Caterpillars with n vertices are reconstructible from subgraphs with at most $n/2+1$ vertices

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Abstract

The *m-deck* of an n-vertex graph is the multiset of unlabeled induced subgraphs with m vertices. Caterpillars are trees in which all nonleaf vertices lie on a single path. We prove for $n \geq 48$ that any n-vertex caterpillar is reconstructible (up to isomorphism) from its m-deck when $m > n/2$. The result is sharp, since for $n \geq 6$ there are two n-vertex caterpillars having the same $\lfloor n/2 \rfloor$-deck. Our result proves the special case for caterpillars of a 1990 conjecture by Nýdl about trees.

1 Introduction

The *m-deck* of a graph G is the multiset of its m-vertex induced subgraphs (as isomorphism classes); each member is an *m-card*, but the vertices are unlabeled. We write the m-deck as $\mathcal{D}_m(G)$ or simply as $\mathcal{D}_m$. Generalizing the classical problem of graph reconstruction, we say that an n-vertex graph G is *ℓ-reconstructible* if it is determined by $\mathcal{D}_{n-\ell}(G)$, meaning that no graph not isomorphic to G has the same $(n-\ell)$-deck.

Since every member of $\mathcal{D}_{m-1}$ arises $n-m+1$ times by deleting a vertex from a member of $\mathcal{D}_m$, the m-deck of a graph determines its $(m-1)$-deck (and all decks of smaller subgraphs). The natural problem is then to find for each graph the maximum ℓ such that it is ℓ-reconstructible, or equivalently the minimum m such that it is reconstructible from its m-deck. In this context, Manvel [11, 12] extended the classical Reconstruction Conjecture of Kelly [6] and Ulam [15].

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Conjecture 1.1 (Manvel [11, 12]). *For $\ell \in \mathbb{N}$, there exists a threshold M_ℓ such that every graph with at least M_ℓ vertices is ℓ-reconstructible.*

Manvel named this "Kelly's Conjecture" in honor of the final sentence in Kelly [7], which suggested studying reconstruction from the $(n-2)$-deck. The statement that $\mathcal{D}_m$ determines $\mathcal{D}_j$ when $j < m$, which Kelly noted for $m = n-1$, is often called "Kelly's Lemma".

The classical Reconstruction Conjecture is $M_1 = 3$. Since Manvel's conjecture is open for all ℓ, researchers have approached the question by studying thresholds on the number of vertices for ℓ-reconstructibility of various classes or properties of graphs. The survey by Kostochka and West [10] describes some of these results. Of particular interest is the threshold for trees, to extend the seminal result of Kelly [7] that trees with at least three vertices are 1-reconstructible. Nýdl [13] stated a conjecture for general ℓ.

Conjecture 1.2 ([13]). *For $n \geq 2\ell + 1$, no two n-vertex trees have the same $(n-\ell)$-deck.*

One isolated counterexample to Conjecture 1.2 is known; Groenland et al. [4] found two 13-vertex trees with the same 7-deck. For general ℓ, it is known that $n \geq 2\ell$ does not suffice. Nýdl [13] provided two trees with 2ℓ vertices having the same ℓ-deck; a short proof is given in [10] using the results of [14]. The two trees arise from a path with $2\ell - 1$ vertices by adding one vertex adjacent to the central vertex of the path or to a neighbor of that vertex. We generalize this example in Theorem 1.6.

Many arguments for reconstructibility of the graphs in a particular class have two parts, as articulated by Bondy and Hemminger [1] for the case $\ell = 1$. First, one proves that the class is *ℓ-recognizable*; this means that for the $(n-\ell)$-deck of any n-vertex graph, all graphs or no graphs having that deck belong to the class. Separately, one proves that the family is *weakly ℓ-reconstructible*, meaning that no two graphs in the family have the same deck. Thus Nýdl's conjecture states that trees with at least $2\ell + 1$ vertices are weakly ℓ-reconstructible. For the other step, Kostochka, Nahvi, West, and Zirlin [9] proved the following result.

Theorem 1.3 ([9]). *For $n \geq 2\ell + 1$, except when $(n, \ell) = (5, 2)$, the family of n-vertex acyclic graphs is ℓ-recognizable.*

The case $(n, \ell) = (5, 2)$ is excluded because the tree T_5 obtained from the complete bipartite graph $K_{1,3}$ by subdividing one edge has the same 3-deck as the disjoint union of a 4-cycle with an isolated vertex.

By Kelly's Lemma, the $(n-\ell)$ deck determines the 2-deck and hence the number of edges. Since n-vertex trees are the n-vertex acyclic graphs with $n-1$ edges, Theorem 1.3 implies that proving Conjecture 1.2 (excluding $(n, \ell) \in \{(5, 2), (13, 6)\}$) would show that trees with at least $2\ell + 1$ vertices are ℓ-reconstructible (except when $\ell \in \{2, 6\}$). Due to T_5 not being 2-reconstructible, at least six vertices are needed to guarantee 2-reconstructibility of trees, and Giles [2] proved that this suffices.

A general threshold for ℓ-reconstructibility of n-vertex trees has been proved. Groenland, Johnston, Scott, and Tan [4] proved that n-vertex trees are ℓ-reconstructible when $n \geq 9\ell + 24\sqrt{2\ell} + o(\sqrt{\ell})$. Kostochka, Nahvi, West and Zirlin [8] improved the sufficient condition to $n \geq 6\ell + 11$.

Since finding the least number of vertices to guarantee ℓ-reconstructibility of trees seems to be quite difficult, it is natural to consider ℓ-reconstructibility of trees with special structure. A *caterpillar* is a tree in which the subgraph induced by the nonleaf vertices is a path called the *spine*. Z. Hunter [5] proved that an n-vertex caterpillar is reconstructible from its m-deck when $m > (n/2) + C$ for some constant C. Our result reduces C to 0, proving Nýdl's conjecture for the special case of caterpillars when n is sufficiently large.

Theorem 1.4. *When $n \geq 48$ and $m > n/2$, every n-vertex caterpillar is reconstructible from its m-deck, and this is sharp.*

We will prove the sharpness of Theorem 1.4 at the end of this section. Because the number of vertices in subgraphs is always an integer, this phrasing for m in terms of n is equivalent to the threshold on n in terms of ℓ in Conjecture 1.2. Similarly, the statement of the result in [4] was that an n-vertex tree is reconstructible from its m-deck when $m = 8n/9 + (4/9)\sqrt{8n+5}$.

In light of Kelly's Lemma, we can speak of "the deck" of given information as being the multiset of all induced subgraphs with at most $(n+2)/2$ vertices. We will write it as $\mathcal{D}$. The individual subgraphs are the "cards" in the deck. We aim to retrieve the original caterpillar when given $\mathcal{D}$.

It is easy to see that a tree is a caterpillar if and only if it does not contain the 7-vertex tree obtained from $K_{1,3}$ by subdividing each edge. Hence when $n \geq 12$ we can tell from $\mathcal{D}$ whether the unknown tree is a caterpillar, and it then remains only to determine which caterpillar has the given deck.

It is still possible that when $n \geq 6$ the conclusion of Theorem 1.4 has no exceptions. Giles [2] proved that all trees are 2-reconstructible, handling the case $n = 6$. When $n = 7$ there are 11 isomorphism classes of trees, and they are distinguished by their numbers of 4-vertex paths and independent 4-sets. Only one is not a caterpillar. The threshold $n \geq 48$ in the statement of Theorem 1.4 is an artifact of the proof. Some of our lemmas invoke thresholds on n to ensure that certain subgraphs are visible in cards. The only one that relies on $n \geq 48$ is Lemma 7.8; otherwise $n \geq 27$ would suffice. We will include "$n \geq 48$" in statements as a flag indicating that a threshold on n will be invoked; varying the threshold in various lemmas would lead to unnecessary distraction from the main result.

Within the family of caterpillars, a particular caterpillar is specified by its ordered list of vertex degrees along the spine (the reverse list specifies the same caterpillar). Hence it is natural to seek first the multiset of vertex degrees. Groenland et al. [4] used algebraic arguments about polynomial equations (obtained by counting cards that are stars of various sizes) to prove that the multiset of vertex degrees of an n-vertex graph is determined by its

m-deck when $m \geq \sqrt{2n \ln 2n}$. Although this is the strongest known result for large n for reconstructing vertex degrees, note that $(n+1)/2 \geq \sqrt{2n \ln 2n}$ only when $n \geq 67$, so for n of moderate size this result does not give us the degree list. Meanwhile, Kostochka et al. [9] gave a combinatorial argument implying that the degree list of any n-vertex acyclic graph is determined by its m-deck when $m > n/2$ (again for $n \geq 6$). Hence we may assume that the multiset of vertex degrees is known from the deck.

In particular, given the deck $\mathcal{D}$ of an unknown caterpillar G, we know the number of nonleaf vertices in G; we call this r. Let $d_1, \ldots, d_r$ denote the degrees of the nonleaf vertices in nonincreasing order. Let $v_1, \ldots, v_r$ be the vertices of the spine in order as a path, and let $d(v)$ denote the degree of vertex v. Although $d_1, \ldots, d_r$ and $d(v_1), \ldots, d(v_r)$ form the same multiset, they are rarely in the same order; our task is to use $\mathcal{D}$ to discover the second order.

Before closing this section, we note one fundamental counting tool that will be useful at many points in the paper.

Lemma 1.5. *The vertex degrees of an n-vertex caterpillar with $n-r$ leaves satisfy*

$$n - r - 2 = \sum_{i=1}^{r} (d_i - 2). \tag{1}$$

With m vertices of maximum degree,

$$d_1 \leq \left\lfloor \frac{n-r-2}{m} \right\rfloor + 2. \tag{2}$$

Proof. A longest path has $r+2$ vertices. Counting the $n-r-2$ vertices not on the path according to their neighbors on the spine yields (1). Keeping only the m vertices of maximum degree in (1) yields (2). $\square$

In Section 2 we consider the "low-diameter" case $r \leq (n-6)/2$. In the subsequent sections we consider the more difficult "high-diameter" case, beginning with a general strategy in Section 3. We then consider cases in terms of the number of vertices of maximum degree and the number of vertices of degree at least 3.

As mentioned after Conjecture 1.2, for even n Nýdl [13] presented two n-vertex caterpillars having the same $n/2$-deck. A proof of this appears in [10] using a special case of the results of [14]. Here we present a direct self-contained proof generalizing the sharpness claim of Theorem 1.4.

Let $T_{a,b}$ denote the tree that is the union of paths of lengths 1, a, and b having a common vertex. Note that $T_{a,b}$ is a caterpillar with $a+b+2$ vertices. A special case of the result below is that when $k = \lfloor n/2 \rfloor$, the caterpillars $T_{k-1,n-k-1}$ and $T_{k-2,n-k}$ have the same k-deck, which proves sharpness of Theorem 1.4 for all n at least 6.

Theorem 1.6. *When $b > a \geq k-2$, the caterpillars $T_{a,b}$ and $T_{a+1,b-1}$ have the same k-deck.*

Proof. Let $n = a+b+2$; both caterpillars are n-vertex trees. We use induction on n. When $n = 3$, we have $(b, a, k) = (1, 0, 2)$ and the claim holds because $T_{0,1} \cong T_{1,0}$.

For $n \geq 4$, always the trees are the same when $b = a+1$, so we may assume $b \geq a+2 \geq k$. Let $G = T_{a,b}$ and $H = T_{a+1,b-1}$, and let v denote in both graphs the leaf at the end of the path of length b or $b-1$ from the branch vertex. For $j \leq k$, in both G and H there is exactly one j-vertex path containing v, because $b \geq k$.

Let $\mathcal{D}_{k,j}(G)$ and $\mathcal{D}_{k,j}(H)$ be the subsets of the two k-decks in which the component containing v has j vertices, for $0 \leq j \leq k$ ($j = 0$ corresponds to v being omitted). When $j = k$, both $\mathcal{D}_{k,j}(G)$ and $\mathcal{D}_{k,j}(H)$ consist of a single card that is a k-vertex path. When $j < k$, we have specified j vertices to be used and one vertex to be omitted. When $j = k-1$, one more vertex must be picked from the remaining $n-k$ vertices, so both $\mathcal{D}_{k,j}(G)$ and $\mathcal{D}_{k,j}(H)$ consist of $n-k$ cards, each a $(k-1)$-vertex path plus an isolated vertex.

For $j \leq k-2$, the cards in the two subsets correspond to induced subgraphs with $k-j$ vertices chosen from G' or H', where since $b \geq k \geq j+2$ we can write $G' = T_{a,b-j-1}$ and $H' = T_{a+1,b-j-2}$. Since $b \geq k$, we have $b-j-2 \geq k-j-2$. Hence $\min\{a, b-j-2\} \geq (k-j)-2$, so the induction hypothesis applies to yield $\mathcal{D}_{k-j}(G') = \mathcal{D}_{k-j}(H')$. Adding a j-vertex path to each card yields $\mathcal{D}_{k,j}(G) = \mathcal{D}_{k,j}(H)$, and the union over all j completes the proof. $\square$

2 Low-diameter Caterpillars

Definition 2.1. Given the deck of an unknown caterpillar G, let $\#H$ denote the number of induced subgraphs of G isomorphic to H; we are given $\#H$ whenever $|V(H)| \leq (n+2)/2$. Let P_t denote an t-vertex path, and let $\langle u_1, \ldots, u_t \rangle$ denote a path with vertices $u_1, \ldots, u_t$ in order. Let the spine of our unknown caterpillar G with r nonleaf vertices be $\langle v_1, \ldots, v_r \rangle$. Let $d'(v_i)$ denote the number of leaf neighbors of v_i; that is, $d'(v_i) = d(v_i) - 1$ when $i \in \{1, r\}$, and otherwise $d'(v_i) = d(v_i) - 2$. For $1 \leq i \leq r$, let $\bar{i} = r + 1 - i$; thus v_i and $v_{\bar{i}}$ are the same distance from the nearest end of the spine. The ith *level pair* is the unordered pair $\{d(v_i), d(v_{\bar{i}})\}$ of vertex degrees. The motivation for the term "level pair" is picturing the spine as in Figure 3 in Section 6.

Theorem 2.2. *When $r \leq (n-6)/2$, every n-vertex caterpillar G with r nonleaf vertices is reconstructible from $\mathcal{D}$.*

Proof. Longest paths have $r+2$ vertices and contain the spine. Consider all cards consisting of a longest path plus one more leaf. Since we cannot distinguish the two ends, we index a longest path in such a card as $\langle u_0, \ldots, u_{r+1} \rangle$ so that the neighbor u_i of the extra leaf satisfies $1 \leq i \leq \lceil r/2 \rceil$. Let Y_i denote the resulting graph. Note that $u_i \in \{v_i, v_{\bar{i}}\}$.

We first determine the level pairs $\{d(v_i), d(v_{\bar{i}})\}$. For $i = 1$, we have $\#P_{r+2} = d'(v_1)d'(v_r)$ and $\#Y_1 = \binom{d'(v_1)}{2}d'(v_r) + \binom{d'(v_r)}{2}d'(v_1)$. Substituting $d'(v_r) = \#P_{r+2}/d'(v_1)$ into the second equation yields a quadratic equation, the two roots of which are $d'(v_1)$ and $d'(v_r)$. We thus obtain the level pair $\{d(v_1), d(v_r)\}$.

Let $M = d'(v_1)d'(v_r)$; note that M is the number of longest paths in G. Consider i with $2 < i \le \lfloor r/2 \rfloor$. Recall that $d'(v_i) = d(v_i) - 2$ for $i \notin \{1, r\}$, so $\#Y_i = [d'(v_i) + d'(v_{\bar{i}})]M$. We also seek the product $d'(v_i)d'(v_{\bar{i}})$. Let Z_i be the graph with $r + 4$ vertices obtained from the path $\langle u_0, \ldots, u_{r+1} \rangle$ with $r + 2$ vertices by giving each of u_i and $u_{\bar{i}}$ one leaf neighbor. Note that $\#Z_i = d'(v_i)d'(v_{\bar{i}})M$. Since $r + 4 \le (n + 2)/2$, we now know the sum and product of $d'(v_i)$ and $d'(v_{\bar{i}})$ and find the pair as the zeros of a quadratic polynomial. We thus obtain $\{d(v_i), d(v_{\bar{i}})\}$.

If $d(v_i) = d(v_{\bar{i}})$ for $1 \le i \le \lfloor r/2 \rfloor$, then G is symmetric and the level pairs suffice to determine G. Otherwise, let j be the least index such that $d(v_j) \ne d(v_{\bar{j}})$; note that $j \le r/2$. By symmetry, we may assume $d(v_j) < d(v_{\bar{j}})$. Let $X_{i,k}$ be the graph with $r + 4$ vertices obtained from $u_0, \ldots, u_{r+1}$ by giving each of u_i and u_k one leaf neighbor; thus $Z_i = X_{i,\bar{i}}$. With j as above, we focus on $X_{j,k}$ with $j < k \le r/2$. If $j \ge 2$, then $\#X_{j,k} = [d'(v_j)d'(v_k) + d'(v_{\bar{j}})d'(v_{\bar{k}})]M$, but $\#X_{1,k} = \binom{d'(v_1)}{2}d'(v_k) + \binom{d'(v_r)}{2}d'(v_{\bar{k}})$.

If $d'(v_k)$ and $d'(v_{\bar{k}})$ are equal, then we already know them; suppose they differ. Since $d'(v_j)$ and $d'(v_{\bar{j}})$ are known and unequal, we can determine which of the pair $\{d'(v_k), d'(v_{\bar{k}})\}$ is $d'(v_k)$, because $ac + bd$ differs from $ad + bc$ when $a \ne b$ and $c \ne d$. Choosing $d'(v_k)$ correctly is the only way to obtain $\#X_{j,k}/M$ (or $\#X_{1,k}$ when $j = 1$) as the value. Doing this for each k completes the reconstruction (when r is odd, $d(v_{(r+1)/2})$ is the remaining degree). $\square$

Henceforth we may assume $r \ge (n - 5)/2$.

3 High-diameter Strategy

The low-diameter case is easy because we can see the entire spine on a card. We used longest paths to anchor the cards so that we knew (up to reflecting the spine) the identity of any vertex of degree 3 on the card. In the high-diameter case $r \ge (n - 5)/2$, we will instead use high-degree vertices as anchors.

Remark 3.1. *The plan.* Knowing r and the vertex degrees, we first seek the unordered degree pairs for all pairs of vertices separated by a fixed distance j, aiming to do so when $j \le (r - 1)/2$. From that, we will determine the level pairs $\{d(v_k), d(v_{\bar{k}})\}$ for all k. In some cases, we also need triples of vertex degrees, which we will describe more precisely later. From this information we will determine the ordered list $(d(v_1), \ldots, d(v_r))$ to complete the reconstruction (up to reflection).

In order to obtain degree pairs and degree triples, we will apply a general counting tool.

Definition 3.2. Given a family $\mathcal{F}$ of graphs, an *$\mathcal{F}$-subgraph* of a graph G is an induced subgraph of G belonging to $\mathcal{F}$. Let $s(F,G)$ denote the number of occurrences of F as an induced subgraph in G. Let $\hat{s}(F,G)$ be the number of occurrences of F as a maximal $\mathcal{F}$-subgraph in G (maximality with respect to induced subgraphs).

The special case of the next lemma for classical reconstruction ($\ell = 1$) is due to Greenwell and Hemminger [3]. Similar statements for general ℓ appear for example in [4, 9, 8] and elsewhere. For completeness, we include the proof from [9] rephrased for m-decks; it is slightly simpler than proofs in the literature involving inclusion chains of subgraphs because we do not need an explicit formula for $\hat{s}(F,G)$.

Lemma 3.3. *Fix an n-vertex graph G, and let $\mathcal{F}$ be a family of graphs such that every $\mathcal{F}$-subgraph of G is an induced subgraph of a unique maximal $\mathcal{F}$-subgraph in G. If the value of $\hat{s}(F,G)$ is known for every $F \in \mathcal{F}$ with at least m vertices, then for all $F \in \mathcal{F}$ the m-deck of G determines $\hat{s}(F,G)$.*

Proof. Let $t = |V(G)| - |V(F)|$; we use induction on t. When $t \le n-m$, the value $\hat{s}(F,G)$ is given in the hypothesis. When $t > n-m$, each induced subgraph of G isomorphic to F lies in a unique maximal $\mathcal{F}$-subgraph H of G. Grouping the induced copies of F according to the choices for the isomorphism class of H yields

$$s(F,G) = \sum_{H \in \mathcal{F}} s(F,H)\hat{s}(H,G).$$

Since $|V(F)| < m$, we know $s(F,G)$ from the deck, and we know $s(F,H)$ when F and H are known. By the induction hypothesis, we know all values of the form $\hat{s}(H,G)$ when F is an induced subgraph of H except $\hat{s}(F,G)$. Therefore, we can solve for $\hat{s}(F,G)$. $\square$

Applying Lemma 3.3 to determine all the maximal $\mathcal{F}$-subgraphs from the m-deck requires determining those with at least m vertices. This is often the difficult part. We refer to the use of Lemma 3.3 to iteratively obtain the smaller maximal $\mathcal{F}$-subgraphs (with multiplicity) by discarding those contained in larger maximal $\mathcal{F}$-subgraphs as the **exclusion argument**. We then find $\hat{s}(F,G)$ for each $F \in \mathcal{F}$ in nonincreasing order of $|V(F)|$.

We will apply Lemma 3.3 to several families of subgraphs.

Definition 3.4. A *baton* is a caterpillar obtained from a path by attaching some positive number of leaves at both ends. A *triton* is obtained from a baton by also appending a positive number of leaves at one internal vertex. A *j-baton* is a baton in which the original path has j edges, and $B_{j:a,b}$ is the j-baton with $j+a+b-1$ vertices having vertices of degrees a and b (adjacent to the leaves and called the *key vertices*) separated by distance j. For a baton we require $a, b \geq 2$. A *j, j'-triton* is a triton in which the distances from the internal specified vertex to the other vertices where leaves have been added are j and j'. Let $B_{j,j':a,b,c}$ denote the j, j'-triton having vertices of degrees a, b, and c (called the *key vertices*) in order as shown in Figure 1 ($a, c \geq 2$); it has $j + j' + a + b + c - 3$ vertices.

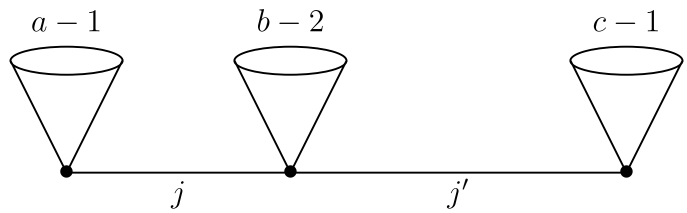

Figure 1: The j, j'-triton $B_{j,j':a,b,c}$

Lemma 3.5. *Fix $j, j' \in \mathbb{N}$, with r being the number of nonleaf vertices in the unknown caterpillar G. If $\mathcal{D}_m$ determines the maximal j-batons having at least m vertices (with multiplicity), then it determines the multiset of $r - j$ unordered degree pairs $\{d(v_i), d(v_{i+j})\}$ for $1 \leq i \leq r - j$, without specifying i in any such pair. Similarly, if $\mathcal{D}_m$ determines the maximal j, j'-tritons having at least m vertices (with multiplicity), then it determines the multiset of degree triples consisting of $(d(v_i), d(v_{i+j}), d(v_{i+j+j'}))$ for $1 \leq i \leq r - j - j'$ and $(d(v_i), d(v_{i-j}), d(v_{i-j-j'}))$ for $j + j' + 1 \leq i \leq r$, without knowing any i or whether such a triple is increasing or decreasing in subscripts.*

Proof. In a caterpillar, every j-baton occurs in a unique maximal j-baton. Thus we can apply Lemma 3.3 when $\mathcal{F}$ is the family of j-batons. The graph induced by $v_i, \ldots, v_{i+j}$ and the other neighbors of v_i and v_{i+j} in G is a maximal j-baton in G, and these are all the maximal j-batons. Given that we know $\hat{s}(F, G)$ for every j-baton $F \in \mathcal{F}$ with at least m vertices, Lemma 3.3 implies that $\mathcal{D}_m$ determines the multiset of maximal j-batons. Each maximal j-baton gives one of the desired degree pairs.

The same argument holds for j, j'-tritons, applying Lemma 3.3 if we know the maximal ones having at least m vertices. Note that when $j \neq j'$ there are $2(r - j - j')$ maximal j, j'-tritons, but when $j = j'$ there are only $r - j - j'$ corresponding subgraphs and we listed them twice, each triple and its reverse. $\square$

The idea of the next two lemmas, which are quite similar, is due to Hunter [5].

Lemma 3.6. *Fix a value q with $q \leq (r-1)/2$. If for each j with $j \leq q$ the multiset of degree pairs $\{d(v_i), d(v_{i'})\}$ with $|i' - i| = j$ is known, then for $1 \leq k \leq q$ the level pair $\{d(v_k), d(v_{\overline{k}})\}$ is known. If $q = \lfloor (r-1)/2 \rfloor$, then the remaining degrees provide the remaining level.*

Proof. We apply the same argument iteratively for k from 1 to q. Suppose that the pairs $\{d(v_i), d(v_{\overline{i}})\}$ are known for $1 \le i < k$; the assumption is vacuous when $k = 1$.

When $k < i < \overline{k}$, the nonleaf vertex v_i is a key vertex in two maximal k-batons, but for i outside that range v_i is a key vertex in only one maximal k-baton. (When $i = k$ and $d(v_1) = 2$, the maximal baton containing $\langle v_0, \ldots, v_i \rangle$ and all neighbors of v_i and v_0 is a $(k-1)$-baton but not a k-baton.)

Consider the multiset of nonleaf vertex degrees, with the degree of each nonleaf vertex written twice. For each maximal k-baton B, delete from this multiset two entries that are the degrees of the key vertices in B. Over the entire process we delete $d(v_i)$ twice if $k < i < \overline{k}$ and once if i is outside that range. From the remaining multiset, delete one copy of each entry in $\{d(v_i), d(v_{\overline{i}})\}$ for $1 \le i < k$, determined earlier. This leaves only $d(v_k)$ and $d(v_{\overline{k}})$, each listed twice and deleted once.

When $q = \lfloor (r-1)/2 \rfloor$ and we have done this for $k \le q$, only one or two values (depending on the parity of r) remain in the multiset. They are $d(v_{(r+1)/2})$ or $\{d(v_{r/2}), d(v_{r/2+1})\}$. $\square$

Lemma 3.7. *Fix values s and q with $s < q \le (r-1)/2$. If for each j with $j \le s$ the multiset of maximal j-batons is known, and for each pair (j, j') with $j + j' \le q$ the multiset of maximal j, j'-tritons is known, then for all k and s with $2 \le k + s \le q$ the following set of two ordered degree pairs is known:*

$$\{(d(v_k), d(v_{k+s})), (d(v_{\overline{k}}), d(v_{\overline{k+s}}))\}.$$

Proof. The argument is similar to that of Lemma 3.6. Fixing s, we obtain the pair for k iteratively, beginning with $k = 1$. Suppose that the pairs $\{(d(v_i), d(v_{i+s})), (d(v_{\overline{i}}), d(v_{\overline{i+s}}))\}$ are known for $1 \le i < k$; the assumption is vacuous when $k = 1$.

