Kempe Changes in H-Free Graphs

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December 2, 2025

Abstract

Given a k-colouring of a graph G and two of the colours, a *Kempe chain* is a connected component of the subgraph of G induced by the vertices coloured with one of these two colours. A *Kempe swap* changes one colouring into another by interchanging the colours of the vertices in a Kempe chain. Two colourings are *Kempe equivalent* if each can be obtained from the other by a series of Kempe swaps; the set of Kempe equivalent colourings is called a *Kempe class*. For a graph G, let $\chi(G)$ denote its chromatic number and let $\mathcal{C}_k(G)$ denote the set of all k-colourings of G. We say G is *Kempe connected* if for all $k \geq \chi(G)$, $\mathcal{C}_k(G)$ forms a Kempe class. For a graph H, graph G is called *H-free* if no induced subgraph of G is isomorphic to H.

We prove that every H-free graph is Kempe connected if and only if H is an induced subgraph of the path on four vertices, P_4.

The graph $2K_2$ consists of four vertices and two edges which are not adjacent. We prove that for all $p \geq 0$, there is a k-colourable $2K_2$-free graph G such that $\mathcal{C}_{k+p}(G)$ does not form a Kempe class.

Keywords: Kempe chain, Kempe swap, forbidden induced subgraph, k-colouring, frozen colouring, P_4-free graph, $2K_2$-free graph.

1 Introduction

In this paper, all graphs are finite and simple. For a positive integer k, a k-colouring of G is a mapping from the vertex-set $V(G)$ of G to a set of colours $\{1, 2, \ldots, k\}$ such that no pair of adjacent vertices receive the same colour. A graph G is called k-colourable if it admits a k-colouring, and the *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colours required to colour G. Let $\mathcal{C}_k(G)$ denote the set of all k-colourings of G. A set of vertices of a graph G is called an *independent set* if no two of them are adjacent. A colouring of a graph partitions the vertices into independent sets called *colour classes*. The subgraph of a graph $G = (V, E)$ *induced* by $U \subseteq V$ is the graph with vertex-set U and edge-set all edges of G with both ends in U.

Given a k-colouring of a graph G and two of the colours, a *Kempe chain* is a connected component of the subgraph of G induced by the union of the two corresponding colour classes. A *Kempe swap* changes one colouring of G into another by interchanging the colours of the vertices in a Kempe chain. Two colourings are *Kempe equivalent* if each can be obtained from the other by a series of Kempe swaps; the set of Kempe equivalent colourings is called a *Kempe class*. We say a graph G is *Kempe connected* if for all $k \geq \chi(G)$, $\mathcal{C}_k(G)$ forms a Kempe class.

Kempe swaps were introduced by Kempe in his incorrect proof of the Four Colour Theorem [10]. They are used in the proof of the Five Colour Theorem and were important in the first and second proofs of the Four Colour Theorem [1, 18]. They were used by Meyniel and Las Vergnas to prove that all 5-colourings of a planar graph [13] and, more generally, of a K_5-minor-free

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graph [12] are Kempe equivalent. They are also useful in the study of edge-colouring (see for example, [6, 8, 11, 16, 20, 21]). For more background and problems on Kempe swaps, see a survey by Mohar [14] and a recent paper by Cranston and Feghali [5].

For a graph H, graph G is called *H-free* if no induced subgraph of G is isomorphic to H. For a collection $\mathcal{H}$ of graphs, graph G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$; $\mathcal{H}$-free classes of graphs are called *hereditary*. Let P_n, C_n, and K_n denote the path, cycle, and complete graph on n vertices, respectively. The complete graph K_3 is also known as a *triangle*. The *complement* $\overline{G}$ of a graph G is obtained from G by interchanging edges and non-edges. For two vertex-disjoint graphs G and H, the *disjoint union* of G and H, denoted by $G+H$, is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. For a positive integer r, the disjoint union of r copies of G is denoted by rG. For a vertex v of graph G, the open neighbourhood of v, denoted $N(v)$, is the set of vertices adjacent to v in G. The closed neighbourhood $N[v]$ of v is $N(v) \cup \{v\}$.

