Minimally tough series-parallel graphs with toughness at least 1/2

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Abstract

Let t be a positive real number. A graph is called *t-tough* if the removal of any vertex set S that disconnects the graph leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t-tough. A graph is minimally t-tough if the toughness of the graph is t, and the deletion of any edge from the graph decreases the toughness. Series–parallel graphs are graphs with two distinguished vertices called terminals, formed recursively by two simple composition operations, series and parallel joins. They can be used to model series and parallel electric circuits.

We characterize the minimally t-tough series-parallel graphs for all $t \geq 1/2$. It is clear that there is no minimally t-tough series-parallel graph if $t > 1$. We show that for $1 \geq t > 1/2$, most of the series-parallel graphs with toughness t are minimally t-tough, but most of the series-parallel graphs with toughness 1/2 are not minimally 1/2-tough.
Keywords: toughness, series-parallel graph

1 Introduction

All graphs considered in this paper are undirected, but they may contain loops and parallel edges. Let $c(G)$ denote the number of components, and $\kappa(G)$ the connectivity number of the graph G. For a connected graph G, a vertex set $S \subseteq V(G)$ is called a *cutset* if $c(G - S) > 1$. The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. In the present paper, we only use this notation when $E_1 \cap E_2 = \emptyset$ and $|V_1 \cap V_2| \leq 2$. For an edge uv where $u, v \in V_1$ the union of G_1 and uv is $G_1 \cup uv = (V_1, E_1 \cup uv)$.

The notion of toughness was introduced by Chvátal in [2].

Definition 1. *Let t be a real number. A graph G is called* t-tough *if $|S| \geq t \cdot c(G-S)$ for any set $S \subseteq V(G)$ with $c(G - S) > 1$. The* toughness *of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$. We say that a cutset $S \subseteq V(G)$ is a* tough set *if $|S| = \tau(G) \cdot c(G - S)$.*

Note that a graph is disconnected if and only if its toughness is 0.

This notion is one of the widely used measures in network reliability, it is mentioned in various applications for different types of networks. So it is a natural question to investigate the following concept of minimally t-tough graphs, introduced by Broersma in [1].

Definition 2. *Let t be a real number. A graph G is said to be* minimally t-tough *if $\tau(G) = t$ and $\tau(G - e) < t$ for every $e \in E(G)$.*

It follows directly from the definition that every t-tough noncomplete graph is $2t$-connected, implying $\kappa(G) \geq 2\tau(G)$ for noncomplete graphs. Therefore, the minimum degree of any t-tough noncomplete graph is at least $\lceil 2t \rceil$.

The following conjecture is motivated by a theorem of Mader [10], which states that every minimally k-connected graph has a vertex of degree k.

Conjecture 1 (Kriesell [5])**.** *Every minimally* 1*-tough graph has a vertex of degree* 2*.*

This conjecture can be naturally generalized.

Conjecture 2 (Generalized Kriesell's Conjecture [6])**.** *Every minimally t-tough graph has a vertex of degree* $\lceil 2t \rceil$*.*

Recently, Conjecture 2 was disproved. In [12] Zheng and Sun constructed minimally $\left(1+\frac{1}{2k-1}\right)$-tough 4-regular graphs for every $k \geq 2$. Since $\left\lceil 2\left(1+\frac{1}{2k-1}\right)\right\rceil < 3$, the Conjecture 2 is not true for all values of t, but it still might be true for all other values of t, or for all irregular graphs [12].

On the other hand, Conjecture 2 was verified for several graph classes and various values of t. In [6] Conjecture 2 was proved for minimally t-tough split graphs for all t, minimally t-tough chordal graphs when $t \leq 1$, and minimally t-tough claw-free graphs when $t \leq 1$. Also, in [9] Ma et al. showed that for minimally $\frac{3}{2}$-tough, claw-free graphs the conjecture is true.

This paper focuses on series–parallel graphs, which hold significant importance in graph theory and computer science due to their structured and hierarchical properties. This inherent organization simplifies their analysis compared to general graphs. Moreover, series-–parallel graphs are particularly valuable in applications such as network design, circuit layout, and algorithm development, as their structure allows many computationally intensive algorithms to run more efficiently.

There are several ways to define series–parallel graphs. The following definition basically follows the one used in [8].

Definition 3 (Series–Parallel Graph)**.** *A graph $G(s,t)$ is a* series–parallel graph *(sp-graph for short) with terminals s and t, if either G consists of one edge connecting s and t, or G is derived from two or more series–parallel graphs by one of the following two operations.*

- *Series join: Given k series–parallel graphs $G_1(s_1,t_1), G_2(s_2,t_2), \ldots, G_k(s_k,t_k)$, form a new graph $G(s,t)$ by identifying the vertices t_i and s_{i+1} for all $1 \leq i \leq k-1$ and setting $s = s_1$ and $t = t_k$.*
- *Parallel join: Given k series–parallel graphs $G_1(s_1,t_1), G_2(s_2,t_2), \ldots, G_k(s_k,t_k)$, construct a new graph $G(s,t)$ by identifying the vertices $s_1, s_2, \ldots, s_k$ as a single vertex s, and similarly identifying the vertices $t_1, t_2, \ldots, t_k$ as a single vertex t.*

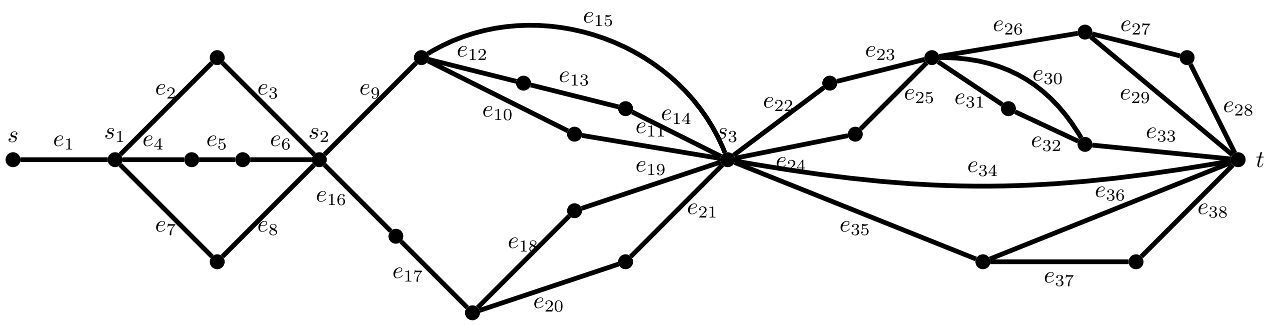

Figure 1: An example of a series-parallel graph with terminals s and t.

In either operation, instead of joining k graphs in one step, we could make $k-1$ joins each time joining precisely 2 graphs.

Note that the ordering $G_1(s_1,t_1), G_2(s_2,t_2), \ldots, G_k(s_k,t_k)$ matters for the series join, while it does not matter for the parallel join.

The recursive definition of the series–parallel graph above naturally gives a rooted, leveled, vertex–ordered tree, called a *series–parallel tree* (sp-tree for short), for each series–parallel graph $G(s,t)$. To avoid confusion, we refer to the vertices of this tree as *nodes*, while the word *vertex* will always refer to a vertex of the series–parallel graph.

Each leaf in an sp-tree corresponds to an edge of $G(s,t)$, and each non-leaf node in an sp-tree corresponds to either a series or a parallel join. We call such nodes *normal, series*, or *parallel*, respectively.

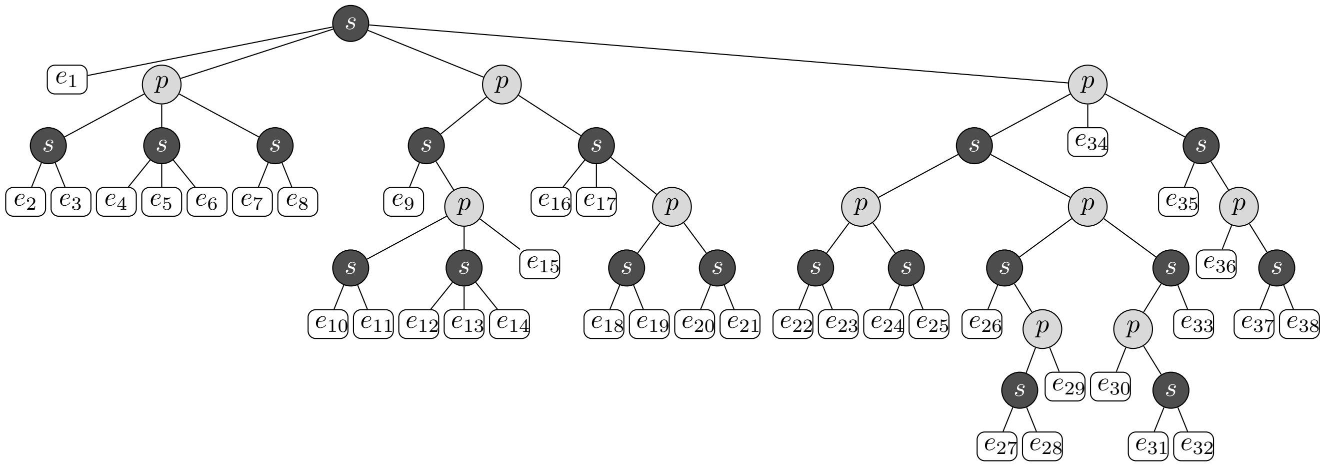

Figure 2: The series parallel tree of the graph of Figure 1.

