

Anisotropic elliptic equations involving unbounded coefficients and singular nonlinearities

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ABSTRACT

In this paper, we study the existence and regularity of solutions for a class of nonlinear singular elliptic equations involving unbounded coefficients and a singular right-hand side. Specifically, we are interested to problem whose simplest model is

$$-\sum_{j=1}^N \partial_j \left([1 + u^q] |\partial_j u|^{p_j-2} \partial_j u \right) = \frac{f}{u^\gamma} \text{ in } \mathcal{D}, \quad u > 0 \text{ in } \mathcal{D}, \quad u = 0 \text{ on } \partial\mathcal{D},$$

where $\mathcal{D}$ is a bounded open subset of $\mathbb{R}^N$ with $N > 2$, $\gamma \geq 0$, $q > 0$, $p_j > 2$ for all $j = 1, \dots, N$ and the source term f belongs to $L^1(\mathcal{D})$, with $f \geq 0$ and $f \not\equiv 0$.

KEYWORDS

Anisotropic elliptic equations, Unbounded coefficients, Existence and regularity, Singular term, L^1 data.

1. Introduction

In the present work we investigate the boundary value problem, given by

$$\begin{cases} -\sum_{j \in \mathcal{E}} \partial_j \left([b(x) + u^q] |\partial_j u|^{p_j-2} \partial_j u \right) = \frac{f}{u^\gamma} & \text{in } \mathcal{D}, \\ u > 0 & \text{in } \mathcal{D}, \\ u = 0 & \text{on } \partial\mathcal{D}, \end{cases} \quad (1)$$

where $\mathcal{D}$ is a bounded open subset of $\mathbb{R}^N$ ($N \geq 3$), $\mathcal{E} = \{j \in \mathbb{N} : 1 \leq j \leq N\}$, $q, \gamma > 0$, and p_j satisfies

$$p_N \geq p_{N-1} \geq \dots \geq p_2 \geq p_1 \geq 2 \quad \text{and} \quad N > \bar{p} \geq 2, \quad (2)$$

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where $\bar{p}$ is the harmonic mean of p_i , defined as

$$\bar{p} = \left(\frac{1}{N} \sum_{j \in \mathcal{E}} \frac{1}{p_j} \right)^{-1}.$$

$a : \mathcal{D} \rightarrow \mathbb{R}$ is a measurable function such that

$$\beta \geq b(x) \geq \alpha \quad \text{a.e. in } \mathcal{D}, \quad (3)$$

with $\alpha, \beta > 0$, and

$$f \in L^1(\mathcal{D}), \quad \text{with} \quad f \geq 0, \quad \text{and} \quad f \not\equiv 0. \quad (4)$$

There is by now a large number of papers and an increasing interest about anisotropic problems. With no hope of being complete, let us mention some pioneering works on anisotropic Sobolev spaces [5, 14, 19, 20] and some more recent existence and regularity results for anisotropic boundary value problems [1, 15, 24]. Interest in anisotropic problems has significantly increased in recent years due to their wide range of applications in mathematical modelling of natural phenomena, particularly in biology and fluid mechanics. For instance, anisotropy plays a crucial role in the mathematical modelling of fluid dynamics in anisotropic media, where the conductivities of the material differ in different directions (see, e.g., [6]). Additionally, anisotropic models are used in biology to describe the spread of epidemic diseases in heterogeneous environments (see, e.g., [7]). Numerous results have been achieved in the study of anisotropic problems across various fields. For further details, the reader is referred to [2, 3, 5, 21], as well as the references cited therein.

From the mathematical point of view, the anisotropic quasilinear elliptic equations of type (1) naturally arise within variational frameworks, for instance when considering integral functionals like

$$\mathcal{J}(v) := \sum_{j \in \mathcal{E}} \frac{1}{p_j} \int_{\mathcal{D}} [b(x) + |v|^q] |\partial_j v|^{p_j} dx - \int_{\mathcal{D}} F(x, v) dx,$$

where $F(x, v)$ suitably captures singular nonlinearities. Problems of this kind, whether or not directly linked to a variational formulation, have been extensively studied in recent literature under various hypotheses on data and coefficients (see, for example, [8, 9]).

We have to mention that the investigation of an anisotropic elliptic problem involving a singular nonlinearity was initiated in [16]; specifically, the authors focus on proving the existence and regularity of solutions to the following boundary value problem

$$-\sum_{j=1}^N \partial_j (|\partial_j u|^{p_j-2} \partial_j u) = \frac{f}{u^\gamma} \quad \text{in } \mathcal{D}, \quad u > 0 \quad \text{in } \mathcal{D}, \quad u = 0 \quad \text{on } \partial\mathcal{D}, \quad (5)$$

The singular nonlinearity arises from the term $u^{-\gamma}$, where the exponent γ can vary. The paper investigates several cases of γ , including $\gamma = 1$, $\gamma < 1$, and $\gamma > 1$. They showed that:

- (**A** $_\gamma$) If $\gamma < 1$, then $u \in W_0^{1, (s_j)}(\mathcal{D})$ with $s_j < \frac{N[\bar{p} - 1 + \gamma]}{[N - 1 + \gamma]\bar{p}} p_j$, $\forall j \in \mathcal{E}$.
 (**B** $_\gamma$) If $\gamma > 1$, then $u \in W_0^{1, (p_j)}(\mathcal{D}) \cap L^{r(\gamma)}(\mathcal{D})$, where $r(\gamma) = \frac{N(\gamma - 1 + \bar{p})}{N - \bar{p}}$.

($\mathbf{C}_\gamma$) If $\gamma = 1$, then $u \in W_0^{1,(p_j)}(\mathcal{D})$.

These results provide a foundation for understanding the regularity of solutions to anisotropic elliptic problems, particularly when singular non-linearities are present. In this paper, we extend and improve upon these findings by deeply explore the interplay between an additional term of the form

$$-\sum_{j \in \mathcal{E}} \partial_j (u^q |\partial_j u|^{p_j-2} \partial_j u)$$

and the singular nonlinearity $u^{-\gamma}$ in presence of data with really poor summability, namely $f \in L^1(\mathcal{D})$. In particular we deal with the regularizing effect, in terms of Sobolev regularity, provided by the lower order terms to the solutions of problems as (5).

These kinds of regularizing effects given by the gradient terms with natural growth in isotropic elliptic problems are nowadays quite classical see for instance [1,8,9].

Namely, by *distributional solution* for problem (1) a function $u \in W_0^{1,1}(\mathcal{D})$ that satisfies

$$(b(x) + u^q) |\partial_j u|^{p_j-1} \in L^1(\mathcal{D}), \quad \forall j \in \mathcal{E}, \quad (6)$$

and there exists $\tilde{C} > 0$ such that

$$u \geq \tilde{C} \text{ in } \mathcal{D}, \quad (7)$$

Moreover, for all test functions $\varphi \in C_c^1(\mathcal{D})$, the weak formulation holds

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} [b(x) + u^q] |\partial_j u|^{p_j-2} \partial_j u \partial_j \varphi \, dx = \int_{\mathcal{D}} \frac{f}{u^\gamma} \varphi \, dx. \quad (8)$$

Our main result is the following.

