Matrix Quasi-tree Theorem

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Abstract

Building on prior work that established Matrix Quasi-tree Theorems for special embedded graphs, in this paper, we develop a comprehensive theory applicable to all embedded graphs. We introduce symbolic skew-adjacency matrices and reduction maps as key innovations, and prove that a specific polynomial derived from these matrices encodes all spanning quasi-trees of a bouquet. This result provides a complete analogue of the Matrix Tree Theorem for topological graph theory, with applications to quasi-tree enumeration in both orientable and non-orientable embedded graphs.

1 Introduction

The classical Matrix Tree Theorem, established by Kirchhoff [11] and further developed by Tutte [19], provides an elegant combinatorial interpretation of determinants for enumerating spanning trees in graphs.

Let G be a finite loopless multigraph with vertex set $V(G)$ and edge set $E(G)$. For an edge $e \in E(G)$, we denote its set of endpoints by $v(e) \subseteq V(G)$. We associate a variable x_e to each edge e, and for any subset $F \subseteq E(G)$, we define the monomial

$$x_F = \prod_{e \in F} x_e.$$

The *Kirchhoff matrix* (or Laplacian matrix) $\mathbf{K} = (\ell_{ij})$ is defined as a symmetric matrix indexed

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by the vertices of G, with entries given by

$$\ell_{ij} = -\sum_{\substack{e \in E(G) \\ v(e)=\{i,j\}}} x_e, \quad \text{if } i \neq j, \quad \text{and} \quad \ell_{ii} = \sum_{\substack{e \in E(G) \\ i \in v(e)}} x_e.$$

Since each row of $\mathbf{K}$ sums to zero, its determinant vanishes. However, the determinant of the submatrix $\mathbf{K}^{(p)}$ obtained by deleting the pth row and column is independent of the choice of p. This determinant defines the *Kirchhoff polynomial* $\mathcal{K}_G$ in the variables x_e.

The Matrix Tree Theorem [19, Theorem VI.29] states that the Kirchhoff polynomial expands as

$$\mathcal{K}_G = \det \mathbf{K}^{(p)} = \sum_T x_{E(T)},$$

where the sum is taken over all spanning trees of G. Each monomial $x_{E(T)}$ appears with coefficient 1 and corresponds to a spanning tree of the graph. For further details, we refer to [15].

Recent developments in graph polynomials [4, 5, 20] and delta-matroid [7, 8] reveal a fundamental correspondence: spanning quasi-trees in topological graph theory are the natural analogue of spanning trees in abstract graph theory.

Merino et al. [16] established an analogue of Kirchhoff's Matrix Tree Theorem for orientable embedded graphs, replacing the count of spanning trees with that of spanning quasi-trees. This result can alternatively be derived from the work of Macris and Pule [14, Theorem 1] on Eulerian circuit enumeration (see also [12, 13]).

Theorem 1.1 ([2, 12, 14, 16]). *Let B be an orientable bouquet with edge set $[n] = \{1, 2, \ldots, n\}$ and signed rotation σ, and let $\mathbf{A}^{\mathbf{u}}_{\sigma}$ be its unsymbolic skew-adjacency matrix. Then the number of spanning quasi-trees of B is*

$$\tau(B) = \det(I_n + \mathbf{A}^{\mathbf{u}}_{\sigma}).$$

This result was subsequently extended by Deng et al. [9] to bouquets containing exactly one non-orientable loop using delta-matroid theory. This progression naturally leads to the following problem:

Problem 1.2. *Establish a Matrix Quasi-tree Theorem for all embedded graphs.*

In this paper, we introduce the novel concepts of symbolic skew-adjacency matrices and reduction maps f (defined in Section 3), and use them to establish a comprehensive Matrix Quasi-tree Theorem for all embedded graphs.

Throughout this paper, for any polynomial $P \in \mathbb{Z}[x_A : A \subseteq [n]]$, we define $P \bmod 2$ to be the polynomial obtained by reducing each coefficient modulo 2.

Theorem 1.3 (Matrix Quasi-tree Theorem). *Let B be a bouquet with edge set $[n]$ and signed rotation σ, and let $\mathbf{A}^{\mathbf{s}}_{\sigma}$ be its symbolic skew-adjacency matrix. Then*

$$f(\det(I_n + \mathbf{A}^{\mathbf{s}}_{\sigma})) \bmod 2 = \sum_X x_X,$$

where the sum is taken over all the edge sets of spanning quasi-trees in B. Consequently,

$$\tau(B) = f(\det(I_n + \mathbf{A}^{\mathbf{s}}_{\sigma})) \bmod 2\big|_{x_X=1}.$$