The parent of a normal node can be either a series or a parallel node. The parent of a series or a parallel node could be a series or a parallel node. However, it is easy to see that the operations can always be chosen so that the parent of a series node is always a parallel node, and the parent of a parallel node is always a series node. We can also assume that the children of series and parallel nodes are ordered from left to right in the same order as the corresponding operations were performed. We denote this unique series–parallel tree of G by T_G.

In Section 2 we will prove some valuable lemmas, in Section 3 we give a characterization of minimally t-tough sp-graphs with $\frac{1}{2} < t \leq 1$, in Section 4 we give a characterization of minimally $\frac{1}{2}$-tough sp-graphs, the main result is stated in Theorem 26. Finally, in Section 5 we have some open questions.

2 Useful Lemmas

The first lemma establishes that we can restrict our attention to simple graphs. While it might seem natural to assume that we are working exclusively with simple graphs, sp-graphs are not always simple. Although sp-graphs do not contain loops, they can include parallel edges.

Lemma 1. *If there are loops or parallel edges in graph G, then G is not minimally t-tough for any value of $t > 0$.*

Proof. It is easy to see that removing a loop does not change the toughness.

Let e_1 and e_2 be parallel, and S be a cutset. If S is incident to e_1 (or e_2) then $c(G - e_1 - S) = c(G - S)$, so $\tau(G - e_1) = \tau(G)$, thus G is not minimally t-tough.

Moreover, if S is not incident with e_1 and e_2 then $c(G - e_1 - S) = c(G - S)$, since e_1 connects two vertices that are in the same component of $G - S$, i.e. they are connected with e_2. Therefore, G is not minimally t-tough. □

The next lemma provides a useful inequality involving the *mediant* of two fractions. The *mediant* of $\frac{a}{b}$ and $\frac{c}{d}$ is defined as $\frac{a+c}{b+d}$. Lemma 2 was first introduced long ago in [3], and a straightforward proof can be found in [11].

Lemma 2 (Mediant Inequality)**.** *Let a, b, c, d be positive integers. If $\frac{a}{b} \leq \frac{c}{d}$ then*

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}.$$

Moreover, equalities hold if and only if $\frac{a}{b} = \frac{c}{d}$.

In the following, we show that if a graph has a cutset of size 1 or 2, then its tough sets satisfy some useful properties.

Lemma 3. *Let $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v\}$. If S is a tough set that does not contain v, then S must be fully contained in either $V(G_1) - \{v\}$ or $V(G_2) - \{v\}$. (See Fig. 3.)*

Note that G is essentially the series join of G_1 and G_2, but we do not assume that G_1 and G_2 are sp-graphs.

Proof. Let S be a tough set of G so that $v \notin S$, and let $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$. Suppose, to the contrary, that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. *(We will use the notation S_1 and S_2 in a similar way in the next few lemmas, without giving the explicit definition.)* Since S is a tough set in G, we have $\tau(G) = \frac{|S|}{c(G-S)}$. We claim

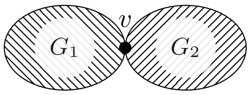

Figure 3: The structure of the graphs studied in Lemma 3.

that S_i is a cutset of G i.e., $c(G - S_i) \geq 2$ for $i = 1, 2$. If S_1 were not a cutset then $c(G - S_2) = c(G - S)$; thus

$$\frac{|S_2|}{c(G - S_2)} < \frac{|S|}{c(G - S)} = \tau(G),$$

which would contradict the fact that S is a tough set. By symmetry, we obtain that both S_1 and S_2 both are cutsets of G, which implies that $\tau(G) \leq \frac{|S_i|}{c(G-S_i)}$ for $i = 1, 2$. It is easy to see that $c(G - S) = c(G - S_1) + c(G - S_2) - 1 > 0$ holds. Without loss of generality, we can assume that $\frac{|S_2|}{c(G-S_2)} \leq \frac{|S_1|}{c(G-S_1)}$. Now the Mediant Inequality (Lemma 2) implies that

$$\frac{|S_2|}{c(G - S_2)} \leq \frac{|S_1| + |S_2|}{c(G - S_1) + c(G - S_2)} \leq \frac{|S_1|}{c(G - S_1)},$$

and hence

$$\tau(G) \leq \frac{|S_2|}{c(G - S_2)} < \frac{|S_1| + |S_2|}{c(G - S_1) + c(G - S_2) - 1} = \tau(G),$$

a contradiction. □

Lemma 4. *Let $G = G_1 \cup G_2 \cup v_1v_2$, where $v_i \in V(G_i)$ and $V(G_1) \cap V(G_2) = \emptyset$. If S is a tough set, then S must be fully contained in either $V(G_1)$ or $V(G_2)$.*

Note that, unlike in Lemma 3, now S may contain v_1 or v_2 or even both.

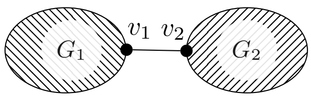

Figure 4: The structure of the graphs studied in Lemma 4.

Proof. If $v_1 \notin S$ or $v_2 \notin S$ then we can apply Lemma 3 by setting v_1 or v_2 as v. Hence, we can assume that $v_1, v_2 \in S$.

Suppose to the contrary that $S_1 = S \cap V(G_1) \neq \emptyset$ and $S_2 = S \cap V(G_2) \neq \emptyset$. Since S is a tough set in G, we have $\tau(G) = \frac{|S|}{c(G-S)}$. Since S is a tough set, it is easy to see that $V(G_1) \nsubseteq S$ and $V(G_2) \nsubseteq S$. Therefore S_1 and S_2 are both cutsets in G, which implies that $\tau(G) = \frac{|S_i|}{c(G-S_i)}$ holds for $i = 1, 2$.

Note that in this case $c(G - S) = c(G - S_1) + c(G - S_2) - 2$. Without loss of generality, we can assume that $\frac{|S_2|}{c(G-S_2)} \leq \frac{|S_1|}{c(G-S_1)}$. Now the Mediant Inequality (Lemma 2) implies that

$$\frac{|S_2|}{c(G - S_2)} \leq \frac{|S_1| + |S_2|}{c(G - S_1) + c(G - S_2)} \leq \frac{|S_1|}{c(G - S_1)}$$

and hence

$$\tau(G) \leq \frac{|S_2|}{c(G - S_2)} < \frac{|S_1| + |S_2|}{c(G - S_1) + c(G - S_2) - 2} = \tau(G),$$

a contradiction. □

Lemma 5. *Let $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{u, v\}$. Assume that $\tau(G) < 1$. If S is tough set of minimum size, and $S \cap \{u, v\} = \emptyset$, then S must be fully contained in either $V(G_1) - \{u, v\}$ or $V(G_2) - \{u, v\}$. Moreover, if S is a tough set, $S \cap \{u, v\} = \emptyset$, and $|S_1|, |S_2| > 0$ (hence S is not minimum size by the first claim), then $\frac{|S_1|}{c(G-S_1)} = \frac{|S_2|}{c(G-S_2)} = \tau(G)$.*

Note that there may be an edge in G between u and v. In this case, we can assume that this edge belongs to G_1.

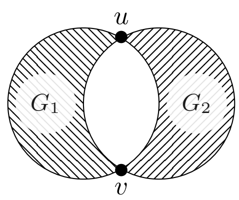

Figure 5: The structure of the graphs studied in Lemma 5.

Proof. Suppose to the contrary that S is tough set in G, such that $S \cap \{u, v\} = \emptyset$, but $S_1 = S \cap V(G_1) \neq \emptyset$ and $S_2 = S \cap V(G_2) \neq \emptyset$. This implies that $|S| \geq 2$. Since S is a tough set in G, we have $\tau(G) = \frac{|S|}{c(G-S)}$.

If $c(G - S_1) = 1$ and $c(G - S_2) = 1$, then each vertex of $G - S_1$ and $G - S_2$ is connected with a path to either u or v. This implies that $c(G - S) \leq 2$, hence $\tau(G) \geq 1$, a contradiction.

If $c(G - S_1) = 1$ and $c(G - S_2) \geq 2$, with $|S_1|, |S_2| \geq 1$, then $c(G - S) \leq c(G - S_2) + 1$ holds, since in $G - S_2$ the vertices u and v belong to the same component, but in $G - S_1$ each vertex is connected to either u or v. So if we remove S_1 from $G - S_2$, then the number of components may only increase with 1.

If $\frac{|S_2|}{c(G-S_2)} < 1$, then the Mediant Inequality (Lemma 2) implies that

$$\frac{|S_2|}{c(G - S_2)} < \frac{|S_2| + 1}{c(G - S_2) + 1} < \frac{1}{1}.$$

Therefore we have

$$\begin{aligned} \frac{|S_2|}{c(G - S_2)} < \frac{|S_2| + 1}{c(G - S_2) + 1} &\leq \frac{|S_1| + |S_2|}{c(G - S_2) + 1} \\ &\leq \frac{|S|}{c(G - S)} = \tau(G), \end{aligned}$$

so S cannot be a tough set.

If $\frac{|S_2|}{c(G-S_2)} \geq 1$, then the Mediant Inequality (Lemma 2) implies that

$$\frac{1}{1} \leq \frac{|S_2| + 1}{c(G - S_2) + 1} \leq \frac{|S_2|}{c(G - S_2)}.$$

Now we have

$$\begin{aligned} 1 \leq \frac{|S_2| + 1}{c(G - S_2) + 1} \leq \frac{|S_1| + |S_2|}{c(G - S_2) + 1} &\leq \\ \frac{|S|}{c(G - S)} &= \tau(G), \end{aligned}$$

contradicting the assumption.