Theorem 1.1. *Let $f \in L^1(\mathcal{D})$ be nonnegative, and assume conditions (2)–(3) hold. Then, problem (1) admits a nonnegative weak solution u , with the following regularity*

($\mathbf{A}_{\gamma,q}$) If $0 < \gamma < 1$:

- (i) When $q > 1 - \gamma$, the solution belongs to $W_0^{1,(p_j)}(\mathcal{D})$.
- (ii) When $q \leq 1 - \gamma$, the solution belongs to $W_0^{1,(\eta_j)}(\mathcal{D})$, where

$$\eta_j = \frac{N(q + \gamma + \bar{p} - 1)}{[N - (1 - q - \gamma)]\bar{p}} p_j, \quad \forall j \in \mathcal{E}.$$

($\mathbf{B}_{\gamma,q}$) If $\gamma > 1$ and $q \geq 0$, then $u \in W_0^{1,(p_j)}(\mathcal{D}) \cap L^{r(\gamma,q)}(\mathcal{D})$, where

$$r(\gamma, q) = \frac{N(q + \gamma - 1 + \bar{p})}{N - \bar{p}}.$$

($\mathbf{C}_{\gamma,q}$) If $\gamma = 1$ and $q \geq 0$, then $u \in W_0^{1,(p_j)}(\mathcal{D}) \cap L^{(1,q)}(\mathcal{D})$.

Remark 1. We observe that

$$q > 0 \Leftrightarrow r(\gamma, q) > r(\gamma),$$

and

$$\bar{p} < N \Leftrightarrow W_0^{1,\eta_j}(\mathcal{D}) \subseteq W_0^{1,s_j}(\mathcal{D}).$$

Remark 2. Note that

$$\gamma = 0 \Rightarrow (\mathbf{A}_{\gamma,q}) \equiv \text{Theorem 3.7 in [10] (with } \lambda = p). \quad (9)$$

So, our result links up continuously with the result in the isotropic case without the singular term in the right hand side.

Remark 3. In this paper, we restrict our investigation of the problem (1) to the case $f \in L^1(\mathcal{D})$. However, inspired by the results in the isotropic framework presented in [1], it is possible to extend the regularization effect established here to the case with purely summable data, namely to functions $f \in L^m(\mathcal{D})$ with $1 < m < (\bar{p}^*)'$.

Notation. Hereafter we use the following standard notations: for $k > 0$ and all $s \in \mathbb{R}$, we define the truncation functions $T_k(s)$ and $G_k(s)$ as follows

$$T_k(s) = \min\{k, \max\{-k, s\}\}, \quad \text{and} \quad G_k(s) = s - T_k(s).$$

The paper is organized as follows. In Section 2 we provide some fundamental information for the theory of anisotropic Sobolev spaces since it is our work space. Section 3 is devoted to prove our main Theorem.

2. Anisotropic Sobolev spaces

The anisotropic Sobolev spaces $W^{1,(p_j)}(\mathcal{D})$ and $W_0^{1,(p_j)}(\mathcal{D})$ provide the functional framework for Problem (1); see [19,20,22] for details. Let $\mathcal{D}$ be an open bounded subset of $\mathbb{R}^N$, where $N \geq 2$, and let p_j satisfy the conditions (2). The anisotropic Sobolev spaces $W^{1,(p_j)}(\mathcal{D})$ and $W_0^{1,(p_j)}(\mathcal{D})$ are defined as follows

$$\begin{aligned} W^{1,(p_j)}(\mathcal{D}) &= \left\{ z \in W^{1,1}(\mathcal{D}) : \int_{\mathcal{D}} |\partial_j z|^{p_j} dx < \infty, \quad j \in \mathcal{E} \right\}, \\ W_0^{1,(p_j)}(\mathcal{D}) &= \left\{ z \in W_0^{1,1}(\mathcal{D}) : \int_{\mathcal{D}} |\partial_j z|^{p_j} dx < \infty, \quad j \in \mathcal{E} \right\}. \end{aligned}$$

Alternatively, $W_0^{1,(p_j)}(\mathcal{D})$ is the closure of $C_0^\infty(\mathcal{D})$ with respect to the norm

$$\|z\|_{W_0^{1,(p_j)}(\mathcal{D})} = \sum_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} |\partial_j z|^{p_j} dx \right)^{\frac{1}{p_j}},$$

with this norm, $W_0^{1,(p_j)}(\mathcal{D})$ is a separable and reflexive Banach space, and its dual is $\left(W_0^{1,(p_j)}(\mathcal{D})\right)^*$, where p'_j is the conjugate of p_j , i.e., $\frac{1}{p_j} + \frac{1}{p'_j} = 1$ for all $j \in \mathcal{E}$.

Lemma 2.1. [22] For any $z \in W_0^{1,(p_j)}(\mathcal{D})$ with $\bar{p} < N$, there exists a constant $C > 0$ depending only on $\mathcal{D}$ such that

$$\left(\int_{\mathcal{D}} |z|^\delta dx \right)^{\frac{1}{\delta}} \leq C \prod_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} |\partial_j z|^{p_j} dx \right)^{\frac{1}{N p_j}}, \quad \forall \delta \in [1, \bar{p}^*], \quad \bar{p}^* = \frac{N \bar{p}}{N - \bar{p}}, \quad (10)$$

$$\left(\int_{\mathcal{D}} |z|^{\bar{p}^*} dx \right)^{\frac{p_j}{\bar{p}^*}} \leq C \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} |\partial_j z|^{p_j} dx. \quad (11)$$

Lemma 2.2. [22] For any $z \in W_0^{1,(\delta_j)}(\mathcal{D}) \cap L^\infty(\mathcal{D})$ with $\bar{\delta} < N$, there exists a constant $C > 0$ depending only on $\mathcal{D}$ such that

$$\left(\int_{\mathcal{D}} |z|^r dx \right)^{\frac{N}{\bar{\delta}} - 1} \leq C \prod_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} |\partial_j z|^{\delta_j} |v|^{t_j \delta_j} dx \right)^{\frac{1}{\delta_j}}, \quad (12)$$

for any r and $t_j \geq 0$ satisfying

$$\frac{1}{r} = \frac{b_j(N-1) - 1 + \frac{1}{\delta_j}}{t_j + 1} \quad \text{and} \quad \sum_{j \in \mathcal{E}} b_j = 1.$$

To prove that the solution z in $\mathcal{D}$ is positive, we need the theorem, which we will discuss later. As a preliminary to this theorem, we take into account the following problem

$$\begin{cases} -\sum_{j \in \mathcal{E}} \partial_j [|\partial_j z|^{p_j-2} \partial_j z] = \lambda |z|^{q-2} z & \text{in } \mathcal{D}, \\ z = 0 & \text{on } \partial \mathcal{D}, \end{cases} \quad (13)$$

here $\lambda > 0$ and $p_1 < q < p_N$.