Hereditary classes $\mathcal{G}$ of graphs where every graph in the class is known to be Kempe connected include chordal graphs [15], bipartite graphs [14], and P_4-free graphs (which are also called cographs) [4]. The last class is most important for this paper:

Theorem 1 ([4])**.** *Every P_4-free graph is Kempe connected.*

In this paper, we prove:

Theorem 2. *Every H-free graph is Kempe connected if and only if H is an induced subgraph of P_4.*

We also prove:

Theorem 3. *For all $p \geq 0$, there is a k-colourable $2K_2$-free graph G such that $\mathcal{C}_{k+p}(G)$ does not form a Kempe class.*

We have previously studied what can be considered a specialization of Kempe swaps: In reconfiguration of graph colourings, we seek to transform one k-colouring into another by changing the colour of a single vertex at a time, always maintaining a k-colouring (thus always changing the colour of a vertex to a colour which does not appear in its neighbourhood). Note that such a single vertex recolouring corresponds to a Kempe swap on a Kempe chain consisting of just one vertex; this is sometimes called a *trivial Kempe swap*. For more on reconfiguration, see the surveys by van den Heuvel [19] and Nishimura [17].

A graph G is *k-mixing* if any k-colouring can be transformed into any other by single-vertex recolouring steps described above. In a k-colouring of K_k, it is not possible to change the colour of any vertex, so it is common to study reconfiguration of colourings with at least one colour more than the chromatic number. We call a graph G *recolourable* if it is k-mixing for every $k > \chi(G)$.

In [3], with Owen Merkel, we proved:

Theorem 4 ([3])**.** *Every H-free graph G is recolourable if and only if H is an induced subgraph of P_4 or of $P_3 + P_1$.*

Our main result, Theorem 2, on Kempe connectedness is a parallel result to Theorem 4 on reconfiguration of colourings.

2 Proof of Theorem 2

A k-colouring of a graph G is called *frozen* if for every vertex $v \in V(G)$, v is adjacent to a vertex of every colour different from its colour; in other words, a k-colouring is frozen if for each vertex v and for each of the k colours c, the closed neighbourhood $N[v]$ of v contains a vertex of colour c. A k-colouring of a graph G is called *Kempe frozen* if each of the k colours appears on some vertex and for any two colours, the subgraph induced by the union of the corresponding colour classes is connected.

Observation 1. *If a k-colouring of a graph is Kempe frozen, then it is frozen.*

Lemma 1. *Let G be a graph which has at least two k-colourings which give different partitions of the vertices. If G has a k-colouring which is Kempe frozen, then the k-colourings of G do not form a Kempe class.*

Proof. Assume the hypotheses. Let γ be a k-colouring which is Kempe frozen. Then, starting with γ, any Kempe swap only interchanges the colours in two of the colour classes, so we cannot obtain a colouring whose partition into colour classes is different from that of the starting colouring. □

Lemma 2. *There exists a triangle-free graph which is not Kempe connected.*

Proof. In [7], a class of triangle-free graphs each of which is Kempe frozen is given. The smallest graph in that class has 15 vertices. We give a graph on 14 vertices: see Figure 1 for a triangle-free graph G with $\chi(G) = 3$ such that $\mathcal{C}_3(G)$ does not form a Kempe class.

In the colouring on the left, for any two of the colours, the subgraph induced by the union of the two colour classes is connected; i.e., the colouring is Kempe frozen. The two colourings give different partitions of the vertices into colour classes. Thus by Lemma 1, the set of all 3-colourings does not form a Kempe class. □

A *Hamiltonian path* in a graph G is a path which contains all the vertices of G and a *Hamiltonian cycle* in G is a cycle which contains all the vertices of G.

Lemma 3. *There exists a C_4-free graph which is not Kempe connected.*

Proof. See Figure 2 for a C_4-free graph G with $\chi(G) = 3$ such that $\mathcal{C}_3(G)$ does not form a Kempe class.

In the colouring on the left, for any two of the colours, the subgraph induced by the union of the two colour classes is a Hamiltonian path and thus connected; i.e., the colouring is Kempe frozen. The two colourings give different partitions of the vertices into colour classes. Thus by Lemma 1, the set of 3-colourings does not form a Kempe class.

To see that the graph does not contain a C_4, first note that for the colouring on the left, any two colour classes induce a Hamiltonian path, so there can't be a C_4 that uses vertices of only two colours. So any C_4 would have two vertices, say a and b, of one colour and one of each of the other two colours, say vertices c and d; this would mean that c and d have the same neighbours in a's colour class, but that never happens. □

Corollary 1. *For all $k \geq 3$ there exists a k-colourable C_4-free graph H_k such that $\mathcal{C}_k(H_k)$ does not form a Kempe class.*

Proof. Let G be the graph of Figure 2. By Lemma 1, $\chi(G) = 3$ and $\mathcal{C}_3(G)$ does not form a Kempe class. For $k \geq 4$, by taking the join of G and K_{k-3}, we obtain a k-colourable graph H_k for which $\mathcal{C}_k(H_k)$ does not form a Kempe class. □

Lemma 4. *Suppose H is a graph on at most four vertices. Every H-free graph is Kempe connected if and only if H is an induced subgraph of P_4.*

Proof. By Theorem 1, if H is an induced subgraph of P_4 then every H-free graph is Kempe connected. So we may assume that H is not an induced subgraph of P_4.

