ON RESTRICTED SUMS OF FOUR SQUARES AND ZHI-WEI SUN'S $x + 24y$ CONJECTURE

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Abstract. In this paper, by using the arithmetic theory of ternary quadratic forms, we study some refinements on Lagrange's four-square theorem. For example, given positive integers a, b satisfying some algebraic conditions and a positive integer $C \geq 3$, we will show that for any sufficiently large integer n with $\mathrm{ord}_2(n) \leq C$, there exist non-negative integers x, y, z, w such that

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = n, \\ ax + by \in \mathcal{S}, \end{cases}$$

where $\mathcal{S}$ is the set of all squares over $\mathbb{Z}$. In particular, we obtain some progress on Zhi-Wei Sun's $x + 24y$ conjecture.

1. Introduction

1.1. Background and Motivation.

In 1770, Lagrange established the celebrated four-square theorem (cf. [13, pp. 5–7]), which states that each number $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$ can be written as

$$n = x^2 + y^2 + z^2 + w^2$$

with $x, y, z, w \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the ring of integers. Along this line, in 1917, Ramanujan [15] proved that there are at most 55 possible quadruples (a, b, c, d) of positive integers with $a \leq b \leq c \leq d$ such that the quadratic form $ax^2 + by^2 + cz^2 + dw^2$ is *universal* over $\mathbb{Z}$, i.e.,

$$\left\{ax^2 + by^2 + cz^2 + dw^2 : \ x, y, z, w \in \mathbb{Z}\right\} = \mathbb{N}.$$

Ten years later, Dickson [2] confirmed that 54 quadratic forms on Ramanujan's list are indeed universal over $\mathbb{Z}$, and pointed out that for the remaining one quadruples $(1, 2, 5, 5)$, we have

$$\left\{x^2 + 2y^2 + 5z^2 + 5w^2 : \ x, y, z, w \in \mathbb{Z}\right\} = \mathbb{N} \setminus \{15\}.$$

Cauchy first considered a refinement of Lagrange's four-square theorem and obtained the following result, which was later known as Cauchy's lemma (cf. [13, p. 31]). Let n and m be positive odd integers with $m^2 < 4n$ and $3n < m^2 + 2m + 4$. Then there exist $x, y, z, w \in \mathbb{N}$

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such that
$$\begin{cases} n = x^2 + y^2 + z^2 + w^2, \\ m = x + y + z + w. \end{cases}$$
Along this line, for positive integers m, n, t, Kloosterman [7] considered
$$r_t(n, m) := \# \left\{ (x_1, x_2, \ldots, x_t) \in \mathbb{Z}^t : \ x_1^2 + x_2^2 + \cdots + x_t^2 = n \text{ and } x_1 + x_2 + \cdots + x_t = m \right\},$$
and obtained an asymptotic formula via circle method, where $\#S$ denotes the cardinality of a finite set S. Later Lomadze [10] and Van der Blij [1] also obtained some developments on this topic.

Inspired by the above results, in 2017, Z.-W. Sun [20] initiated the study of the following refinement of Lagrange's four-square theorem. Let $P(X, Y, Z, W) \in \mathbb{Z}[X, Y, Z, W]$. Then by Sun's definition, the polynomial $P(X, Y, Z, W)$ is said to be *suitable*, if for any $n \in \mathbb{N}$, there are $x, y, z, w \in \mathbb{N}$ such that
$$\begin{cases} n = x^2 + y^2 + z^2 + w^2, \\ P(x, y, z, w) \in \mathcal{S}, \end{cases}$$
where
$$\mathcal{S} = \{x^2 : \ x \in \mathbb{Z}\}$$
is the set of all squares over $\mathbb{Z}$.

The most well-known problem in this topic is Sun's 1-3-5 conjecture (see [20, Conjecture 4.3(i)]). This challenging conjecture states that for any integer $n \in \mathbb{N}$, there exist $x, y, z, w \in \mathbb{N}$ such that
$$\begin{cases} n = x^2 + y^2 + z^2 + w^2, \\ x + 3y + 5z \in \mathcal{S}, \end{cases}$$
Later Machiavelo and his collaborators [11, 12] confirmed this conjecture by using quaternion arithmetic in the ring of Lipschitz integers. Furthermore, for $n \in \mathbb{N}$, $m \in \mathbb{Z}$ and $\boldsymbol{v} = (a, b, c, d) \in \mathbb{Z}^4$, we use $r_{\boldsymbol{v}}(n, m)$ or $r_{(a\text{-}b\text{-}c\text{-}d)}(n, m)$ to denote the number of solutions $(x, y, z, w) \in \mathbb{Z}^4$ for the system
$$\begin{cases} n = x^2 + y^2 + z^2 + w^2, \\ m = ax + by + cz + dw. \end{cases}$$
In 2024, by using the theory of Siegel modular forms, D. Li and H. Zhou [8] gave an explict formula for
$$r_{(1\text{-}3\text{-}5\text{-}0)}(n, m) + r_{(1\text{-}3\text{-}3\text{-}4)}(n, m),$$
which involves the Hurwitz class numbers. Later, for any $t \in \mathbb{N}$, J. Li and H. Wang [9] obtained a formula for
$$\sum_{a^2+b^2+c^2+d^2=t} r_{\boldsymbol{v}}(n, m).$$
However, it remains difficult to compute the explicit value of $r_{\boldsymbol{v}}(n, m)$ separately for general $\boldsymbol{v} \in \mathbb{Z}^4$.

