ON CHERNIKOV-BY-NILPOTENT GROUPS

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In memory of Francesco de Giovanni

Abstract

Let $\gamma_k = [x_1, \ldots, x_k]$ be the k-th lower central group-word. Given a group G, we write $X_k(G)$ for the set of γ_k-values and $\gamma_k(G)$ for the k-th term of the lower central of G. This paper deals with groups in which $\langle g^{X_k(G)}\rangle$ is a Chernikov group of size at most (m, n) for all $g \in G$. The main result is that $\gamma_{k+1}(G)$ is a Chernikov group and its size is (k, m, n)-bounded.

1. Introduction

Let G be a group. If X and Y are non-empty subsets of G, we write X^Y to denote the set $\{y^{-1}xy : x \in X, y \in Y\}$. Given $k \geq 1$ and elements $x_1, \ldots, x_k \in G$, we write $[x_1, \ldots, x_k]$ for the left-normed commutator $[\ldots[[x_1, x_2], x_3] \ldots, x_k]$. Elements that can be written as $[x_1, \ldots, x_k]$ for suitable $x_1, \ldots, x_k \in G$ will be called *γ_k-values*, and the set of all γ_k-values in G will be denoted by $X_k(G)$. Of course, the subgroup generated by the γ_k-values is the k-th term of the lower central series of G, usually denoted by $\gamma_k(G)$.

Of course, if the conjugacy class x^G of an element $x \in G$ is finite, we have $|x^G| = [G : C_G(x)]$. A group is said to be an *FC-group* if x^G is finite for every $x \in G$ and a *BFC-group* if there is an integer n such that $|x^G| \leq n$ for every $x \in G$. One of the most famous of B. H. Neumann's theorems says that in a BFC-group the commutator subgroup is finite [9]. Later J. Wiegold showed that if $|x^G| \leq n$ for every $x \in G$, then the order of G' is bounded by a number depending only on n. Moreover, Wiegold found a first explicit bound for the order of G' [13], and the best known was obtained in [6] (see also [10, 11]).

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In recent years, Neumann's theorem has been extended in several directions. In particular, groups in which conjugacy classes containing commutators are bounded were treated in [3, 4]. Several authors have considered so-called *BCC-groups*, that are groups in which $G/C_G(\langle x^G \rangle)$ is a Chernikov group of uniformly bounded size for every $x \in G$ (see [1]).

Recall that a group G is *Chernikov* if it contains a radicable abelian normal subgroup R (the finite residual) which is the direct product of a finite number $m(G)$ of groups of Prüfer type, such that the factor group G/R is finite, of order $n(G)$ say. In general a group G is called *radicable* if the equation $x^n = a$ has a solution in G for every positive integer n and every $a \in G$. By a deep result obtained independently by Shunkov [12] and Kegel and Wehrfritz [7], Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup. Furthermore, if G is a Chernikov group with $m = m(G)$ and $n = n(G)$, the pair (m, n) is usually called the *size* of G. Of course, a Chernikov group G is finite if and only if $m(G) = 0$, and it is a radicable abelian group if and only if $n(G) = 1$.

It is well-known that the class of Chernikov groups is closed with respect to subgroups, homomorphic images and extensions. In particular, if G is a group and N is a normal subgroup of G such that N is a Chernikov group of size (m, n) and G/N is a Chernikov group of size (m', n'), then G is a Chernikov group of size at most $((m + m' + n + n')^2, nn')$ and so bounded in terms of m, n, m' and n'.

Using the concept of verbal conjugacy classes introduced in [5], an extension of Neumann's theorem was obtained in [2]. It was shown that if k, n are positive integers and G is a group in which $|g^{X_k(G)}| \leq n$ for any $g \in G$, then $\gamma_{k+1}(G)$ has finite (k, n)-bounded order.

Throughout the article, we use the expression "$(a, b, \ldots)$-bounded" to mean that a quantity is finite and bounded by a certain number depending only on the parameters $a, b, \ldots$.

In this paper we consider groups in which the subgroup $\langle g^{X_k(G)} \rangle$ is a Chernikov group of bounded size for every $g \in G$. We establish the following theorem.

Theorem 1.1. *Let k, m, n be non-negative integers (with $k, n \neq 0$), and let G be a group such that $\langle g^{X_k(G)} \rangle$ is a Chernikov group of size at most (m, n) for every $g \in G$. Then $\gamma_{k+1}(G)$ is a Chernikov group and its size is (k, m, n)-bounded.*

Obviously, this is an extension of the aforementioned result from [2], which can be recovered from Theorem 1.1 as a particular case where $m = 0$.

2. Proofs

We start the section with the following well-known lemmas. The interested reader can find their proofs for example in [8].

Lemma 2.1. *In a periodic nilpotent group G every radicable abelian subgroup R is central.*

Let A be a group acting on a group G. As usual, $[G, A]$ denotes the subgroup generated by all elements of the form $x^{-1}x^a$, where $x \in G$, $a \in A$. It is well-known that $[G, A]$ is an A-invariant normal subgroup of G.

