Multiplicity of normalized solutions to the upper critical fractional Choquard equation with L^2-supercritical perturbation

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Abstract

We investigate normalized solutions with prescribed L^2-norm for the upper critical fractional Choquard equation

$$(-\Delta)^s u + V(\varepsilon x)u = \lambda u + \big(I_\alpha * |u|^p\big)|u|^{p-2}u + \big(I_\alpha * |u|^q\big)|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

where $N > 2s$, $0 < s < 1$, $(N-4s)^+ < \alpha < N$, and the nonlocal exponents satisfy

$$\frac{N+2s+\alpha}{N} < q < p = \frac{N+\alpha}{N-2s},$$

so that both nonlinearities are L^2-supercritical and the p–term has upper critical growth of Hartree type. Under standard assumptions on the slowly varying potential V, we develop a constrained variational approach on the L^2-sphere, based on a truncation–penalization of the critical term in the energy functional, to overcome the lack of compactness. We prove that, for all sufficiently small $\varepsilon > 0$, the problem admits at least $\mathrm{cat}_{M_\delta}(M)$ distinct normalized solutions, where M is the set of global minima of V and these solutions concentrate near M as $\varepsilon \to 0$.

Keywords: Fractional Choquard equation, Critical growth, Multiple normalized solutions.

MSC2020: 35A15, 35B40, 35J20.

1 Introduction and main results

This paper is concerned with the existence of multiple normalized solutions to the upper critical fractional Choquard equation with mixed L^2-supercritical Hartree nonlinearities

$$(-\Delta)^s u + V(\varepsilon x)u = \lambda u + \big(I_\alpha * |u|^p\big)|u|^{p-2}u + \big(I_\alpha * |u|^q\big)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

under the prescribed mass condition

$$\int_{\mathbb{R}^N} |u|^2 dx = a,$$

where $N > 2s$, $0 < s < 1$, $(N-4s)^+ < \alpha < N$, $\varepsilon > 0$ is a small parameter and $\lambda \in \mathbb{R}$ is the Lagrange multiplier associated with the L^2-constraint. The nonlocal exponents satisfy

$$\frac{N+2s+\alpha}{N} < q < p = 2^*_{\alpha,s} = \frac{N+\alpha}{N-2s},$$

so that both Hartree terms are L^2-supercritical and the p–nonlinearity has Hardy–Littlewood–Sobolev upper critical growth.

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The fractional Laplace operator $(-\Delta)^s$ for $0 < s < 1$ is defined by

$$(-\Delta)^s u(x) = \frac{C(N,s)}{2} \, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where P.V. stands for the Cauchy principal value and $C(N,s)$ is a suitable normalizing constant; we refer to [4] for further properties. The kernel

$$I_\alpha(x) = \frac{A_{N,\alpha}}{|x|^{N-\alpha}}, \qquad x \in \mathbb{R}^N \setminus \{0\},$$

is the Riesz potential of order $(N-4s)^+ < \alpha < N$, with

$$A_{N,\alpha} = \frac{\Gamma\left(\frac{N-2}{2}\right)}{\pi^{\frac{N}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}.$$

The exponent $2^*_{\alpha,s}$ is the upper critical Hardy–Littlewood–Sobolev exponent associated with the following classical Hardy–Littlewood–Sobolev inequality [15].

Proposition 1.1. *Let $t, r > 1$ and $0 < \alpha < N$ be such that $1/t + 1/r = 1 + \alpha/N$, and let $f \in L^t(\mathbb{R}^N)$, $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t,r,\alpha,N) > 0$, independent of f and h, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\alpha}} \, dx \, dy \le C(t,r,\alpha,N) \|f\|_{L^t} \|h\|_{L^r}.$$

In particular, if $t = r = 2N/(N+\alpha)$, then

$$C(t,r,\alpha,N) = C(N,\alpha) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \right\}^{-\frac{\alpha}{N}}.$$

As a consequence, for $u \in H^s(\mathbb{R}^N)$ the nonlocal functional

$$B_r(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^r |u(y)|^r}{|x-y|^{N-\alpha}} \, dx \, dy$$

is well defined whenever

$$\underline{p} := \frac{N+\alpha}{N} \le r \le 2^*_{\alpha,s}.$$

The endpoint $r = 2^*_{\alpha,s}$ is called the Hardy–Littlewood–Sobolev upper critical exponent, in complete analogy with the usual Sobolev critical exponent for local nonlinearities. The presence of this upper critical exponent plays a crucial role in the loss of compactness for problem (1.1).

On the other hand, under the L^2-preserving scaling

$$u_t(x) = t^{\frac{N}{2}} u(tx),$$

the kinetic term and the Hartree energies scale as

$$\|(-\Delta)^{s/2} u_t\|_2^2 = t^{2s} \, \|(-\Delta)^{s/2} u\|_2^2, \qquad B_r(u_t) = t^{\delta_r} \, B_r(u), \quad \delta_r = Nr - (N+\alpha).$$

Equating δ_r with $2s$ gives the L^2-critical Hartree exponent

$$\bar{p} = \frac{N+2s+\alpha}{N}.$$

Accordingly, we say that the Hartree nonlinearity is L^2-subcritical if $r < \bar{p}$, L^2-critical if $r = \bar{p}$, and L^2-supercritical if $r > \bar{p}$. In this terminology, the mixed exponents in (1.1) satisfy

$$\bar{p} < q < p = 2^*_{\alpha,s},$$

so both Hartree terms are L^2-supercritical, while only the p–term is HLS upper critical.

Equations involving nonlinearities of the form

$$\big(I_\alpha * |u|^r\big)|u|^{r-2}u$$

are usually referred to as Choquard equations (or Hartree-type equations). Historically, Choquard proposed in 1976 the nonlinear equation

$$-\Delta u + u = \big(I_2 * |u|^2\big)u \quad \text{in } \mathbb{R}^3$$

as an effective model in Hartree–Fock theory for a one-component plasma [14]. Since then, Choquard-type equations have appeared in several other contexts in quantum physics. In polaron theory, Pekar's continuum model leads to a Coulomb-type self-interaction and yields precisely the Choquard–Pekar equation for the electron's effective wave function [21]. In semiclassical gravity and in models of self-gravitating bosonic matter, the same Hartree structure with the Newton kernel gives the Schrödinger–Newton equation and its stationary states [5, 18]. For a modern mathematical account on existence, qualitative properties and decay of ground states for nonlinear Choquard equations, we refer to [19, 20] and the references therein.

From a physical point of view, one is often interested in normalized solutions, that is, standing waves with prescribed L^2-norm. The L^2-norm typically encodes a conserved physical quantity (mass, charge, number of particles) and is closely related to the orbital stability of standing waves. A foundational variational approach to normalized solutions for nonlinear Schrödinger equations was initiated by Jeanjean [9], who studied constrained critical points of the energy functional on the L^2-sphere. This strategy was further developed by Bartsch and de Valeriola [1] via a compactness threshold and a deformation argument on the mass constraint, and has since been refined for various combined nonlinearities; see, for instance, Soave [23, 24] and the references therein.

In the framework of fractional quantum mechanics, the stationary states of a particle subject to a long-range self-interaction are often described by the fractional Choquard equation

$$(-\Delta)^s u + V(x)u = \lambda u + \big(I_\alpha * |u|^p\big)|u|^{p-2}u.$$

Fractional powers of the Laplacian arise naturally in Laskin's fractional quantum mechanics, where Lévy flights replace Brownian motion in Feynman's path integral formulation [11]. Motivated by this, Li and Luo [12] studied normalized solutions of the purely fractional Choquard equation

$$(-\Delta)^s u - \lambda u = \big(|x|^{\alpha-N} * |u|^p\big)|u|^{p-2}u,$$

with $N \geq 3$, $(N-4s)^+ < \alpha < N$ and

$$p \in \big(\max\{2, \bar{p}\},\, 2^*_{\alpha,s}\big),$$

which corresponds to an L^2-supercritical but HLS-subcritical regime. By searching for critical points of the energy functional constrained on the L^2-sphere, and combining a Pohozaev-type natural constraint with genus-type arguments, they proved the existence of normalized ground states and infinitely many normalized solutions.

The lower critical fractional Choquard case $p = \underline{p}$ was systematically investigated by Yu, Tang and Zhang [26]. They considered

$$(-\Delta)^s u = \lambda u + \gamma\big(I_\alpha * |u|^{\underline{p}}\big)|u|^{\underline{p}-2}u + \mu|u|^{q-2}u,$$

with $2 < q \le 2_s^* = 2N/(N-2s)$, and obtained nonexistence, existence and symmetry results for normalized ground states depending on the parameters a, γ, μ. Non-autonomous versions with spatially varying Hartree kernels at the lower critical exponent were later analyzed by Liu, Sun and Zhang [16], who studied normalized solutions for a fractional HLS lower critical Choquard equation with nonconstant weights, proving existence, multiplicity and a detailed description of the asymptotic behavior as the potential concentrates.

The HLS upper critical situation $p = 2^*_{\alpha,s}$ has also attracted considerable attention. Lan, He and Meng [10] investigated a critical fractional Choquard equation perturbed by a nonlocal term and established the existence of normalized solutions by combining sharp HLS inequalities with concentration–compactness arguments. He, Rădulescu and Zou [8] considered the critical fractional Choquard equation with an additional local power nonlinearity

$$(-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + \big(I_\alpha * |u|^{2^*_{\alpha,s}}\big)|u|^{2^*_{\alpha,s}-2}u \quad \text{in } \mathbb{R}^N, \qquad \|u\|_2^2 = a^2,$$

with $2 < q \le 2_s^*$. They proved the existence of normalized ground states in the L^2-subcritical, mass-critical and mass-supercritical regimes by a delicate analysis of the constrained functional, using truncation and a refined characterization of the Pohozaev manifold to overcome the lack of compactness at the HLS critical level. More recently, Meng and He [17] focused on the pure HLS upper critical Hartree term for the fractional Choquard equation and constructed normalized solutions at the exact exponent $p = 2^*_{\alpha,s}$, clarifying the borderline between existence and nonexistence in the mass-critical and mass-supercritical regimes.

There has also been rapid progress on fractional Choquard equations with mixed Hartree nonlinearities and general potentials, where the normalization interacts with the geometry of $V(x)$. Zhang, Thin and Liang [27] considered a critical Choquard equation involving the fractional p-Laplacian and proved the existence of multiple normalized solutions by means of a refined Nehari-type decomposition adapted to the nonlocal operator. Chen, Yang and Zhang [3] studied more general weighted Hartree nonlinearities of the form

$$(-\Delta)^s u + V(x)u = \lambda u + f(x)\big(I_\alpha * (f|u|^q)\big)|u|^{q-2}u + g(x)\big(I_\alpha * (g|u|^p)\big)|u|^{p-2}u,$$

with

$$\frac{N+\alpha}{N} \le q < p \le \bar{p},$$

and established existence results for normalized solutions on the mass constraint by combining refined compactness and a careful use of the HLS inequality. In particular, in the unweighted case $f \equiv g \equiv 1$ they considered, for $a > 0$, the mass sphere

$$S(a) = \{u \in H^s(\mathbb{R}^N) : \ \|u\|_2^2 = a\},$$

and, for $\varepsilon > 0$, the energy functional

$$\begin{aligned} J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N}\Big(|(-\Delta)^{s/2}u|^2 + V(\varepsilon x)|u|^2\Big)\,dx - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx \\ - \frac{1}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx, \end{aligned}$$

whose constrained critical points on $S(a)$ provide normalized solutions to (1.1). The mixed exponents are assumed to satisfy

$$\frac{N+\alpha}{N} < q < \frac{N+2s+\alpha}{N} < p \leq 2^*_{\alpha,s},$$

and the potential $V \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is nonnegative, with $V(0) = 0$ and

$$V_\infty := \liminf_{|x|\to\infty} V(x) \in (0, \infty). \tag{V}$$

Let $M = \{x \in \mathbb{R}^N : \ V(x) = 0\} \neq \emptyset$ and, for $\delta > 0$, $M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\}$. Under a smallness condition on the mass a, Chen, Yang and Zhang [2] proved that problem (1.1) admits at least $\operatorname{cat}_{M_\delta}(M)$ pairs of weak solutions $(u_i, \lambda_i) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ with $\|u_i\|_2^2 = a$.

This naturally raises the question whether analogous multiplicity and concentration phenomena persist when both exponents p and q lie strictly above the L^2-critical threshold $\bar{p}$, and in particular when the larger exponent is fixed at the HLS upper critical value $2^*_{\alpha,s}$. In this paper we address precisely this fully L^2-supercritical regime, taking

$$\bar{p} < q < p = 2^*_{\alpha,s},$$

and we show that, under a suitable smallness assumption on the mass a, one can still obtain multiple normalized solutions which concentrate around the global minima set of V as $\varepsilon \to 0$. Our existence and multiplicity results are achieved by developing a constrained variational approach on $S(a)$ based on a careful truncation of the critical Hartree term and a precise analysis of the associated autonomous limit problem.

Our main result can be stated as follows.

Theorem 1.2. *Assume that* (V) *holds and that the mass $a > 0$ satisfies*

$$a \leq \left(K_q^{-1} S_\alpha^{\vartheta_q}\right)^{\frac{1}{q(1-\gamma_{q,s})}},$$

where the constants $K_q > 0$, $S_\alpha > 0$, $\vartheta_q > 0$ and $\gamma_{q,s} \in (0,1)$ are defined in Section 2. Then there exists $\varepsilon_0 > 0$ such that, for each $\delta > 0$ and all $0 < \varepsilon < \varepsilon_0$, equation (1.1) *admits at least* $\operatorname{cat}_{M_\delta}(M)$ *distinct solutions (u_i, λ_i) with $\|u_i\|_2^2 = a$. Moreover, each u_i is a normalized solution whose mass concentrates near M as $\varepsilon \to 0$.*

The rest of the paper is organized as follows. In Section 2 we collect the preliminary material. Section 3 is devoted to the analysis of the autonomous case, where we study the truncated functional and the associated constrained Palais–Smale sequences. In Section 4 we pass to the non-autonomous problem. Finally, in Section 5 we apply Lusternik–Schnirelmann category theory to obtain the multiplicity result stated in Theorem 1.2.

Notation. Throughout the paper we use the following notation.

- $B_R(x)$ denotes the open ball with radius $R > 0$ centered at $x \in \mathbb{R}^N$.
- The symbols C and C_i ($i \in \mathbb{N}^+$) denote positive constants whose value may change from line to line.
- The arrows "$\to$" and "$\rightharpoonup$" stand for strong convergence and weak convergence, respectively.
- $o_n(1)$ denotes a quantity that tends to 0 as $n \to \infty$.

- For $r \geq 1$, $\|u\|_r = \left(\int_{\mathbb{R}^N} |u|^r\,dx\right)^{1/r}$ is the norm of u in $L^r(\mathbb{R}^N)$.
- $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|$ is the norm of u in $L^\infty(\mathbb{R}^N)$.
- $2_s^* := \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent.

2 Preliminaries

In this section we collect the functional setting and several auxiliary inequalities that will be used throughout the paper. For any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ can be characterized in two equivalent ways:

$$\begin{aligned} H^s(\mathbb{R}^N) &= \Big\{u \in L^2(\mathbb{R}^N) :\ \frac{u(x)-u(y)}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\Big\} \\ &= \Big\{u \in L^2(\mathbb{R}^N) :\ \int_{\mathbb{R}^N} (1+|\xi|^{2s})|\mathcal{F}(u)(\xi)|^2\,d\xi < \infty\Big\}, \end{aligned}$$

where $\mathcal{F}(u)$ denotes the Fourier transform of u.

The Gagliardo seminorm of u is given by

$$[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,$$

and a standard computation using [4, Propositions 3.4 and 3.6] yields, for $u \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u\right|^2 dx = \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}(u)(\xi)|^2\,d\xi = \frac{1}{2}C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,$$

where $C(N,s) > 0$ is the normalizing constant in the definition of $(-\Delta)^s$.

Throughout the paper we endow $H^s(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^2\,dx + \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u\right|^2 dx\right)^{\frac{1}{2}},$$

and with the associated inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u\,(-\Delta)^{s/2}v\,dx + \int_{\mathbb{R}^N} uv\,dx.$$

We also make use of the homogeneous fractional Sobolev space

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \Big\{u \in L^{2_s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy < \infty\Big\},$$

equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u\right|^2 dx\right)^{\frac{1}{2}},$$

where $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent.

We also recall a sharp interpolation inequality that controls the Hartree energy by the H^s-norm and the L^2-norm.

