ON THE MAXIMALITY OF THE λ-INVARIANTS OF MAZUR–TATE ELEMENTS

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Abstract. Let E be an elliptic curve with good ordinary reduction at an odd prime p. Assuming that Greenberg's $\mu = 0$ conjecture holds, we show that the λ-invariants of the Mazur–Tate elements attached to E either stabilise to the λ-invariant of the p-adic L-function or they attain the largest possible value at all finite levels. We characterise the latter phenomenon: it occurs if and only if $\mathrm{ord}_p\left(\frac{L(E',1)}{\Omega_{E'}}\right)$ is negative for some E' that is isogenous to E. Furthermore, we relate this condition to congruences with boundary symbols coming from Eisenstein series. We also study the extension of these results to Hecke eigenforms of weight two.

1. Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$ and let f_E be the weight two cusp form of level N_E attached to E. In this article, we are interested in the modular elements of Mazur and Tate attached to E introduced in [MT87], which we call Mazur–Tate elements in this article. These are defined as elements of group rings of the form $\mathbb{Q}[\mathrm{Gal}(M/\mathbb{Q})]$, where M is an abelian extension of $\mathbb{Q}$. Let p be an odd prime. We shall focus on the p-adic Mazur–Tate elements $\theta_n(E)$, which interpolate the twisted L-values $L(E,\chi,1)$ for Dirichlet characters χ of p-power conductor.

Let k_n denote the unique sub-extension of $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}$ that is of degree p^n. We shall regard the Mazur–Tate element $\theta_n(E)$ for the extension $k_n/\mathbb{Q}$ as an element of $\mathbb{Q}_p[\mathrm{Gal}(k_n/\mathbb{Q})]$. One can define μ- and λ-invariants of elements of $\mathbb{Q}_p[\mathrm{Gal}(k_n/\mathbb{Q})]$ as one does for elements of the Iwasawa algebra (see Definition 3.1). The λ-invariant of $\theta_n(E)$ can be arbitrarily large, as illustrated by the following example.

Example 1.1. *Consider $E = X_0(11)$ and $p = 5$. For $n \geq 0$,*

$$\mu(\theta_n(E)) = 0 \quad \text{and} \quad \lambda(\theta_n(E)) = p^n - 1.$$

This behaviour can be explained by the existence of a congruence between the modular symbol of E with a boundary symbol (a modular symbol corresponding to an Eisenstein series, see §5.1; in particular, Remark 5.3) modulo p, which leads to the above expression for the λ-invariants.

Note that the λ-invariant of a non-zero element in $\mathbb{Q}_p[\mathrm{Gal}(k_n/\mathbb{Q})]$ is at most $p^n - 1$. We shall refer to the property $\lambda(\theta_n(E)) = p^n - 1$ as the *maximality* of the λ-invariants of $\theta_n(E)$. Note that this phenomenon is expected to occur only when the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module $E[p]$ is reducible, as when $E[p]$ is irreducible, we have:

Theorem 1.2. *Let $E/\mathbb{Q}$ be an elliptic curve with good ordinary reduction at p such that $E[p]$ is irreducible as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. Let $L_p(E)$ be the p-adic L-function of Mazur–Swinnerton-Dyer attached to E constructed in* [MSD74]. *If $\mu(L_p(E)) = 0$, then, for n large enough*

$$\mu(\theta_n(E)) = 0 \quad \text{and} \quad \lambda(\theta_n(E)) = \lambda(L_p(E)).$$

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Proof. See [PW11, Proposition 3.7]. □

Greenberg's $\mu = 0$ conjecture predicts that $\mu(L_p(E)) = 0$ when $E[p]$ is an irreducible $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. In other words, we expect that $\lambda(\theta_n(E))$ should always be bounded in this case. See also Theorem 4.4 for a slight generalization of Theorem 1.2, where we relax the hypothesis on the irreducibility of $E[p]$.

One of the main goals of this article is to characterise the Iwasawa invariants of $\theta_n(E)$ at good ordinary primes p for $n \gg 0$ when $E[p]$ is reducible. In fact, we show that the previous two kinds of behaviour, those exhibited in Example 1.1 and Theorem 1.2, are the only possibilities for the Iwasawa invariants of $\theta_n(E)$. Furthermore, our result characterises exactly when these two behaviours happen in terms of the p-adic valuation of the normalised L-value at $s = 1$.

1.1. **Main results.** In this section, we describe the main results of this article. First, we show that the maximality of the Iwasawa invariants of the Mazur–Tate elements observed in Example 1.1 can be generalised to elliptic curves E satisfying the condition $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$, where Ω_E is the real Néron period of E.

Theorem A (Theorems 4.1 and 4.2)**.** *Let E be an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at p.*

(1) If $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$, then

$$\mu(\theta_n(E)) = \mathrm{ord}_p(L(E,1)/\Omega_E) \quad \text{and} \quad \lambda(\theta_n(E)) = p^n - 1$$

for all $n \geq 0$.

(2) If $\mu(L_p(E)) = 0$ and $\lambda(\theta_n(E)) = p^n - 1$ for $n \gg 0$, then

$$\mathrm{ord}_p\left(\frac{L(E,1)}{\Omega_E}\right) < 0.$$

Remark 1.3. Greenberg's $\mu = 0$ conjecture [Gre99, Conjecture 1.11] predicts that there exists at least one curve E' in the isogeny class of E satisfying $\mu(L_p(E')) = 0$. Furthermore, $\lambda(\theta_n(E))$ is constant within an isogeny class. Thus, admitting Greenberg's conjecture, Theorem A implies that the λ-invariants of $\theta_n(E)$ are maximal if and only if E is isogenous to a curve E' such that $\mathrm{ord}_p(L(E',1)/\Omega_{E'}) < 0$.

The proof of Theorem A relies crucially on a result of Wuthrich [Wut14] on the integrality of the p-adic L-function of E. More precisely, it asserts that $L_p(E)$ is an element of the Iwasawa algebra Λ when E is good ordinary at p (see Theorem 2.6). The condition $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$ implies that $\theta_0(E)$ is non-integral. Combining these facts, we deduce that $\theta_n(E)$ is non-integral for all n. The exact formula for $\lambda(\theta_n(E))$ given in part (a) is obtained by analyzing the effect on λ-invariants under the natural trace/corestriction map $\mathbb{Q}_p[\mathrm{Gal}(k_n/\mathbb{Q})] \to \mathbb{Q}_p[\mathrm{Gal}(k_{n+1}/\mathbb{Q})]$ (see Lemma 3.3 for details). For the converse implication (part (b) of the theorem), we exploit the norm relations satisfied by $\theta_n(E)$ to show that the maximality of λ-invariants and the vanishing of $\mu(L_p(E))$ imply that $\theta_n(E)$ is non-integral for all n.

Assuming the Birch and Swinnerton-Dyer conjecture for $E/\mathbb{Q}$, the condition $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$ implies the reducibility of $E[p]$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module as $|E_{\mathrm{tor}}(\mathbb{Q})|^2$ is the only term appearing in the denominator of the BSD quotient. However, the converse is not true (see Example 5.8). Our next result shows that the condition $\mathrm{ord}_p\big(L(E,1)/\Omega_E\big) < 0$ induces a dichotomy in the behavior of Mazur-Tate elements: either their μ- and λ-invariants stabilise to the corresponding invariants of the p-adic L-function or their λ-invariants are maximal.

Theorem B (Theorem 4.6)**.** *Let E be an elliptic curve over $\mathbb{Q}$ with good ordinary reduction at an odd prime p. Assume E is isogenous over $\mathbb{Q}$ to an elliptic curve $E'/\mathbb{Q}$ with $\mu(L_p(E')) = 0$. Then, we have the following dichotomy:*

(a) $\theta_n(E') \in \Lambda_n$ for all $n \geq 0$. In this case,

$$\mu(\theta_n(E)) = \mu(L_p(E)) \quad \text{and} \quad \lambda(\theta_n(E)) = \lambda(L_p(E))$$

for n sufficiently large, or

(b) $\theta_n(E') \notin \Lambda_n$ for all $n \geq 0$. In this case,

$$\mu(\theta_n(E)) = \mathrm{ord}_p(L(E,1)/\Omega_E) \quad \text{and} \quad \lambda(\theta_n(E)) = p^n - 1$$

for all $n \geq 0$.