We define weak supersolutions for the given problem (13); for further details, see [12].

Definition 2.3. A function $z \in W_0^{1,(p_j)}(\mathcal{D})$ that is nonnegative a.e. in $\mathcal{D}$ is a positive weak supersolution of (13) if

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} |\partial_j z|^{p_j-2} \partial_j z \partial_j \phi dx \geq 0, \quad \forall \phi \in C_0^\infty(\mathcal{D}), \phi \geq 0. \quad (14)$$

The following weak Harnack inequality for weak super-solutions is the key result.

Theorem 2.4. [12, 23] Suppose that z is a weak non-negative supersolution of (13) with $z < M$ in $\mathcal{D}$, and that $p_1 \geq 2$. Then

$$\rho^{-\frac{N}{\beta}} \|z\|_{L^\beta(K(2\rho))} \leq C \min_{K(\rho)} z, \quad (15)$$

for $\beta < \frac{N(p_1-1)}{N-p_1}$, if $p_1 \leq N$, for any β , if $p_1 > N$.

The following is obvious from the previous Theorem.

Corollary 2.5. [12] Under the assumption that $p_1 \geq 2$, any weak non-negative solution of (13) must either be strictly positive in $\mathcal{D}$ or identically zero.

3. Proof of the Main result

3.1. Approximation of problem (1)

For $n \in \mathbb{N}$, we approximate problem (1) by considering the following regularized (non-singular) problem

$$\begin{cases} -\sum_{j \in \mathcal{E}} \partial_j ([b(x) + u_n^q] |\partial_j u_n|^{p_j-2} \partial_j u_n) = \frac{f_n}{(u_n + \frac{1}{n})^\gamma} & \text{in } \mathcal{D}, \\ u_n > 0 & \text{in } \mathcal{D}, \\ u_n = 0 & \text{on } \partial\mathcal{D}, \end{cases} \quad (16)$$

where $f_n = T_n(f) \in L^\infty(\mathcal{D})$ is a sequence of bounded functions that converges to $f > 0$ in $L^1(\mathcal{D})$.

Lemma 3.1. *Under assumptions (3) and (4), problem (16) admits a non-negative solution $u_n \in W_0^{1,(p_j)}(\mathcal{D}) \cap L^\infty(\mathcal{D})$ satisfying*

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} [b(x) + u_n^q] |\partial_j u_n|^{p_j-2} \partial_j u_n \partial_j \varphi \, dx = \int_{\mathcal{D}} \frac{f_n \varphi \, dx}{(u_n + \frac{1}{n})^\gamma}, \quad (17)$$

for all $\varphi \in W_0^{1,(p_j)}(\mathcal{D}) \cap L^\infty(\mathcal{D})$.

Proof. For fixed $n \geq 1$ and $k \geq 0$, let $v \in L^{\bar{p}}(\mathcal{D})$ and define $w = \mathcal{P}(v)$ as the unique solution of

$$\begin{cases} -\sum_{j \in \mathcal{E}} \partial_j ([b(x) + T_k(v)^q] |\partial_j w|^{p_j-2} \partial_j w) = \frac{f_n}{(|v| + \frac{1}{n})^\gamma} & \text{in } \mathcal{D}, \\ w = 0 & \text{on } \partial\mathcal{D}, \end{cases} \quad (18)$$

Since problem (18) has a unique solution [13], the operator $\mathcal{P}$ is well-defined. We aim to establish the existence of a fixed point for $\mathcal{P}$ via Schauder's fixed point theorem [17].

Picking $\varphi = w$ as a test function in (18), and making use of (3) along with the fact that

$$\left| \frac{f_n}{(|v| + \frac{1}{n})^\gamma} \right| \leq n^{\gamma+1}, \quad |T_k(v)|^q \geq 0,$$

we have

$$\alpha \sum_{j \in \mathcal{E}} \|\partial_j w\|_{L^{p_j}(\mathcal{D})}^{p_j} \leq n^{\gamma+1} \int_{\mathcal{D}} |w| \, dx.$$

Using (11) on the left and Hölder's inequality with exponent $\bar{p}^*$ on the right, we obtain

$$\|w\|_{L^{\bar{p}^*}(\mathcal{D})}^{p_N} \leq \sum_{j \in \mathcal{E}} \|\partial_j w\|_{L^{p_j}(\mathcal{D})}^{p_j} \leq C(n, \gamma, \alpha) |\mathcal{D}|^{\frac{1}{(\bar{p}^*)^\gamma}} \|w\|_{L^{\bar{p}^*}(\mathcal{D})}. \quad (19)$$

Since $p_N > 1$, there exists a positive constant $R(n, |\mathcal{D}|, \gamma, \alpha)$ that is independent of v and w , such that

$$\|w\|_{L^{\bar{p}^*}(\mathcal{D})} \leq R(n, |\mathcal{D}|, \gamma, \alpha). \quad (20)$$

Observing that $\bar{p} < \bar{p}^*$, then

$$\|w\|_{L^{\bar{p}}(\mathcal{D})} \leq R(n, |\mathcal{D}|, \gamma, \alpha). \quad (21)$$

Therefore, equation (21) implies that the ball B in $L^{\bar{p}}(\mathcal{D})$, with radius $R(n, |\mathcal{D}|, \gamma, \alpha)$, remains invariant under the mapping $\mathcal{P}$.

Observation 1: $\mathcal{P}(L^p(\mathcal{D}))$ has relative compactness in $L^{\bar{p}}(\mathcal{D})$.

Using equations (19) and (20), we can deduce that

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} |\partial_j w|^{p_j} dx = \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} |\partial_j \mathcal{P}(v)|^{p_j} dx \leq R(n, |\mathcal{D}|, \gamma, \alpha), \quad \forall v \in L^{\bar{p}}(\mathcal{D}).$$

By Sobolev embedding, $\mathcal{P}(L^{\bar{p}}(\mathcal{D}))$ can be shown to be compact in $L^{\bar{p}}(\mathcal{D})$.

Observation 2: $\mathcal{P}$ is a continuous operator.