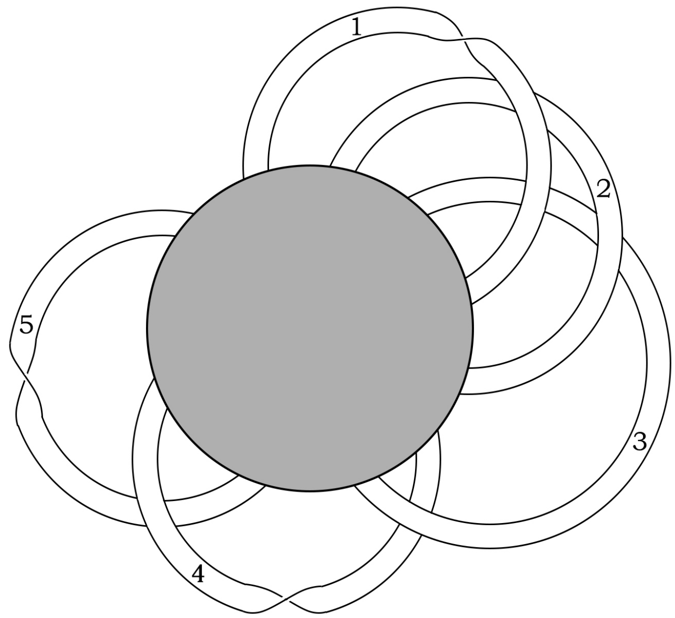


Figure 1: A signed rotation of the bouquet is $[-1^a, 2^a, 3^a, 1^b, 2^b, -4^a, 3^b, -5^a, 4^b, 5^b]$.

An example demonstrating Theorem 1.3 is given as follows.

Example 1.4. *Let B be a bouquet with signed rotation σ as depicted in Figure 1:*

$$\sigma = [-1^a, 2^a, 3^a, 1^b, 2^b, -4^a, 3^b, -5^a, 4^b, 5^b].$$

Its symbolic skew-adjacency matrix is

$$\mathbf{A}^{\mathbf{s}}_\sigma = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 \\ -x_{12} & 0 & x_{23} & 0 & 0 \\ -x_{13} & -x_{23} & 0 & x_{34} & 0 \\ 0 & 0 & -x_{34} & x_{44} & x_{45} \\ 0 & 0 & 0 & -x_{45} & x_{55} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(I_5 + \mathbf{A}^{\mathbf{s}}_\sigma) = {} & x_{11}x_{23}^2x_{44}x_{55} + x_{11}x_{23}^2x_{44} + x_{11}x_{23}^2x_{45}^2 + x_{11}x_{23}^2x_{55} + x_{11}x_{23}^2 + x_{11}x_{34}^2x_{55} \\ & + x_{11}x_{34}^2 + x_{11}x_{44}x_{55} + x_{11}x_{44} + x_{11}x_{45}^2 + x_{11}x_{55} + x_{11} + x_{12}^2x_{34}^2x_{55} \\ & + x_{12}^2x_{34}^2 + x_{12}^2x_{44}x_{55} + x_{12}^2x_{44} + x_{12}^2x_{45}^2 + x_{12}^2x_{55} + x_{12}^2 + x_{13}^2x_{44}x_{55} \\ & + x_{13}^2x_{44} + x_{13}^2x_{45}^2 + x_{13}^2x_{55} + x_{13}^2 + x_{23}^2x_{44}x_{55} + x_{23}^2x_{44} + x_{23}^2x_{45}^2 \\ & + x_{23}^2x_{55} + x_{23}^2 + x_{34}^2x_{55} + x_{34}^2 + x_{44}x_{55} + x_{44} + x_{45}^2 + x_{55} + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} f(\det(I_5 + \mathbf{A}_\sigma^{\mathbf{s}})) &= x_{12345} + x_{1234} + x_{12345} + x_{1235} + x_{123} + x_{1345} \\ &\quad + x_{134} + x_{145} + x_{14} + x_{145} + x_{15} + x_1 + x_{12345} \\ &\quad + x_{1234} + x_{1245} + x_{124} + x_{1245} + x_{125} + x_{12} + x_{1345} \\ &\quad + x_{134} + x_{1345} + x_{135} + x_{13} + x_{2345} + x_{234} + x_{2345} \\ &\quad + x_{235} + x_{23} + x_{345} + x_{34} + x_{45} + x_4 + x_{45} + x_5 + 1 \\ &= 1 + x_1 + x_{12} + x_{123} + 2x_{1234} + 3x_{12345} + x_{1235} \\ &\quad + x_{124} + 2x_{1245} + x_{125} + x_{13} + 2x_{134} + 3x_{1345} + x_{135} \\ &\quad + x_{14} + 2x_{145} + x_{15} + x_{23} + x_{234} + 2x_{2345} + x_{235} \\ &\quad + x_{34} + x_{345} + x_4 + 2x_{45} + x_5. \end{aligned}$$

Reducing modulo 2, we obtain

$$\begin{aligned} f(\det(I_5 + \mathbf{A}_\sigma^{\mathbf{s}})) \bmod 2 &= 1 + x_1 + x_{12} + x_{123} + x_{12345} + x_{1235} + x_{124} + x_{125} \\ &\quad + x_{13} + x_{1345} + x_{135} + x_{14} + x_{15} + x_{23} + x_{234} \\ &\quad + x_{235} + x_{34} + x_{345} + x_4 + x_5. \end{aligned}$$