Lemma 2.1. *[6] Let $N > 2s$, $0 < s < 1$ and $(N-4s)^+ < \alpha < N$. Let t satisfy*

$$\frac{N+\alpha}{N} < t < 2^*_{\alpha,s}, \qquad 2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}.$$

Then there exists $C_{\alpha,t} > 0$ such that, for all $u \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^t)|u|^t\,dx \le C_{\alpha,t}\,\|u\|^{2t\gamma_{t,s}}\|u\|_2^{2t(1-\gamma_{t,s})},$$

where

$$\gamma_{t,s} = \frac{Nt-N-\alpha}{2st} \in (0,1),$$

and the optimal constant $C_{\alpha,t}$ is given by

$$C_{\alpha,t} = \frac{2st}{2st-Nt+N+\alpha}\Big(\frac{2st-Nt+N+\alpha}{Nt-N-\alpha}\Big)^{\frac{Nt-N-\alpha}{2s}}\|U\|_2^{2-2t},$$

where U is a ground state solution of

$$(-\Delta)^s U + U - \big(I_\alpha * |U|^t\big)|U|^{t-2}U = 0 \quad \text{in } \mathbb{R}^N.$$

At the HLS upper critical exponent $2^*_{\alpha,s}$, the optimal constant can be expressed in terms of the best constant in the critical Choquard inequality.

Lemma 2.2. *[7] Let $N > 2s$, $0 < s < 1$ and $(N-4s)^+ < \alpha < N$. Define*

$$\mathcal{S}_\alpha = \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2}u\big|^2\,dx}{\Big(\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\,dx\Big)^{\frac{1}{2^*_{\alpha,s}}}}.$$

Then $\mathcal{S}_\alpha$ is achieved if and only if u is of the form

$$u(x) = C\Big(\frac{\varepsilon}{\varepsilon^2+|x-x_0|^2}\Big)^{\frac{N-2s}{2}}, \qquad x \in \mathbb{R}^N,$$

for some $x_0 \in \mathbb{R}^N$, $C > 0$ and $\varepsilon > 0$.

Remark 2.3. *If we take $\gamma_{2^*_{\alpha,s},s} = 1$ and*

$$C_{\alpha,2^*_{\alpha,s}} = \big(\mathcal{S}_\alpha^{-1}\big)^{2^*_{\alpha,s}},$$

then, in view of Lemma 2.2, Lemma 2.1 remains valid for all

$$t \in \Big(\frac{N+\alpha}{N},\, 2^*_{\alpha,s}\Big].$$

It is convenient to summarize here the exponents and constants that will appear in the small mass condition of Theorem 1.2. We recall that the L^2-critical Hartree exponent is

$$\bar{p} = \frac{N+2s+\alpha}{N}, \qquad 2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}, \qquad \gamma_{t,s} = \frac{Nt-N-\alpha}{2st}, \quad t > 1.$$

For endpoint bookkeeping we also introduce

$$\sigma = \frac{N+\alpha}{2(\alpha+2s)}.$$

When dealing with the mixed nonlinearity at exponent $q \in (\bar{p}, 2^*_{\alpha,s})$, we set

$$K_q = \frac{(2 - 2q\gamma_{q,s})\, C_{\alpha,q}\, 2^*_{\alpha,s}}{q\big(2\, 2^*_{\alpha,s} - 2\big)}, \qquad \vartheta_q = 2\sigma\big(2^*_{\alpha,s} - q\,\gamma_{q,s}\big) - 2^*_{\alpha,s}.$$

In these terms, the smallness condition on the prescribed mass $a > 0$ appearing in Theorem 1.2 reads

$$a \leq \Big(K_q^{-1}\, S_\alpha^{\vartheta_q}\Big)^{\frac{1}{q(1-\gamma_{q,s})}}.$$

The rôle of this condition will be clarified in the variational analysis of Section 3.

We conclude this section by recalling two compactness results that are crucial for handling the nonlocal nonlinearities and the lack of compactness in $\mathbb{R}^N$.

Lemma 2.4. *[25] Let $N > 2s$, $0 < s < 1$, and let (u_n) be a sequence bounded in $H^s(\mathbb{R}^N)$ such that for some $R > 0$,*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2\, dx = 0. \tag{2.1}$$

Then, for every $r \in (2, 2^*_s)$ with $2^*_s = \frac{2N}{N-2s}$, we have

$$\|u_n\|_{L^r(\mathbb{R}^N)} \;\to\; 0 \qquad \text{as } n \to \infty.$$

Lemma 2.5. *[19] Let $N > 2s$, $(N - 4s)^+ < \alpha < N$, and let $p = 2^*_{\alpha,s}$ with $s \in (0, 1)$. If (u_n) is a sequence bounded in $L^{2^*_s}(\mathbb{R}^N)$ and $u_n \to u$ almost everywhere in $\mathbb{R}^N$, then*

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)\, |u_n|^p\, dx = \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^p)\, |u_n - u|^p\, dx + \int_{\mathbb{R}^N} (I_\alpha * |u|^p)\, |u|^p\, dx + o(1).$$

These lemmas will be repeatedly used in the analysis of Palais–Smale sequences for the constrained functionals associated with problem (1.1).

3 The Autonomous Case

In this section, we consider the autonomous problem

$$(-\Delta)^s u + \mu u = \lambda u + \big(I_\alpha * |u|^p\big)|u|^{p-2}u + \big(I_\alpha * |u|^q\big)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

with $\mu \in [0, V_\infty]$ and mixed L^2-supercritical exponents

$$\frac{N + 2s + \alpha}{N} < q < p = 2^*_{\alpha,s} = \frac{N + \alpha}{N - 2s}.$$

We analyze the constrained critical points of the autonomous energy functional

$$J_\mu(u) = \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{\mu}{2}\, a - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q,$$

on the mass sphere

$$S(a) = \{u \in H^s(\mathbb{R}^N) :\ \|u\|_2^2 = a\}.$$

We adapt to our setting the truncation method of [13, 22], freezing only the critical term outside a fixed ball of H^s and is used to prevent escape to infinity along the constraint.

Definition 3.1. *Let $0 < R_0 < R_1$ and fix $\tau \in C^\infty([0,\infty);[0,1])$ with $\tau \equiv 1$ on $[0,R_0]$, $\tau \equiv 0$ on $[R_1,\infty)$, and nonincreasing on $[0,\infty)$. Define*

$$J_{\varepsilon,T}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 dx$$
$$- \frac{\tau(\|u\|_{H^s})}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx - \frac{1}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx,$$

and, in the autonomous case $V \equiv \mu \geq 0$,

$$J_{\mu,T}(u) = \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{\mu}{2}\|u\|_2^2 - \frac{\tau(\|u\|_{H^s})}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx - \frac{1}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx.$$

Throughout, we set

$$\|u\|_{H^s}^2 = \|(-\Delta)^{s/2}u\|_2^2 + \|u\|_2^2.$$

Lemma 3.2. *Assume $N > 2s$, $0 < s < 1$, $(N-4s)^+ < \alpha < N$, and*

$$\frac{N+2s+\alpha}{N} < q < p = 2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}.$$

Fix $a > 0$ and $\mu \in [0, V_\infty]$. Then there exist $u_0, u_1 \in S(a)$ such that

$$J_\mu(u_0) > 0 \qquad \text{and} \qquad J_\mu(u_1) < 0.$$

Consequently, with

$$\Gamma = \big\{\gamma \in C([0,1], S(a)) :\ \gamma(0) = u_0,\ \gamma(1) = u_1\big\},$$

the mountain-pass value

$$b_{\mu,a} = \inf_{\gamma\in\Gamma}\ \max_{t\in[0,1]} J_\mu\big(\gamma(t)\big)$$

is well defined and satisfies $b_{\mu,a} \geq J_\mu(u_0) > 0$.

Proof. Pick any nonzero $w \in S(a)$. For $t > 0$ consider the L^2-mass preserving scaling

$$w_t(x) = t^{\frac{N}{2}} w(tx) \in S(a).$$

Set

$$A = \big\|(-\Delta)^{s/2}w\big\|_2^2, \qquad B_p = \int_{\mathbb{R}^N}(I_\alpha * |w|^p)|w|^p\,dx, \qquad B_q = \int_{\mathbb{R}^N}(I_\alpha * |w|^q)|w|^q\,dx,$$

and write the homogeneity exponents

$$P = Np - (N+\alpha), \qquad Q = Nq - (N+\alpha).$$

Then

$$J_\mu(w_t) = \frac{1}{2}\,t^{2s}A + \frac{\mu}{2}a - \frac{1}{2p}\,t^P B_p - \frac{1}{2q}\,t^Q B_q.$$

Since $q > \frac{N+2s+\alpha}{N}$, we have $Q > 2s$; with $p = 2^*_{\alpha,s}$ one also has

$$P = Np - (N+\alpha) = \frac{N(N+\alpha)}{N-2s} - (N+\alpha) = \frac{2s\,(N+\alpha)}{N-2s} > 2s,$$

and clearly $P > Q$. Hence, as $t \to 0^+$,

$$J_\mu(w_t) = \frac{\mu}{2}a + \frac{1}{2}\,t^{2s}A + o(t^{2s}) > 0,$$

so choosing $t_0 > 0$ sufficiently small and setting $u_0 = w_{t_0}$ gives $J_\mu(u_0) > 0$.

As $t \to +\infty$, the negative terms dominate because $P > Q > 2s$, whence $J_\mu(w_t) \to -\infty$. Therefore there exists $t_1 \gg 1$ with $J_\mu(w_{t_1}) < 0$; set $u_1 = w_{t_1}$. The path

$$\gamma : [0, 1] \to S(a), \qquad \gamma(\theta) = w_{(1-\theta)t_0+\theta t_1},$$

is continuous and connects u_0 to u_1. For any $\gamma \in \Gamma$, $\max_{[0,1]} J_\mu(\gamma) \geq J_\mu(u_0) > 0$, so

$$b_{\mu,a} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\mu(\gamma(t)) \;\geq\; J_\mu(u_0) \;>\; 0.$$

□

Remark 3.3. *Let $J_{\mu,T}$ be the truncated companion where only the critical term is truncated:*

$$J_{\mu,T}(u) = \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{\mu}{2}\, a - \frac{\tau(\|u\|_{H^s})}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx - \frac{1}{2q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \, dx.$$

Choose $R_0 < R_1$ with

$$R_0 > \sup_{t\in[t_0,t_1]} \|w_t\|_{H^s}$$

and any $R_1 > R_0$. Then

$$J_{\mu,T}(u_0) = J_\mu(u_0) > 0, \qquad J_{\mu,T}(u_1) = J_\mu(u_1) < 0,$$

and for all $u \in S(a)$ one has $J_{\mu,T}(u) \geq J_\mu(u)$. Consequently, with the same path class

$$\Gamma = \big\{\gamma \in C([0, 1], S(a)) : \; \gamma(0) = u_0, \; \gamma(1) = u_1\big\},$$

the min–max value

$$b_{\mu,T,a} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\mu,T}\big(\gamma(t)\big)$$

is well defined and satisfies $b_{\mu,T,a} \geq b_{\mu,a} > 0$.

For $u \in S(a)$ and $t > 0$ set

$$u_t(x) = t^{\frac{N}{2}} u(tx) \in S(a).$$

Define the autonomous fibering curve

$$\begin{aligned}\phi_{\mu,u}(t) = J_\mu(u_t) &= \frac{1}{2}\, t^{2s}\, \|(-\Delta)^{s/2}u\|_2^2 + \frac{\mu}{2}\, a \\ &\quad - \frac{t^{Np-(N+\alpha)}}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx - \frac{t^{Nq-(N+\alpha)}}{2q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \, dx.\end{aligned}$$

The next lemma shows that if $u \in S(a)$ is a constrained critical point of J_μ, then the fibering derivative vanishes at $t = 1$, i.e. $\phi'_{\mu,u}(1) = 0$, yielding a Pohozaev-type identity. Here and below $p = 2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}$, so $\frac{Np-(N+\alpha)}{2p} = s$.

Lemma 3.4. *Assume $N > 2s$, $0 < s < 1$, $(N - 4s)^+ < \alpha < N$, and q satisfies $\frac{N+2s+\alpha}{N} < q < p$ with $p = 2^*_{\alpha,s}$. Then:*

(i) If $u \in S(a)$ is a constrained critical point of J_μ, then $\phi'_{\mu,u}(1) = 0$, equivalently

$$\begin{aligned}\mathcal{P}(u) = 2s\, \|(-\Delta)^{s/2}u\|_2^2 &- \frac{Np - (N+\alpha)}{p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx \\ &- \frac{Nq - (N+\alpha)}{q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \, dx = 0.\end{aligned} \tag{3.2}$$

(ii) Conversely, for any fixed $u \in S(a)$, every critical point $t_* > 0$ of $\phi_{\mu,u}$ satisfies $\mathcal{P}(u_{t_*}) = 0$.

Proof. The mass-preserving scaling yields

$$\|(-\Delta)^{s/2}u_t\|_2^2 = t^{2s}\|(-\Delta)^{s/2}u\|_2^2, \qquad \int_{\mathbb{R}^N}(I_\alpha * |u_t|^r)|u_t|^r\,dx = t^{Nr-(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u|^r)|u|^r\,dx.$$

Hence

$$\begin{aligned}\phi'_{\mu,u}(t) = s\,t^{2s-1}\|(-\Delta)^{s/2}u\|_2^2 - \frac{Np-(N+\alpha)}{2p}\,t^{Np-(N+\alpha)-1}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx \\ - \frac{Nq-(N+\alpha)}{2q}\,t^{Nq-(N+\alpha)-1}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx.\end{aligned}$$

Evaluating at $t = 1$ gives

$$\phi'_{\mu,u}(1) = s\,\|(-\Delta)^{s/2}u\|_2^2 - \frac{P}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx - \frac{Q}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx,$$

where $P = Np - (N+\alpha)$ and $Q = Nq - (N+\alpha)$. Thus $\phi'_{\mu,u}(1) = 0$ is equivalent to (3.2).

To connect constrained criticality with $\phi'_{\mu,u}(1)$, note that $t \mapsto u_t \in S(a)$, so

$$0 = \frac{d}{dt}\|u_t\|_2^2\Big|_{t=1} = 2\int_{\mathbb{R}^N} u\,\psi\,dx, \qquad \psi = \frac{d}{dt}u_t\Big|_{t=1} \in T_uS(a).$$

If u is a constrained critical point, there exists $\lambda \in \mathbb{R}$ such that

$$J'_\mu(u)[v] = \lambda\int_{\mathbb{R}^N} u\,v\,dx \qquad \forall\, v \in H^s(\mathbb{R}^N).$$

Choosing $v = \psi$ and using $\int u\psi\,dx = 0$ gives $J'_\mu(u)[\psi] = 0$. By the chain rule,

$$\phi'_{\mu,u}(1) = \frac{d}{dt}J_\mu(u_t)\Big|_{t=1} = J'_\mu(u)\Big[\frac{d}{dt}u_t\Big|_{t=1}\Big] = J'_\mu(u)[\psi] = 0,$$

hence (3.2). For (ii), apply the same computation to u_{t_*} and evaluate at $t = 1$ to conclude $\mathcal{P}(u_{t_*}) = 0$. □

As a consequence, the min–max level can be characterized by maximizing along rays and then minimizing over directions.