On the other hand when $P(X,Y,Z,W)=aX+bY$, Sun [20, Conjecture 4.1] posed the following conjecture.

Conjecture 1.1 (Z.-W. Sun). *Let $a,b\in\mathbb{Z}^+$ with $\gcd(a,b)$ square-free and $a\le b$. Then $aX+bY$ is suitable if and only if*

$$(a,b)=(1,2),(1,3),(1,24).$$

For this conjecture, some partial results were obtained.

- Y.-C. Sun and Z.-W. Sun [19, Theorem 1.1(ii)] showed that $X+2Y$ is suitable.
- Y.-F. She and H.-L. Wu [18, Theorem 1.4] proved that $X+3Y$ is suitable.

In addition, as mentioned by Sun, confirming the suitability of $X+24Y$ might be very difficult.

Motivated by the above results, in this paper, we will concentrate on the case when the polynomial $P(X,Y,Z,W)$ is a binary homogeneous integral polynomial of degree one.

1.2. **Main Results.** For any positive integer $l\ge 2$, let

$$(\mathbb{Z}/l\mathbb{Z})^\times=\{x \bmod l\mathbb{Z}:\ x\in\mathbb{Z} \text{ and } \gcd(x,l)=1\}$$

be the multiplicative group of all units in the residue ring $\mathbb{Z}/l\mathbb{Z}$. Now we state our main result.

Theorem 1.1. *Let $a,b\in\mathbb{Z}^+$ with $\gcd(a,b)=1$ such that $(\mathbb{Z}/(a^2+b^2)\mathbb{Z})^\times$ is a cyclic group. Let $C\ge 3$ be a positive integer. Then for any sufficiently large integer n with $\mathrm{ord}_2(n)\le C$, there exist $x,y,z,w\in\mathbb{N}$ such that*

$$\begin{cases}x^2+y^2+z^2+w^2=n,\\ ax+by\in\mathcal{S}.\end{cases}$$

Remark 1.1. (i) Clearly $(\mathbb{Z}/(a^2+b^2)\mathbb{Z})^\times$ is a cyclic group if and only if

$$a^2+b^2\in\{p^r:\ r\in\mathbb{Z}^+ \text{ and } p \text{ is an odd prime}\}\cup\{2p^k:\ k\in\mathbb{N} \text{ and } p \text{ is an odd prime}\}. \tag{1.1}$$

(ii) The condition $\mathrm{ord}_2(n)\le C$ can not be removed. For example, let $(a,b)=(3,10)$, which satisfies the condition (1.1). It is known that each positive integer of the form $4^{2r+1}\cdot 6$ has a unique partition into four integral squares (see [6, p. 86]), that is,

$$4^{2r+1}\cdot 6=(4^{r+1})^2+(2\cdot 4^r)^2+(2\cdot 4^r)^2+0^2.$$

However, it is easy to verify that for any positive integer of the form $4^{2r+1}\cdot 6$, there are no $x,y,z,w\in\mathbb{N}$ such that

$$\begin{cases}x^2+y^2+z^2+w^2=4^{2r+1}\cdot 6,\\ 3x+10y\in\mathcal{S}.\end{cases}$$

As $(a,b)=(1,24)$ satisfies the condition (1.1), we directly obtain the following corollary, which makes some progress on Sun's $x+24y$ conjecture.

Corollary 1.1. *Let $C \geq 3$ be a positive integer. Then for any sufficiently large integer n with $\mathrm{ord}_2(n) \leq C$, there exist $x, y, z, w \in \mathbb{N}$ such that*

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = n, \\ x + 24y \in \mathcal{S}. \end{cases}$$

1.3. **Outline of this paper.** In Section 2, we will introduce some basic notations and necessary lemmas concerning lattices on quadratic spaces. The proof of our main result will be given in Section 3.