Setting all $x_X = 1$ for $X \subseteq [n]$ recovers $\tau(B) = 20$ and the set of spanning quasi-trees of B is

$$\{\emptyset, \{1\}, \{12\}, \{123\}, \{12345\}, \{1235\}, \{124\}, \{125\}, \{13\}, \{1345\},$$
$$\{135\}, \{14\}, \{15\}, \{23\}, \{234\}, \{235\}, \{34\}, \{345\}, \{4\}, \{5\}\}.$$

This example clearly demonstrates that the reduction modulo 2 in Matrix Quasi-tree Theorem is essential. As seen in the computation, the polynomial $f(\det(I_5 + \mathbf{A}_\sigma^{\mathbf{s}}))$ contains coefficients greater than 1 (specifically, coefficients 2 and 3 appear in various terms). The modulo 2 operation ensures that each spanning quasi-tree contributes exactly once to the count, eliminating the overcounting that would otherwise occur due to these higher coefficients. This phenomenon underscores the necessity of the mod 2 reduction in the main theorem statement.

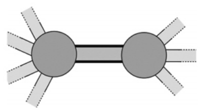
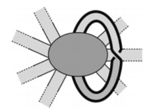
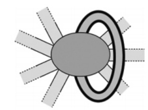
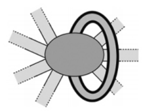
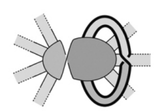
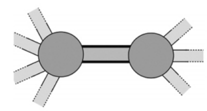
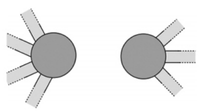
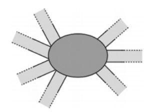
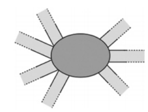
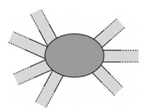
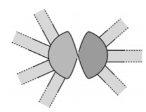
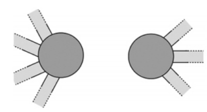
A bouquet is a ribbon graph with a single vertex. Crucially, any connected ribbon graph G can be transformed into a bouquet by applying partial duality with respect to the edge set of any spanning quasi-tree [10]. Since partial duality preserves the number of spanning quasi-trees [8], we obtain $\tau(G) = \tau(B)$ for the resulting bouquet B. This transformation allows us to extend our Matrix Quasi-tree Theorem to general connected ribbon graphs, as detailed in Corollary 4.8.

2 Preliminaries

There are different ways to describe an embedded graph. In this paper, we will use the concept of a ribbon graph.

Definition 2.1 ([1])**.** *A ribbon graph $G = (V(G), E(G))$ is a (orientable or non-orientable) surface with boundary, represented as the union of two sets of topological discs, a set $V(G)$ of vertices, and a set $E(G)$ of edges, subject to the following restrictions.*

Table 1: Operations on an edge e (highlighted in bold) of a ribbon graph

	Non-loop	Non-orientable loop	Orientable loop
G			
$G^{\delta(e)}$			
$G\backslash e$			
$G/e = G^{\delta(e)}\backslash e$			

(1) *The vertices and edges intersect in disjoint line segments;*

(2) *Each such line segment lies on the boundary of precisely one vertex and precisely one edge;*

(3) *Every edge contains exactly two such line segments.*

A ribbon graph $H = (V(H), E(H))$ is a *ribbon subgraph* of $G = (V(G), E(G))$ if H can be obtained from G by deleting vertices and edges. If $V(H) = V(G)$, then H is a *spanning ribbon subgraph* of G. For a subset $A \subseteq E(G)$, the ribbon subgraph *induced by* A, denoted $G|_A$, is obtained from G by deleting all edges not in A, and then deleting any isolated vertices. A ribbon graph is *non-orientable* if it contains a subgraph homeomorphic to a Möbius band; otherwise, it is *orientable*. An edge e is a *loop* if it is incident to only one vertex. A loop is *non-orientable* if its corresponding ribbon, together with the vertex, forms a Möbius band; otherwise, it is *orientable*.

Partial duality, introduced by Chmutov [6], provides a powerful tool for studying signed Bollobás-Riordan polynomials and knot polynomials. For a ribbon graph G and $A \subseteq E(G)$, the *partial dual*, $G^{\delta(A)}$, of G with respect to A is the ribbon graph obtained from G by gluing a disc to G along each boundary component of the spanning ribbon subgraph $(V(G), A)$ (such discs will be the vertex-discs of $G^{\delta(A)}$), removing the interiors of all vertex-discs of G and keeping the edge-ribbons unchanged.

For an edge $e \in E(G)$, the *contraction* of e is defined as:

$$G/e := G^{\delta(e)}\backslash e.$$

Table 1 from [10] illustrates the local transformations of partial duality, deletion, and contraction on an edge of a ribbon graph.

Quasi-trees serve as the topological counterpart to trees in ribbon graph theory, and the substitution of "tree" by "quasi-tree" in the cycle matroid definition naturally gives rise to a delta-matroid.

