ON THE STABILITY OF EINSTEIN METRICS CARRYING A SPECIAL TWISTED SPINOR

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Abstract. We prove linear semi-stability for a large class of Einstein metrics of non-positive scalar curvature. More precisely, we show that any Einstein n-manifold with non-positive scalar curvature carrying a parallel twisted pure spin^r spinor is linearly semi-stable, under mild restrictions on n and r. We thus extend the parallel spin and spin^c stability results of Dai–Wang–Wei. As an application, our result implies linear semi-stability for all negative quaternion-Kähler manifolds of dimension greater than 8.

1. Introduction

A Riemannian metric g on a smooth manifold M is said to be Einstein if its Ricci curvature is a constant multiple of the metric,

$$\mathrm{Ric}_g = E\, g,$$

for some real constant E [Bes87]. For compact manifolds, Einstein metrics are precisely the critical points of the Einstein–Hilbert functional

$$\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}, \qquad \mathcal{S}(g) := \int_M \mathrm{scal}_g |\mathrm{vol}_g|$$

on the space $\mathcal{M}_1$ of unit-volume Riemannian metrics, where $|\mathrm{vol}_g|$ denotes the volume density determined by g.

Given such a metric g, a fundamental problem in geometric analysis is to understand the second variation of $\mathcal{S}$ at g. We refer to the recent survey [SS25] for a comprehensive overview. An Einstein metric is called *stable* if it is a local maximum of $\mathcal{S}$ restricted to the space

$$\mathcal{C} := \{g \in \mathcal{M}_1 \mid \mathrm{scal}_g \text{ is constant on } M\}.$$

The linearisation of this notion leads to the definition of *linear (semi-)stability*: an Einstein metric g is *linearly stable* (resp. *semi-stable*) if the second variation $\mathcal{S}''_g$ is negative-definite (resp. negative semi-definite) on the space of transverse-traceless tensors

$$\mathcal{S}^2_{\mathrm{tt}} := \{h \in \mathcal{S}^2(M) \mid \mathrm{tr}_g h = 0,\ \delta_g h = 0\},$$

where $\mathcal{S}^2(M) := \Gamma(\mathrm{Sym}^2 T^*M)$ and δ_g denotes the divergence.

It is well known that linear semi-stability is governed by the spectrum of the so-called *linearised Einstein operator*; namely, it corresponds to

$$\nabla^*\nabla - 2\mathring{R} \geq 0$$

on $\mathcal{S}^2_{\mathrm{tt}}$, where ∇ is the Levi-Civita connection of g and $\mathring{R}$ denotes the curvature action on symmetric 2-tensors – see (1). Linear stability corresponds to the inequality being strict.

Several conditions for stability have been obtained in terms of the curvature [Koi78, Fuj79, BCG91, IN05, Krö15, BK25]. A long-standing conjecture in the field asserts that every Einstein metric of non-positive scalar curvature is stable. This problem has attracted significant attention, with substantial progress in several geometric settings. For example:

- in the presence of a parallel spinor, Dai–Wang–Wei [DWW05] proved linear semi-stability by exploiting a spinorial Bochner identity; in fact, full $\mathcal{S}$-stability holds in this setting – see [SS25, Cor. 5.7];
- in the Kähler–Einstein case, an extension of this method using spin^c geometry yields linear semi-stability in the negative scalar curvature regime [DWW07];
- more recently, Kröncke–Semmelmann [KS24] proved strict stability for negative quaternion-Kähler manifolds using representation-theoretic methods.

In this paper, we prove linear semi-stability for a broad class of Einstein metrics of non-positive scalar curvature, namely those admitting a *parallel twisted pure spin^r spinor*. Our main result is as follows.

Theorem 1. Let (M^n, g) be a Riemannian manifold carrying a parallel twisted pure spin^r spinor, for some $r \geq 3$, $r \neq 4$, $n \neq 8$, and $n + 4r - 16 \neq 0$. Then g is Einstein and, if $\mathrm{scal}_g \leq 0$, it is linearly semi-stable. $\square$

The class of manifolds that carry parallel twisted pure spin^r spinors will be introduced in Section 2. For now, let us remark that this is a vast family: it includes, for example, all even-dimensional manifolds with special Riemannian holonomy [HS19, Sec. 4].