2. Preliminaries concerning lattices on quadratic spaces

2.1. **Notation.** We first adopt standard notations appeared in [14]. Let R be a principal ideal domain and let $\mathbb{F}$ be the fractional field of R with $\mathrm{char}(\mathbb{F}) \neq 2$, where $\mathrm{char}(\mathbb{F})$ is the characteristic of $\mathbb{F}$. Let (U, B, Q) be a non-degenerated quadratic space over $\mathbb{F}$ with $\dim_{\mathbb{F}} U = m$, where B is a symmetric bilinear form on U and Q is the corresponding quadratic map. The group of all rotations of U is defined by

$$O^+(U) := \{\sigma \in \mathrm{Hom}_{\mathbb{F}}(U, U) : \ \det(\sigma) = 1 \text{ and } Q(\sigma(\boldsymbol{x})) = Q(\boldsymbol{x}) \text{ for any } \boldsymbol{x} \in U\}.$$

Also, let

$$\theta_{\mathbb{F}} : \ O^+(U) \longrightarrow \mathbb{F}^{\times}/\mathbb{F}^{\times 2}$$

be the spinor norm map, where

$$\mathbb{F}^{\times} = \mathbb{F} \setminus \{0\},$$

and

$$\mathbb{F}^{\times 2} = \left\{x^2 : \ x \in \mathbb{F} \setminus \{0\}\right\}.$$

The kernel of $\theta_{\mathbb{F}}$ is denoted by $O'(U)$. Let

$$M = R\boldsymbol{e}_1 + R\boldsymbol{e}_2 + \cdots + R\boldsymbol{e}_m$$

be a non-degenerated lattice on U, where $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m$ is an R-basis of M. Then the gram matrix A of M with respect to $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m$ is defined by

$$A = (B(\boldsymbol{e}_i, \boldsymbol{e}_j))_{1 \leq i, j \leq m}.$$

Conversely, for any $m \times m$ symmetric matrix A over $\mathbb{F}$, we write $M \cong A$, if there is an R-basis of M such that A is the gram matrix of M with respect to this basis. For any symmetric matrices $A_1, A_2, \cdots, A_s$ over $\mathbb{F}$, we simply write

$$M \cong A_1 \perp A_2 \perp \cdots \perp A_s \tag{2.1}$$

if

$$M \cong \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix}.$$

In particular, if $A_i = (a_i)$ is a 1×1 matrix for $i = 1, 2, \cdots, s$, then (2.1) is abbreviated as

$$M \cong \langle a_1 \rangle \perp \langle a_2 \rangle \cdots \perp \langle a_s \rangle.$$

Throughout the remaining part of this paper, $\mathbb{Q}$ denotes the field of rational numbers. Let $\Omega_{\mathbb{Q}}$ be the set of all places of $\mathbb{Q}$. By the Ostrowski theorem, each non-archimedean place arises from the p-adic valuation $|\cdot|_p$, where p is a rational prime. Also, the unique archimedean place of $\mathbb{Q}$, denoted by ∞, arises from the ordinary absolute value $|\cdot|_\infty$ on $\mathbb{Q}$. Hence, for simplicity, we write

$$\Omega_{\mathbb{Q}} = \{p : \ p \text{ is a rational prime}\} \bigcup \{\infty\}.$$

For any $p \in \Omega_{\mathbb{Q}} \setminus \{\infty\}$, let $\mathbb{Q}_p$, $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$ denote the p-adic number field, the ring of p-adic integers and the multiplicative group of all p-adic units over $\mathbb{Z}_p$ respectively. Also, for the archimedean place ∞, we let $\mathbb{Q}_\infty = \mathbb{R}$ be the field of real numbers.

Now consider a non-degenerated quadratic space (V, B, Q) over $\mathbb{Q}$ with $\dim_{\mathbb{Q}}(V) = 3$. Given a non-degenerated lattice L on V with $\operatorname{rank}(L) = 3$, for any $p \in \Omega_{\mathbb{Q}} \setminus \{\infty\}$, let

$$V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p,$$

and

$$L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

be the p-adic completions of V and L respectively. For $\infty \in \Omega_{\mathbb{Q}}$, let

$$V_\infty = L_\infty = V \otimes_{\mathbb{Q}} \mathbb{Q}_\infty = V \otimes_{\mathbb{Q}} \mathbb{R}.$$

The adelization of $O^+(V)$ is defined by

$$O_{\mathbb{A}}^+(V) := \left\{ \sigma = (\sigma_v) \in \prod_{v \in \Omega_{\mathbb{Q}}} O^+(V_v) : \ \sigma_v(L_v) = L_v \text{ for almost all } v \in \Omega_{\mathbb{Q}} \right\}.$$

Clearly $O^+(V)$ can be viewed as a subgroup of $O_{\mathbb{A}}^+(V)$. Also, let

$$O'_{\mathbb{A}}(V) := \left\{ \sigma = (\sigma_v) \in O_{\mathbb{A}}^+(V) : \ \sigma_v \in O'(V_v) \text{ for any } v \in \Omega_{\mathbb{Q}} \right\},$$

and

$$O_{\mathbb{A}}^+(L) := \left\{ \sigma = (\sigma_v) \in O_{\mathbb{A}}^+(V) : \ \sigma_v(L_v) = L_v \text{ for any } v \in \Omega_{\mathbb{Q}} \right\}.$$