Lemma 3.5. *Let $\mu \geq 0$ and*

$$\mathcal{R}(u) = \max_{t>0} J_{\mu,T}(u_t), \qquad u_t(x) = t^{\frac{N}{2}}u(tx), \quad u \in S(a).$$

Then

$$b_{\mu,T,a} = \inf_{u\in S(a)} \mathcal{R}(u). \tag{3.3}$$

Moreover, for every fixed $u \in S(a)$ there exists $t(u) > 0$ such that $\mathcal{R}(u) = J_{\mu,T}(u_{t(u)})$ and

$$\frac{d}{dt}\,J_{\mu,T}(u_t)\Big|_{t=t(u)} = 0. \tag{3.4}$$

If, in addition, $\|u_{t(u)}\|_{H^s} \leq R_0$, then $t(u)$ is unique and is characterized by the scaling identity

$$\mathcal{P}(u_{t(u)}) = 0. \tag{3.5}$$

In the general truncated case, (3.4) *is equivalent to* $\mathcal{P}_T(u_{t(u)}) = 0$, *where*

$$\mathcal{P}_T(u_t) = 2s\,\|(-\Delta)^{s/2}u\|_2^2 - \frac{Np-(N+\alpha)}{p}\,\tau(\|u_t\|_{H^s})\,t^{Np-(N+\alpha)-2s}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx \quad (3.6)$$
$$- \frac{Nq-(N+\alpha)}{q}\,t^{Nq-(N+\alpha)-2s}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx$$
$$- \frac{s\,\tau'(\|u_t\|_{H^s})}{\|u_t\|_{H^s}}\,\|(-\Delta)^{s/2}u\|_2^2\left[\frac{t^{Np-(N+\alpha)}}{p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx\right],$$

and one has the exact identity

$$\frac{d}{dt}\,J_{\mu,T}(u_t) = \frac{t^{2s-1}}{2}\,\mathcal{P}_T(u_t) \qquad (t>0). \tag{3.7}$$

Proof. Fix $u \in S(a)$ and set $A = \|(-\Delta)^{s/2}u\|_2^2 > 0$,

$$B_p = \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx, \qquad B_q = \int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx, \qquad \delta_r = Nr-(N+\alpha) \quad (r\in\{p,q\}).$$

By the scaling,

$$\|(-\Delta)^{s/2}u_t\|_2^2 = t^{2s}A, \qquad \int_{\mathbb{R}^N}(I_\alpha * |u_t|^r)|u_t|^r\,dx = t^{\delta_r}B_r, \quad r\in\{p,q\},$$

and, writing $R(t) = \|u_t\|_{H^s} = \left(t^{2s}A + a\right)^{1/2}$,

$$\phi_{\mu,T;u}(t) = J_{\mu,T}(u_t) = \frac{1}{2}t^{2s}A + \frac{\mu}{2}a - \frac{\tau(R(t))}{2p}\,t^{\delta_p}B_p - \frac{1}{2q}\,t^{\delta_q}B_q.$$

Since $q,p > \frac{N+2s+\alpha}{N}$, we have $\delta_q, \delta_p > 2s$. Hence as $t\to 0$,

$$\phi_{\mu,T;u}(t) = \frac{\mu}{2}a + \frac{1}{2}t^{2s}A + o(t^{2s}) > \frac{\mu}{2}a.$$

On the other hand, when $R(t) \ge R_1$ we have $\tau(R(t)) = 0$, so

$$\phi_{\mu,T;u}(t) = \frac{1}{2}t^{2s}A + \frac{\mu}{2}a - \frac{1}{2q}t^{\delta_q}B_q \;\to\; -\infty \qquad (t\to+\infty).$$

Therefore, by continuity of $\phi_{\mu,T;u}$ and the above asymptotics, there exists $t(u) > 0$ such that

$$\mathcal{R}(u) = \max_{t>0} J_{\mu,T}(u_t) = J_{\mu,T}(u_{t(u)}).$$

If $t(u)$ is an interior maximizer, then

$$\frac{d}{dt}J_{\mu,T}(u_t)\Big|_{t=t(u)} = 0,$$

which is (3.4). For arbitrary $t>0$, using $R'(t) = \dfrac{sAt^{2s-1}}{R(t)}$ and differentiating term by term, we obtain

$$\phi'_{\mu,T;u}(t) = st^{2s-1}A - \frac{1}{2p}\Big[\tau'(R(t))R'(t)\,t^{\delta_p}B_p + \tau(R(t))\,\delta_p\,t^{\delta_p-1}B_p\Big] - \frac{1}{2q}\,\delta_q\,t^{\delta_q-1}B_q$$
$$= \frac{t^{2s-1}}{2}\left(2sA - \frac{\delta_p}{p}\,\tau(R(t))\,t^{\delta_p-2s}B_p - \frac{\delta_q}{q}\,t^{\delta_q-2s}B_q - \frac{s\,\tau'(R(t))}{R(t)}\,A\Big[\frac{t^{\delta_p}}{p}B_p\Big]\right),$$

which is exactly (3.7), with $\mathcal{P}_T$ given by (3.6). When $\|u_{t(u)}\|_{H^s} \leq R_0$ so that $\tau \equiv 1$,

$$\phi'_{\mu,T;u}(t) = st^{2s-1}A - \frac{\delta_p}{2p} t^{\delta_p - 1} B_p - \frac{\delta_q}{2q} t^{\delta_q - 1} B_q.$$

Set

$$\Psi(t) = \frac{2}{t^{2s-1}} \phi'_{\mu,T;u}(t) = 2sA - \frac{\delta_p}{p}\, t^{\delta_p - 2s} B_p - \frac{\delta_q}{q}\, t^{\delta_q - 2s} B_q.$$

Since $\delta_q, \delta_p > 2s$, we have

$$\Psi'(t) = -\frac{\delta_p}{p}(\delta_p - 2s) t^{\delta_p - 2s - 1} B_p - \frac{\delta_q}{q}(\delta_q - 2s) t^{\delta_q - 2s-1} B_q < 0 \quad \text{for all } t > 0.$$

Thus Ψ is strictly decreasing and admits a unique zero, which yields the unique maximizer and $\mathcal{P}(u_{t(u)}) = 0$; this proves (3.5) and uniqueness.

We now prove (3.3). Let

$$\Gamma = \Big\{ \gamma \in C([0,1], S(a)) : \ J_{\mu,T}(\gamma(0)) > 0, \quad J_{\mu,T}(\gamma(1)) < 0 \Big\}, \qquad b_{\mu,T,a} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{\mu,T}(\gamma(t)).$$

By Lemma 3.2 and Remark 3.3, we can choose the truncation so that there exist $w \in S(a)$ and $0 < t_0 < t_1$ with $\tau \equiv 1$ on $[t_0, t_1]$ and

$$J_{\mu,T}(w_{t_0}) > 0 > J_{\mu,T}(w_{t_1}).$$

Then the fiber path $\gamma(\theta) = w_{(1-\theta)t_0 + \theta t_1}$ belongs to Γ. Since

$$\mathcal{R}(w) = \max_{t>0} J_{\mu,T}(w_t) \ \geq \ \max_{\theta \in [0,1]} J_{\mu,T}(\gamma(\theta)),$$

we obtain

$$\mathcal{R}(w) \ \geq \ \inf_{\gamma \in \Gamma} \ \max_{\theta \in [0,1]} J_{\mu,T}(\gamma(\theta)) \ = \ b_{\mu,T,a}.$$

Therefore $\inf_{u \in S(a)} \mathcal{R}(u) \ \geq \ b_{\mu,T,a}$.

Conversely, let $\gamma \in \Gamma$. For each $\theta \in [0,1]$, by definition

$$\mathcal{R}(\gamma(\theta)) = \sup_{t>0} J_{\mu,T}(\gamma(\theta)_t) \ \geq \ J_{\mu,T}(\gamma(\theta)).$$

Hence

$$\max_{\theta \in [0,1]} J_{\mu,T}(\gamma(\theta)) \ \leq \ \max_{\theta \in [0,1]} \mathcal{R}(\gamma(\theta)).$$

Taking the infimum over $\gamma \in \Gamma$ yields

$$b_{\mu,T,a} \ \leq \ \inf_{\gamma \in \Gamma} \ \max_{\theta \in [0,1]} \mathcal{R}(\gamma(\theta)) \ \leq \ \inf_{u \in S(a)} \mathcal{R}(u).$$

Combining the two inequalities gives (3.3). The proof is complete. $\square$

Lemma 3.6. *For any $\mu \geq 0$,*

$$b_{\mu,T,a} = b_{0,T,a} + \frac{\mu a}{2}, \qquad b_{\infty,T,a} - b_{0,T,a} = \frac{V_\infty a}{2} > 0.$$

Here $b_{\infty,T,a} = b_{\mu,T,a}\big|_{\mu = V_\infty}$. In particular, $\mu \mapsto b_{\mu,T,a}$ is strictly increasing and affine on $[0, \infty)$.

Proof. Fix $u \in S(a)$ and $t > 0$. By definition of $J_{\mu,T}$ and the fact that $\|u_t\|_2^2 = a$,

$$J_{\mu,T}(u_t) = J_{0,T}(u_t) + \frac{\mu a}{2}.$$

Taking the maximum over $t > 0$ and then the infimum over $u \in S(a)$, Lemma 3.5 yields

$$\begin{aligned} b_{\mu,T,a} &= \inf_{u\in S(a)} \max_{t>0} J_{\mu,T}(u_t) = \inf_{u\in S(a)} \max_{t>0} \Big(J_{0,T}(u_t) + \frac{\mu a}{2} \Big) \\ &= \Big(\inf_{u\in S(a)} \max_{t>0} J_{0,T}(u_t) \Big) + \frac{\mu a}{2} = b_{0,T,a} + \frac{\mu a}{2}. \end{aligned}$$

Specializing to $\mu = V_\infty$ gives

$$b_{\infty,T,a} - b_{0,T,a} = \frac{V_\infty a}{2},$$

which is strictly positive because $V_\infty \in (0,\infty)$ and $a > 0$. The affine (hence strictly increasing) dependence on μ follows from the first identity. $\square$

Lemma 3.7. *For the autonomous problem, the map $\rho \mapsto b_{0,\rho}$ is nonincreasing on $(0,\infty)$. Consequently, for the truncated level one also has $\rho \mapsto b_{0,T,\rho}$ nonincreasing and*

$$0 < m \le a \Rightarrow b_{0,T,m} \ \ge \ b_{0,T,a}.$$

Proof. Fix $u \in S(1)$ and, for $\sigma > 0$, set

$$\phi_\sigma(t) = J_0\big((\sqrt{\sigma}\, u)_t\big) = \frac{1}{2}\, \sigma\, t^{2s} A - \frac{1}{2p}\, \sigma^p t^{\delta_p} B_p - \frac{1}{2q}\, \sigma^q t^{\delta_q} B_q,$$

with $A = \|(-\Delta)^{s/2} u\|_2^2$, $B_r = \int_{\mathbb{R}^N} (I_\alpha * |u|^r)\, |u|^r\, dx$, and $\delta_r = Nr - (N+\alpha)$.

Define $F_u(\sigma) = \max_{t>0} \phi_\sigma(t)$ and let t_σ be a maximizer. Using $\partial_t \phi_\sigma(t_\sigma) = 0$, we obtain

$$F_u'(\sigma) = \partial_\sigma \phi_\sigma(t_\sigma) = \frac{1}{2}\, t_\sigma^{2s} A - \frac{1}{2}\, \sigma^{p-1} t_\sigma^{\delta_p} B_p - \frac{1}{2}\, \sigma^{q-1} t_\sigma^{\delta_q} B_q.$$

The stationarity condition $\partial_t \phi_\sigma(t_\sigma) = 0$ reads

$$s\, \sigma\, t_\sigma^{2s-1} A - \frac{\delta_p}{2p}\, \sigma^p t_\sigma^{\delta_p - 1} B_p - \frac{\delta_q}{2q}\, \sigma^q t_\sigma^{\delta_q - 1} B_q = 0,$$

which implies

$$\frac{1}{2}\, t_\sigma^{2s} A = \sum_{r\in\{p,q\}} \frac{\delta_r}{4sr}\, \sigma^{\,r-1} t_\sigma^{\delta_r} B_r.$$

Substituting into $F_u'(\sigma)$ gives

$$F_u'(\sigma) = \sum_{r\in\{p,q\}} \frac{\delta_r - 2sr}{4sr}\, \sigma^{\,r-1} t_\sigma^{\delta_r} B_r.$$

Because $p = 2^*_{\alpha,s}$, we have $\delta_p - 2sp = (N-2s)p - (N+\alpha) = 0$, while $\delta_q - 2sq = (N-2s)q - (N+\alpha) < 0$. Therefore

$$F_u'(\sigma) = \frac{\delta_q - 2sq}{4sq}\, \sigma^{\,q-1} t_\sigma^{\delta_q} B_q \ < \ 0,$$

so F_u is strictly decreasing in σ.

For any $\rho_2 > \rho_1 > 0$, take $\sigma = \rho_2/\rho_1 > 1$. For every direction $u \in S(1)$, $F_u(\rho_2) < F_u(\rho_1)$; taking the infimum over all directions yields

$$b_{0,\rho_2} = \inf_{v\in S(\rho_2)} \max_{t>0} J_0(v_t) = \inf_{u\in S(1)} F_u(\rho_2) \ \le \ \inf_{u\in S(1)} F_u(\rho_1) = b_{0,\rho_1}.$$

Hence $\rho \mapsto b_{0,\rho}$ is nonincreasing.

Finally, if the truncation radii $R_0 < R_1$ are chosen so that the minimax level is attained where $\tau \equiv 1$, then $b_{0,T,\rho} = b_{0,\rho}$ and the same monotonicity follows. In particular, for $0 < m \leq a$,

$$b_{0,T,m} = b_{0,m} \;\geq\; b_{0,a} = b_{0,T,a}.$$

$\square$

Lemma 3.8. *Let $N > 2s$, $0 < s < 1$, $(N-4s)^+ < \alpha < N$, and let*

$$\frac{N+2s+\alpha}{N} < q < p.$$

Suppose (u_n) is bounded in $H^s(\mathbb{R}^N)$ and vanishes. Then

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)\, |u_n|^q \, dx \to 0 \quad \textit{and} \quad \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)\, |u_n|^p \, dx \to 0.$$

Proof. By Lemma 2.4, for every $r \in (2, 2_s^*)$,

$$\|u_n\|_{L^r(\mathbb{R}^N)} \;\to\; 0 \qquad (n \to \infty). \tag{3.8}$$

Set $r_q = q \cdot \frac{2N}{N+\alpha} < 2_s^*$. Using the Hardy–Littlewood–Sobolev mapping

$$I_\alpha :\; L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \;\to\; L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N),$$

and Hölder inequality with conjugate exponents $\frac{2N}{N-\alpha}$ and $\frac{2N}{N+\alpha}$, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)\, |u_n|^q \;\leq\; C\, \|u_n\|_{L^{r_q}}^{2q} \;\xrightarrow[n\to\infty]{}\; 0,$$

by (3.8) since $r_q < 2_s^*$.

The endpoint $p = 2_{\alpha,s}^*$ cannot be concluded from (3.8) alone. Define the nonnegative densities

$$\mu_n(x) = |u_n(x)|^p.$$

Since (u_n) is bounded in $H^s(\mathbb{R}^N)$, the Sobolev embedding yields

$$\|\mu_n\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = \|u_n\|^p_{L^{p\cdot\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = \|u_n\|^p_{L^{2_s^*}(\mathbb{R}^N)} \;\leq\; C,$$

so (μ_n) is uniformly bounded in $L^{\frac{2N}{N+\alpha}}$.

We claim that (μ_n) vanishes locally, i.e.

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \mu_n \, dx = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^p \, dx = 0 \qquad \text{for every fixed } R > 0. \tag{3.9}$$

Indeed, fix $R > 0$ and choose $\theta \in (0,1)$ by

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{2_s^*} \quad \Longleftrightarrow \quad \theta = \frac{N}{2s}\Big(1 - \frac{2}{p}\Big) \in (0,1),$$

since $2 < p < 2_s^*$. Interpolation between L^2 and $L^{2_s^*}$ gives, for any $y \in \mathbb{R}^N$,

$$\|u_n\|_{L^p(B_R(y))} \;\leq\; C\, \|u_n\|^{1-\theta}_{L^2(B_R(y))}\, \|u_n\|^{\theta}_{L^{2_s^*}(\mathbb{R}^N)} \;\leq\; C\, \|u_n\|^{1-\theta}_{L^2(B_R(y))},$$

using the uniform $L^{2^*_s}$-bound. Taking the supremum in y and using vanishing condition (2.1) yields

$$\sup_{y\in\mathbb{R}^N} \|u_n\|_{L^p(B_R(y))} \;\to\; 0,$$

which implies (3.9).