Hence, we can assume that $c(G - S_i) \geq 2$ holds for $i = 1, 2$. Since both S_1 and S_2 are cutsets in G, we have $\tau(G) \leq \frac{|S_i|}{c(G-S_i)}$ for $i = 1, 2$.

Consider the components of $G - S$.

(i) The components that are disjoint from $V(G_1)$, i.e. components of $G - S_2$ not containing u or v. There are $c(G - S_2) - 1$ such components.

(ii) The components that are disjoint from $V(G_2)$, i.e. components of $G - S_1$ not containing u or v. There are $c(G - S_1) - 1$ such components.

(iii) The components of u and v. These may be separate components, but if there is a path in $G - S$ between u and v then they are the same. So there are 1 or 2 such components.

Thus

$$c(G - S) = c(G - S_1) - 1 + c(G - S_2) - 1 + 1 = \\ c(G - S_1) + c(G - S_2) - 1,$$

if u and v are in one component of $G - S$, and

$$c(G - S) = c(G - S_1) - 1 + c(G - S_2) - 1 + 2 = \\ c(G - S_1) + c(G - S_2),$$

if u and v are in different components of $G - S$. Therefore we have

$$c(G - S_1) + c(G - S_2) \geq c(G - S) \geq \\ c(G - S_1) + c(G - S_2) - 1.$$

Without loss of generality, we can assume that $\frac{|S_1|}{c(G-S_1)} \leq \frac{|S_2|}{c(G-S_2)}$. Now the Mediant Inequality (Lemma 2) implies that

$$\frac{|S_1|}{c(G - S_1)} \leq \frac{|S_1| + |S_2|}{c(G - S_1) + c(G - S_2)} \leq \frac{|S_2|}{c(G - S_2)}, \tag{2.1}$$

and hence

$$\frac{|S_1|}{c(G - S_1)} \leq \frac{|S|}{c(G - S)}, \tag{2.2}$$

so S cannot be a minimum-size tough set. If the inequality in 2.2 is strict, then S cannot even be a tough set. If equality holds in 2.2, then there is also equality in 2.1, so

$$\frac{|S_1|}{c(G - S_1)} = \tau(G) = \frac{|S_2|}{c(G - S_2)}$$

holds, completing the proof. $\square$

Lemma 6. *Let $G = G_1 \cup G_2 \cup u_1u_2 \cup v_1v_2$ where $\{u_i, v_i\} = V(G_i)$ holds for $i = 1, 2$. If S is a minimum-size tough set, then S must be fully contained in either $V(G_1)$ or $V(G_2)$ (see Fig. 6). Moreover, if S is a tough set with $|S_1|, |S_2| > 0$ (hence S is not minimal), then $\frac{|S_1|}{c(G-S_1)} = \frac{|S_2|}{c(G-S_2)} = \tau(G)$.*

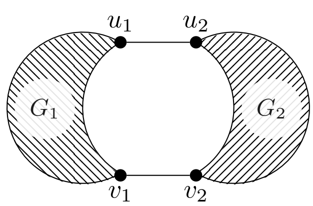

Figure 6: The structure of the graphs studied in Lemma 6.

Proof. We can argue similarly as in the proof of Lemma 5. Notice that in $G - S$, the vertices u_1 and u_2 (resp. v_1 and v_2) cannot belong to different components. On the other hand, one or both of them may not be in the graph $G - S$. So there may be 0 or 1 component containing u_1 or u_2 (resp. v_1 or v_2). Therefore, there may be $0, 1$, or 2 components of type (iii).

This leads to the inequality

$$c(G - S_1) + c(G - S_2) \geq c(G - S) \geq \\ c(G - S_1) + c(G - S_2) - 2.$$

Now, the proof can be finished in the same way. $\square$

Our next goal is to show that it is enough to consider graphs that do not contain and induced path of length longer than 2.

Lemma 7. *Let $t < 1$, and $u, w_1, w_2, \ldots, w_k, v$ be an induced path in a minimally t-tough graph G such that neither u or v are cut vertices. If S is a minimum size tough set of G, then $S \cap \{w_1, w_2, \ldots, w_k\} = \emptyset$.*

Proof. We can apply Lemma 6 by setting $u_1u_2 = uw_1$ and $v_1v_2 = w_kv$. Let G_1 be the path, and G_2 the rest of G.

If $\emptyset \neq S \subseteq V(G_1)$ is a minimum size tough set then it is easy to see that $\frac{|S_1|}{c(G-S_1)} = 1$, which contradicts the assumption $t < 1$.

If S is a tough set with $|S_1|, |S_2| > 0$ (hence not minimal), then $\frac{|S_2|}{c(G-S_2)} = \frac{|S_1|}{c(G-S_1)} = 1$, a contradiction again. ☐

Lemma 8. *Let $t < 1$, and $u, w_1, w_2, \ldots, w_k, v$ be an induced path in a minimally t-tough graph G such that u is a cut vertex. Then there exists a tough set S with $S \cap \{w_1, w_2, \ldots, w_k\} = \emptyset$.*

Proof. Since u is a cut vertex $\tau(G) \leq \frac{1}{2}$ holds. If $\tau(G) = \frac{1}{2}$ then $\{u\}$ is the desired tough set. If $\tau(G) < \frac{1}{2}$ then we can apply Lemma 4 for the edge uw_1 and notice that if G_1 is the path then $\frac{|S_1|}{c(G-S_1)} = 1$. ☐

Lemma 9. *Suppose that G' is obtained from G by replacing an induced path of length at least 3 with a path of length 2. Assume that $t < 1$ holds. G' is minimally t-tough if and only if G is minimally t-tough.*

Proof. Suppose we replace the path $u, w_1, w_2, \ldots, w_k, v$ with the path u, w, v.

Claim 10. *$\tau(G) = \tau(G')$.*

Proof. By Lemma 7 and 8 there exist a tough set S in G with $S \cap \{w_1, w_2, \ldots, w_k\} = \emptyset$. Thus $c(G'-S) = c(G-S)$, since in $G-S$ all w_i belong to the same component, which corresponds to the component of $G'-S$ that contains w. This implies that $\tau(G) = \frac{|S|}{C(G-S)} = \frac{|S|}{C(G'-S)} \geq \tau(G')$.

For the other direction, notice that G' must contain a tough set that does not contain w. Now, the proof can be finished in the same way. ☐

Claim 11. *G' is minimally t-tough if and only if G is minimally t-tough.*

Proof. Let e be an arbitrary edge of G. If e is not on the path, then we can apply Claim 10 for $G - e$ as well. This implies that $\tau(G - e) = \tau(G' - e)$, so the removal of e decreases the toughness of G iff it decreases the toughness of G'.

If e is on the path in G, then we can choose uw in G' as e'. $G - e$ and $G' - e'$ are both graphs with one or two induced paths from u and v. Hence by Lemma 7 and 8 there exist a tough set which is disjoint from $\{w_1, w_2 \ldots, w_k\}$. Thus $\tau(G - e) = \tau(G' - e')$, completing the proof. ☐

☐

Definition 4. *Let $r(G)$, the reduced graph of G, be the graph obtained from G by repeatedly replacing an induced path of length at least 3 with a path of length 2 until there is no induced path of length at least 3.*

Lemma 9 proves that if $t < 1$ then G is minimally t-tough if and only if $r(G)$ is minimally t-tough. Therefore, if we can characterize the reduced minimally tough graphs, then we can also characterize all such graphs.

We show yet another useful claim.

Claim 12. *Let G be a minimally t-tough graph. For any edge $e = uv$ in G, no tough set of $G - e$ can contain either u or v.*

Proof. Let S be a tough set of $G - S$. If $u \in S$ or $v \in S$, then $(G - e) - S$ is the same graph as $G - S$, so removing e does not change the toughness, a contradiction. ☐

The edges e, whose parent in the sp-tree (i.e. its first join) is a parallel join, are special. Note that in this case, the siblings of e can not be edges since we assume that there are no parallel edges. So, all siblings of e must be series nodes. We will see that some edges have different properties from others, so we define two special types of edges.

Definition 5. *An edge e is a* leap-edge *if its parent is a parallel node, which is the root of the sp-tree, and it has only one sibling (which is a series node). An edge is a* jump-edge *if its parent is a parallel node that is not the root or the parent is the root, but there are at least two series node siblings (i.e. it is not a leap-edge).*

Notice that if two edges are parallel, then both of them are either leap-edges or jump-edges.

Lemma 13. *Let G be a graph with $\frac{1}{2} \leq \tau(G) \leq 1$, and let e be a jump-edge in G. Then $\tau(G-e) = \tau(G)$ in either of the following cases:*

(i) $\tau(G) > \frac{1}{2}$; or

(ii) $\tau(G) = \frac{1}{2}$ and the grandparent of e in the SP-tree (a series node) is not the root.

Proof. Let $e = s_1t_1$, thus the parent of e is a parallel node with joining vertices s_1 and t_1.

Case 1; e has a grandparent:

The grandparent must be a series node. If it is the root, then removing a joining vertex gives at least 2 components, thus $\tau(G) \leq \frac{1}{2}$, implying that neither (i) nor (ii) holds. Therefore, we can assume that e has a great-grandparent that is a parallel node.

