular for $N, L \in \mathbb{Z}_{\geq 0}$, the spectrum (4.7) is generated by $\mathbb{P}^l_B$ for $l \in \mathbb{Z}$.

Then the claim follows for by Theorem 4.4 applied to (4.7). □

## 5. Construction of the $\phi$-motivic homotopy category

In this section we present the definition of $\phi$-motivic homotopy category, and outline its properties.

5.0.1. *Motivic category of "compactified" schemes.*

**Definition 5.1.** $\mathrm{Sch}^{\mathrm{comp}}_B$ is the category with objects
$$(\mathsf{X}, \partial),$$
where

- $\mathsf{X} \in \mathrm{Sch}_B$, and $\partial \in \mathcal{S}(\mathsf{X})$ invertible at each generic point of $\mathsf{X}$ and admits locally a presentation as a product $\prod_\alpha \partial_\alpha$ for some $\partial_\alpha \in \mathcal{S}(\mathsf{X})$ such that for any subset $A'$ of $A$, the connected components of $\cap_{A'\subset A} Z(\partial_\alpha)$ are irreducible and have codimension $\#(A')$ in $\mathsf{X}$,

and morphisms
$$(\mathsf{X}, \partial) \to (\mathsf{X}', \partial')$$
defined by

- a morphism $f\colon \mathsf{X} \to \mathsf{X}'$ in $\mathrm{Sch}_B$ such that
- $\partial \leq f^*(\partial')$.

**Definition 5.2.** The $(\tau, (\mathbb{P}^1, \infty), Bl)$-motivic homotopy category of compactified schemes over $B$ is the $\infty$-category
$$\mathrm{Spc}^\bullet_{\tau,(\mathbb{P}^1,\infty),Bl}(\mathrm{Sch}^{\mathrm{comp}}_B)$$
that is the subcategory of $\mathrm{Spc}(\mathrm{Sch}^{\mathrm{comp}}_B)$, Definition 5.1, spanned by $(\tau, (\mathbb{P}^1, \infty), Bl)$-invariant objects. Here

- $\tau$ indicates local equivalences with respect to the inverse image of the topology $\tau$ on $\mathrm{Sch}_B$ along the functor from Definition 5.5;
- $(\mathbb{P}^1, \infty)$ indicates invariance with respect to morphisms
$$(\mathsf{X}, \partial) \leftarrow (\mathsf{X}, \partial) \times (\mathbb{P}^1, \infty);$$
- $Bl$ indicates the invariance with respect to the blow-ups of objects $(X, \partial)$ that centers $C \subset X$ that have locally on $X$ a presentation $C = Z(ec_0, \dots ec_l)$ for some sections $ec_\alpha$ of $\mathcal{S}$ such that $\mathrm{codim}_X C = l + 1$, and $ec_0 \leq \partial$.

*Remark* 22. The argument of Theorem 4.3 applies to $S$ in $\mathrm{Spc}^\bullet_{\mathrm{Zar},(\mathbb{P}^1,\infty),Bl}(\mathrm{Sch}^{\mathrm{comp}})$ generated by $\mathbb{P}^{\wedge l}_B$ via shifts and colimits and any
$$(U, \partial), \quad Z \subset U,$$
so that $U$ is local henselian and $(U, Z(\prod I(\partial) \cdot I(Z)) \in \mathrm{Sch}^{\mathrm{comp}}_{\mathrm{reg}}$, where $I(\partial)$ and $I(Z)$ denote the sheaf of ideals defined by the divisor $\partial$ and the closed subscheme $Z$, and $\mathrm{Sch}^{\mathrm{comp}}_{\mathrm{reg}}$ is the subcategory of objects $(\mathsf{X}, \partial) \in \mathrm{Sch}^{\mathrm{comp}}$ such that $\mathsf{X}$ is regular, and $Z(\partial)$ is regular.

5.0.2. $(Bl_\rho, \underline{\mathbb{A}^1}, \phi, r, \tau)$*-motivic homotopy category.*

**Definition 5.3.** We define the following classes of morphisms in the category $A_B$, and consequently, $O_B$:

- $\tau$-local equivalences, where $\tau$ is the topology on $A_B$ induced by a topology $\tau$ on $\mathrm{Sch}_B$ as the inverse image along the functor $\mathrm{Sch}^U_B$,
- $\downarrow$-equivalences defined by the counit of the adjunction $\vec{A}_B \simeq (A_B)^{\Delta^1} \leftrightarrows A_B$, i.e. morphisms $\vec{Y} \to x$ such that the induced morphism $Y \to X$ is an equivalence,
- $r$-equivalences are defined by morphisms $x$ given by (2.5) in $A_B$ such that $\mathrm{Sch}^U_B(x)$ is an isomorphism and $\partial = \mathrm{Sch}^U_B(x)^*(\partial')$,

and define the following class of morphisms in the $\infty$-category $\mathrm{Spc}(O_B)$:

- $\varsigma$-equivalences are morphisms cofiber $\to X$, where $X \in A_B$ is given by (2.4), and cofibre $= \mathrm{cofib}((\mathsf{X},(\rho,(\partial\cdot\varsigma,1,\beta))) \to (\mathsf{X},(\rho,(\partial,1,\beta))))$.