Now use the bilinear form of the Choquard energy,

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)\,|u_n|^p\,dx = C_{N,\alpha} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\mu_n(x)\,\mu_n(y)}{|x-y|^{N-\alpha}}\,dx\,dy.$$

We estimate the Choquard energy

$$\mathcal{I}_n = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\mu_n(x)\,\mu_n(y)}{|x-y|^{\,N-\alpha}}\,dx\,dy$$

by splitting the interaction into near and far ranges:

$$\mathcal{I}_n = \iint_{|x-y|\le R} \frac{\mu_n(x)\mu_n(y)}{|x-y|^{N-\alpha}}\,dx\,dy + \iint_{|x-y|>R} \frac{\mu_n(x)\mu_n(y)}{|x-y|^{N-\alpha}}\,dx\,dy =: \mathcal{I}_n^{\text{near}}(R) + \mathcal{I}_n^{\text{far}}(R).$$

For $|x-y|>R$, $|x-y|^{-(N-\alpha)} \le R^{-(N-\alpha)}$, hence

$$\mathcal{I}_n^{\text{far}}(R) \;\le\; R^{-(N-\alpha)}\Big(\int_{\mathbb{R}^N} \mu_n\Big)^2.$$

Using global interpolation between L^2 and $L^{2^*_s}$, $\|u_n\|_{L^p} \le \|u_n\|_{L^2}^{1-\theta}\|u_n\|_{L^{2^*_s}}^{\theta}$ for some $\theta\in(0,1)$, we get $\sup_n \int \mu_n = \sup_n \|u_n\|_{L^p}^p < \infty$. Thus, for any $\varepsilon>0$, choose $R=R_\varepsilon$ large so that

$$\sup_n \mathcal{I}_n^{\text{far}}(R_\varepsilon) \;\le\; \frac{\varepsilon}{2}.$$

For $|x-y|\le R$, by Fubini theorem and translation invariance,

$$\begin{aligned}
\mathcal{I}_n^{\text{near}}(R) &= \int_{\mathbb{R}^N} \mu_n(x)\bigg(\int_{B_R(x)} \frac{\mu_n(y)}{|x-y|^{N-\alpha}}\,dy\bigg)\,dx \\
&\le \bigg(\int_{B_R(0)} \frac{dz}{|z|^{N-\alpha}}\bigg)\bigg(\int_{\mathbb{R}^N} \mu_n(x)\,dx\bigg) \sup_{x\in\mathbb{R}^N} \int_{B_R(x)} \mu_n(y)\,dy.
\end{aligned}$$

Since $\int_{B_R(0)} |z|^{-(N-\alpha)}dz = C_N R^\alpha$ and $\sup_n \int \mu_n < \infty$, we obtain

$$\mathcal{I}_n^{\text{near}}(R) \;\le\; C\,R^\alpha\Big(\sup_{x\in\mathbb{R}^N}\int_{B_R(x)} \mu_n\Big).$$

Fix the above $R=R_\varepsilon$. By local vanishing of (μ_n), there exists n_ε such that for all $n\ge n_\varepsilon$, $\sup_x \int_{B_{R_\varepsilon}(x)} \mu_n \le \varepsilon/(2CR_\varepsilon^\alpha)$, hence

$$\mathcal{I}_n^{\text{near}}(R_\varepsilon) \;\le\; \frac{\varepsilon}{2} \qquad (n\ge n_\varepsilon).$$

Combining the two ranges with this choice of R_ε gives

$$\mathcal{I}_n \;\le\; \mathcal{I}_n^{\text{near}}(R_\varepsilon) + \mathcal{I}_n^{\text{far}}(R_\varepsilon) \;\le\; \varepsilon \qquad \text{for all } n\ge n_\varepsilon,$$

i.e.

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_n(x)|^p\,|u_n(y)|^p}{|x-y|^{\,N-\alpha}}\,dx\,dy \;\to\; 0.$$

Equivalently, $\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)\,|u_n|^p\,dx \to 0.$

□

Lemma 3.9. *Assume $N > 2s$, $0 < s < 1$, $(N-4s)^+ < \alpha < N$ and $\frac{N+2s+\alpha}{N} < q < p$. For $\mu \in [0, V_\infty]$, every $(PS)_{b_{\mu,T,a}}$ sequence of $J_{\mu,T}\big|_{S(a)}$ is bounded in $H^s(\mathbb{R}^N)$ and does not vanish in the sense of Lions.*

Proof. Let $\{u_n\} \subset S(a)$ be a $(PS)_{b_{\mu,T,a}}$ sequence, i.e.

$$J_{\mu,T}(u_n) = b_{\mu,T,a} + o(1), \qquad \|(J_{\mu,T}\big|_{S(a)})'(u_n)\|_{(T_{u_n}S(a))^*} = o(1).$$

Set

$$A_n = \|(-\Delta)^{s/2}u_n\|_2^2, \quad B_{p,n} = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \, dx, \quad B_{q,n} = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \, dx,$$

and $\delta_r = Nr - (N+\alpha)$ for $r \in \{p, q\}$. Since $p = 2^*_{\alpha,s}$, we have $\delta_p/p = 2s$.

For $t > 0$, define $u_{n,t}(x) = t^{\frac{N}{2}} u_n(tx)$ and $R_n(t) = \|u_{n,t}\|_{H^s} = (t^{2s}A_n + a)^{1/2}$. By Lemma 3.5,

$$\frac{d}{dt}\, J_{\mu,T}(u_{n,t}) = \frac{t^{2s-1}}{2}\, \mathcal{P}_T(u_{n,t}),$$

with

$$\mathcal{P}_T(u_{n,t}) = 2s\, A_n - \frac{\delta_p}{p}\, \tau(R_n(t))\, t^{\delta_p - 2s} B_{p,n} - \frac{\delta_q}{q}\, t^{\delta_q - 2s} B_{q,n} - \frac{2s\, \tau'(R_n(t))}{R_n(t)}\, A_n \Big[\frac{t^{\delta_p}}{p} B_{p,n}\Big].$$

Since $t \mapsto u_{n,t} \in S(a)$, its velocity at $t = 1$, $\psi_n = \frac{d}{dt} u_{n,t}\big|_{t=1}$, lies in $T_{u_n}S(a)$. By the (PS) condition,

$$0 = (J_{\mu,T}\big|_{S(a)})'(u_n)[\psi_n] = \frac{d}{dt} J_{\mu,T}(u_{n,t})\Big|_{t=1} = \frac{1}{2}\, \mathcal{P}_T(u_n) + o(1),$$

hence

$$\mathcal{P}_T(u_n) = 2s\, A_n - 2s\, \tau(\|u_n\|_{H^s})\, B_{p,n} - \frac{\delta_q}{q}\, B_{q,n} - \frac{2s\, \tau'(\|u_n\|_{H^s})}{\|u_n\|_{H^s}}\, A_n \Big[\frac{1}{p} B_{p,n}\Big] = o(1). \tag{3.10}$$

From the energy identity

$$\frac{1}{2}\, A_n + \frac{\mu}{2} a - \frac{\tau(\|u_n\|_{H^s})}{2p}\, B_{p,n} - \frac{1}{2q}\, B_{q,n} = b_{\mu,T,a} + o(1), \tag{3.11}$$

and Lemma 2.1, there exist constants $C_1, C_2 > 0$ (independent of n) such that

$$B_{p,n} \;\le\; C_1\, A_n^p, \qquad B_{q,n} \;\le\; C_2\, A_n^{\theta_q}\, a^{\eta_q},$$

with $\theta_q > 1$ and $\eta_q \ge 0$ determined by scaling. Hence, using $\tau \in [0,1]$,

$$b_{\mu,T,a} + o(1) = \frac{1}{2} A_n + \frac{\mu}{2} a - \frac{\tau}{2p} B_{p,n} - \frac{1}{2q} B_{q,n} \ge \frac{1}{2} A_n + \frac{\mu}{2} a - C\big(A_n^p + A_n^{\theta_q}\big).$$

If $A_n \to \infty$, the right-hand side tends to $-\infty$ (since $p > 1$ and $\theta_q > 1$), contradicting the boundedness of $J_{\mu,T}(u_n)$. Therefore $\{A_n\}$ is bounded, and so $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$.

Assume by contradiction that $\{u_n\}$ vanishes in the sense of Lions. By Lemma 2.4, the subcritical Choquard term vanishes:

$$B_{q,n} = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \, dx \;\to\; 0.$$

For the critical term, Lemma 3.8 gives

$$B_{p,n} = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \, dx \;\to\; 0.$$

Returning to (3.10), the last term is nonnegative, while the $B_{p,n}$ and $B_{q,n}$ terms tend to 0; hence

$$0 = \lim_{n\to\infty} \mathcal{P}_T(u_n) \ \geq\ \limsup_{n\to\infty} 2s\, A_n,$$

so $A_n \to 0$. Plugging $A_n \to 0$ and $B_{p,n}, B_{q,n} \to 0$ into (3.11) gives

$$\lim_{n\to\infty} J_{\mu,T}(u_n) = \frac{\mu a}{2}.$$

But by Lemma 3.6, $b_{\mu,T,a} = b_{0,T,a} + \frac{\mu a}{2}$ with $b_{0,T,a} > 0$, a contradiction to $J_{\mu,T}(u_n) \to b_{\mu,T,a}$. Therefore vanishing cannot occur, and the sequence is nonvanishing in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. □

Lemma 3.10. *If $u \in S(a)$ is a constrained critical point of $J_{\mu,T}$ with*

$$J_{\mu,T}(u) \ \leq\ b_{\mu,T,a},$$

then its Lagrange multiplier satisfies

$$\lambda - \mu < 0 \quad \textit{equivalently,} \quad \lambda < \mu.$$

Proof. By Lemma 3.5, with our choice of truncation radii the mountain–pass critical points are attained where $\tau \equiv 1$. Hence the truncation is inactive at u, i.e. $\tau(\|u\|_{H^s}) = 1$.

We may work with the full constrained Lagrangian

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{\mu}{2}\|u\|_2^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p - \frac{1}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q - \frac{\lambda}{2}\big(\|u\|_2^2 - a\big),$$

so that the Euler–Lagrange equation on $S(a)$ reads

$$(-\Delta)^s u + \mu u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + (I_\alpha * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N.$$

Let

$$A = \|(-\Delta)^{s/2}u\|_2^2, \qquad B_p = \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p, \qquad B_q = \int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q,$$

and $a = \|u\|_2^2$. Then

$$A + \mu a = \lambda a + B_p + B_q. \tag{3.12}$$

By Lemma 3.4 and $\delta_r = Nr - (N+\alpha)$,

$$2s\, A - \frac{\delta_p}{p}\, B_p - \frac{\delta_q}{q}\, B_q = 0.$$

Since $p = 2^*_{\alpha,s}$ we have $\frac{\delta_p}{p} = 2s$, hence

$$A - B_p = \frac{\delta_q}{2s\, q}\, B_q = \frac{(N-2s)q - (N+\alpha)}{2s\, q}\, B_q. \tag{3.13}$$

From (3.12),

$$\lambda a = (A - B_p) - B_q + \mu a.$$

Using (3.13),

$$\lambda a = \Big(\frac{\delta_q}{2s\, q} - 1\Big) B_q + \mu a = \mu a - \frac{(N+\alpha) - (N-2s)q}{2s\, q}\, B_q.$$

Because $q < p = \frac{N+\alpha}{N-2s}$, the coefficient of B_q is strictly positive; since $B_q > 0$ for $u \not\equiv 0$, we conclude

$$\lambda a < \mu a \qquad \to \qquad \lambda < \mu,$$

i.e. $\lambda - \mu < 0$. □

Lemma 3.11. *Assume the hypotheses of Theorem 1.2 hold. If $(u_n) \subset S(a)$ is a $(PS)_{b_{\mu,T,a}}$ sequence for $J_{\mu,T}\big|_{S(a)}$, then there exist translations $y_n \in \mathbb{R}^N$ and $w_\mu \in S(a)$ such that $v_n = u_n(\cdot + y_n) \to w_\mu$ strongly in $H^s(\mathbb{R}^N)$. In particular $b_{\mu,T,a}$ is attained by a positive constrained critical point $w_\mu \in S(a)$.*

Proof. Fix $u \in S(a)$ and let $t(u) > 0$ be the fiber maximizer given by Lemma 3.5; set

$$X = \|(-\Delta)^{s/2}(u_{t(u)})\|_2^2.$$

At $t = t(u)$ we have $\frac{d}{dt}J_{\mu,T}(u_t)\big|_{t=t(u)} = 0$, i.e. $\mathcal{P}_T(u_{t(u)}) = 0$. Since $\tau \le 1$ and $\tau' \le 0$, from the explicit formula of $\mathcal{P}_T$ we obtain the inequality

$$2s\,X \;\le\; \frac{\delta_p}{p}\,B_p(u_{t(u)}) \;+\; \frac{\delta_q}{q}\,B_q(u_{t(u)}), \qquad \delta_r = Nr - (N+\alpha). \tag{3.14}$$

By Proposition 1.1 and Lemma 2.1,

$$B_p(w) \;\le\; S_\alpha\,\|(-\Delta)^{s/2}w\|_2^{2p}, \qquad B_q(w) \;\le\; C_{\alpha,q}\,\|(-\Delta)^{s/2}w\|_2^{2q\gamma_{q,s}}\,\|w\|_2^{2q(1-\gamma_{q,s})},$$

and recalling that $p = 2^*_{\alpha,s}$ and $\frac{\delta_q}{2s} = q\gamma_{q,s}$, inequality (3.14) gives

$$X \;\le\; S_\alpha X^p \;+\; K_q\,S_\alpha^{-\vartheta_q}\,a^{q(1-\gamma_{q,s})}\,X^{q\gamma_{q,s}}. \tag{3.15}$$

Define

$$h(X) = S_\alpha X^{p-1} + K_q\,S_\alpha^{-\vartheta_q}\,a^{\,q(1-\gamma_{q,s})}\,X^{\,q\gamma_{q,s}-1} - 1 \quad \text{on } (0,\infty).$$

Since $p > 1$ and $q\gamma_{q,s} > 1$, the equation $h(X) = 0$ has a unique positive root $X_*(a)$. By (3.15), any fiber maximizer satisfies $X \le X_*(a)$. By the small mass hypothesis, a is chosen so that

$$X_*(a) \;\le\; R_*^2 \quad \text{for some fixed } R_* > 0,$$

equivalently,

$$S_\alpha\,R_*^{2(p-1)} \;+\; K_q\,S_\alpha^{-\vartheta_q}\,a^{\,q(1-\gamma_{q,s})}\,R_*^{2(q\gamma_{q,s}-1)} \;\le\; 1.$$

Therefore, for every fiber maximizer $t(u)$,

$$\|u_{t(u)}\|_{H^s}^2 \;=\; X + a \;\le\; R_*^2 + a.$$

Fix the truncation radii so that

$$R_0 \;>\; \sqrt{R_*^2 + a} \qquad \text{and} \qquad R_1 > R_0.$$

Then necessarily $\|u_{t(u)}\|_{H^s} < R_0$, hence the truncation is inactive at the maximizing point: $\tau(\|u_{t(u)}\|_{H^s}) \equiv 1$ and $\tau'(\|u_{t(u)}\|_{H^s}) = 0$. By Lemma 3.5, this forces the mountain pass critical level to be attained entirely in the region $\{\tau \equiv 1\}$, and in particular along any $(PS)_{b_{\mu,T,a}}$ sequence.

It is convenient to use the full energy

$$\mathcal{E}_\mu(u) = \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{\mu}{2}\|u\|_2^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx - \frac{1}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx,$$

for which $\mathcal{E}_\mu(u_n) = b_{\mu,T,a} + o(1)$ and $(\mathcal{E}_\mu|_{S(a)})'(u_n) = o(1)$. By Lemma 3.9, there exist $R, \beta > 0$ and $y_n \in \mathbb{R}^N$ such that, with $v_n = u_n(\cdot + y_n)$,

$$\int_{B_R(0)} |v_n|^2\,dx \ge \beta, \qquad v_n \in S(a), \qquad \mathcal{E}_\mu(v_n) = b_{\mu,T,a} + o(1), \qquad (\mathcal{E}_\mu|_{S(a)})'(v_n) = o(1).$$

Up to a subsequence,

$$v_n \rightharpoonup w_\mu \ \text{ in } H^s, \qquad v_n \to w_\mu \ \text{ in } L^r_{\mathrm{loc}} \ (2 \le r < 2^*_s), \qquad v_n(x) \to w_\mu(x) \ \text{ a.e.},$$

and $w_\mu \not\equiv 0$ by the local L^2 lower bound.