For each maximal s-baton B, add to a multiset L two ordered pairs, consisting of the degrees of the key vertices in B written in both orders (the two pairs may be the same). These pairs have the form $(d(v_i), d(v_{i+s}))$ or $(d(v_i), d(v_{i-s}))$. We aim to cancel most of L, leaving only the ordered pairs $(d(v_k), d(v_{k+s}))$ and $(d(v_{\overline{k}}), d(v_{\overline{k+s}}))$. When $k < i < \overline{k} - s$, the maximal s-baton containing $\{v_i, v_{i+s}\}$ occurs in two maximal k, s-tritons, with key vertices (v_{i-k}, v_i, v_{i+s}) and $(v_{i+s+k}, v_{i+s}, v_i)$. For each maximal k, s-triton B, delete from L the two ordered degree pairs at distance s that arise from B. (When $s = k$, we get four ordered pairs from B to delete.)

The deletions leave in L one copy each of the ordered pairs (v_i, v_{i+s}) with $i \le k$ and (v_{i+s}, v_i) with $i + s \ge \overline{k}$. We have previously determined those with $i < k$ and $i + s > \overline{k}$ (not knowing which come from which end). Deleting them leaves us with the two ordered pairs $(d(v_k), d(v_{k+s}))$ and $(d(v_{\overline{k}}), d(v_{\overline{k+s}}))$, not knowing which is which. $\square$

Say that the *length* of $B_{j:a,b}$ is j, and the *length* of $B_{j,j':a,b,c}$ is $j + j'$.

Theorem 3.8. *Fix q with $1 \le q \le (r-1)/2$. If the maximal batons and tritons in G that have length at most q and at least $\lfloor (n+2)/2 \rfloor$ vertices are known, then the lists $(d(v_1), \ldots, d(v_q))$ and $(d(v_{\overline{1}}), \ldots, d(v_{\overline{q}}))$ (with specified subscripts) are known.*

Proof. By Lemmas 3.5, 3.6, and 3.7, we know the degree pairs $\{d(v_k), d(v_{\overline{k}})\}$ for $1 \le k \le q$ and the sets of two ordered pairs $\{(d(v_k), d(v_{k+s})), (d(v_{\overline{k}}), d(v_{\overline{k+s}}))\}$ for $k + s \le q$.

If $d(v_k) = d(v_{\overline{k}})$ for all k at most q, then there is nothing to prove. Otherwise, from the level pairs $\{d(v_k), d(v_{\overline{k}})\}$ we know the least index h such that $d(v_h) \ne d(v_{\overline{h}})$, and it satisfies $h \le q$. We may assume $d(v_h) < d(v_{\overline{h}})$ by symmetry. Since we know the unordered pairs of *ordered* pairs of the form $(d(v_h), d(v_{h+s}))$ and $(d(v_{\overline{h}}), d(v_{\overline{h+s}}))$ for $h+s \le q$ and $d(v_h) \ne d(v_{\overline{h}})$, we also know $d(v_{h+s})$ and $d(v_{\overline{h+s}})$. For the levels below h, the two choices are the same. □

The difficulty in applying Theorem 3.8 is finding the maximal batons and tritons having at least $\lfloor (n+2)/2 \rfloor$ vertices and length up to $(r-1)/2$. If the hypotheses of Theorem 3.8 are known with $q = \lceil r/2 \rceil$, then the caterpillar is known. If we only know this for some smaller q, then we can still determine the list of degrees if we know the middle part of the list.

Lemma 3.9. *Fix an integer q with $q \le (r+1)/2$. Suppose that the maximal batons with length less than q and the lists of degrees $(d(v_1), \ldots, d(v_{q-1}))$ and $(d(v_{\overline{1}}), \ldots, d(v_{\overline{q-1}}))$ (with known subscripts) are determined by the deck. If also the "middle portion" $(d(v_q), \ldots, d(v_{\overline{q}}))$ is known up to reversal, then the unknown caterpillar is determined.*

Proof. Let $W = \{v_q, \ldots, v_{\overline{q}}\}$, and let U be the set of vertices outside W. We may assume $d(v_q) \ne d(v_{\overline{q}})$, since otherwise we can move level q into U and apply the argument below with $q-1$ instead of q. Let $a = \max\{d(v_q), d(v_{\overline{q}})\}$.

If $d(v_i) = d(v_{\overline{i}})$ for all $v_i \in U$, then both choices for ordering W yield the same caterpillar. Hence we may let h be the maximum index less than q such that $d(v_h) \ne d(v_{\overline{h}})$. Let $j = q-h$, and let $c = \max\{d(v_h), d(v_{\overline{h}})\}$. By symmetry, we may label the indices so that $c = d(v_h)$.

Let L be the multiset of unordered degree pairs of the form $\{d(v_i), d(v_{i+j})\}$, known from the maximal batons (here j is fixed). Delete from L all members that arise when both vertices are in W or both in U; these pairs are known. The pairs that remain in L arise from a vertex in W and a vertex in U in some level i with $h \le i < q$.

Let t be the number of copies of $\{a, c\}$ that remain in L (note that a may equal c). If $t = 0$, then $a = d(v_{\overline{q}})$, which completes the reconstruction.

If $t \ge 1$, then we eliminate from L instances of $\{a, c\}$ at distance j that do not arise as $\{d(v_h), d(v_q)\}$. Such an instance involves a vertex with level between h and q. For $h < i < q$, we have $d(v_i) = d(v_{\overline{i}})$, so we get the same number of such pairs no matter which of the two orderings of W is used. Hence we know the number of copies of $\{a, c\}$ to eliminate.

We then check again whether $\{a, c\}$ remains in L. If it does, then $a = d(v_q)$. Otherwise, $a = d(v_{\overline{q}})$. This determines the order on W and completes the reconstruction. □

The first application of Lemma 3.9 is a slight reduction in the length of the maximal tritons we need to know.

Corollary 3.10. *If the maximal batons with length at most $(r-1)/2$ and the tritons with length at most $(r-2)/2$ are determined by the deck, then the unknown caterpillar is reconstructible from the deck.*

Proof. Let $q = \lfloor r/2 \rfloor$. The middle portion of the spine between v_q and $v_{\overline{q}}$ has only two vertices (when r is even) or three vertices (when r is odd). By Lemma 3.6 determining the level pairs, we thus know $(v_q, \ldots, v_{\overline{q}})$ up to reversal.

By Theorem 3.8, to determine the lists $(d(v_1), \ldots, d(v_{q-1}))$ and $(d(v_{\overline{1}}), \ldots, d(v_{\overline{q-1}}))$ we only need the maximal batons and tritons with length at most $q-1$ that have at least $n-\ell$ vertices. Lemma 3.9 will then complete the reconstruction. $\square$

Next we further reduce the length of tritons needed to complete the reconstruction. Let R denote the set $\{v_1, \ldots, v_r\}$ of nonleaf vertices.

Lemma 3.11. *For $n \geq 48$, if the maximal batons with length at most $(r-1)/2$ are known and the lists $(d(v_1), \ldots, d(v_{q-1}))$ and $(d(v_r), \ldots, d(v_{\overline{q-1}}))$ (with known subscripts) are known, where $q \geq (r-3)/2$, then the unknown caterpillar is determined by the deck.*

Proof. We may assume $q = \lfloor (r-2)/2 \rfloor$. Unknown between the lists $(d(v_1), \ldots, d(v_{q-1})$ and $(d(v_{\overline{1}}), \ldots, d(v_{\overline{q-1}}))$ is the segment $(d(v_q), \ldots, d(v_{\overline{q}}))$ in the middle consisting of four vertex degrees (when r is even) or five vertex degrees (when r is odd). Let $W = \{v_q, \ldots, v_{\overline{q}}\}$.

When we know all the short maximal batons, by Lemma 3.6 we know the level pairs. Here each level pair $\{d(v_k), d(v_{\overline{k}})\}$ for $k < q$ has already been allocated to v_k and $v_{\overline{k}}$ (and $d(v_{(r+1)/2}$ is known when r is odd). We must allocate the remaining two level pairs.

Let $A = \{v_q, v_{\overline{q}}\}$ and $B = \{v_{q+1}, v_{\overline{q+1}}\}$. Let U denote the set of spine vertices outside $A \cup B$; the degrees of all vertices in U are known by hypothesis. If equality holds for the two degrees in the level pair from A or in the level pair from B, then we know the degrees of the vertices in W in order, up to reversal. In this case Lemma 3.9 applies to complete the reconstruction. Letting $\{a, a'\} = \{d(v)\colon\ v \in A\}$ and $\{b, b'\} = \{d(v)\colon\ v \in B\}$, we may therefore assume $a \neq a'$ and $b \neq b'$. Hence we may also assume (after switching the names of a and a', if needed)

$$a \neq b' \neq b \neq a' \neq a. \tag{3}$$

With this notation, the ordered pairs (a, b) and (a', b') are symmetric in the argument.

With $q = \lfloor (r-2)/2 \rfloor$, we have $q = (r-3)/2$ when r is odd, and $q = (r-2)/2$ when r is even. By symmetry, we may assume that the list of degrees of the middle vertices from v_q to $v_{\overline{q}}$ on the spine is a, b, c, b', a' (Type 1) or a', b, c, b', a (Type 2). Here c is an actual value when r is odd, but c is nonexistent (and b is next to b') when r is even.

For every short baton H, we know the number of copies of H with both key vertices in U, with both key vertices in A, with both key vertices in B, and (when r is odd) with $v_{(r+1)/2}$ as a key vertex. We know the last category because (1) we know the level pairs and (2) the distances from $v_{(r+1)/2}$ to the two vertices involved in a level pair are the same.

These known copies of H we call "old"; the remaining copies are "new". Under Type 1 or Type 2, the number of old copies of any short baton H is the same. If for some short baton H the number of new copies of H is different under Type 1 and Type 2, then we know the list of middle degrees up to reversal, and Lemma 3.9 applies to complete the reconstruction. Hence we may assume that for each short baton H the number of new copies of H is the same for both types.

First consider $B_{1:a,b}$. Type 1 has a new copy of $B_{1:a,b}$ with key vertices v_q and v_{q+1}. Type 2 does not have such a baton intersecting B. Since (3) guarantees $a', b' \notin \{a, b\}$, having a new copy of $B_{1:a,b}$ in Type 2 requires $d(v_{\overline{q}+1}) = b$. By the symmetry noted after (3), considering new copies of $B_{1:a',b'}$ yields $d(v_{q-1}) = b'$. Since $b' \neq b$, this fixes the arrangement of B with respect to U. For the remainder of the argument, we move the batons with both key vertices in $B \cup U$ into the "old" category and restrict new batons to be precisely those having one key vertex in A and the other in $B \cup U$. At this point, up to reversal, the central portion of degrees from v_{q-1} to $v_{\overline{q}+1}$ (with c nonexistent for even r) is

$$b', a, b, c, b', a', b \text{ (Type 1)} \quad \text{or} \quad b', a', b, c, b', a, b \text{ (Type 2)}. \tag{4}$$

Let $p = \overline{q} - q$, so $p = 4$ when r is odd and $p = 3$ when r is even. The point is that the distance between the vertices in A is p. By induction on i we now prove for $-1 \leq i \leq \lfloor (r-1)/2 \rfloor$ the following **Claim:**

$$d(v_{q-i}) = b \text{ and } d(v_{\overline{q}+i}) = b' \text{ when } i \equiv -1 \mod p, \tag{5}$$

and

$$d(v_{q-i}) = b' \text{ and } d(v_{\overline{q}+i}) = b \text{ when } i \equiv 1 \mod p. \tag{6}$$

The base cases $i = -1$ and $i = 1$ hold in both cases described in (4). For the induction step, note that since $i \leq (r-1)/2$, we have $q - (i-p) \geq 1$, and symmetrically $\overline{q} + i - p \leq r$. Hence the vertices v_{q-i+p} and $v_{\overline{q}+i-p}$ exist on the spine, and the induction hypothesis applies to them. For the details of the induction step, we consider the two congruence classes for i separately; it may help to refer to (4).

First suppose $i \equiv -1 \mod p$. By (5), $d(v_{q-(i-p)}) = b$. Type 2 has a new copy of $B_{i:a,b}$ with key vertices $v_{\overline{q}}$ and $v_{q-(i-p)}$. Since $a' \notin \{a, b\}$, one key vertex of a new copy of $B_{i:a,b}$ in Type 1 must be v_q, so its other key vertex must have degree b. Among the two vertices in R at distance i from v_q, the vertex $v_{\overline{q}+i-p}$ has degree b', by (5). Therefore, $d(v_{q-i}) = b$.

Symmetrically, $d(v_{\overline{q}+i-p}) = b'$, by (5). Thus Type 2 has a new copy of $B_{i:a',b'}$ with key vertices v_q and $v_{\overline{q}+i-p}$. One key vertex of a new copy of $B_{i:a',b'}$ in Type 1 must be $v_{\overline{q}}$. Among the two vertices in R at distance i from $v_{\overline{q}}$, the vertex $v_{q-(i-p)}$ has degree b, by (5). Therefore, $d(v_{\overline{q}+i}) = b'$. This finishes the induction step when $i \equiv -1 \mod 4$.

Now suppose $i \equiv 1 \mod p$. We have $d(v_{q-(i-p)}) = b'$, by (6). Now Type 1 has a new copy of $B_{i:a',b'}$ with key vertices v_{qb} and $v_{q-(i-p)}$. Since $a \notin \{a', b'\}$, one key vertex of a new copy of $B_{i:a',b'}$ in Type 2 must be v_q. The other key vertex has degree b'. Among the two vertices in R at distance i from v_q, the vertex $v_{\overline{q}+i-p}$ has degree b, by (5). Therefore, $d(v_{q-i}) = b'$.

Symmetrically, $d(v_{\overline{q}+i-p}) = b$, by (6). Thus Type 1 has a new copy of $B_{i:a,b}$ with key vertices v_q and $v_{\overline{q}+i-p}$. One key vertex of a new copy of $B_{i:a,b}$ in Type 2 must be $v_{\overline{q}}$. Among the two vertices in R at distance i from $v_{\overline{q}}$, the vertex $v_{q-(i-p)}$ has degree b', by (6). Therefore, $d(v_{\overline{q}+i}) = b$. This completes the proof of the claim.

Under the assumption that in both structures Type 1 and Type 2 the number of new copies of each short baton is the same, we have obtained for i equal to $\lfloor (r-1)/2 \rfloor$ or $\lfloor (r-1)/2 \rfloor - 1$ (whichever is odd when r is odd or whichever is not divisible by 3 when r is even) a vertex v_{q-i} with degree in $\{b, b'\}$. When r is odd and $q = (r-3)/2$, this yields $q - i \le 0$, so there is no such vertex. When r is even and $(r-2)/2$ is not divisible by 3, again $q - i - 0$ and there is no such vertex. In these cases, the contradiction implies that somewhere along the way we discover that only one of Type 1 and Type 2 for the central vertices can generate the deck.

This leaves the case $3 \mid (r-2)/2$ (that is, $r \equiv 2 \mod 6$), where we must work much harder. Because we are only guaranteed knowing the maximal batons with length at most $(r-1)/2$, the last step we can perform in the iteration is $i = (r-4)/2 \equiv -1 \mod 3$, which yields $(d(v_1), d(v_r)) = (b, b')$. The previous step $i = (r-6)/2 \equiv 1 \mod 3$ told us $(d(v_2), d(v_{r-1})) = (b', b)$.

We know all old batons of all lengths, since we know the entire caterpillar except for the assignment of $\{a, a'\}$ to $\{v_q, v_{\overline{q}}\}$. After discarding old batons, the only new batons with length $r/2$ are those with key vertices $\{v_q, v_{r-1}\}$ and $\{v_{\overline{q}}, v_2\}$. In Type 1 the new maximal batons of length $r/2$ are $B_{r/2:a,b}$ and $B_{r/2:a',b'}$. In Type 2 we instead have $B_{r/2:a',b}$ and $B_{r/2:a,b'}$. It suffices to determine which of these pairs of batons occurs.

Let $b_0 = \min\{b, b'\}$ and $b_1 = \max\{b, b'\}$. Let $a_0 = \min\{b, b'\}$ and $a_1 = \max\{b, b'\}$. We look for $B_{r/2:a_0+1,b_0+1}$ as a new baton. If it occurs, then the correct choice of the unknown caterpillar is the Type containing $B_{r/2:a_1,b_1}$ as a new baton. Otherwise, the caterpillar is the Type not having $B_{r/2:a_1,b_1}$ as a new baton.

It suffices to show that $B_{r/2:a_0+1,b_0+1}$ is small enough to fit in a card. The number of vertices in $B_{r/2:a_0+1,b_0+1}$ is $r/2 + a_0 + b_0 + 1$. Thus we want to prove $a_0 + b_0 \le (n - r + 1)/2$. We have at least $(r+1)/3$ vertices each with degrees b and b', plus at least one vertex each with degrees a and a'. From the equation $\sum_{i=1}^{r} (d(v_i) - 2) = n - r - 2$ we obtain

$$2(a_0 - 2) + 2\frac{r+1}{3}(b_0 - 2) \le n - r - 3 - \frac{r+1}{3}. \tag{7}$$

When either b_0 or a_0 reaches its maximum under (7), the other is restricted to value 2. Since we want to bound $a_0 + b_0$, the more dangerous extreme is when $b_0 = 2$ and a_0 is maximized. In this case (7) requires $a_0 \le (n-r+1)/2 - (r+1)/6$. With $b_0 = 2$, we now have $b_0 + a_0 \le (n-r+1)/2$ as long as $2-(r+1)/6 \le 0$, which holds when $r \ge 11$. This is guaranteed by $n \ge 27$, since $r \ge (n-5)/2$.

Now consider the non-extreme choices for a_0 and b_0. By (7), every increase of 1 in b_0 (starting from $b_0 = 2$) requires a_0 to decrease by $(r+1)/3$. That amount is an integer and is at least 1. Hence all other choices of (a_0, b_0) also result in $B_{r/2:a_0+1,b_0+1}$ fitting in a card.

Since $r \equiv 2 \mod 6$, the only case not covered is $r = 8$ and $b_0 = 2$ (the gain discussed above will be good enough when $b_0 \ge 3$). The degree list in order along the spine in this remaining case is $(b, b', a, b, b', a', b, b')$ (Type 1) or $(b, b', a', b, b', a, b, b')$ (Type 2). We fail only if $a_0 = (n-r-2)/2$ (so n is even), with $\{b, b'\} = \{2, 3\}$ and $\{a, a'\} = \{(n-10)/2, (n-8)/2\}$. By symmetry, we may assume $b = 2$.

The baton $B_{r/2:a_0+1,b_0+1}$ that we were seeking has $(n+4)/2$ vertices. This is truly too big only when $(n+4)/2 > n-\ell$, which is equivalent to $n < 2\ell+4$. Since n here is even, we have $n = 2\ell+2$. Also, $(n-10)/2 = a_0 \ge 2$, so $n \ge 14$. For the remaining cases, let $(n, \ell) = (14+2t, 6+t)$, where $t \ge 0$. We seek a subgraph with $8+t$ vertices whose multiplicity differs in the decks of the Type 1 and Type 2 caterpillars.

The remaining cases are shown in Figure 2 (recall $q = 3$ and $\overline{q} = 6$). We count copies of the triton $B_{2,2:2,3,2+t}$, which has $8+t$ vertices (when $t = 0$, it consists of a 7-vertex path plus one pendant edge at the center). In the display below, we list the number of copies according to which vertex plays the role of the central key vertex. Although $\#B_{2,2:2,3,2+t}$ depends on t, in each case the Type 2 caterpillar has more copies of $B_{2,2:2,3,2+t}$ than the Type 1 caterpillar does. Hence the deck distinguishes the two Types. $\square$

	Type 1				Type 2	
	v_3	v_5	v_6		v_3	v_5	v_6
$t=0$	0	1	2	$t=0$	2	2	0
$t=1$	1	1	2	$t=1$	2	2	1
$t>1$	0	1	0	$t>1$	0	t+1	0

Remark 3.12. At this point we summarize what the results of this section imply about the goal of proving that n-vertex caterpillars are reconstructible from the multiset of subgraphs with at most $(n+2)/2$ vertices. We will consider cases for the degree list; we know when these cases occur because we know the degree list.

If in a particular class of caterpillars the deck determines all maximal batons with length at most $(r-1)/2$ and all maximal tritons with length at most $(r-4)/2$, then caterpillars in that class are ℓ-reconstructible.

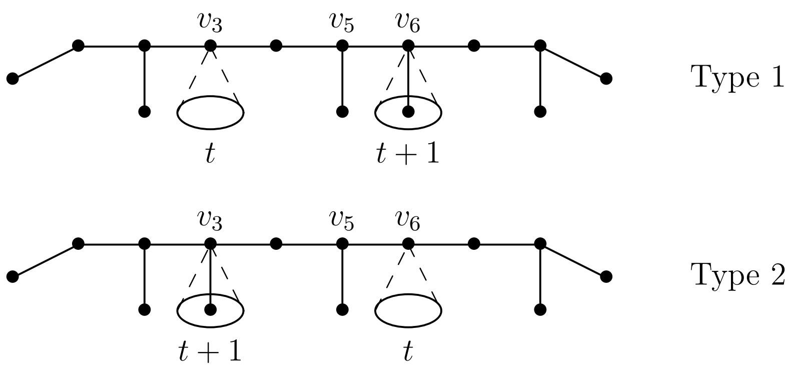

Figure 2: Caterpillars with $r = 8$

To summarize how this works, the maximal batons give us the degree pairs for each fixed distance between the vertices on the spine. The distance pairs then give us the level pairs as in Lemma 3.6. Incorporating the knowledge of the tritons then gives us the lists $(d(v_1), \ldots, d(v_q))$ and $(d(v_{\overline{1}}), \ldots, d(v_{\overline{q}}))$ for $q = \lfloor (r-4)/2 \rfloor$, as in Theorem 3.8. Finally, we apply Lemma 3.11 to stitch the lists together.

To apply this method, we need to determine the short maximal batons and shorter maximal tritons, where we define a baton to be *short* if it has length at most $(r-1)/2$ and define a triton to be *shorter* if it has length at most $(r-4)/2$.

When there is a unique vertex of maximum degree, the technique is different. We will be able to determine exactly where on the spine that vertex is and then use it to find the other vertex degrees. See Section 8.

In these arguments it will also be helpful to have at least four branch vertices, in order to obtain better upper bounds on the high degrees. For the case where G has at most three branch vertices, see Section 9.

4 Short Batons when $d_1 = d_2 \geq d_4 \geq 3$

The first step in applying the plan in Remark 3.12 is to determine the short maximal batons. In order to obtain the desired threshold, we must access only subgraphs with at most $(n+2)/2$ vertices (these will automatically have at most $(n+1)/2$ vertices when n is odd, since the number of vertices is an integer. In this section we find the short maximal batons when there are at least two vertices of maximum degree and at least four branch vertices, so throughout this section we assume $d_1 = d_2 \geq d_4 \geq 3$.

We first summarize what we know and introduce more notation.

Remark 4.1. We have the deck $\mathcal{D}$ of an n-vertex tree G, the multiset of all induced subgraphs with at most $(n+2)/2$ vertices. With $n \geq 12$, we know that G is a caterpillar.

We know the nonincreasing degree list $d_1, \ldots, d_n$ with r nonleaves and may assume the case $r \geq (n-5)/2$. Let R be the set of nonleaf vertices; index R as $v_1, \ldots, v_r$ in order along the spine and also as $w_1, \ldots, w_r$ with $d(w_i) = d_i$.

The baton $B_{j:a,b}$ is *short* if $j \leq (r-1)/2$. Throughout this section, let $\alpha = \lfloor (n-r+1)/4 \rfloor$. The number of vertices in a short baton whose key vertices have degree at most $\alpha+1$ is at most $(r-1)/2 + 2\alpha + 1$, which is at most $(n+2)/2$. Thus *all short batons with maximum degree at most $\alpha+1$ fit in cards, so we can count them.*

By Lemma 3.3, to determine all the short maximal batons it suffices to determine all the short maximal batons with at least $\lfloor (n+2)/2 \rfloor$ vertices.

Lemma 4.2. *If $d_1 = d_2$ and $d_4 > 2$ and $d_3 \leq \alpha$, then the deck determines all short batons.*

Proof. If $d_2 \leq \alpha$, then by Remark 4.1 every short baton fits in a card, which suffices. Hence we may assume $d_1 = d_2 > \alpha \geq d_3$. By Remark 4.1, we see all copies of $B_{j:\alpha+1,\alpha+1}$ Every copy of $B_{j:\alpha+1,\alpha+1}$ in the deck (if any exist) is contained in the unique copy of $B_{j:d_1,d_1}$ in G, and $B_{j:d_1,d_1}$ contains $\binom{d_1-1}{\alpha}^2$ copies of $B_{j:\alpha+1,\alpha+1}$. Thus we know the number of copies of $B_{j:d_1,d_1}$ when $j \leq (r-1)/2$ (it is always 0 or 1).

If $\#B_{j:d_1,d_1} = 1$ with $j \leq (r-1)/2$, then the unique copy of $B_{j:d_1,d_1}$ in G contains $\binom{d_1-1}{a-1}\binom{d_1-1}{b-1}$ copies of $B_{j:a,b}$, for each a and b with $2 \leq a, b \leq \alpha+1$. Delete this many copies of $B_{j:a,b}$ from the deck; they do not contribute to the count of maximal batons.

Each remaining copy of $B_{j:\alpha+1,d_3}$ occurs in some copy of $B_{j:d_1,d_3}$ in G, and each copy of $B_{j:d_1,d_3}$ contains $\binom{d_1-1}{\alpha}$ copies of $B_{j:\alpha+1,d_3}$. This tells us the number of maximal batons in G that are copies of $B_{j:d_1,d_3}$, for $j \leq (r-1)/2$. As in the previous paragraph, when G has q_j copies of $B_{j:d_1,d_3}$ as maximal batons, we delete from the remaining deck $q_j\binom{d_1-1}{a-1}\binom{d_3-1}{b-1}$ copies of $B_{j:a,b}$, for $2 \leq a, b \leq \alpha+1$.