Let v_r , for any $r \in \mathbb{N}$, be a sequence converging to v in $L^{\bar{p}}(\mathcal{D})$. By the dominated convergence theorem, it follows that

$$\frac{f_n}{(|v_r| + \frac{1}{n})^\gamma} \rightarrow \frac{f_n}{(|v| + \frac{1}{n})^\gamma} \text{ strongly in } L^p(\mathcal{D}).$$

By uniqueness, $w_r := \mathcal{P}(v_r)$ converges to $w := \mathcal{P}(v)$ in $L^{\bar{p}}(\mathcal{D})$. Applying Schauder's fixed point theorem, for each fixed n , there exists $u_{n,k} \in W_0^{1,p}(\mathcal{D})$ such that $\mathcal{P}(u_{n,k}) = u_{n,k}$, with $u_{n,k} \in L^\infty(\mathcal{D})$ for all $n, k \in \mathbb{N}$. Indeed, for fixed $h \geq 1$, using $G_h(u_{n,k})$ as a test function and noting that $u_{n,k} + \frac{1}{n} \geq h \geq 1$ on $\{u_{n,k} \geq h\}$, we obtain

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} |\partial_j G_h(u_{n,k})|^{p_j} dx \leq \int_{\mathcal{D}} f_n G_h(u_{n,k}) dx.$$

We apply the standard technique from [11] to obtain $u_{n,k} \in L^\infty(\mathcal{D})$. Moreover, the L^∞ estimate is independent of k , so for sufficiently large k and fixed n , $u_n \in W_0^{1,(p_j)}(\mathcal{D}) \cap L^\infty(\mathcal{D})$ solves (16). Choosing $\varphi = u_n^- = \min\{u_n, 0\}$ in (16) and applying (3), we observe that $\frac{f_n}{(u_n + \frac{1}{n})^\gamma}$ is nonnegative. Thus,

$$\alpha \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} |\partial_j u_n^-|^{p_j} dx \leq \int_{\mathcal{D}} \frac{f_n}{(u_n + \frac{1}{n})^\gamma} u_n^- dx \leq 0.$$

This implies $u_n^- = 0$ a.e. in $\mathcal{D}$, leading to $u_n \geq 0$ a.e. in $\mathcal{D}$. □

We show in the following Lemma that u_n remains strictly positive in $\mathcal{D}$.

Lemma 3.2. *Let u_n solve (16). Then, there exists a constant $\tilde{C} > 0$, independent of n , such that $u_n \geq \tilde{C}$ a.e. in $\mathcal{D}$ for all $n \in \mathbb{N}$.*

Proof. For $s \geq 0$, consider the function $\Theta_\sigma(s)$ defined as follows

$$\Theta_\sigma(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1, \\ \frac{1}{\sigma}(1 + \sigma - s) & \text{if } 1 < s \leq \sigma + 1, \\ 0 & \text{if } s > \sigma + 1, \end{cases}$$

Using $\Theta_\sigma(s)\phi$ as a test function in (16), where $\phi \in W_0^{1,(p_j)}(\mathcal{D}) \cap L^\infty(\mathcal{D})$ with $\phi \geq 0$, we obtain

$$\begin{aligned} & \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + u_n^q) |\partial_j u_n|^{p_j-2} \partial_j u_n \partial_j \phi \Theta_\sigma(u_n) dx \\ &= \frac{1}{\sigma} \int_{\mathcal{D} \cap \{1 \leq u_n \leq \sigma+1\}} (b(x) + u_n^q) |\partial_j u_n|^{p_j} \phi dx + \int_{\mathcal{D}} \frac{f_n}{(u_n + \frac{1}{n})^\gamma} \Theta_\sigma(u_n) \phi dx. \end{aligned}$$

Omitting the non-negative term on the right-hand side and letting $\sigma \rightarrow 0$, we derive

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + u_n^q) |\partial_j u_n|^{p_j-2} \partial_j u_n \partial_j \phi \chi_{\{0 \leq u_n \leq 1\}} dx \geq \int_{\mathcal{D}} \frac{f_n dx}{(u_n + \frac{1}{n})^\gamma} \phi \chi_{\{0 \leq u_n \leq 1\}}.$$

Using the fact that $\frac{f_n}{(u_n + \frac{1}{n})^\gamma} \geq \frac{T_1(f)}{2^\gamma}$ on the set $\{0 \leq u_n \leq 1\}$, we have

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + T_1(u_n)^q) |\partial_j T_1(u_n)|^{p_j-2} \partial_j T_1(u_n) \partial_j \phi dx \geq \frac{1}{2^\gamma} \int_{\mathcal{D}} T_1(f) \phi \chi_{\{0 \leq u_n \leq 1\}} dx.$$

Since $b(x) + T_1(u_n)^q \leq \beta + 1$, we get

$$(\beta + 1) \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} |\partial_j T_1(u_n)|^{p_j-2} \partial_j T_1(u_n) \partial_j \phi dx \geq 0.$$

This yields that $T_1(u_n)$ is a super-solution solution of the following problem

$$\begin{cases} -\sum_{j \in \mathcal{E}} \partial_j (|\partial_j Z|^{p_j-2} \partial_j Z) = 0 & \text{in } \mathcal{D}, \\ Z = 0 & \text{on } \partial \mathcal{D}, \end{cases}$$

It follows from Theorem 2.4 and Corollary 2.5 that $Z > 0$ in $\mathcal{D}$, and therefore, $T_1(u_n) > 0$ in $\mathcal{D}$. By the strict monotonicity of T_1 , there exists $\tilde{C} > 0$ such that $u_n \geq \tilde{C}$ in $\mathcal{D}$ for all $n \in \mathbb{N}$. $\square$

3.2. A priori estimates

In this subsection, C denotes a positive constant independent of n , which may vary between lines.

Lemma 3.3. *Let u_n be a solution of (16), with $f \in L^1(\mathcal{D})$. Then*

(1) *If $0 < \gamma < 1$:*

- (i) *u_n is bounded in $W_0^{1,(p_j)}(\mathcal{D})$ for $q > 1 - \gamma$.*
- (ii) *u_n is bounded in $W_0^{1,(p_j)}(\mathcal{D})$ for $q = 1 - \gamma$.*
- (iii) *u_n is bounded in $W_0^{1,(\eta_j)}(\mathcal{D})$ for $q < 1 - \gamma$, with*

$$\eta_j = \frac{N(q + \gamma + \bar{p} - 1)}{[N - (1 - (q + \gamma))]\bar{p}} p_j \quad \forall j \in \mathcal{E}.$$

(2) If $\gamma > 1$ and $q \geq 0$, then u_n is bounded in $W_0^{1,(p_j)}(\mathcal{D}) \cap L^{r(\gamma)}(\mathcal{D})$, with

$$r(\gamma) = \frac{N(q + \gamma - 1 + \bar{p})}{N - \bar{p}}.$$

(3) If $\gamma = 1$ and $q \geq 0$, then u_n is bounded in $W_0^{1,p}(\mathcal{D}) \cap L^{r(1)}(\mathcal{D})$.