Suppose that $\tau(G-e) < \tau(G)$ and let S be a tough set for $G-e$. If $s_1 \in S$ or $t_1 \in S$ then $C((G-e)-S) = C(G-S)$ implying $\tau(G-e) \geq \tau(G)$. Otherwise, the only difference between the components of $(G-e)-S$ and $G-S$ is that in $(G-e)-S$, the vertices s_1 and t_1 may be in different components, but e connects them in $G-S$.

$\{s_1, t_1\}$ is clearly a cutset. Let G_1 denote the subgraph containing the series node sibling of e, and G_2 the rest of the graph. Applying Lemma 5 by setting $u = s_1$ and $v = t_1$ gives that S is contained in either $(G_1 - s_1 - t_1)$ or $(G_2 - s_1 - t_1)$.

If S is contained in $(G_1 - s_1 - t_1)$ then there is a path in $G-e$ between s_1 and t_1 through G_2, thus $C(G-e) = C(G)$, which implies $\tau(G-e) \geq \tau(G)$, a contradiction.

If S is contained in $G_2 - s_1 - t_1$ then there is a path in $G-e$ between s_1 and t_1 through G_1, thus $C(G-e) = C(G)$ which implies $\tau(G-e) \geq \tau(G)$, a contradiction again.

Case 2; e has no grandparent:

If the parent of e is a parallel join, which is the root, but it has ≥ 2 siblings that are series nodes, then the argument is similar; just let G_1, G_2 be the subgraphs corresponding to the two series siblings. □

3 Characterization of minimally t-tough sp-graphs with $\frac{1}{2} < t \leq 1$

Theorem 14. *If G is a minimally 1-tough sp-graph, then G is a cycle.*

Proof. If the root of the sp-tree is a series node, then there is a cut vertex, so $\tau(G) \leq \frac{1}{2} < 1$, a contradiction. So, the root must be a parallel node. It cannot have two children that are edges since they would be parallel.

Case 1; The root has two children, an edge and a series node.

Subcase 1a; all children of the series node are edges.

In other words, the graph is a parallel join of an edge and a series join of some edges (i.e., a path) and an edge. So it is a cycle.

Subcase 1b; the series node has a parallel node child.

Let us denote this parallel node by x, and the two terminals of this join by s_x, t_x. First, note that x must have at least two children. If it has at least two series nodes, then $G - s_x - t_x$ has at least 3 components: two components from the series node children of x and one more containing the rest of the vertices. This implies that $\tau(G) \leq \frac{2}{3} < 1$, a contradiction.

x cannot have more than one child, which is an edge, so it remains to be handled when x has two children: an edge and a series node. However, this means that the edge is a jump-edge. Now Lemma 13 implies that G is not minimally 1-tough.

Case 2; the root has two children that are series nodes.

First, note that the root cannot have children that are edges, since they would be jump-edges, and then Lemma 13 implies that G cannot be minimally 1-tough.

Subcase 2a; all children of both series nodes are edges.

In other words, the graph is a parallel join of two series joins of some edges (i.e. two paths). So it is a cycle.

Subcase 2b; one of the series nodes has a parallel node child.

This case is essentially the same as Subcase 1b.

Case 3; the root has at least 3 children that are series nodes.

Removing the two terminals of the root gives at least 3 components. This implies that $\tau(G) \leq \frac{2}{3} < 1$, a contradiction. $\square$

Now, we are ready to state the main result of this section.

Theorem 15. *An sp-graph is minimally t-tough with $\frac{1}{2} < t < 1$ if and only if there are no jump-edges.*

Proof. If there is a jump-edge, then the graph is not minimally tough by Lemma 13.

Now, suppose there are no jump-edges. We show that for any edge e, we have $\tau(G - e) < \tau(G)$. In fact, we show that for any edge $\tau(G - e) \leq \frac{1}{2} < \tau(G)$.

Case 1: The parent of e is a series node.

Subcase 1a; The parent of e is the root.

In this case, $G - e$ has 2 components. This is true even in the special cases when G is just an edge or if e is incident to one of the terminals of the series node. In these cases, one or both of the components is a single vertex. This implies that $\tau(G) \leq \frac{1}{2}$, a contradiction.

Subcase 1b; The parent of e is not the root.

Therefore, its parent must be a parallel node. Let s and t be the terminals of this parallel join. Also, let G' be the subgraph of the series node with terminals s and t. It is clear that all paths connecting s and t that use the edges of G' must contain e. If e is not incident to either s or t, then it is easy to see that $G' - e$ has two components, one containing s, the other containing t. It is clear that both s and t are cut vertices in $G - e$. On the other hand, if e is incident to say s (it cannot be incident to both), then t is still a cut-vertex. This implies that $\tau(G) \leq \frac{1}{2}$, a contradiction.

Case 2: The parent of e is a parallel node.

If the parent is not the root, then it is a jump-edge, contradicting our assumption. If the parent is the root, then it may only have one sibling, since otherwise, it would be a jump-edge. Thus, $G - e$ is a series join of at least two sp-graphs, so any of the internal join vertices are cut vertices. This implies that $\tau(G - e) \leq \frac{1}{2} < \tau(G)$. $\square$

4 Characterization of minimally $\frac{1}{2}$-tough sp-graphs

In this section, we first characterize all reduced minimally $\frac{1}{2}$-tough sp-graphs. By Lemma 9 this will provide a characterization of all minimally $\frac{1}{2}$-tough sp-graphs.

For this, we define a few specific *substructures* of the sp-tree. A substructure is like a subtree, but (only) the root may have other siblings in the sp-tree. A subtree is always a substructure as well. A special case is when the substructure is the whole graph.

Definition 6. *Let P_2 denote the substructure consisting of a series connection of two edges, and let Q_2 denote the parallel join of P_2 with a single edge. For $i \geq 2$, define R_i to be the substructure given by the parallel join of i copies of P_2. Then, for $i, j \geq 1$, let $R_{i,j}$ denote the series join of R_i and R_j, where we define R_1 to be a single edge. (In particular, $R_{1,1} = P_2$.) See Figure 7 for illustrations of these substructures.*

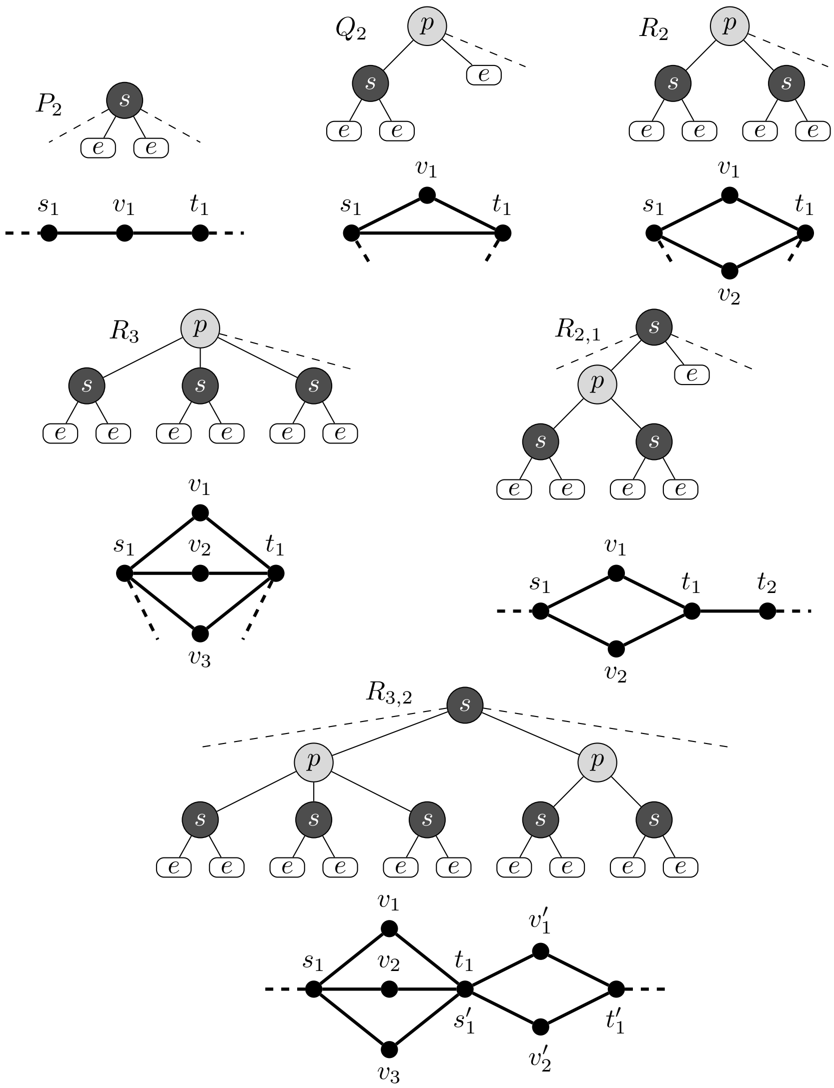

Figure 7: Substructures $P_2, Q_1, R_2, R_3, R_{2,1}, R_{3,2}$ and the corresponding subgraphs.