Since $(\mathcal{E}_\mu|_{S(a)})'(v_n) = o(1)$, there are $\lambda_n \in \mathbb{R}$ with

$$\mathcal{E}'_\mu(v_n) = \lambda_n v_n + o(1) \quad \text{in } H^{-s}.$$

Thus

$$(-\Delta)^s v_n + \mu v_n = \lambda_n v_n + (I_\alpha * |v_n|^p)|v_n|^{p-2}v_n + (I_\alpha * |v_n|^q)|v_n|^{q-2}v_n + o(1).$$

Testing by v_n gives

$$A(v_n) + \mu a = \lambda_n a + B_p(v_n) + B_q(v_n), \tag{3.16}$$

where $A(u) = \|(-\Delta)^{s/2}u\|_2^2$ and $B_r(u) = \int_{\mathbb{R}^N}(I_\alpha * |u|^r)|u|^r\,dx$. From (3.16) and boundedness of $A(v_n), B_r(v_n)$ we get $\{\lambda_n\}$ bounded. Passing to a subsequence, $\lambda_n \to \lambda \in \mathbb{R}$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Since $v_n \rightharpoonup w_\mu$ in $H^s(\mathbb{R}^N)$ and $v_n \to w_\mu$ a.e., by local Sobolev compactness we have

$$v_n \to w_\mu \quad \text{in } L^r(\operatorname{supp}\varphi) \ \text{ for all } 2 \le r < 2^*_s.$$

Because

$$(p-1)\,\frac{2N}{N+\alpha} < \frac{2N}{N-2s} = 2^*_s,$$

it follows that

$$|v_n|^{p-2}v_n\,\varphi \to |w_\mu|^{p-2}w_\mu\,\varphi \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N),$$

and similarly

$$|v_n|^{q-2}v_n\,\varphi \to |w_\mu|^{q-2}w_\mu\,\varphi \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Moreover $|v_n|^p$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, hence

$$|v_n|^p \ \rightharpoonup\ |w_\mu|^p \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N),$$

and by the linear continuity of the Riesz potential

$$I_\alpha : L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \ \to\ L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$$

we get

$$(I_\alpha * |v_n|^p) \ \rightharpoonup\ (I_\alpha * |w_\mu|^p) \quad \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).$$

Therefore, by weak–strong pairing,

$$\int_{\mathbb{R}^N}(I_\alpha * |v_n|^p)\,|v_n|^{p-2}v_n\varphi dx \ \to\ \int_{\mathbb{R}^N}(I_\alpha * |w_\mu|^p)\,|w_\mu|^{p-2}w_\mu\varphi dx.$$

Passing to the limit in the weak formulation yields

$$(-\Delta)^s w_\mu + \mu w_\mu = \lambda w_\mu + (I_\alpha * |w_\mu|^p)|w_\mu|^{p-2}w_\mu + (I_\alpha * |w_\mu|^q)|w_\mu|^{q-2}w_\mu. \tag{3.17}$$

By Lemma 3.4, w_μ satisfies

$$\mathcal{P}(w_\mu) = 2s\,A(w_\mu) - \frac{\delta_p}{p}B_p(w_\mu) - \frac{\delta_q}{q}B_q(w_\mu) = 0, \qquad \delta_r = Nr - (N+\alpha), \tag{3.18}$$

and since $p = 2^*_{\alpha,s}$, $\delta_p/p = 2s$.

Set $z_n = v_n - w_\mu$. Then:

$$A(v_n) = A(w_\mu) + A(z_n) + o(1), \tag{3.19}$$

$$B_p(v_n) = B_p(w_\mu) + B_p(z_n) + o(1) \tag{3.20}$$

and for q we argue as follows

$$B_q(v_n) = B_q(w_\mu) + B_q(z_n) + o(1). \tag{3.21}$$

Moreover, $\|v_n\|_2^2 = \|w_\mu\|_2^2 + \|z_n\|_2^2 + o(1) = a$. As $(\mathcal{E}_\mu|_{S(a)})'(v_n) = o(1)$ and the μ-term is constant along the mass-preserving ray, Lemma 3.5 yields

$$\mathcal{P}(v_n) = o(1).$$

Using (3.18), (3.19), (3.20), (3.21) we get

$$\mathcal{P}(z_n) = o(1). \tag{3.22}$$

By the Brezis–Lieb splittings,

$$\mathcal{E}_\mu(v_n) = \mathcal{E}_\mu(w_\mu) + \mathcal{E}_\mu(z_n) + o(1).$$

Since $\delta_p/p = 2s$, a direct computation gives

$$2\,\mathcal{E}_\mu(z_n) - \frac{1}{2s}\,\mathcal{P}(z_n) = \mu\|z_n\|_2^2 + \Big(\frac{\delta_q}{2s\,q} - \frac{1}{q}\Big) B_q(z_n) + o(1), \tag{3.23}$$

where $\frac{\delta_q}{2s\,q} - \frac{1}{q} = \frac{1}{q}\big(\frac{\delta_q}{2s} - 1\big) > 0$ because $q > \frac{N+2s+\alpha}{N} \iff \delta_q > 2s$. With $\mathcal{P}(z_n) \to 0$ from (3.22), we deduce

$$\mu\|z_n\|_2^2 + \Big(\frac{\delta_q}{2s\,q} - \frac{1}{q}\Big) B_q(z_n) \;\to\; 0 \quad\to\quad \begin{cases} \|z_n\|_2 \to 0 & \text{if } \mu > 0, \\ B_q(z_n) \to 0 & \text{always.} \end{cases} \tag{3.24}$$

From $\mathcal{P}(z_n) \to 0$ and $\delta_p/p = 2s$ we also get

$$A(z_n) - B_p(z_n) \;\to\; 0. \tag{3.25}$$

Comparing (3.16) with the test of (3.17) by w_μ and using the splittings,

$$(\lambda_n - \mu)\,a - (\lambda - \mu)\,m = \big(A(z_n) - B_p(z_n) - B_q(z_n)\big) + o(1), \qquad m = \|w_\mu\|_2^2 \in (0, a].$$

Letting $n \to \infty$ and using (3.24)–(3.25) yields $(\lambda - \mu)(a - m) = 0$. By Lemma 3.10, $\lambda < \mu$; hence $m = a$. In particular, when $\mu > 0$ this also follows from (3.24) and the L^2-splitting. Therefore

$$\|w_\mu\|_2^2 = a, \qquad \|z_n\|_2 \to 0. \tag{3.26}$$

Since $w_\mu \in S(a)$ and $\mathcal{P}(w_\mu) = 0$, by Lemma 3.5 the fiber map $t \mapsto \mathcal{E}_\mu((w_\mu)_t)$ has a unique critical point at $t = 1$, which is its global maximum on the ray; hence

$$b_{\mu,T,a} \;\le\; \max_{t>0} \mathcal{E}_\mu((w_\mu)_t) = \mathcal{E}_\mu(w_\mu) \;\le\; \liminf_n \mathcal{E}_\mu(v_n) = b_{\mu,T,a}.$$

Thus

$$\mathcal{E}_\mu(w_\mu) = b_{\mu,T,a}, \qquad \mathcal{E}_\mu(z_n) \to 0. \tag{3.27}$$

Combining (3.23), (3.22), (3.27) and (3.26) gives

$$A(z_n) \to 0, \qquad B_p(z_n) \to 0, \qquad B_q(z_n) \to 0,$$

so $z_n \to 0$ strongly in $H^s(\mathbb{R}^N)$; i.e. $v_n \to w_\mu$ in H^s.

Since $\mathcal{E}_\mu(|u|) \le \mathcal{E}_\mu(u)$ and $|w_\mu| \in S(a)$, replacing w_μ by $|w_\mu|$ preserves the level. The Euler–Lagrange equation (3.17) and the fractional strong maximum principle then imply $w_\mu > 0$ in $\mathbb{R}^N$. □

4 The nonautonomous case

In this section we work with the slowly varying potential $V(\varepsilon x)$:

$$(-\Delta)^s u + V(\varepsilon x)u = \lambda u + \big(I_\alpha * |u|^p\big)|u|^{p-2}u + \big(I_\alpha * |u|^q\big)|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

with

$$\frac{N+2s+\alpha}{N} < q < p = 2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}.$$

and we keep the mass constraint $S(a) = \{u \in H^s(\mathbb{R}^N) : \|u\|_2^2 = a\}$.

Lemma 4.1. *Let $\{u_n\} \subset S(a)$ be a $(PS)_c$ sequence for $J_{\varepsilon,T}\big|_{S(a)}$ with $c < b_{\infty,T,a}$. Then $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$.*

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence on the constraint $S(a)$, we have

$$J_{\varepsilon,T}(u_n) = c + o(1) \qquad \text{and} \qquad \|(J_{\varepsilon,T}|_{S(a)})'(u_n)\|_{(T_{u_n}S(a))^*} \to 0.$$

In particular, the energies $\{J_{\varepsilon,T}(u_n)\}$ are bounded: there exists $M > 0$ such that

$$|J_{\varepsilon,T}(u_n)| \le M \qquad \text{for all } n. \tag{4.1}$$

Write $\|u\|_{H^s}^2 = \|(-\Delta)^{s/2}u\|_2^2 + \|u\|_2^2 = \|(-\Delta)^{s/2}u\|_2^2 + a$ for $u \in S(a)$. We prove by contradiction that $\{\|u_n\|_{H^s}\}$ is bounded.

Assume $\|u_n\|_{H^s} \to \infty$. Consequently, for all sufficiently large n, $\|u_n\|_{H^s} \ge R_1$, hence by the definition of τ we have $\tau(\|u_n\|_{H^s}) = 0$ and, since $\tau \equiv 0$ on $[R_1, \infty)$, also $\tau'(\|u_n\|_{H^s}) = 0$.

Because $(J_{\varepsilon,T}|_{S(a)})'(u_n) \to 0$, there exist $\lambda_n \in \mathbb{R}$ such that

$$J'_{\varepsilon,T}(u_n) = \lambda_n u_n + o(1) \quad \text{in } H^{-s}.$$

Testing by u_n and using that the truncated p-term and its derivative vanish on this region, we get

$$A_n + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2\,dx = \lambda_n a + B_{q,n} + o(1). \tag{4.2}$$

Moreover, the energy identity reduces to

$$\begin{aligned} J_{\varepsilon,T}(u_n) &= \frac{1}{2}A_n + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2\,dx - \frac{1}{2q}\,B_{q,n} \\ &= \frac{q-1}{2q}\Big(A_n + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2\,dx\Big) + \frac{\lambda_n a}{2q} + o(1), \end{aligned} \tag{4.3}$$

where we used (4.2) to eliminate $B_{q,n}$.

By Proposition 1.1,

$$B_{q,n} \;\le\; C_{\alpha,q}\, a^{q(1-\gamma_{q,s})}\, A_n^{q\gamma_{q,s}}, \qquad q\gamma_{q,s} > 1.$$

Since V is bounded above by V_∞, we have $\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2\,dx \le V_\infty a$. Thus from (4.2),

$$\lambda_n a = A_n + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2\,dx - B_{q,n} + o(1) \;\le\; A_n + V_\infty a - c\,A_n^{q\gamma_{q,s}} + o(1) \;\xrightarrow[n\to\infty]{}\; -\infty,$$

because $q\gamma_{q,s} > 1$ and $A_n \to \infty$. Plugging this into (4.3) yields $J_{\varepsilon,T}(u_n) \to -\infty$, which contradicts the uniform bound $|J_{\varepsilon,T}(u_n)| \le M$. Hence $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. $\square$

Lemma 4.2. *Let $\{u_n\} \subset S(a)$ be a $(PS)_c$ sequence for $J_{\varepsilon,T}\big|_{S(a)}$ with $c < b_{\infty,T,a}$. If $\{u_n\}$ vanishes, then*

$$\liminf_{n\to\infty} J_{\varepsilon,T}(u_n) \;\geq\; \frac{V_\infty a}{2}.$$

Proof. By Lemma 4.1, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Since it vanishes and is bounded, Lemma 3.8 yields

$$B_q(u_n) \to 0 \quad \text{and} \quad B_p(u_n) \to 0,$$

where $B_t(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^t)\,|u|^t\,dx$.

By the definition of $J_{\varepsilon,T}$,

$$J_{\varepsilon,T}(u_n) = \frac{1}{2}\|(-\Delta)^{s/2}u_n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)\,|u_n|^2\,dx - \frac{\tau(\|u_n\|_{H^s})}{2p}\,B_p(u_n) - \frac{1}{2q}\,B_q(u_n).$$

Using $B_p(u_n) \to 0$ and $B_q(u_n) \to 0$,

$$J_{\varepsilon,T}(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)\,|u_n|^2\,dx + o(1).$$

Fix $R > 0$. Vanishing implies $\int_{B_R(0)} |u_n|^2\,dx \to 0$, hence

$$\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2\,dx \;\geq\; \Big(\inf_{|x|\geq R} V(\varepsilon x)\Big)\Big(a - \int_{B_R(0)} |u_n|^2\,dx\Big) = \Big(\inf_{|z|\geq \varepsilon R} V(z)\Big)\,(a - o(1)).$$

Let $n \to \infty$ and then $R \to \infty$. Since $\inf_{|z|\geq\varepsilon R} V(z) \to V_\infty$, we obtain

$$\liminf_{n\to\infty} J_{\varepsilon,T}(u_n) \;\geq\; \frac{V_\infty a}{2}.$$

□

Lemma 4.3. *Let $\{u_n\} \subset S(a)$ be a $(PS)_c$ sequence for $J_{\varepsilon,T}\big|_{S(a)}$ with $c < b_{\infty,T,a}$. Then, up to a subsequence, either*

1. *$u_n \to u$ strongly in $H^s(\mathbb{R}^N)$ for some $u \in S(a)$, or*
2. *there exist $\beta > 0$ and a sequence $y_n \in \mathbb{R}^N$ such that for all large n*

$$\int_{B_1(y_n)} |u_n - u|^2\,dx \;\geq\; \beta.$$

Proof. By Lemma 4.1, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. By the small-mass hypothesis and Lemma 3.5, the truncation is inactive along $\{u_n\}$, hence $J'_{\varepsilon,T}(u_n) = (J_\varepsilon)'(u_n)$ and $(J_{\varepsilon,T}|_{S(a)})'(u_n) = (J_\varepsilon|_{S(a)})'(u_n)$.

Passing to a subsequence there exists $u \in H^s(\mathbb{R}^N)$ with

$$u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^N), \qquad u_n \to u \text{ in } L^r_{\text{loc}}(\mathbb{R}^N) \ \text{ for } 2 \leq r < 2_s^*, \qquad u_n(x) \to u(x) \ \text{ a.e.}$$

Set $v_n = u_n - u$. If

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^2\,dx = 0,$$

then by Lemma 2.4 one has $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for all $2 < r < 2_s^*$. Since $\{u_n\} \subset S(a)$ is $(PS)_c$ on the constraint, there exist $\lambda_n \in \mathbb{R}$ such that

$$J'_{\varepsilon,T}(u_n) - \lambda_n u_n \to 0 \quad \text{in } H^{-s}(\mathbb{R}^N).$$

Testing by v_n yields

$$\|(-\Delta)^{s/2}v_n\|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^2\,dx = \lambda_n\|v_n\|_2^2 + o(1). \tag{4.4}$$

Fix $R>0$. From the vanishing of v_n, $\int_{B_R(0)}|v_n|^2\,dx \to 0$. Hence

$$\int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^2\,dx \;\geq\; \Big(\inf_{|x|\geq R} V(\varepsilon x)\Big)\Big(\|v_n\|_2^2 - \int_{B_R(0)}|v_n|^2\,dx\Big),$$

so taking $\liminf$ in n and then $R\to\infty$,

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^2\,dx \;\geq\; V_\infty\;\liminf_{n\to\infty}\|v_n\|_2^2. \tag{4.5}$$

From (4.4) and (4.5),

$$\limsup_{n\to\infty}\Big(\|(-\Delta)^{s/2}v_n\|_2^2 + (V_\infty-\lambda_n)\|v_n\|_2^2\Big) \;\leq\; 0.$$

Let $\lambda_n\to\lambda$. If $\lambda<V_\infty$, then $\|(-\Delta)^{s/2}v_n\|_2\to 0$ and $\|v_n\|_2\to 0$, hence $u_n\to u$ in $H^s(\mathbb{R}^N)$, which is (i).

If $\lambda\geq V_\infty$, then by Lemma 4.2 and the energy splitting,

$$c = \lim_n J_{\varepsilon,T}(u_n) = J_{\varepsilon,T}(u) + \lim_n J_{\varepsilon,T}(v_n) \;\geq\; \frac{V_\infty}{2}\liminf_n\|v_n\|_2^2.$$

By Lemma 3.6 and Lemma 3.7, $b_{\infty,T,a} = b_{0,T,a} + \frac{V_\infty a}{2} > \frac{V_\infty a}{2}$. Since $c<b_{\infty,T,a}$, it follows that $\|v_n\|_2\to 0$, hence $u_n\to u$ in $H^s(\mathbb{R}^N)$, which is (i).