Proof. Proof of (1)-(i): Taking $u_n^\gamma (1 - (1 + u_n))^{1-(q+\gamma)}$ as a test function in (16), leads to

$$\begin{aligned} & \gamma \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + u_n^q) |\partial_j u_n|^{p_j} u_n^{\gamma-1} (1 - (1 + u_n))^{1-(q+\gamma)} dx \\ & + (q + \gamma - 1) \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + u_n^q) u_n^\gamma \frac{|\partial_j u_n|^{p_j}}{(1 + u_n)^{q+\gamma}} dx \\ & = \int_{\mathcal{D}} \frac{f_n}{(u_n + \frac{1}{n})^\gamma} u_n^\gamma (1 - (1 + u_n))^{1-(q+\gamma)} dx. \end{aligned}$$

By neglecting the first positive term on the left-hand side and using the fact that

$$\frac{f_n}{(u_n + \frac{1}{n})^\gamma} u_n^\gamma (1 - (1 + u_n))^{1-(q+\gamma)} \leq f,$$

we obtain

$$\int_{\mathcal{D}} (b(x) u_n^\gamma + u_n^{q+\gamma}) \frac{|\partial_j u_n|^{p_j}}{(1 + u_n)^{q+\gamma}} dx \leq C \|f\|_{L^1(\mathcal{D})}, \quad \forall j \in \mathcal{E}. \quad (22)$$

By working in $\{u_n \geq 1\}$, we have for all $j \in \mathcal{E}$

$$\int_{\mathcal{D}} (\alpha u_n^\gamma + u_n^{q+\gamma}) \frac{|\partial_j u_n|^{p_j}}{(1 + u_n)^{q+\gamma}} dx \geq \frac{\min(1, \alpha)}{2^{q+\gamma-1}} \int_{\{u_n \geq 1\}} |\partial_j u_n|^{p_j} dx. \quad (23)$$

Then it follows from (22) and (23) that

$$\int_{\{u_n \geq 1\}} |\partial_j u_n|^{p_j} dx \leq C \|f\|_{L^1(\mathcal{D})}, \quad \forall j \in \mathcal{E}. \quad (24)$$

After that, we will be testing (16) by $T_k^\gamma(u_n)$ for all $k > 0$, in order to obtain

$$\gamma \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + u_n^q) \frac{|\partial_j T_k(u_n)|^{p_j}}{[T_k(u_n)]^{1-\gamma}} dx = \int_{\mathcal{D}} \frac{f_n [T_k(u_n)]^\gamma}{(u_n + \frac{1}{n})^\gamma} dx$$

For the left hand side, we dropping the positive term and use (3), as for the right hand side, using that $T_k(u_n) \leq u_n$, we have

$$\int_{\mathcal{D}} \frac{|\partial_j T_k(u_n)|^{p_j}}{[T_k(u_n)]^{1-\gamma}} dx \leq C \|f\|_{L^1(\mathcal{D})}, \quad \forall j \in \mathcal{E}. \quad (25)$$

On the other hand, knowing that $T_k(u_n) \leq k$ and according to (25), we have

$$\begin{aligned} \int_{\mathcal{D}} |\partial_j T_k(u_n)|^{p_j} dx &= \int_{\mathcal{D}} \frac{|\partial_j T_k(u_n)|^{p_j}}{[T_k(u_n)]^{1-\gamma}} [T_k(u_n)]^{1-\gamma} dx \\ &\leq k^{1-\gamma} C \|f\|_{L^1(\mathcal{D})}, \quad \forall j \in \mathcal{E}. \end{aligned} \quad (26)$$

According to (24) and (26), it follows that

$$\int_{\mathcal{D}} |\partial_j u_n|^{p_j} dx \leq C, \quad \forall j \in \mathcal{E}.$$

Meaning, we were able to show the boundedness of u_n in $W_0^{1,(p_j)}(\mathcal{D})$ as an outcome of the previous estimate.

Proof of (1)-(ii): Taking u_n^γ as a test function in (16), we arrive at

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x)u_n^{\gamma-1} + u_n^{q+\gamma-1}) |\partial_j u_n|^{p_j} dx = \int_{\mathcal{D}} \frac{f_n}{(u_n + \frac{1}{n})^\gamma} u_n^\gamma dx.$$

Since $b(x)u_n^{\gamma-1} \geq \alpha u_n^{\gamma-1} \geq 0$ and $\frac{f_n}{(u_n + \frac{1}{n})^\gamma} u_n^\gamma \leq f$, we obtain

$$\int_{\mathcal{D}} \frac{|\partial_j u_n|^{p_j}}{u_n^{1-q-\gamma}} dx \leq \|f\|_{L^1(\mathcal{D})}, \quad \forall j \in \mathcal{E}. \quad (27)$$

If $q = 1 - \gamma$, then from (27) we get u_n is bounded in $W_0^{1,(p_j)}(\mathcal{D})$.

Proof of (1)-(iii): If $q < 1 - \gamma$, we select η_j , such that $1 < \eta_j \leq p_j$ for all $j \in \mathcal{E}$ and employing Hölder's inequality with exponents $\left(\frac{p_j}{\eta_j}, \left(\frac{p_j}{\eta_j}\right)'\right)$ along with (27), we derive

$$\begin{aligned} \int_{\mathcal{D}} |\partial_j u_n|^{\eta_j} dx &= \int_{\mathcal{D}} \frac{|\partial_j u_n|^{\eta_j}}{u_n^{(1-q-\gamma)\frac{\eta_j}{p_j}}} u_n^{(1-q-\gamma)\frac{\eta_j}{p_j}} dx \\ &\leq \left(\int_{\mathcal{D}} \frac{|\partial_j u_n|^{p_j}}{u_n^{1-q-\gamma}} dx \right)^{\frac{\eta_j}{p_j}} \left(\int_{\mathcal{D}} u_n^{(1-q-\gamma)\frac{\eta_j}{p_j-\eta_j}} dx \right)^{\frac{p_j-\eta_j}{p_j}} \\ &\leq C \left(\int_{\mathcal{D}} u_n^{(1-q-\gamma)\frac{\eta_j}{p_j-\eta_j}} dx \right)^{\frac{p_j-\eta_j}{p_j}}. \end{aligned} \quad (28)$$

We set $\eta_j = \varrho p_j$, where $\varrho \in [0, 1)$, yielding

$$(1 - q - \gamma) \frac{\eta_j}{p_j - \eta_j} = (1 - q - \gamma) \frac{\varrho}{1 - \varrho} = \frac{\varrho \bar{p} N}{N - \varrho \bar{p}} = \bar{\eta}^*. \quad (29)$$

This inequality, along with the condition $\bar{p} < N$, indicates that

$$\varrho = \frac{N(\bar{p} + q + \gamma - 1)}{\bar{p}(N + q + \gamma - 1)} < 1. \quad (30)$$

By (28) and (29), we have

$$\int_{\mathcal{D}} |\partial_j u_n|^{\eta_j} dx \leq C \left(\int_{\mathcal{D}} u_n^{\bar{\eta}^*} dx \right)^{1-\varrho} dx, \quad (31)$$

hence,

$$\begin{aligned} \prod_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} |\partial_j u_n|^{\eta_j} dx \right)^{\frac{1}{N\eta_j}} &\leq C \prod_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} u_n^{\bar{\eta}^*} dx \right)^{\frac{1-\varrho}{\eta_j N}} \\ &= \left(\int_{\mathcal{D}} u_n^{\bar{\eta}^*} dx \right)^{\frac{1-\varrho}{\bar{\eta}}}. \end{aligned}$$

Then, by applying (10) from Lemma 2.1 with $w = u_n$ and $\tau = \bar{\eta}^*$, one obtains

$$\|u_n\|_{L^{\bar{\eta}^*}(\mathcal{D})} \leq C \|u_n\|_{L^{\frac{(1-\varrho)\bar{\eta}^*}{\bar{\eta}}}(\mathcal{D})}.$$

Since $1 > \frac{(1-\varrho)\bar{\eta}^*}{\bar{\eta}}$ (since $\bar{p} < N$), we have u_n is bounded in $L^{\bar{\eta}^*}(\mathcal{D})$. Hence, from (30) and (31), we deduce that u_n is bounded in $W_0^{1,(\eta_j)}(\mathcal{D})$.