Lemma 16. *Let G be an sp-graph containing the substructure R_2 or R_3. If S is a tough set in G, then the middle vertices v_i of these substructures are not in S.*

Proof. We present the proof for R_2, the argument is almost the same for R_3. It is easy to see that if at least one of s_1 and t_1 belongs to S as well as at least one of v_1 and v_2 then by setting $S' = S - \{v_1, v_2\}$ we have $|S'| < |S|$ and $c(G - S') \geq c(G - S) \geq 2$. Thus S cannot be a tough set. Hence, we can assume that $s_1, t_1 \notin S$.

Now suppose that $v_1 \in S$ and $v_2 \notin S$. In this case, s_1 and t_1 belong to the same component of $G - S$ since the path $s_1 v_2 t_1$ connects them. By setting $S' = S - v_1$ we have $|S'| < |S|$ and $c(G - S') = c(G - S)$. Thus S cannot be a tough set, again.

Finally, suppose $v_1, v_2 \in S$, and let $S' = S - \{v_1, v_2\} \cup t_1$. Clearly, $|S'| < |S|$. Now consider the components of $G - S'$; how are these different from the components of $G - S$? We can see that v_1 and v_2 will belong to the same component as s_1. Since any path of G between different components of $G-S$ containing v_1 or v_2 also contains t_1, it is impossible that two vertices in different components of $G-S$ are connected with a path in $G-S'$. On the other hand, adding t_1 to the cutset may separate some other components. Therefore, $c(G-S') \geq c(G-S) \geq 2$, so S cannot be a tough set. $\square$

Definition 7. *A* necklace graph *is a series join of edges and R_2 substructures, arranged so that both the first and last are single edges. (See Figure 8.)*

A path is a special case of necklace graphs.

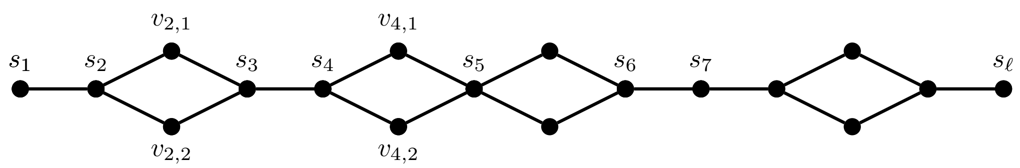

Figure 8: A necklace graph.

Lemma 17. *The necklace graphs are minimally $\frac{1}{2}$-tough.*

Proof. The toughness is $\leq \frac{1}{2}$ since there is a cut vertex, for example, the non-leaf vertex of the first edge. Suppose that $\tau(G) < \frac{1}{2}$ and let S be a tough set.

By Lemma 16 S does not contain any of v_1 and v_2 vertex of the R_2 substructures. Therefore, S consists of some of the joining vertices $s_1, \ldots, s_\ell$ (see Figure 8 for the notation).

Suppose $s_i \in S$, but $s_j \notin S$ for all $1 \leq j < i$. Then in $G-S$ there will be one component to the left of s_i, this contains $s_1, \ldots, s_{i-1}$. A symmetric situation occurs on the right.

If $s_i, s_{i+1} \in S$, then there are two components between s_i, s_{i+1}, the isolated vertices $v_{i,1}$ and $v_{i,2}$. If $s_i, s_j \in S$ with $j > i+1$ and $s_{i+1}, \ldots, s_{j-1} \notin S$, then a single component $G-S$ contains all its vertices between s_i and s_j. So to summarize, $G-S$ contains at most two components between two consecutive s_i vertices in S, and there may be one additional component on each end. Hence, we obtain

$$\frac{|S|}{c(G-S)} \geq \frac{k}{(2k-2)+2} = \frac{1}{2}.$$

Now we show that for any edge $\tau(G-e) < \tau(G)$. If the edge is a cut edge, then $\tau(G-e) = 0 < \tau(G) = \frac{1}{2}$.

Otherwise, without loss of generality, we can assume that $e = (s_i, v_{i,1})$ for some R_2 subgraph. In this case, $c((G-e)-s_{i+1}) = 3$, so the toughness is at most $\frac{1}{3} < \frac{1}{2}$. $\square$

The next Lemma is crucial in the following arguments. We will only use it for $k=2$, but it might be interesting in this more general form. Note that the claim is not necessarily true if the toughness is not in the form $\frac{1}{k}$.

Lemma 18. *Let G be a graph with $\tau(G) = \frac{1}{k}$, for some integer $k \geq 2$. Let e be an edge so that $\tau(G-e) < \tau(G)$ and S be a tough set in $G-e$. Now S is a tough set in G as well.*

Proof. If $\tau(G) = \frac{1}{k}$ then $\tau(G-e) = \frac{|S|}{c((G-e)-S)} < \frac{1}{k}$ by the choice of e. This implies that $c((G-e)-S) > 2$.

Adding e back to $(G-e)-S$ either leaves the number of components unchanged or decreases it by 1. Therefore, $1 < c((G-e)-S) - 1 \leq c(G-S) \leq c((G-e)-S)$, which shows that S is a cutset in G.

Now observe that

$$\tau(G) = \frac{1}{k} \leq \frac{|S|}{c(G-S)} \leq \frac{|S|}{c((G-e)-S)-1}$$

while from the assumption on $\tau(G-e)$, we also have

$$\frac{|S|}{c((G-e)-S)} < \frac{1}{k} \leq \frac{|S|}{c((G-e)-S)-1}.$$

These inequalities imply that $c((G-e)-S)-1 \leq k|S| < c((G-e)-S)$, so $k|S|$ lies between two consecutive integers and is therefore itself an integer. Hence, we conclude that $c((G-e)-S)-1 = k|S|$.

Putting everything together, we have

$$\frac{1}{k} \leq \frac{|S|}{c(G-S)} \leq \frac{|S|}{c((G-e)-S)-1} = \frac{1}{k},$$

which shows that $\frac{|S|}{c(G-S)} = \frac{1}{k}$. Therefore, S is a tough set in G. □

Next, we show that minimally $\frac{1}{2}$-tough graphs cannot contain certain substructures.

Lemma 19. *If an sp-graph G contains R_4 as a substructure, then it is not minimally $\frac{1}{2}$-tough.*

Proof. First, suppose that $G = R_4$ or G is the parallel join of R_4 and an edge. It is easy to verify that in both cases $\tau(G) = \frac{1}{2} = \tau(G-e)$ holds for any edge e of the graph. Hence G is not minimally $\frac{1}{2}$-tough.

In any other case, G contains at least one vertex apart from the vertices of R_4. Setting $S = \{s_1, t_1\}$ gives $c(G-S) \geq 4+1 = 5$, so $\tau(G) \leq \frac{2}{5}$, therefore G is not even $\frac{1}{2}$-tough. □

This implies that we can assume that G does not contain the substructure R_4. Note that this also implies that G does not contain any R_i with $i \geq 4$.

Lemma 20. *If an sp-graph G contains the substructure Q_2, then G is not minimally $\frac{1}{2}$-tough.*

Proof. Let $e = s_1t_1$, we show that $\tau(G-e) = \frac{1}{2}$. Suppose on the contrary that $\tau(G-e) < \frac{1}{2}$, then there exists a tough set S for $G-e$. By Claim 12 we know that $s_1, t_1 \notin S$. By Lemma 5, either $S = \{v_1\}$ or S is disjoint from Q_2. The first is impossible since Lemma 18 implies that S must be a tough set in G as well, but it is not even a cutset. In the second case, $c((G-e)-S) = c(G-S)$ since s_1 and t_1 is connected by the path $s_1v_1t_1$. Hence, $\tau(G-e) < \frac{1}{2}$ cannot hold, a contradiction. □

Lemma 21. *If an sp-graph G contains $R_{3,3}$ as a substructure, then it is not minimally $\frac{1}{2}$-tough.*

Proof. First suppose that $G = R_{3,3}$ or G is the parallel join of $R_{3,3}$ and an edge. It is easy to verify that in both cases $\tau(G) = \frac{1}{2} = \tau(G-e)$ holds for $e = v_1t_1$. Hence G is not minimally $\frac{1}{2}$-tough. (Notice that for $e = s_1v_1$ the vertex set $\{s_1', t_1'\}$ is a tough set in $G-e$, showing that $\tau(G-e) = \frac{2}{5}$.)

In any other case, G contains at least one vertex apart from the vertices of $R_{3,3}$. Setting $S = \{s_1, t_1, t_1'\}$ gives $c(G-S) \geq 6+1 = 7$, so $\tau(G) \leq \frac{3}{7}$, therefore G is not even $\frac{1}{2}$-tough. □

Lemma 22. *If an sp-graph G contains $R_{3,2}$ as a substructure, then it is not minimally $\frac{1}{2}$-tough.*

Proof. First suppose that $G = R_{3,2}$ or G is the parallel join of $R_{3,2}$ and an edge. It is easy to verify that in both cases $\tau(G) = \frac{1}{2} = \tau(G-e)$ holds for $e = v_1t_1$. Hence G is not minimally $\frac{1}{2}$-tough.

In any other case, G contains at least one vertex apart from the vertices of $R_{3,2}$. Let $e = v_1t_1$, we show that $\tau(G-e) = \tau(G) = \frac{1}{2}$, which proves our claim.

Suppose to the contrary that $\tau(G-e) < \tau(G)$ and let S be a tough set of $G-e$. By Lemma 16, none of the vertices v_i and v_i' belong to S. By Claim 12 we have $v_1 \notin S$ and $t_1 \notin S$. This implies that $s_1 \in S$, otherwise v_1 and t_1 belong to the same component of $G-e$, which contradicts the assumption that S is a tough set.