Furthermore, case (b) occurs if and only if $\mathrm{ord}_p(L(E',1)/\Omega_{E'}) < 0$.

We remind the reader that Greenberg conjectures that our assumption on the μ-invariant always holds.

Our methods can be easily extended to obtain analogues of Theorems A and B for p-ordinary Hecke eigenforms of weight 2 and level coprime to p (see Theorems 4.3 and 4.8). For higher weight p-ordinary Hecke eigenforms, if their modular symbols are normalised using the cohomological periods (see Remark 2.5), then under the assumption that the μ-invariant of the associated p-adic L-function vanishes, it can be shown that the first case of Theorem B always holds (see [PW11, proof of Proposition 3.7]).

In the present article, we concentrate on good ordinary primes. For multiplicative primes, $\theta_n(E)$ have the same λ-invariants as the p-adic L-function for $n \gg 0$. When p is a supersingular prime, we also have a good understanding of the growth of $\lambda(\theta_n(E))$; see [PW11, Theorem 4.1]. In the case of additive primes, different patterns can emerge. This has been studied in [LPP25] and [DL21]. Interestingly, the maximality of λ-invariants can occur for additive primes; see Example 5.6.

1.2. **Interpreting our results in terms of Eisenstein congruences.** As mentioned above, the Birch and Swinnerton-Dyer conjecture predicts that $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$ implies that $E(\mathbb{Q})[p]$ is non-trivial. In this case, when $E(\mathbb{Q})[p] \neq 0$, the semi-simplification of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $E[p]$ is isomorphic to $1 \oplus \omega$, where ω denotes the mod p cyclotomic character. Thus, the associated cusp form $f_E = \sum_n a_n(E)q^n$ satisfies

$$a_\ell(E) \equiv \ell + 1 \mod p$$

for all primes $\ell \nmid N_E$. Furthermore, $a_\ell(E) = -\epsilon_\ell \in \{1, -1\}$ for primes $\ell \,||\, N_E$, where ϵ_ℓ is the eigenvalue of the Atkin–Lehner operator W_ℓ acting on f_E. Equivalently, since f_E is a newform of level N_E, one may interpret $a_\ell(E)$ as the eigenvalue of the U_ℓ-operator when $\ell \,||\, N_E$ since $U_\ell = -W_\ell$ on new forms.

Note that there exists an Eisenstein series $E_{2,N_E} = \sum a_n(E_{2,N_E})q^n$ on $\Gamma_0(N_E)$ of weight 2 such that $a_n(E_{2,N_E}) \equiv a_n(E) \mod p$ for all n when E has non-trivial p-torsion over $\mathbb{Q}$ and $p \nmid N_E$. As mentioned earlier, Eisenstein series give rise to boundary symbols. One might hope to deduce from the aforementioned congruence to a congruence relation between the modular symbols attached to f_E and the boundary symbol defined using E_{2,N_E}.

The space of weight 2 modular symbols of level N can be canonically identified with the compactly supported cohomology group $H^1_c(Y_0(N), \mathbb{C})$. Thus, in order to pass from a congruence of Hecke eigenvalues to a congruence of the respective modular symbols (equivalently, a congruence of the corresponding cohomology classes), one usually requires a mod p 'multiplicity one' result for a cohomology group. Such a result would imply that a modular symbol is uniquely determined (up

to scaling) by the Hecke eigenvalues. Multiplicity one results of this kind are well established in the residually irreducible case, see, for example, [Edi92, Theorem 9.2]. In the residually reducible case, the situation becomes more subtle, and one usually has to study Eisenstein completions of the Hecke algebra and the *Eisenstein ideal*, as illustrated by the following example.

Example 1.4 ($X_0(11)$ at $p = 5$, revisited)**.** *Let $\mathbb{T}_N$ denote the Hecke algebra acting on modular forms of weight 2 and prime level N generated by the usual Hecke operators T_ℓ for $\ell \nmid N$ and U_N. Let $\mathbb{T}_N^{\mathrm{Eis}}$ denote the completion of $\mathbb{T}_N$ at the ideal $(T_\ell - (\ell+1), U_N - 1, p)$.*

In his celebrated work [Maz77]*, Mazur studied various properties of the Hecke algebra $\mathbb{T}_N^{\mathrm{Eis}}$. He showed that the cuspidal quotient $\mathbb{T}_N^{\mathrm{Eis},0}$ of $\mathbb{T}_N^{\mathrm{Eis}}$ is a Gorenstein ring under certain explicit conditions on (N, p). In particular, his results apply to $(N, p) = (11, 5)$. Let $\mathfrak{m}$ denote the maximal ideal of $\mathbb{T}_N^{\mathrm{Eis},0}$ (this corresponds to the residual representation $1 \oplus \omega$). The Gorenstein property allows one to show $\dim_{\mathbb{F}_p} J_0(N)(\overline{\mathbb{Q}}_p)[\mathfrak{m}] = 2$, which implies, using* [Maz77, (7.5)]*,*

$$\dim_{\mathbb{F}_p} H^1(X_0(N), \mathbb{F}_p)^{\pm}[\mathfrak{m}] = 1,$$

i.e., the space of cuspidal modular symbols with system of mod p Hecke eigenvalues corresponding to $\mathfrak{m}$ is one dimensional. In this case, $\mathbb{T}_{Np}^{\mathrm{Eis}}$ is in fact Gorenstein, which can be used to show that the full space of modular symbols satisfies

$$\dim_{\mathbb{F}_p} H^1(Y_0(N), \mathbb{F}_p)^{\pm}[\mathfrak{m}] = 1.$$

We refer to this property as mod p multiplicity one for level N. Thus, up to a non-zero constant, the mod 5 reduction of ϕ_E equals the mod 5 reduction of a boundary symbol, since they are both $\mathbb{F}_p$-valued modular symbols that are annihilated by $\mathfrak{m}$.

Remark 1.5. In [DL24], the λ-invariants for the Mazur–Tate elements attached to the weight 12 modular form Δ arising from the Ramanujan tau function were studied for the primes $p \in \{3, 5, 7\}$. These primes are Eisenstein, i.e., Δ satisfies a congruence modulo p with certain Eisenstein series in weight 2. It is shown in *loc. cit.* that $\lambda(\theta_n(\Delta))$ is maximal (i.e., equal to $p^n - 1$) for $p = 5, 7$ for $n \geq 1$ by verifying computationally that the modular symbols of Δ are congruent to a weight 2 boundary symbol modulo p. In this setting, mod p multiplicity one results were not available to establish these congruences theoretically. This motivates our study of the maximality phenomenon for elliptic curves in the present article.

In Theorem A, we deduce the maximality of $\lambda(\theta_n)$ without invoking a congruence with a boundary symbol (thus avoiding questions of mod p multiplicity one). However, our data suggest that the maximality of $\lambda(\theta_n)$ occurs for an elliptic curve of square-free conductor N (and p good ordinary) if and only if mod p multiplicity one holds for level N. Using the results of Wake–Wang-Erickson [WWE21], which studies the appropriate generalisation of Mazur's results on $\mathbb{T}_N$ for prime N to squarefree N, we are able to verify in each of the examples we have considered that the relevant Hecke algebra is indeed Gorenstein. Furthermore, congruences with boundary symbols are also observed, indicating that a mod p multiplicity one result might actually hold. However, proving that the Gorenstein property implies mod p multiplicity one in our current setting has eluded us so far.