A set system is a pair $D = (E, \mathcal{F})$, where E is a finite set called the *ground set*, and $\mathcal{F}$ is a collection of subsets of E. The elements of $\mathcal{F}$ are called *feasible sets*. A set system is *proper* if $\mathcal{F} \neq \emptyset$.

Following Bouchet [3], a *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ satisfying the following symmetric exchange axiom: for any $X, Y \in \mathcal{F}$ and any $u \in X\Delta Y$, there exists $v \in X\Delta Y$ (possibly $v = u$) such that $X\Delta\{u, v\} \in \mathcal{F}$. Here, $X\Delta Y = (X \cup Y) \setminus (X \cap Y)$ denotes the symmetric difference of sets.

For $A \subseteq E$, the *twist* of D with respect to A, denoted $D * A$, is defined as:

$$D * A = (E, \{A\Delta X \mid X \in \mathcal{F}\}).$$

A ribbon graph is called a *quasi-tree* if it has exactly one boundary component. The number of spanning quasi-trees in a connected ribbon graph G is denoted by $\tau(G)$. Remarkably, the collection of edge sets of all spanning quasi-trees in a ribbon graph forms the feasible sets of a delta-matroid on $E(G)$.

Definition 2.2 ([7])**.** *Let $G = (V(G), E(G))$ be a connected ribbon graph, and define*

$$\mathcal{F}(G) := \{F \subseteq E(G) \mid F \textit{ is the edge set of a spanning quasi-tree of } G\}.$$

The delta-matroid of G is defined as $D(G) = (E(G), \mathcal{F}(G))$.

Chun et al. [8] established the equivalence between twisting and partial duality at the delta-matroid level.

Proposition 2.3 ([8])**.** *Let G be a ribbon graph and $A \subseteq E(G)$. Then*

$$D(G) * A = D(G^{\delta(A)}).$$

3 Symbolic skew-adjacency matrices

Definition 3.1. Let B be a bouquet with edge set $[n]$. A *signed rotation* of B is a cyclic sequence of length $2n$ encoding the order of half-edges around the vertex. Each loop $i \in [n]$ appears twice in the sequence, with one occurrence labeled i^a and the other occurrence labeled i^b. The signs assigned to these labels are determined by the orientability of the corresponding loop: for an orientable loop, i^a and i^b are assigned identical signs, while for a non-orientable loop, they are assigned opposite signs.

Remark 3.2. *In the signed rotation representation, the positive sign is conventionally omitted in the notation. A bouquet may admit multiple signed rotations due to different choices in the starting point of the cyclic sequence, the direction of rotation (clockwise or counterclockwise), and the assignment of a and b labels to the two occurrences of each loop. These variations all yield equivalent representations of the same bouquet. See Figure 1 for an example.*

Definition 3.3. Let B be a bouquet with edge set $[n]$ and signed rotation σ. The *symbolic skew-adjacency matrix* $\mathbf{A}_\sigma^\mathbf{s}$ of B with respect to σ is an $n \times n$ matrix over the polynomial ring $\mathbb{Z}[x_{ij}]$, where x_{ij} are commuting indeterminates for all $i, j \in [n]$. The entries of $\mathbf{A}_\sigma^\mathbf{s}$ are defined as follows:

- **Diagonal entries** $(i = j)$**:**

$$\mathbf{A}^\mathbf{s}_{\sigma ii} = \begin{cases} x_{ii}, & \text{if the signs of } i^a \text{ and } i^b \text{ are opposite,} \\ 0, & \text{if the signs of } i^a \text{ and } i^b \text{ are the same.} \end{cases}$$

- **Off-diagonal entries** $(i < j)$**:** Consider the relative cyclic order of i^a, i^b, j^a, j^b in σ (ignoring their signs):

$$\mathbf{A}^\mathbf{s}_{\sigma ij} = \begin{cases} x_{ij}, & \text{if the cyclic order is } i^a, j^a, i^b, j^b, \\ -x_{ij}, & \text{if the cyclic order is } i^a, j^b, i^b, j^a, \\ 0, & \text{otherwise.} \end{cases}$$

- **Skew-symmetry:** For $i > j$, define $\mathbf{A}^\mathbf{s}_{\sigma ji} = -\mathbf{A}^\mathbf{s}_{\sigma ij}$.

We define two related matrices:

- The *unsymbolic skew-adjacency matrix* $\mathbf{A}_\sigma^\mathbf{u}$ is obtained from $\mathbf{A}_\sigma^\mathbf{s}$ by substituting $x_{ij} = 1$ for all $i, j \in [n]$.

- The *adjacency matrix* $\mathbf{M}$ is defined by $\mathbf{M}_{ij} = |\mathbf{A}^\mathbf{u}_{\sigma ij}|$ for all $i, j \in [n]$.