Lemma 18 implies that S is a tough set of G as well. Let $S' = S \cup \{t_1\}$. It is clear that in $G-S$, the vertices v_1, v_2, v_3, t_1, v_1', and v_2' all belong to the same component. Now, if t_1 is deleted from $G-S$, then v_1, v_2, v_3 become isolated vertices, and there will be a component that contains v_1'. This component will contain v_2' as well if $t_1' \notin S$; otherwise, v_1' and v_2' also become isolated vertices. In either way, we obtain $c(G-S') \geq c(G-S)+3$. By the Mediant Inequality (Lemma 2) we have

$$\frac{1}{3} < \frac{|S|+1}{c(G-S)+3} < \frac{|S|}{c(G-S)} = \frac{1}{2},$$

thus

$$\tau(G) \leq \frac{|S'|}{c(G-S')} \leq \frac{|S|+1}{c(G-S)+3} < \frac{1}{2},$$

which contradicts that $\tau(G) = \frac{1}{2}$, since S' is clearly a cutset. □

Lemma 23. *If an sp-graph G contains $R_{3,1}$ as a substructure, then it is not minimally $\frac{1}{2}$-tough.*

Proof. First suppose that $G = R_{3,1}$ or G is the parallel join of $R_{3,1}$ and an edge. It is easy to verify that in both cases $\tau(G) = \frac{1}{2} = \tau(G-e)$ holds for $e = v_1t_1$. Hence G is not minimally $\frac{1}{2}$-tough.

Now suppose that the parent of $R_{3,1}$ is the root of G, so G is a series join of $R_{3,1}$ and some other subgraphs. If there is a series join on both sides (i.e., s_1 and t_2 ar both join vertices) then $S = \{s_1, t_1\}$ is clearly a cutset with $c(G-S) = 5$, a contradiction. If there is no series join at s_1, then let $e = v_1t_1$ and S be a tough set of $G-e$. By Lemma 18 we know that S is a tough set in G as well. By Claim 12 we have $t_1, v_1 \notin S$. Lemma 16 implies that $v_2, v_3 \notin S$. The path $t_1v_2s_1v_1$ connects t_1 and v_1 in $G-e$, therefore $s_1 \in S$ must hold, otherwise t_1 and v_1 belong to the same component of $(G-e)-S$, a contradiction. It is clear, that in this case $S \neq \{s_1\}$ since it would not be a cutset in G. Thus, it is easy to verify that $S' = S - s_1$ is still a cutset in G, and that $c(G-S') = c(G-S)$ holds. This contradicts the assumption that S is a tough set in G.

The above argument proves that the root of $R_{3,1}$ has a parent, it must be a parallel join. Let $e = s_1v_1$, we show that $\tau(G-e) \geq \tau(G)$, which implies that it is not minimally tough.

Let S be a tough set in $G-e$. By Claim 12 we have $s_1, v_1 \notin S$. Lemma 18 and 16 implies that $v_2, v_3 \notin S$. The path $s_1v_2t_1v_1$ connects s_1 and v_1 in $G-e$, therefore $t_1 \in S$ must hold, otherwise s_1 and v_1 belong to the same component of $(G-e)-S$, a contradiction.

Thus, the remaining case is when $t_1 \in S$ and $s_1, v_1, v_2, v_3 \notin S$. Let $S' = S - t_1$, hence $|S'| = |S| - 1$. We claim that $c(G-S') \geq c(G-S) - 1$. Consider the components of $G-S$. Three of the neighbors of t_1 namely v_1, v_2, v_3 are in the same component, since $s_1, v_1, v_2, v_3 \notin S$. t_2 may be in a different component, but removing t_1 from S may only connect these two components, so the number of components may only decrease by 1.

We also claim that $c(G-S') > 1$, i.e., S' is a cutset. Lemma 18 implies that S is a tough set also in G. Since the parent of the root of $R_{3,1}$ is a parallel node, there is a path between s_1 and t_2 that avoids the substructure. This implies that $G - t_1$ is connected, so $S \neq \{t_1\}$, therefore $|S| \geq 2$ must hold. Since $\tau(G) = \frac{1}{2}$, this gives that $c(G-S) \geq 4$ and $c(G-S') \geq 3 > 1$.

Using the above claims we obtain

$$\tau(G) \leq \frac{|S'|}{c(G-S')} \leq \frac{|S|-1}{c(G-S)-1}.$$

However, $\frac{1}{2} = \frac{|S|}{c(G-S)}$ is the mediant of $\frac{|S|-1}{c(G-S)-1}$ and $\frac{1}{1}$. Thus, the Mediant Inequality implies that

$$\tau(G) \leq \frac{|S'|}{c(G-S')} \leq \frac{|S|-1}{c(G-S)-1} < \frac{|S|}{c(G-S)} = \frac{1}{2} < 1,$$

which contradicts the assumption that $\tau(G) = \frac{1}{2}$. □

Lemma 24. *If a minimally $\frac{1}{2}$-tough sp-graph G contains the substructure $R_{2,1}$ then the root of $R_{2,1}$ is the root of G and s_1 is a cut-vertex.*

Notice that a minimally $\frac{1}{2}$-tough sp-graph may contain $R_{2,1}$ substructures, some of the necklace graphs have this property.

Proof. The proof is essentially the same as the proof of Lemma 23. The main difference is that if s_1 is a cut-vertex, then the above argument does not work, hence we have the exceptional case. □

So far, we have shown that a minimally $\frac{1}{2}$-tough sp-graph cannot contain certain substructures. This also holds for the reduced $\frac{1}{2}$-tough sp-graphs, which implies that their structure is fairly simple. Now we turn our attention to the height of the sp-tree of the reduced graph. Our goal is to prove that the height is at most 3. In fact, we show that it cannot have a subtree of height 4 or more. To achieve this, we characterize the possible subtrees of height 0–3. (Note that now we are considering subtrees, not substructures.)

Lemma 25. *If G is a minimally $\frac{1}{2}$-tough reduced sp-graph, then the height of its sp-tree can only be 1 or 3. If the height is 1, then $G = P_3$, otherwise, G is a necklace graph different from P_3.*

Proof. Let us recall a few facts about sp-trees:

- The height of a tree is the number of edges on the longest path from the root to a leaf.
- Every non-leaf vertex has at least two children.
- If the height of a tree is k, then at least one of the children of the root is the root of a subtree of height $k-1$.

- The parent of a series node is a parallel node and vice versa.

If v is a node of the sp-tree having children $w_1, \dots, w_k$, then we call the subtrees with roots $w_1, \dots, w_k$ the *direct subtrees* of v.

Height 0**:**

The sp-tree has only one vertex, the root. This must be an edge. Since the toughness of a single edge is ∞, this cannot be the whole graph, it can only be a proper subtree.

Height 1**:**

The root cannot be a parallel node, since this would give parallel edges, contradicting Lemma 1. Thus, the root is a series node. It must have at least two children, all children must be leafs, i.e., edges. On the other hand, it cannot have more than two children that are edges, because we have a reduced graph. This subtree corresponds to P_3 (i.e., the path on 3 vertices with 2 edges). The graph P_3 itself is clearly minimally $\frac{1}{2}$-tough.

Height 2**:**

At least one direct subtree must have height 1, such a subtree can only be a P_3.

Since the root of P_3 is a series node, the root of the height 2 tree must be a parallel node. This node cannot have more than one edge as a child, as parallel edges are not allowed. Moreover, if it has both an edge child and a P_3 child, then a Q_2 substructure would arise, contradicting Lemma 20.

Thus, all the children of the root must be P_3 graphs. Having four or more such children would yield an R_4 substructure, which is excluded by Lemma 19. Hence, the subtree must have exactly two or three children, each being P_3, so the subtree is either R_2 or R_3.

Since R_2 has toughness 1 and R_3 has toughness $\frac{2}{3}$, neither can be the entire graph, they can only occur as proper subtrees.

Height 3**:**

At least one direct subtree must have height 2, and such a subtree can only be an R_2 or R_3. Since their roots are parallel nodes, the root of the height 3 tree must be a series node. This, implies that no direct subtree can have a series node as its root, which rules out subtrees of height 1. Thus, all children of the root are either R_2 or R_3 subtrees or single edges. That is, the entire graph is a series join of R_2, R_3 substructures, and edges, in some order.

We now show that if this subtree is the whole graph, then it is minimally $\frac{1}{2}$-tough if and only if it contains no R_3 substructure, and both the first and last graphs in the series composition are edges, not R_2.

Let $S = \{s_1, t_1\}$, where s_1 and t_1 are the terminals connecting the R_3 substructure to the rest of the graph. Then S forms a cutset of size 2, and the removal of S results in 5 components: $c(G - S) = 5$. Therefore, $\tau(G) \le \frac{2}{5}$, a contradiction.

Next, suppose that the first direct subtree is an R_2 or R_3 with terminals s_1 and t_1, where s_1 is the initial terminal of the graph (i.e., not a cut-vertex) and t_1 is an internal terminal (i.e., a cut-vertex). Using arguments similar to those above, one can show that removing the edge $e = t_1v_1$ does not decrease the toughness, implying that G is not minimally tough.

Hence, the graph must be a necklace graph.

Height 4 **or more:**

If the tree has height at least 4, then it must contain at least one subtree of height exactly 4. If there is more than one such subtree, we select the one that is at maximum distance from the root of the entire tree. Let T_4 denote this subtree, and let r_4 be its root. This choice guarantees that if r_4 has siblings, then the subtrees rooted at those siblings have height at most 4.