1.3. **Organisation.** In §2, we review the definitions of modular symbols and Mazur–Tate elements. In §3, we prove preliminary results that will be used to prove our main theorems. We prove Theorems A and B in §4.1 and 4.2, respectively. Both Theorems A and B are followed by the appropriate extension of our results to modular forms in 4. In §5, we discuss partial progress in relating Theorem A to congruences with boundary symbols and mod p multiplicity one for modular symbols. We also give explicit examples related to our results, one of which illustrates an interesting utility of our results in detecting when mod p multiplicity one fails.

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2. Modular symbols, boundary symbols and Mazur–Tate elements

2.1. **Modular symbols.** Let R be any commutative ring. Let $\mathbb{\Delta}$ denote the abelian group of divisors on $\mathbb{P}^1(\mathbb{Q})$, and let $\mathbb{\Delta}^0$ denote the subgroup of degree 0 divisors. Let $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{\Delta}^0$ by linear fractional transformations, which allows us to endow $\mathrm{Hom}(\mathbb{\Delta}^0, R)$ with a right action of $\mathrm{SL}_2(\mathbb{Z})$ via

$$(\varphi\,|_\gamma)(D) = (\varphi(\gamma \cdot D))\,|_\gamma,$$

where $\varphi \in \mathrm{Hom}(\mathbb{\Delta}^0, V_g(R))$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $D \in \mathbb{\Delta}^0$.

Definition 2.1. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. We define $\mathrm{Hom}_\Gamma(\mathbb{\Delta}^0, R)$ to be the space of R-valued **modular symbols** (of weight two), level Γ for some commutative ring R, and we denote this space by $\mathrm{Symb}(\Gamma, R)$.*

Remark 2.2. There is a canonical isomorphism

$$\mathrm{Symb}(\Gamma, R) \cong H^1_c(\Gamma, R)$$

where $H^1_c(\Gamma, R)$ denotes the compactly supported cohomology group (see [AS86, Proposition 4.2]).

For $f \in S_2(\Gamma)$, we define the **modular symbol associated with** f as

$$\xi_f : \{s\} - \{r\} \to 2\pi i \int_s^r f(z)dz,$$

which is an element of $\mathrm{Symb}(\Gamma, \mathbb{C})$ as f is a holomorphic cusp form. Let A_f be the field of Fourier coefficients of f and fix a prime p. The matrix $\iota := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts as an involution on $\mathrm{Symb}(\Gamma, \mathbb{C})$ and we decompose $\xi_f = \xi_f^+ + \xi_f^-$ with $\xi_f^\pm$ in the ± 1-eigenspace of ι respectively. By a theorem of Shimura, there exist $\Omega^\pm \in \mathbb{C}$ such that $\xi_f^\pm/\Omega^\pm$ take values in A_f, and in $\overline{\mathbb{Q}}_p$ upon fixing an embedding of $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (which we fix for the rest of the article). Define $\phi_f^\pm := \psi_f^\pm/\Omega^\pm$, and $\phi_f := \phi_f^+ + \phi_f^-$, which we regard as an element of $\mathrm{Symb}(\Gamma, \overline{\mathbb{Q}}_p)$. Let K_f denote the completion of the image of A_f in $\overline{\mathbb{Q}}_p$ and let $\mathcal{O}_f$ denote the ring of integers of K_f. We can choose Ω^+ and Ω^- so that each of ϕ_f^+ and ϕ_f^- takes values in $\mathcal{O}_f$ and that each takes on at least one value in $\mathcal{O}_f^\times$. We denote these periods $\Omega_f^\pm$; they are called **cohomological periods** of f, which are well-defined up to p-adic units (see [PW11, Definition 2.1]).

For an elliptic curve E defined over $\mathbb{Q}$, we are supplied with the real and imaginary **Néron periods**, which we denote by Ω_E^+ and Ω_E^-, respectively. We shall use Ω_E to denote the real Néron period Ω_E^+ in our arguments. Let ϕ_E denote the modular symbol attached to E normalised by $\Omega_E^\pm$, so that

$$\phi_E := \frac{\xi_{f_E}^+}{\Omega_E^+} + \frac{\xi_{f_E}^-}{\Omega_E^-}.$$

It takes values in $\mathbb{Q}_p$ but *a priori* is not guaranteed to take values in $\mathbb{Z}_p$.

2.2. **Mazur–Tate elements and p-adic L-functions.** For a non-negative integer n, let $\mathcal{G}_n := \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$. Recall from the introduction that k_n denotes the unique subextension of $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}$ that is of degree p^n. We write $G_n = \mathrm{Gal}(k_n/\mathbb{Q})$. For $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, we write $\sigma_a \in \mathcal{G}_n$ for the element that satisfies $\sigma_a(\zeta) = \zeta^a$ for $\zeta \in \mu_{p^n}$.

Definition 2.3. *For a modular symbol $\varphi \in \mathrm{Symb}(\Gamma, R)$, define the associated Mazur–Tate element of level $n \geq 1$ by*

$$\vartheta_n(\varphi) = \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \varphi(\{\infty\} - \{a/p^n\}) \cdot \sigma_a \in R[\mathcal{G}_n].$$

Let $f \in S_2(\Gamma)$. We define the p-adic Mazur–Tate element *of level $n \geq 0$ associated with f to be the image of $\vartheta_{n+1}(\phi_f)$ under the natural projection $\overline{\mathbb{Q}}_p[\mathcal{G}_{n+1}] \to \overline{\mathbb{Q}}_p[G_n]$. We denote this element by $\theta_n(f)$.*

For an elliptic curve $E/\mathbb{Q}$, we normalise $\theta_n(E) := \theta_n(\phi_E) \in \mathbb{Q}_p[G_n]$ using the Néron periods. We define $\theta_n(\phi_{E,\mathrm{Coh}})$ to be the corresponding elements under the normalisation by the cohomological periods of f_E.

Remark 2.4. Suppose E has good reduction at a prime $p > 2$ and let $f_E \in S_2(\Gamma_0(N_E))$ denote the associated cusp form. As $\theta_0(E)$ is the image of $\vartheta_1(\phi_f)$ under the natural map $\overline{\mathbb{Q}}_p[\mathcal{G}_1] \to \overline{\mathbb{Q}}_p[G_0]$, we have $\theta_0(E) = \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} \phi_E(\{\infty\} - \{a/p\})\sigma_1$, where σ_1 is the identity automorphism on $\mathbb{Q}$. Since ϕ_E is an eigenvector for the Hecke operator $T_p = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{u=0}^{p-1} \left(\begin{smallmatrix} 1 & u \\ 0 & p \end{smallmatrix}\right)$, we have

$$a_p(E)\phi_E = \phi_E(\{\infty\} - \{0\}) + \sum_{a=0}^{p-1} \phi_E(\{\infty\} - \{a/p\}),$$

which implies

$$\theta_0(E) = (a_p(E) - 2)\phi_E(\{\infty\} - \{0\})\sigma_1 = \left((a_p(E) - 2)\frac{L(E,1)}{\Omega_E}\right)\sigma_1. \tag{2.1}$$

We have implicitly used the fact that $\{\infty\} - \{a\}$ is $\Gamma_0(N_E)$-equivalent to $\{\infty\} - \{0\}$ for all $a \in \mathbb{Z}/p\mathbb{Z}$. The last equality above follows from $\phi_E(\{\infty\} - \{0\}) \cdot \Omega_E = 2\pi i \int_0^\infty f_E(z)dz = L(E,1)$. Similarly, for $f \in S_2(\Gamma_1(N), \epsilon_f)$, we have

$$\theta_0(f) = (a_p(f) - \epsilon_f(p) - 1)\phi_f(\{\infty\} - \{0\}).$$

We briefly recall the construction of the p-adic L-function of a p-ordinary Hecke eigenform $f = \sum_n a_n(f)e^{2\pi inz} \in S_2(\Gamma_1(N), \epsilon_f)$ when $p \nmid N$. Let α denote the unique p-adic unit root of the Hecke polynomial $X^2 - a_p(f)X + \epsilon_f(p)p$. We consider the p-stabilisation

$$f_\alpha(z) := f(z) - \frac{\epsilon_f(p)p}{\alpha} f(pz). \tag{2.2}$$

Define ϕ_f^α as the 'p-stabilised' modular symbol attached to f normalised by some periods $\Omega^\pm$, i.e.,

$$\phi_f^\alpha := \phi_f - \frac{\epsilon_f(p)}{\alpha}\phi_f|\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right). \tag{2.3}$$

This gives us a norm-compatible system $\left\{\frac{1}{\alpha^{n+1}}\theta_n(\phi_f^\alpha)\right\}_n$ since ϕ_f^α is an eigenvector for the U_p-operator with eigenvalue α.