**Definition 5.4.** $(Bl_\rho, \overline{\underline{\mathbb{A}^1}}, \phi, r, \downarrow, \tau)$-motivic homotopy category defined as the subcategory

$$\mathrm{Spc}_{\tau,\varsigma,\downarrow,r,\phi,\overline{\underline{\mathbb{A}}},Bl_\rho}(O_B^{\varsigma=1})$$

of $(\tau,\downarrow,r,\phi,\overline{\underline{\mathbb{A}^1}},Bl_\rho)$-invariant objects in $\mathrm{Spc}(O_B^{\varsigma=1})$.

**Definition 5.5.** $\mathrm{Sch}^{\mathrm{comp}}$ is the functor

$$\begin{array}{rclcl}\mathrm{Sch}^{\mathrm{comp}} & : & O_B^c & \to & \mathrm{Sch}_B^{\mathrm{comp}} \\ & & (x\colon X \to \underline{X}) & \mapsto & (\mathsf{X},\partial).\end{array}$$

**Definition 5.6.** $Q^{\mathrm{comp}}$ is the functor

$$\begin{array}{rclcl}Q^{\mathrm{comp}} & : & O_B^c & \to & \mathrm{Spc}(\mathrm{Sch}_B^{\mathrm{comp}}) \\ & & (\mathsf{X},(\rho,(\partial,\varsigma,\beta))) & \mapsto & (\mathsf{X},\partial)/(\mathsf{X},\partial\cdot\varsigma).\end{array} \tag{5.1}$$

*Remark* 23. The claim of Theorem 4.3 holds for $(Bl_\rho, \overline{\underline{\mathbb{A}^1}}, \phi, r, \mathrm{Nis})$-motivic homotopy category, Definition 5.4, in the sense that the fibre of the map

$$F((U,(\rho,1,\rho))) \to F((U,(\rho,1,1)))$$

is trivial for any $F \in \mathrm{Spc}_{\mathrm{Nis},r,\phi,\overline{\underline{\mathbb{A}}},Bl_\rho}(O_B)$.

5.1. **Sketch for Comparison Theorem.** The subcategory of continuous objects in

$$\mathrm{Spc}_{\tau,\varsigma,\downarrow,r,\phi,\overline{\underline{\mathbb{A}}},Bl_\rho}(O_B)$$

is equivalent to the $\infty$-category

$$\mathrm{Spc}_{\tau,(\mathbb{P}^1,\infty),Bl}(\mathrm{Sch}_B^{\mathrm{tcomp}}).$$

Here $\mathrm{Sch}_B^{\mathrm{tcomp}}$ is the intersection $\mathrm{Sch}_B^{\mathrm{comp}}$ and the subcategory $\mathrm{limSch}_B^{\mathrm{comp}}$ of $\mathrm{Spc}^{\mathrm{Sch}_B^{\mathrm{comp}}}$ defined by analogous conditions as for $O_B$ in Definition 2.43. We outline the proof.

Firstly, we introduce notation $O_B^{\varsigma=1}$ for the subcategory of $O_B$ spanned by the objects defined by diagrams in $A_B$ of objects of the form (2.4) such that $\varsigma = 1$ is the trivial element in $\mathcal{S}(\mathsf{X})$, For any $\tau \supset \mathrm{Zar}$, the adunction

$$\begin{array}{lcl}O_B^{\varsigma=1} & \leftrightarrows & O_B \\ (\mathsf{X},(\rho,(\partial,1,\beta))) & \leftarrow & (\mathsf{X},(\rho,(\partial,1,\beta))), \\ (\mathsf{X},(\rho,(\partial,1,\beta))) & \leftarrow & (\mathsf{X},(\rho,(\partial,\varsigma,\beta))).\end{array}$$

induces an equivalence

$$\mathrm{Spc}_{\tau,\varsigma,\downarrow,r,\phi,\overline{\underline{\mathbb{A}}},Bl_\rho}(O_B) \simeq \mathrm{Spc}_{\tau,\downarrow,r,\phi,\overline{\underline{\mathbb{A}}},Bl_\rho}(O_B^{\varsigma=1}).$$

Further, there is an equivalence

$$\mathrm{Spc}_{\tau,\downarrow,r,\overline{\underline{\mathbb{A}}},Bl_\rho}(O_B^{\varsigma=1}) \simeq \mathrm{Spc}_{\tau,(\mathbb{P}^1,\infty),Bl}(\mathrm{limSch}_B^{\mathrm{comp}}) \tag{5.2}$$

of the localisations of the categories of presheaves on $O_B$ and $\mathrm{limSch}_B^{\mathrm{comp}}$ with respect to the equivalences of the types mentioned in the subscripts. Without providing the required definitions, we explain that the arrows category $\vec{A}_B = A_B^{\Delta^1}$ and the data $\rho$ can be reduced in the construction of the left side because $\downarrow$-equivalences equalise $A_B$ and $\vec{A}_B$, and because $(\downarrow,r)$-equivalences allow modifying $\rho$ in arbitrary way.

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