One of the direct subtrees of r_4 must have height 3, and hence is a series join of R_2, R_3 substructures, and edges. Since its root is a series node, r_4 must be a parallel node. Consequently, r_4 cannot have a direct subtree of height 2.

Now suppose that the height 3 subtree contains an R_3 substructure. Consider its neighboring subtree in the series join. This neighbor must be either an edge, an R_2, or another R_3, resulting in an $R_{3,1}$, $R_{3,2}$, or

$R_{3,3}$ substructure, respectively. However, Lemmas 23, 22, and 21 rule out these configurations, leading to contradictions in all cases. Therefore, the height 3 subtree cannot contain an R_3 substructure.

Now suppose that the height 3 subtree contains at least one R_2 and at least one edge among the children of the series node. Then, necessarily, there exists an R_2 substructure and an edge that are adjacent in the series join, forming an $R_{2,1}$ substructure. In this configuration, the common ancestor of these substructures is the parallel node r_4. Therefore, by Lemma 24, such a configuration is not allowed, a contradiction.

This implies that any height 3 subtree must be a series join of two or more R_2 graphs. The other direct subtrees of r_4 may have height 1 or 0; that is, they may be P_3 subgraphs or single edges.

Let us now analyze the possible configurations. In each case, the argument differs depending on whether T_4 is a proper subtree or whether it is the whole graph. Subcases (a) always handles the first case, and subcases (b) the second one.

We define *a bracelet of length* ℓ, denoted by B_ℓ, as a series join of ℓ copies of the R_2 graph. Let $s_1, s_2, \dots, s_{\ell+1}$ denote the joining vertices of the consecutive R_2 subgraphs, in natural order. The terminals of the bracelet are s_1 and $s_{\ell+1}$. Observe that removing the set $\{s_1, s_2, \dots, s_{\ell+1}\}$ from the bracelet leaves 2ℓ components, each of which is an isolated vertex.

Case 1(a); One child of r_4 is B_ℓ, another child is an edge e, and r_4 is not the root.

Since e is a jump-edge and its parent is not the root, Lemma 13 implies that the graph is not minimally $\frac{1}{2}$-tough.

Case 1(b); One child of r_4 is B_ℓ, another child is an edge e, and r_4 is the root.

If r_4 has at least three children, then e is again a jump-edge. By Lemma 13, this implies that the graph is not minimally $\frac{1}{2}$-tough.

Now suppose that r_4 has exactly two children. Then the graph is a parallel join of a bracelet B_ℓ and a single edge e. We claim that in this case $\tau(G) > \frac{1}{2}$.

Let S be a minimum size tough set of G. By Lemma 16, $S \subseteq \{s_1, s_2, \dots, s_{\ell+1}\}$. If $S = \{s_1, s_2, \dots, s_{\ell+1}\}$, then $\tau(G) = \frac{\ell+1}{2\ell} > \frac{1}{2}$. Suppose instead that two vertices $s_p, s_q \notin S$ are such that $G - \{s_p, s_q\}$ has two nonempty components, each intersecting S. This contradicts Lemma 5, which rules out such a configuration in a minimal tough set. Therefore, the only possible forms of S are either: $S = \{s_i, s_{i+1}, \dots, s_j\}$, for some $1 \le i < j \le \ell + 1$; or $S = \{s_1, \dots, s_i, s_j, s_{j+1}, \dots, s_{\ell+1}\}$, for some indices $1 \le i < j \le \ell + 1$. In the first case,

$$\tau(G) = \frac{j - i + 1}{2(j - i)} > \frac{1}{2},$$

and in the second case,

$$\tau(G) = \frac{(\ell + 1 - j + 1 + i)}{2(\ell + 1 - j) + 2(i - 1)} > \frac{1}{2}.$$

In both cases, we conclude that $\tau(G) > \frac{1}{2}$, so the graph cannot be minimally $\frac{1}{2}$-tough.

Case 2(a); One child of r_4 is B_ℓ, another child is B_k, and r_4 is not the root.

Let $\{s_1, \dots, s_{\ell+1}\}$ and $\{s'_1, \dots, s'_{k+1}\}$ denote the joining vertices of the bracelets B_ℓ and B_k, respectively, chosen so that $s_1 = s'_1$ and $s_{\ell+1} = s'_{k+1}$. Define the set $S = \{s_1, \dots, s_{\ell+1}\} \cup \{s'_1, \dots, s'_{k+1}\}$.

In the graph $G - S$, all the middle vertices of both bracelets become isolated, contributing $2\ell + 2k$ components. Moreover, since r_4 is not the root, there is at least one additional component in $G - S$. Therefore,

$$\tau(G) \le \frac{|S|}{c(G - S)} \le \frac{\ell + k + 1}{2\ell + 2k + 1} < \frac{1}{2},$$

which contradicts the assumption that $\tau(G) \ge \frac{1}{2}$.

Case 2(b); One child of r_4 is B_ℓ, another child is B_k, and r_4 is the root.

If r_4 has any additional child besides B_ℓ and B_k, then the argument from Case 2(a) gives the same contradiction. Therefore, it is enough to consider the case where r_4 has exactly two children.

In this case, the graph consists of a cyclic structure formed by $\ell + k$ copies of the R_2 substructures. Let the joining vertices be $\{s_1, s_2, \dots, s_{\ell+k}\}$, indexed cyclically. Consider the edge $e = s_1 v_1$, where v_1 is one of the middle vertices of the first R_2, which has terminals s_1 and s_2. (By symmetry, any such edge behaves similarly.) We claim that $\tau(G-e)=\frac{1}{2}$, leading to a contradiction.

Suppose that $\tau(G-e)<\frac{1}{2}$, and let S be a minimum-size tough set of $G-e$. Using similar reasoning as in the previous cases and applying, we can assume that $s_1 \notin S$ and $s_2 \in S$. By Lemmas 16 and 5 this implies that $S=\{s_2,s_3,\dots,s_i\}$ for some $3 \le i < \ell+k$.

By Lemma 18 S is a tough set in G as well. Now observe that removing $i-1$ vertices from G disconnects the graph into $2(i-2)+1$ components. Therefore,

$$\tau(G) \le \frac{i-1}{2(i-2)+1} < \frac{1}{2},$$

which contradicts the assumption that S is a tough set in G. Hence, $\tau(G-e)=\frac{1}{2}$, and the graph is not minimally $\frac{1}{2}$-tough.

Case 3(a); One child of r_4 is B_ℓ, two other children are P_3, and r_4 is not the root.

Let $S=\{s_1,s_2,\dots,s_{\ell+1}\}$. In the graph $G-S$, all the middle vertices of the bracelet B_ℓ become isolated, contributing 2ℓ components. Additionally, each of the two P_3 subgraphs contributes an isolated middle vertex, adding 2 more components. Since r_4 is not the root, there is at least one additional component in $G-S$. Therefore,

$$\tau(G) \le \frac{|S|}{c(G-S)} \le \frac{\ell+1}{2\ell+3} < \frac{1}{2},$$

which contradicts the assumption that G is minimally $\frac{1}{2}$-tough.

Case 3(b); One child of r_4 is B_ℓ, two other children are P_3, and r_4 is the root.

The two P_3 subgraphs together form an R_2, so the graph is a cyclic structure consisting of $\ell+1$ copies of R_2. If r_4 has no additional children, then the argument from Case 2(a) applies and shows that the graph is not minimally $\frac{1}{2}$-tough. On the other hand, if r_4 has any additional children, then it is straightforward to verify that $\tau(G)<\frac{1}{2}$, a contradiction.

Case 4(a); r_4 has only two children B_ℓ and P_3, and r_4 is not the root.

This case is the most complicated one, we need to handle several subcases. Let $\{s_1,\dots,s_{\ell+1}\}$ denote the joining vertices of B_ℓ and let $s_1,w,s_{\ell+1}$ be the P_3 substructure.

Since r_4 is not the root, its parent is a series node which must have at least two children, therefore r_4 must have at least one sibling. Without loss of generality, we may assume that it is joined to T_4 at $s_{\ell+1}$, let us denote it by r_4', and let T_4' be its subtree. By the choice of r_4, we know that T_4' has height at most 4. Since r_4' is either an edge or a parallel node, T_4' can only be the following subtrees: an edge, R_2, R_3, or a height 4 subtree. However, in the previous cases we already ruled out most of the height 4 subtrees, the only remaining possibility is a parallel join of B_k and P_3. Now we consider each of these subcases.

Subcase (i): T_4' is an edge. Let $s_{\ell+1}t_1'$ be the edge T_4'. If there is no other vertex or edge in the graph, then let $e=ws_{\ell+1}$. Thus $G-e$ is a necklace graph, therefore its toughness is $\frac{1}{2}$, a contradiction. If G also contains the leap edge $f=s_1t_1'$ then $\tau(G-f)\ge\tau(G-f-e)\ge\frac{1}{2}$, a contradiction again.

Subsubcase (i:α): The parent r_5 of r_4 is the root. This means that r_5 is a series node, all its children are parallel nodes or edges, one of the children is r_4, the next child is r_4'. There may be other children on either side of them.