Our proof generalises the strategy initiated by Dai–Wang–Wei: we construct an isometric bundle map from symmetric 2-tensors to twisted spinor-valued 1-forms, and use it to relate the linearised Einstein operator to the square of a Dirac-type operator. We do this in Section 3. The key new ingredient comes from the algebraic identities satisfied by the 2-forms associated to twisted pure spinors [HS19], which yield a curvature relation that allows us to control the extra terms appearing in a general Bochner-like formula that we obtain in Section 3. This makes it possible to derive the required lower bound for the linearised Einstein operator in the non-positive scalar curvature case, which we accomplish in Section 4.

2. Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold. Throughout the paper, we will work locally: we fix a point $x \in M$ and a local orthonormal frame $(e_1, \ldots, e_n)$ around it such that $\nabla e_i(x) = 0$ for all i.

For vector fields X, Y, the Riemannian curvature endomorphism is given by

$$R_{XY} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

and for $i, j, k, l \in \{1, \dots n\}$

$$R_{ijkl} := g\left(R_{e_i e_j} e_k, e_l\right).$$

The Ricci and scalar curvature are given, respectively, by

$$\mathrm{Ric}_{ij} := \mathrm{Ric}_g(e_i, e_j) = R_{ikkj}, \qquad \mathrm{scal}_g = R_{ikki},$$

where we adopt the convention that a summation is taken whenever indices are repeated.

The curvature operator acts on symmetric 2-tensors by

$$(\mathring{R}h)(X, Y) := h\left(R_{Xe_i} e_i, Y\right). \tag{1}$$

We say that M is *spin^r* if there exists an oriented Euclidean rank-r vector bundle $F \to M$ such that $TM \oplus F$ is spin [EH16, AM21]. Such an F is called an *auxiliary vector bundle* for TM. If ΣM and ΣF denote the locally defined complex spinor bundles of TM and F respectively, the fact that $TM \oplus F$ is spin implies that

$$\Sigma(M, F, m) := \Sigma M \otimes (\Sigma F)^{\otimes m}$$

is an honest vector bundle over M for all odd values of m. If M is itself spin, even values of m are also allowed. This bundle is equipped with a natural Hermitian product, which we denote by $\langle \cdot, \cdot \rangle$. Sections of these bundles are called *spin^r spinors*, *twisted spinors* or simply *spinors*, and they can be Clifford-multiplied by sections of $TM \oplus F$ in a natural way.

Each metric connection θ on F, together with the Levi-Civita connection on TM, uniquely determines a Hermitian connection ∇^θ on $\Sigma(M, F, m)$.

We now fix a local orthonormal frame $(f_1, \dots, f_r)$ of F around x. Each spinor $\psi \in \Gamma(\Sigma(M, F, m))$ determines a 2-form on M for each $1 \le k, l \le r$:

$$\eta^\psi_{kl}(X, Y) := \Re \left\langle X \wedge Y f_k f_l \cdot \psi, \psi \right\rangle,$$

where $X \wedge Y := XY + g(X, Y)$ and $\Re$ denotes the real part.

Every section of ω of $T^*M \otimes T^*M$ naturally defines a section of $\mathrm{End}(TM) = T^*M \otimes TM$ by $\hat{\omega}(X) := \omega(X, \cdot)^\sharp$, which locally reads

$$g(\hat{\omega}(e_i), e_j) = \omega(e_i, e_j).$$

We now define the central property of spinors that we study in this paper:

Definition 2 ([HS19, Def. 3.5])**.** A spinor $\psi \in \Gamma(\Sigma(M, F, m))$ is said to be *twisted pure* if the following two conditions are satisfied for all $1 \le k < l \le r$:

- $\hat{\eta}^\psi_{kl} \circ \hat{\eta}^\psi_{kl} = -\mathrm{Id}$, and
- $r = 2$ or $\left(\eta^\psi_{kl} + 2 f_k f_l\right) \cdot \psi = 0$.

Twisted pure spinors are of interest because they can be used to characterise Riemannian manifolds with special holonomy, as shown in [HS19, Sec. 4]. They generalise the classical notion of pure spinor for $r = 2$ used to describe Kähler holonomy.