Then there is a group action of $O_{\mathbb{A}}^+(V)$ on the set of all lattices in the genus of L. It is also known that

$$\operatorname{gen}(L) = O_{\mathbb{A}}^+(V)L,$$

and

$$\operatorname{spn}(L) = \operatorname{spn}^+(L) = O^+(V)O'_{\mathbb{A}}(V)O_{\mathbb{A}}^+(L)L,$$

where $\operatorname{gen}(L)$ denotes the set of all lattices in the genus of L, $\operatorname{spn}(L)$ is the set of all lattices in the spinor genus of L and $\operatorname{spn}^+(L)$ is the set of all lattices in the proper spinor genus of L(since $\operatorname{rank}(L) = 3$, we clearly have $\operatorname{spn}(L) = \operatorname{spn}^+(L)$).

For any rational number $a \in \mathbb{Q}$, we say that a is represented by $\mathrm{gen}(L)$, denoted by $a \in Q(\mathrm{gen}(L))$, if there is a lattice $M \in \mathrm{gen}(L)$ such that

$$a \in Q(M) := \{Q(\boldsymbol{x}) : \ \boldsymbol{x} \in M\}\,.$$

It is known that

$$a \in Q(\mathrm{gen}(L)) \Leftrightarrow a \in Q(L_v) := \{Q(\boldsymbol{x}) : \ \boldsymbol{x} \in L_v\} \text{ for any } v \in \Omega_{\mathbb{Q}}.$$

Also, we say that a is represented by $\mathrm{spn}(L)$, denoted by $a \in Q(\mathrm{spn}(L))$, if there exists a lattice $M \in \mathrm{spn}(L)$ such that $a \in Q(M)$.

2.2. **Some necessary lemmas.** Let $p \in \Omega_{\mathbb{Q}} \setminus \{\infty\}$ be a place and (U, B, Q) be a non-degenerated ternary quadratic space over $\mathbb{Q}_p$. We say that U is isotropic if there exists an $\boldsymbol{x} \in U \setminus \{\boldsymbol{0}\}$ such that $Q(\boldsymbol{x}) = 0$. Otherwise, U is said to be anisotropic.

For any non-degenerated ternary lattice

$$M = \mathbb{Z}_p\boldsymbol{e}_1 + \mathbb{Z}_p\boldsymbol{e}_2 + \mathbb{Z}_p\boldsymbol{e}_3 \subseteq U,$$

the scale of M, denoted by $\mathfrak{s}(M)$, is a fractional ideal of $\mathbb{Z}_p$ generated by the set

$$\{B(\boldsymbol{e}_i, \boldsymbol{e}_j) : \ 1 \le i, j \le 3\}\,.$$

Then the lattice M is said to be a unimodular lattice over $\mathbb{Z}_p$ if $\mathfrak{s}(M) = \mathbb{Z}_p$ and $d(M) \in \mathbb{Z}_p^{\times} \bmod \mathbb{Z}_p^{\times 2}$, where $d(M)$ is the discriminant of M and $\mathbb{Z}_p^{\times 2} = \left\{x^2 : \ x \in \mathbb{Z}_p^{\times}\right\}$.

For unimodular lattices over $\mathbb{Z}_p$, we have the following results (see [14, 92:1 and 93:11]).

Lemma 2.1. *Let $p \in \Omega_{\mathbb{Q}} \setminus \{\infty\}$ be a place and (U, B, Q) be a non-degenerated ternary quadratic space over $\mathbb{Q}_p$. Let*

$$M \cong \langle \varepsilon_1 \rangle \perp \langle \varepsilon_2 \rangle \perp \langle \varepsilon_3 \rangle \subseteq U$$

be a ternary unimodular lattice over $\mathbb{Z}_p$ with $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{Z}_p^{\times}$.