Suppose H contains a triangle, then see Figure 1 for an H-free graph which is not Kempe connected. Suppose H contains a $3K_1$, then see Figure 3 for an H-free graph which is not Kempe connected. There are two (triangle, $3K_1$)-free graphs on at most four vertices which are not induced subgraphs of P_4. They are $2K_2$ and C_4. See Figure 3 for a $2K_2$-free graph which is not Kempe connected, and see Figure 2 for a C_4-free graph which is not Kempe connected. □

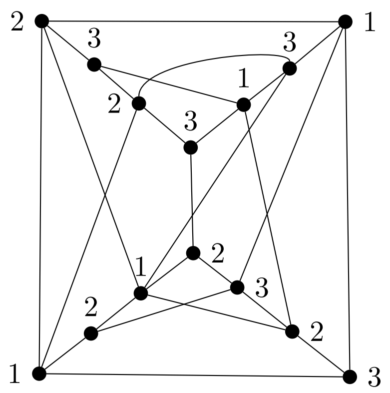

Figure 1: Two 3-colourings of a triangle-free graph which are not Kempe equivalent

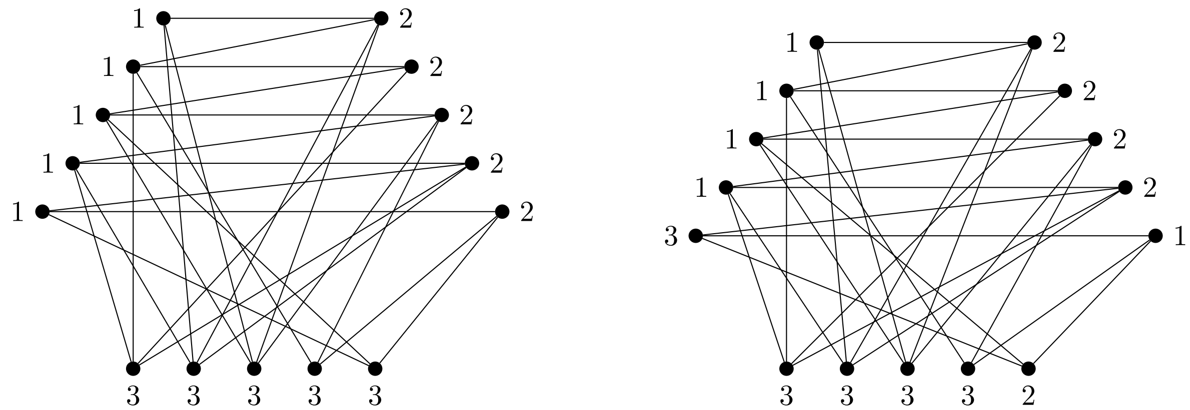

Figure 2: Two 3-colourings of a C_4-free graph which are not Kempe equivalent

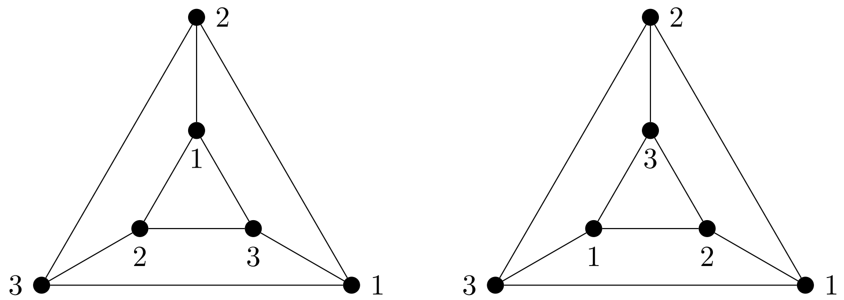

Figure 3: Two 3-colourings of the triangular prism which are not Kempe equivalent [19].