3. A Bochner-type formula

Throughout this section, (M^n, g) is a compact spin^r Einstein manifold with auxiliary vector bundle F that admits a non-zero spin^r spinor $\psi \in \Gamma(\Sigma(M, F, m))$ that is parallel with respect to an auxiliary connection θ with curvature 2-form Θ. For simplicity, we denote $\nabla = \nabla^\theta$. Since ψ necessarily has constant length, we may assume without loss of generality that $|\psi| = 1$.

Following the classical ideas of [DWW05, DWW07], we define a bundle map from the space of symmetric 2-tensors to the space of $\Sigma(M, F, m)$-valued 1-forms:

$$\begin{aligned} \Phi\colon \mathrm{Sym}^2(T^*M) &\to \Sigma(M, F, m) \otimes T^*M \\ h &\mapsto \left(X \mapsto \hat{h}(X) \cdot \psi\right). \end{aligned} \tag{2}$$

With respect to a local orthonormal frame $(e_1, \ldots, e_n)$, it takes the form

$$\Phi(h) = h_{ij} e_i \cdot \psi \otimes e^j,$$

This map is easily seen to be parallel and isometric.

Let $\mathcal{D}$ be the Dirac operator on $\Sigma M \otimes T^*M$, given locally by

$$\mathcal{D} = e_i \cdot \nabla_{e_i}.$$

We now prove a relationship between the linearised Einstein operator $\nabla^*\nabla - 2\mathring{R}$ on symmetric 2-tensors and $\mathcal{D}^*\mathcal{D}$ via the bundle map Φ.

Lemma 3. For each symmetric 2-tensor h on M,

$$\mathcal{D}^*\mathcal{D}\Phi(h) = \Phi\left(\nabla^*\nabla h - 2\mathring{R}h + (\mathrm{Ric} \circ h)\right) - \frac{1}{2} h_{ip} \left(\hat{\Theta}_{kl}\right)_{pj} e_i f_k f_l \cdot \psi \otimes e^j,$$

where Φ is interpreted as in (2), extended in the obvious way to general (not necessarily symmetric) 2-tensors.

Proof. Using that $\mathcal{D}^* = \mathcal{D}$ and that Φ is parallel, we compute

$$\begin{aligned}
\mathcal{D}^*\mathcal{D}(\Phi(h)) &= \mathcal{D}\left(e_l \cdot \nabla_{e_l}\Phi(h)\right) = e_k \cdot \nabla_{e_k}\left(e_l \cdot \Phi(\nabla_{e_l}h)\right)\\
&= e_k \cdot \left((\nabla_{e_k}e_l)\cdot \Phi(\nabla_{e_l}h) + e_l \cdot \nabla_{e_k}\Phi(\nabla_{e_l}h)\right)\\
&= e_ke_l \cdot \nabla_{e_k}\Phi(\nabla_{e_l}h) = e_ke_l \cdot \Phi(\nabla_{e_k}\nabla_{e_l}h)\\
&= \left(\nabla_{e_k}\nabla_{e_l}h\right)(e_i,e_j)e_ke_le_i \cdot \psi \otimes e^j\\
&= \left(-\left(\nabla_{e_k}\nabla_{e_k}h\right)(e_i,e_j)e_i + \sum_{k<l}\left(\nabla_{e_k}\nabla_{e_l}h - \nabla_{e_l}\nabla_{e_k}h\right)(e_i,e_j)e_ke_le_i\right)\cdot \psi \otimes e^j\\
&= \left(\left(\nabla^*\nabla h\right)(e_i,e_j)e_i + \sum_{k<l}\left(h\left(R_{e_le_k}e_i,e_j\right) + h\left(e_i,R_{e_le_k}e_j\right)\right)e_ke_le_i\right)\cdot \psi \otimes e^j\\
&= \Phi(\nabla^*\nabla h) + \frac{1}{2}h(R_{e_le_k}e_i,e_j)e_ke_le_i \cdot \psi \otimes e^j + \frac{1}{2}h(e_i,R_{e_le_k}e_j)e_ke_le_i \cdot \psi \otimes e^j\\
&= \Phi(\nabla^*\nabla h) + \frac{1}{2}h_{jp}R_{lkip}e_ke_le_i \cdot \psi \otimes e^j + \frac{1}{2}h_{ip}R_{lkjp}e_ke_le_i \cdot \psi \otimes e^j.
\end{aligned}$$