If the above vanishing fails, then there exist $\beta>0$ and $y_n\in\mathbb{R}^N$ with $\int_{B_1(y_n)}|u_n-u|^2\,dx\geq\beta$ for all large n, which is (ii). □

Lemma 4.4. *There exists $\rho_0>0$ (independent of ε) such that, for any*

$$c<b_{0,T,a}+\rho_0 \quad \textit{with} \quad \rho_0 \;\leq\; \frac{\beta}{a}\Big(b_{\infty,T,a}-b_{0,T,a}\Big) \;=\; \frac{\beta}{a}\cdot\frac{V_\infty a}{2},$$

every $(PS)_c$ sequence of $J_{\varepsilon,T}\big|_{S(a)}$ is precompact in $H^s(\mathbb{R}^N)$.

Proof. Let $\{u_n\}\subset S(a)$ be a $(PS)_c$ sequence with $c<b_{0,T,a}+\rho_0$. By Lemma 4.1, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Assume by contradiction that $\{u_n\}$ does not converge strongly in H^s. By Lemma 4.3, there exist $\beta>0$ and $y_n\in\mathbb{R}^N$ such that, up to a subsequence,

$$\int_{B_1(y_n)}|u_n-u|^2\,dx \;\geq\; \beta,$$

where u is the weak limit of u_n in H^s. Set

$$v_n(x) = (u_n-u)(x+y_n).$$

Then $v_n\rightharpoonup v$ in $H^s(\mathbb{R}^N)$, $v_n\to v$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and $m=\|v\|_2^2\geq\beta$. The Brezis–Lieb splittings give

$$J_{\varepsilon,T}(u_n) = J_{\varepsilon,T}(u) + J^{(n)}_{\varepsilon,T}(v_n) + o(1),$$

where $J^{(n)}_{\varepsilon,T}$ denotes $J_{\varepsilon,T}$ with potential $x\mapsto V(\varepsilon(x+y_n))$.

By Lemma 3.6 and Lemma 4.2,

$$\liminf_{n\to\infty} J^{(n)}_{\varepsilon,T}(v_n) \;\ge\; b_{\infty,T,m} \;=\; b_{0,T,m} + \frac{V_\infty}{2}\, m.$$

Moreover $J_{\varepsilon,T}(u) \ge J_{0,T}(u) \ge b_{0,T,\|u\|_2^2}$. Using the mass splitting $\|u\|_2^2 + (\lim \|v_n\|_2^2) = a$, subadditivity of $b_{0,T,\cdot}$ in the mass, and passing to the limit, we obtain

$$c = \lim_n J_{\varepsilon,T}(u_n) \;\ge\; b_{0,T,\|u\|_2^2} + b_{\infty,T,m} \;\ge\; \big(b_{0,T,a} - b_{0,T,m}\big) + \big(b_{0,T,m} + \tfrac{V_\infty}{2} m\big) \;=\; b_{0,T,a} + \frac{V_\infty}{2}\, m.$$

Since $m \ge \beta$,

$$c \;\ge\; b_{0,T,a} + \frac{V_\infty}{2}\,\beta \;\ge\; b_{0,T,a} + \rho_0,$$

which contradicts $c < b_{0,T,a} + \rho_0$. Therefore every such $(PS)_c$ sequence is precompact in $H^s(\mathbb{R}^N)$. □

Lemma 4.5. *There exists $c_* > 0$ (independent of ε) such that any constrained critical point $u \in S(a)$ of $J_{\varepsilon,T}$ with $J_{\varepsilon,T}(u) \le c_*$ satisfies $\|u\|_{H^s} < R_0$. In particular, $\tau(\|u\|_{H^s}) = 1$ in a neighborhood of u, and u is also a constrained critical point of $J_\varepsilon\big|_{S(a)}$.*

Proof. For $t \in \{p, q\}$ set

$$B_t(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^t)\,|u|^t\,dx, \qquad A(u) = \|(-\Delta)^{s/2} u\|_2^2.$$

By Lemma 2.1 and $\|u\|_2^2 = a$,

$$B_t(u) \;\le\; S_{\alpha,t}\, A(u)^{\theta_t/2}\, a^{\eta_t/2}, \qquad \theta_t > 2.$$

Case A: $\|u\|_{H^s} \ge R_1$. Then $\tau(\|u\|_{H^s}) = 0$, hence

$$J_{\varepsilon,T}(u) = \frac12 A(u) + \frac12 \int_{\mathbb{R}^N} V(\varepsilon x)|u|^2\,dx \;\ge\; \frac12 A(u) = \frac12\big(\|u\|_{H^s}^2 - a\big) \;\ge\; \frac12 (R_1^2 - a).$$

Set $c_1 = \frac12 (R_1^2 - a) > 0$.

Case B: $\|u\|_{H^s} \in [R_0, R_1]$. Here $\tau(\|u\|_{H^s}) \in (0,1]$, so

$$J_{\varepsilon,T}(u) \;\ge\; \frac12 A(u) - \frac{S_{\alpha,p}}{2p}\, a^{\eta_p/2} A(u)^{\theta_p/2} - \frac{S_{\alpha,q}}{2q}\, a^{\eta_q/2} A(u)^{\theta_q/2}.$$

Fix $\varepsilon_p, \varepsilon_q \in (0, 1/8]$. Since $\theta_t/2 > 1$, Young's inequality yields

$$\frac{S_{\alpha,t}}{2t}\, a^{\eta_t/2} A(u)^{\theta_t/2} \;\le\; \varepsilon_t\, A(u) + C_t \quad (t \in \{p,q\})$$

for suitable constants C_t. Thus

$$J_{\varepsilon,T}(u) \;\ge\; \Big(\frac12 - \varepsilon_p - \varepsilon_q\Big) A(u) - (C_p + C_q) \;\ge\; \frac14 A(u) - C_0,$$

after choosing $\varepsilon_p, \varepsilon_q$ so that the bracket is $\ge \frac14$ and setting $C_0 = C_p + C_q$. Since $\|u\|_{H^s}^2 = A(u) + a$ and $\|u\|_{H^s} \ge R_0$, one has $A(u) \ge R_0^2 - a$, hence

$$J_{\varepsilon,T}(u) \;\ge\; \frac14 (R_0^2 - a) - C_0.$$

Let $c_2 = \frac14 (R_0^2 - a) - C_0$. Choosing R_0 large (with $R_1 > R_0$ fixed) gives $c_2 > 0$.

Set

$$c_* = \min\{c_1, c_2\} = \min\Big\{\frac12 (R_1^2 - a),\ \frac14 (R_0^2 - a) - C_0\Big\} > 0.$$

If $J_{\varepsilon,T}(u) \le c_*$ and $\|u\|_{H^s} \ge R_0$, Case A or Case B is contradicted. Hence $\|u\|_{H^s} < R_0$, so $\tau(\|u\|_{H^s}) = 1$ near u, and the Euler–Lagrange equations of $J_{\varepsilon,T}$ and J_ε agree at u. □

5 Multiplicity result

Let $w \in S(a)$ be a constrained critical point of $J_{0,T}\big|_{S(a)}$ at level $b_{0,T,a}$ (Lemma 3.11). By the autonomous ray analysis, the truncation is inactive at w, that is, $\tau(\|w\|_{H^s}) = 1$, and we may fix a margin

$$\|w\|_{H^s} \;\le\; R_0 - \delta_0 \qquad \text{for some } \delta_0 \in (0, R_0). \tag{5.1}$$

Choose $\chi \in C_c^\infty(\mathbb{R}^N; [0,1])$ radial with $\chi \equiv 1$ on $B_1(0)$ and $\chi \equiv 0$ outside $B_2(0)$. Pick a scaling $R_\varepsilon \to \infty$ such that $\varepsilon R_\varepsilon \to 0$ as $\varepsilon \to 0$. For $y \in M$, set

$$\widetilde{w}_{\varepsilon,y}(x) = \chi\Big(\frac{x - y/\varepsilon}{R_\varepsilon}\Big)\, w\Big(x - \frac{y}{\varepsilon}\Big), \qquad \Phi_\varepsilon(y) = \sqrt{\frac{a}{\|\widetilde{w}_{\varepsilon,y}\|_2^2}}\;\widetilde{w}_{\varepsilon,y}\;\in S(a).$$

Lemma 5.1. *With $\Phi_\varepsilon(y) \in S(a)$ defined from $w \in S(a)$ as above, one has*

$$\sup_{y\in M} \big|J_{\varepsilon,T}(\Phi_\varepsilon(y)) - b_{0,T,a}\big| \to\; 0 \qquad \textit{as } \varepsilon \to 0.$$

Proof. By the change of variables $x = y/\varepsilon + z$,

$$\|\widetilde{w}_{\varepsilon,y}\|_2^2 = \int_{\mathbb{R}^N} \chi^2\Big(\frac{z}{R_\varepsilon}\Big)|w(z)|^2\,dz \;\xrightarrow[\varepsilon\to 0]{}\; \int_{\mathbb{R}^N} |w|^2\,dz = a,$$

hence the prefactor $\sqrt{a/\|\widetilde{w}_{\varepsilon,y}\|_2^2} \to 1$, uniformly in y. Moreover, the standard cut-off approximation in H^s gives

$$\Big\|\chi\Big(\frac{\cdot}{R_\varepsilon}\Big)\, w - w\Big\|_{H^s} \;\xrightarrow[\varepsilon\to 0]{}\; 0.$$

Using translation invariance of the H^s–norm, this yields

$$\|\widetilde{w}_{\varepsilon,y} - w(\cdot - y/\varepsilon)\|_{H^s} \;\to 0 \quad \text{and} \quad \|\Phi_\varepsilon(y) - w(\cdot - y/\varepsilon)\|_{H^s} \;\to 0,$$

uniformly in $y \in M$. In particular,

$$\sup_{y\in M} \|\Phi_\varepsilon(y)\|_{H^s} \;\xrightarrow[\varepsilon\to 0]{}\; \|w\|_{H^s} \;\le\; R_0 - \delta_0, \tag{5.2}$$

so for ε small, independent of y, one has $\|\Phi_\varepsilon(y)\|_{H^s} \le R_0 - \frac{\delta_0}{2}$; hence the truncation is inactive:

$$\tau\big(\|\Phi_\varepsilon(y)\|_{H^s}\big) = 1 \qquad \text{for all } y \in M.$$

Write $B_t(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^t)|u|^t\,dx$ for $t \in \{p, q\}$. From the continuous embeddings $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $2 \le r \le 2_s^*$, we have

$$\Phi_\varepsilon(y) \;\to\; w(\cdot - y/\varepsilon) \quad \text{in } H^s \cap L^{t\frac{2N}{N+\alpha}},$$

uniformly in y. By continuity of the maps $u \mapsto \|(-\Delta)^{s/2}u\|_2^2$ and $u \mapsto B_t(u)$ along these convergences,

$$\|(-\Delta)^{s/2}\Phi_\varepsilon(y)\|_2^2 \;\to\; \|(-\Delta)^{s/2}w\|_2^2, \qquad B_t\big(\Phi_\varepsilon(y)\big) \;\to\; B_t(w),$$

uniformly in $y \in M$.

With $x = y/\varepsilon + z$,

$$\int_{\mathbb{R}^N} V(\varepsilon x)\,|\Phi_\varepsilon(y,x)|^2\,dx = \int_{\mathbb{R}^N} V\big(y + \varepsilon z\big)\,|\Phi_\varepsilon(y, y/\varepsilon + z)|^2\,dz.$$

By Step 1, $|\Phi_\varepsilon(y,y/\varepsilon+z)|^2 \to |w(z)|^2$ in $L^1(\mathbb{R}^N)$ uniformly in y. Since M is compact, V is continuous, and $\varepsilon R_\varepsilon \to 0$,

$$\sup_{\substack{y\in M\\ |z|\le 2R_\varepsilon}} \big|V(y+\varepsilon z)-V(y)\big| \xrightarrow[\varepsilon\to 0]{} 0, \qquad V(y)=0 \ \text{ for } y\in M,$$

whence by dominated convergence

$$\sup_{y\in M} \int_{\mathbb{R}^N} V(\varepsilon x)\,|\Phi_\varepsilon(y,x)|^2\,dx \ \to\ 0.$$

Since $\tau\equiv 1$ on $\{\Phi_\varepsilon(y): y\in M\}$ for ε small by (5.2), we may drop the truncation and write

$$J_{\varepsilon,T}(\Phi_\varepsilon(y)) = \frac{1}{2}\|(-\Delta)^{s/2}\Phi_\varepsilon(y)\|_2^2 - \frac{1}{2p}B_p\big(\Phi_\varepsilon(y)\big) - \frac{1}{2q}B_q\big(\Phi_\varepsilon(y)\big) + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|\Phi_\varepsilon(y)|^2\,dx.$$

By above the first three terms converge uniformly in y to the corresponding autonomous value at w, and the last term tends to 0 uniformly. Hence

$$\sup_{y\in M}\big|J_{\varepsilon,T}(\Phi_\varepsilon(y)) - J_{0,T}(w)\big| \ \to\ 0.$$

Finally $J_{0,T}(w)=J_0(w)=b_{0,T,a}$ by (5.1), which gives the claim. $\square$

To convert these profiles into distinct critical points, we encode their location through a barycenter map adapted to the rescaled potential.

Definition 5.2. *Fix $\zeta\in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ with $\zeta(x)=x$ for $|x|\le 1$ and $\zeta(x)=0$ for $|x|\ge 2$. For $u\in S(a)$ and $\varepsilon>0$ set*

$$\beta_\varepsilon(u) = \frac{1}{a}\int_{\mathbb{R}^N} \zeta(\varepsilon x)\,|u(x)|^2\,dx \in \mathbb{R}^N.$$

The next statement shows that the barycenter of each model profile concentrates at its anchor point.

Lemma 5.3. *Let $\Phi_\varepsilon(y)\in S(a)$ be defined from $w\in S(a)$ as above, with $R_\varepsilon\to\infty$ and $\varepsilon R_\varepsilon\to 0$. Assume that there exists $\delta_M\in(0,1)$ with $M\subset B_{1-\delta_M}(0)$. Then*

$$\sup_{y\in M}|\beta_\varepsilon(\Phi_\varepsilon(y))-y| \ \to\ 0 \qquad \text{as } \varepsilon\to 0.$$

Proof. Recall

$$\widetilde{w}_{\varepsilon,y}(x) = \chi\Big(\frac{x-y/\varepsilon}{R_\varepsilon}\Big)\, w\Big(x-\frac{y}{\varepsilon}\Big), \qquad \Phi_\varepsilon(y) = \sqrt{\frac{a}{\|\widetilde{w}_{\varepsilon,y}\|_2^2}}\ \widetilde{w}_{\varepsilon,y}\in S(a).$$

By construction, $\operatorname{supp}\widetilde{w}_{\varepsilon,y}\subset B_{2R_\varepsilon}(y/\varepsilon)$, hence $\operatorname{supp}\Phi_\varepsilon(y)\subset B_{2R_\varepsilon}(y/\varepsilon)$ as well.

With the change of variables $x=\frac{y}{\varepsilon}+z$,

$$\beta_\varepsilon\big(\Phi_\varepsilon(y)\big) = \frac{1}{a}\int_{\mathbb{R}^N}\zeta(\varepsilon x)\,|\Phi_\varepsilon(y,x)|^2\,dx = \frac{1}{a}\int_{\mathbb{R}^N}\zeta\big(y+\varepsilon z\big)\,|\Phi_\varepsilon(y,y/\varepsilon+z)|^2\,dz.$$

Set $g_{\varepsilon,y}(z)=|\Phi_\varepsilon(y,y/\varepsilon+z)|^2$. Then $\operatorname{supp} g_{\varepsilon,y}\subset B_{2R_\varepsilon}(0)$ for all y.