Proof of (2): We choose u_n^γ as a test function in (16) and apply (3), which gives us

$$\alpha\gamma \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} u_n^{\gamma-1} |\partial_j u_n|^{p_j} dx + \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} u_n^{q+\gamma-1} |\partial_j u_n|^{p_j} dx \leq \|f\|_{L^1(\mathcal{D})}. \quad (32)$$

Dropping the positive term in (32), we get

$$\int_{\mathcal{D}} u_n^{q+\gamma-1} |\partial_j u_n|^{p_j} dx \leq \|f\|_{L^1(\mathcal{D})} \quad \forall j \in \mathcal{E},$$

then

$$\prod_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} u_n^{q+\gamma-1} |\partial_j u_n|^{p_j} dx \right)^{\frac{1}{p_j}} \leq \|f\|_{L^1(\mathcal{D})}^{\frac{N}{\bar{p}}} \quad (33)$$

Using (33) and (12), we obtain

$$\left(\int_{\mathcal{D}} u_n^r dx \right)^{\frac{N}{\bar{p}}-1} \leq C, \quad (34)$$

where

$$\begin{cases} r = \frac{1+t_j}{b_j(N-1)-1+\frac{1}{p_j}}, & t_j \geq 0, \quad \forall j \in \mathcal{E}, \\ t_j p_j = q + \gamma - 1, & \forall j \in \mathcal{E}, \\ \sum_{j \in \mathcal{E}} b_j = 1, & b_j \geq 0, \quad \forall j \in \mathcal{E}. \end{cases}$$

This implies that

$$r = \frac{N(q + \gamma - 1 + \bar{p})}{N - \bar{p}} \quad (35)$$

From (34) and (35), it follows the boundedness of u_n in $L^r(\mathcal{D})$.
Therefore, by (32), we obtain

$$\int_{\mathcal{D}} u_n^{\gamma-1} |\partial_j u_n|^{p_j} dx \leq \frac{\|f\|_{L^1(\mathcal{D})}}{\gamma\alpha}, \quad \forall j \in \mathcal{E}.$$

Using Lemma 3.2, we get

$$\int_{\mathcal{D}} |\partial_j u_n|^{p_j} dx \leq \frac{\|f\|_{L^1(\mathcal{D})}}{\tilde{C}^{\gamma-1} \gamma \alpha}, \quad \forall j \in \mathcal{E},$$

this implies that u_n is bounded in $W_0^{1,(p_j)}(\mathcal{D})$.

Proof of (3): The proof of (3) is similar to the one of (2) with set $\gamma = 1$. □

Lemma 3.4. *Let u_n be a solution to (16). Assume that $q \geq 0$ and $f \in L^1(\mathcal{D})$, then*

- (1) *If $0 < \gamma \leq 1$, then $u_n^q |\partial_j u_n|^{p_j-1}$ is bounded in $L^{\delta_j}(\mathcal{D})$ for every $1 < \delta_j < p'_j$ and for all $j \in \mathcal{E}$.*
- (2) *If $\gamma > 1$ and $q < (\gamma - 1)(p_j - 1)$ for all $j \in \mathcal{E}$, then $u_n^q |\partial_j u_n|^{p_j-1}$ is bounded in $L^{p'_j}(\mathcal{D})$ for all $j \in \mathcal{E}$.*

Proof. *Proof of (1):* Testing (16) with

$$(T_1(u_n))^\gamma \left(1 - \frac{1}{(1 + u_n)^{\lambda-1}} \right), \quad \text{with } \lambda > 1,$$

we get

$$\begin{aligned} & \gamma \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (T_1(u_n))^{\gamma-1} \left(1 - \frac{1}{(1 + u_n)^{\lambda-1}} \right) (b(x) + u_n^q) |\partial_j T_1(u_n)|^{p_j} dx \\ & + (\lambda - 1) \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (T_1(u_n))^\gamma (b(x) + u_n^q) \frac{|\partial_j u_n|^{p_j}}{(1 + u_n)^\lambda} dx \\ & = \int_{\mathcal{D}} \frac{f_n}{(u_n + \frac{1}{n})^\gamma} (T_1(u_n))^\gamma \left(1 - \frac{1}{(1 + u_n)^{\lambda-1}} \right) dx. \end{aligned}$$

Neglecting the positive terms on the left-hand side, use (3) and the fact that

$$\frac{b(x) + u_n^q}{(1 + u_n)^\lambda} \geq \frac{C_\alpha}{(1 + u_n)^{\lambda-q}}, \quad (T_1(u_n))^\gamma \left(1 - \frac{1}{(1 + u_n)^{\lambda-1}} \right) \leq u_n^\gamma,$$

we obtain

$$C_\alpha (\lambda - 1) \int_{\mathcal{D}} T_1(u_n)^\gamma \frac{|\partial_j u_n|^{p_j}}{(1 + u_n)^{\lambda-q}} dx \leq \|f\|_{L^1(\mathcal{D})}, \quad \forall j \in \mathcal{E}.$$

Now working on $\{u_n \geq 1\}$, we have

$$\int_{\{u_n \geq 1\}} \frac{|\partial_j u_n|^{p_j}}{(1+u_n)^{\lambda-q}} dx \leq C, \quad \forall j \in \mathcal{E}.$$

Consequently, using (26) and the above estimate, we reach

$$\int_{\mathcal{D}} \frac{|\partial_j u_n|^{p_j}}{(1+u_n)^{\lambda-q}} dx = \int_{\{u_n \geq 1\}} \frac{|\partial_j u_n|^{p_j}}{(1+u_n)^{\lambda-q}} dx + \int_{\{u_n < 1\}} \frac{|\partial_j T_1(u_n)|^{p_j}}{(1+u_n)^{\lambda-q}} dx \leq C. \quad (36)$$