Remark 3.4. (1) *It is important to note that the symbolic skew-adjacency matrix* $\mathbf{A}_\sigma^\mathbf{s}$ *and the unsymbolic skew-adjacency matrix* $\mathbf{A}_\sigma^\mathbf{u}$ *depend not only on the bouquet* B *but also on the specific signed rotation* σ*. Different signed rotations of the same bouquet may yield different skew-adjacency matrices.*

Specifically, the dependence on σ *manifests as follows:*

- *Changing the starting point of the cyclic sequence leaves* $\mathbf{A}_\sigma^\mathbf{s}$ *and* $\mathbf{A}_\sigma^\mathbf{u}$ *unchanged.*
- *Reversing the cyclic direction flips the signs of off-diagonal entries (i.e.,* x_{ij} *becomes* $-x_{ij}$ *and* $-x_{ij}$ *becomes* x_{ij} *in* $\mathbf{A}_\sigma^\mathbf{s}$*, while* 1 *becomes* -1 *and* -1 *becomes* 1 *in* $\mathbf{A}_\sigma^\mathbf{u}$*).*
- *Swapping the* a *and* b *labels for a loop multiplies the corresponding row and column by* -1 *in both* $\mathbf{A}_\sigma^\mathbf{s}$ *and* $\mathbf{A}_\sigma^\mathbf{u}$*.*

However, the adjacency matrix $\mathbf{M}$ *is independent of the choice of signed rotation. Since this matrix is defined by taking absolute values of the entries of* $\mathbf{A}_\sigma^\mathbf{u}$*, it depends only on the bouquet* B *itself. Notably,* $\mathbf{M}$ *is precisely the adjacency matrix of the signed intersection graph of* B *[17, 21].*

(2) *Let B be a bouquet with edge set $[n]$ and signed rotation σ, and let $\mathbf{A}_\sigma^{\mathbf{s}}$ be its symbolic skew-adjacency matrix. By the Leibniz formula, the determinant of $\mathbf{A}_\sigma^{\mathbf{s}}$ is given by*

$$\det(\mathbf{A}_\sigma^{\mathbf{s}}) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} \mathbf{A}_{\sigma\, i,\pi(i)}^{\mathbf{s}},$$

where S_n denotes the symmetric group on $[n]$, and each monomial corresponds to a permutation π. Since each entry $\mathbf{A}_{\sigma ij}^{\mathbf{s}}$ is an element of the ring $\mathbb{Z}[x_{ij}]$, it follows that $\det(\mathbf{A}_\sigma^{\mathbf{s}}) \in \mathbb{Z}[x_{ij}]$. Furthermore, the determinant of $I_n + \mathbf{A}_\sigma^{\mathbf{s}}$ can be expanded as a sum over all principal submatrices:

$$\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}}) = \sum_{X \subseteq [n]} \det(\mathbf{A}_\sigma^{\mathbf{s}}[X]) \in \mathbb{Z}[x_{ij}],$$

where $\mathbf{A}_\sigma^{\mathbf{s}}[X]$ denotes the principal submatrix of $\mathbf{A}_\sigma^{\mathbf{s}}$ induced by $X \subseteq [n]$, with the convention that $\det(\mathbf{A}_\sigma^{\mathbf{s}}[\emptyset]) = 1$.

Definition 3.5. We define a reduction map $f : \mathbb{Z}[x_{ij}] \to \mathbb{Z}[x_A : A \subseteq [n]]$ as follows. The target ring has generators $\{x_A : A \subseteq [n]\}$ with multiplication $x_A \cdot x_B = x_{A \cup B}$ and unit $x_\emptyset$. For a monomial $m = \prod_{r=1}^{k} x_{i_r j_r}^{e_r}$ with $e_r \in \mathbb{Z}_{\geq 1}$, define

$$f(m) = x_A, \quad \text{where } A = \bigcup_{r=1}^{k} \{i_r, j_r\}.$$

This extends linearly to polynomials: for $P = \sum_m \alpha_m m \in \mathbb{Z}[x_{ij}]$ with $\alpha_m \in \mathbb{Z}$,

$$f(P) = \sum_m \alpha_m f(m).$$

The map f preserves coefficients while replacing each monomial by a single variable indexed by the union of all indices appearing in its factors.

4 Proof of the Matrix Quasi-tree Theorem

Before proving the Matrix Quasi-tree Theorem (Theorem 1.3), we establish some key lemmas. The next lemma combines results from two references. For orientable bouquets, Merino et al. [16] established the following result using directed connected maps and principally unimodular matrices. Deng et al. [9] later extended this result to bouquets with exactly one non-orientable loop.

Lemma 4.1 ([9], [16])**.** *Let B be a bouquet with edge set $[n]$ and signed rotation σ, and let $\mathbf{A}_\sigma^{\mathbf{u}}$ be its unsymbolic skew-adjacency matrix. For any subset $X \subseteq [n]$, if B has at most one non-orientable loop, then*

$$\det(\mathbf{A}_\sigma^{\mathbf{u}}[X]) = \begin{cases} 1, & \text{if } X \text{ is the edge set of a spanning quasi-tree of } B, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.2. *A direct consequence of Lemma 4.1 is that for bouquets with at most one non-orientable loop, the polynomial $f(\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}}))$ has all coefficients equal to 1. Therefore, the modulo 2 reduction in Theorem 1.3 becomes unnecessary in these cases.*

The pivot operation is defined for matrices over arbitrary fields (see, e.g., [18]).