Let $\Lambda_{n,\mathcal{O}_f}$ denote $\mathcal{O}_f[G_n]$ and define $\Lambda_{\mathcal{O}_f} = \varprojlim \Lambda_{n,\mathcal{O}_f}$, where the connecting maps are the natural projections. When $\mathcal{O}_f = \mathbb{Z}_p$, we shall write Λ_n for $\Lambda_{n,\mathbb{Z}_p}$ for simplicity. Then,

$$L_p(f) = \varprojlim_n \frac{1}{\alpha^{n+1}} \theta_n(\phi_f^\alpha) \in \Lambda_{\mathcal{O}_f} \otimes \overline{\mathbb{Q}}_p$$

is the trivial isotypic component of the p-adic L-function attached to f. We use the notations $L_p(f)$ and $L_p(f,T)$ interchangeably, where $L_p(f,T)$ denotes the image of $L_p(f)$ under our fixed isomorphism $\Lambda_{\mathcal{O}_f} \otimes \overline{\mathbb{Q}}_p \cong \mathcal{O}_f[[T]] \otimes \overline{\mathbb{Q}}_p$. One can also define the p-adic L-function as an element of $\mathcal{O}_f[[\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}]] \otimes \overline{\mathbb{Q}}_p$ by considering the norm-compatible system built from $\frac{1}{\alpha^n}\vartheta_n(\phi_f^\alpha)$ directly. We denote this inverse limit by $\mathcal{L}_p(f)$.

Remark 2.5 (**On periods**). Our methods in this article rely on the integrality of the associated p-adic L-function. From the existence of the cohomological periods, it is clear, for a p-ordinary Hecke eigenform, that there exist periods that ensure the p-adic L-function is an element of $\Lambda_{\mathcal{O}_f}$. We denote such periods by by $\Omega_{f,\bullet}^{\pm}$. For simplicity, we shall write $\Omega_{f,\bullet}$ for $\Omega_{f,\bullet}^{+}$.

From now on, whenever we refer to ϕ_f for a p-ordinary Hecke eigenform f, we assume that the ϕ_f is normalised by a choice of $\Omega_{f,\bullet}^{\pm}$. The cohomological periods associated with the ordinary p-stabilisation f_α of f (see (2.2) for the definition of f_α) are possible candidates for $\Omega_{f,\bullet}^{\pm}$. While such a normalisation ensures integrality of the p-adic L-function, the modular symbol ϕ_f takes values in K_f, and not necessarily in $\mathcal{O}_f$.

A choice of $\Omega_{f,\bullet}^{\pm}$ ensures $\mathcal{L}_p(f) \in \mathcal{O}_f[[\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}]]$ and $L_p(f) \in \Lambda_{\mathcal{O}_f}$. For elliptic curves, since we normalise ϕ_E^α by the Neron periods $\Omega_E^{\pm}$, the corresponding p-adic L-functions, which we denote by $L_p(E)$ and $\mathcal{L}_p(E)$ are *a priori* elements of $\Lambda \otimes \mathbb{Q}_p$ and $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}]] \otimes \mathbb{Q}_p$, respectively. Nonetheless, we have:

Theorem 2.6. *Let E be an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at an odd prime p. Then the p-adic L-function $\mathcal{L}_p(E)$ normalised by $\Omega_E^{\pm}$ lies in $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]]$.*

Proof. This is [Wut14, Corollary 18]. □

Remark 2.7. As a corollary of the above theorem, we deduce that the trivial isotypic component of $\mathcal{L}_p(E)$, i.e., $L_p(E)$ lies in $\Lambda \cong \mathbb{Z}_p[[T]]$.

Theorem 2.6 allows us to deduce the following:

Lemma 2.8. *Let E be an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at a prime $p > 2$. Then*

$$\phi_E^\alpha(\{\infty\} - \{a/p^n\}) \in \mathbb{Z}_p$$

for all $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and $n \geq 0$.

Proof. Theorem 2.6 says that $\mathcal{L}_p(E)$ is an element of

$$\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]] \cong \varprojlim_n \mathbb{Z}_p[\mathcal{G}_n] \cong \mathbb{Z}_p[[\mathbb{Z}_p^\times]].$$

The projection of $\mathcal{L}_p(E)$ to $\mathbb{Z}_p[\mathcal{G}_n]$ is $\vartheta_n(\phi_E^\alpha)/\alpha^n$, where $\vartheta_n(\phi_E^\alpha)$ is defined as

$$\vartheta_n(\phi_E^\alpha) := \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \phi_E^\alpha(\{\infty\} - \{a/p^n\}) \cdot \sigma_a$$

for $\sigma_a \in \mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$. As $\alpha \in \mathbb{Z}_p^\times$, we have $\vartheta_n(\phi_E^\alpha) \in \mathbb{Z}_p[\mathcal{G}_n]$, implying $\phi_E^\alpha(\{\infty\} - \{a/p^n\}) \in \mathbb{Z}_p$ for all $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and $n \geq 0$. □

We note that the same proof works for a p-ordinary eigenform $f \in S_k(\Gamma_1(N), \epsilon_f)$ with $p \nmid N$ and implies that $\phi_f^\alpha(\{\infty\} - \{a/p^n\}) \in \mathbb{Z}_p$ for all $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and $n \geq 0$, since we normalise the modular symbols by $\Omega_{f,\bullet}^{\pm}$.

3. General results on Iwasawa invariants

Let γ_n be a generator of G_n and let K be a finite extension of $\mathbb{Q}_p$ with a fixed uniformizer ϖ. Let $\mathcal{O}_K$ denote the ring of integers of K. For any element $F \in K[G_n]$, we may write it as a polynomial $\sum_{i=0}^{p^n-1} a_i T^i$ with $T = \gamma_n - 1$ and $a_i \in K$.

Definition 3.1 (Iwasawa invariants)**.** *The μ and λ-invariants of $F = \sum_{i=0}^{p^n-1} a_i T^i \in K[G_n]$ are defined as*

$$\mu(F) = \min_i\{\mathrm{ord}_\varpi(a_i)\},$$
$$\lambda(F) = \min\{i : \mathrm{ord}_\varpi(a_i) = \mu(F)\},$$

where ord_ϖ is the valuation such that $\mathrm{ord}_\varpi(\varpi) = 1$.

These invariants are independent of the choice of γ_n. One can directly define μ and λ-invariants for an element of the finite level group algebra $K[G_n]$ without reference to a generator γ_n; for more details, see [PW11, § 3.1].

We begin with a lemma involving the Iwasawa invariants of a sum of elements in $K[G_n]$, which will be utilised in the proof of Theorem A (Theorem 4.2).

Lemma 3.2. *Let $F_1, F_2 \in K[G_n]$ with $\lambda(F_1 + F_2) < \lambda(F_1), \lambda(F_2)$. Then*

$$\mu(F_1 + F_2) > \mu(F_1) = \mu(F_2) \quad \text{and} \quad \lambda(F_1) = \lambda(F_2).$$

Proof. Suppose that $\mu(F_1) > \mu(F_2)$. Then $\mu(F_1 + F_2) = \mu(F_2)$ and $\lambda(F_1 + F_2) = \lambda(F_2)$. This contradicts $\lambda(F_1 + F_2) < \lambda(F_2)$. Thus, we deduce $\mu(F_1) = \mu(F_2)$.