Next, for $1 < \rho_j < p'_j = \frac{p_j}{p_j-1}$, applying Hölder's inequality and (36), we obtain the following for all $j \in \mathcal{E}$

$$\begin{aligned} \int_{\mathcal{D}} u_n^{q\rho_j(p_j-1)} |\partial_j u_n|^{\rho_j(p_j-1)} dx &\leq \int_{\mathcal{D}} (1+u_n)^{q\rho_j(p_j-1)} |\partial_j u_n|^{\rho_j(p_j-1)} dx \\ &= \int_{\mathcal{D}} \frac{|\partial_j u_n|^{\rho_j(p_j-1)}}{(1+u_n)^{\frac{(\lambda-q)\rho_j(p_j-1)}{p_j}}} (1+u_n)^{\frac{(\lambda-q)\rho_j(p_j-1)}{p_j} + \rho_j q(p_j-1)} dx \\ &= \int_{\mathcal{D}} \frac{|\partial_j u_n|^{\rho_j(p_j-1)}}{(1+u_n)^{\frac{(\lambda-q)\rho_j(p_j-1)}{p_j}}} (1+u_n)^{\frac{\rho_j(p_j-1)(\lambda-q+p_j q)}{p_j}} dx \\ &\leq \left(\int_{\mathcal{D}} \frac{|\partial_j u_n|^{p_j}}{(1+u_n)^{\lambda-q}} dx \right)^{\frac{\delta_j}{p_j}} \left(\int_{\mathcal{D}} (1+u_n)^{\frac{\rho_j(p_j-1)(\lambda-q+p_j q)}{p_j - \rho_j(p_j-1)}} dx \right)^{\frac{p_j - \rho_j(p_j-1)}{p_j}} \\ &\leq C \left(\int_{\mathcal{D}} (1+u_n)^{\frac{\rho_j(p_j-1)(\lambda-q+p_j q)}{p_j - \rho_j(p_j-1)}} dx \right)^{\frac{p_j - \rho_j(p_j-1)}{p_j}}, \end{aligned}$$

which implies for all $j \in \mathcal{E}$

$$\int_{\mathcal{D}} u_n^{q\rho_j(p_j-1)} |\partial_j u_n|^{\rho_j(p_j-1)} dx \leq C \left(\int_{\mathcal{D}} (1+u_n)^{\frac{\rho_j(p_j-1)(\lambda-q+p_j q)}{p_j - \rho_j(p_j-1)}} dx \right)^{\frac{p_j - \rho_j(p_j-1)}{p_j}}. \quad (37)$$

We define $\rho_j = \theta \frac{p_j}{p_j-1}$ with $\theta \in (0, 1)$, which means that for all $j \in \mathcal{E}$

$$\frac{p_j - \rho_j(p_j-1)}{p_j} = 1 - \theta, \quad \text{and} \quad \frac{\rho_j(p_j-1)(\lambda-q+p_j q)}{p_j - \rho_j(p_j-1)} = \frac{\theta}{1-\theta} [\lambda + q(p_j-1)]. \quad (38)$$

From (37) and (38), we obtain for all $j \in \mathcal{E}$

$$\int_{\mathcal{D}} u_n^{q\rho_j(p_j-1)} |\partial_j u_n|^{\rho_j(p_j-1)} dx \leq C \left(\int_{\mathcal{D}} (1+u_n)^{\frac{\theta}{1-\theta} [\lambda + q(p_j-1)]} dx \right)^{1-\theta}. \quad (39)$$

Hence

$$\prod_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} u_n^{q\rho_j(p_j-1)} |\partial_j u_n|^{\rho_j(p_j-1)} dx \right)^{\frac{1}{\theta p_j}} \leq C \prod_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} (1+u_n)^{\frac{\theta}{1-\theta} [\lambda + q(p_j-1)]} dx \right)^{\frac{1-\theta}{\theta p_j}}.$$

Using the inequality anisotropic (12) with $\delta_j = \rho_j(p_j-1) \leq p_j$ (since $\theta < 1$), $t_j = q \geq 0$, $\bar{\delta} = \theta \bar{p}$,

we have

$$\left(\int_{\mathcal{D}} u_n^r dx \right)^{\frac{N}{\theta \bar{p}} - 1} \leq C \prod_{j \in \mathcal{E}} \left(\int_{\mathcal{D}} (1 + u_n)^{\frac{\theta}{1-\theta} [\lambda + q(p_j - 1)]} dx \right)^{\frac{1-\theta}{\theta p_j}}. \quad (40)$$

Since we require that $r = \frac{\theta}{1-\theta} [\lambda + q(p_j - 1)]$ in (40) for all $j \in \mathcal{E}$, we must solve the following system

$$\begin{cases} r = \frac{1+q}{b_j(N-1) - 1 + \frac{1}{\theta p_j}}, & \forall j \in \mathcal{E}, \end{cases} \quad (41)$$

$$\begin{cases} r = \frac{\theta}{1-\theta} [\lambda + q(p_j - 1)], & \forall j \in \mathcal{E}, \end{cases} \quad (42)$$

$$\begin{cases} \sum_{j \in \mathcal{E}} b_j = 1, b_j \geq 0, & \forall j \in \mathcal{E}. \end{cases} \quad (43)$$

From (41) and (43), we obtain

$$r = \frac{N(1+q)\theta \bar{p}}{N - \theta \bar{p}}, \quad (44)$$

and by (42), we get

$$\theta = \frac{N(\bar{p} - \lambda + q)}{\bar{p}[N(1+q) - q\bar{p} - \lambda + q]}. \quad (45)$$

Since $q \geq 0$ and $\lambda > 1$ we obtain $0 < \theta < 1$. Combining (44) and (45), we get

$$r = \frac{N(\bar{p} - \lambda + q)}{N - \bar{p}}. \quad (46)$$

By (40) and (46), we find

$$\left(\int_{\mathcal{D}} u_n^r dx \right)^{\frac{N}{\theta \bar{p}} - 1} \leq C \left(\int_{\mathcal{D}} u_n^r dx \right)^{\frac{(1-\theta)N}{\theta \bar{p}}}, \quad (47)$$

and the assumption $\bar{p} < N$ implies that $\frac{N}{\theta \bar{p}} - 1 > \frac{(1-\theta)N}{\theta \bar{p}}$. Thus, from (47), we obtain the boundedness of u_n in $L^r(\mathcal{D})$, which, according to (39), implies that

$$\int_{\mathcal{D}} u_n^{q\rho_j(p_j-1)} |\partial_j u_n|^{\rho_j(p_j-1)} dx \leq C, \quad \forall j \in \mathcal{E}. \quad (48)$$

As $p_j > 2$ and $\rho_j > 1$ for all $j \in \mathcal{E}$, according to (48), and using Hölder's inequality, we derive

$$\begin{aligned}
\int_{\mathcal{D}} u_n^{q\rho_j} |\partial_j u_n|^{\rho_j(p_j-1)} dx &= \int_{\{u_n < 1\}} u_n^{q\rho_j} |\partial_j u_n|^{\rho_j(p_j-1)} dx + \int_{\{u_n \geq 1\}} u_n^{q\rho_j} |\partial_j u_n|^{\rho_j(p_j-1)} dx \\
&\leq \int_{\mathcal{D}} |\partial_j T_1(u_n)|^{\rho_j(p_j-1)} dx + \int_{\mathcal{D}} u_n^{q\rho_j(p_j-1)} |\partial_j u_n|^{\rho_j(p_j-1)} dx \\
&\leq C \left(\int_{\mathcal{D}} |\partial_j T_1(u_n)|^{p_j} dx \right)^{\frac{\rho_j(p_j-1)}{p_j}} + C \\
&\leq C.
\end{aligned}$$

Then $u_n^q |\partial_j u_n|^{p_j-1}$ is bounded in $L^{\rho_j}(\mathcal{D})$ for every $1 < \rho_j < p'_j$ and for all $j \in \mathcal{E}$.