Every remaining short baton lies in a maximal baton (of the same length) having a key vertex with degree at most $\min\{d_4, d_3-1\}$, and its maximum degree is at most d_1. Letting N be its number of vertices, we have

$$N \leq \frac{r-1}{2} + d_1 + \min\{d_4, d_3-1\} - 1 \leq \frac{r+1}{2} + (d_2 + d_4 - 2). \tag{8}$$

Since $d_2 = d_1$ and $d_4 \leq d_3$, (1) yields $d_2 - 2 + d_4 - 2 \leq (n-r-2)/2$. If $d_3 > d_4$, then this improves to $d_2 - 2 + d_4 - 2 \leq (n-r-3)/2$, and then $N \leq (r+1)/2 + (n-r+1)/2 = (n+2)/2$. Otherwise, $d_3 - 1 = d_4 - 1$, which improves the upper bound in (8) by 1, yielding

$$N \leq \frac{r+1}{2} + (d_2 + d_4 - 3) \leq \frac{r+1}{2} + \frac{n-r}{2} = \frac{n+1}{2}.$$

In both cases, all remaining short maximal batons have at most $(n+2)/2$ vertices, and Lemma 3.3 applies. □

Let $W_k = \{w_1, \dots, w_k\}$ (k specified vertices of highest degrees).

Lemma 4.3. *If $d_1 = d_2$ and $2 < d_4 \le \alpha \le d_3 - 1$, then the deck determines all short batons.*

Proof. Our main task is to obtain the short maximal batons whose key vertices are both in W_3, after which an exclusion argument determines the other short maximal batons.

If $d_2 = d_3$, then any copy of $B_{j:\alpha+1,\alpha+1}$ in the deck lies in a copy of $B_{j:d_1,d_1}$ in G, of which there are at most two. Each copy of $B_{j:d_1,d_1}$ contains $\binom{d_1-1}{\alpha}^2$ copies of $B_{j:\alpha+1,\alpha+1}$. Thus the number of copies of $B_{j:\alpha+1,\alpha+1}$ in the deck tells us the number of copies of $B_{j:d_1,d_1}$ in G. Thus we know the numbers of all short maximal batons having both key vertices in W_3.

Now suppose $d_2 > d_3$. Since $d_3 > d_4$, in this case only one vertex has degree d_3. Each copy of $B_{j:\alpha+1,\alpha+1}$ in the deck (if exists) is contained either in a copy of $B_{j:d_1,d_1}$ (there is at most one) or in a copy of $B_{j:d_1,d_3}$ (there are at most two). A copy of $B_{j:d_1,d_1}$ (if it exists) contains $\binom{d_1-1}{\alpha}^2$ copies of $B_{j:\alpha+1,\alpha+1}$, and each copy of $B_{j:d_1,d_3}$ contains $\binom{d_1-1}{\alpha}\binom{d_3-1}{\alpha}$ copies of $B_{j:\alpha+1,\alpha+1}$. So, if over all j with $j \le (r-1)/2$ the total number of short batons in $\mathcal{D}$ with key vertices of degree $\alpha+1$ is $\binom{d_1-1}{\alpha}^2 + 2\binom{d_1-1}{\alpha}\binom{d_3-1}{\alpha}$, then we know all short batons whose key vertices are both in W_3.

We also know these batons if the total is $\binom{d_1-1}{\alpha}^2 + \binom{d_1-1}{\alpha}\binom{d_3-1}{\alpha}$ or $\binom{d_1-1}{\alpha}\binom{d_3-1}{\alpha}$. The only confusion is when the total equals both $\binom{d_1-1}{\alpha}^2$ and $2\binom{d_1-1}{\alpha}\binom{d_3-1}{\alpha}$, which requires $\binom{d_1-1}{\alpha} = 2\binom{d_3-1}{\alpha}$. If this happens and $d_4 < \alpha$, then we consider copies of $B_{j:\alpha,\alpha}$, which now must be contained in a copy of $B_{j:d_1,d_1}$-baton or $B_{j:d_1,d_3}$ in G. Since $\binom{d_1-1}{\alpha}/\binom{d_1-1}{\alpha-1} \ne \binom{d_3-1}{\alpha}/\binom{d_3-1}{\alpha-1}$, this distinguishes the two possibilities.

On the other hand, if $d_4 = \alpha$, then $d_1 > d_3 > \alpha$. Now, since $d_1 \ge \alpha+2$, (1) yields

$$n - r - 2 \ge \sum_{i=1}^{4}(d_i - 2) \ge 2d_1 + 2\alpha + 1 - 8 \ge 4\alpha - 3 \ge n - r - 5.$$

Keeping $\sum_{i=1}^4 (d_i - 2) \le n - r - 2$ thus requires $d_1 \le \alpha + 3$. If ambiguity occurs, then

$$2 = \binom{d_1-1}{\alpha}\Big/\binom{d_3-1}{\alpha} \ge \binom{d_1-1}{\alpha}\Big/\binom{d_1-2}{\alpha} = \frac{d_1-1}{d_1-1-\alpha} \ge \frac{\alpha+2}{2}.$$

Since $\alpha = d_4 \ge 3$, we have $(\alpha+2)/2 > 2$, so the ambiguity does not occur.

Thus in all cases we know all short maximal batons whose key vertices lie in W_3. More simply than in Lemma 4.2 (since here $d_4 < d_3$), we derive that every short baton in G with at most one key vertex in W_3 has at most $(n+2)/2$ vertices. Again Lemma 3.3 applies. $\square$

Lemma 4.4. *If $d_1 = d_2$ and $d_4 \ge \max\{3, \alpha+1\}$ with $2\alpha + 1 \le (n-r+1)/2$, then the deck determines all short batons.*

Proof. When $j \le (r-1)/2$, the number of vertices in a copy of $B_{j:\alpha+2,\alpha+1}$ is at most $(r-1)/2+2\alpha+2$, which in the specified case is at most $(n+2)/2$. This is small enough to fit in cards, so we see all such batons.

If $\sum_{i=1}^{4} d_i \ge 4\alpha+9$, then by (1) we have

$$n-r-2 \ge \sum_{i=1}^{4}(d_i-2) \ge 4\alpha+1 = 4\left\lfloor \frac{n-r+1}{4} \right\rfloor + 1 \ge n-r-2+1,$$

a contradiction. Thus $\sum_{i=1}^{4} d_i \le 4\alpha+8$. Since $d_4 \ge \alpha+1$, either $\alpha+2 \ge d_1 \ge d_4 \ge \alpha+1$ or $\alpha+3 = d_1 = d_2 = d_3+2 = d_4+2$. Note also that when $d_4 = \alpha+1$ we have $\alpha \ge 2$, since we have assumed $d_4 > 2$.

Suppose first that $\alpha+3 = d_1 = d_2 = d_3+2 = d_4+2$. Each copy of $B_{j:\alpha+2,\alpha+1}$ lies either in the unique copy of $B_{j:d_1,d_1}$ in G or in some copy of $B_{j:d_1,\alpha+1}$ that is a maximal baton in G. The copy of $B_{j:d_1,d_1}$ (if it exists) contains $2\binom{d_1-1}{\alpha+1}\binom{d_1-1}{\alpha}$ copies of $B_{j:\alpha+2,\alpha+1}$, and every copy of $B_{j:d_1,d_3}$ that is a maximal baton in G contains $\binom{d_1-1}{\alpha+1}$ copies of $B_{j:\alpha+2,\alpha+1}$. Since for a given j, each of w_1 and w_2 is in at most two copies of $B_{j:d_1,d_3}$, the total number of such batons in G is at most 4. On the other hand, $2\binom{d_1-1}{\alpha} \ge 2(\alpha+1) \ge 6$, since $\alpha \ge 2$. It follows that the number of copies of $B_{j:\alpha+2,\alpha+1}$ in $\mathcal{D}$ uniquely determines the numbers of maximal batons in G that are copies of $B_{j:d_1,d_2}$ and $B_{j:d_1,d_3}$.

Thus, we know all short batons in G with both key vertices having degree at least d_4. The number of vertices in a short baton in G having at least one key vertex of degree less than d_4 is at most $d_1+d_4-2+(r-1)/2$, and we compute

$$d_1+d_4-2+\frac{r-1}{2} \le 2\alpha+2+\frac{r-1}{2} \le \frac{n-r+1}{2}+1+\frac{r-1}{2} = \frac{n+2}{2}.$$

Thus such a baton has at most $\lfloor (n+2)/2 \rfloor$ vertices, which is small enough to fit in a card.

We now know all short maximal batons with at least $\lfloor (n+2)/2 \rfloor$ vertices, which by Lemma 3.3 is enough to determine all short maximal batons in G.

Now suppose $\alpha+2 \ge d_1 \ge d_4 \ge \alpha+1$. If $d_1 = d_4 = \alpha+2$, then $d_5 = 2$, and no copy of $B_{j:\alpha+2,\alpha+1}$ is a maximal baton in G. Thus $\#B_{j:\alpha+2,\alpha+1}$ tells us all maximal batons with two branch vertices, which suffices.

Hence we may assume $\alpha+1 = d_4 \ge 3$, so $2(\alpha+1) \ge 6$. Each copy of $B_{j:\alpha+2,\alpha+1}$ in $\mathcal{D}$ lies in a copy of $B_{j:d_1,d_1}$ in G or is itself a copy of $B_{j:d_1,d_4}$ that is a maximal baton in G. Now every copy of $B_{j:d_1,d_1}$ in G contains $2(\alpha+1)$ copies of $B_{j:\alpha+2,\alpha+1}$.

If $d_1 = d_3 = \alpha+2 > \alpha+1 = d_4$, then $d_6 = 2$ and $d_5 \le 3$. In this case only w_4 and w_5 can have degree d_4, so at most four copies of $B_{j:d_1,d_4}$ can be maximal batons in G, while every copy of $B_{j:d_1,d_1}$ in G contains $2(\alpha+1)$ copies of $B_{j:d_1,d_4}$ as cards in $\mathcal{D}$. Since $2(\alpha+1) \ge 6$, the number of copies of $B_{j,\alpha+2,\alpha+1}$ in $\mathcal{D}$ tells us how many copies of $B_{j:d_1,d_1}$ and $B_{j:d_1,d_4}$ occur as maximal batons in G. All other short batons fit in cards.

If $d_1 = d_2 = \alpha + 2 > \alpha + 1 = d_3 = d_4$, then only w_1 and w_2 have degree $\alpha + 2$. They are the only candidates for key vertices of degree d_1, so again G can only have four maximal batons that are copies of $B_{j:d_1,d_4}$. Still any copy of $B_{j:d_1,d_1}$ in G (there is at most one) contains $2(\alpha + 1)$ copies of $B_{j:d_1,d_4}$. Since $2(\alpha + 1) \geq 6$, again we know how many maximal batons in G are copies of $B_{j:d_1,d_1}$ or $B_{j:d_1,d_4}$. Again, all other short batons fit in cards.

Finally, when $d_1 = d_2 = d_3 = d_4 = \alpha + 1$, all short batons fit in cards. □

Lemma 4.5. *If $d_1 = d_2$ and $d_4 \geq \alpha+1$ with $2\alpha+1 \geq (n-r+2)/2$, then the deck determines all short batons.*

Proof. The hypothesis yields $\alpha \geq (n-r)/4$ and $d_4 \geq (n-r+4)/4$. As before, by Remark 4.1, $d_1 = \alpha + 1$ implies that all short batons fit in cards, so we may assume $d_1 \geq \alpha + 2$.

By (1) and the lower bounds on d_1 and d_4,

$$\begin{aligned} n - r - 2 = \sum_{i=1}^{r}(d_i - 2) &\geq 2(d_1 - 2) + 2(d_4 - 2) + \sum_{i=5}^{r}(d_i - 2) \\ &\geq 2 + 4(\alpha - 1) + \sum_{i=5}^{r}(d_i - 2) \geq (n - r - 2) + \sum_{i=5}^{r}(d_i - 2). \end{aligned}$$

This requires $d_5 = 2$, $\alpha = (n - r)/4$ and $d_1 = d_3 + 1 = d_4 + 1 = \alpha + 2$. Since $d_4 = \alpha + 1$, again $\alpha \geq 2$. Let $\beta = \alpha + 1$.

By Remark 4.1, each copy of $B_{j:\beta,\beta}$ with $j \leq (r-1)/2$ fits in a card. With $d_5 = 2$, both branch vertices in each such baton are in W_4. The unique copy of $B_{j:d_1,d_1}$ in G (if it exists) contains $\binom{d_1-1}{\alpha}^2$ (that is, β^2) copies of $B_{j:\beta,\beta}$. The unique copy of $B_{j:d_3,d_3}$ that is a maximal baton in G (if it exists) is a copy of $B_{j:\beta,\beta}$. Also, each copy of $B_{j:d_1,d_4}$ that is a maximal baton in G contains $\binom{d_1-1}{\alpha}$ (that is, β) copies of $B_{j:\beta,\beta}$.

Say that W_4 is *equally spaced* if there is some i such that $W_4 = \{v_i, v_{i+j}, v_{i+2j}, v_{i+3j}\}$. If $B_{j:d_1,d_1} \subseteq T$, then there may be two, one, or no copies of $B_{j:d_1,d_4}$ that are maximal batons in G (when there are two, W_4 must be equally spaced). In these cases the numbers of copies of $B_{j:\beta,\beta}$ in $\mathcal{D}$ in addition to the β^2 contained in $B_{j:d_1,d_1}$ are 2β, ($\beta + 1$ or β), or (1 or 0), respectively. Since $\beta \geq 3$, these values are all different and distinguish the cases.

If $B_{j:d_1,d_1} \not\subseteq T$, then there may be up to three copies of $B_{j:d_1,d_4}$ that are maximal batons in G. Having three such copies requires W_4 to be equally spaced and prohibits $B_{j:d_4,d_4}$ as a maximal baton in G, yielding 3β copies of $B_{j:\beta,\beta}$ in $\mathcal{D}$. With two such maximal batons in G, the number of copies of $B_{j:\beta,\beta}$ in $\mathcal{D}$ is $2\beta + 1$ or 2β, depending on whether W_4 is equally spaced. With at most one copy of $B_{j:d_1,d_4}$ as a maximal baton in G, the numbers of copies of $B_{j:\beta,\beta}$ can be $\beta + 1$, β, 1, or 0, depending on whether $B_{j:d_1,d_4}$ and $B_{j:d_4,d_4}$ occur as maximal batons in G. These values are all different and distinguish the cases.

The only ambiguity in using $\#B_{j:\beta,\beta}$ to determine the maximal batons in G having at least $n-\ell$ vertices is that when $\beta = 3$ the case with $B_{j:d_1,d_1} \subseteq T$ generating all β^2 copies of $B_{j:\beta,\beta}$ may be confused with the case with $B_{j:d_1,d_1} \not\subseteq T$ in which W_4 is equally spaced and $\#B_{j:\beta,\beta} = 3\beta$. To have such confusion, we must have $j \leq (r-1)/3$. Now the number of vertices in $B_{j:d_1,d_4}$ is bounded by $(r-1)/3+(n-r)/2+2$, since $d_1 = \alpha+2$ and $\alpha = (n-r)/2$. This bound is at most $(n+2)/2$ when $r \geq 4$. If $r \leq 3$, then we cannot have four branch vertices. Hence $B_{j:d_1,d_4}$ fits in a card, we can use $\#B_{j:d_1,d_4}$ to distinguish the two cases (there are $2\beta+2$ copies in $B_{j:d_1,d_1}$ and only three copies in the case where W_4 is equally spaced.

Thus we find all short maximal batons in G whose key vertices are in W_4, and all other short batons fit in cards. □

5 Four or More Vertices of Maximum Degree

In Section 3, we proved that for reconstruction of the unknown n-vertex caterpillar G it suffices to know the short maximal batons (length at most $(r-1)/2$) and the shorter maximal tritons (length at most $(r-4)/2$). In Section 4, we proved that the short maximal batons are known when G has at least two vertices of maximum degree. Here we begin by observing that the shorter tritons are also determined when the maximum degree is small. Keep in mind that the deck $\mathcal{D}$ is the multiset of induced subgraphs with at most $(n+2)/2$ vertices.

Lemma 5.1. *If $d_1 \leq (n-r+12)/6$, then $\mathcal{D}$ determines all tritons of length at most $(r-4)/2$. Thus n-vertex caterpillars with $d_1 = d_6$ are reconstructible from $\mathcal{D}$.*

Proof. Tritons with length at most $(r-4)/2$ have at most $(r-4)/2+3d_1-3$ vertices. This is at most $(n+2)/2$ when $d_1 \leq (n-r+12)/6$. Hence all shorter tritons are then known.

If $d_1 = d_6$, then the short maximal batons are known by Section 4. Also, by (1), $n-r-2 \geq 6(d_1-2)$, so $d_1 \leq (n-r+10)/6$. □

We may henceforth assume $d_1 > d_6$. The next lemmas provide sufficient conditions for tritons to fit in cards. Recall that $B_{j:a,b}$ has $j+a+b-1$ vertices and $B_{j,j':a,b,c}$ has $j+j'+a+b+c-3$ vertices.

Lemma 5.2. *Suppose $d_1 = d_4 > d_6$. If $j+j' \leq (r-4)/2$, then the triton $B_{j,j':d_6+1,d_6+1,d_6+1}$ fits in a card.*

Proof. The triton has at most $(r-4)/2+3d_6$ vertices. Since $d_4 \geq d_6+1$, we obtain $6d_6 \leq -4+\sum_{i=1}^{6} d_i$. Now for the number N of vertices in the triton we use (1) to obtain

$$N \leq \frac{r-4}{2}+3d_6 \leq \frac{r-4}{2}-2+\frac{1}{2}\sum_{i=1}^{6} d_i \leq \frac{r-8}{2}+\frac{n-r+10}{2} = \frac{n+2}{2}. \qquad \square$$

Lemma 3.11 enables us to handle the case $d_1 = d_5$.

Corollary 5.3. *If the unknown caterpillar has exactly five vertices of maximum degree (that is, $d_1 = d_5 > d_6$), then it is determined by the deck.*

Proof. It suffices to discover the shorter tritons. A vertex of degree at least $d_6 + 1$ in a subgraph of G has degree d_1 in G. On the other hand, a j, j'-triton with three branch vertices of degree d_1 is a maximal triton and contains exactly M copies of $B_{j,j':d_6+1,d_6+1,d_6+1}$, where $M = \binom{d_1-1}{d_6}^2\binom{d_1-2}{d_6-1}$. Thus $\#B_{j,j':d_1,d_1,d_1} = \#B_{j,j':D,D,D}/M$, where $D = d_6 + 1$. By Lemma 5.2, $B_{j,j':D,D,D}$ fits in a card.

We must also find the other shorter maximal tritons with at least $\lfloor (n + 2)/2\rfloor$ vertices. These may exist with one or two vertices of maximum degree. Since we know the number of these with three vertices of maximum degree, and those with at most two vertices are contained in $B_{j,j':D,D,d_6}$, we can find the remaining ones via an exclusion argument. $\square$

Now suppose $d_1 = d_4 > d_5$. As before, it suffices to determine the shorter tritons. We split the task into two cases, depending on whether $d_5 \le 1 + (n - r)/6$.

Lemma 5.4. *Suppose $d_1 = d_4 > d_5$. If $d_5 \le (n - r + 6)/6$, then the deck determines all shorter maximal tritons (length at most $(r - 4)/2$) having at least $\lfloor (n + 2)/2\rfloor$ vertices.*

Proof. Let $L = 1 + d_5$, so any triton containing a vertex of degree L is a proper subgraph of a triton in which that vertex has degree d_1. Also, since $B_{j,j':a,b,c}$ has $j + j' + a + b + c - 3$ vertices, a shorter triton with maximum degree L has at most $(r - 4)/2 + 3L - 3$ vertices. We compute

$$\frac{r-4}{2} + 3L - 3 \le \frac{r-4}{2} + 3d_5 \le \frac{r-4}{2} + 3\frac{n-r+6}{6} = \frac{n+2}{2}.$$

Thus every shorter triton with maximum degree L fits in a card, and we can count the copies of each. We use these counts to determine the shorter maximal tritons containing at least one vertex of maximum degree.

Every copy of $B_{j,j':L,L,L}$ lies in a copy of $B_{j,j':d_1,d_1,d_1}$, which is the unique maximal triton containing it, and every copy of $B_{j,j':d_1,d_1,d_1}$ contains M copies of $B_{j,j':L,L,L}$, where $M = \binom{d_1-1}{L-1}^2\binom{d_1-2}{L-2}$. Hence $\#B_{j,j':d_1,d_1,d_1} = \#B_{j,j':L,L,L}/M$.

Now consider the copies of $B_{j,j':a,L,L}$, where $a \le d_5$. We know $\#B_{j,j':a,L,L}$, since such subgraphs fit in cards. Each copy of $B_{j,j':a,L,L}$ occurs in a unique copy of $B_{j,j':a,d_1,d_1}$, but that copy of $B_{j,j':a,d_1,d_1}$ need not be a maximal triton.

We start with $B_{j,j':d_5,L,L}$. Each copy of $B_{j,j':d_1,d_1,d_1}$ contains $2\binom{d_1-1}{d_5-1}\binom{d_1-2}{L-2}\binom{d_1-1}{L-1}$ copies of $B_{j,j':d_5,L,L}$. When we delete these from the count $\#B_{j,j':d_5,L,L}$, the remaining copies lie in distinct copies of $B_{j,j':d_5,d_1,d_1}$, and those copies of $B_{j,j:d_5,d_1,d_1}$ are maximal tritons.

We proceed inductively with this exclusion argument as a decreases. From $\#B_{j,j':a,L,L}$ we must delete all copies that arise as subgraphs of maximal tritons that are copies of $B_{j,j':a',d_1,d_1}$ for $a < a' \le d_5$ or $B_{j,j':d_1,d_1,d_1}$, all of which have already been counted. The remaining count is the number of maximal tritons that are copies of $B_{j,j':a,d_1,d_1}$.

The same technique enables us to count the maximal tritons that are copies of $B_{j,j':d_1,b,d_1}$, for $2 \le b \le d_5$. Having determined the maximal j,j'-tritons having two vertices of maximum degree, we can then use the same exclusion approach to determine the maximal j,j'-tritons having one vertex of maximum degree, starting with copies of $B_{j,j':d_5,d_5,d_1}$ and $B_{j,j':d_5,d_1,d_5}$ and working down to smaller j,j'-tritons.

Since all shorter tritons having no vertices of degree d_1 fit in cards, the exclusion argument now permits finding all the maximal tritons among them. $\square$

Lemma 5.5. *Suppose $d_1 = d_4 > d_5$. If $d_5 \ge (n-r+7)/6$, then the deck determines all shorter maximal tritons having at least $\lfloor (n+2)/2 \rfloor$ vertices.*

Proof. We first prove $d_5 \ge 3$. Otherwise, $2 \ge (n-r+7)/6$, so $n-r \le 5$. Now (1) implies

$$3 \ge n-r-2 \ge \sum_{i=1}^{4}(d_i-2) \ge 4.$$

The inequality $d_6 < d_5$ is now immediate if $d_6 \le 2$. When $d_6 \ge 3$, we prove a stronger inequality. Let $\alpha = d_5 - (n-r+6)/6$, so $\alpha \ge 1/6$. Since $d_1 \ge d_5+1$, using (1) yields

$$\begin{aligned} d_6 - 2 &\le n-r-2-5(d_5-2)-4 \le (n-r-1)-5(d_5-1) \\ &\le n-r-1-5\left(\frac{n-r}{6}+\alpha\right) = \frac{n-r+6}{6} - 2 - 5\alpha. \end{aligned} \tag{9}$$

In particular, $d_6 \le (n-r+6)/6 - 5\alpha \le d_5 - 1$.

Again let $W_k = \{w_1, \dots, w_k\}$. Given j and j' with $j+j' \le (r-4)/2$, vertex w_5 can be in at most two maximal tritons that are copies of $B_{j,j':d_1,d_5,d_1}$ in G. Moreover, if w_5 is in two such maximal tritons, then $j' \ne j$ and w_5 has two vertices in W_4 at distance j in G and two vertices in W_4 at distance j'. Since we know all short batons, we can determine whether this occurs. If it does, then we know all shorter tritons whose key vertices are all in W_5. Otherwise, we know that at most one maximal triton is a copy of $B_{j,j':d_1,d_5,d_1}$.

A similar statement holds for copies of $B_{j,j':d_5,d_1,d_1}$, with the only difference being that then two such maximal tritons may also occur when $j = j'$. We still see whether this happens from the deck. The case of $B_{j,j':d_1,d_1,d_5}$ is symmetric.

Having restricted to one of these j,j'-tritons occurring at most once as a maximal triton, we want to determine whether it in fact appears. Let β be the number of vertices in $B_{j,j':d_5+1,d_6+1,d_5+1}$. Using (9), we compute that it fits in a card. That is,

$$\beta \le \frac{r-4}{2} + 2d_5 + d_6 \le \frac{r-4}{2} + 2\alpha + 3\frac{n-r+6}{6} - 5\alpha < \frac{n+2}{2}. \tag{10}$$

Every such triton is contained in a maximal triton in G that is a copy of (i) $B_{j,j':d_1,d_1,d_1}$ or (ii) $B_{j,j':d_1,d_5,d_1}$. Each triton of type (i) in G contains $\binom{d_1-1}{d_5}^2\binom{d_1-2}{d_6-1}$ copies of $B_{j,j':d_5+1,d_6+1,d_5+1}$, while each triton of type (ii) contains fewer copies, only $\binom{d_1-1}{d_5}^2\binom{d_5-2}{d_6-1}$. Since we have restricted to the case where there is at most one triton of type (ii), we decide whether there is such a triton by checking whether the number of copies of $B_{j,j':d_5+1,d_6+1,d_5+1}$ is divisible by $\binom{d_1-1}{d_5}^2\binom{d_1-2}{d_6-1}$; if so, then all the copies of $B_{j,j':d_5+1,d_6+1,d_5+1}$ are contained in copies of $B_{j,j':d_1,d_1,d_1}$. We thus find the number of maximal j,j'-tritons of types (i) and (ii) in G.

Now, consider shorter tritons of the form $B_{j,j':d_6+1,d_5+1,d_5+1}$ that are not subgraphs of the tritons of types (i) and (ii) in G. As in (10), such shorter tritons fit in cards, so we can count them. The copies we have not excluded are contained in copies of $B_{j,j':d_5,d_1,d_1}$. We know how many copies of $B_{j,j':d_6+1,d_5+1,d_5+1}$ lie in each copy of $B_{j,j':d_5,d_1,d_1}$, so we now know how the number of copies of $B_{j,j':d_5,d_1,d_1}$ that are maximal tritons in G. Thus, we know the all the shorter maximal tritons in G whose key vertices all lie in W_5.