After multiplying by a scalar if necessary, we may assume that $\mu(F_1) = \mu(F_2) = 0$. Suppose that $\lambda(F_1) > \lambda(F_2)$. Then we have $\lambda(F_1 + F_2) = \lambda(F_2)$. This contradicts $\lambda(F_1 + F_2) < \lambda(F_2)$. Hence, we deduce $\lambda(F_1) = \lambda(F_2)$.

Let $d = \lambda(F_1) = \lambda(F_2)$. There exists $u_i \in \mathcal{O}^\times$ such that

$$F_i \equiv u_i T^d \mod \varpi\mathcal{O}[G_n], \quad i = 1, 2.$$

In particular, $F_1 + F_2 \equiv (u_1 + u_2)T^d \mod \varpi\mathcal{O}[G_n]$. As $\lambda(F_1 + F_2) < d$, this implies that $u_1 + u_2 \equiv 0 \mod \varpi$. Hence, $\mu(F_1 + F_2) > 0$, as desired. □

Let $\pi_{n-1}^n : G_n \to G_{n-1}$ be the natural projection map. For $\sigma \in G_{n-1}$, define the corestriction map

$$\mathrm{cor}_{n-1}^n(\sigma) := \sum_{\substack{\pi_{n-1}^n(\tau) = \sigma \\ \tau \in \mathrm{Gal}(k_n/\mathbb{Q})}} \tau \in K[G_n]$$

which extends K-linearly to a map $K[G_{n-1}] \to K[G_n]$. We recall some standard facts about Iwasawa invariants of Mazur–Tate elements under the natural projection and corestriction maps.

Lemma 3.3. *(1) For an integer $N \geq 1$ and a prime p, let $\phi \in \mathrm{Symb}(\Gamma_0(N), \mathcal{O})$ be a modular symbol. For $n \geq 1$, we have*

$$\theta_n(\phi \mid \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)) = \mathrm{cor}_{n-1}^n(\theta_{n-1}(\phi)).$$

(2) For $\theta \in K[G_{n-1}]$, we have

$$\mu(\mathrm{cor}_{n-1}^{n}(\theta)) = \mu(\theta) \text{ and } \lambda(\mathrm{cor}_{n-1}^{n}(\theta)) = p^n - p^{n-1} + \lambda(\theta).$$

(3) For $\theta \in \mathcal{O}[G_n]$, if $\mu(\pi_{n-1}^{n}(\theta)) = 0$, then $\mu(\theta) = 0$.

Proof. Part (1) is [PW11, Lemma 2.6]; (2) and (3) are discussed in [Pol05, § 4] as well as [PW11, Lemmas 3.2 and 3.3]. □

We prove the main theorem of this section that involves an arbitrary sequence of elements in $K[G_n]$, which will be crucial in the proofs of our main results in this article.

Theorem 3.4. *Let $\{\theta_n\}_{n\geq 0}$ be a sequence in $K[G_n]$ such that*

(1) $\mu(\theta_n)$ is bounded from below as $n \to \infty$, and
(2) there exists $\alpha \in \mathcal{O}^\times$ such that for all $n \geq 1$

$$\theta_n^{\alpha} := \theta_n - \alpha^{-1}\mathrm{cor}_{n-1}^{n}(\theta_{n-1}) \in \mathcal{O}[G_n].$$

If there exists an integer n such that $\mu(\theta_n) < 0$, then

$$\lambda(\theta_n) = p^n - 1 \quad \text{and} \quad \mu(\theta_n) = \mu(\theta_0) < 0$$

for all $n \geq 0$.

Proof. Define $t := -\min_n\{\mu(\theta_n)\} > 0$, which exists by assumption (1). For each n, define $\theta'_n := \varpi^t\theta_n$, which belongs to $\mathcal{O}[G_n]$ by the definition of t. From (2), we have

$$\theta'_n \equiv \frac{1}{\alpha}\mathrm{cor}_{n-1}^{n}(\theta'_{n-1}) \mod \varpi^t\mathcal{O}[G_n]$$

for all $n \geq 1$. Iterating this congruence, we obtain

$$\theta'_n \equiv \frac{1}{\alpha^n}\mathrm{cor}_{0}^{n}(\theta'_0) \mod \varpi^t\mathcal{O}[G_n] \tag{3.1}$$

for all $n \geq 0$.

Let m be an integer such that $\mu(\theta_m) = -t$. Then $\mu(\theta'_m) = 0$. Setting n to be m in (3.1), we deduce from Lemma 3.3(2): $\mu(\mathrm{cor}_0^m(\theta'_0)) = \mu(\theta'_0) = 0$. Consequently, both sides of the congruence in (3.1) are non-zero. Hence, from Lemma 3.3(2), we deduce the equations

$$\lambda(\theta'_n) = p^n - p^{n-1} + \lambda(\theta'_{n-1}) = \cdots = p^n - 1 + \lambda(\theta'_0) = p^n - 1$$

and $\mu(\theta'_n) = \mu(\theta'_0) = 0$ for all $n \geq 0$. Hence, $\mu(\theta_n) = \mu(\theta_0) = t < 0$ and $\lambda(\theta_n) = p^n - 1$ for all n. □

Remark 3.5. We shall apply Theorem 3.4 to the sequence of p-adic Mazur–Tate elements $\{\theta_n(f)\}_{n\geq 0}$ attached to a p-ordinary Hecke eigenform $f \in S_2(\Gamma_1(N), \epsilon_f)$ with $p \nmid N$. Assumption (1) is satisfied since $\mu(\theta_n(f)) \geq \mathrm{ord}_{\varpi}(\Omega_f/\Omega_{f,\bullet})$. Taking α to be the p-adic unit root of $x^2 - a_p(f)x + \epsilon_f(p)p$ and $\theta_n^{\alpha} = \epsilon_f(p)\theta_n(\phi_f^{\alpha})$, by (2.3) and Lemma 3.3(1), assumption (2) is satisfied due to our choice of normalisation by the period $\Omega_{f,\bullet}$.

For an elliptic curve E defined over $\mathbb{Q}$ that has good ordinary reduction at p, the assumptions can be verified similarly for the sequence $\{\theta_n(E)\}_{n\geq 0}$: For (1), note that if $r, s \in \mathbb{P}^1(\mathbb{Q})$, we have $\phi_E(\{r\} - \{s\}) \in \delta_E^{-1}\mathbb{Z}$ for an integer δ_E independent of r and s (see [MR23, Lemma 1.2(i)]). Thus, $\mu(\theta_n(E)) \geq -\mathrm{ord}_p(\delta_E)$ for all $n \geq 0$. For (2), let α be the p-adic unit root of $x^2 - a_p(E)x + p$. Then Lemma 2.8 says that $\theta_n(\phi_E^{\alpha}) \in \mathbb{Z}_p[G_n]$. Hence, we can take $\theta_n^{\alpha} = \theta_n(\phi_E^{\alpha})$ by (2.3) and Lemma 3.3(1).

4. Proofs of main results

In this section, we prove Theorems A and B.

4.1. **Proof of Theorem A.** We begin with part (1) of Theorem A.

Theorem 4.1. *Let E be an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at p such that $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$. Then we have*

$$\mu(\theta_n(E)) = \mathrm{ord}_p(L(E,1)/\Omega_E) \quad \text{and} \quad \lambda(\theta_n(E)) = p^n - 1$$

for all $n \geq 0$.

Proof. If $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$, then $\alpha \equiv 1 \mod p$ since

$$L_p(E,T)|_{T=0} = \left(1 - \frac{1}{\alpha}\right)^2 \frac{L(E,1)}{\Omega_E} \in \mathbb{Z}_p$$

by Theorem 2.6. Thus, $a_p(E) \equiv 1 \mod p$. Consequently, from (2.1) it follows that

$$\mu(\theta_0(E)) = \mathrm{ord}_p((a_p - 2)L(E,1)/\Omega_E) < 0.$$

Therefore, Theorem 3.4 applies to the sequence $\{\theta_n(E)\}_{n\geq 0}$ due to the verification carried out in Remark 3.5, concluding the proof of the theorem. □

Next, we prove the converse to Theorem 4.1 under the additional assumption that the μ-invariant of the p-adic L-function attached to E vanishes, i.e., part (2) of Theorem A.