Proof of (1): We consider u_n^γ as a test function in (16), applying (3), $\frac{u_n^\gamma}{(u_n + \frac{1}{n})^\gamma} \leq 1$, $f_n \leq f$, and ignoring the positive term, we get

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} u_n^{q+\gamma-1} |\partial_j u_n|^{p_j} dx \leq \|f\|_{L^1(\mathcal{D})}.$$

Therefore, we have

$$\int_{\{u_n \geq 1\}} u_n^{q+\gamma-1} |\partial_j u_n|^{p_j} dx \leq C, \quad \forall j \in \mathcal{E}. \quad (49)$$

As $q \leq (p_j - 1)(\gamma - 1)$ for all $j \in \mathcal{E}$, it follows that

$$qp'_j \leq q + \gamma - 1. \quad (50)$$

Using (49) and (50), we deduce for all $j \in \mathcal{E}$

$$\begin{aligned}
\int_{\mathcal{D}} |u_n^q |\partial_j u_n|^{p_j-2} \partial_j u_n|^{p'_j} dx &= \int_{\mathcal{D}} u_n^{qp'_j} |\partial_j u_n|^{p_j} dx \\
&= \int_{\{u_n > 1\}} u_n^{qp'_j} |\partial_j u_n|^{p_j} dx + \int_{\{u_n \leq 1\}} u_n^{qp'_j} |\partial_j u_n|^{p_j} dx \\
&\leq \int_{\{u_n > 1\}} u_n^{q+\gamma-1} |\partial_j u_n|^{p_j} dx + \int_{\mathcal{D}} |\partial_j T_1(u_n)|^{p_j} dx \\
&\leq C.
\end{aligned}$$

Then the last inequality ensure that $u_n^q |\partial_j u_n|^{p_j-1}$ is bounded in $L^{p'_j}(\mathcal{D})$ for all $j \in \mathcal{E}$. $\square$

3.3. Passing to the limit

Because the proof of the cases $(\mathbf{B}_{\gamma,q})$ and $(\mathbf{C}_{\gamma,q})$ in Theorem 1.1 are similar to that of the case $(\mathbf{A}_{\gamma,q})$, we restrict to that of the case (1). By applying Lemma 3.3, we conclude that u_n is bounded in $W_0^{1,(p_j)}(\mathcal{D})$ (or equivalently in $W_0^{1,(\eta_j)}(\mathcal{D})$). It follows that there exists a function u such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,(p_j)}(\mathcal{D}) \text{ (or in } W_0^{1,(\eta_j)}(\mathcal{D})) \text{ a.e. in } \mathcal{D}. \quad (51)$$

Furthermore, according to [4], it follows that

$$\partial_j u_n \rightharpoonup \partial_j u \quad \text{a.e. in } \mathcal{D}. \quad (52)$$

For the first term, by (52), we observe that $b(x)|\partial_j u_n|^{p_j-2}\partial_j u_n$ converge to $b(x)|\partial_j u|^{p_j-2}\partial_j u$ almost everywhere in $\mathcal{D}$. Furthermore, since u_n is bounded in $W_0^{1,(\eta_j)}(\mathcal{D})$ (or equivalently in $W_0^{1,(p_j)}(\mathcal{D})$), we can apply Hölder's inequality to obtain, for any measurable subset $E \subset \mathcal{D}$, the estimate

$$\begin{aligned} \int_E b(x)|\partial_j u_n|^{p_j-2}\partial_j u_n \, dx &\leq \beta \int_E |\partial_j u_n|^{p_j-1} \, dx \\ &\leq \beta \left(\int_{\mathcal{D}} |\partial_j u_n|^{\eta_j} \, dx \right)^{\frac{p_j-1}{\eta_j}} |E|^{1-\frac{p_j-1}{\eta_j}} \\ &\leq \beta C |E|^{\frac{\eta_j-p_j+1}{\eta_j}}. \end{aligned}$$

This indicates that the sequence $\{b(x)|\partial_j u_n|^{p_j-2}\partial_j u_n\}_n$ is equi-integrable. By applying Vitali's theorem, we infer that for every test function $\varphi \in C_0^1(\mathcal{D})$, the result is

$$\lim_{n \rightarrow +\infty} \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} b(x)|\partial_j u_n|^{p_j-2}\partial_j u_n \partial_j \varphi \, dx = \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} b(x)|\partial_j u|^{p_j-2}\partial_j u \partial_j \varphi \, dx. \quad (53)$$

Alternatively, by Lemma 3.4 and (51), we know that the sequence $\{u_n^q|\partial_j u_n|^{p_j-2}\partial_j u_n\}_n$ converges weakly to $u^q|\partial_j u|^{p_j-2}\partial_j u$ in $L^{\rho_j}(\mathcal{D})$, for every $1 < \rho_j < p'_j$ and for all indices $j \in \mathcal{E}$. By combining this weak convergence with (53), we derive

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + u_n^q) |\partial_j u_n|^{p_j-2}\partial_j u_n \partial_j \varphi \, dx \\ &= \sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + u^q) |\partial_j u|^{p_j-2}\partial_j u \partial_j \varphi \, dx, \quad \forall \varphi \in C_0^1(\mathcal{D}). \end{aligned} \quad (54)$$

Regarding the limit of the term on the right-hand side of (17), by Lemma 3.2, for every $\varphi \in C_c^1(\mathcal{D})$,

$$\left| \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} \right| \leq \frac{\|\varphi\|_{L^\infty(\mathcal{D})} f}{\tilde{C}^\gamma}.$$

Then, from the previous estimate, (51), and by applying Lebesgue's Theorem, we get

$$\lim_{n \rightarrow +\infty} \int_{\mathcal{D}} \frac{f_n}{(u_n + \frac{1}{n})^\gamma} \varphi \, dx = \int_{\mathcal{D}} \frac{f}{u^\gamma} \varphi \, dx, \quad \forall \varphi \in C_c^1(\mathcal{D}). \quad (55)$$

Consider $\varphi \in C_c^1(\mathcal{D})$ as a test function in (17). By the convergence results (54), (55), and taking the limit as $n \rightarrow +\infty$, we obtain

$$\sum_{j \in \mathcal{E}} \int_{\mathcal{D}} (b(x) + u^q) |\partial_j u|^{p_j-2}\partial_j u \partial_j \varphi \, dx = \int_{\mathcal{D}} \frac{f}{u^\gamma} \varphi \, dx, \quad \forall \varphi \in C_c^1(\mathcal{D}).$$

Acknowledgements

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