The tritons $B_{j,j':d_5+1,d_5+1,d_6}$, $B_{j,j':d_5+1,d_6,d_5+1}$, and $B_{j,j':d_6,d_5+1,d_5+1}$ fit in cards, since they are smaller than $B_{j,j':d_6+1,d_5+1,d_5+1}$. They tell us how many maximal j,j'-tritons in G have exactly two key vertices in W_4 and one key vertex of degree d_6. Now, using exclusion, we obtain all the shorter maximal tritons. □

Theorem 5.6. *For $n \geq 48$, every n-vertex caterpillar with at least four vertices of maximum degree is reconstructible from the deck.*

Proof. We have shown that the short maximal batons and the shorter maximal tritons are determined by the deck. Theorem 3.8 and Lemma 3.11 complete the reconstruction. □

6 Three Maximum-degree Vertices

In this section we reconstruct high-diameter caterpillars with exactly three vertices of maximum degree and at least four branch vertices: $d_1 = d_3 > d_4 \geq 3$. We recognize being in this case from the degree list.

Our approach has three main steps. First we find the positions of the three vertices of maximum degree. Next we consider the position of a branch vertex whose degree appears only once in the degree list. Finally, we show that the deck determines the shorter tritons. As we have observed, that suffices for the reconstruction.

By Section 4 we know all short maximal batons. Hence by Lemma 3.6 we know all level pairs. Throughout this section, we use the following terminology and notation.

Definition 6.1. Recall that $w_1, \ldots, w_r$ indexes vertices on the spine P in order of degree, so that $d(w_i) = d_i$, and $W_t = \{w_1, \ldots, w_t\}$. In particular, with $d_1 = d_3$, we index $\{w_1, w_2, w_3\}$

in nondecreasing order from the center of P. A W_3*-baton* is a baton whose key vertices are both in W_3. The *depth* $f(v)$ of a vertex v on P is the distance of v from the "middle" or "top" of P. That is, for the level pair $\{v_k, v_{\overline{k}}\}$ we have $f(v_k) = f(v_{\overline{k}}) = |s-k|$, where $s = (r+1)/2$. The depth of a vertex is a half-integer when r is even. To describe the depths of the vertices in W_3, we use a, b, c as follows:

$$f(w_1) = a \le f(w_2) = a + b \le f(w_3) = a + b + c \le \frac{r-1}{2}. \tag{11}$$

Also let $h = s - a - b - c$, so $w_3 \in \{v_h, v_{\overline{h}}\}$. The notation appears in Figure 3, showing a difficult case in the next lemma, where we aim to decide which square is the location of w_2.

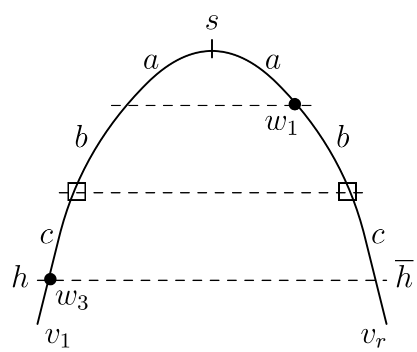

Figure 3: Notation for Section 6

Since $r \ge (n-5)/2$ in the high-diameter case, the condition $r \ge 9$ is ensured by $n \ge 23$.

Lemma 6.2. *In the high-diameter case with exactly three vertices of maximum degree (and $r \ge 9$), the deck determines the positions of the three maximum-degree vertices.*

Proof. By Sections 4 and 3, we know the short maximal batons and the level pairs. Hence we know the differences a, b, c in Definition 6.1. When two vertices in W_3 have the same level, the claim is trivial (up to reflection). Hence we may assume $b, c > 0$. If $a = 0$, then we see W_3-batons with lengths b and $b + c$. If we also see a W_3-baton with length c, then w_2 and w_3 are in the same half of P; otherwise, they are in different halves. Thus henceforth we may also assume $a > 0$.

Our goal is to determine the "Type" of W_3, where W_3 is *Type 0* if all of W_3 is in the same half of P and *Type* i for $i \in \{1, 2, 3\}$ if w_i is in the half opposite $W_3 - \{w_i\}$. There are always three W_3-batons; we consider cases by the number of short W_3-batons. Since W_3 has at least two vertices on the same side of P, there is always at least one short W_3-baton with length b, c, or $b + c$. Short W_3-batons may also occur with length $b + 2a$, $b + c + 2a$, or $c + 2a + 2b$ when the key vertices are in opposite halves of P.

Case 1: *All three W_3-batons are short.* We see all three W_3-batons and their lengths. The triples of lengths are as follows:

Type 0:	$\{b, c, b+c\}$	Type 2:	$\{b+c, 2a+b, 2a+2b+c\}$
Type 1:	$\{c, 2a+b, 2a+b+c\}$	Type 3:	$\{b, 2a+b+c, 2a+2b+c\}$.

The triples of lengths are distinct in the four cases, so they distinguish the cases. In all cases these are the triples of lengths of the W_3-batons, but in other cases we don't see them all.

Case 2: *There are exactly two short W_3-batons.* In each triple of lengths listed above, the third length is the sum of the other two, so to have only two short W_3-batons they must be the two shorter lengths in each triple. For the four Types, the lengths of the two shortest W_3-batons are distinct pairs, so the lengths of the two short W_3-batons distinguish the cases.

Case 3: *There is only one short W_3-baton, B.* Type 0 has three short W_3-batons, so that is excluded. Among the other types, the two vertices of W_3 in the same half of P guarantee a short W_3-baton with length in $\{b, c, b+c\}$. If B has length $b+c$, then we see no W_3-batons with the shorter lengths b or c, and W_3 is Type 2. If B has length b or c, and $b \neq c$, then W_3 is Type 3 or Type 1, respectively. Hence we may assume that B has length b and $b = c$, and we need to distinguish between Type 3 and Type 1.

In both types, w_1 and w_3 are in opposite halves of P, and we need to determine which half contains w_2 (see Figure 3 in Section 6). By reflection, we may assume $w_1 = v_{s+a}$ and $w_3 = v_h$. If $2a + b \leq (r-1)/2$, then W_3 is Type 3, since B being the only short W_3-baton forbids a W_3-baton of length $2a + b$.

Hence we may assume $2a+b \geq r/2$. Now let $H = B_{c,h:d_4+1,d_4+1,2}$ and $H' = B_{c,h:d_4+1,d_4+1,3}$. Since $d_4 \geq 3$, any copy of H or H' in G has exactly two vertices in W_3. Let $M = \binom{d_1-1}{d_4}\binom{d_1-2}{d_4-1}$; there are M ways of choosing the neighbors of the high-degree vertices when forming such tritons. If W_3 is Type 1, then $w_2 = v_{h+c}$, and the third key vertex of copies of H or H' is v_{2h+c}. If W_3 is Type 3, then $w_2 = v_{s+a+b}$ and the third key vertex may be $v_{s+a+b+h}$ or v_{s+a-h}. In the two cases we obtain the following counts.

	Type 1	Type 3
$\#B_{c,h:d_4+1,d_4+1,2}/M$	$d(v_{2h+c}) - 1$	$d(v_{s+a+b+h}) - 1 + d(v_{s+a-h}) - 1$
$\#B_{c,h:d_4+1,d_4+1,3}/M$	$\binom{d(v_{2h+c})-1}{2}$	$\binom{d(v_{s+a+b+h})-1}{2} + \binom{d(v_{s+a-h})-1}{2}$

The counts distinguish the two cases, because if positive integers x and y sum to z, then $\binom{x}{2} + \binom{y}{2} < \binom{z}{2}$.

It remains only to show that H' (and hence also H) fits in a card, so we can count the copies. By (1) and $d_3 > d_4$, we have $n - r - 2 \geq 4d_4 - 5$, so $d_4 \leq (n-r+3)/4$. Also $h + c = s - a - b$, so $2a + b \geq r/2$ yields $2h + 2c + b \leq 2s - r/2 = (r+2)/2$. Since $b \geq 1$, this implies $h + c \leq r/4$. Now

$$|V(H')| = c + h + 2d_4 + 2 \leq \frac{r}{4} + \frac{n-r+3}{2} + 2 = \frac{n+7}{2} - \frac{r}{4}.$$

We need $|V(H')| < (n+3)/2$, which is guaranteed when $r \geq 9$. $\square$

Lemma 6.3. *In the high-diameter case with $d_1 = d_3 > d_4 > d_5$, if we know the positions of the vertices of W_3 on P, then we know the position of w_4.*

Proof. We continue using the notation of Definition 6.1. Also, let a W_4-*baton* be a maximal baton whose key vertices have degrees d_4 and d_1; the lengths of W_4-batons are the distances from w_4 to vertices of W_3.

Again we know the short maximal batons and the level pairs; we also know the W_3-batons. If W_3 has two vertices on the same level, then the pair of distances from w_4 to these vertices is known. From the level pairs, $f(w_4)$ is known, so we have two options for the length of the third W_4-baton. At least one of the options is short, so we can tell whether w_4 is in the same half of P with the third vertex of W_3.

Hence we may again assume that the vertices of W_3 have distinct depths, so $b, c > 0$. If $f(w_4) = f(w_j)$ for some $1 \le j \le 3$, then we know the position of w_4. Hence we may assume strict inequalities in specify the case below for the depth of w_4.

Case 1: *$f(w_4) < a$ or $f(w_4) > a+b+c$.* In this case the length of the shortest W_4-baton determines which half of P contains w_4. If we see no short W_4-baton, then w_4 is in the opposite half of P from all of W_3.

Case 2: *$a + b < f(w_4) < a + b + c$.* If $f(w_4) \ne a + b + c/2$ or w_2 and w_3 are in the same half of P, then the length of the shortest W_4-baton determines the position of w_4. Hence we may assume $f(w_4) = a + b + c/2$ and that w_2 and w_3 are in opposite halves of P.

One of the distances from w_4 to $\{w_2, w_3\}$ is $c/2$, and the other is at least $2a + 2b + c/2$. We determine the position of w_4 by testing for W_4-batons of length $b+c/2$, which is between the two distances from w_4 to $\{w_2, w_3\}$. If such a W_4-baton exists, then w_4 is in the same half of P as w_1; otherwise, it is in the opposite half.

Case 3: *$a < f(w_4) < a+b$.* This begins like Case 2 but becomes harder. If $f(w_4) \ne a+b/2$ or w_1 and w_2 are in the same half of P, then again the length of the shortest W_4-baton determines the position of w_4. Hence we may assume $f(w_4) = a + b/2$ and that w_1 and w_2 are in opposite halves of P.

One of the distances from w_4 to $\{w_1, w_2\}$ is $b/2$, and the other is either $2a+b/2$ or $2a+3b/2$. If no W_4-baton has length $b/2 + c$, then w_4 is the opposite half of P from w_3. If there are two W_4-batons with length $b/2 + c$, or if there is one and $b/2 + c \notin \{2a + b/2, 2a + 3b/2\}$, then w_4 is in the same half of P as w_3.

Hence we may assume one W_4-baton has length $b/2+c$ and $b/2+c \in \{2a+b/2, 2a+3b/2\}$. First suppose $b/2 + c = 2a + 3b/2$. This requires $2a + 3b/2 < r/2$, so we can see whether there is a W_4-baton with length $2a + 3b/2$. If not, then w_4 is in the same half of P as w_1; otherwise, w_4 is in the same half of P as w_2.

Finally, suppose that one W_4-baton has length $b/2 + c$ and $b/2 + c = 2a + b/2$, which requires $c = 2a$. Since there is exactly one W_4-baton of this length, w_3 must be in the same half of P as w_1 (and w_2 is in the opposite half). If $2a + 3b/2 < r/2$, then we can check the existence of a W_4-baton with length $2a + 3b/2$; it exists if and only if w_4 is on the opposite side of P from w_2.

Hence we may assume $2a + 3b/2 \geq r/2$. Let $H = B_{b/2+c,h:d_4,d_4+1,2}$. Note that the key vertex of degree d_4 in any copy of H in G must be w_4, since no two vertices in W_3 are at distance $b/2 + c$. Now w_4 can be on the same side of P as w_2 at distance $b/2 + c$ from w_1, or on the opposite side from w_2 at distance $b/2 + c$ from w_3. However, the path does not extend far enough beyond w_3 to complete a copy of H in the latter case. Hence if G contains H, then w_4 is in the same half of P as w_2; otherwise it is not.

It remains only to show that H fits in a card in order to test whether it appears. Let N be its number of vertices. Again (1) implies $d_4 \leq (n - r + 3)/4$. Using $h = s - a - b - c$, and letting $x = a + b/2$, we have

$$N = \frac{b}{2} + c + h + 2d_4 = \frac{r+1}{2} - a - \frac{b}{2} + 2d_4 \leq \frac{r+1}{2} - x + \frac{n-r+3}{2} = \frac{n+2}{2} + 1 - x.$$

It thus suffices to have $x \geq 1$. Because $2a = c \geq 1$, we have $a \geq 1/2$. Also b is a positive integer, so $a + b/2 \geq 1$. $\square$

Lemma 6.4. *In the high-diameter case with $d_1 = d_3 > d_4 > 2$, let i and j be indices such that both d_i and d_j are at least 3 and occur only once in the degree list. If the deck determines the position of w_i on P, then the deck also determines the position of w_j.*

Proof. We know the short maximal batons, and these give us the level pairs. Hence along with the position of w_i we also know the depth of w_j. Let $h = |f(w_i) - f(w_j)|$. If the deck has $B_{h:d_i,d_j}$ as a maximal baton, then w_j is in the same half of P as w_i; otherwise, they are in opposite halves. $\square$

Theorem 6.5. *In the high-diameter case with $d_1 = d_3 > d_4 > 2$ and $n \geq 48$, all n-vertex caterpillars are reconstructible from their decks.*

Proof. By Section 4 we know the short maximal batons. By Section 3 we know the level pairs and that it suffices to determine the shorter maximal tritons.

Let t be the smallest index at least 4 such that $d_t = d_{t+1}$. By the lemmas above, we know the positions of all vertices in W_{t-1}, so we also know all shorter maximal tritons whose three key vertices all lie in W_{t-1}, though these tritons may not all fit in cards. Our goal is to determine all shorter maximal tritons with at most two vertices in W_{t-1}.

Case 1: *$t = 4$, so $d_1 = d_3 > d_4 = d_5$.* Let $d_6^* = \min\{d_6, d_5 - 1\}$, so $d_6^* \leq d_1 - 2$. For all j and j' with $j + j' \leq (r-4)/2$, consider the tritons H and H' defined by $H = B_{j,j':d_4+1,d_4+1,d_6^*+1}$

and $H = B_{j,j':d_4+1,d_6^*+1,d_4+1}$. We first show that H and H' fit in cards. Since $d_1 > d_4$, $d_2 > d_5$, and $d_3 \geq d_6^* + 2$, (1) yields

$$n - r - 2 \geq \sum_{i=1}^{6}(d_i - 2) \geq -12 + 2(d_4 + d_5 + d_6^*) + 4.$$

For the number N of vertices, with $d_4 = d_5$, we have

$$N \leq \frac{r-4}{2} + d_4 + d_4 + d_6^* \leq \frac{r-4}{2} + \frac{n-r+6}{2} = \frac{n+2}{2}.$$

Thus H and H' fit in cards.

Each copy of H or H' has at least two key vertices in W_3, and the third key vertex has degree at least d_4, since $d_4 = d_5$. Since H and H' fit in cards, we can compute $\#H$ and $\#H'$. After excluding the copies that are contained in $B_{j,j':d_1,d_1,d_1}$, we obtain the number of copies of $B_{j,j':d_1,d_1,d_4}$ and $B_{j,j':d_1,d_4,d_1}$ that are maximal tritons. All other shorter maximal tritons are contained in H or H' and thus also fit in cards. Hence by Lemma 3.3 we can find all the shorter maximal tritons.

Case 2: $t \geq 5$. If $d_t = 2$, then we already know the positions of all branch vertices and hence know the caterpillar. Hence we may assume $d_{t+1} \geq 3$. For all j, j' with $j+j' \leq (r-4)/2$, consider the tritons H and H' given by $H = B_{j,j':d_4+1,d_4+1,d_{t+1}}$ and $H = B_{j,j':d_4+1,d_{t+1},d_4+1}$. Since $t+1 \geq 6$ and $d_1 > d_4$, (1) yields

$$n - r - 2 \geq \sum_{i=1}^{t+1}(d_i - 2) \geq 3(d_1 - 2) + (d_4 - 2) + 2(d_{t+1} - 2) \geq 3 + 4(d_4 - 2) + 2(d_{t+1} - 2),$$

which implies $4d_4 + 2d_{t+1} \leq n - r - 7$. Now,

$$|V(H)| = |V(H')| \leq \frac{r-4}{2} + 2d_4 + d_{t+1} - 1 \leq \frac{r-4}{2} + \frac{n-r+7}{2} - 1 = \frac{n+1}{2}.$$

Hence these tritons fit in cards.

As in Case 1, we know the shorter maximal tritons with key vertices in W_4'. Again we can exclude from the count of H and H' the copies that arise from such larger tritons to obtain the number of copies of H and H' that are maximal tritons. All other shorter maximal tritons are contained in H or H', and as in Case 1 we can find them. □

7 Two Maximum-degree Vertices

In this section we reconstruct high-diameter caterpillars with exactly two vertices of maximum degree and at least four branch vertices: $d_1 = d_2 > d_3 \geq d_4 \geq 3$. We recognize being in this case from the degree list. Throughout this section we consider only such caterpillars.

Lemma 7.1. *In this section we may assume that the number r of nonleaf vertices satisfies*

$$\frac{n-5}{2} \le r \le n-8, \tag{12}$$

and also we may assume $d_3 \le (n-r+1)/3$.

Proof. When $r \le (n-6)/2$ the small-diameter case in Section 2 applies, so we may assume $r \ge (n-5)/2$. Given $d_4 \ge 3$, with exactly two vertices of maximum degree we also have $d_2 \ge 4$. Thus by (1) we have $6 \le \sum_{i=1}^{4}(d_i - 2) \le n-r-2$, so $r \le n-8$.

With $d_2 > d_3$ and $d_4 \ge 3$, we also have $3d_3 + 2 + 3 - 8 \le n - r - 2$, which simplifies to $d_3 \le (n-r+1)/3$. $\square$

Let b_1 and b_2 be the indices of the two vertices of degree d_1; that is, $d(v_{b_1}) = d(v_{b_2}) = d_1$. By symmetry, we may assume that v_{b_1} is earlier along the spine and is at no higher a level than v_{b_2}; that is, $b_1 \le \min\{b_2, \overline{b}_2\}$. (Among two level pairs, the one giving degrees of vertices closer to the center of the spine is "higher" in the sense of Figure 3 in Section 6.)

Lemma 7.2. *All short maximal batons and the level pairs are determined by the deck, as are the indices b_1 and b_2 of the maximum-degree vertices.*

Proof. Since $d_1 = d_2$, by Section 4 we know the multiset of short maximal batons. This gives us the multiset of degree pairs for each distance at most $(r-1)/2$ along the spine. By Lemma 3.6, we thus also know the level pairs $\{d(v_k), d(v_{\overline{k}})\}$ for all k.

The value b_1 is the least k such that the level pair $\{d(v_k), d(v_{\overline{k}})\}$ contains d_1. When this level pair consists of two copies of d_1, we have $b_2 = \overline{b}_1$ and call this the *symmetric case.* When $b_2 \ne \overline{b}_1$, another level pair contains d_1. That pair has the degrees of two vertices, at least one of which is within distance $(r-1)/2$ of v_{b_1}. Since we know the short maximal batons, we know whether G contains $B_{j:d_1,d_1}$ for some j bounded by $(r-1)/2$, even though this subgraph may be too big to see in a card. If such a baton exists, then we know $b_2 = b_1 + j$. If no such baton exists, then $b_2 > b_1 + (r-1)/2$, and b_2 is the larger index in its level pair. $\square$

We will split the analysis into cases depending on the values of b_1 and b_2. We summarize the cases here.

Case 1: $b_2 - b_1 \le r/3 - 7/2$ (the vertices of maximum degree are close together).

Case 2: $b_1 \le r/4 - 2$ and $b_2 \ne \overline{b}_1$ (asymmetric with b_1 near the end of the spine).

Case 3: $r/4 - 2 < b_1 < r/3 + 9/4$ (asymmetric and b_1 somewhat central).

Case 4: $b_1 < r/3 + 9/4$ and $b_2 = \overline{b}_1$ (b_1 and b_2 symmetric and near the ends).

Note that $b_1 \le \overline{b}_2$ implies $b_2 \le r+1-b_1$, and hence $b_2 - b_1 \le r+1-2b_1$. Thus if $b_1 \ge r/3+9/4$, then $b_2 - b_1 \le r/3 - 7/2$ and Case 1 applies. Therefore, the upper bounds on b_1 in Cases 3 and 4 cover all possibilities. We will find that the argument for Case 3 needs $n \ge 48$.

We next describe additional information that will suffice to reconstruct the unknown caterpillar. We call this the "Climbing Lemma" because we iteratively discover the allocation of higher (that is, as in Figure 3, more central) level pairs.

Lemma 7.3 (Climbing Lemma)**.** *In the asymmetric case $b_1 < \min\{b_2, \overline{b}_2\}$, the unknown caterpillar can be reconstructed in either of the following cases:*
(1) $b_2 \le (r+1)/2$ and $(d(v_1), \ldots, d(v_{b_2}))$ is known.
(2) $b_2 > (r+1)/2$ and $(d(v_{b_2}), \ldots, d(v_r))$ is known.

Proof. The key technique is to detect vertex degrees at fixed distances from b_1 and b_2. We prove the second case in detail; the first is similar. Since we know $(d(v_{b_2}), \ldots, d(v_r))$, the level pairs also tell us $(d(v_1), \ldots, d(v_{\overline{b}_2}))$. We aim to discover $d(v_{b_2-h})$, which tells us also $d(v_{\overline{b}_2+h})$. We do this in order from $h = 1$ to $h = \lceil b_2 - (r+1)/2 \rceil$. We use maximal batons of length h, which by Section 4 we know since $h \le (r-1)/2$.

Having done this for distances from 1 through $h-1$, consider the maximal batons of the form $B_{h:a,d_1}$. These exist when a is the degree of a vertex at distance h from v_{b_1} or v_{b_2}. Over all choices for a, since there are exactly two vertices of degree d_1, there are three or four such maximal batons (only three if $h \ge b_1$). This tells us the multiset of four values in the list $(d(v_{b_1-h}), d(v_{b_1+h}), d(v_{b_2+h}), d(v_{b_2-h}))$, where we treat the first as 0 if $h \ge b_1$. Since $b_1 < \min\{b_2, \overline{b}_2\}$, we already know the first three values in the list, so from the multiset of four we obtain $d(v_{b_2-h})$. Now we also know $d(v_{\overline{b}_2+h})$ because we know the level pairs.

In the case where $b_2 \le (r+1)/2$, we just obtain $d(v_{b_2+h)})$ before $d(v_{\overline{b}_2-h})$. $\square$

We proceed to prove reconstructibility from the deck in the four cases listed after Lemma 7.2 for the position of the two maximum-degree vertices v_{b_1} and v_{b_2} (with $b_1 \le \min\{b_2, \overline{b}_2\}$).

Lemma 7.4. *If $b_2 - b_1 \le r/3 - 7/2$ (Case 1), then the caterpillar is reconstructible.*

Proof. We first obtain the list $(d(v_{b_1}), \ldots, d(v_{b_2}))$, up to reversal. The idea is that a certain non-maximal baton whose key vertices can only be v_{b_1} and v_{b_2} fits in a card with room for two more vertices. We then use the small-diameter method from Section 2.

Let $h = b_2 - b_1$. The number of vertices in $B_{h:d_3+1,d_3+1}$ is $h + 1 + 2d_3$. The baton fits in a card with room for two more vertices if $h + 3 + 2d_3 \le (n+2)/2$. From $r \ge (n-5)/2$ we obtain $n - 4r \le -2r + 5$. Using the bound on d_3 from Lemma 7.1, we compute

$$4d_3 \le n + \frac{n - 4r + 4}{3} \le n - \frac{2r}{3} + 3.$$

Now when $h \le r/3 - 7/2$, we have the desired bound:

$$2h + 6 + 4d_3 \le \frac{2r}{3} - 7 + 6 + n - \frac{2r}{3} + 3 = n + 2.$$

Hence we can see and count the desired subgraphs.

Let $H = B_{h:d_3+1,d_3+1}$, and let P denote the path in H connecting the two key vertices. Let H_i be the graph obtained from H by adding a pendant edge to the vertex at distance i along P. Let $H_{i,j}$ be the graph obtained from H by adding one pendant edge each to the vertices at distances i and j along P from one end.

We can see copies of H_i and $H_{i,j}$ in the deck, and in each copy the key vertices must be v_{b_1} and v_{b_2}. Let $M = \binom{d_1-1}{d_3}^2$. Letting $d'(v)$ denote the number of leaf neighbors of v as in Section 2, we have $\#H_i = (d'(v_{b_1+i}) + d'(v_{b_2-i}))M$ and $\#H_{i,h-i} = d'(v_{b_1+i})d'(v_{b_2-i})M$ (except $\#H_i = d'(v_{b_1+h/2})M$ when $i = h/2$). Knowing the sum and product, we can find the pairs $\{d'(v_{b_1+i}), d'(v_{b_2-i})\}$. Let x_i and x'_i denote these two values.

If $x_i = x'_i$ for $1 \le i \le h/2$, then we already know the list, which is unchanged by reversal. Otherwise, it suffices to know, for each i and j with $1 \le i < j \le \lfloor h/2 \rfloor$ such that $x_i \ne x'_i$ and $x_j \ne x'_j$, which of $\{x_i, x'_i\}$ is in the same half of the segment $(d'(v_{b_1}), \dots, d'(v_{b_2}))$ with which of $\{x_j, x'_j\}$. To determine this for a given pair (i, j), consider $\#H_{i,j}/M$. The value is either $x_i x_j + x'_i x'_j$ or $x_i x'_j + x'_i x_j$. Since $x_i \ne x'_i$ and $x_j \ne x'_j$, the value when the larger elements in each pair are multiplied is bigger, so we can determine the correct pairing. We now know the list $(d(v_{b_1}), \dots, d(v_{b_2}))$ up to reversal.

We want to extend this list outward, still only up to reversal. Having extended to $(v_{b_1-i+1}, \dots, v_{b_2+i-1})$, consider the next step. Since $i < b_1 \le (r-1)/2$ and by Section 4 we know the short maximal batons, we know the values in the set

$$\{d(v_{b_1+i}), d(v_{b_1-i}), d(v_{b_2+i}), d(v_{b_2-i})\}.$$

Since $b_1 < b_1 + i < b_2 + i$ and $b_1 - i < b_2 - i < b_2$, in this set we already know the pair $\{d(v_{b_1+i}), d(v_{b_2-i})\}$. Hence we only need to determine which value is which in the known pair $\{d(v_{b_1-i}), d(v_{b_2+i})\}$, subject to the ordering of $(d(v_{b_1-i+1}), \dots, d(v_{b_2+i-1}))$. If the two values are equal, or if the list before this point is symmetric, then there is nothing to do, since we are only determining the list up to reversal.