Definition 4.3. *Let $\mathbf{A}$ be a square matrix over a field $\mathbb{K}$, with rows and columns labelled (in the same order) by a set E. For any $X \subseteq E$, let $\overline{X} = E \setminus X$. Assume without loss of generality (after reordering if needed) that X labels the first $|X|$ rows and columns. Then $\mathbf{A}$ has the block form*

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{array}{c} X \\ \overline{X} \end{array} & \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \end{array}$$

where $P = \mathbf{A}[X]$. If P is non-singular, the pivot of $\mathbf{A}$ on X, denoted $\mathbf{A} * X$, is defined as the matrix

$$\mathbf{A} * X = \begin{array}{c c} & \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{array}{c} X \\ \overline{X} \end{array} & \begin{bmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & S - RP^{-1}Q \end{bmatrix} \end{array}.$$

Lemma 4.4 ([17])**.** *Let B be a bouquet with edge set $[n]$, and let $\mathbf{M}(B)$ be the adjacency matrix of B over $\mathbb{F}_2$. If $e_1 \in [n]$ is a non-orientable loop, then the pivot matrix $\mathbf{M}(B) * \{e_1\}$ coincides with the adjacency matrix of $B^{\delta(e_1)}$, i.e.,*

$$\mathbf{M}(B) * \{e_1\} = \mathbf{M}(B^{\delta(e_1)}),$$

where $\mathbf{M}(B^{\delta(e_1)})$ is the adjacency matrix of $B^{\delta(e_1)}$.

Theorem 4.5. *Let B be a bouquet with edge set $[n]$ and signed rotation σ, and let $\mathbf{M}$ be the adjacency matrix of B. Then for any subset $X \subseteq [n]$,*

$$\det(\mathbf{M}[X]) \bmod 2 = \begin{cases} 1, & \textit{if } X \textit{ is the edge set of a spanning quasi-tree of } B, \\ 0, & \textit{otherwise.} \end{cases} \tag{1}$$

Proof. We first consider the case when B is orientable. Let $\mathbf{A}^{\mathbf{u}}_{\sigma}$ be the unsymbolic skew-adjacency matrix of B with respect to σ. Since $\mathbf{M}_{ij} = |\mathbf{A}^{\mathbf{u}}_{\sigma ij}|$ and $\mathbf{A}^{\mathbf{u}}_{\sigma ij} \in \{-1, 0, 1\}$, we have $\mathbf{M}_{ij} \equiv \mathbf{A}^{\mathbf{u}}_{\sigma ij}$ (mod 2) for all i, j. Hence, $\mathbf{A}^{\mathbf{u}}_{\sigma} \equiv \mathbf{M}$ (mod 2). This implies that for any subset $X \subseteq [n]$, the principal submatrices satisfy $\mathbf{A}^{\mathbf{u}}_{\sigma}[X] \equiv \mathbf{M}[X]$ (mod 2), and therefore

$$\det(\mathbf{A}^{\mathbf{u}}_{\sigma}[X]) \equiv \det(\mathbf{M}[X]) \pmod 2.$$

By Lemma 4.1, $\det(\mathbf{A}^{\mathbf{u}}_{\sigma}[X])$ is 1 if and only if X is the edge set of a spanning quasi-tree of B, and 0 otherwise. Thus, the same holds for $\det(\mathbf{M}[X]) \bmod 2$.

For the case when B has at least one non-orientable loop, we proceed by induction on $k = |X|$, the size of the edge subset.

Base Cases:

Case $k = 0$: When $X = \emptyset$, the matrix $\mathbf{M}[X]$ is empty and its determinant is defined to be 1 by convention. The empty edge set corresponds to the spanning quasi-tree of the bouquet consisting of a single isolated vertex (which has one boundary component). Thus, $\det(\mathbf{M}[X]) \bmod 2 = 1$, and equation (1) holds.

Case $k = 1$**:** When X contains a single edge, $\mathbf{M}[X]$ is a 1×1 matrix. If the edge is a non-orientable loop, then $\mathbf{M}[X] = [1]$ and $\det(\mathbf{M}[X]) = 1$. If the edge is an orientable loop, then $\mathbf{M}[X] = [0]$ and $\det(\mathbf{M}[X]) = 0$. A single edge of a bouquet forms a spanning quasi-tree if and only if it is non-orientable. Therefore, $\det(\mathbf{M}[X]) \bmod 2 = 1$ if and only if X is the edge set of a spanning quasi-tree, satisfying equation (1).

Inductive Step: Assume the result holds for all edge subsets of size less than k, where $k \geq 2$. Now consider a set X with $|X| = k$.

Let $B|_X$ denote the bouquet obtained by restricting B to the subset X of edges. We consider two subcases based on the orientability of $B|_X$.