(i) *Suppose $p \neq 2$. Then*

$$M \cong \langle 1 \rangle \perp \langle 1 \rangle \perp \langle d(M) \rangle.$$

Moreover, if U is isotropic, then

$$M \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -d(M) \rangle.$$

(ii) *Suppose $p = 2$. If U is isotropic, then*

$$M \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -d(M) \rangle.$$

If U is anisotropic, then

$$M \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \langle -3d(M) \rangle.$$

Moreover,

$$Q\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right) = \left\{2x^2 + 2xy + 2y^2 : \ x, y \in \mathbb{Z}_2\right\}$$

$$= \{x \in \mathbb{Z}_2 : \ \mathrm{ord}_2(x) \equiv 1 \pmod 2\} \bigcup \{0\}.$$

Let $v \in \Omega_{\mathbb{Q}}$. Then the Hilbert symbol

$$(\cdot, \cdot)_v : \ \mathbb{Q}_v^{\times}/\mathbb{Q}_v^{\times 2} \times \mathbb{Q}_v^{\times}/\mathbb{Q}_v^{\times 2} \longrightarrow \{1, -1\}$$

is defined by

$$(a,b)_v = \begin{cases} 1 & \text{if } a \in \mathrm{Norm}_{\mathbb{Q}_v(\sqrt{b})/\mathbb{Q}_v}(\mathbb{Q}_v(\sqrt{b})^{\times}), \\ -1 & \text{otherwise,} \end{cases}$$

where

$$\mathrm{Norm}_{\mathbb{Q}_v(\sqrt{b})/\mathbb{Q}_v}(\mathbb{Q}_v(\sqrt{b})^{\times}) = \left\{x^2 - by^2 : \ x, y \in \mathbb{Q}_v\right\} \setminus \{0\}.$$

We also need the following result (see [14, 58:6]).

Lemma 2.2. *Let $v \in \Omega_{\mathbb{Q}}$ and (U, B, Q) be a non-degenerated ternary quadratic space over $\mathbb{Q}_v$. Let*

$$M \cong \langle a_1 \rangle \perp \langle a_2 \rangle \perp \langle a_3 \rangle \subseteq U$$

be a non-degenerated lattice on U with $a_1, a_2, a_3 \in \mathbb{Q}_v$. Then

$$U \text{ is isotropic } \Leftrightarrow S_v(U) = (-1,-1)_v,$$

where

$$S_v(U) = \prod_{1 \le i \le j \le 3} (a_i, a_j)_v$$

is the Hasse invariant of U.

The next lemma determines the number of spinor genera in $\mathrm{gen}(L)$ (see [14, 102:7 and 102:9]).

Lemma 2.3. *Let (V, B, Q) be a non-degenerated quadratic space over $\mathbb{Q}$ with $\dim_{\mathbb{Q}}(V) = 3$, and let L be a non-degenerated ternary lattice on V. Then*

$$\begin{aligned} \# \{\mathrm{spn}(M) : \ M \in \mathrm{gen}(L)\} &= \left[O_{\mathbb{A}}^{+}(V) : \ O^{+}(V) O'_{\mathbb{A}}(V) O_{\mathbb{A}}^{+}(L)\right] \\ &= \left[J_{\mathbb{Q}} : \ \mathbb{Q}^{\times} J_L\right], \end{aligned}$$

where

$$J_{\mathbb{Q}} = \left\{ (i_v)_v \in \prod_{v \in \Omega_{\mathbb{Q}}} \mathbb{Q}_v^{\times} : \ |i_v|_v = 1 \text{ for almost all } v \in \Omega_{\mathbb{Q}} \setminus \{\infty\} \right\}$$

is the group of all idèles, and

$$J_L = \left\{ x = (x_v) \in J_{\mathbb{Q}} : \ x_v \in \theta_{\mathbb{Q}_v}(O^{+}(L_v)) \text{ for any } v \in \Omega_{\mathbb{Q}} \right\}.$$

In particular, if $\mathbb{Z}_p^{\times} \subseteq \theta_{\mathbb{Q}_p}(O^{+}(L_p))$ for any $p \in \Omega_{\mathbb{Q}} \setminus \{\infty\}$, then $\mathrm{gen}(L) = \mathrm{spn}(L)$.

We next state some necessary results to compute spinor norms for lattices over $\mathbb{Z}_2$. Let (U,B,Q) be a non-degenerated quadratic space over $\mathbb{Q}_2$ and W be a non-degenerated $\mathbb{Z}_2$-lattice on U. We say that W has *even order* (resp. *odd order*) if $\mathrm{ord}_2(Q(\boldsymbol{x}))$ is even (resp. odd) for each primitive vector $\boldsymbol{x}\in W$ which gives rise to a $\mathbb{Z}_2$-integral symmetry of W.

Suppose now that a non-degenerated lattice M over $\mathbb{Z}_2$ has the following Jordan splitting

$$M\cong 2^{r_1}M_{(1)}\perp 2^{r_2}M_{(2)}\perp\cdots\perp 2^{r_s}M_{(s)},$$

where each $M_{(i)}$ is a unimodular lattice over $\mathbb{Z}_2$ and $0=r_1<r_2<\cdots<r_s$. In this paper, we will use the following result due to Earnest [4, 1.6].