Let j be the least value such that $d(v_{b_1+j}) \ne d(v_{b_2-j})$. (If there is no such value of j at most $h/2$, then see the variation discussed below.) By analogy with the earlier use of $H_{i,j}$, let $F_{i,j}$ be the triton $B_{i,j:2,d_3+1,2}$. Because $j < h/2$ and $b_1 < \min\{b_2, \overline{b}_2\}$, we have $i+j < (r-1)/2$. Since also $d_3 \le (n-r+1)/3$, the triton $F_{i,j}$ fits in a card, and we can count its occurrences.

The central key vertex in a copy of $F_{i,j}$ must be v_{b_1} or v_{b_2}. Letting in general $x_t = d(v_t) - 1$, when $i \ne j$ we have

$$\#F_{i,j} = \binom{d_1-2}{d_3-1}[x_{b_1-i}x_{b_1+j} + x_{b_1+i}x_{b_1-j} + x_{b_2-i}x_{b_2+j} + x_{b_2+i}x_{b_2-j}],$$

while

$$\#F_{j,j} = \binom{d_1-2}{d_3-1}[x_{b_1-j}x_{b_1+j} + x_{b_2-j}x_{b_2+j}].$$

Since $x_{b_1+j} \neq x_{b_2-j}$, the count $\#F_{j,j}$ distinguishes between x_{b_1-j} and x_{b_2+j}. Since we also know the set $\{d(v_{b_1-i}), d(v_{b_2+i})\}$ and the ordered pair $(d(v_{b_1+i}), d(v_{b_2-i}))$ for the fixed ordering $(d(v_{b_1}), \dots, d(v_{b_2}))$, from the count $\#F_{i,j}$ we can now determine which way to allocate $\{d(v_{b_1-i}), d(v_{b_2+i})\}$.

If there is no such j, then the portion of the caterpillar between v_{b_1} and v_{b_2} is symmetric. Let j be the least index such that $d(v_{b_1-j}) \neq d(v_{b_2+j})$. Letting i be a larger index such that $d(v_{b_1-i}) \neq d(v_{b_2+i})$, we use the triton $B_{i-j,j:2,3,d_3+1}$ in a argument like that above.

When we reach $i = b_1$, there is no vertex v_{b_1-i}, and the term $x_{b_1-i}x_{b_1+j}$ in the count for $F_{i,j}$ is replaced by 0. This allows us to fix the orientation (forward or backward) for $(d(v_1), \dots, d(v_{b_2+b_1-1}))$. If $b_2 > (r+1)/2$, then the level pairs now complete the degree list, while if $b_2 \le (r+1)/2$ then the Climbing Lemma (Lemma 7.3) applies. $\square$

Lemma 7.5. *If $b_1 \le r/4 - 2$ and $b_2 \neq \overline{b}_1$ (Case 2), then the caterpillar is reconstructible.*

Proof. By Lemma 7.2, we know b_1 and b_2. By their definition, $b_2 < \overline{b}_1$. Again by Sections 4 and 3 we know the short maximal batons and the level pairs. In particular, we know the short maximal batons having a key vertex of degree d_1. Therefore, for $i \le (r-1)/2$ we know the existing values in the multiset

$$\{d(v_{b_1+i}), d(v_{b_1-i}), d(v_{b_2+i}), d(v_{b_2-i})\}. \tag{13}$$

By "existing", we mean that we drop $d(v_{b_1-i})$ from the set when $i \ge b_1$, we drop $d(v_{b_2} - i)$ when $b_2 \le i \le (r-1)/2$, and we drop $d(v_{b_2+i})$ when $b_2 > (r+1)/2$ and $i \ge \overline{b}_2$.

Let $h = \min\{\lfloor r/4 - 1 \rfloor, \overline{b}_2 - 1\}$; note that $h > b_1$. We can assume $b_2 - b_1 \ge r/3 - 3$ by not being in Case 1, so automatically $h < b_2$. If we allowed $h \ge \overline{b}_2$ (that is, $b_2 + h > r$), then we could not have a baton or triton with key vertex v_{b_2+h}.

Consider the triton H given by $H = B_{h,h:2,d_3+1,2}$. The high-degree vertex in a copy of H cannot be v_{b_1} because v_{b_1} is too close to v_1. Hence it must be must be v_{b_2}, and $\#H = \binom{d_1-2}{d_3-1}(d(v_{b_2-h}) - 1)(d(v_{b_2+h}) - 1)$. We can compute $\#H$ because H is small enough to fit in a card. Using $d_3 \le (n-r+1)/3$ and $r \le n-8$ from Lemma 7.1,

$$|V(H)| = 2h + 2 + d_3 \le \frac{r}{2} + \frac{n-r+1}{3} \le \frac{n+1}{3} + \frac{r}{6} \le \frac{n+1}{3} + \frac{n-8}{6} = \frac{n}{2} - 1. \tag{14}$$

Thus in fact we can also count subgraphs obtained from H by adding two vertices.

Let $M = \#H/\binom{d_1-2}{d_3-1}$; we use M to determine $d(v_{b_1+h})$. From (13), we know the three values in the multiset $\{d(v_{b_1+h}) - 1, d(v_{b_2-h}) - 1, d(v_{b_2+h}) - 1\}$. Taking their product and dividing by M produces $d(v_{b_1+h}) - 1$. Knowing $d(v_{b_1+h})$, we also obtain $\{d(v_{b_2+h}), d(v_{b_2-h})\}$.

Next we find $\{d(v_{b_2+i}), d(v_{b_2-i})\}$ for $1 \le i < h$. Let $x = d(v_{b_2-i}) - 1$ and $y = d(v_{b_2+i}) - 1$. Let H' be obtained from H by adding a pendant edge at one vertex at distance i along the spine from the vertex of degree d_3+1. Let H'' be obtained from H by adding a pendant edge at both such vertices. As observed above, H' and H'' both fit in cards. Now $x+y = \#H'/\#H$ and $xy = \#H''/\#H$. Knowing their sum and product, we find x and y and the desired degree pair.

Knowing these degree pairs for vertices at fixed distance from v_{b_2}, we next determine the list $(d(v_{b_2-h}), \ldots, d(v_{b_2+h}))$, up to reversal. Let i be the least positive index such that $i < h$ and $d(v_{b_2-i}) \neq d(v_{b_2+i})$ (if there is no such index, then there is nothing to do). For j with $i < j < h$ and $d(v_{b_2-j}) \neq d(v_{b_2+j})$, we need to determine which of $\{d(v_{b_2-j}), d(v_{b_2+j})\}$ to associate with which of $\{d(v_{b_2-i}), d(v_{b_2+i})\}$ as degrees of vertices at distance i and j from v_{b_2} on the same side of v_{b_2}. Consider the graph $H_{i,j}$ obtained from H by adding one pendant edge each at the two vertices at distances i and j in the same direction from the vertex of degree $d_3 + 1$. We know that $H_{i,j}$ fits in a card, and

$$\#H_{i,j}/\#H = (d(v_{b_2-i}) - 2)(d(v_{b_2-j}) - 2) + (d(v_{b_2+i}) - 2)(d(v_{b_2+j}) - 2).$$

We can determine which among the values $\{d(v_{b_2-i}), d(v_{b_2+i}), d(v_{b_2-j}), d(v_{b_2+j})\}$ belong to vertices on the same side of b_2, because $ac + bd$ differs from $ad + bc$ when $a \neq b$ and $c \neq d$. With this we have the list $(d(v_{b_2-h+1}, \ldots, d(v_{b_2+h-1}))$, up to reversal.

If $d(v_{b_2-h}) = d(v_{b_2+h})$, or if $(d(v_{b_2-h+1}, \ldots, d(v_{b_2+h-1}))$ is unchanged under reversal, then we can extend the list to include v_{b_2-h} and v_{b_2+h}. Otherwise, we consider slightly different cards. Using the same value i as in the previous paragraph, we have $x \neq x'$ and $y \neq y'$, where $\{x, x'\} = \{d(v_{b_2-h}) - 1, d(v_{b_2+h}) - 1\}$ and $\{y, y'\} = \{d(v_{b_2-i}) - 2, d(v_{b_2+i}) - 2\}$. Let H^* be the graph obtained from the triton $B_{h,h:3,d_3+1,2}$ by adding a pendant edge at the vertex having distance i from the central key vertex in the direction away from the key vertex of degree 3. Note that H^* has two more vertices than H and hence fits in a card. Also, the high-degree vertex in any copy of H^* must be v_{b_2}. Thus

$$\#H^*/\binom{d_1 - 2}{d_3 - 1} = \binom{x}{2}y'x' + xy\binom{x'}{2} = \frac{xx'}{2}[(x-1)y' + (x'-1)y].$$

Again since $ac + bd$ differs from $ad + bc$ when $a \neq b$ and $c \neq d$, this computation tells us the proper assignment of $d(v_{b_2-h})$ and $d(v_{b_2+h})$ to the ends of the list, yielding $(d(v_{b_2-h}), \ldots, d(v_{b_2+h}))$ up to reversal.

Now we can determine the exact list $d(v_1), \ldots, d(v_{b_1+h})$. From the set described in (13) for $i < b_1$, delete the pair $\{d(v_{b_2+i}), d(v_{b_2-i})\}$ to obtain the pair $\{d(v_{b_1+i}), d(v_{b_1-i})\}$. For $b_1 \leq i \leq h$, in (13) we have triples rather than 4-sets, and deleting the pair yields $d(v_{b_1+i})$ exactly. For $i < b_1$, let $H_i = B_{i,h:2,d_3+1,2}$. Since H_i is smaller than H, it fits in a card and we know $\#H_i$. In copies of H_i in G, the high-degree vertex must be v_{b_1} or v_{b_2}. Because we know $\{(d(v_{b_2-i}), d(v_{b_2+h})), (d(v_{b_2+i}), d(v_{b_2-h}))\}$, we know the number N of copies of H_i in which the high-degree vertex is v_{b_2}. After subtracting those, in the remaining copies the key vertex at distance h from v_{b_1} must be v_{b_1+h}, whose degree we know. The other key vertex must be $d(v_{b_1-i})$. Since we already know the pair $\{d(v_{b_1-i}), d(v_{b_1+i})\}$, from $\#H_i - N$ we obtain $d(v_{b_1-i})$ and $d(v_{b_1+i})$. Over all such i, we now know $d(v_1), \dots, d(v_{b_1+h})$.

We want to expand our knowledge to $d(v_1), \dots, d(v_{\lfloor (r-4)/2 \rfloor})$. For $b_1 < b_1+i \le \lfloor (r-4)/2 \rfloor$, let F_i be the triton $B_{b_1,i:2,d_3+1,2}$. Since $|V(F_i)| = b_1 + i + 2 + d_3 \le r/2 + d_3$, and we proved $r/2 + d_3 \le (n-2)/2$ in (14), subgraphs consisting of F_i plus two extra vertices fit in cards. Since there is only one nonleaf vertex at distance b_1 from v_{b_1}, the high-degree vertex in any copy of F_i must be b_2. Thus

$$\#F_i / \binom{d_1-2}{d_3-1} = (d(v_{b_2-b_1}) - 1)(d(v_{b_2+i}) - 1) + (d(v_{b_2-i}) - 1)(d(v_{b_2+b_1}) - 1). \tag{15}$$

Let F_i' be the graph obtained from F_i by adding one pendant edge to the vertex at distance b_1 from the high-degree vertex on the path of length i from that vertex. Again the high-degree vertex in a copy of F_i' in G must be b_2. Thus $\#F_i'/\binom{d_1-2}{d_3-1}$ equals

$$(d(v_{b_2-b_1})-1)(d(v_{b_2+b_1})-2)(d(v_{b_2+i})-1)+(d(v_{b_2+b_1})-1)(d(v_{b_2-b_1})-2)(d(v_{b_2-i})-1). \tag{16}$$

Summing the counts and dividing by the common factors, we have

$$\frac{\#F_i + \#F_i'}{\binom{d_1-2}{d_3-1}(d(v_{b_2-b_1}) - 1)(d(v_{b_2+b_1}) - 1)} = d(v_{b_2+i}) - 1 + d(v_{b_2-i}) - 1. \tag{17}$$

Since $b_1 < h$, we know $\{d(v_{b_2-b_1}), d(v_{b_2+b_1})\}$, and hence we obtain $x+y$ where $x = d(v_{b_2+i})-1$ and $y = d(v_{b_2-i}) - 1$. By making similar counts starting with the triton $B_{b_1,i:2,d_3+1,3}$ and a subgraph obtained by adding one vertex to it (which has two vertices more than $B_{b_1,i:2,d_3+1,2}$), we obtain $\binom{x}{2} + \binom{y}{2}$. When $x + y = a$ and $\binom{x}{2} + \binom{y}{2} = b$, we have $x^2 + y^2 = 2b + a$ and $(x + y)^2 = a^2$, so $xy = (a^2 - 2b - a)/2$. Knowing the sum and product, we obtain $\{x, y\}$.

Thus we obtain $\{d(v_{b_2+i}), d(v_{b_2-i})\}$. Deleting these two from the triple provided by (13) when $i \ge b_1$ tells us $d(v_{b_1+i})$. If $b_2 + i > r$, then the terms involving v_{b_2+i} in (15) and (16) disappear, and the ratio in (17) simplifies to $d(v_{b_2-i}) - 1$. Now (13) simplifies to the pair $\{d(v_{b_1+i}), d(v_{b_2-i})\}$, and we still obtain $d(v_{b_1+i})$.

Since we can do this for $b_1 + i \le (r-4)/2$, we now know $(d(v_1), \dots, d(v_{\lfloor (r-4)/2 \rfloor}))$. Since we know the level pairs, we also know the degrees in order for the last $\lfloor (r-1)/2 \rfloor$ vertices on the spine. Now Lemma 3.11 applies to complete the reconstruction. $\square$

We next consider Case 3. We will need algebraic lemmas. We mention only the special cases we use, forgoing more general statements. The first lemma is an instance of what are called Newton's Identities or Newton–Girard Formulae, expressing elementary symmetric functions in terms of sums of powers (see https://en.wikipedia.org/wiki/Newton%27s_identities).

Lemma 7.6. *If the sum of the kth powers of nonnegative integers x, y, z are known for $k \in \{1, 2, 3\}$, then xyz is known.*

Proof. We have already used the analogue for two variables: $x + y = \alpha$ and $x^2 + y^2 = \beta$ yields $xy = (\alpha^2 - \beta)/2$.

For three variables, let $x + y + z = \alpha$, $x^2 + y^2 + z^2 = \beta$, and $x^3 + y^3 + z^3 = \gamma$. Straightforward computation yields $6xyz = \alpha^3 - 3\alpha\beta + 2\gamma$. $\square$

Lemma 7.7. *The product $a^s b^t$ can be expressed as a linear combination of products of the form $\binom{a}{i}\binom{b}{j}$ with $i \le s$ and $j \le t$. In particular, we use the following instances:*

$$\begin{aligned} a^2b^2 &= \left[2\binom{a}{2} + a\right]\left[2\binom{b}{2} + b\right], \\ a^3b^3 &= \left[6\binom{a}{3} + 6\binom{a}{2} + a\right]\left[6\binom{b}{3} + 6\binom{b}{2} + b\right], \\ a^2b^4 &= \left[2\binom{a}{2} + a\right]\left[24\binom{b}{4} + 36\binom{b}{3} + 14\binom{b}{2} + b\right]. \end{aligned}$$

Proof. The pure powers $\{z^k\colon\ k \ge 0\}$ and the binomial coefficients $\{\binom{z}{k}\colon\ k \ge 0\}$ both form bases for the set of polynomials in z. Furthermore, within the subspace of polynomials of degree at most k, we can write z_k as a linear combination of $\{\binom{z}{j}\colon\ 0 \le j \le k\}$. The linear combination is found by successively matching the leading remaining coefficient and restricting to smaller degree. $\square$

Lemma 7.8. *If $b_1 \ge (r-7)/4$ and $b_2 \ne \overline{b}_1$ (Case 3) and $n \ge 48$, then the unknown caterpillar is reconstructible.*

Proof. By Lemma 7.4, we may assume $b_2 - b_1 > r/3 - 7/2$. From this we obtain $b_1 \le \overline{b}_2 - 1 < 2r/3 + 7/2 - b_1$, so $b_1 < r/3 + 7/4$, which yields $b_1 \le (r+5)/3$. With the lower bound on b_1, we also have $b_2 > 7r/12 - 21/4 > (r+1)/2$.

As in Case 2 (Lemma 7.5), we know the multiset of degrees of four vertices as listed in (13). Our initial aim is to find the multiset of three degrees $\{d(v_{b_1+i}), d(v_{b_2-i}), d(v_{b_2+i})\}$ in order to determine $d(v_{b_1-i})$, where i is a fixed positive integer less than b_1.

Define integers x_j and y_j as follows:

$$\begin{array}{llll} x_1 = d(v_{2b_1}) - 1 & x_2 = d(v_{b_2-b_1}) - 1 & x_3 = d(v_{b_2+b_1}) - 1 & \\ y_1 = d(v_{b_1+i}) - 2 & y_2 = d(v_{b_2-i}) - 2 & y_3 = d(v_{b_2+i}) - 2 & y_4 = d(v_{b_2-i}) - 2. \end{array}$$

Note that each x_j is positive and each y_j is nonnegative. As noted above, from the batons of length b_1 we know the multiset $\{x_1, x_2, x_3\}$, and from the batons of length i we know the multiset $\{y_1, y_2, y_3, y_4\}$. If we can determine $x_1y_1x_2y_2x_3y_3$ and it is nonzero, then we can divide it into the product of all seven numbers to obtain y_4.

We use special tritons. Let $F_{s,t} = B_{b_1-i,i:1+s,2+t,d_3+1}$ for $s,t \in \{1,2,3\}$. Note that

$$|V(F_{1,1})| = b_1 + d_3 + 3 < \frac{r+5}{3} + 3 + \frac{n-r+1}{3} \le \frac{n}{3} + 5.$$

Thus $F_{1,1}$ fits in a card when $n \ge 24$, and in general $F_{s,t}$ fits when $n \ge 6(2+s+t)$. We use instances with $s+t \le 6$, which can be counted when $n \ge 48$.

The high-degree vertex in a copy of $F_{s,t}$ in G can only be v_{b_1} or v_{b_2}. If v_{b_1} then the other key vertices are v_{b_1+i} and v_{2b_1}. If v_{b_2}, then the triton may extend to $v_{b_2+b_1}$ or to $v_{b_2-b_1}$. The parameters s and t indicate the number of leaf neighbors of two key vertices in the triton.

Let $M = \binom{d_1-1}{d_3}$. We compute

$$\#F_{s,t}/M = \binom{x_1}{s}\binom{y_1}{t} + \binom{x_2}{s}\binom{y_2}{t} + \binom{x_3}{s}\binom{y_3}{t}. \tag{18}$$

In particular, $\#F_{1,1}/M = x_1y_1 + x_2y_2 + x_3y_3$. By Lemma 7.7,

$$\Big[4\#F_{2,2} + 2\#F_{1,2} + 2\#F_{2,1} + \#F_{1,1}\Big]/M = x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2. \tag{19}$$

and $x_1^3y_1^3 + x_2^3y_2^3 + x_3^3y_3^3$ is given by

$$\Big[36(\#F_{3,3} + \#F_{2,3} + \#F_{3,2} + \#F_{2,2}) + 6(\#F_{1,3} + \#F_{3,1} + \#F_{1,2} + \#F_{2,1}) + \#F_{1,1}\Big]/M.$$

We can compute these terms because $F_{3,3}$ and its subgraphs fit in cards when $n \ge 48$.

Letting α, β, γ respectively denote the resulting values using first, second, and third powers, Lemma 7.6 yields

$$x_1y_1x_2y_2x_3y_3 = (\alpha^3 - 3\alpha\beta + 2\gamma)/6.$$

Since we know $\{x_1, x_2, x_3\}$ from the batons of length b_1, we can divide by their product to obtain the product $y_1y_2y_3$. We also know $\{y_1, y_2, y_3, y_4\}$ from the batons of length i. If $y_1y_2y_3 \ne 0$, then we can divide to obtain y_4 and thus $d(v_{b_1-i})$.

If $y_1y_2y_3 = 0$, then we cannot perform this division, and at least one factor is 0. Let p and q be the other two indices. Our counts of $F_{s,t}$ give expressions involving indices p and q, since the third term becomes 0. We obtain $x_py_p + x_qy_q$ from $\#F_{1,1}$ and $x_p^2y_p^2 + x_q^2y_q^2$ using the linear combination in (19). Now using the identity $2ab = (a+b)^2 - (a^2+b^2)$ we obtain $x_py_px_qy_q$.

We seek also $x_py_p^2x_qy_q^2$; dividing it by $x_py_px_qy_q$ yields y_py_q, and dividing this into the product of the three nonzero entries in the multiset $\{y_1, y_2, y_3, y_4\}$ gives us y_4.

To obtain $x_py_p^2x_qy_q^2$, first let $a = x_py_p^2$ and $b = x_qy_q^2$ We have $[2\#F_{1,2} + \#F_{1,1}]/M = x_py_p^2 + x_qy_q^2 = a+b$. Once we determine a^2+b^2, we again apply $2ab = (a+b)^2 - (a^2+b^2)$. To obtain a^2+b^2, we start with $\#F_{2,4}/M$ as in (18). We obtain terms involving x^2y^4 and must eliminate lower order terms. We can compute these terms because, as observed earlier, $F_{2,4}$ and its subgraphs fit in cards when $n \geq 48$. Using the expression for a^2b^4 in Lemma 7.7,

$$\Big[48\#F_{2,4}+24\#F_{1,4}+72\#F_{2,3}+36\#F_{1,3}+28\#F_{2,2}+14\#F_{1,2}+2\#F_{2,1}+\#F_{1,1}\Big]/M = x_p^2y_p^4+x_q^2y_q^4.$$

If two entries in $\{y_1, y_2, y_3\}$ are 0, then we discover this when computing $x_py_px_qy_q$. If y_p remains positive, then then $\#F_{1,1}/M = x_py_p$ and $[2\#F_{1,2} + \#F_{1,1}]/M = x_py_p^2$. Now the ratio of these two numbers is y_p, and the remaining nonzero entry in $\{y_1, y_2, y_3, y_4\}$ is y_4. If all entries in $\{y_1, y_2, y_3\}$ are 0, then $\#F_{1,1}/M = 0$ and again we obtain y_4 as $\max\{y_1, y_2, y_3, y_4\}$.

This completes the discovery of y_4 and hence $d(v_{b_2-i})$.

After performing these computations for i from 1 to b_1-1, we have the list $d(v_1), \dots, d(v_{b_1})$ in order. Next we want to extend our knowledge to determine $d(v_{b_1+i})$ for $1 \leq i \leq \overline{b}_2 - b_1$. Letting $F'_{s,t} = B_{b_1,i:1+s,d_3+1,1+t}$, we apply arguments similar to those above using $F'_{s,t}$ instead of $F_{s,t}$. This portion is simpler because the -2 in the definition of y_j is replaced by -1, and all y_j are positive. Thus the arguments to obtain y_4 for each i remain the same as the case above where all y_j are positive.

Since we know the short maximal batons, and those give us the level pairs, we now know not only $(d(v_1), \dots, d(v_{\overline{b}_2}))$ but also $(d(v_{b_2}, \dots, d(v_r))$. With this, the Climbing Lemma (Lemma 7.3) completes the reconstruction. $\square$

Lemma 7.9. *If $b_1 < r/3 + 9/4$ and $b_2 = \overline{b}_1$ (Case 4), then the caterpillar is reconstructible.*

Proof. We know the short maximal batons and the level pairs. Our first goal is to determine $(d(v_{b_1}), \dots, d(v_{b_2}))$. If $d(v_k) = d(v_{\overline{k}})$ for $b_1 < k \leq (r + 1)/2$, then there is nothing to do. Otherwise, let h be the least positive integer such that $d(v_{b_1+h}) \neq d(v_{b_2-h})$. Since $d(v_{b_1}) = d(v_{b_2}) = d_1$, we may assume by symmetry that $d(v_{b_1+h}) < d(v_{b_2-h})$.

We want to distinguish, for h' such that $d(v_{b_1+h'}) \neq d(v_{b_2-h'})$, which degree belongs to which vertex. There is no such h' between 1 and h; consider $h' > h$. For such h' with $b_1 \leq h' \leq (r - 2)/2$, let $H = B_{h,h'-h:d_3+1,d(v_{b_1+h})+1,2}$. In any copy of H, the key vertex of degree $d_3 + 1$ must be v_{b_1} or v_{b_2}. Since H has length at least b_1, the second key vertex must be v_{b_1+h} or v_{b_2-h}. Since it has degree greater than the known value $d(v_{b_1+h})$, it must be v_{b_2-h}. Hence the key vertex having degree 2 in H must be $v_{b_2-h'}$. Letting $M = \binom{d_1-1}{d_3}\binom{d(v_{b_2-h})-2}{d(v_{b_1+h})-1}$, we have $\#H = M[d(v_{b_2-h'}) - 1]$, which tells us $d(v_{b_2-h'})$.

To perform the computation, we must guarantee that H fits in a card. Note that $|V(H)| = h' + d_3 + d(v_{b_1+h}) + 1$. Since $d(v_{b_1+h}) < d(v_{b_2-h})$, we have $d(v_{b_1+h}) < d_3$. By (1), any four degrees sum to at most $n - r + 6$. Hence

$$2d_3 + 2d(v_{b_1+h}) \leq d_1 - 1 + d_2 - 1 + d_3 - 1 + d_4 \leq n - r + 3.$$

If equality holds throughout, then there are only four branch vertices and we already know the full caterpillar. Hence we can reduce the upper bound by 1. Thus $2|V(H)| \leq r+(n-r+2) =$

$n+2$, and H fits in a card. Having assigned the degrees in the level pairs from level $2b_1$ up to b_1+h' when $h' \le (r-2)/2$, we obtain all of $(d(v_{2b_1}), \ldots, d(v_{b_2-b_1}))$, since $b_1+(r-2)/2 \ge r/2$. (If r is odd and the largest h' is $(r-3)/2$ with $b_1 = 1$, then there is only one element in the remaining level $(r+1)/2$.)

To enlarge the known list to $(d(v_{b_1}), \ldots, d(v_{b_2}))$, consider $h < h' < b_1$ with $d(v_{b_1+h'}) \ne d(v_{b_2-h'})$. Let H' be the graph obtained from $B_{h,b_1-h:d_3+1,d(v_{b_1+h})+1,2}$ by adding a pendant edge at the vertex whose distance from the key vertex of degree d_3+1 is h'.