Case 1: $B|_X$ **is orientable.** By the argument at the beginning of the proof, equation (1) holds.

Case 2: $B|_X$ **is non-orientable.** Let $e_1 \in X$ be a non-orientable loop. Note that the adjacency matrix of the restricted bouquet $B|_X$ is exactly the principal submatrix $\mathbf{M}[X]$.

We partition the matrix $\mathbf{M}[X]$, placing e_1 first:

$$\mathbf{M}[X] = \begin{bmatrix} P & Q \\ Q^\top & R \end{bmatrix},$$

where $P = \mathbf{M}[X][\{e_1\}] = [1]$ (since e_1 is non-orientable). Hence, P is invertible. Using the block matrix determinant formula:

$$\det \begin{bmatrix} P & Q \\ Q^\top & R \end{bmatrix} = \det(P) \cdot \det(R - Q^\top P^{-1} Q).$$

Since $P = [1]$, we have $P^{-1} = [1]$, so

$$\det(\mathbf{M}[X]) = \det(R - Q^\top Q).$$

Thus,

$$\det(\mathbf{M}[X]) \bmod 2 = \det(R - Q^\top Q) \bmod 2. \tag{2}$$

Recall that by Lemma 4.4, the pivot operation on $\mathbf{M}[X]$ with respect to $\{e_1\}$ yields the adjacency matrix of $(B|_X)^{\delta(e_1)}$ over $\mathbb{F}_2$. In our partitioned matrix $\mathbf{M}[X] = \begin{bmatrix} P & Q \\ Q^\top & R \end{bmatrix}$, the pivot operation gives:

$$\mathbf{M}[X] * \{e_1\} = \begin{bmatrix} P^{-1} & -P^{-1}Q \\ Q^\top P^{-1} & R - Q^\top P^{-1} Q \end{bmatrix}.$$

Since $P = [1]$ and $P^{-1} = [1]$, the bottom-right block becomes $R - Q^\top Q$. This submatrix $R - Q^\top Q$ corresponds to the adjacency matrix of the bouquet obtained by first taking the partial dual of $B|_X$ with respect to e_1 to get $(B|_X)^{\delta(e_1)}$, and then deleting e_1 to obtain $(B|_X)^{\delta(e_1)} \backslash \{e_1\}$, that is, $B|_X / \{e_1\}$. Therefore, over $\mathbb{F}_2$, $R - Q^\top Q$ is the adjacency matrix of $B|_X / \{e_1\}$.

By the induction hypothesis, equation (1) holds for the bouquet $B|_X/\{e_1\}$ on the smaller edge set $X \setminus \{e_1\}$, that is,

$$\det(R - Q^\top Q) \bmod 2 = \begin{cases} 1, & \text{if } B|_X/\{e_1\} \text{ is a spanning quasi-tree,} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

From Table 1, we see that $F \in \mathcal{F}(B|_X)$ and $e_1 \in F$ if and only if $F\backslash e_1 \in \mathcal{F}((B|_X)^{\delta(e_1)})$ if and only if $F\backslash e_1 \in \mathcal{F}((B|_X)^{\delta(e_1)}\backslash e_1) = \mathcal{F}((B|_X)/e_1)$. Therefore, $B|_X$ is a spanning quasi-tree if and only if $B|_X/\{e_1\}$ is a spanning quasi-tree. Then by equations (2) and (3), we conclude

$$\det(\mathbf{M}[X]) \bmod 2 = \begin{cases} 1, & \text{if } B|_X \text{ is a spanning quasi-tree,} \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\det(\mathbf{M}[X]) \bmod 2 = \begin{cases} 1, & \text{if } X \text{ is the edge set of a spanning quasi-tree of } B, \\ 0, & \text{otherwise.} \end{cases}$$

□

Lemma 4.6. *Let B be a bouquet with edge set $[n]$ and signed rotation σ, and let $\mathbf{A}^{\mathbf{s}}_{\sigma}$ be its symbolic skew-adjacency matrix. For any subset $X \subseteq [n]$, we have*

$$f\big(\det(\mathbf{A}^{\mathbf{s}}_{\sigma}[X])\big) \bmod 2 = \begin{cases} x_X, & \textit{if } X \textit{ is the edge set of a spanning quasi-tree of } B, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By the Leibniz formula:

$$\det(\mathbf{A}^{\mathbf{s}}_{\sigma}[X]) = \sum_{\pi \in S_X} \operatorname{sgn}(\pi) \prod_{i \in X} \mathbf{A}^{\mathbf{s}}_{\sigma\, i,\pi(i)},$$

where S_X denotes the symmetric group on X. Applying the reduction map f to each term:

$$f\left(\operatorname{sgn}(\pi) \prod_{i \in X} \mathbf{A}^{\mathbf{s}}_{\sigma\, i,\pi(i)}\right) = \operatorname{sgn}(\pi) \left(\prod_{i \in X} \mathbf{A}^{\mathbf{u}}_{\sigma\, i,\pi(i)}\right) x_X.$$