First, suppose that r_4' has a sibling joined to t_1'; possibly, there are further siblings joined to that. Let H denote the subgraph spanned by all these siblings, so it is a sequence of series-joined substructures. Thus $s_{\ell+1}t_1'$ is a bridge connecting T_4 and H. Let $e=ws_{\ell+1}$ again, we show that $\tau(G-e)=\tau(G)\le\frac{1}{2}$. Suppose, to the contrary, that $\tau(G-e)<\tau(G)$ and let S be a tough set of $G-e$. By Lemma 4 S is either contained in T_4 or in H. In the latter case, there is a path in $G-e$ between the two ends of e, thus $c((G-e)-S)=c(G-S)$. So we may suppose that S is contained in T_4 and $\frac{|S|}{c(G-e)}<\tau(G)$. However, it is easy to see that if we contract H into one vertex (t_1') to obtain G', then $\frac{|S|}{c(G'-e)}<\tau(G')$ holds as well. This is a contradiction since $G'-e$ is a necklace graph, so it cannot contain such a cutset S.

Now suppose that r_4 has a sibling joined to s_1; possibly, there are further siblings joined to that. Let H denote the subgraph spanned by all these siblings. In this case, s_1 is a cut-vertex. Let $S=\{s_1,s_2,\dots,s_{\ell+1}\}$.

Now $G-S$ consists of all middle vertices of the bracelet B_ℓ, the isolated vertices w and t_1', plus the nonempty component $H-s_1$. This implies that $\frac{|S|}{c(G-S)}=\frac{\ell+1}{2\ell+3}<\frac{1}{2}$, a contradiction.

We obtain the same contradiction if r_4 has a sibling joined to both s_1 and t_1' (i.e., to both ends of the substructure).

Subsubcase (i:β): The parent r_5 of r_4 is not the root. Thus, the parent of r_5 is a parallel node, which means that there is a path connecting s_1 and t_1' that does not contain any other vertices of T_4 and T_4'.

Let $e=s_1w$, suppose that $\tau(G-e)<\tau(G)$ and let S be a minimum size tough set. By Claim 12 we know that $s_1,w\notin S$.

First, consider the case when $s_{\ell+1}\notin S$. We can apply Lemma 5 with $u=s_1$ and $v=s_{\ell+1}$, which implies that S is either fully contained in T_4 or is fully outside of T_4. In either case, $G-e$ contains a path between s_1 and w, a contradiction.

Thus, we can assume that $s_{\ell+1}\in S$. Suppose also that $s_i\in S$ and $s_j\notin S$ for some $1<i<i+1<j<\ell+1$, this contradicts Lemma 5 with $u=s_1$ and $v=s_j$. Thus for some $1\le i\le \ell+1$ we have $s_1,\dots,s_{i-1}\notin S$ and $s_i,\dots,s_{\ell+1}\in S$. However, this S cannot be minimal if $i<\ell+1$, because if we remove $s_i,\dots,s_\ell$ from S then the size of the cutset decreases by $\ell-i+1$ and the number of components decreases by $2(\ell-i+1)$, so the ratio (that was less than $\frac{1}{2}$) decreases. (It is easy to see that the smaller set is still a cutset.) Therefore $s_1,\dots,s_\ell\notin S$. This implies that in $G-S$ the vertices $w,s_1,\dots,s_\ell$ and the two middle vertices between s_ℓ and $s_{\ell+1}$ (denote them by v_1 and v_2) are all in one component.

By Lemma 18 S is a tough set in G as well, so $\frac{|S|}{c(G-S)}=\frac{1}{2}$. Let $S'=S-s_{\ell+1}$. Clearly $|S'|=|S|-1$. The vertex $s_{\ell+1}$ has 4 neighbors: w,v_1,v_2,t_1'. Since the first 3 are in one component of $G-S$, we have $c(G-S')\ge c(G-S)-1$. Using the Median Inequality it can be verified that $\frac{|S'|}{c(G-S')}<\frac{1}{2}$, a contradiction.

Subcase (ii): T_4' is R_2. If there is no other vertex or edge in the graph, then let $e=ws_{\ell+1}$. Thus $G-e$ is a nearly a necklace graph, the last edge in the series connection is missing here. However, its toughness is still $\frac{1}{2}$, which can be proved by the same argument that we used in the proof of Lemma 17. This again gives a contradiction.

If G also contains the leap edge $f=s_1t_1'$ then $\tau(G-f)\ge\tau(G-f-e)\ge\frac{1}{2}$, a contradiction again.

Subsubcase (ii:α): The parent r_5 of r_4 is the root. We obtain a contradiction in essentially the same way as Subsubcase (i:α), using again that the toughness of the graph obtained from a necklace graph by deleting the last edge is still $\frac{1}{2}$.

Subsubcase (ii:β): The parent r_5 of r_4 is not the root. We obtain a contradiction in essentially the same way as Subsubcase (i:β).

Subcase (iii): T_4' is R_3. If there is no other vertex or edge in the graph, then let $e=ws_{\ell+1}$. Thus, $G-e$ is still very similar to a necklace graph. With a slight adjustment in the argument of Lemma 17 one can prove that its toughness is still $\frac{1}{2}$, a contradiction.

If G also contains the leap edge $f=s_1t_1'$ then $\tau(G-f)\ge\tau(G-f-e)=\frac{1}{2}$, a contradiction again.

Now suppose that G contains a vertex z outside of T_4 and T_4'. Let $S=\{s_1,\dots,s_{\ell+1},t_1'\}$. Thus, in $G-S$ all the middle vertices of the bracelets become isolated vertices, as well as the 3 middle vertices of $T_4'=R_3$, and w. There is at least one more component that contains z. Therefore, $\tau(G)\le\frac{\ell+2}{2\ell+5}<\frac{1}{2}$, a contradiction.

Subcase (iv): T_4' is a parallel join of B_k and P_3. Let $\{s_1,\dots,s_{\ell+1}\}$ and $\{s_1',\dots,s_{k+1}'\}$ denote the joining vertices of B_ℓ and B_k respectively so that $s_1'=s_{\ell+1}$, and let w and w' denote the middle vertices of the two P_3 substructures.

If there is no other vertex or edge in the graph, then let $e=ws_{\ell+1}$ and $e=s_{\ell+1}w'$. Now $G-e-f$ is a necklace graph, which implies that $\tau(G-e)\ge\tau(G-e-f)=\frac{1}{2}$, a contradiction again.

If G also contains the leap edge $g=s_1s_{k+1}'$ then $\tau(G-g)\ge\tau(G-f-e-g)=\frac{1}{2}$, a similar contradiction.

Thus, we can assume that G contains a vertex z outside of T_4 and T_4'. Let $S=\{s_1,\dots,s_{\ell+1}\}\cup\{s_1',\dots,s_{k+1}'\}$. Now in $G-S$ all the middle vertices of the bracelets become isolated vertices, as well as w and w'. There is at least one more component that contains z. So, $\tau(G)\le\frac{\ell+k+1}{2\ell+2k+5}<\frac{1}{2}$, a contradiction.

Case 4(b); r_4 has only two children B_ℓ and P_3, and r_4 is the root.

This graph is nearly identical to the one in Case 1(b), except that in this case, a P_3 replaces an edge. Nevertheless, we can apply essentially the same argument here to establish that $\tau(G)>\frac{1}{2}$.

This completes the proof that no subtree can have height 4 or greater.

To summarize, we ruled out the possibility of the sp-tree having height 0, 2, or 4 above, and determined the only possible graphs when the sp-tree has height 1 or 3. This concludes the proof of Lemma 25. ☐

Since P_3 is a necklace graph, Lemma 25 implies that a reduced sp-graph is minimally $\frac{1}{2}$-tough if and only if it is a necklace graph. It is easy to describe the graphs whose reduced graph is a necklace graph: they can be obtained from necklace graphs by repeatedly replacing induced P_3 subgraphs with longer paths of length at least 3.

We define a *pearl* as the parallel join of two paths, each of length at least 2. Now the above Lemmas and Theorems imply the following characterization of minimally $\frac{1}{2}$-tough sp-graphs, which is the main result of our paper.

Theorem 26. *A sp-graph G is minimally $\frac{1}{2}$-tough if and only if it is a series connection of an arbitrary number of pearls and at least two edges, with the first and last components being edges. (See Figure 9.)*

It is clear the all necklace graph contain a vertex of degree 1. Therefore an easy consequence of Theorem 26 is the following.

Theorem 27. *The Generalized Kriesell conjecture holds for t-tough series-parallel graphs if $t \geq \frac{1}{2}$.*

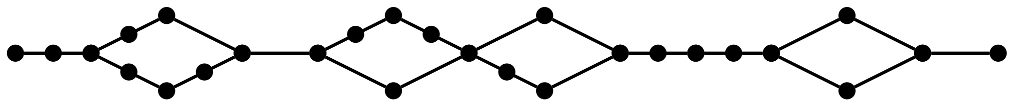

Figure 9: A minimally $\frac{1}{2}$-tough graph

5 Open questions

The most evident open problem is whether the Generalized Kriesell conjecture holds for minimally t-tough graphs when $t < \frac{1}{2}$. Additionally, obtaining a characterization of such graphs would be valuable. Our methods may be applicable when $t = \frac{1}{k}$ for some integer $k \geq 3$, as Lemma 18 can be used in these cases. However, we do not yet see how to characterize these graphs.

Another open problem is the characterization of minimally tough *generalized sp-graphs*. In these graphs, apart from the series and parallel join, one can also use the *source merge* operation, i.e., identify the sources of two smaller generalized sp-graphs.

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