Arguing as before, copies of H' have key vertices v_{b_2}, v_{b_2-h}, and $v_{b_2-b_1}$, and the added pendant edge is at $v_{b_2-h'}$. Thus $\#H' = M[d(v_{b_2-b_1})-1][d(v_{b_2-h'})-2]$. Since we already know $d(v_{b_2-b_1})$ from the previous step, $\#H'$ determines $d(v_{b_2-h'})$. Also $|V(H')| = b_1+d_3+d(v_{b_1+h})+1$. We have the same bound on the degrees as for H, and the bound on b_1 is smaller than the earlier bound $(r-3)/2$ on h, so H' also fits in a card.

Now consider the levels below b_1. That is, we may have h' with $1 < h' < b_1$ such that $d(v_{b_1-h'}) \ne d(v_{b_2+h'})$. Let $H'' = B_{h',h:2,d_3+1,2}$. In any copy of H'', the central key vertex must be v_{b_1} or v_{b_2}, but there may be several ways for H'' to appear. Note first that $h'+h \le (r-2)/2$, by the definition of h, so

$$|V(H'')| \le \frac{r-2}{2} + d_3 + 1 \le \frac{r}{2} + \frac{n-r+1}{3} = \frac{n+1}{3} + \frac{r}{6} \le \frac{n+1}{3} + \frac{n-8}{6} = \frac{n-2}{2}.$$

Next, let $x_i = d(v_i) - 1$. When $h \ge b_1$, the path of length h from v_{b_1} or v_{b_2} in the triton must extend to higher levels. In that case

$$\#H'' = \binom{d_1-2}{d_3-1}[x_{b_1+h}x_{b_1-h'} + x_{b_2-h}x_{b_2+h'}] \tag{20}$$

Since $x_{b_1+h} \ne x_{b_2-h}$ and $x_{b_1-h'} \ne x_{b_2+h'}$, we learn which of $\{x_{b_1-h'}, x_{b_2+h'}\}$ belongs to the vertex on the same side of the caterpillar as v_{b_1+h}.

Now suppose $h < b_1$. In the special case $h' = h$, whether $x_{b_1-h'}$ equals $x_{b_2+h'}$ or not, again H'' can appear in only two ways, and the formula for $\#H''$ is again (20). Thus we determine x_{b_1-h} and x_{b_2+h}. Finally, in the general case with $h < b_1$, we have

$$\#H'' = \binom{d_1-2}{d_3-1}[x_{b_1+h}x_{b_1-h'} + x_{b_1-h}x_{b_1+h'} + x_{b_2-h}x_{b_2+h'} + x_{b_2+h}x_{b_2-h}].$$

We know everything in this formula except which of $\{x_{b_1-h'}, x_{b_2+h'}\}$ is which. Again since also $x_{b_1+h} \ne x_{b_2-h}$, matching the actual count tells us that information.

Resolving this for $1 \le h' < b_1$ completes the reconstruction in the case where the parameter h that we have defined actually exists; that is, where there is a level pair $\{d(v_{b_1+h}), d(v_{b_2-h})\}$ with $h > 0$ consisting of distinct degrees. If $(d(v_{b_1}), \ldots, d(v_{b_2}))$ is unchanged under reversal, then the argument above does not distinguish the degrees in the level pairs lower than b_1.

If no level pairs have distinct degrees, then there is nothing to do. Hence in the remaining case we can choose the least h such that $\{d(v_{b_1-h}), d(v_{b_2+h})\}$ has distinct degrees. Since all levels above that are symmetric, we may index these so that $d(v_{b_1-h}) < d(v_{b_2+h})$. Now consider any h' with $h < h' < b_1$ such that $\{d(v_{b_1-h'}), d(v_{b_2+h'})\}$ has distinct degrees. Let $F = B_{h,h'-h:d_3+1,3,2}$; note that F fits in a card. In any copy of F in G, the key vertex of high degree must be v_{b_1} or v_{b_2}. The other key vertices may occur in four ways. With $x_i = d(v_i)-1$ and $y_i = d(v_i) - 2$, we have

$$\#F = \binom{d_1-1}{d_3}[y_{b_1-h}x_{b_1-h'} + y_{b_1+h}x_{b_1+h'} + y_{b_2+h}x_{b_2+h'} + y_{b_2-h}x_{b_2-h'}].$$

The only unknown quantities in the formula are $x_{b_1-h'}$ and $x_{b_2+h'}$, though we know the unordered pair. Since $x_{b_1-h'} \neq x_{b_2+h'}$ and $y_{b_1-h} \neq y_{b_2+h}$, matching the actual count tells us which elements of the pairs belong to vertices in the same half of the caterpillar. $\square$

8 Unique Maximum-degree Vertex

In this section, we consider the case $d_1 > d_2 \geq d_4 \geq 3$ and assume these inequalities throughout the section, along with being in the high-diameter case $r \geq (n-5)/2$. We recognize these properties from the known degree list $d_1, \ldots, d_n$ in nonincreasing order. Notation is as in the preceding sections. One subcase requires a restriction to $n \geq 17$.

Lemma 8.1. *Fix q with $q \leq (r-1)/2$. If the deck determines all maximal batons in G having w_1 as a key vertex and length at most q, then it determines all maximal batons with length at most q.*

Proof. Let $\alpha = \lfloor (n-r+1)/4 \rfloor$. If $j \leq (r-1)/2$, then

$$|V(B_{j:\alpha+1,\alpha+1})| = j + 2\alpha + 1 \leq \frac{r-1}{2} + \frac{n-r+3}{2} = \frac{n+2}{2}. \tag{21}$$

Thus every short baton with maximum degree at most $\alpha+1$ fits in a card. If $d_2 \leq \alpha+1$, then we see all short batons not given by the hypothesis. Hence we may assume $d_2 > \alpha + 1$.

Case 1: $d_3 < d_2$. Here $d_1 + d_2 + d_3 + d_4 \geq 3d_3 + 6$. Thus by (1), $n - r - 2 \geq 3d_3 - 2$, so $n - r \geq 3d_3 \geq 9$. Let $\alpha' = \lfloor (n-r)/3 \rfloor$. Now when $j \leq (r-1)/2$ we have

$$|V(B_{j:\alpha'+1,3})| \leq \frac{r-1}{2} + 3 + \frac{n-r}{3} = \frac{r+5}{2} + \frac{n-r}{2} - \frac{n-r}{6} \leq \frac{n+5}{2} - \frac{3}{2} = \frac{n+2}{2}. \tag{22}$$

Thus $B_{j:\alpha'+1,3}$ fits in a card, and we can count its copies in G.

Since $\alpha' \geq d_3$, every copy of $B_{j:\alpha'+1,3}$ in G lies in a copy of $B_{j:d_1,3}$ or $B_{j:d_2,3}$, and similarly for copies of $B_{j:\alpha'+1,2}$. For $j \leq q$, by the hypothesis we know the batons having w_1 as a key vertex. Hence we know the number of copies of $B_{j:\alpha'+1,3}$ and $B_{j:\alpha'+1,2}$ contained in such batons. Let N_3 and N_2, respectively, be the number of these batons remaining (not containing w_1 as a key vertex). These are contained in maximal j-batons with w_2 as a key vertex. Since only one vertex has degree d_2, there are at most two such maximal batons.

Suppose that vertices v_i and $v_{i'}$ occur at distance j from w_2. We obtain $N_2/\binom{d_2-1}{\alpha'} = (d(v_i)-1)+(d(v_{i'})-1)$ and $N_3/\binom{d_2-1}{\alpha'} = \binom{d(v_i)-1}{2}+\binom{d(v_{i'})-1}{2}$. Hence we obtain $\{d(v_i), d(v_{i'})\}$. If there is no solution, then only one of $\{v_i, v_{i'}\}$ exists, and N_2 determines its degree.

Now we know also the maximal j-batons having w_2 as a key vertex. If $d_3 \le \alpha+1$, then all remaining short batons fit in cards, so we may assume $d_3 \ge \alpha+2$.

If also $d_4 \ge \alpha+2$, then $d_1+d_2+d_3+d_4 \ge 4(\alpha+2)+3$. By (1), we have

$$n-r-2 \ge 4\alpha+3 \ge 4(n-r-2)/4+3 = n-r+1.$$

By this contradiction, we conclude $d_4 \le \alpha+1$, and hence $d_4 < d_3$.

Now w_3 is the only vertex of degree d_3. By repeating the earlier argument with d_3 in place of d_2 and d_4+1 in place of d_3+1, we obtain the maximal j-batons whose key vertex of largest degree is w_3. Since $d_4 \le \alpha+1$, we then see all the remaining j-batons.

Case 2: $d_3 = d_2$. If $d_4 \ge \alpha+2$, then $d_1+d_2+d_3+d_4 \ge 4\alpha+9$. By (1),

$$n-r-2 \ge 4\alpha+1 \ge 4(n-r-2)/4+1 = n-r-1, \tag{23}$$

a contradiction. Thus $d_4 \le 1+\alpha < d_2 = d_3$.

If $d_4 \le \alpha$, then each copy of $B_{j:\alpha+1,\alpha+1}$ in G has both key vertices in W'. By hypothesis, we know how many of these have w_1 as a key vertex. Any remaining copy of $B_{j:\alpha+1,\alpha+1}$ must be contained in the unique maximal j-baton with key vertices of degree d_2. Therefore, when some copy of $B_{j:\alpha+1,\alpha+1}$ exists beyond those with key vertex w_1, we know that the distance between w_2 and w_3 in G is j. The unique copy of $B_{j:\alpha+1,\alpha+1}$ with key vertices $\{w_2, w_3\}$ contains 2α copies of $B_{j:\alpha+1,\alpha}$. Excluding those 2α copies (and those arising from maximal batons with key vertex w_1) allows us to find the maximal j-batons with key vertices of degrees d_2 and α. Continuing the exclusion argument allows us to find all the maximal j-batons having a key vertex of degree d_2. All remaining short batons fit in cards.

Finally, suppose $d_4 = \alpha+1$. We still have the same lower bounds on d_1, d_2, d_3 as in (23), so instead of that computation we get

$$n-r-2 \ge d_1+d_2+d_3+d_4-8 \ge 4\alpha \ge n-r-2.$$

This equality requires all of the following: $r' = 4$, $d_1 = 1+d_2 = 1+d_3 = 2+d_4 = \alpha+3$ and $\alpha = (n-r-2)/4$.

In this case

$$|V(B_{j:\alpha+2,\alpha+1})| = j+2\alpha+2 \le \frac{r-1}{2} + \frac{n-r+2}{2} = \frac{n+1}{2},$$

so these batons fit in cards. Every such baton in G not having w_1 as an key vertex is contained either in $B_{j:d_2,d_2}$ (which occurs at most once) or in a copy of $B_{j:d_2,\alpha+1}$ that is a maximal baton (since $r'=4$, there are at most two of these. Since $d_2=\alpha+2$, the instance of $B_{j:d_2,d_2}$ (if it exists) contains $2(\alpha+1)$ copies of $B_{j:\alpha+2,\alpha+1}$, and every instance of $B_{j:d_2,\alpha+1}$ contains only one copy of $B_{j:\alpha+2,\alpha+1}$ (namely, itself). Note that $2(\alpha+1)\geq 6$, since w_4 is a branch vertex. Excluding the copies that come from larger batons, the number of remaining copies of $B_{j:\alpha+2,\alpha+1}$ thus tells us whether we have a copy of $B_{j:d_2,d_2}$ as a maximal baton and how many maximal batons are copies of $B_{j:d_2,\alpha+1}$. The remaining short maximal batons fit in cards. $\square$

In light of Lemma 8.1, our first aim is to find the short maximal batons containing w_1.

Lemma 8.2. *If $d_2 \leq (n-r-3)/2$, then the unknown caterpillar is reconstructible from $\mathcal{D}$.*

Proof. Initially, the level of the maximum-degree vertex w_1 is unknown. By symmetry, we may call it h, so that $w_1 = v_h$ with $h \leq (r+1)/2$.

Fix j with $1 \leq j \leq (r-1)/2$, and let $H = B_{j:d_2+1,2}$ and $H' = B_{j:d_2+1,3}$. Since

$$\frac{r-1}{2} + \frac{n-r-3}{2} + 3 = \frac{n+2}{2},$$

the batons H' and H fit in cards. Since $d_2 \geq 3$, each copy of H or H' has w_1 as its higher-degree key vertex. The other key vertex is at distance j from w_1. Let $M = \binom{d_1-1}{d_2}$. When $j < h$, we have $\#H = x-1+y-1$ and $\#H' = \binom{x-1}{2} + \binom{y-1}{2}$. where x and y are the degrees of the two vertices at distance j from w_1. When $j \geq h$, we have $\#H = x-1$ and $\#H' = \binom{x-1}{2}$, where $x = d(v_{h+j})$. In particular, we have $\binom{\#H}{2} = \#H'$ when $j \geq h$, but $\binom{\#H}{2} > \#H'$ when $j < h$, so these cases are distinguishable.

Considering this over all such j, we find not only h (including when $h = \lceil r/2 \rceil$, by satisfying the case $j < h$ for all j up to $\lfloor (r-1)/2 \rfloor$), but also the maximal baton(s) of length j containing w_1. By Lemma 8.1, the deck determines all short maximal batons. Hence by Lemma 3.6 it also determines the level pairs $\{d(v_k), d(v_{\overline{k}})\}$ for $1 \leq k \leq r/2$. We now consider three cases depending on the value of h. Let $j_1 = \lfloor (r-1)/2 \rfloor$.

Case 1: $h \leq (r-1)/2$. As observed above, we already know $d(v_{2h}), \ldots, d(v_{h+j_1})$; note that $h+j_1 \geq (r+1)/2$. Let $H_i = B_{i,j_1-i:d_2+1,3,2}$ for $1 \leq i < h$. Note that the triton H_i has the same number of vertices as the baton H' and hence also fits in a card. Because H_i has length at least h, in any copy of it the key vertices are v_h, v_{h+i}, and v_{h+j_1}. Since we already know the degree of v_h and v_{h+j_1}, the count $\#H_i$ tells us $d(v_{h+i})$. Since we knew $\{d(v_{h+i}), d(v_{h-i})\}$ from $\#H$ and $\#H'$ when $j = i$, we now also know $d(v_{h-i})$. After doing this for $1 \leq i < h$, we know $(d(v_1), \ldots, d(v_{\lceil r/2 \rceil}))$, and the level pairs complete the reconstruction.

Case 2: $h = r/2$. Since $|V(B_{r/2:d_2+1,2})| = r/2+d_2+2 \le r/2+(n-r+1)/2 = (n+1)/2$, this baton fits in a card. Its key vertices can be only $v_{r/2}$ and v_r. The count thus tells us $d(v_r)$ and hence $d(v_1)$. Next we iterate the following from $i = 1$ to $i = r/2-1$. The count $\#B_{r/2-i:d_2+1,2}$ tells us $d(v_i) + d(v_{r-i})$ (minus 2). We already know $d(v_i)$ from the previous step, so we learn $d(v_{r-i})$ from the count and then $d(v_{i+1})$ from the level pair (since $\overline{r-i} = i+1$). Continuing, we learn the degrees of all vertices.

Case 3: $h = (r+1)/2$. If at most one level pair has distinct values, then we know the caterpillar. Let k be the highest level with unequal values and i be another such level; thus $d(v_i) \ne d(v_{\overline{i}})$ and $d(v_k) \ne d(v_{\overline{k}})$ with $i < k \le (r-1)/2$. Let $F = B_{h-k,k-i:d_2+1,3,2}$. Since

$$|V(F)| = h - i + d_2 + 3 \le \frac{r+1}{2} - 1 + \frac{n-r-3}{2} + 3 \le \frac{n+2}{2},$$

F fits in a card. The key vertices in any copy of F are $\{v_h, v_k, v_i\}$ or $\{v_h, v_{\overline{k}}, v_{\overline{i}}\}$. Thus,

$$\#F = \binom{d_1-1}{d_2}[(d(v_k)-2)(d(v_i)-1) + (d(v_{\overline{k}})-2)(d(v_{\overline{i}})-1)].$$

Again since $ac+bd$ differs from $ad+bc$ when $a \ne b$ and $c \ne d$, this computation tells us which of $\{d(v_i), d(v_{\overline{i}})\}$ in level i lies in the same half of the caterpillar with the larger of $\{d(v_k), d(v_{\overline{k}})\}$ in level k. Doing this over all levels i with unequal values completes the reconstruction. □

When d_2 is too big for Lemma 8.1, the vertices w_1 and w_2 absorb almost all the leaves, and the remaining possibilities for the other branch vertices are very restricted.

Lemma 8.3. *If $d_2 \ge (n-r-2)/2$ and $n \ge 17$, then the unknown caterpillar is reconstructible from $\mathcal{D}$.*

Proof. First consider the possible degrees of branch vertices. With $d_1 > d_2$ and $d_3 \ge d_4 \ge 3$, (1) and the lower bound on d_2 yield

$$n-r-2 \ge 1 + 2(d_2-2) + \sum_{i=3}^{r}(d_i-2) \ge 1 + 2(d_2-2) + 2 \ge n-r-3. \qquad (24)$$

In particular, $d_2 = \lfloor (n-r-1)/2 \rfloor$. Because the difference between the expressions on the ends of (24) is 1, we obtain

$$(d_1-d_2) + \sum_{i=3}^{r}(d_i-2) \le 4. \qquad (25)$$

With at least four branch vertices, we now have limited choices in satisfying (25). In each case below there are two main steps. First we find the short maximal batons where w_1

is a key vertex, and thus by Lemma 8.1 we know all the short maximal batons. We then find the shorter maximal tritons, and hence by Lemma 3.11 the caterpillar is reconstructible from $\mathcal{D}$. Let R' denote the set of branch vertices.

Case 1: $d_2 = 3$. By (25) and $d_4 \geq 3$, there are only three possibilities for the degrees of vertices in R': $(5,3,3,3)$, $(4,3,3,3)$ and $(4,3,3,3,3)$. In all these cases, $n-r-2 = \sum_{i=1}^{r}(d_i-2) \geq 5$. Hence for $j \leq (r-1)/2$ we have

$$|V(B_{j:3,3})| = j+5 \leq \frac{r-1}{2}+5 = \frac{n+2}{2} - \frac{n-r-7}{2} \leq \frac{n+2}{2}.$$

Thus $B_{j:3,3}$ fits in a card. Similarly, for $j \leq (r-3)/2$ the baton $B_{j:4,3}$ fits in a card. The vertex of degree 4 in any copy of $B_{j:4,3}$ must be w_1 (which may have degree 5 in G). Thus, $\#B_{j:4,3}$ tells us $\#B_{j:d_1,3}$ when $j \leq (r-3)/2$.

It remains to consider the one integer j in $\{(r-2)/2, (r-1)/2\}$. If $\#B_{j:3,3} \leq 2$, then $B_{j:4,3} \not\subseteq G$, since a copy of $B_{j:4,3}$ yields three copies of $B_{j:3,3}$ in the deck. If $\#B_{j:3,3} = 3$, then $\#B_{j:4,3} = 1$, since we have at most four vertices of degree 3, and forming three copies of $B_{j:3,3}$ using them would put distance $3j$ between the farthest vertices of degree 3. That requires $r \geq 1+3j \geq (3r-4)/2$, simplifying to $r \leq 4$, leaving no room on the spine for w_1.

When $\#B_{j:3,3} \geq 4$, in all cases we can determine $\#B_{j:d_1,3}$. Using an exclusion argument we also find $\#B_{j:d_1,2}$, since $B_{j:4,2}$ fits in a card.

Subcase 1a: *Vertex degrees in R' are* $(4,3,3,3)$. Consider the shorter tritons, with $j+j' \leq (r-4)/2$ and $j \leq j'$. Let $H = B_{j,j':3,3,3}$, $F = B_{j,j':4,3,3}$, $\hat{F} = B_{j,j':3,3,4}$, and $F' = B_{j,j':3,4,3}$, Here $n-r-2 = \sum_{i=1}^{r}(d_i-2) = 5$, so when $j+j' \leq (r-4)/2$ we have

$$|V(H)| = \frac{r-4}{2}+6 = \frac{n+2}{2} - \frac{n-r-6}{2} < \frac{n+2}{2},$$

and hence we know $\#H$. The only larger tritons that may contain H are F and F', and they contain three and two copies of H, respectively. Because G has only three vertices of degree 3, there can only be one copy of H not contained in F or F'. Using these observations, if $\#H = 1$, then neither F nor F' occurs in G. If $\#H = 2$, then F does not appear and F' must appear. If $\#H = 3$, then F (or $B_{j,j':3,3,4}$) appears (once) and F' does not, except in the special situation where the degrees appear in the order $(3,4,3,3)$ with distances $(j, j'-j, j)$, where one copy of F' appears. These possibilities are distinguished by $\#B_{j:4,3}$ and $\#B_{j':4,3}$.

If $\#H = 4$, then G has one copy of F or $\hat{F}$ and one copy of H that is a maximal triton. The branch degrees in order are $(4,3,3,3)$ with distances (j,j',j) or (j',j,j'), and whether G has F or $\hat{F}$ is distinguished by $\#B_{j:4,3}$.

There are two ways that $\#H = 5$ can arise. If the branch degrees in order are $(3,4,3,3)$ with distances (j,j',j), then $\#F' = 1$ and $\#\hat{F} = 1$; if the distances are (j',j,j'), then $\#F' = 1$ and $\#F = 1$. These two possibilities are distinguished by which of $B_{j:3,3}$ and $B_{j':3,3}$ occurs as a maximal baton (not using w_1).

When $j = j'$, the possibilities with $\#H \in \{3, 4, 5\}$ each reduce to one case.

It is also possible that $\#H = 6$: if the branch degrees in order are $(4, 3, 3, 3)$ with distances $(j, j' - j, j)$, then $\#F = 1$ and $\#\hat{F} = 1$, each contributing three copies of H.

These cases tell us the shorter maximal tritons that do not fit in cards (actually, only those with length $(r-4)/2$ fail to fit in cards). All shorter tritons whose key degrees are not $(4, 3, 3)$ fit in cards, so by an exclusion argument we know them.

Subcase 1b: *Vertex degrees in* R' *are* $(4, 3, 3, 3, 3)$ *or* $(5, 3, 3, 3)$. Define $F, \hat{F}, F'$ as above. Here we have $n - r - 2 = \sum_{i=1}^{r}(d_i - 2) = 6$, so

$$|V(F)| = |V(\hat{F})| = |V(F')| \le \frac{r-4}{2} + 7 = \frac{n+2}{2} - \frac{n-r-8}{2} \le \frac{n+2}{2}.$$

Hence each of these tritons fits in a card, and when any one of them appears it either is already a maximal triton (case $(4, 3, 3, 3, 3)$) or groups with a fixed number of other copies into a fixed maximal triton (case $(5, 3, 3, 3)$). Hence we obtain the shorter maximal tritons whose key vertices are w_1 and two vertices of degree 3. All other tritons are smaller and fit in cards, so by an exclusion argument we know them.

Case 2: $d_2 \ge 4$ *and* $d_3 = 3$. Since $d_2 \le (n - r - 1)/2$, for $1 \le j \le (r-1)/2$ we have

$$|V(B_{j:d_2,2})| < |V(B_{j:d_2,3})| \le j + d_2 + 3 - 1 \le \frac{r-1}{2} + \frac{n-r-1}{2} + 2 = \frac{n+2}{2}. \tag{26}$$

Thus these batons fit in cards. The key vertex of degree d_2 in a copy of $B_{j:d_2,3}$ in $\mathcal{D}$ can only be w_1 or w_2. The vertex w_2 lies in at most two copies of $B_{j:d_2,3}$ whose other key vertex has degree 3 in G. Since $d_1 > d_2 \ge 4$, the vertex w_1 belongs to either 0 or at least four such copies of $B_{j:d_2,3}$. If w_1 and w_2 are at distance j in G, then $\#B_{j:d_2,3} \ge (d_1 - 1)\binom{d_2-1}{2} + \binom{d_1-1}{2}$. Thus from $\#B_{j:d_2,3}$ we find the number of copies of $B_{j:d_1,d_2}$ and $B_{j:d_1,3}$ that are maximal batons. Using exclusion we also find $\#B_{j:d_1,2}$ by looking at $\#B_{j:d_2+1,2}$, since $B_{j:d_2+1,2}$ has the same size as $B_{j:d_2,3}$ and fits in a card.

Now that we know all the short maximal batons, we also know the level pairs, by Lemma 3.6. We consider two subcases.

Subcase 2a: $\sum_{i=1}^{r}(d_i - 2) \le 2d_2 - 1$. Since $d_1 > d_2$, the first two terms of the sum already contribute at least $2d_2 - 3$, and $d_3 \ge d_4 \ge 3$ contributes at least 2, so in this subcase the degrees of the branch vertices must be $(d_2 + 1, d_2, 3, 3)$.

Knowing the level pairs, we know the levels of the four branch vertices. Using symmetry, we may assume that $w_3 = v_b$ and $w_4 = v_{b'}$ with $b \le \min\{b', \overline{b}'\}$ (similar to the use of b_1 and b_2 in Section 7). We know b but don't know whether $v_{b'}$ is also in the first half of P.

If $b' = \overline{b}$, then knowing whether w_1 and w_2 are in the same half of P determines the caterpillar. Since we know the short maximal batons, we can test this by the presence or absence of $B_{h:d_1,d_2}$, where h is the difference between the levels of w_1 and w_2 (if $h = 0$, then we also know the caterpillar). Hence we may assume $b' < \overline{b}$.

For $i \in \{1,2\}$, let j_i be a difference between the level of w_i and the level of w_3 or w_4 (there are two choices for each i). If v_b and $v_{b'}$ are both in the first half of P, then checking for batons of the form $B_{j_1:d_1,3}$ as short maximal batons tells us whether w_1 is also in that half. We can then similarly place w_2, completing the reconstruction.

Thus we may assume $\overline{b} > b' > (r+1)/2$. Let $k = (\overline{b}' + b)/2$; note that k is the level halfway between the levels of w_3 and w_4, if they have the same parity. If w_1 or w_2 (say w_i) is not in $\{v_k, v_{\overline{k}}, v_{(r+1)/2}\}$, then again we can use the existence or absence of $B_{j_i:d_i,3}$ as a short maximal baton to locate it. We can then locate the remaining branch vertex by using the same trick with $B_{j_{3-i}:d_{3-i},3}$ or by using the presence or absence of $B_{j':d_1,d_2}$, where j' is the difference between the levels of w_1 and w_2.

The remaining possibility is $w_1, w_2 \in \{v_k, v_{\overline{k}}, v_{(r+1)/2}\}$. If $b'-k \le (r-1)/2$, then checking the existence of $B_{b'-k:d_1,3}$ and $B_{b'-k:d_2,3}$ as maximal batons in G tells us whether w_1 or w_2 is v_k (or $v_{\overline{k}}$), and then we can also place the other branch vertex (since we know the levels).