Lemma 5. *For every graph H on five vertices, there exists an H-free graph which is not Kempe connected.*

Proof. Note that there are 34 graphs on 5 vertices. We partition them into three groups: Group 1 consists of the 20 graphs which contain a triangle. Group 2 consists of the 13 graphs which are triangle-free but contain a $3K_1$. Group 3 consists of C_5. These facts can be easily verified (for example, on houseofgraphs.org).

If H is in Group 1, then the lemma follows from Lemma 2. The triangular prism $\overline{C_6}$ in Figure 3 is a ($3K_1$, C_5)-free graph which is not Kempe connected. Thus, if H is in Group 2 or Group 3, then there is an H-free graph which is not Kempe connected. □

Theorem 2 follows from Lemmas 4 and 5.

3 Kempe Frozen Colourings of $2K_2$-free graphs

In [2], several classes of $2K_2$-free graphs with frozen colourings are given. We will show that for some of these classes, the frozen colourings are also Kempe frozen. Also in [2], an operation was given which transforms a k-colourable graph with a frozen $(k+1)$-colouring into a $(k+1)$-colourable graph with a frozen $(k+2)$-colouring, and preserves being $2K_2$-free. (There are some restrictions on the graph, the colouring, and the frozen colouring.) We will show that the operation does the same for Kempe frozen colourings.

A set of vertices of a graph G is called a *clique* if every pair of them is adjacent. A *clique partition* of a graph is a partition of its vertices into cliques, and a partition into at most k cliques is called a *k-clique partition.* A *frozen k-clique partition* of a graph G is a partition of its vertices into k nonempty cliques such that each vertex v has a nonneighbour in every clique of the partition other than the clique containing it; in other words, $V(G) \setminus N(v)$ contains a vertex of each of the k cliques. A *Kempe frozen k-clique partition* of a graph G is a partition of the vertices into k nonempty cliques, such that the subgraph induced by any two of the cliques is a graph whose complement is connected.

Because the classes of graphs of [2] are dense, it is easier to visualize their complements, which are C_4-free graphs which admit frozen clique partitions.

Remark 1. A Kempe frozen k-colouring of a graph corresponds to a Kempe frozen k-clique partition of its complement. In the frozen clique partitions in this section, all cliques have size 2, so for the clique partitions to be Kempe frozen, for any two cliques of the clique partition, there must be at most one edge joining them, thus the subgraph induced by the vertices of the two cliques is either $2K_2$ or P_4.

For an integer $q \geq 2$, $\overline{D_q}$ is the graph with $4q+2$ vertices $\{u_i : i = 0, 1, \ldots, q+1\} \cup \{\cup\{v_{i1}, v_{i2}, v_{i3}\} : i = 1, 2, \ldots, q\}$
whose edges are:

- the edges of the Hamiltonian cycle C: u_0, u_1, $\ldots$, u_{q+1}, v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{33},$\ldots$, v_{q1}, v_{q2}, v_{q3}, u_0.
- edges $u_i v_{i2}$ for $i = 1, 2, \ldots, q$
- edges $v_{i1} v_{i3}$ for $i = 1, 2, \ldots, q$

See Figure 4 for $\overline{D_2}$ and and Figure 5 for $\overline{D_3}$.

Consider the $(2q+1)$-colouring ψ of D_q which partitions the vertices into the following colour classes:
$\{u_1, v_{12}\}, \{u_2, v_{22}\}, \ldots, \{u_q, v_{q2}\}, \{u_{q+1}, v_{11}\}, \{v_{13}, v_{21}\}, \{v_{23}, v_{31}\}, \ldots, \{v_{q-1\,3}, v_{q\,1}\}, \{v_{q3}, u_0\}$

Theorem 5 ([2])**.** *For $q \geq 2$, D_q is a $2K_2$-free graph such that*

$$\chi(D_q) = \omega(D_q) = \begin{cases} (3q+2)/2 & \textit{if } q \textit{ is even} \\ (3q+3)/2 & \textit{if } q \textit{ is odd} \end{cases}$$

and ψ is a frozen colouring of D_q.