If $z\in B_{2R_\varepsilon}(0)$ and $y\in M\subset B_{1-\delta_M}(0)$, then

$$|y+\varepsilon z| \ \le\ |y|+\varepsilon|z| \ \le\ 1-\delta_M+2\varepsilon R_\varepsilon.$$

Since $\varepsilon R_\varepsilon \to 0$, there exists $\varepsilon_0 > 0$ such that $|y + \varepsilon z| \le 1 - \delta_M/2 < 1$ for all $y \in M$, $z \in B_{2R_\varepsilon}(0)$ and $0 < \varepsilon < \varepsilon_0$. By the definition of ζ, it follows that

$$\zeta\big(y + \varepsilon z\big) = y + \varepsilon z \qquad \text{for all } z \in \operatorname{supp} g_{\varepsilon,y},\ y \in M,\ 0 < \varepsilon < \varepsilon_0.$$

Using $\int_{\mathbb{R}^N} g_{\varepsilon,y}(z)\,dz = \|\Phi_\varepsilon(y)\|_2^2 = a$, we have

$$\beta_\varepsilon\big(\Phi_\varepsilon(y)\big) = \frac{1}{a}\int_{\mathbb{R}^N} \big(y + \varepsilon z\big)\, g_{\varepsilon,y}(z)\,dz = y + \varepsilon\,\frac{1}{a}\int_{\mathbb{R}^N} z\, g_{\varepsilon,y}(z)\,dz.$$

Hence

$$\big|\beta_\varepsilon(\Phi_\varepsilon(y)) - y\big| \ \le\ \varepsilon\,\frac{1}{a}\int_{\mathbb{R}^N} |z|\, g_{\varepsilon,y}(z)\,dz \ \le\ \varepsilon\,\frac{1}{a}\,(2R_\varepsilon)\int_{\mathbb{R}^N} g_{\varepsilon,y}(z)\,dz \ =\ 2\,\varepsilon R_\varepsilon.$$

The right-hand side is independent of y and tends to 0 as $\varepsilon \to 0$. Therefore

$$\sup_{y\in M} |\beta_\varepsilon(\Phi_\varepsilon(y)) - y| \ \le\ 2\,\varepsilon R_\varepsilon \ \xrightarrow[\varepsilon\to 0]{}\ 0,$$

which proves the claim. □

We now restrict attention to a thin sublevel where the functional is close to the autonomous ground energy. Define the near-peak sublevel set

$$\widetilde{S}_\varepsilon(a) = \Big\{u \in S(a) :\ J_{\varepsilon,T}(u) \le b_{0,T,a} + h(\varepsilon)\Big\}, \qquad h(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Lemma 5.4. *For all sufficiently small $\varepsilon > 0$,*

$$\Phi_\varepsilon(M) \subset \widetilde{S}_\varepsilon(a).$$

Proof. Set

$$H(\varepsilon) = \sup_{y\in M} \Big|J_{\varepsilon,T}(\Phi_\varepsilon(y)) - b_{0,T,a}\Big|.$$

By Lemma 5.1 we have $H(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, for every $\varepsilon > 0$ and every $y \in M$,

$$J_{\varepsilon,T}(\Phi_\varepsilon(y)) \ \le\ b_{0,T,a} + H(\varepsilon).$$

Since $h(\varepsilon) \to 0$, choose $\varepsilon_0 > 0$ such that $H(\varepsilon) \le h(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$. Then, for every $y \in M$ and $0 < \varepsilon < \varepsilon_0$,

$$J_{\varepsilon,T}(\Phi_\varepsilon(y)) \ \le\ b_{0,T,a} + H(\varepsilon) \ \le\ b_{0,T,a} + h(\varepsilon),$$

that is, $\Phi_\varepsilon(y) \in \widetilde{S}_\varepsilon(a)$. Taking the supremum over $y \in M$ gives the claim. □

Within this thin sublevel, every function has barycenter close to the zero set of the potential; thus low-energy mass concentrates near M.

Lemma 5.5. *For every fixed $\delta > 0$,*

$$\sup_{u\in \widetilde{S}_\varepsilon(a)}\ \inf_{z\in M_\delta} |\beta_\varepsilon(u) - z| \ \xrightarrow[\varepsilon\to 0]{}\ 0.$$

Proof. Since $M = \{x : V(x) = 0\}$ is compact by assumption (V) and $\chi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ equals the identity near the origin, choose χ so that

$$\chi(\xi) = \xi \quad \text{for all } \xi \in M_\delta.$$

Let

$$v_\delta \;=\; \inf\{\, V(x):\ \operatorname{dist}(x,M)\ge \delta/2\,\} \;>\; 0,$$

which is positive by continuity of V and compactness of $\{x : \operatorname{dist}(x,M)\ge\delta/2\}$.

Assume there exist $\delta_0>0$, $\varepsilon_n\to 0$, and $u_n\in\widetilde{S}_{\varepsilon_n}(a)$ such that

$$\operatorname{dist}\big(\beta_{\varepsilon_n}(u_n), M_\delta\big) \;\ge\; \delta_0 \qquad \forall n. \tag{5.3}$$

By Ekeland's variational principle on $S(a)$ for $J_{\varepsilon_n,T}\big|_{S(a)}$, there exist $v_n\in S(a)$ with

$$J_{\varepsilon_n,T}(v_n) \;\le\; J_{\varepsilon_n,T}(u_n) \le b_{0,T,a}+h(\varepsilon_n), \qquad \|(J_{\varepsilon_n,T}\big|_{S(a)})'(v_n)\|_{H^{-s}} \;\le\; \frac{1}{n},$$

and $\|v_n-u_n\|_{H^s}\le \frac{1}{n}$. Since

$$\big|\beta_\varepsilon(u)-\beta_\varepsilon(v)\big| \;\le\; \frac{\|\chi\|_\infty}{a}\,\big\||u|^2-|v|^2\big\|_{L^1} \;\le\; \frac{2\|\chi\|_\infty}{\sqrt{a}}\,\|u-v\|_{L^2},$$

we obtain, for n large,

$$\operatorname{dist}\big(\beta_{\varepsilon_n}(v_n), M_\delta\big) \;\ge\; \frac{\delta_0}{2}. \tag{5.4}$$

Thus $\{v_n\}$ is a $(PS)_{c_n}$ sequence with levels $c_n = J_{\varepsilon_n,T}(v_n)\le b_{0,T,a}+h(\varepsilon_n)$.

Fix $\rho_0>0$ as in Lemma 4.4. Since $h(\varepsilon_n)\to 0$, for n large one has $c_n < b_{0,T,a}+\rho_0$. By Lemma 4.4, up to a subsequence,

$$v_n \;\to\; v \quad \text{strongly in } H^s(\mathbb{R}^N), \qquad v\in S(a). \tag{5.5}$$

Write

$$J_{\varepsilon_n,T}(v_n) = J_{0,T}(v_n) + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x)\,|v_n|^2\,dx,$$

the negative parts of $J_{0,T}$ and $J_{\varepsilon_n,T}$ being identical. By Lemma 3.5 and Lemma 3.4, the limit v satisfies $\mathcal{P}_T(v)=0$. By Lemma 4.5 the truncation is inactive near $b_{0,T,a}$, hence v lies on the Pohozaev manifold of J_0 and

$$J_{0,T}(v) \;\ge\; b_{0,T,a}.$$

Passing to the limit using (5.5) gives

$$\liminf_{n\to\infty} J_{\varepsilon_n,T}(v_n) \;\ge\; J_{0,T}(v) + \frac{1}{2}\liminf_{n\to\infty}\int_{\mathbb{R}^N} V(\varepsilon_n x)\,|v_n|^2\,dx \;\ge\; b_{0,T,a} + \frac{1}{2}\liminf_{n\to\infty}\int V(\varepsilon_n x)|v_n|^2.$$

Since $J_{\varepsilon_n,T}(v_n)\le b_{0,T,a}+h(\varepsilon_n)\to b_{0,T,a}$, it follows that

$$\int_{\mathbb{R}^N} V(\varepsilon_n x)\,|v_n|^2\,dx \;\xrightarrow[n\to\infty]{}\; 0 \quad\text{and}\quad J_{0,T}(v)=b_{0,T,a}. \tag{5.6}$$

Decompose $\mathbb{R}^N=\Omega_n^{\mathrm{near}}\cup\Omega_n^{\mathrm{far}}$ where

$$\Omega_n^{\mathrm{near}} = \{x:\ \operatorname{dist}(\varepsilon_n x, M)<\delta/2\}, \qquad \Omega_n^{\mathrm{far}} = \mathbb{R}^N\setminus\Omega_n^{\mathrm{near}}.$$

By the choice of $v_\delta>0$,

$$\int_{\mathbb{R}^N} V(\varepsilon_n x)\,|v_n|^2 \;\ge\; v_\delta\int_{\Omega_n^{\mathrm{far}}}|v_n|^2.$$

Together with (5.6) this yields

$$\int_{\Omega_n^{\mathrm{far}}}|v_n|^2 \;\xrightarrow[n\to\infty]{}\; 0, \qquad \int_{\Omega_n^{\mathrm{near}}}|v_n|^2 \;\xrightarrow[n\to\infty]{}\; a. \tag{5.7}$$

Since $\chi = \mathrm{id}$ on M_δ, for $x \in \Omega_n^{\mathrm{near}}$ one has $\chi(\varepsilon_n x) \in M_\delta$ and therefore $\mathrm{dist}(\chi(\varepsilon_n x), M_\delta) = 0$. On Ω_n^{far}, $\mathrm{dist}(\chi(\varepsilon_n x), M_\delta) \le C_\chi$ with $C_\chi = \|\chi\|_\infty + \mathrm{diam}(M_\delta)$. Hence

$$\mathrm{dist}\big(\beta_{\varepsilon_n}(v_n), M_\delta\big) \;\le\; \frac{1}{a}\int_{\mathbb{R}^N} \mathrm{dist}\big(\chi(\varepsilon_n x), M_\delta\big)\,|v_n|^2\,dx \;\le\; \frac{C_\chi}{a}\int_{\Omega_n^{\mathrm{far}}} |v_n|^2 \;\xrightarrow[n\to\infty]{}\; 0,$$

by (5.7). This contradicts (5.4). The assumption (5.3) is false. The argument does not depend on the choice of $u_n \in \widetilde{S}_{\varepsilon_n}(a)$. Therefore

$$\sup_{u\in\widetilde{S}_\varepsilon(a)}\ \inf_{z\in M_\delta}\ |\beta_\varepsilon(u) - z| \;\to 0 \quad \text{as } \varepsilon \to 0.$$

□

Lemma 5.6. *Let $\varepsilon_n \to 0$ and $u_n \in S(a)$ satisfy $J_{\varepsilon_n,T}(u_n) \to b_{0,T,a}$. Then there exist translations $x_n \in \mathbb{R}^N$ and a ground state $w \in S(a)$ of the autonomous problem with $J_{0,T}(w) = b_{0,T,a}$ such that, up to a subsequence,*

$$u_n(\cdot + x_n) \;\to\; w(\cdot - y_0) \quad \textit{strongly in } H^s(\mathbb{R}^N)$$

for some $y_0 \in \mathbb{R}^N$. Moreover,

$$\mathrm{dist}\big(\beta_{\varepsilon_n}(u_n), M\big) \;\to\; 0.$$

Proof. By Ekeland's variational principle on $S(a)$, replace u_n by $v_n \in S(a)$ with

$$J_{\varepsilon_n,T}(v_n) \le J_{\varepsilon_n,T}(u_n) + \frac{1}{n}, \qquad \big\|(J_{\varepsilon_n,T}\big|_{S(a)})'(v_n)\big\|_{H^{-s}} \le \frac{1}{n}, \qquad \|v_n - u_n\|_{H^s} \le \frac{1}{n}.$$

Relabel v_n as u_n. Then $c_n = J_{\varepsilon_n,T}(u_n) \to b_{0,T,a}$ and $\{u_n\}$ is a $(PS)_{c_n}$ sequence.

By Lemma 4.1, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Vanishing is impossible by Lemma 4.2, since otherwise $\liminf_n J_{\varepsilon_n,T}(u_n) \ge \frac{V_\infty a}{2} > b_{0,T,a}$. Because $c_n \to b_{0,T,a}$, for any $h(\varepsilon) \to 0$ one has $u_n \in \widetilde{S}_{\varepsilon_n}(a)$ for all large n. Lemma 5.5 gives, for every fixed $\delta > 0$,

$$\mathrm{dist}\big(\beta_{\varepsilon_n}(u_n), M_\delta\big) \;\to\; 0.$$

Letting $\delta \to 0$ along a sequence yields

$$\mathrm{dist}\big(\beta_{\varepsilon_n}(u_n), M\big) \;\to\; 0. \tag{5.8}$$

Choose $y_n \in M$ with $|\beta_{\varepsilon_n}(u_n) - y_n| = \mathrm{dist}(\beta_{\varepsilon_n}(u_n), M) \to 0$ and set

$$w_n(x) = u_n\Big(x + \frac{y_n}{\varepsilon_n}\Big) \in S(a).$$

Define the translated functional

$$J^{(y_n)}_{\varepsilon_n,T}(v) = \frac{1}{2}\|(-\Delta)^{s/2}v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|v|^2\,dx - \frac{\tau(\|v\|_{H^s})}{2p}A_p(v) - \frac{1}{2q}A_q(v),$$

so that $J_{\varepsilon_n,T}(u_n) = J^{(y_n)}_{\varepsilon_n,T}(w_n)$. Then $\{w_n\}$ is a $(PS)_{c_n}$ sequence for $J^{(y_n)}_{\varepsilon_n,T}\big|_{S(a)}$, bounded and nonvanishing.

By compactness of M, up to a subsequence $y_n \to y_* \in M$. For each fixed $R > 0$, continuity of V gives $\sup_{|x|\le R} V(\varepsilon_n x + y_n) \to V(y_*) = 0$, hence

$$\int_{B_R(0)} V(\varepsilon_n x + y_n)\,|w_n|^2 \;\to\; 0.$$

On $B_R(0)^c$, tightness of $\{|w_n|^2\}$ follows from boundedness and the concentration–compactness exclusion of dichotomy at level $b_{0,T,a}$ as in the proof of Lemma 3.11; thus $\sup_n \int_{|x|>R} |w_n|^2$ can be made arbitrarily small by choosing R large. Therefore

$$\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\, |w_n|^2 \;\to\; 0. \tag{5.9}$$

Decompose

$$J^{(y_n)}_{\varepsilon_n,T}(w_n) = J_{0,T}(w_n) + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\, |w_n|^2\, dx.$$

Since $c_n \to b_{0,T,a}$ and (5.9) holds,

$$J_{0,T}(w_n) \;\to\; b_{0,T,a}.$$

Moreover,

$$\big\|(J_{0,T}|_{S(a)})'(w_n)\big\|_{H^{-s}} \;\le\; \big\|(J^{(y_n)}_{\varepsilon_n,T}|_{S(a)})'(w_n)\big\|_{H^{-s}} + C\Big(\int V(\varepsilon_n x + y_n)|w_n|^2\Big)^{1/2} \;\to\; 0,$$

so $\{w_n\}$ is a $(PS)_{b_{0,T,a}}$ sequence for $J_{0,T}|_{S(a)}$.

By Lemma 3.11, up to a subsequence and a fixed translation $y_0 \in \mathbb{R}^N$,

$$w_n \;\to\; w(\cdot - y_0) \quad \text{strongly in } H^s(\mathbb{R}^N),$$

where $w \in S(a)$ and $J_{0,T}(w) = b_{0,T,a}$. Since $w_n(x) = u_n(x + y_n/\varepsilon_n)$, set $x_n = y_n/\varepsilon_n$. Then $u_n(\cdot + x_n) = w_n \to w(\cdot - y_0)$ in $H^s(\mathbb{R}^N)$. The barycenter assertion follows from (5.8). □

The maps $\Phi_\varepsilon : M \to \widetilde{S}_\varepsilon(a)$ and $\beta_\varepsilon : \widetilde{S}_\varepsilon(a) \to M_\delta$ yield a Lusternik–Schnirelmann lower bound for the category of the near-peak sublevel set. We first recall the basic definitions of the Lusternik–Schnirelmann category.

Definition 5.7. *Let X be a topological space.*

1. *The (absolute) category of X is*

$$\operatorname{cat}(X) = \min\Big\{m \in \mathbb{N} :\; X = A_1 \cup \cdots \cup A_m,\; A_j \text{ closed in } X,\; A_j \text{ contractible in } X\Big\}.$$

   A set $A \subset X$ is contractible in X if there exist $x_* \in X$ and a homotopy $H : [0,1] \times A \to X$ with $H(0,a) = a$ and $H(1,a) = x_*$ for all $a \in A$.