Hence we may assume $b' - k \ge r/2$. Since $b \le \min\{b', \overline{b}'\}$, this yields $k \le r/4$. Now let $B' = B_{k-b,b:d_2+1,3,2}$. We compute

$$|V(B')| \le \frac{r}{4} + d_2 + 3 \le \frac{r}{4} + \frac{n-r-1}{2} + 3 = \frac{n+2}{2} - \frac{r-6}{4}.$$

Thus B' fits in a card when $r \ge 6$, which holds when $n \ge 17$, since $r \ge (n-5)/2$. Since B' has length k, it cannot appear with key vertices v_k and v_b. Hence $w_1 = v_{\overline{k}}$ if B' appears, and $w_1 = v_k$ if it does not. This completes the reconstruction.

Subcase 2b: $\sum_{i=1}^{r}(d_i - 2) \ge 2d_2$. In this subcase we find all the shorter maximal tritons and apply Lemma 3.11. By (1), our hypothesis yields $d_2 \le (n-r-2)/2$. Also, since $d_1 > d_2 \ge 4$ and $d_3 = 3$, by (25) we have $d_6 = 2$, meaning that G has at most three vertices of degree 3. Also, if G has three vertices of degree 3, then $d_1 = d_2 + 1$.

Consider j, j'-tritons, where $j + j' \le (r-4)/2$. Let $H = B_{j,j':d_2+1,3,3}$ and $H' = B_{j,j':3,d_2+1,3}$. Each of the shorter tritons H and H' has at most N vertices, where

$$N \le \frac{r-4}{2} + d_2 + 4 = \frac{r-4}{2} + \frac{n-r-2}{2} + 4 = \frac{n+2}{2}. \tag{27}$$

Hence H and H' fit in cards. In any copy of H or H' in G, the vertex of degree d_2+1 must be w_1. We use $\#H$ and $\#H'$ to determine the shorter maximal j, j'-tritons that may have more than $(n+2)/2$ vertices; their key vertices have degrees $\{d_1, 3, 3\}$ or $\{d_1, d_2, 3\}$. We determine the larger ones first.

Let h be the distance in G between w_1 and w_2, and let h' be the difference between their levels. We have $h = h'$ if and only if G has $B_{h':d_1,d_2}$ as a maximal baton. Otherwise, $h = h' + r + 1 - 2k$, where k is the higher of the levels of w_1 and w_2. In particular, we know h.

If $h \notin \{j, j', j+j'\}$, then G contains no j, j'-triton having key vertices with degrees d_1 and d_2. If $h = j + j'$ and $B_{j,j':d_1,3,d_2} \not\subseteq G$, then since G has at most three vertices of degree

3, we have $\#H \le 2$, while $B_{j,j':d_1,3,d_2} \subseteq G$ yields $\#H \ge \binom{d_2-1}{2} \ge 3$. Thus $\#H$ determines whether $B_{j,j':d_1,3,d_2}$ is present.

Now suppose $h = j$. If $B_{j,j':d_2,d_1,3} \not\subseteq G$, then H' can only occur with key vertices having degrees $3, d_1, 3$ in G, yielding $\#H' \le \binom{d_1-2}{d_2-1}$. On the other hand, $B_{j,j':d_2,d_1,3} \subseteq G$ yields $\#H' \ge \binom{d_2-1}{2}\binom{d_1-2}{d_2-1}$. Since $d_2 \ge 4$, we conclude that $\#H'$ determines whether $B_{j,j':d_2,d_1,3}$ is present.

If $B_{j,j':d_1,d_2,3} \subseteq G$, then $\#H \ge \binom{d_1-1}{d_2}(d_2-2)$. To prove that $\#H$ determines whether $B_{j,j':d_1,d_2,3}$ is present, we show that $\#H$ is smaller when $B_{j,j':d_1,d_2,3} \not\subseteq G$. The vertex of degree d_2+1 in a copy of H must be w_1. Since w_2 has distance j from w_1, using w_2 in a copy of H would produce $B_{j,j':d_1,d_2,3}$, which is forbidden. Hence we can only produce a copy of H moving in one direction from w_1, distance j and then j' to reach two key vertices having degree 3 in G. The result is $\#H \le \binom{d_1-1}{d_2}$, which is less than the earlier lower bound since $d_2 \ge 4$.

The cases with $h = j'$ are symmetric to those with $h = j$. Thus we have determined all shorter maximal tritons whose key vertices have degrees $\{d_1, d_2, 3\}$.

Knowing h and the levels, we actually know the positions of w_1 and w_2, where we may assume $w_1 = v_b$ with $b \le (r+1)/2$. Since $d_1 \ne d_2$, there is a j,j-triton with initial key vertices of degrees d_1 and then d_2 if and only if the distances allow it to fit along the spine. The third key vertex will have degree 2 if and only if it does not have degree 3. Since we know the shorter maximal tritons whose key vertices have degrees $\{d_1, d_2, 3\}$, we know this answer. The same argument applies to determine all shorter maximal tritons whose key vertices have degrees $\{d_1, d_2, 2\}$.

The remaining tritons that don't fit in cards have key vertices of degrees $\{d_1, 3, 3\}$. Furthermore, by (27), these fail to fit in cards only if the degrees of the branch vertices are $\{d_2+2, d_2, 3, 3\}$. Since in this case there are only two branch vertices of degree 3, there can only be one copy of $B_{j,j':d_1,3,3}$ or $B_{j,j':3,d_1,3}$ as a maximal triton, and there cannot also be a j,j'-triton with key vertices of degrees $\{d_1, d_2, 3\}$. After checking that we have no j,j'-triton with degrees $\{d_1, d_2, 3\}$, we have $B_{j,j':d_1,3,3}$ or $B_{j,j':3,d_1,3}$ in G if and only if $\#H = \binom{d_1-1}{d_2} = d_2+1$ or $\#H' = \binom{d_1-2}{d_2-1} = d_2$.

All remaining shorter tritons fit in cards. Hence the exclusion argument allows us to determine the remaining shorter maximal tritons.

Case 3: $d_2 > d_3 \ge 4$. By (25) and $d_4 \ge 3$, there are four branch vertices, with degrees $(d_2+1, d_2, 4, 3)$, all distinct. Using (1), $d_2 = (n-r-2)/2$. By the same computation as (26) in Case 2 (improved by 1/2), the batons $B_{j:d_2,2}$ and $B_{j:d_2,3}$ fit in cards when $j \le (r-1)/2$.

Since $d_1 = d_2+1$, the number of copies of $B_{j:d_2,3}$ whose key vertices are branch vertices at distance j is $d_2\binom{d_2-1}{2} + \binom{d_2}{2}$ if the branch vertices are $\{w_1, w_2\}$; this count simplifies to $\binom{d_2}{2}(d_2-1)$, which is at least 40 since $d_2 \ge 5$. The count is $3d_2$ for $\{w_1, w_3\}$, is d_2 for $\{w_1, w_4\}$, is 3 for $\{w_2, w_3\}$, and is 1 for $\{w_2, w_4\}$. These five numbers are distinct, and indeed no two subsets of them have the same sum. Hence we know which short maximal batons with key vertices of degree at least 3 are present.

All other short batons fit in cards (note that $B_{j:d_1,2} = B_{j:d_2+1,2}$, and $B_{j:d_2+1,2}$ has the same size as $B_{j:d_2,3}$, which fits in a card). Hence by the exclusion argument we know all short maximal batons.

To complete the reconstruction, since $w_1, \ldots, w_4$ have distinct degrees and we know their levels, we simply use short maximal batons to test for each of w_2, w_3, w_4 whether it is in the same half of the spine as w_1.

Case 4: $d_2 = d_3 \geq 4$. By (25) and $d_4 \geq 3$, the degrees of the branch vertices are $(5, 4, 4, 3)$. Thus $n - r - 2 = \sum_{i=1}^{r}(d_i - 2) = 8$. For $j \leq (r-1)/2$, we have

$$|V(B_{j:4,2})| < |V(B_{j:4,3})| = j + 6 \leq \frac{r-1}{2} + 6 = \frac{n+2}{2} - \frac{n-r-9}{2} = \frac{n+1}{2}.$$

Thus $B_{j:4,2}$ and $B_{j:4,3}$ fit in cards.

To find the short maximal batons that don't fit in cards, consider $\#B_{j:4,3}$. It is $4\binom{3}{2} + \binom{4}{2}$ (that is, 18) for batons with key vertices $\{w_1, w_2\}$ or $\{w_1, w_3\}$, is 4 for $\{w_1, w_4\}$, is $3 + 3$ for $\{w_2, w_3\}$, and is 1 for $\{w_2, w_4\}$ or $\{w_3, w_4\}$. Distinct subsets of these larger batons give distinct values of $\#B_{j:4,3}$, except for possibly $4+1+1 = 6$. However, the values $4, 1, 1$ cannot all occur, because they require w_4 to be at distance j from all of $\{w_1, w_2, w_3\}$. Hence the numbers $\#B_{j:4,3}$ determine which batons occur with both key vertices of degree at least 3.

All other short batons fit in cards ($B_{j:5,2}$ has the same size as $B_{j:4,3}$). Hence we know the short maximal batons (using exclusion) and also the level pairs. We now locate the branch vertices as in Subcase 2a, with the roles of w_2 and w_4 exchanged. That is, we first place w_3 and w_2, with $w_3 = v_b$ and $w_2 = v_{b'}$, where $b \leq \min\{b', \overline{b}'\}$. Using batons and tritons as in Subcase 2a, we then place w_1 and w_4 relative to w_3 and w_2. □

Lemmas 8.1 and 8.3 complete the proof of the theorem for all high-diameter cases with a unique vertex of maximum degree and at least four branch vertices.

9 At Most Three Branch Vertices

Here we reconstruct caterpillars with few branch vertices. Trees with no branch vertices are paths, which are reconstructible from the degree list. The next case is also fairly easy.

Lemma 9.1. *When $n \geq 2\ell + 1$, caterpillars with one branch vertex are ℓ-reconstructible.*

Proof. We know the degree list, so we recognize this case, with a branch vertex having degree d_1 and $r = n - d_1$. Since $r \geq (n-5)/2$ in the high-diameter case, we may assume $d_1 \leq (n+5)/2$. We seek i such that $d(v_i) = d_1$. We may assume $i \leq (r+1)/2$. We have $i = 1$ if and only if the deck does not contain the 6-vertex caterpillar obtained by adding a pendant edge at the central vertex of the path P_5.

Hence we may assume $1 < i \le (r+1)/2$. We count copies in the deck of the path P_j with j vertices. There are $r - j + 1$ copies of P_j contained in the spine $\langle v_1, \ldots, v_r \rangle$. Excluding those, there remain $d_1 - 2$ copies of P_j if $j > i+2$ (through indices greater than i), and there are $2(d_1 - 2)$ copies of P_j if $j \le i+2$. Thus, as j increases from 1, the value of i is the value of $j - 2$ when number of copies of P_j not along the spine decreases.

We must check that P_j fits in a card for the needed values of j. If we can see P_j when $j \le (r+5)/2$,then we can detect the branch vertex v_i for $i \le (r-1)/2$ in this way. Indeed we have $(r+5)/2 \le (n+2)/2$, because $r \le n - 3$, which holds since $d_1 \ge 3$.

If we do not find the branch vertex in this way, then $i = \lceil r/2 \rceil$. $\square$

Lemma 9.2. *When $n \ge 2\ell + 1$, caterpillars with two branch vertices are ℓ-reconstructible.*

Proof. Let v_b and $v_{b'}$ be the two branch vertices along $v_1, \ldots, v_r$. By symmetry, we may assume $b \le \min\{b', \overline{b}'\}$. Let $h = b' - b$. If $j \le (n-8)/2$, then $|V(B_{j:3,3})| \le (n+2)/2$, and we see whether G contains $B_{j:3,3}$. If it does, then $h = j$. Hence we begin by determining h if $h \le (n-8)/2$, and otherwise we know $h \ge (n-7)/2$. Note also that $b - 1 \le (r - 1 - h)/2$; this means that when we don't know h we know that it is greater than b.

The spine is the path $\langle v_1, \ldots, v_r \rangle$ of nonleaf vertices. Specifying v_0 and v_{r+1} as arbitary leaf neighbors gives us the *augmented spine* $\langle v_0, \ldots, v_{r+1} \rangle$. With this definition, any branch vertex with degree d has $d - 2$ leaf neighbors outside the augmented spine. Let Q_j be the path with $j + 3$ vertices; note that $Q_j = B_{j:2,2}$ when $j \ge 1$. There are always $r - j$ copies of Q_j contained in the augmented spine. Other copies of Q_j start at leaf neighbors of v_b or $v_{b'}$ and continue in either direction along the spine. The maximum value of $\#Q_j$ is achieved when the augmented spine is long enough in both directions from both branch vertices to fit Q_j. For $1 < j \ne h$, that value is $r - j + 2d_1 + 2d_2 - 8$. When $j = h$, there is another contribution where both endpoints are leaves outside the augmented spine, and the value is $r - j + 2d_1 + 2d_2 - 8 + (d_1 - 2)(d_2 - 2)$. In both cases, let g_j denote this greatest value.

We claim that $\#Q_j = g_j$ when $j < b$. In particular, we know whether j equals h when $j \le b$, and in that case the contribution of $(d_1 - 2)(d_2 - 2)$ appears in both $\#Q_j$ and g_j. However, $\#Q_b < g_b$, since the distance from v_b to v_1 is less than b. Thus we obtain b as the least value of j such that $\#Q_j < g_j$. In addition, $g_b - \#Q_b$ tells us whether the degree of v_b is d_1 or d_2; if the difference is $d_i - 2$, then the degree is d_i. There is also the possibility that $g_b - \#Q_b = d_1 + d_2 - 4$; this happens if and only if $\overline{b} = b'$, which determines G.

Having determined v_b and its degree, we again consider h. Recall that we have determined h if it is at most $(n-8)/2$. In that case, we set $b' = b + h$ and know all of G.

In the remaining case, we have $h \ge (n-7)/2$. If $b + (n-7)/2 \ge (r+1)/2$, then we need only find the level of b' to complete the reconstruction of G. Let $a = g_b - \#Q_b$. The deficiency of a continues as we increase j beyond b. Let k be the least value of j greater than b such that $\#Q_j < g_j - a$; note that $k < b + h$. If $b + (n-7)/2 \ge (r+1)/2$, then we set

$b' = \overline{k}$ and have determined G. In order to be guaranteed seeing such Q_j, it suffices for Q_j to fit in a card when $j \leq (r-1)/2$, since if we don't find k by then k is the only remaining level, $r/2$ or $(r+1)/2$. Thus it suffices to have $(r+5)/2 \leq (n+2)/2$, which simplifies to $n - r \geq 3$, which holds when G has two branch vertices.

When $b+(n-7)/2 < (r+1)/2$, we still need to find $b' \in \{k, \overline{k}\}$, but very few possibilities remain. Since $b \geq 1$, the condition requires $r \geq n-5$. By (1), $n - r = d_1 + d_2 - 2$. With two branch vertices, $n - r \geq 4$, so $r \leq n-4$.

Let $n - r = 4 + \varepsilon$, where $\varepsilon \in \{0, 1\}$. We have $(d_1, d_2) = (3+\varepsilon, 3)$. We may assume $(b, k) = (1, (r-1+\varepsilon)/2)$, since $(n-7)/2 = (r-3+\varepsilon)/2$ and $b + h \geq (r-1+\varepsilon)/2$. We count copies of Q_j, where $j = (r-3+\varepsilon)/2$, which we have observed fit in a card. Let $D = \#Q_j - (r-j)$ in order to ignore the $r-j$ copies contained in the augmented spine in all cases. When $d_1 = 3$ we have $D = 5$ if $b' = v_k$ and $D = 4$ if $b' = v_{\overline{k}}$. When $d_1 = 4$ we have $D \in \{7, 8\}$ if $b' = v_k$ and $D \in \{5, 6\}$ if $b' = v_{\overline{k}}$, depending on whether v_1 has degree 4 or 3. Hence we determine b'. □

The case when G has exactly three branch vertices takes quite a bit more work.

Lemma 9.3. *For $n \geq 19$, let G be an n-vertex caterpillar having exactly three branch vertices. If the levels of the branch vertices are known and the sum of the degrees of the branch vertices is* 10, *then G is reconstructible from the deck. The same holds when those degrees sum to* 9 *if r is odd or $\#B_{(r-2)/2:3,3}$ is known.*

Proof. Let R' the set of branch vertices. By (1), $n - r = 2 + \sum_{i=1}^{3}(d_i - 2) \geq 5$. Furthermore, $n - r \geq 6$ unless the degrees in R' all equal 3; call that the "3-case". Since the inequalities $r/2 + 4 \leq (n+2)/2$ and $(r-2)/2 + 5 \leq (n+2)/2$ reduce to $n - r \geq 6$, except in the 3-case we have

$$B_{j:3,2} \text{ fits in a card when } j \leq r/2, \tag{28}$$

and

$$B_{j:3,3} \text{ fits in a card when } j \leq (r-2)/2. \tag{29}$$

In the 3-case the requirements are $j \leq (r-1)/2$ for $B_{j:3,2}$ and $j \leq (r-3)/2$ for $B_{j:3,3}$, but in that case we are given $\#B_{(r-2)/2:3,3}$, so we know that quantity in all cases.

Let $s = (r+1)/2$. Let the *depth* $f(v)$ of a vertex v in P be the distance of v from the middle of P. That is, the level k of v_i is $\min\{i, \overline{i}\}$, and $f(v_i) = s - k$; note that $f(v_i)$ is a half-integer when r is even. By hypothesis, we are given the depths of all three branch vertices. Let an R'-*baton* be a maximal baton whose key vertices are both branch vertices.

An easy case is $v_s \in R'$. Let b be the difference between the levels of the other two branch vertices. Since we know all the depths, we know the R'-batons having v_s as a key vertex. Excluding those, we check whether $B_{b:3,3}$ occurs to determine whether the other two branch vertices are in the same half of P, completing the reconstruction. Since the other branch vertices are not v_s, we have $b \le (r-3)/2$, so $B_{b:3,3}$ fits in a card, by (29).

Another easy case is when two branch vertices have the same depth, with degrees d and d', and the third branch vertex (with degree d^*) has some positive depth. When $d = d'$ we already know G, but when they differ we must determine which half of the spine contains the third vertex. Letting b be the difference between the two depths, we simply check whether $\#B_{b:3,3}$ is $(d-1)(d^*-1)$ or $(d'-1)(d^*-1)$. We can do this when $b \le (r-3)/2$ by the computation above. It is also possible that $b = (r-2)/2$ and $|V(B_{b:3,3})| = (n+3)/2$, but this occurs only in the 3-case, and then $d = d'$.

Hence we may assume that the vertices in R' have distinct depths, all positive. Analogously to Definition 6.1, rename the vertices of R' as u_1, u_2, u_3 in order of depth, with

$$f(u_1) = a < f(u_2) = a + b < f(u_3) = a + b + c$$

(see Figure 3). Let $h = s - a - b - c$, so $u_3 \in \{v_h, v_{\overline{h}}\}$.

As in Lemma 6.2, say that R' is *Type* 0 if all of R' is in the same half of P and *Type* i for $i \in \{1, 2, 3\}$ if u_i is in the half opposite $R' - \{u_i\}$. There must be at least two vertices in the same half of the spine. Since $a \ge 1/2$, any R'-baton with key vertices in the same half of the spine has length at most $(r-2)/2$. By (29), we can discover the lengths of such R'-batons by checking for $B_{j:3,3}$, or in the 3-case we are given whether $B_{(r-2)/2:3,3}$ occurs. These lengths lie in $\{b, c, b+c\}$.

We consider subcases by the number of R'-batons with length at most $(r-2)/2$. For $j \le (r-2)/2$, we know $\#B_{j:3,3}$ by (29) or the information given in the 3-case. In the 3-case, all copies of $B_{j:3,3}$ are R'-batons. In the case where the branch vertices have degrees $(4, 3, 3)$, we see $B_{j:4,3}$ when it occurs if $j \le (r-4)/2$, and $B_{(r-2)/2:4,3}$ occurs if and only if $\#B_{(r-2)/2:3,3} \ge 3$ (and there are two copies if $\#B_{(r-2)/2:3,3} = 6$).

Case 1: *Only one R'-baton, say B.* Since R' has two vertices in the same half of P, the length of B is b, c, or $b+c$, and R' is respectively Type 3, 1, or 2 if those lengths are distinct. If $b = c$ and B has length b, then we still must distinguish Type 3 and Type 1. In either case, u_1 and u_3 are in different halves of P; we may assume $u_1 = v_{s+a}$ and $u_3 = v_h$. Now Type 3 means $u_2 = v_{s+a+b}$ and Type 1 means $u_2 = v_{s-a-b}$.

If $b + 2a \le (r-2)/2$, then since we do not also have an R'-baton of length $b + 2a$, the arrangement is Type 3. (Note: $c = 2a$ is possible, but the two vertices of R' in the same half already give an R'-baton with length $b + c$, and another with key vertices u_2 and u_1 of length $b + 2a$ would count as a second R'-baton.)

Hence we may assume $b + 2a \ge (r-1)/2$. Now we distinguish Type 1 from Type 3 by counting copies of $B_{c,h:3,3,2}$. When $u_2 = v_{s+a+b}$, the triton can extend from v_{s+a} through increasing indices or from v_{s+a+b} through decreasing indices. When $u_2 = v_{s-a-b}$, the triton occurs only by extending from v_h through increasing indices, since the path through decreasing indices is not long enough. Thus in the 3-case there are two copies when $u_2 = v_{s+a+b}$ and only one when $u_2 = v_{s-a-b}$. In the other case we are given the level of the vertex of degree 4. We have the following counts to distinguish Type 1 from Type 3.

	$d(u_1) = 4$	$d(u_2) = 4$	$d(u_3) = 4$
$\#B_{c,h:3,3,2}$ when $u_2 = v_{s+a+b}$	5	5	2
$\#B_{c,h:3,3,2}$ when $u_2 = v_{s-a-b}$	1	2	3

Thus it suffices to show that $B_{c,h:3,3,2}$ fits in a card. From $b + 2a \geq (r-1)/2$ we obtain

$$2(h-1) + 2c + b = r - 1 - b - 2a \leq (r-1)/2$$

and so $h + c \leq (r+1)/4$ (since $b \geq 1$). Using $n - r \geq 5$, we have

$$|V(B_{c,h:3,3,2})| = c + h + 8 - 3 \leq \frac{r+1}{4} + 5 = \frac{r+11}{4} + \frac{5}{2} \leq \frac{r+11}{4} + \frac{n-r}{2} = \frac{n+2}{2} + \frac{7}{4} - \frac{r}{4}.$$

With $r \geq 7$, which happens when $n \geq 19$ (since $r \geq (n-5)/2$ in the high-diameter case), $B_{c,h:3,3,2}$ fits in a card.

Case 2: *Two R'-batons.* Having only two R'-batons means that the two extreme vertices of R' are too far apart to see their R'-baton in a card. Hence R' is not Type 0. If R' is Type 1, then the lengths of the R'-batons we see are $\{2a + b, c\}$. If Type 2, then they are $\{2a+b, b+c\}$. If Type 3, then they are $\{2a+b+c, b\}$. Since the pairs are distinct, we know G.

Case 3: *Three R'-batons.* If R' is Type 0, then the triple of lengths of the batons is $\{b, c, b+c\}$. If Type 1, then it is $\{c, 2a+b, 2a+b+c\}$. If Type 2, then it is $\{b+c, 2a+b, 2a+2b+c\}$. If Type 3, then it is $\{b, 2a+b+c, 2a+2b+c\}$. Since these triples are distinct, we know G. $\square$

We now consider cases by the sum of the degrees of the three branch vertices. When the sum is at most 10, we apply Lemma 9.3 and hence use the threshold for n assumed there.

Lemma 9.4. *For $n \geq 19$, if the unknown caterpillar G has exactly three branch vertices and their degrees sum to 9, then G is reconstructible from the deck.*

Proof. Let R' denote the set of branch vertices in G; we have $R' = \{w_1, w_2, w_3\}$, with $d(w_i) = 3$. By (1), $\sum_{i=1}^{3} d_i = 9$ implies $n - r = 5$, so $r = n - 5$. Hence

$$B_{j:3,2} \textit{ fits in a card when } j \leq (r-1)/2, \tag{30}$$

and

$$B_{j:3,3} \textit{ fits in a card when } j \leq (r-3)/2. \tag{31}$$

Let $s = (r+1)/2$. Let the *depth* $f(v)$ of a vertex v in P be the distance of v from v_s (depth is $(r+1)/2$ minus level and is a half-integer when r is even). By Lemma 9.3, it suffices to determine the levels of the branch vertices and, if r is even, $\#B_{(r-2)/2:3,3}$.

Let $t = \lfloor (r-1)/2 \rfloor$. The only short baton not fitting in a card is $B_{t:3,3}$. Every such baton contains four copies of $B_{t:3,2}$, which we see in cards. We obtain $\#B_{t:3,3}$ from $\#B_{t:3,2}$ as follows:

$\#B_{t:3,2}$	8	7	6	5	4	3
$\#B_{t:3,3}$	2	1	1	1|0	0	0

A branch vertex with depth at most $1/2$ can belong to copies of $B_{t:3,2}$ extending in both directions along P; branch vertices in other positions extend in one direction. Thus when $B_{t:3,3}$ appears, the third branch vertex contributes at least one more copy of $B_{t:3,2}$. Hence when $\#B_{t:3,3} = 1$ we have $\#B_{t:3,2} \geq 5$. We have $\#B_{t:3,3} = 2$ if and only if copies of $B_{t:3,3}$ extend in both directions from a branch vertex with depth at most $1/2$, and then $\#B_{t:3,2} = 8$.

The only possible confusion is when we see $\#B_{t:3,2} = 5$; we may also have $\#B_{t:3,2} = 0$. This requires two branch vertices that extend to copies of $B_{t:3,2}$ in both directions, which occurs only for branch vertices with depth at most $1/2$. When r is odd the only such vertex is $v_{(r+1)/2}$. Hence $\#B_{t:3,3} = 5$ with $\#B_{t:3,3} = 0$ occurs only when r is even and $v_{r/2}$ and $v_{r/2+1}$ are both branch vertices; call this Type 0. Meanwhile, if $\#B_{t:3,3} = 1$ and there is a branch vertex in $\{v_{r/2}, v_{r/2+1}\}$, then $\#B_{t:3,2} \geq 6$. Therefore, $\#B_{t:3,3} = 5$ with $\#B_{t:3,3} = 1$ when r is even can only occur when $\{v_{r/2}, v_{r/2+1}\}$ contains no branch vertex; call this Type 1. We distinguish between Type 0 and Type 1 by computing $\#B_{t:2,2}$. Note $|V(B_{t:2,2})| = t+3$. Designating leaf neighbors v_0 and v_{r+1} of v_1 and v_r to augment the spine, both types have $r-t$ copies of $B_{t:2,2}$ contained in the augmented spine. A Type 0 caterpillar has five additional copies of $B_{t:2,2}$, while a Type 1 caterpillar has only four additional copies.