Now, using the first Bianchi identity and the Clifford relations, we can simplify the last two terms of the previous sum as follows:

$$\begin{aligned}
R_{lkip}e_ke_le_i &= R_{pikl}e_ke_le_i = -R_{pkli}e_ke_le_i - R_{plik}e_ke_le_i\\
&= -R_{pkli}e_ke_le_i + R_{plik}e_le_ke_i + 2\delta_{kl}R_{plik}e_i\\
&= -R_{pkli}e_ke_le_i - R_{plik}e_le_ie_k - 2\delta_{ik}R_{plik}e_l + 2R_{pkik}e_i\\
&= -2R_{pkli}e_ke_le_i - 2\mathrm{Ric}_{pi}e_i\\
&= -\frac{2}{3}\sum_{\substack{k,l,i\\ \text{distinct}}}\left(R_{pkli} + R_{plik} + R_{pikl}\right)e_ke_le_i\\
&\quad - 2R_{pkki}e_ke_ke_i - 2R_{pklk}e_ke_le_k - 2\mathrm{Ric}_{pi}e_i\\
&= -4R_{pklk}e_l - 2\mathrm{Ric}_{pi}e_i = 2\mathrm{Ric}_{pl}e_l,
\end{aligned}$$

and

$$\begin{aligned}
R_{lkjp}e_ke_le_i &= -R_{jpkl}e_ke_le_i = R_{jpkl}e_ke_ie_l + 2\delta_{il}R_{jpkl}e_k\\
&= -R_{jpkl}e_ie_ke_l - 2\delta_{ik}R_{jpkl} + 2\delta_{il}R_{jpkl}e_k\\
&= -R_{jpkl}e_ie_ke_l - 2R_{jpil}e_l + 2R_{jpki}e_k\\
&= -R_{jpkl}e_ie_ke_l - 4R_{jpik}e_k.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\mathcal{D}^*\mathcal{D}\Phi(h) &= \Phi(\nabla^*\nabla h) + h_{jp}\mathrm{Ric}_{pl}e_l \cdot \psi \otimes e^j + \frac{1}{2}h_{ip}\left(-R_{jpkl}e_ie_ke_l - 4R_{jpik}e_k\right)\cdot\psi\otimes e^j \\ &= \Phi(\nabla^*\nabla h) + \left((\mathrm{Ric}\circ h)_{lj}e_l - 2(\mathring{R}h)_{kj}e_k - \frac{1}{2}h_{ip}R_{jpkl}e_ie_ke_l\right)\cdot\psi\otimes e_j \\ &= \Phi\left(\nabla^*\nabla h + (\mathrm{Ric}\circ h) - 2(\mathring{R}h)\right) + \frac{1}{2}h_{ip}R_{pjkl}e_ie_ke_l\cdot\psi\otimes e^j.\end{aligned}$$

Finally, since ψ is parallel,

$$0 = R_{pjkl}e_ke_l\cdot\psi + (\Theta_{kl})_{pj}\, f_kf_l\cdot\psi,$$

and this finishes the proof. □

Corollary 4. For each symmetric 2-tensor h on M,

$$\left\langle \nabla^*\nabla h - 2\mathring{R}h, h\right\rangle \geq -\langle \mathrm{Ric}\circ h, h\rangle - \frac{1}{2}\left\langle \hat{\eta}^{\psi}_{kl}\circ h\circ\hat{\Theta}_{kl}, h\right\rangle. \tag{3}$$