Theorem 4.2. *Let E be an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at an odd prime p with $\mu(L_p(E)) = 0$. Assume $\lambda(\theta_n(E)) = p^n - 1$ for all n sufficiently large. Then*

$$\mathrm{ord}_p\left(\frac{L(E,1)}{\Omega_E}\right) < 0.$$

Proof. Since $L_p(E)$ can be realized as the limit of $\theta_n(\phi_E^\alpha)$ as $n \to \infty$, we have $\mu(\theta_n(\phi_E^\alpha)) = \mu(L_p(E)) = 0$ and $\lambda(\theta_n(\phi_E^\alpha)) = \lambda(L_p(E)) < p^n - 1$ for n large enough. Recall that

$$\theta_n(\phi_E^\alpha) = \theta_n(E) - \frac{1}{\alpha}\mathrm{cor}_{n-1}^n(\theta_{n-1}(E)). \tag{4.1}$$

Applying Lemma 3.2 gives $0 = \mu(\theta_n(\phi_E^\alpha)) > \mu(\theta_n(E))$. Consequently, Theorem 3.4 applies and we deduce $\mu(\theta_0(E)) < 0$. By (2.1), this is equivalent to $\mathrm{ord}_p((a_p - 2)L(E,1)/\Omega_E) < 0$. As $a_p(E) \in \mathbb{Z}_p$, we have $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$ as desired. □

In a completely analogous manner, we obtain the following extension of Theorems 4.1 and 4.2 to modular forms of weight 2. Recall that the period $\Omega_{f,\bullet}$ is chosen so that $L_p(f)$ is integral.

Theorem 4.3. *Let $f \in S_2(\Gamma_1(N), \epsilon_f)$ be a Hecke eigenform and p an odd prime such that $p \nmid N$ and f is ordinary at p. Let ϖ denote a uniformizer for K_f, the completion of the field of Fourier coefficients of f at a prime above p.*

(1) If $\mathrm{ord}_\varpi(L(f,1)/\Omega_{f,\bullet}) < 0$, then

$$\mu(\theta_n(f)) = \mathrm{ord}_\varpi(L(f,1)/\Omega_{f,\bullet}) \quad \text{and} \quad \lambda(\theta_n(f)) = p^n - 1$$

for all $n \geq 0$.

(2) If $\mu(L_p(f)) = 0$ and $\lambda(\theta_n(f)) = p^n - 1$ for $n \gg 0$, then

$$\mathrm{ord}_\varpi\left(\frac{L(f,1)}{\Omega_{f,\bullet}}\right) < 0.$$

4.2. **Proof of Theorem B.** In order to prove Theorem B, we discuss how the integrality of Mazur–Tate elements influences the corresponding Iwasawa invariants at good ordinary primes.

Theorem 4.4. *Let E be an elliptic curve over $\mathbb{Q}$ with good ordinary reduction at an odd prime p with $\mu(L_p(E)) = 0$. Suppose that $\theta_n(E) \in \Lambda_n$ for all $n \gg 0$. Then*

$$\mu(\theta_n(E)) = 0 \quad \textit{and} \quad \lambda(\theta_n(E)) = \lambda(L_p(E))$$

for all n sufficiently large.

Proof. This proof is based on the argument used in [PW11, proof of Proposition 3.7]. Note that $\theta_{n-1}(E) \in \Lambda_{n-1}$ implies $\mathrm{cor}_{n-1}^{n}(\theta_{n-1}(E)) \in \Lambda_n$. Further, $\mu(\theta_n(\phi_E^\alpha)) = \mu(L_p(E)) = 0$ and $\lambda(\theta_n(\phi_E^\alpha)) = \lambda(L_p(E))$ for all $n \gg 0$ since $\theta_n(\phi_E^\alpha)$ converges to $L_p(E)$ as $n \to \infty$.

Assume that n is chosen so that $\mu(\theta_n(\phi_E^\alpha)) = 0$. As the right-hand side of (4.1) consists of two elements in Λ_n, one of them must have zero μ-invariant. Moreover, $\mu(\mathrm{cor}_{n-1}^{n}(\theta_{n-1}(E))) = \mu(\theta_{n-1}(E))$ by Lemma 3.3(2). Hence, we deduce that $\mu(\theta_{n'}(E)) = 0$ for some n'. Further, n' can be arbitrarily large. Suppose that n' is chosen so that $\mu(\theta_{n'}(E)) = 0$. If $\mu(\theta_{n'+1}(E)) > 0$, this would imply that

$$\theta_{n'+1}(\phi_E^\alpha) \equiv -\frac{1}{\alpha}\mathrm{cor}_{n'}^{n'+1}(\theta_{n'}(E)) \mod p\Lambda_{n'+1}.$$

Since both sides of this congruence have trivial μ-invariants, Lemma 3.3(2) gives

$$\lambda(\theta_{n'+1}(\phi_E^\alpha)) = p^{n'+1} - p^{n'} + \lambda(\theta_{n'}(E)),$$

which cannot hold when n' is sufficiently large since $\lambda(\theta_{n'+1}(\phi_E^\alpha)) = \lambda(L_p(E))$ is bounded. So, we must have $\mu(\theta_{n'+1}(E)) = 0$. In particular, $\mu(\theta_n(E)) = 0$ for $n \gg 0$. In addition,

$$\lambda(L_p(E,T)) = \lambda(\theta_n(\phi_E^\alpha)) < \lambda(\mathrm{cor}_{n-1}^{n}(\theta_{n-1}) = p^n - p^{n-1} + \lambda(\theta_{n-1}(E))$$

for $n \gg 0$. Hence, from Equation (4.1), we deduce

$$\lambda(\theta_n(E)) = \min\left\{\lambda(\theta_n(\phi_E^\alpha)),\ \lambda(\mathrm{cor}_{n-1}^{n}(\theta_{n-1}(E)))\right\} = \lambda(\theta_n(\phi_E^\alpha)) = \lambda(L_p(E.T)),$$

as desired. □

Remark 4.5. Let $f \in S_2(\Gamma_1(N), \epsilon_f)$ be a p-ordinary Hecke eigenform with $p \nmid N$. Let α be the p-adic unit root of $x^2 - a_p(f) + \epsilon_f(p)p$ and let f_α denote the ordinary p-stabilisation of f. Let ϖ denote a uniformizer of K_f and $\mathcal{O}_f$ denote the ring of integers of K_f. Recall that we normalise $\theta_n(f)$ by the periods $\Omega_{f,\bullet}^{\pm}$, which ensures the integrality of $L_p(f)$. If $\mu(L_p(f)) = 0$ and $\theta_n(f) \in \mathcal{O}_f[G_n]$, then

$$\mu(\theta_n(f)) = 0 \quad \text{and} \quad \lambda(\theta_n(f)) = \lambda(L_p(f))$$

for all n sufficiently large. This can be shown by an argument similar to the one used in the proof of Theorem 4.4.

In this article, we focus on weight 2 forms. Mazur–Tate elements can be defined for cusp forms of higher weight (see [PW11, § 2]) and if the modular symbols are normalised by the cohomological periods, one can obtain the same result as in Theorem 4.4. More precisely, if the μ-invariant of the associated p-adic L-function vanishes, the λ-invariant of the Mazur–Tate element always matches the λ-invariant of the p-adic L-function for $n \gg 0$.

We now give the proof of Theorem B.