Lemma 2.4. *Let notations be as above. Then the following results hold.*

(i) *Suppose that at least one Jordan component of M has rank ≥ 3. Then*

$$\theta_{\mathbb{Q}_2}(M)=\begin{cases}\mathbb{Z}_2^{\times}\mathbb{Q}_2^{\times 2} & \textit{if } 2^{r_k}M_{(k)} \textit{ has odd order for each } k=1,2,\cdots,s,\\ \mathbb{Q}_2^{\times} & \textit{otherwise.}\end{cases}$$

(ii) *If $\mathrm{rank}(M_{(k)})\leq 2$ for each $k=1,2,\cdots,s$ and at least one component, say $M_{(k_0)}$ is binary, then $\theta_{\mathbb{Q}_2}(M)\neq\mathbb{Q}_2^{\times}$ if and only if one of the following three cases occur:*

- *all Jordan components have odd order;*
- *all Jordan components have even order;*
- *whenever $\mathrm{rank}(M_{(k)})=2$, then*

$$M_{(k)}\cong\begin{pmatrix}\alpha & 1\\ 1 & 2\beta\end{pmatrix}$$

for some $\alpha,\beta\in\mathbb{Z}_2^{\times}$ and the associated quadratic space of $M_{(k)}$ is isotropic. Also, for any unary component, say $M_{(j)}\cong\langle c_j\rangle$ with $c_j\in\mathbb{Z}_2^{\times}$, the Hilbert symbol

$$\left(2^{r_j-r_{k_0}}\alpha c_j,-d(M_{(j)})\right)_2=1.$$

Meanwhile, $r_{l+1}-r_l\geq 4$ for any $l=1,2,\cdots,s-1$.

We next briefly introduce some basic facts on spinor exceptional integers. Let (V,B,Q) be a non-degenerated quadratic space over $\mathbb{Q}$ with $\dim_{\mathbb{Q}}(V)=3$, and let L be a non-degenerated ternary lattice on V. An integer t is called a *spinor exception* of $\mathrm{gen}(L)$ if $t\in Q(\mathrm{gen}(L))$ and t is represented by exactly half of the spinor genera in $\mathrm{gen}(L)$. The arithmetic properties of spinor exception have been investigated extensively. Readers may refer to [3, 5, 16, 17]. In this paper, we need the following result due to R. Schulze-Pillot [17].

Lemma 2.5. *Let (V,B,Q) be a positive definite quadratic space over $\mathbb{Q}$ with $\dim_{\mathbb{Q}}(V)=3$, and let L be a non-degenerated ternary lattice on V. Let $r\in\mathbb{Z}^+$ and let*

$$Q_r(\mathrm{gen}(L))=\left\{t\in\mathbb{Z}:\ t\in Q(\mathrm{gen}(L)) \textit{ and } \mathrm{ord}_p(t)\leq r \textit{ for any } p\in T\right\},$$

where

$$T=\{p\in\Omega_{\mathbb{Q}}\setminus\{\infty\}:\ V_p \textit{ is anisotropic}\}.$$

Then there is a constant c, depending on r and the level of L, such that all $t \in Q_r(\mathrm{gen}(L))$ with $t \geq c$ are represented by all lattices in $\mathrm{gen}(L)$ unless one of the following conditions are satisfied:

(i) *t is a spinor exception of $\mathrm{gen}(L)$;*

(ii) *t/p^2 is a spinor exception of $\mathrm{gen}(L)$ for some prime p that is inert in the quadratic extension $\mathbb{Q}(\sqrt{-t \cdot d(L)})/\mathbb{Q}$.*

We conclude this section with the following result obtained by Sun [21, Lemma 2.3].

Lemma 2.6. *Let a, b, c, d, m, n be non-negative real numbers with $a^2+b^2+c^2+d^2 \neq 0$. Suppose that x, y, z, w are real numbers satisfying*

$$\begin{cases} x^2+y^2+z^2+w^2 = n, \\ ax+by+cz+dw = m. \end{cases}$$

If

$$m \geq \sqrt{((a^2+b^2+c^2+d^2) - \min(\{a^2, b^2, c^2, d^2\} \setminus \{0\})) \cdot n},$$

then all the numbers ax, by, cz, dw are non-negative.

3. Proof of the main result

Proof of Theorem 1.1. Without loss of generality, we may assume $a \leq b$. Let (V, B, Q) be a positive definite quadratic space over $\mathbb{Q}$ with

$$V = \mathbb{Q}\boldsymbol{e}_1 + \mathbb{Q}\boldsymbol{e}_2 + \mathbb{Q}\boldsymbol{e}_3,$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is an orthogonal basis of V with $Q(\boldsymbol{e}_1) = 1$ and $Q(\boldsymbol{e}_2) = \mathbb{Q}(\boldsymbol{e}_3) = a^2+b^2$. Recall that $\gcd(a,b) = 1$ and $a^2+b^2 = p^r$ or $2p^k$, where p is an odd prime, $r \in \mathbb{Z}^+$ and $k \in \mathbb{N}$. Moreover, one can easily verify that $p \equiv 1 \pmod 4$.