For elements x, y in a group G, we write $[x, {}_l y]$ to denote the long commutator $[x, y, \dots, y]$, where y is repeated l times. If N is a subgroup of G, the symbol $[N, y]$ stands for the subgroup generated by the commutators $[x, y]$, where x ranges over N. For $l \geq 2$, define by induction $[N, {}_l y] = [[N, {}_{l-1} y], y]$. Note that if N is an abelian normal subgroup, $[N, {}_l y]$ is precisely the set of commutators $[x, {}_l y]$, where $x \in N$.

Lemma 2.2. *Let A be a periodic group acting on a periodic radicable abelian group G. Then $[G, A, A] = [G, A]$.*

Lemma 2.3. *Let A be a finite group acting on a periodic radicable abelian group G. Then $[G, A]$ is radicable.*

The following results will be useful later on.

Lemma 2.4. *Let G be a periodic group, and let N be a normal subgroup of G. Assume that $[N, x]$ is a Chernikov group for every $x \in G$, and let R_x be its radicable part. Then the subgroup R generated by the subgroups R_x is a radicable abelian normal subgroup of G.*

Proof. Clearly, for every $x \in G$, the subgroup R_x is normal in N. Therefore R is normal as well. Moreover, for every $x, y \in G$, the product $R_x R_y$ is nilpotent of class at most two and so abelian by Lemma 2.1. Hence, R is abelian as well and the lemma follows. □

Recall that a key property of radicable abelian groups is that any radicable subgroup of an abelian group A is a direct factor of A.

Lemma 2.5. *Let m be a non-negative integer, and let G be a periodic nilpotent group acting on a periodic radicable abelian group R. Suppose that for each $x \in G$ the subgroup $[R, x]$ is a direct product of at most*

m Prüfer subgroups. Then there exists an integer $h = h(m)$ depending only on m such that the subgroup $[R, G]$ is a direct product of at most h Prüfer subgroups.

Proof. For each $x \in G$, let m_x be the number of Prüfer direct factors of the subgroup $[R, x]$. Define $m(G, R)$ to be the maximum of all the numbers m_x. The lemma will be proved by induction on $m(G, R)$.

Clearly, if $m(G, R) = 0$, then $[R, G]$ is trivial. Assume that $m(G, R) \geq 1$ and the existence of $h(m-1)$ is established.

We may assume that G is nontrivial and acts on R faithfully. Choose a nontrivial element $z \in Z(G)$. Observe that $[R, z]$ is G-invariant. In a natural way the group G acts on the quotient $R/[R, z]$, and thus we can consider the quantity $m(G, R/[R, z])$. If $m(G, R/[R, z]) < m(G, R)$, then the subgroup $[R, G]/[R, z]$ is a direct product of at most $h(m-1)$ Prüfer subgroups. Consequently, since $[R, z]$ is a direct factor of $[R, G]$, it follows that $[R, G]$ itself is a direct product of at most $m + h(m-1)$ Prüfer subgroups.

Therefore assume that $m(G, R/[R, z]) = m(G, R) = m$. Then there exists an element $y \in G$ such that the subgroup $[R, y]/([R, y] \cap [R, z])$ is a direct product of exactly m Prüfer subgroups. On the other hand, $m_y \leq m$, and therefore $m_y = m$, which implies that the intersection $[R, y] \cap [R, z]$ does not contain any Prüfer subgroups and hence is finite. Moreover, since $[R, z]$ is y-invariant and $[R, y]$ is z-invariant, we have $[R, y, z] \leq [R, y] \cap [R, z]$ and $[R, z, y] \leq [R, y] \cap [R, z]$.

Clearly, the mapping $f \colon [r, y] \to [r, y, z]$ is a homomorphism from $[R, y]$ to $[R, y, z]$, and since $[R, y]$ has no proper subgroups of finite index, it follows that $[R, y, z] = \{1\}$. In the same way, we also obtain $[R, z, y] = \{1\}$. Then $[R, y, y] = [R, y, zy]$, and Lemma 2.2 shows that $[R, y] = [R, y, y] \leq [R, zy]$. Similarly, $[R, z] \leq [R, yz]$, where of course $[R, yz] = [R, zy]$. Therefore $[R, y][R, z] \leq [R, yz]$, and since $m_{yz} \leq m$ and $m_y = m$, it follows that the subgroup $[R, y][R, z]$ is a direct product of exactly m Prüfer subgroups. Moreover, since $[R, y]$ is a direct factor of $[R, y][R, z]$ and it is also a direct product of exactly m Prüfer subgroups, it follows that $[R, z]$ must be finite and so trivial. Since G acts faithfully on R, it follows that the element z is trivial, which is a contradiction. $\square$

LEMMA 2.6. *Let G be a periodic group, and let $x \in G$ such that $[G, x]$ is a Chernikov group. Let R be the radicable part of $[G, x]$ and assume that $[R, x] = \{1\}$. Then $R = \{1\}$.*