Theorem 4.6. *Let E be an elliptic curve over $\mathbb{Q}$ with good ordinary reduction at an odd prime p. Assume E is isogenous over $\mathbb{Q}$ to an elliptic curve $E'/\mathbb{Q}$ with $\mu(L_p(E')) = 0$. Then, we have the following dichotomy:*

(a) $\theta_n(E') \in \Lambda_n$ *for all* $n \geq 0$. *In this case,*

$$\mu(\theta_n(E)) = \mu(L_p(E)) \quad \textit{and} \quad \lambda(\theta_n(E)) = \lambda(L_p(E))$$

for n *sufficiently large, or*

(b) $\theta_n(E') \notin \Lambda_n$ *for all* $n \geq 0$. *In this case,*

$$\mu(\theta_n(E)) = \mathrm{ord}_p(L(E,1)/\Omega_E) \quad \textit{and} \quad \lambda(\theta_n(E)) = p^n - 1$$

for all $n \geq 0$.

Furthermore, case (b) occurs if and only if $\mathrm{ord}_p(L(E',1)/\Omega_{E'}) < 0$.

Proof. Since $\theta_n(E) = \theta_n(E') \cdot \Omega_{E'}/\Omega_E$ and $L_p(E,T) = L_p(E',T) \cdot \Omega_{E'}/\Omega_E$, we have

$$\lambda(\theta_n(E)) = \lambda(\theta_n(E')) \quad \text{and} \quad \mu(\theta_n(E)) - \mu(L_p(E,T)) = \mu(\theta_n(E')) - \mu(L_p(E',T)).$$

Thus, we may assume $E = E'$.

Suppose $\theta_n(E) \notin \Lambda_n$ for some n. In other words, $\mu(\theta_n(E)) < 0$. Theorem 3.4 combined with Remark 3.5 implies that the conditions described in case (b) hold. In addition, Theorem 4.2 tells us that $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$ in this case.

Otherwise, suppose that $\theta_n(E) \in \Lambda_n$ for all $n \geq 0$. In this case, Theorem 4.4 implies that the conditions described in case (a) hold. Furthermore, the contrapositive of Theorem 4.1 tells us that $\mathrm{ord}_p(L(E,1)/\Omega_E) \geq 0$. □

Remark 4.7. Assume that $\mu(L_p(E)) = 0$. Theorem 4.6 implies that $\theta_n(E) \in \Lambda_n$ for all good ordinary primes $p > 2$ except when $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$. If $p > 7$, then Mazur's theorem on torsion groups of elliptic curves over $\mathbb{Q}$, which says that $E(\mathbb{Q})[p] = 0$. In particular, the Birch–Swinnerton-Dyer conjecture predicts $\mathrm{ord}_p(L(E,1)/\Omega_E) \geq 0$. Therefore, we expect that only case (a) of Theorem 4.6 occurs when $p > 7$.

One can obtain the analogue of Theorem 4.6 for a p-ordinary Hecke eigenform $f \in S_2(\Gamma_1(N), \epsilon_f)$ with $p \nmid N$ in the same way, combining the previous argument with Remark 4.5:

Theorem 4.8. *Let* $f \in S_2(\Gamma_1(N), \epsilon_f)$ *be a* p*-ordinary Hecke eigenform with* $p \nmid N$. *Suppose* $\Omega_{f,\bullet}$ *is chosen so that* $\mu(L_p(f)) = 0$. *We have the following dichotomy:*

(a) $\theta_n(f) \in \mathcal{O}_f[G_n]$ *for all* $n \geq 0$. *In this case,*

$$\mu(\theta_n(f)) = \mu(L_p(f)) = 0 \quad \textit{and} \quad \lambda(\theta_n(f)) = \lambda(L_p(f))$$

for n *sufficiently large, or*

(b) $\theta_n(f) \notin \mathcal{O}_f[G_n]$ *for all* $n \geq 0$. *In this case,*

$$\mu(\theta_n(f)) = \mathrm{ord}_p(L(f,1)/\Omega_{f,\bullet}) < 0 \quad \textit{and} \quad \lambda(\theta_n(f)) = p^n - 1.$$

5. Examples and remarks on congruences with boundary symbols

In this section, we relate the maximality of λ-invariants of Mazur–Tate elements to certain congruences satisfied by the corresponding modular symbol. We begin with the following proposition, which is a reformulation of Theorem 3.4 in terms of a congruence condition on modular symbols at divisors of the form $\{\infty\} - \{a/p^n\}$ for $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and $n \geq 0$.

Proposition 5.1. *For a positive integer* N *and an odd prime* p, *let* $\phi \in \mathrm{Symb}(\Gamma_0(N), \mathbb{Z}_p)$. *Let* $\mathrm{ord}_p(\phi(\{\infty\} - \{0\})) = m$. *If there exists a constant* $\alpha \in \mathbb{Z}_p^\times$ *and an integer* $t > m$ *such that*

$$\phi(\{\infty\} - \{a/p^{n+1}\}) \equiv \alpha\phi(\{\infty\} - \{a/p^n\}) \mod p^t \tag{5.1}$$

for all $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ *and* $n \geq 0$*, then*

$$\mu(\theta_n(\phi)) = m \quad \text{and} \quad \lambda(\theta_n(\phi)) = p^n - 1$$

for all $n \geq 0$*.*

Proof. The condition (5.1) is equivalent to

$$\phi(\{\infty\} - \{a/p^n\}) \equiv \alpha\phi \mid \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)(\{\infty\} - \{a/p^n\}) \mod p^t$$

for all $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and $n \geq 1$. Since $\theta_n(\phi)$ is determined by the values of ϕ on divisors of the form $\{\infty\} - \{a/p^n\}$, we apply Lemma 3.3(1) to deduce

$$\theta_n(\phi) \equiv \alpha \cdot \mathrm{cor}_{n-1}^n(\theta_{n-1}(\phi)) \equiv \cdots \equiv \alpha^n \cdot \mathrm{cor}_0^n(\theta_0(\phi)) \mod p^t.$$

But

$$\theta_0(\phi) = \sum_{a=1}^{p-1} \phi(\{\infty\} - \{a/p\})\sigma_1 \overset{(5.1)}{\equiv} \alpha \sum_{a=1}^{p-1} \phi(\{\infty\} - \{a\})\sigma_1 = \alpha(p-1)\cdot\phi(\{\infty\} - \{0\})\cdot\sigma_1 \not\equiv 0 \mod p^t$$

by hypothesis. The proposition now follows from Lemma 3.3(2). □

Ihara's lemma, in the form described in [PW11, Theorem 3.5], for a Hecke eigensymbol $\varphi_f \in \mathrm{Symb}(\Gamma_0(N), \mathbb{Z}_p)$ (corresponding to a weight 2 eigenform f) implies the following: if, for some constant $c_0 \in \mathbb{Z}_p^\times$,

$$\varphi_f \equiv c_0\varphi_f|\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod p \tag{5.2}$$

then f is congruent modulo p to an Eisenstein series of weight 2. Note that if (5.2) holds and $\varphi_f(\{\infty\} - \{0\}) \in \mathbb{Z}_p^\times$, Proposition 5.1 implies that

$$\lambda(\theta_n(f)) = p^n - 1 \text{ for all } n \geq 0,$$

where the modular symbols are normalised by the cohomological periods $\Omega_f^\pm$ associated with f.

5.1. **Congruences with boundary symbols.** We give a brief review on boundary symbols, following [BD15]. Let R be a commutative ring. We have the exact sequence

$$0 \to R^\Gamma \to \mathrm{Hom}_\Gamma(\mathbb{\Delta}, R) \xrightarrow{b} \mathrm{Symb}(\Gamma, R) \xrightarrow{h} H^1(\Gamma, R). \tag{5.3}$$

Definition 5.2. *The map* b *in* (5.3) *is called the* ***boundary map*** *and its image, denoted by* $\mathrm{BSymb}(\Gamma, R)$*, is called the module of* ***boundary modular symbols*** *(or simply* ***boundary symbols****). We denote the space of weight* 2 *boundary symbols by* $\mathrm{BSymb}(\Gamma, R)$*.*

The exact sequence (5.3) yields an isomorphism of Hecke-modules

$$\mathrm{BSymb}(\Gamma, R) \cong \mathrm{Hom}_\Gamma(\mathbb{\Delta}, R)/R^\Gamma,$$

relating modular symbols to boundary symbols. Furthermore, there is a short exact sequence

$$0 \to \mathrm{BSymb}_\Gamma(R) \to \mathrm{Symb}(\Gamma, R) \to H^1(\Gamma, R).$$

This space of boundary symbols can be identified with the space of weight 2 Eisenstein series under the Eichler–Shimura isomorphism (see Remark 5.3 below and note that a notion of modular symbols that is dual to the one discussed here is utilized therein). These symbols can be considered as Γ-invariant maps on the set of divisors $\mathbb{\Delta}$, greatly simplifying the computation.