Lemma 6. *For $q \geq 2$, the $(2q+1)$-colouring ψ of D_q is Kempe frozen.*

Proof. Let $q \geq 2$. It is easily seen by considering $\overline{D_q}$ and Remark 1 that for any two colours, a and b, of the colouring ψ of D_q, the subgraph induced by the vertices coloured a or b in D_q is connected. □

Lemma 7. *For $q \geq 2$, the set $\mathcal{C}_{2q+1}(D_q)$ of all $(2q+1)$-colourings of D_q does not form a Kempe class.*

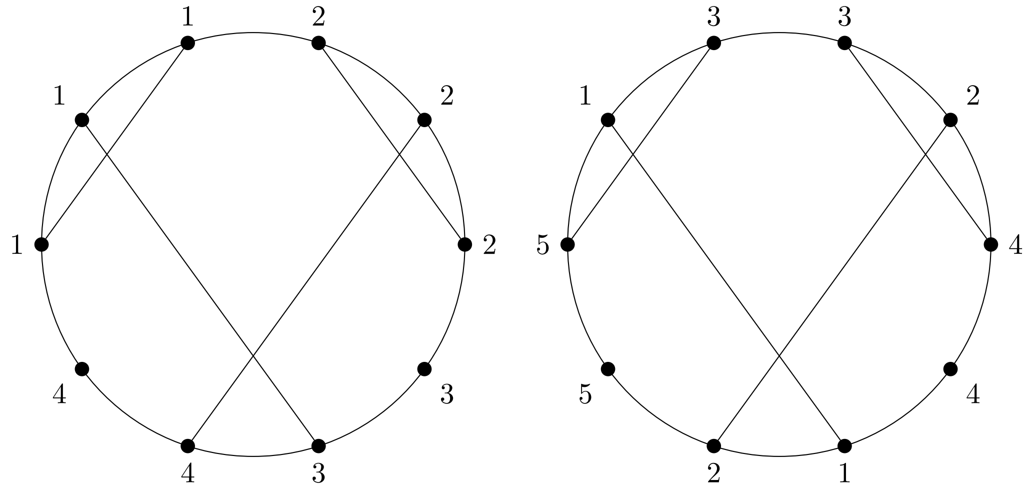

Figure 4: A C_4-free graph $\overline{D_2}$ with a 4-clique-partition (left) and a Kempe frozen 5-clique-partition (right). The numbers indicate which clique a vertex is in. Equivalently, the numbers indicate a 4-colouring of the complement D_2 of the graph shown (left) and a Kempe frozen 5-colouring of D_2 (right).

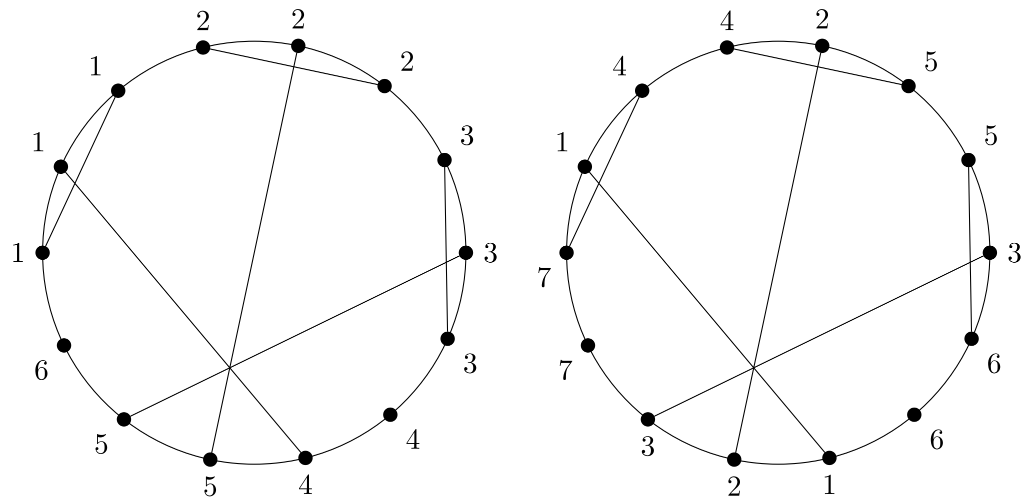

Figure 5: A C_4-free graph $\overline{D_3}$ with a 6-clique-partition (left) and a Kempe frozen 7-clique-partition (right). The numbers indicate which clique a vertex is in. Equivalently, the numbers indicate a 6-colouring of the complement D_3 of the graph shown (left) and a Kempe frozen 7-colouring of D_3 (right).