2. *If $A \subset Z$, the relative category of A in Z is*

$$\operatorname{cat}_Z(A) = \min\Big\{m \in \mathbb{N} :\; A = B_1 \cup \cdots \cup B_m,\; B_j \text{ closed in } A,\; B_j \text{ contractible in } Z\Big\}.$$

Remark 5.8. *We record the properties used later:*

1. *(Monotonicity) If $A \subset B \subset Z$, then $\operatorname{cat}_Z(A) \le \operatorname{cat}_Z(B)$.*
2. *(Homotopy invariance) If $i : A \hookrightarrow Z$ is homotopic in Z to $g \circ f$ with $f : A \to Y$, $g : Y \to Z$, then $\operatorname{cat}_Z(A) \le \operatorname{cat}(Y)$.*
3. *(Deformation retract) If A is a strong deformation retract of Z (in particular, for the tubular neighborhood M_δ), then $\operatorname{cat}_Z(A) = \operatorname{cat}(A)$.*

Lemma 5.9. *For all sufficiently small $\varepsilon > 0$,*

$$\operatorname{cat}\big(\widetilde{S}_\varepsilon(a)\big) \;\ge\; \operatorname{cat}_{M_\delta}(M).$$

Proof. Fix $\delta > 0$ so small that the tubular neighborhood $M_\delta = \{x : \operatorname{dist}(x, M) \le \delta\}$ admits a strong deformation retraction

$$\mathscr{R} : [0,1] \times M_\delta \to M_\delta, \qquad \mathscr{R}(0,z) = z, \quad \mathscr{R}(1,z) = r(z) \in M, \quad \mathscr{R}(t,y) = y \text{ for } y \in M,$$

with $r : M_\delta \to M$ the nearest–point projection.

By Lemma 5.4 one has $\Phi_\varepsilon(M) \subset \widetilde{S}_\varepsilon(a)$ for ε small. By Lemma 5.5, $\beta_\varepsilon\big(\widetilde{S}_\varepsilon(a)\big) \subset M_\delta$, and by Lemma 5.3 the convergence

$$\sup_{y \in M} \big|\beta_\varepsilon(\Phi_\varepsilon(y)) - y\big| \ \to \ 0$$

holds as $\varepsilon \to 0$. Hence, for ε small,

$$\beta_\varepsilon \circ \Phi_\varepsilon : M \to M_\delta$$

is homotopic in M_δ to the inclusion $i : M \hookrightarrow M_\delta$ by the concatenation

$$[0,1] \times M \ni (t,y) \ \mapsto \ \begin{cases} \mathscr{R}\big(2t,\, \beta_\varepsilon(\Phi_\varepsilon(y))\big), & t \in \big[0, \tfrac{1}{2}\big], \\ \Gamma_\varepsilon(2t-1, y), & t \in \big[\tfrac{1}{2}, 1\big], \end{cases}$$

where $\Gamma_\varepsilon : [0,1] \times M \to M$ is a homotopy in M between $r \circ \beta_\varepsilon \circ \Phi_\varepsilon$ and id_M. Such a homotopy exists because $r \circ \beta_\varepsilon \circ \Phi_\varepsilon \to \mathrm{id}_M$ uniformly on the compact set M.

Let $m = \operatorname{cat}(\widetilde{S}_\varepsilon(a))$. There exist closed sets

$$\widetilde{S}_\varepsilon(a) = A_1 \cup \cdots \cup A_m,$$

and homotopies $H_j : [0,1] \times A_j \to \widetilde{S}_\varepsilon(a)$ with $H_j(0,u) = u$ and $H_j(1,u) \equiv u_j^* \in \widetilde{S}_\varepsilon(a)$. Since $\Phi_\varepsilon : M \to \widetilde{S}_\varepsilon(a)$ is continuous and injective,

$$B_j = \Phi_\varepsilon^{-1}\big(A_j \cap \Phi_\varepsilon(M)\big)$$

is closed in M, and $\{B_j\}_{j=1}^m$ covers M. Define

$$K_j : [0,1] \times B_j \to M_\delta, \qquad K_j(t,y) = \beta_\varepsilon\big(H_j\big(t, \Phi_\varepsilon(y)\big)\big).$$

Then $K_j(0,y) = \beta_\varepsilon(\Phi_\varepsilon(y)) \in M_\delta$ and $K_j(1,y) \equiv z_j^* = \beta_\varepsilon(u_j^*) \in M_\delta$, hence $\beta_\varepsilon \circ \Phi_\varepsilon(B_j)$ is contractible in M_δ to z_j^*. Composing with the homotopy joining $\beta_\varepsilon \circ \Phi_\varepsilon$ to i gives a contraction of B_j in M_δ. Thus each B_j is contractible in M_δ, and by the definition of relative category,

$$\operatorname{cat}_{M_\delta}(M) \ \le \ m \ = \ \operatorname{cat}\big(\widetilde{S}_\varepsilon(a)\big).$$

□

We now work in the energy range where $J_{\varepsilon,T}\big|_{S(a)}$ satisfies the Palais–Smale condition with precompactness, namely all levels $c < b_{0,T,a} + \rho_0$ from Lemma 4.4. On $S(a)$ we then apply the constrained deformation Lemma 5.10 to obtain critical points at the target level.

Lemma 5.10. *Let $J = J_{\varepsilon,T}\big|_{S(a)} : S(a) \to \mathbb{R}$ be C^1 and assume that J satisfies $(PS)_c$. Then for every neighborhood U of the critical set*

$$\mathcal{K}_c = \{u \in S(a) : \ J'(u) = 0, \ J(u) = c\}$$

and every $\eta_0 \in (0,1)$ there exist $\eta \in (0, \eta_0]$ and a continuous map $\mathscr{D} : [0,1] \times S(a) \to S(a)$ such that:

1. $\mathscr{D}(0,u) = u$ *for all* $u \in S(a)$*, and* $t \mapsto J(\mathscr{D}(t,u))$ *is nonincreasing.*

2. $\mathscr{D}(t,u) = u$ *for all* $u \in \left(J^{c-\eta}\right) \cup U$ *and all* $t \in [0,1]$*, where* $J^{\alpha} = \{u \in S(a) : J(u) \le \alpha\}$*.*

3. $\mathscr{D}\big(1,\, J^{c+\eta} \setminus U\big) \ \subset\ J^{c-\eta}$*.*

In particular, if $A \subset S(a)$ *is closed with* $J \le c$ *on* A*, then (shrinking* U *if needed)* $\mathscr{D}$ *is a descending deformation on* $S(a)$ *which strictly lowers* J *on* $A \setminus U$ *and fixes* $A \cap U$*.*

Proof. We work on the C^{∞} Hilbert submanifold $S(a) = \{u \in H^s(\mathbb{R}^N) : \|u\|_2^2 = a\}$ endowed with the ambient H^s metric. Since $J \in C^1(S(a), \mathbb{R})$ and $(PS)_c$ holds, the level-c critical set $\mathcal{K}_c$ is compact.

For $\delta \in (0,1)$ and $\rho > 0$ set

$$\mathcal{A}_{\rho,\delta} = \big\{u \in S(a) :\ |J(u) - c| \le \rho,\ \operatorname{dist}(u, \mathcal{K}_c) \ge \delta\big\}.$$

There exist $\rho, \sigma > 0$ such that

$$\|\nabla_S J(u)\|_{H^s} \ \ge\ \sigma \qquad \forall\, u \in \mathcal{A}_{\rho,\delta}, \tag{5.10}$$

where $\nabla_S J$ denotes the gradient of J on $S(a)$.

On $\Omega = \{u \in S(a) :\ \nabla_S J(u) \ne 0\}$ define

$$Y(u) = \frac{\nabla_S J(u)}{\|\nabla_S J(u)\|_{H^s}} \in T_u S(a).$$

Mollify Y via a smooth partition of unity on $S(a)$ to obtain a locally Lipschitz tangent field $V : \Omega \to TS(a)$ and constants $1 \le C_1 \le C_2$ with

$$C_1^{-1}\, \|\nabla_S J(u)\|_{H^s} \ \le\ \langle J'(u), V(u)\rangle \ \le\ C_2\, \|\nabla_S J(u)\|_{H^s} \qquad \forall\, u \in \Omega,$$

and $V(u) = 0$ when $\nabla_S J(u) = 0$.

Fix a neighborhood U of $\mathcal{K}_c$. Take $\delta > 0$ so small that $\overline{B_\delta(\mathcal{K}_c)} \subset U$ and pick $\eta \in (0, \rho)$. Choose $\xi \in C^1(\mathbb{R}; [0,1])$ with

$$\xi(t) = 0 \ \text{ if } \ t \le c - \eta, \qquad \xi(t) = 1 \ \text{ if } \ t \ge c - \frac{\eta}{2}, \qquad \xi' \ge 0.$$

Define

$$W(u) = \begin{cases} \xi\big(J(u)\big)\, \vartheta(u)\, V(u), & u \notin U, \\ 0, & u \in U, \end{cases}$$

where $\vartheta \in C^1(S(a); [0,1])$ satisfies $\vartheta \equiv 1$ on $\{u :\ |J(u) - c| \le \rho\} \setminus U$ and vanishes outside a slightly larger neighborhood. Then W is locally Lipschitz, tangent to $S(a)$, and supported in $\{u :\ J(u) \ge c - \eta\} \setminus U$. For $u \in \{\, |J - c| \le \rho \,\} \setminus U$,

$$\langle J'(u), W(u)\rangle \ \ge\ c_0\, \|\nabla_S J(u)\|_{H^s}, \qquad c_0 = C_1^{-1}/2 > 0. \tag{5.11}$$

Since W is locally Lipschitz on $S(a)$, the Cauchy problem

$$\dot\gamma(t) = -W(\gamma(t)), \qquad \gamma(0) = u \in S(a),$$

admits a unique solution on some interval; by the support of W solutions exist up to time $T = \eta / c_1$ for a suitable $c_1 > 0$. Define $\mathscr{D}(t,u) = \gamma(tT, u)$ on $[0,1] \times S(a)$. Then $\mathscr{D}(0,u) = u$, $\mathscr{D}(t,\cdot) \in S(a)$, and

$$\frac{d}{dt} J(\mathscr{D}(t,u)) \ = \ -T\, \langle J'(\mathscr{D}), W(\mathscr{D})\rangle \ \le 0.$$

If $u \in U$ or $J(u) \le c - \eta$, then $W(u) = 0$ and $\mathscr{D}(t,u) = u$ for all t, proving item (2).

While $\mathscr{D}(t,u) \in \{\,|J - c| \le \rho\,\} \setminus U$, by (5.10) and (5.11),

$$\frac{d}{dt} J(\mathscr{D}(t,u)) \;\le\; -T\, c_0\, \sigma.$$

Choose $T = \frac{2\eta}{c_0 \sigma}$. Then starting from any $u \in J^{\,c+\eta} \setminus U$ we obtain

$$J\big(\mathscr{D}(1,u)\big) \;\le\; J(u) - 2\eta \;\le\; c - \eta,$$

which gives item (3). The final statement follows by restricting $\mathscr{D}$ to a closed $A \subset S(a)$ with $J \le c$. $\square$

We are now in position to prove the main theorem.

Proof of Theorem 1.2. Fix $\delta > 0$. By Lemma 3.6,

$$b_{\infty,T,a} - b_{0,T,a} = \frac{V_\infty a}{2} > 0.$$

Choose $\rho_0 > 0$ as in Lemma 4.4 and then $\varepsilon_0 > 0$ so small that, for all $0 < \varepsilon < \varepsilon_0$,

$$\sup_{y \in M} \big|J_{\varepsilon,T}(\Phi_\varepsilon(y)) - b_{0,T,a}\big| \;\le\; \frac{1}{2}\rho_0, \qquad h(\varepsilon) \;\le\; \frac{1}{2}\rho_0.$$

By Lemma 5.1 and the definition of $\widetilde{S}_\varepsilon(a)$,

$$\Phi_\varepsilon(M) \;\subset\; \widetilde{S}_\varepsilon(a) \;\subset\; \big\{u \in S(a) : J_{\varepsilon,T}(u) \le b_{0,T,a} + \rho_0\big\} \qquad (0 < \varepsilon < \varepsilon_0).$$

Set $c := b_{0,T,a} + h(\varepsilon)$. Since $c < b_{0,T,a} + \rho_0$, Lemma 4.4 yields that $J_{\varepsilon,T}\big|_{S(a)}$ satisfies $(PS)_c$ on the whole strip $[\,b_{0,T,a},\, c\,]$. By Lemma 5.9,

$$\operatorname{cat}\big(\widetilde{S}_\varepsilon(a)\big) \;\ge\; \operatorname{cat}_{M_\delta}(M) =: k.$$

Introduce the sublevel $J^{\,c} := \{u \in S(a) : J_{\varepsilon,T}(u) \le c\}$. Since $\Phi_\varepsilon(M) \subset \widetilde{S}_\varepsilon(a) \subset J^{\,c}$, monotonicity of category gives

$$\operatorname{cat}\big(J^{\,c}\big) \;\ge\; \operatorname{cat}\big(\widetilde{S}_\varepsilon(a)\big) \;\ge\; k.$$

Define the Lusternik–Schnirelmann families and levels

$$\mathcal{F}_j := \{A \subset J^{\,c} :\; A \text{ closed in } S(a),\; \operatorname{cat}(A) \ge j\}, \qquad d_j := \inf_{A \in \mathcal{F}_j} \sup_{u \in A} J_{\varepsilon,T}(u), \quad j = 1, \dots, k.$$

Then

$$b_{0,T,a} \;\le\; d_1 \;\le\; \cdots \;\le\; d_k \;\le\; c.$$

If some d_j were not a critical value of $J_{\varepsilon,T}\big|_{S(a)}$, the constrained deformation lemma 5.10, applicable at level c under $(PS)_c$, would produce a descending deformation that strictly lowers the energy on $J^{\,d_j+\eta} \setminus U$ while fixing a neighborhood of the critical set at level d_j, contradicting the minmax characterization of d_j. Hence each d_j is a critical value. Therefore there exist at least $k = \operatorname{cat}_{M_\delta}(M)$ distinct constrained critical points

$$\mathcal{C}_\varepsilon \subset S(a), \qquad J_{\varepsilon,T}(u) \in [\,b_{0,T,a},\, b_{0,T,a} + h(\varepsilon)\,] \;\text{ for } u \in \mathcal{C}_\varepsilon.$$

By Lemma 3.10, every $u \in \mathcal{C}_\varepsilon$ has a Lagrange multiplier $\lambda < 0$. Fix the truncation radii with R_0 so large that the threshold c_* in Lemma 4.5 satisfies $c_* > b_{0,T,a} + 1$. For $0 < \varepsilon < \varepsilon_0$ small, $b_{0,T,a} + h(\varepsilon) < c_*$, hence Lemma 4.5 applies to each $u \in \mathcal{C}_\varepsilon$ and yields

$$\|u\|_{H^s} < R_0, \qquad \tau(\|u\|_{H^s}) = 1 \text{ near } u,$$

so that u is also a constrained critical point of the untruncated functional $J_\varepsilon\big|_{S(a)}$. Together with $\lambda < 0$, this gives normalized solutions (u, λ) of (1.1) with $\|u\|_2^2 = a$ and $\lambda < 0$. The number of distinct solutions is at least $\mathrm{cat}_{M_\delta}(M)$.

By construction of Φ_ε and Lemma 5.3, $\beta_\varepsilon(\Phi_\varepsilon(y)) \to y$ uniformly in $y \in M$ as $\varepsilon \to 0$. Lemma 5.5 then implies that for ε small the barycenters of all low-energy critical points lie arbitrarily close to M:

$$\mathrm{dist}\big(\beta_\varepsilon(u), M\big) \;\to\; 0 \qquad (u \in \mathcal{C}_\varepsilon).$$

Finally, choose any selection $u_\varepsilon \in \mathcal{C}_\varepsilon$ with $J_{\varepsilon,T}(u_\varepsilon) \to b_{0,T,a}$ along a sequence $\varepsilon \to 0$. By Lemma 5.6 there exist translations $x_\varepsilon \in \mathbb{R}^N$ and a ground state $w \in S(a)$ of the autonomous problem such that, up to a subsequence,

$$u_\varepsilon(\cdot + x_\varepsilon) \;\to\; w(\cdot - y_0) \quad \text{strongly in } H^s(\mathbb{R}^N),$$

for some $y_0 \in \mathbb{R}^N$, and $\mathrm{dist}(\beta_\varepsilon(u_\varepsilon), M) \to 0$.

This completes the proof of Theorem 1.2. $\square$

Acknowledgment

We express our gratitude to the anonymous referee for their meticulous review of our manuscript and valuable feedback provided for its enhancement. This work is supported by National Natural Science Foundation of China (12301145,12161007,12261107), Guangxi Natural Science Foundation Project (2023GXNSFAA026190) and Yunnan Fundamental Research Projects (202301AU070144, 202401AU070123). Y. Aikyn is supported by the Bolashak Government Scholarship of the Republic of Kazakhstan and M. Ruzhansky is supported by FWO Odysseus 1 Grant G.0H94.18N: Analysis and Partial Differential Equations and the Methusalem programme of the Ghent University Special Research Fund (Grant Number 01M01021).

Data availability: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of Interests: The author declares that there is no conflict of interest.

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