Hence we know all short maximal batons. By Lemma 3.6, this yields the level pairs $\{d(v_k), d(v_{\overline{k}})\}$ for $1 \leq k \leq (r-1)/2$. The remaining degrees belong to the remaining level, $\{d(v_s)\}$ or $\{d(v_{r/2}), d(v_{r/2+1})\}$. As noted earlier, Lemma 9.3 applies to complete the reconstruction. □

Lemma 9.5. *For $n \geq 19$, if the unknown caterpillar G has exactly three branch vertices and their degrees sum to 10, then G is reconstructible from the deck.*

Proof. The set R' of branch vertices is $\{w_1, w_2, w_3\}$, with degrees $4, 3, 3$ in order. By (1), $\sum_{i=1}^{3} d_i = 10$ implies $n - r = 6$; in particular, $r = n - 6$. Hence

$$B_{j:3,2} \textit{ fits in a card when } j \leq r/2, \tag{32}$$

and

$$B_{j:3,3} \textit{ fits in a card when } j \leq (r-2)/2. \tag{33}$$

Since $d_1 = 4$, there are one (or two) copies of $B_{j:4,3}$ if and only if G has at least three (or six) copies of $B_{j:3,3}$, respectively, and the number of copies of $B_{j:3,3}$ that are maximal batons is at most 2. Thus $\#B_{j:3,3}$ determines $\#B_{j:4,3}$. By Lemma 9.3, it suffices to determine the levels of the branch vertices.

Using (33) and exclusion, we obtain all maximal batons with length at most $(r-2)/2$. By Lemma 3.6, this yields the level pairs $\{d(v_k), d(v_{\overline{k}})\}$ for $1 \le k \le (r-2)/2$.

When r is even, the only remaining level is $\{v_{r/2}, v_{(r+2)/2}\}$, and its vertices have the remaining degrees. When r is odd, the remaining set is $\{v_{(r-1)/2}, v_{(r+1)/2}, v_{(r+3)/2}\}$; call this I. We know $|R' \cap I|$ by knowing the level pairs outside I. If $|R' \cap I| = 3$, then we learn the position of w_1 within I by whether $\#B_{1:4,3}$ is 1 or 2, and this determines G. If $|R' \cap I| = 0$, then all vertices in I have degree 2.

Let $s = (r+1)/2$. When $|R' \cap I| \in \{1, 2\}$, knowing $d(v_s)$ determines the remaining level pair $\{d(v_{s-1}), d(v_{s+1})\}$, since we know the level pairs outside I. By (32), $B_{s-1:3,2}$ fits in a card; we use $\#B_{s-1:3,2}$ to determine $d(v_s)$. The nonzero contributions to $\#B_{s-1:3,2}$ from key vertices separated by distance $s-1$ are 9, 4, 3, or 1 when the degrees of the key vertices are $(4,3)$, $(3,3)$, $(4,2)$, or $(3,2)$, respectively. Vertex v_s may generate such batons in both directions, but any other vertex can be a key vertex in a copy of $B_{s-1:3,2}$ extending in only one direction along the spine.

Let k and k' with $k \le k'$ be the two least levels of branch vertices (if $|R' \cap I| = 2$, then $k' = s-1$). The columns below are distinct cases, mostly since we know k and k'. The second column occurs when the distance between two branch vertices not including v_s is $s-1$; in the first column that does not occur. Those two cases are distinguished by $\#B_{s-1:3,2}$.

$k > 1$	usual	$k > 1$	extra	$k' > k = 1$		$k' = k = 1$	
$\#B_{s-1:3,2}$	$d(v_s)$	$\#B_{s-1:3,2}$	$d(v_s)$	$\#B_{s-1:3,2}$	$d(v_s)$	$\#B_{s-1:3,2}$	$d(v_s)$
8	4	10	4	13	4	18	4
6	3	11	3	11 or 8	3	13	3
5	2	7 or 10	2	10 or 7 or 5	2	5	2

The "or" choices depend on which branch vertex other than v_s has degree 4. The only ambiguity is in column 2 with $k > 1$ and $\#B_{s-1:3,2} = 10$. If $d(v_s) = 4$, then the branch vertices at levels k and k' both have degree 3; in the other case their degrees are $\{4, 3\}$. Since we know the level pairs except in I, ambiguity thus requires $k = 2$ and $k' = s-1$, with $d(v_2) = 3$ and $v_{s+1} \in R'$ (by symmetry). Since $|R' \cap I|$ is known, the third branch vertex is v_s or v_{s-1}. Now we distinguish the two cases by which of $B_{1:4,3}$ and $B_{2:4,3}$ occurs. $\square$

We are now left with the case where the unknown caterpillar has exactly three branch vertices and their degrees sum to at least 11. By (1), this yields $n - r = 2 + \sum_{i=1}^{3}(d_i - 2) \ge 7$, so $r \le n - 7$. Thus $(r-1)/2 \le (n-8)/2$. Again let R' denote the set of branch vertices.

Definition 9.6. Let u_1, u_2, u_3 be the indices of the branch vertices (not the vertices themselves) along the spine, where $1 \le u_1 < u_2 < u_3 \le r$ and the ordering $v_1, \ldots, v_r$ of the spine is chosen so that $u_2 - u_1 \le u_3 - u_2$. For $i, j \in \{1, 2, 3\}$, let $\delta_{i,j} = |u_i - u_j|$.

Being able to see R'-batons in the deck depends to a large extent on their lengths, which are the distances between the branch vertices. We distinguish three "distance cases":
Case 1. $\delta_{1,2}, \delta_{2,3}, \delta_{1,3} \le (r-1)/2$.
Case 2. $\delta_{1,2}, \delta_{2,3} \le (r-1)/2$ and $\delta_{1,3} > (r-1)/2$.
Case 3. $\delta_{1,2} \le (r-1)/2$ and $\delta_{2,3}, \delta_{1,3} > (r-1)/2$.

We first prove that the distance case of the unknown caterpillar is recognizable from the deck. We then use the properties of each case to determine the exact indices and complete the reconstruction.

Lemma 9.7. *In the situation where the unknown caterpillar G has exactly three branch vertices and their degree sum to at least 11, the deck determines the distance case of G, as defined in Definition 9.6.*

Proof. We will use the counts $\#B_{j:3,3}$ for $1 \le j \le (r-1)/2$. Since $|V(B_{j,3,3})| = j + 5 \le (r-1)/2 + 5 \le (n+2)/2$, all such batons fit in cards, and we can count them. Note that the key vertices in a copy of $B_{j:3,3}$ must be branch vertices.

For simplicity of notation in the computations, let $y_s = d(u_s) - 1$ and $x_s = \binom{y_s}{2}$ for $s \in \{1, 2, 3\}$. For various lengths of the batons, the exact counts are as follows:

$$\#B_{j:3,3} = \begin{cases} 0 & j \notin \{\delta_{1,2}, \delta_{2,3}, \delta_{1,3}\} \\ x_1x_2 & j = \delta_{1,2} \neq \delta_{2,3} \\ x_2x_3 & j = \delta_{2,3} \neq \delta_{1,2} \\ x_1x_2 + x_2x_3 & j = \delta_{1,2} = \delta_{2,3} \\ x_1x_3 & j = \delta_{1,3} \end{cases}$$

Consider the number of distinct values of j such that $j \le (r-1)/2$ and $\#B_{j:3,3} \neq 0$. There is at least one, since there must be two branch vertices in one half of the spine. If there are three such values, say j_1, j_2, j_3 with $j_1 < j_2 < j_3$, then $j_1 = \delta_{1,2}$, $j_2 = \delta_{2,3}$, and $j_3 = j_1 + j_2 = \delta_{1,3}$. Thus Case 1 holds.

Next, suppose that there are two such values, j_1 and j_2 with $j_1 < j_2$. If $j_2 \neq 2j_1$, then we must have $j_1 = \delta_{1,2}$, $j_2 = \delta_{2,3}$, and $\delta_{1,3} > (r-1)/2$. Thus Case 2 holds.

If $j_2 = 2j_1$, then we must look closer. Two types of caterpillars produce such counts: we may have $\delta_{1,2} = \delta_{2,3} = j_1$ and $\delta_{1,3} = j_2$ (Case 1) or $\delta_{1,2} = j_1$, $\delta_{2,3} = 2j_1$ and $\delta_{1,3} > (r-1)/2$ (Case 2). We distinguish these possibilities by comparing the counts. In the Case 1 instance, $\#B_{j_1:3,3} = x_1x_2 + x_2x_3$ and $\#B_{j_2:3,3} = x_1x_3$, while the Case 2 instance yields $\#B_{j_1:3,3} = x_1x_2$ and $\#B_{j_2:3,3} = x_2x_3$. Although we do not know the order of the degrees of branch vertices along the spine, we know the set of their degrees, so we know the value $x_1x_2 + x_1x_3 + x_2x_3$. If Case 1 holds here, then this value equals $\#B_{j_1:3,3} + \#B_{j_2:3,3}$; otherwise it is larger than $\#B_{j_1:3,3} + \#B_{j_2:3,3}$. This distinguishes Case 1 and Case 2 when $j_2 = 2j_1$.

Finally, suppose that j_1 is the only value at most $(r-1)/2$ such that $\#B_{j:3,3} > 0$. By our choice of indexing, $j_1 = \delta_{1,2}$. It may happen that $\delta_{2,3} > (r-1)/2$ (Case 3) or that also $\delta_{2,3} = j_1$ and $\delta_{1,3} > (r-1)/2$ (Case 2). These two possibilities appear as T_3 and T_2, respectively, in Figure 4.

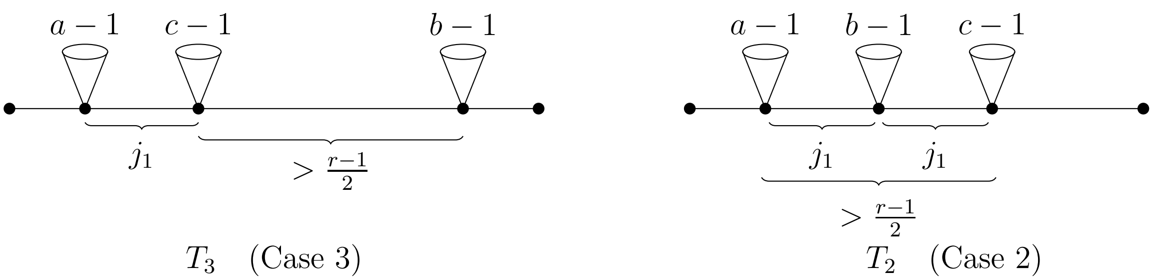

Figure 4: Only one visible length of R'-baton

In Case 3 we have $\#B_{j_1:3,3} = x_1x_2$, in Case 2 $\#B_{j_1:3,3} = x_1x_2 + x_2x_3$. However, we do not know the indexing of the values in $\{x_1, x_2, x_3\}$. If the product of two of those values never equals the sum of the other two products, then $\#B_{j_1:3,3}$ distinguishes Case 3 and Case 2.

Hence we may assume otherwise. In particular, the product of the two largest values in $\{x_1, x_2, x_3\}$ equals the sum of the two products involving the smallest of the three numbers. To compare the values, let $b = \min\{y_1, y_2, y_3\}$, and let a and c be the other two values. In Figure 4, the position of the branch vertex with degree $b+1$ is fixed in the two possibilities, while the positions of those with degrees $a+1$ and $c+1$ may be interchanged. If Case 3 and Case 2 are indistinguishable, then

$$\binom{a}{2}\binom{c}{2} = \binom{b}{2}\left[\binom{a}{2} + \binom{c}{2}\right]. \tag{34}$$

If $j_1 \le (r-3)/2$, then $B_{j_1:4,3}$ fits in a card. Now $\#B_{j_1:4,3} = \binom{a}{3}\binom{c}{2} + \binom{a}{2}\binom{c}{3}$ in Case 3, while $\#B_{j_1:4,3} = \binom{b}{2}\left[\binom{a}{3} + \binom{c}{3}\right] + \binom{b}{3}\left[\binom{a}{2} + \binom{c}{2}\right]$ in Case 2. If Case 3 and Case 2 are indistinguishable, then

$$\binom{a}{3}\binom{c}{2} + \binom{a}{2}\binom{c}{3} = \binom{b}{2}\left[\binom{a}{3} + \binom{c}{3}\right] + \binom{b}{3}\left[\binom{a}{2} + \binom{c}{2}\right]. \tag{35}$$

Since $a, b, c \ge 2$, we can divide (35) by (34), which yields

$$\frac{a-2}{3} + \frac{c-2}{3} = \frac{\binom{a}{3} + \binom{c}{3}}{\binom{a}{2} + \binom{c}{2}} + \frac{b-2}{3}.$$

Multiplying both sides by $\binom{a}{2}+\binom{c}{2}$ and canceling like terms yields

$$\frac{a-2}{3}\binom{c}{2}+\frac{c-2}{3}\binom{a}{2}=\frac{b-2}{3}\left[\binom{a}{2}+\binom{c}{2}\right].$$

This is a contradiction, since $b<\min\{a,c\}$. Hence (34) and (35) cannot both hold, and $\#B_{j_1:3,3}$ or $\#B_{j_1:4,3}$ distinguishes Case 3 from Case 2.

We conclude that ambiguity between Case 3 and Case 2 requires $j_1 \geq (r-2)/2$. From Case 3, this requires $\delta_{1,3}=r-1$, so $j_1=(r-2)/2$. Case 2 then requires $\delta_{1,3}=r-2$. The baton $B_{j_1-1:3,2}$ is small enough to fit in a card, so we can obtain $\#B_{j_1-1:3,2}$ from the deck. This baton is too short to have two branch vertices as key vertices and too long to extend in both directions from the outermost branch vertices. Since the vertices of degrees $a+1$ and $c+1$ can be switched, in Case 3 we have $B_{j_1-1:3,2}\in\{\binom{a}{2}+2\binom{c}{2}+\binom{b}{2},\binom{c}{2}+2\binom{a}{2}+\binom{b}{2}\}$. In Case 2, $\#B_{j_1-1:3,2}=\binom{a}{2}+2\binom{b}{2}+\binom{c}{2}$. Since $b<\min\{a,c\}$, there are more copies in Case 3 than in Case 2, so $\#B_{j_1-1:3,2}$ distinguishes the two cases. □

In addition to knowing the distance case from Lemma 9.7, from the batons in the deck we also know the actual distances, except that in Case 3 we do not know the differences between u_3 and the other indices. Our next tast is to determine the order of the degrees of the branch vertices along the spine. After that we will have the smallest subtree containing the branch vertices, and reconstruction will be completed by determining how close that configuration is to one of the ends.

Lemma 9.8. *For an unknown caterpillar having exactly three branch vertices, whose degrees sum to at least* 11*, the deck determines the degrees of the branch vertices in order, except that in Case* 3 *it determines* $d(u_3)$ *and leaves the remaining two degrees unordered.*

Proof. We maintain the same notation as in the proof of Lemma 9.7. Since we know the degree list, we know the unordered set $\{x_1,x_2,x_3\}$. Therefore, we know α and β, defined by $\alpha=x_1x_2+x_2x_3+x_1x_3$ and $\beta=x_2x_2x_3$. Let $j_1=\delta_{1,2}$, $j_2=\delta_{2,3}$, and $j_3=\delta_{1,3}$; we know the values in (j_1,j_2,j_3) that are at most $(r-1)/2$. Let $b_i=\#B_{j_i:3,3}$ for $i\in\{1,2,3\}$; we know b_i when $j_i\leq(r-1)/2$.

In Case 1 or Case 2 of the distance cases, we know j_1,j_2,b_1,b_2 If $j_1<j_2$, then $b_1=x_1x_2$ and $b_2=x_2x_3$. Hence we obtain $x_3=\beta/b_1$, $x_1=\beta/b_2$, and then $x_2=\beta/(x_1x_3)$.

If $\delta_{1,2}=\delta_{2,3}$, then $b_1=b_2=x_1x_2+x_2x_3$ and $b_3=x_1x_3$. We can obtain b_3 as $\alpha-b_1$, even if $j_3>(r-1)/2$ (that is, the computation is valid in Case 1 or Case 2). We then have $x_2=\beta/b_2$. The ordering of x_1 and x_3 can be chosen arbitrarily (see Figure 4); to make the choice definite, we make it so that $x_1\leq x_3$.

In Case 3, we have learned only $b_1=x_1x_2$ and obtain $x_3=\beta/b_1$. The set $\{x_1,x_2\}$ remains unordered at this point (see Figure 4).

We retrieve the degrees of the branch vertices using $x_i = \binom{d(v_{u_i})-1}{2}$. $\square$

In Case 1 or Case 2 of the distance cases, we now know the subgraph induced by the portion of the spine from v_{u_1} to v_{u_3} and the neighboring leaves. To complete the reconstruction, it suffices to determine the index of one end of this portion, u_1 or u_3. Case 3 will be somewhat harder, since there we do not yet completely know the ordering of the degrees of branch vertices.

Lemma 9.9. *If G is a caterpillar having exactly three branch vertices, whose degrees sum to at least* 11*, then G is determined by the deck.*

Proof. We know $\#B_{j:3,3}$ when $j \le (r-1)/2$ and $\#B_{j:3,2}$ when $j \le (r+1)/2$.

Let $\#B'_{j:3,2}$ denote the number of copies of $B_{j:3,2}$ in which the leaf neighbor of the key vertex of degree 2 lies on the augmented spine. We don't see $\#B'_{j:3,2}$ directly in the deck, but we will be able to compute these values when needed using the expressions below in (36). Recall the notation $y_i = d(v_{u_i})-1$ and $x_i = \binom{y_i}{2}$. The computation uses that the key vertices in copies of $B_{j:3,2}$ not counted by $\#B'_{j:3,2}$ must both be branch vertices. By convention, let $\#B'_{0:3,2} = 2x_1 + 2x_2 + 2x_3$.

$$\#B'_{j:3,2} = \begin{cases} \#B_{j:3,2} & j \notin \{\delta_{1,2}, \delta_{2,3}, \delta_{1,3}\} \\ \#B_{j:3,2} - x_1(y_2-1) - x_2(y_1-1) & j = \delta_{1,2} \ne \delta_{2,3} \\ \#B_{j:3,2} - x_2(y_3-1) - x_3(y_2-1) & j = \delta_{2,3} \ne \delta_{1,2} \\ \#B_{j:3,2} - x_1(y_3-1) - x_3(y_1-1) & j = \delta_{1,3} \\ \#B_{j:3,2} - x_1(y_2-1) - x_2(y_1-1) - x_2(y_3-1) - x_3(y_2-1) & j = \delta_{1,2} = \delta_{2,3} \end{cases} \tag{36}$$

We cannot compute $\#B'_{j:3,2}$ when $j > (r+1)/2$, because then $B_{j:3,2}$ does not fit in a card (such as when $j = \delta_{1,3} > (r+1)/2$. It does fit when $j \le (r+1)/2$, since $\sum_{i=1}^{3} d_i \ge 11$ implies $n - r \ge 7$ and $(r+1)/2 + 4 \le (n+2)/2$.

When $j < u_1$ and $j < \overline{u}_3$, we have $\#B'_{j:3,2} = 2x_1 + 2x_2 + 2x_3$, because there is room for $B_{j:3,2}$ to extend in either direction from any branch vertex. When the length j exceeds a value equal to the distance between a branch vertex u_1 and v_1 or v_r, we lose the x_i batons that had been extending toward one end of the caterpillar. That is, the values j such $\#B'_{j:3,2} < \#B'_{j-1:3,2}$ are the levels j such that there is a branch vertex in $\{v_j, v_{\overline{j}}\}$. Since there are three branch vertices, there are two or three values of j where such a difference occurs. (Our convention for $\#B'_{0:3,2}$ makes the computation valid also when $j = 1$.) We consider cases as in Definition 9.6.

Cases 1&2: $\delta_{1,2} \le \delta_{2,3} \le (r-1)/2$. By Lemma 9.8 we know the values y_i and x_i for $i \in \{1,2,3\}$, so we obtain $\#B_{j:3,2}$ from $\#B_{j:3,2}$ as in (36) for $j \le (r+1)/2$.

Let h be the least j such that $\#B'_{h:3,2} < 2x_1+2x_2+2x_3$. Since $u_2 > \min\{u_1, \overline{u}_3\}$, we have $h = \min\{u_1, \overline{u}_3\}$. To determine whether h is u_1 or $\overline{u}_3$ (or both) we compute the difference. If $x_1 \neq x_3$, then

$$\#B'_{h-1:3,2} - \#B'_{h:3,2} = \begin{cases} x_1 & u_1 = h \\ x_1 + x_3 & u_1 = \overline{u}_3 = h \\ x_3 & \overline{u}_3 = h \end{cases}. \tag{37}$$

The outcome specifies the position along the spine for at least one of $\{v_{u_1}, v_{u_3}\}$, completing the reconstruction.

When $x_1 = x_3$, the difference D in (37) does not distinguish $u_1 = h$ and $\overline{u}_3 = h$. If $u_1 = \overline{u}_3 = h$, which we recognize by $D = x_1 + x_3$, then the caterpillar is determined up to symmetry. Hence we may assume $u_1 \neq \overline{u}_3$.

Since we know x_1, x_2, x_3, we know the three levels of the branch vertices, two of which may be equal. If $\delta_{1,2} = \delta_{2,3}$, then setting $u_1 = h$ or $u_3 = \overline{h}$ yields the same caterpillar, since $x_1 = x_3$. Hence we may assume $\delta_{1,2} < \delta_{2,3}$. Now setting $u_1 = h$ or $u_3 = \overline{h}$ interchanges the levels of v_{u_1} and v_{u_3}, yielding the same contributions to the differences $\#B'_{j-1:3,2} - \#B'_{j:3,2}$ from these two vertices. However, when $u_1 = h$ the vertex v_{u_2} has level $h + \delta_{1,2}$, and when $\overline{u}_3 = h$ the level of v_{u_2} is higher than $h + \delta_{1,2}$. We detect this difference, since we know the levels j such that $\#B'_{j:3,2} < \#B'_{j-1:3,2}$. This distinguishes which branch vertex has level h, completing the reconstruction.

Case 3: $\delta_{1,2} \leq (r-1)/2 < \delta_{2,3}$. By Lemma 9.8, we know x_3 and hence also the set $\{x_1, x_2\}$. Since we do not know $\delta_{2,3}$, fixing u_1 does not complete the reconstruction. However, since $\delta_{2,3} > (r-1)/2$, we know that v_{u_3} is in the second half of the spine and that v_{u_1} and v_{u_2} are in the first half ($\delta_{2,3} > (r-1)/2$ yields $u_2 \leq r/2$). Hence it suffices to determine the levels of the three branch vertices and assign the values in $\{x_1, x_2\}$ to the correct indices.

To find the levels of branch vertices, we seek j such that $\#B'_{j:3,2} < \#B'_{j-1:3,3}$ and $j \leq r/2$. Because $\delta_{2,3} > (r-1)/2$, the only lines in (36) that are relevant for this computation are the first two. When $j = \delta_{1,2}$, we know the value of $\#B'_{j:3,2}$ because the formula is unchanged when (x_1, y_1) and (x_2, y_2) are interchanged. For all other relevant values, $\#B'_{j:3,2} = \#B_{j:3,2}$.

Let h be the least value of j such that $\#B'_{j:3,2} < \#B'_{j-1:3,2}$, and let h' be the next. These are the only two such values if and only if two branch vertices are on the same level. The possibilities are then $u_1 = \overline{u}_3 = h$ or $u_2 = \overline{u}_3 = h'$. We obtain

$$\#B'_{h-1:3,2} - \#B'_{h:3,2} = \begin{cases} x_1 + x_3 & u_1 = h = \overline{u}_3 \\ x_1 & u_1 = h \neq \overline{u}_3 \end{cases} \tag{38}$$

We also have

$$\#B'_{h'-1:3,2} - \#B'_{h':3,2} = \begin{cases} x_2 & u_2 = h' \neq \overline{u}_3 \\ x_2 + x_3 & u_2 = h' = \overline{u}_3 \end{cases} \tag{39}$$

Thus the ordered pair of values in (38) and (39) is $(x_1 + x_3, x_2)$ (two branch vertices at level h) or $(x_1, x_2 + x_3)$ (two branch vertices at level h'). If $x_1 = x_2$, then the ordered pairs are distinguished by which coordinate is larger, completing the reconstruction. If $x_1 \neq x_2$, then we count appearances of $B_{\delta_{1,2},h:3,3,2}$, which fits in a card since its length is at most $(r-2)/2$. We have $\#B_{\delta_{1,2},h:3,3,2} = x_1(y_2 - 1)$, and that value differs from $x_2(y_1 - 1)$ when $x_1 \neq x_2$. Thus this count tells us whether the degrees of v_{u_1} and v_{u_2} respect the order (x_1, x_2) or the order (x_2, x_1). Having determined which of x_1 and x_2 is smaller, we can determine which of them we see in the ordered pair from (38) and (39), completing the reconstruction.

It remains only to consider the subcase where the three branch vertices have distinct levels. That is, there are distinct values h_1, h_2, h_3 with $1 \leq h_1 < h_2 < h_3 \leq r/2$ such that $\#B'_{h_i:3,2} < \#B'_{h_i-1:3,2}$. The caterpillar is now one of three possibilities, given by (u_1, u_2, u_3) being $(h_1, h_2, \overline{h}_3)$, $(h_1, h_3, \overline{h}_2)$, or $(h_2, h_3, \overline{h}_1)$.

Let $\beta_i = \#B'_{h_i-1:3,2} - \#B'_{h_i:3,2}$. In the three options for G, the triple $(\beta_1, \beta_2, \beta_3)$ is (x_1, x_2, x_3), (x_1, x_3, x_2), or (x_3, x_1, x_2).

We first distinguish the second possibility from the others using $\delta_{1,2}$, a value we know. In the three options, $\delta_{1,2}$ is $h_2 - h_1$, $h_3 - h_1$, or $h_3 - h_2$, respectively. Since $h_3 - h_1$ is strictly largest, that option cannot be confused with either of the others. It also distinguishes x_1 as the difference associated with h_1.

Ambiguity among the other two options requires $(x_1, x_2, x_3) = (x_3, x_1, x_2)$. Here x_3 must equal both x_1 and x_2, so all three branch vertices have the same degree, d. Now we consider a triton. Let $x = \binom{d-1}{2}$. We have $\#B_{\delta_{1,2},h:3,3,2} = x(d-2)$ if $u_1 = h_1$, and $\#B_{\delta_{1,2},h:3,3,2} = 2x(d-2)$ if $u_1 = h_2$. Hence we can distinguish the cases. The value β_2 then tells us which of $\{x_1, x_2\}$ is which, completing the reconstruction. □

We have completed the consideration of all cases and proved Theorem 1.4.

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