Proof. Let $q \geq 2$. Note that $\chi(D_q) < 2q+1$. By Lemma 6, the colouring ψ of D_q is Kempe frozen and thus not Kempe equivalent to any minimum colouring of D_q. So the set of all $(2q+1)$-colourings of D_q does not form a Kempe class. $\square$

We now define a second class of graphs. For $r \geq 1$, $\overline{Y_r}$ is the graph with $6r$ vertices
$\{\cup\{v_{i1}, v_{i2}, v_{i3}\} : i = 1, 2, \ldots, 2r\}$
whose edges are:

- the edges of a Hamiltonian cycle C: $v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{33}, \ldots, v_{2r\,1}, v_{2r\,2}, v_{2r\,3}, v_{11}$
- edges $v_{i1}v_{i3}$ for $i = 1, 2, \ldots, 2r$
- edges $v_{i2}v_{i+r\,2}$ for $i = 1, 2, \ldots, r$

We refer to $\{v_{i1}, v_{i2}, v_{i3}\}$ as *triangle* i. Note that $\overline{Y_r}$ consists of a Hamiltonian cycle C together with $2r$ edges which induce $2r$ vertex-disjoint triangles with consecutive pairs of edges of C, and r more edges pairing the middle vertices v_{i2} of "opposite" triangles.

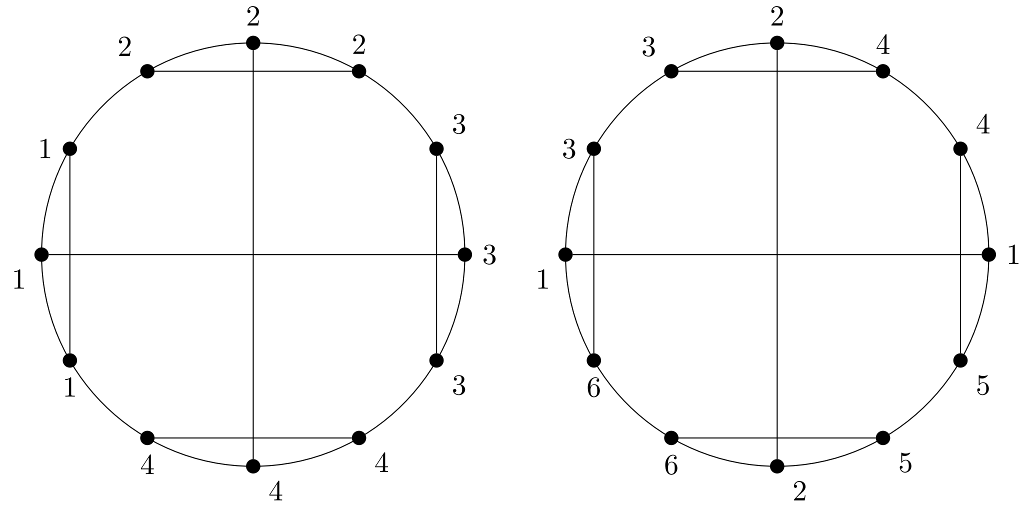

Figure 6: A C_4-free graph $\overline{Y_2}$ with a 4-clique-partition (left) and a Kempe frozen 6-clique-partition (right). Equivalently, a 4-colouring of the complement Y_2 (left) and a Kempe frozen 6-colouring (right).

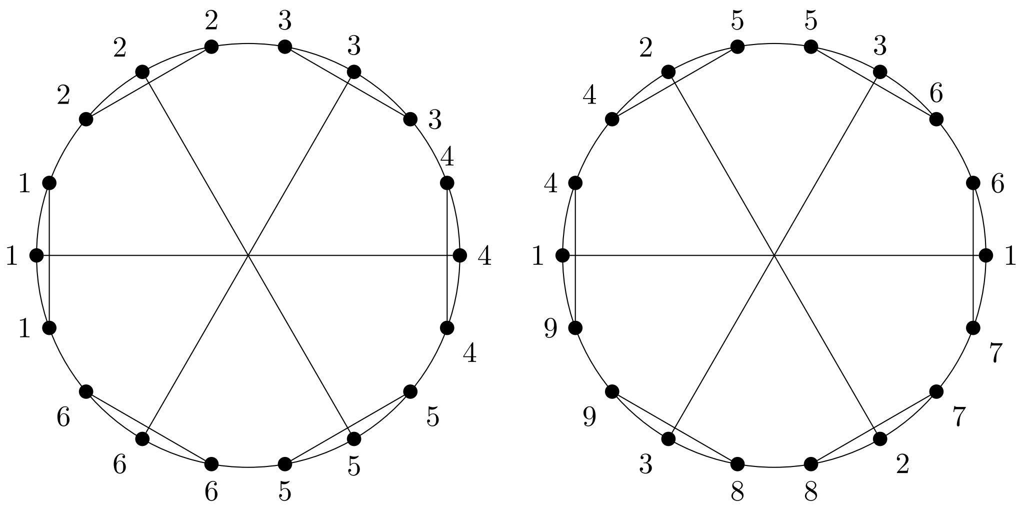

Figure 7: A C_4-free graph $\overline{Y_3}$ with a 6-clique-partition (left) and a Kempe frozen 9-clique-partition (right). Equivalently, a 6-colouring of the complement Y_3 (left) and a Kempe frozen 9-colouring (right).