Summing over all $\pi \in S_X$ yields:

$$f(\det(\mathbf{A}^{\mathbf{s}}_{\sigma}[X])) = \det(\mathbf{A}^{\mathbf{u}}_{\sigma}[X]) \cdot x_X.$$

Taking modulo 2:

$$f(\det(\mathbf{A}^{\mathbf{s}}_{\sigma}[X])) \bmod 2 = (\det(\mathbf{A}^{\mathbf{u}}_{\sigma}[X]) \bmod 2) \cdot x_X.$$

Since $\det(\mathbf{A}^{\mathbf{u}}_{\sigma}[X]) \equiv \det(\mathbf{M}[X]) \pmod 2$, and by Theorem 4.5, it follows that

$$f(\det(\mathbf{A}^{\mathbf{s}}_{\sigma}[X])) \bmod 2 = \begin{cases} x_X, & \text{if } X \text{ is the edge set of a spanning quasi-tree of } B, \\ 0, & \text{otherwise.} \end{cases}$$

□

Proof of Matrix Quasi-tree Theorem (Theorem 1.3). By the Leibniz formula, $\det(I_n+\mathbf{A}_\sigma^{\mathbf{s}})$ expands as a sum over all principal submatrices:

$$\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}}) = \sum_{X\subseteq[n]} \det(\mathbf{A}_\sigma^{\mathbf{s}}[X]),$$

where $\mathbf{A}_\sigma^{\mathbf{s}}[X]$ denotes the principal submatrix indexed by X and $\det(\mathbf{A}_\sigma^{\mathbf{s}}[\emptyset]) = 1$.

Since the map f is linear, it commutes with summation:

$$f\left(\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}})\right) = f\left(\sum_{X\subseteq[n]} \det(\mathbf{A}_\sigma^{\mathbf{s}}[X])\right) = \sum_{X\subseteq[n]} f\left(\det(\mathbf{A}_\sigma^{\mathbf{s}}[X])\right).$$

Reducing both sides modulo 2 yields:

$$f\left(\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}})\right) \bmod 2 = \sum_{X\subseteq[n]} \left(f\left(\det(\mathbf{A}_\sigma^{\mathbf{s}}[X])\right) \bmod 2\right).$$

By Lemma 4.6, $f\left(\det(\mathbf{A}_\sigma^{\mathbf{s}}[X])\right) \bmod 2$ is nonzero (and equal to x_X) if and only if X is the edge set of a spanning quasi-tree. Consequently, in the sum above, the coefficient of x_X is 1 precisely for such subsets X, and 0 otherwise.

The number of spanning quasi-trees, $\tau(B)$, is the number of subsets X for which the coefficient of x_X is 1. This is obtained by evaluating the polynomial at $x_X = 1$ for all $X \subseteq [n]$, which sums these coefficients:

$$\tau(B) = f(\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}})) \bmod 2\big|_{x_X=1}.$$

$\square$

Remark 4.7. *The polynomial $f(\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}}))$ is invariant under the choice of signed rotation σ, depending only on the bouquet B. This follows from Theorem 1.3, since the set of spanning quasi-trees of B is independent of the specific signed rotation representation.*

Corollary 4.8. *Let G be a connected ribbon graph with edge set $[n]$, T be the edge set of any spanning quasi-tree of G, and σ be a signed rotation of $G^{\delta(T)}$. Let $\mathbf{A}_\sigma^{\mathbf{s}}$ be the symbolic skew-adjacency matrix of $G^{\delta(T)}$ with respect to σ. Then*

$$f(\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}})) \bmod 2 = \sum_X x_X,$$

where the sum is taken over all subsets $X \subseteq [n]$ such that $X\Delta T$ is the edge set of a spanning quasi-tree of G. Consequently, the number of spanning quasi-trees is

$$\tau(G) = f(\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}})) \bmod 2\big|_{x_X=1}.$$

Proof. Since T is the edge set of a spanning quasi-tree of G, the partial dual $G^{\delta(T)}$ is a bouquet. By Theorem 1.3 applied to $G^{\delta(T)}$, the coefficient of x_X in $f(\det(I_n + \mathbf{A}_\sigma^{\mathbf{s}})) \bmod 2$ is 1 if and only if X is the edge set of a spanning quasi-tree of $G^{\delta(T)}$.

By Proposition 2.3, $D(G^{\delta(T)}) = D(G) * T$, so the edge sets of spanning quasi-trees of $G^{\delta(T)}$ are exactly the sets $F\Delta T$ where F is the edge set of a spanning quasi-tree of G. Therefore, X is the edge set of a spanning quasi-tree of $G^{\delta(T)}$ if and only if $X\Delta T$ is the edge set of a spanning quasi-tree of G.

By Theorem 1.3, the polynomial evaluation at $x_X = 1$ for all $X \subseteq [n]$ gives the number of spanning quasi-trees of $G^{\delta(T)}$. This number equals $\tau(G)$ via the bijection. $\square$

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