Let

$$L = \mathbb{Z}\boldsymbol{e}_1 + \mathbb{Z}\boldsymbol{e}_2 + \mathbb{Z}\boldsymbol{e}_3$$

be a non-degenerated lattice on V, that is,

$$L \cong \langle 1 \rangle \perp \langle a^2+b^2 \rangle \perp \langle a^2+b^2 \rangle.$$

For any integer m, define

$$l_{a,b,m}(n) = (a^2+b^2)n - m^2.$$

We next consider the representations of $l_{a,b,m}(n)$ by the lattice L_v, where $v \in \Omega_{\mathbb{Q}}$.

(i) For the archimedean place $v = \infty \in \Omega_{\mathbb{Q}}$ and each $n \in \mathbb{N}$, clearly

$$\left\{ l_{a,b,m}(n) : \ 0 \leq m \leq \sqrt{(a^2+b^2)n} \right\} \subseteq Q(L_\infty). \tag{3.1}$$

(ii) For the place $v = q \in \Omega_{\mathbb{Q}} \setminus \{2, p, \infty\}$, the lattice L_q is a unimodular lattice over $\mathbb{Z}_q$. For any $\varepsilon \in \mathbb{Z}_q^\times$, the extension $\mathbb{Q}_q(\sqrt{\varepsilon})/\mathbb{Q}_q$ is unramified. Thus, by the local class field theory,

$$\mathrm{Norm}_{\mathbb{Q}_q(\sqrt{\varepsilon})/\mathbb{Q}_q}(\mathbb{Q}_q(\sqrt{\varepsilon})^\times) = \begin{cases} \mathbb{Q}_q^\times & \text{if } \varepsilon \in \mathbb{Z}_q^{\times 2}, \\ \{q^{2\delta} \cdot u : \ \delta \in \mathbb{Z} \text{ and } u \in \mathbb{Z}_q^\times\} & \text{if } \varepsilon \notin \mathbb{Z}_q^{\times 2}. \end{cases}$$

By this, we obtain

$$S_q(V_q) = (-1,-1)_q = 1,$$

and hence V is isotropic by Lemma 2.2. Using Lemma 2.1 again,

$$L_q \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle -1 \rangle.$$

This implies $Q(L_q) = \mathbb{Z}_q$. Hence for any $n \in \mathbb{N}$,

$$\{l_{a,b,m}(n) :\ m \in \mathbb{Z}\} \subseteq Q(L_q). \tag{3.2}$$

In addition, as L_q is a ternary unimodular lattice over $\mathbb{Z}_q$, we have

$$\theta_{\mathbb{Q}_q}(L_q) = \mathbb{Z}_q^{\times}\mathbb{Q}_q^{\times 2}. \tag{3.3}$$

(iii) For the place $v = p$, note that

$$L_p \cong \langle 1 \rangle \perp \langle p^r\eta \rangle \perp \langle p^r\eta \rangle,$$

where $r = \mathrm{ord}_p(a^2+b^2) \geq 1$ and $\eta = (a^2+b^2)/p^r \in \mathbb{Z}_p^{\times}$. Since $p \equiv 1 \pmod 4$, we have $-1 \in \mathbb{Z}_p^{\times 2}$. By [14, 92:1],

$$\langle p^r\eta \rangle \perp \langle p^r\eta \rangle \cong \begin{pmatrix} 0 & p^r \\ p^r & 0 \end{pmatrix}.$$

Thus,

$$L_p \cong \langle 1 \rangle \perp \begin{pmatrix} 0 & p^r \\ p^r & 0 \end{pmatrix}.$$

This implies that

$$Q(L_p) = \left\{x^2 + p^r y :\ x, y \in \mathbb{Z}_p\right\}.$$

By this and noting that $-1 \in \mathbb{Z}_p^{\times 2}$, for any $n \in \mathbb{N}$, we obtain

$$\{l_{a,b,m}(n) :\ m \in \mathbb{Z}\} \subseteq Q(L_p). \tag{3.4}$$

On the other hand, as $\langle \eta \rangle \perp \langle \eta \rangle$ is a binary unimodular lattice on $\mathbb{Z}_p$, we have

$$\mathbb{Z}_p^{\times}\mathbb{Q}_p^{\times 2} = \theta_{\mathbb{Q}_p}(\langle \eta \rangle \perp \langle \eta \rangle) = \theta_{\mathbb{Q}_p}(\langle p^r\eta \rangle \perp \langle p^r\eta \rangle) \subseteq \theta_{\mathbb{Q}_p}(L_p). \tag{3.5}$$

(iv) We next consider the dyadic place $v = 2$. Note that $\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2$ is a totally ramified extension. By the local class field theory, it is known that

$$\mathrm{Norm}_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{-1})^{\times}) = \left\{2^{\delta} \cdot u :\ \delta \in \mathbb{Z} \text{ and } u \in \mathbb{Z}_2^{\times} \text{ with } u \equiv 1 \ (\mathrm{mod}\ 4\mathbb{Z}_2)\right\}. \tag{3.6}$$

We next consider the following two cases.