Proof. Using Lemma 3 and the fact that Φ is isometric, we compute

$$\begin{aligned}\left\langle \nabla^*\nabla h - 2\mathring{R}h, h\right\rangle &= \left\langle \Phi(\nabla^*\nabla h - 2\mathring{R}h), \Phi(h)\right\rangle \\ &= \langle \mathcal{D}^*\mathcal{D}\Phi(h), \Phi(h)\rangle - \langle \Phi(\mathrm{Ric}\circ h), \Phi(h)\rangle \\ &+ \frac{1}{2}h_{ip}\left(\hat{\Theta}_{kl}\right)_{pj}\langle e_if_kf_l\cdot\psi\otimes e^j, \Phi(h)\rangle \\ &\geq -\langle \mathrm{Ric}\circ h, h\rangle + \frac{1}{2}h_{ip}\left(\hat{\Theta}_{kl}\right)_{pj}\left\langle e_if_kf_l\cdot\psi\otimes e^j, h_{st}e_s\cdot\psi\otimes e^s\right\rangle \\ &= -\langle \mathrm{Ric}\circ h, h\rangle + \frac{1}{2}h_{ip}h_{qj}\left(\hat{\Theta}_{kl}\right)_{pj}\left\langle e_if_kf_l\cdot\psi\otimes e^j, e_q\cdot\psi\otimes e^j\right\rangle \\ &= -\langle \mathrm{Ric}\circ h, h\rangle - \frac{1}{2}h_{ip}h_{qj}\left(\hat{\Theta}_{kl}\right)_{pj}\langle e_qe_if_kf_l\cdot\psi, \psi\rangle \\ &= -\langle \mathrm{Ric}\circ h, h\rangle - \frac{1}{2}h_{ip}h_{qj}\left(\hat{\Theta}_{kl}\right)_{pj}\left(\eta^{\psi}_{kl}\right)_{qi} \\ &= -\langle \mathrm{Ric}\circ h, h\rangle - \frac{1}{2}\left\langle \hat{\eta}^{\psi}_{kl}\circ h\circ\hat{\Theta}_{kl}, h\right\rangle.\end{aligned}$$

□

Remark 5. Every oriented Riemannian n-manifold admits a natural spin^n structure, given by taking the tangent bundle itself as the auxiliary vector bundle, which we endow with the Levi-Civita connection. This structure carries a non-zero parallel spin^n spinor [EH16, Prop. 3.3]. An elementary calculation applying Corollary 4 to this spinor yields

$$\left\langle \nabla^*\nabla h - 2\mathring{R}h, h\right\rangle \geq -\langle \mathrm{Ric}\circ h, h\rangle - \left\langle \mathring{R}h, h\right\rangle.$$

In particular, if an Einstein metric with Einstein constant E satisfies $\mathring{R} < E$, then it is stable, which recovers a well-known result by Koiso [Koi78, Thm. 3.3].

In the next section, we assume an additional condition on ψ that will establish a relation between $\hat{\eta}^{\psi}_{kl}$ and $\hat{\Theta}_{kl}$ and will allow us to deduce stability in some cases from (3).

4. Stability of metrics with parallel twisted pure spinors

We continue to assume that (M^n, g) is a compact Einstein manifold endowed with a parallel spin^r spinor ψ with $|\psi| = 1$. As mentioned earlier, we will now impose a further condition on ψ, namely that

$$\psi = \frac{1}{|\psi'|}\psi',$$

with ψ' twisted pure – see Definition 2, and note that it is *not* invariant by rescaling.

Noting that there is a sign error in [HS19, eq. (4.1)], we have the following:

Lemma 6 ([HS19, Lem. 4.8])**.** Let (M, g) be a Riemannian manifold carrying a parallel twisted pure spin^r spinor ψ', and let $\Theta = \Theta_{kl} f_k \wedge f_l$ be the curvature form of the auxiliary connection. Then, if $r \geq 3$, $r \neq 4$, $n \neq 8$, and $n + 4r - 16 \neq 0$,

$$\hat{\Theta}_{kl} = \frac{-\mathrm{scal}_g}{n(\frac{n}{4} + 2r - 4)}\hat{\eta}^{\psi'}_{kl},$$

for $1 \leq k < l \leq r$, and g is Einstein. □

We can now prove the main result of this paper, Theorem 1:

Proof of Theorem 1. The fact that g is Einstein follows from Lemma 6. Let E be the Einstein constant, ψ' a parallel twisted pure spin^r spinor, $\psi = \psi'/|\psi'|$, and $\Theta = \Theta_{kl} f_k \wedge f_l$ the curvature 2-form of the auxiliary connection. By Corollary 4,

$$\begin{aligned}\left\langle \nabla^*\nabla h - 2\mathring{R}h, h\right\rangle &\geq -\left\langle \mathrm{Ric}\circ h, h\right\rangle - \frac{1}{2}\left\langle \hat{\eta}^{\psi}_{kl}\circ h\circ \hat{\Theta}_{kl}, h\right\rangle \\ &= -E|h|^2 + \frac{E}{\frac{n}{2} + 4r - 8}\left\langle \hat{\eta}^{\psi}_{kl}\circ h\circ \hat{\eta}^{\psi}_{kl}, h\right\rangle.\end{aligned}$$

If $E = 0$, the result follows immediately, so suppose that $E < 0$. Then, by the triangle inequality, the Cauchy-Schwarz inequality, and the fact that $\hat{\eta}^{\psi'}_{kl}\circ\hat{\eta}^{\psi'}_{kl} = -\mathrm{Id}$,

$$\begin{aligned}\left|\left\langle \hat{\eta}^{\psi}_{kl}\circ h\circ \hat{\eta}^{\psi}_{kl}, h\right\rangle\right| &\leq |h||\hat{\eta}^{\psi}_{kl}\circ h\circ \hat{\eta}^{\psi}_{kl}| = \frac{|h|}{|\psi'|^2}|\hat{\eta}^{\psi'}_{kl}\circ h\circ \hat{\eta}^{\psi'}_{kl}| \\ &= \frac{|h|}{|\psi'|^2}\mathrm{tr}\left(\left(\hat{\eta}^{\psi'}_{kl}\circ h\circ \hat{\eta}^{\psi'}_{kl}\right)\circ\left(\hat{\eta}^{\psi'}_{kl}\circ h\circ \hat{\eta}^{\psi'}_{kl}\right)^t\right)^{1/2} \\ &= \frac{|h|}{|\psi'|^2}\mathrm{tr}\left(\hat{\eta}^{\psi'}_{kl}\circ h\circ \hat{\eta}^{\psi'}_{kl}\circ\hat{\eta}^{\psi'}_{kl}\circ h\circ \hat{\eta}^{\psi'}_{kl}\right)^{1/2} \\ &= r(r-1)\frac{|h|}{|\psi'|^2}\mathrm{tr}\,(h\circ h)^{1/2} = r(r-1)\frac{|h|^2}{|\psi'|^2}.\end{aligned}$$

And, by [EH16, Thm. 3.1],

$$|\psi'|^2\mathrm{Ric} = \hat{\eta}^{\psi'}_{kl} \circ \hat{\Theta}_{kl} = -\frac{E}{\frac{n}{4}+2r-4}\hat{\eta}^{\psi'}_{kl} \circ \hat{\eta}^{\psi'}_{kl} = E\,\frac{r(r-1)}{\frac{n}{4}+2r-4}\mathrm{Id},$$

which implies that

$$|\psi'|^2 = \frac{r(r-1)}{\frac{n}{4}+2r-4}.$$

Putting everything together, we see that

$$\begin{aligned}\left|\frac{E}{\frac{n}{2}+4r-8}\left\langle \hat{\eta}^{\psi}_{kl}\circ h\circ \hat{\eta}^{\psi}_{kl}, h\right\rangle\right| &\leq \frac{|E|}{\frac{n}{2}+4r-8}r(r-1)\frac{|h|^2}{|\psi'|^2}\\ &= \frac{|E|}{\frac{n}{2}+4r-8}r(r-1)|h|^2\frac{(\frac{n}{4}+2r-4)}{r(r-1)} = |E||h|^2,\end{aligned}$$

which concludes the proof. □

As an application of this result, recall that an oriented Riemannian $4m$-dimensional manifold is quaternion-Kähler if and only if it admits a parallel twisted pure spin^3 spinor. Hence, from Theorem 1 one deduces that every quaternion-Kähler manifold of negative scalar curvature of dimension greater than 8 is linearly semi-stable. It has recently been proved by different means that, in fact, quaternion-Kähler manifolds of negative scalar curvature are strictly stable [KS24].

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