Proof. Note that to prove the statement it is enough to show that R is finite. Since R is normal in G, it follows that $[R, G, x] = \{1\}$; then applying the Three Subgroup Lemma we have that R is central in $[G, x]$. Therefore the quotient $[G, x]/Z([G, x])$ is finite, and Schur's Theorem yields that $[G, x]'$ is finite. We pass to the quotient $G/[G, x]'$ and assume that $[G, x]$ is abelian. Clearly, there exists a finite subgroup F of $[G, x]$ such that $[G, x] = R \times F$. Put $E = F^G$. Since E is contained in $[G, x]$, it follows that E is abelian and so it has the same exponent as F. Therefore E is finite, and hence we can replace G by G/E and assume that $[G, x] = R$. In particular, $[G, x, x] = [R, x] = \{1\}$, and so $\langle x^G \rangle$ is abelian. It follows that the exponent of $\langle x^G \rangle$ is exactly the order of x, which is finite; then also $[G, x] = R$ has finite exponent, thus R is trivial. □

We will require the following result whose proof can be found in [8].

Theorem 2.7. *Let k be a positive integer, and let G be a group such that $\langle g^{X_k(G)} \rangle$ is a Chernikov group for every $g \in G$. Then $\langle g^{\gamma_k(G)} \rangle$ is a Chernikov group for every $g \in G$.*

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. In view of Theorem 2.7, $\langle g^{\gamma_k(G)} \rangle$ is a Chernikov group for every $g \in G$, and hence every subgroup $[\gamma_k(G), g]$ is a Chernikov group as well. For every $g \in G$, let R_g denote the radicable part of $[\gamma_k(G), g]$. An application of Lemma 2.4 yields that the subgroup R generated by all the subgroup R_g is a radicable abelian normal subgroup of G.

Let $g \in G$; by Lemma 2.2 and Lemma 2.3, it follows that $[R, g] = [R, {}_{k-1}g]$, where $[R, {}_{k-1}g] \subseteq X_k(G)$. We conclude that

$$[R, g] = [R, g, g] \leq [X_k(G), g] \leq \langle g^{X_k(G)} \rangle,$$

and hence the radicable abelian subgroup $[R, g]$ is contained in the radicable part of $\langle g^{X_k(G)} \rangle$. Therefore, $[R, g]$ is a direct product of at most m Prüfer subgroups.

By the well-known Dedekind's Modular Law, it follows that

$$\langle g^{X_k(G)} \rangle \cap \langle g \rangle [\gamma_k(G), g] = \langle g \rangle (\langle g^{X_k(G)} \rangle \cap [\gamma_k(G), g]) = \langle g^{X_k(G)} \rangle,$$

and therefore $\langle g^{X_k(G)} \rangle \cap [g, \gamma_k(G)]$ has finite index in $\langle g^{X_k(G)} \rangle$, as the element g has finite order. In particular, the radicable part of $\langle g^{X_k(G)} \rangle$ is contained in $\langle g^{X_k(G)} \rangle \cap [g, \gamma_k(G)]$, and consequently in $R_g \leq R \cap [g, \gamma_k(G)]$. Then the quotient $\langle g^{X_k(G)} \rangle / R \cap \langle g^{X_k(G)} \rangle$ is finite of order at most n, so that $\gamma_{k+1}(G)/R$ has finite (k, n)-bounded order by [Corollary 1.3, [2]].

Put $G^* = G/C_G(R)$ and regard G^* as a group of automorphisms of R. Since $R \leq C_G(R)$, it follows that $\gamma_{k+1}(G^*)$ has finite (k, n)-bounded order, and hence the centralizer $C^* = C_{G^*}(\gamma_{k+1}(G^*))$ is nilpotent and

has finite index $t = t(k, n)$, depending only on k and n.
On the other hand, Lemma 2.5 ensures the existence of an integer $h = h(m)$ such that the subgroup $[R, C^*]$ is a direct product of at most h Prüfer subgroups. Now, let $\{g_1^*, \ldots, g_t^*\}$ be a right transversal to C^* in G^*. We deduce that

$$[R, G^*] = [R, C^*\langle g_1^*, \ldots, g_t^* \rangle] = [R, C^*][R, g_1^*] \cdots [R, g_t^*],$$

and hence $[R, G^*]$ is a direct product of at most $h+tm$ Prüfer subgroups, where the quantity $h + tm$ depends only by k, m and n. Observe that $[R, G^*] = [R, G]$.

Finally, we may replace G by the factor group $G/[R, G]$ and assume that R is central. Then, for any $g \in G$, the subgroup $\gamma_k(G)\langle g \rangle$ satisfies the assumption of Lemma 2.6, and therefore we have that $R = \{1\}$. Consequently, $\gamma_{k+1}(G)$ has finite (k, n)-bounded order and the theorem follows. $\square$

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