See Figure 6 for $\overline{Y_2}$ and Figure 7 for $\overline{Y_3}$. Note that $\overline{Y_1}$ is $\overline{C_6}$.

Consider the $3r$-colouring ζ of Y_r which partitions the vertices into the following colour classes:
$\{v_{12}, v_{r+1\,2}\}, \{v_{22}, v_{r+2\,2}\}, \ldots, \{v_{r2}, v_{2r\,2}\}, \{v_{13}, v_{21}\}, \{v_{23}, v_{31}\}, \ldots, \{v_{2r\,3}, v_{11}\}$.

Theorem 6 ([2])**.** *For $r \geq 2$, Y_r is a $2K_2$-free graph with $\chi(Y_q) = \omega(Y_q) = 2r$, and ζ is a frozen $3r$-colouring of Y_r.*

Note that, $\overline{Y_1} \cong \overline{C_6}$ is the triangular prism, whose chromatic number is 3 and the set of 3-colourings $\mathcal{C}_3(\overline{Y_1})$ does not form a Kempe class (see Figure 3).

Lemma 8. *For $r \geq 2$, the $3r$-colouring ζ of Y_r is Kempe frozen.*

The proof is similar to the Proof of Lemma 6.

class	q/r	n	$\chi \quad (= \omega)$	# colours in K-frozen colouring	(# colours in K-frozen colouring) - χ
D_q	$q \geq 2$	$4q+2$	$(3q+2)/2$ for even q; $(3q+3)/2$ for odd q	$n/2 = 2q+1$	$q/2$ for even q; $(q-1)/2$ for odd q
Y_r	$r \geq 2$	$6r$	$2r$	$3r$	r

Table 1: Parameters of $2K_2$-free graph classes

Lemma 9. *For $r \geq 2$, the set $\mathcal{C}_{3r}(Y_r)$ of all $3r$-colourings of Y_r does not form a Kempe class.*

Proof. Let $r \geq 2$. Similar to the proof of Lemma 7, note that $\chi(Y_r) < 3r$. By Lemma 8, the colouring ζ of Y_r is Kempe frozen and thus not Kempe equivalent to any minimum colouring of Y_r. So the set of all $3r$-colourings of Y_r does not form a Kempe class. □

We can now prove Theorem 3: For all $p \geq 0$, there is a k-colourable $2K_2$-free graph G such that $\mathcal{C}_{k+p}(G)$ does not form a Kempe class.

Proof. For $p = 0$, as noted above, $\overline{C_6}$ is a 3-chromatic graph and $\mathcal{C}_3(\overline{C_6})$ does not form a Kempe class.
For $p = 1$, D_2 is a 4-chromatic graph and by Lemma 7, $\mathcal{C}_5(D_2)$ does not form a Kempe class.
For $p \geq 2$, Y_p is a $2p$-chromatic graph and by Lemma 9, $\mathcal{C}_{3p}(Y_p)$ does not form a Kempe class. □

In [2], the following operation was defined and Theorem 7 was proved.

Operation 1. *Given a graph G and nonadjacent vertices x and y in G, we define the following operation to create a new graph G'. Define G' to be the graph G together with two additional vertices u and v and with edges vx, xy and yu; join u and v to all vertices of $G - \{x, y\}$.*

Note that this operation corresponds to subdividing the edge xy of $\overline{G}$, that is, replacing xy by the path x, u, v, y.

Two sets X and Y of vertices of a graph G are called *anticomplete* if no edge of G has an end in each set and called *complete* if every vertex in one set is adjacent to every vertex in the other set. A universal vertex w in a graph G is a vertex which is adjacent to all other vertices of G.

Theorem 7 ([2])**.** *Let G be a k-colourable graph with a k-colouring β and a frozen $(k+1)$-colouring γ, and let x and y be nonadjacent vertices of G such that $\beta(x) \neq \beta(y)$ and such that either*

(1) $\gamma(x) \neq \gamma(y)$, or

(2) $\{x, y\}$ is a colour class of γ.