Remark 5.3. (**On the Eichler–Shimura isomorphism**) The space of (weight 2) modular symbols $\mathrm{Symb}(\Gamma,\mathbb{C}) = \mathrm{Hom}_\Gamma(\mathbb{\Delta},\mathbb{C})$ can be identified with the direct sum $S_2(\Gamma)\oplus S_2^{\mathrm{anti}}(\Gamma)\oplus E_2(\Gamma)$, where $S_2(\Gamma), S_2^{\mathrm{anti}}(\Gamma)$ and $E_2(\Gamma)$ denote the $\mathbb{C}$-vector space of holomorphic cusp forms, anti-holomorphic cusp forms and Eisenstein series, respectively. This is the *Eichler–Shimura isomorphism*. In terms of the cohomology of the modular curve for $\Gamma = \Gamma_0(N)$, we have the following exact sequence of Hecke-modules

$$0 \to \mathrm{BSymb}(\Gamma_0(N),\mathbb{C}) \to H^1_c(Y_0(N),\mathbb{C}) \to H^1(Y_0(N),\mathbb{C}) \to E_2(\Gamma) \to 0$$

(see [BD15, Proposition 2.5]). Recall that there is a canonical isomorphism $\mathrm{Symb}(\Gamma_0(N),\mathbb{C}) \cong H^1_c(Y_0(N),\mathbb{C})$. Thus, $\mathrm{BSymb}(\Gamma,\mathbb{C})$ is the summand corresponding to $E_2(\Gamma)$ inside $\mathrm{Symb}(\Gamma,\mathbb{C})$ and can be identified with the image of $\mathrm{Hom}(\mathbb{\Delta},\mathbb{C})$ inside $\mathrm{Hom}_\Gamma(\mathbb{\Delta}^0,\mathbb{C})$ (via the map $\mathbb{\Delta}\to\mathbb{\Delta}^0$ given by restricting to degree-zero divisors).

Let $\phi_{E,\mathrm{Coh}}$ denote the modular symbol attached to an elliptic curve E normalised by the cohomological periods of E for a given prime p, i.e.,

$$\phi_{E,\mathrm{Coh}} := \frac{\xi^+_{f_E}}{\Omega^+_{f_E}} + \frac{\xi^-_{f_E}}{\Omega^-_{f_E}}.$$

We prove a straightforward consequence of the results from previous sections.

Proposition 5.4. *Let E be an elliptic curve over $\mathbb{Q}$ with good ordinary reduction at an odd prime p and conductor N_E. Assume $L(E,1)/\Omega_{f_E} \in \mathbb{Z}_p^\times$. Suppose $\phi^+_{E,\mathrm{Coh}}$ is congruent modulo p to a boundary symbol $\phi_B \in \mathrm{BSymb}(\Gamma_0(N_E),\mathbb{Z}_p)$ of level $\Gamma_0(N_E)$. Then*

$$\lambda(\theta_n(E)) = p^n - 1 \ \forall\ n \geq 0.$$

Furthermore, if $\mu(L_p(E)) = 0$, we have

$$\mathrm{ord}_p(L(E,1)/\Omega_E) < 0.$$

Proof. The congruence between $\phi^+_{E,\mathrm{Coh}}$ with a weight 2 boundary symbol and the fact that divisors of the form $\{a/p^n\}$ and $\{a/p^{n-1}\}$ are $\Gamma_0(N_E)$-equivalent imply the congruence

$$\phi^+_{E,\mathrm{Coh}}(\{\infty\} - \{a/p^n\}) \equiv \phi^+_{E,\mathrm{Coh}}(\{\infty\} - \{a/p^{n-1}\}) \mod p$$

for all $n \geq 1$. The first assertion of the proposition follows from Proposition 5.1, whereas the second assertion is a consequence of Theorem 4.2. □

This illustrates one way in which case (b) of Theorem 4.6 occurs. Interestingly, all the examples generated via our computations having $\lambda(\theta_n(E)) = p^n - 1$ for a good ordinary prime $p > 2$ satisfy the hypotheses of Proposition 5.4.

5.2. **Examples.**

Example 5.5. *Let $E =$ 26b1 and $p = 7$. It has a non-trivial torsion point of order 7 over $\mathbb{Q}$. From LMFDB, we see that $L(E,1)/\Omega_E = 1/7$. It follows from Theorem 4.1 that*

$$\lambda(\theta_n(E)) = p^n - 1$$

for $n \geq 0$. Furthermore, we see that $\phi^+_{E,\mathrm{Coh}}$ is congruent modulo p to a boundary symbol (taking values in $\mathbb{F}_p$) defined as

$$\psi(\{r\}) = \begin{cases} -1/2 & \text{if } r \in \Gamma_0(26)\{0\} \cup \Gamma_0(26)\{1/13\} \\ 1/2 & \text{if } r \in \Gamma_0(26)\{\infty\} \cup \Gamma_0(26)\{1/2\} \end{cases}$$

for $r \in \mathbb{Q}$. Here $\Gamma_0(26)\{c\}$ denotes the equivalence class of the cusp c.

We have found examples of elliptic curves with additive reduction at p where $\lambda(\theta_n(E))$ is maximal. Interestingly, in all of these examples, we observe that $\mathrm{ord}_p(L(E,1)/\Omega_E) < 0$.

Example 5.6. *Let $E =$50b1 and $p = 5$. This curve admits a non-trivial 5-torsion over $\mathbb{Q}$ and $\phi_{E,\mathrm{Coh}}(\{\infty\} - \{0\}) = 1$. We have $L(E,1)/\Omega_E = 1/5$. We observe, for $n \geq 0$,*

$$\lambda(\theta_n(E)) = p^n - 1.$$

Moreover, $\phi^+_{E,\mathrm{Coh}}$ is indeed congruent to a weight 2 boundary symbol on $\Gamma_0(50)$ modulo 5.

Remark 5.7. Note that when E has p-torsion over $\mathbb{Q}$, the mod p representation is reducible and one can exhibit an Eisenstein series of weight 2 having Hecke eigenvalues congruent modulo p to those of E. In all the examples that exhibit the maximality of $\lambda(\theta_n(E))$ with E semistable and p a prime of good ordinary reduction that we have found, the hypotheses of [WWE21, Theorem 1.5.1] can be verified. So, it follows that, in each of these examples, the relevant Eisenstein completion of the Hecke algebra is a Gorenstein ring. We expect that this implies that the space of $\mathbb{F}_p$-valued modular symbols with Hecke eigenvalues congruent to those of E is one dimensional, implying that there is a congruence with a boundary symbol.

We end with another example to demonstrate a consequence of Proposition 5.4 in this direction.

Example 5.8. *Let $E =$174b1 and $p = 7$. This curve has $\mu(L_p(E)) = 0$, and $L(E,1)/\Omega_{f_E} \in \mathbb{Z}_p^\times$. The curve E admits a non-trivial 7-torsion point over $\mathbb{Q}$, which implies that E is congruent to a weight 2 Eisenstein series of level $\Gamma_0(174)$ modulo 7. However, we have $L(E,1)/\Omega_E = 1$, so Proposition 5.4 tells us that ϕ_E cannot be congruent to a weight 2 boundary symbol on $\Gamma_0(174)$ modulo 7. This implies that mod p multiplicity one fails in this case.*

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