Then the graph G' of Operation 1 is $(k+1)$-colourable and admits a frozen $(k+2)$-colouring. Furthermore,

(3) if G is k-chromatic, then G' is $(k+1)$-chromatic.

(4) if G is $2K_2$-free and if in case (1), there is no edge rs such that $\{r, s\}$ is anticomplete to $\{x, y\}$, then G' is $2K_2$-free.

We now extend Theorem 7 to Kempe frozen colourings:

Theorem 8. *Let G be a k-colourable graph with a k-colouring β and a Kempe frozen $(k+1)$-colouring γ, and let x and y be nonadjacent vertices of G such that $\beta(x) \neq \beta(y)$ and such that either*

(1) $\gamma(x) \neq \gamma(y)$, or

(2) $\{x, y\}$ is a colour class of γ.

Then the graph G' of Operation 1 is $(k+1)$-colourable and admits a Kempe frozen $(k+2)$-colouring. So the set $\mathcal{C}_{k+2}(G')$ of all $(k+2)$-colourings of G' does not form a Kempe class. Furthermore,

(3) if G is k-chromatic, then G' is $(k+1)$-chromatic.

(4) if G is $2K_2$-free and if in case (1), there is no edge rs such that $\{r, s\}$ is anticomplete to $\{x, y\}$, then G' is $2K_2$-free.

A $(k+1)$-colouring of G' is the colouring β of G extended by making $\{u, v\}$ a new colour class. A new Kempe frozen $(k+2)$-colouring γ' of G' is obtained from the Kempe frozen colouring γ of G by

- In Case (1), making $\{u, v\}$ a new colour class.
- In Case (2), give x a new colour and assign this colour to u as well; give v the colour $\gamma(y)$.

Proof. Assume the hypotheses. We only need to prove that γ' is a Kempe frozen $(k+2)$-colouring of G'.
In Case (1): Since γ is a Kempe frozen colouring of G, every pair of colour classes induces a connected subgraph, and this remains true when the edge xy is added.

We only need to check that any colour class of γ together with $\{u, v\}$ induces a connected graph. That is certainly true for any colour class of γ other than the colour class containing x and the colour class containing y since both u and v are complete to $G \setminus \{x, y\}$. In any Kempe frozen colouring, if there is a colour class consisting of a single vertex, say w, then w must be a universal vertex. In G, x and y are nonadjacent, so neither is a universal vertex, and thus there is vertex x' of G different from x in the colour class $I(x)$ of γ containing x and a vertex y' different from y in the colour class $I(y)$ of γ containing y. In G', u and v are complete to x' and to y', and x is adjacent to v, and y is adjacent to u, so $I(x) \cup \{u, v\}$ induces a connected subgraph, and $I(y) \cup \{u, v\}$ induce a connected subgraph.
In Case (2): Since γ is a Kempe frozen colouring of G, any two colour classes of γ induce a connected subgraph of G. Adding the edge xy to G does not change this, although $\{x, y\}$ is no longer a colour class. Thus any two colour classes of γ' induce a connected subgraph of G' except possibly if one of them is $\{x, u\}$ or $\{y, v\}$. The set $\{u, y, x, v\}$ induced a P_4 in G'. Consider a colour class I of γ different from $\{x, y\}$. Since $\{x, y\} \cup I$ induces a connected subgraph of G, there must be a vertex z of I adjacent to both x and y, and every other vertex of I must be adjacent to at least one of x and y. Now consider the union of the vertices of the colour classes $\{x, u\}$ and I of G'. Vertex u is adjacent to every vertex of I and vertex $z \in I$ is adjacent to x. Thus the subgraph induced by $\{x, u\} \cup I$ is connected in G'. Analogously, the subgraph induced by $\{y, v\} \cup I$ is connected in G'. Thus γ' is Kempe frozen. □

4 Open problems

By Theorem 3, there is no constant $c \geq 0$ such that for every $2K_2$-free k-colourable graph G the set of all $(k+c)$-colourings of G form a Kempe class. For $k \geq 3$, graph H_k of [7] is a 3-chromatic triangle-free graph on $5k$ vertices and admits a Kempe frozen k-colouring. We ask the following.

Question 1. *Where H is either C_4 or $3K_1$, is there a constant $c \geq 0$ such that for every k-colourable H-free graph G the set of all $(k+c)$-colourings of G forms a Kempe class?*

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