Case I. $\mathrm{ord}_2(a^2+b^2) = 0$.

In this case,

$$L_2 \cong \langle 1 \rangle \perp \langle \mu \rangle \perp \langle \mu \rangle$$

for some $\mu \in \mathbb{Z}_2^{\times}$ with $\mu \equiv 1 \ (\mathrm{mod}\ 4\mathbb{Z}_2)$. By (3.6), we see that

$$S_2(V_2) = 1 \neq (-1,-1)_2 = -1,$$

and hence V_2 is anisotropic by Lemma 2.2. Applying Lemma 2.1, we obtain

$$L_2 \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \langle 3\rangle \cong \langle 1\rangle \perp \langle 1\rangle \perp \langle 1\rangle.$$

This implies

$$Q(L_2) = \mathbb{Z}_2 \setminus \left\{4^{\delta}(8z+7):\ \delta\in\mathbb{N} \text{ and } z\in\mathbb{Z}_2\right\}. \tag{3.7}$$

By (3.7) one can easily verify that following results hold.

- If $n\in\mathbb{N}$ with $n\equiv 1 \pmod 4$, then

$$\{l_{a,b,m}(n):\ m\in 2\mathbb{Z}\} \subseteq Q(L_2)\cap(4\mathbb{Z}+1)\neq\emptyset. \tag{3.8}$$

- If $n\in\mathbb{N}$ with $n\equiv 3 \pmod 4$, then

$$\{l_{a,b,m}(n):\ m\in 2\mathbb{Z}+1\} \subseteq Q(L_2)\cap(4\mathbb{Z}+2)\neq\emptyset. \tag{3.9}$$

- Suppose $n\in\mathbb{N}$ with $\mathrm{ord}_2(n)=1$, i.e., $n\equiv 2 \pmod 4$. Then

$$\{l_{a,b,m}(n):\ m\in \mathbb{Z}\} \subseteq Q(L_2)\cap((4\mathbb{Z}+1)\cup(4\mathbb{Z}+2))\neq\emptyset. \tag{3.10}$$

- Suppose $n\in\mathbb{N}$ with $\mathrm{ord}_2(n)=2$, i.e., $n\equiv 4 \pmod 8$. Then

$$\{l_{a,b,m}(n):\ m\in 2\mathbb{Z}+1\} \subseteq Q(L_2)\cap(8\mathbb{Z}+3)\neq\emptyset. \tag{3.11}$$

- Suppose $n\in\mathbb{N}$ with $\mathrm{ord}_2(n)=3$, i.e., $n\equiv 8 \pmod{16}$. Then

$$\{l_{a,b,m}(n):\ m\in 4\mathbb{Z}\} \subseteq Q(L_2)\cap(16\mathbb{Z}+8)\neq\emptyset. \tag{3.12}$$

Also, by Lemma 2.4(i),

$$\mathbb{Z}_2^{\times}\mathbb{Q}_2^{\times 2} \subseteq \mathbb{Q}_2^{\times} = \theta_{\mathbb{Q}_2}(L_2). \tag{3.13}$$

Case II. $\mathrm{ord}_2(a^2+b^2)=1$.

Recall that $a^2+b^2=2p^k$ for some prime $p\equiv 1 \pmod 4$ and some $k\in\mathbb{N}$ in this case. Then

$$L_2 \cong \langle 1\rangle \perp \langle 2\mu\rangle \perp \langle 2\mu\rangle$$

for some $\mu\in\mathbb{Z}_2^{\times}$ with $\mu\equiv 1 \pmod 4$. Noting that $\boldsymbol{e}_1, \boldsymbol{e}_2+2\boldsymbol{e}_3, -2\boldsymbol{e}_2+\boldsymbol{e}_3$ is also a $\mathbb{Z}_2$-basis of L_2, we further obtain

$$L_2 \cong \langle 1\rangle \perp \langle 2\mu\rangle \perp \langle 2\mu\rangle \cong \langle 1\rangle \perp \langle 10\mu\rangle \perp \langle 10\mu\rangle \cong \langle 1\rangle \perp \langle 2\rangle \perp \langle 2\rangle.$$

By (3.6), we see that

$$S_2(V_2) = 1 \neq (-1,-1)_2 = -1,$$

and hence V_2 is anisotropic by Lemma 2.2.

Also, one can easily verify that the following results hold.

- If $n\in\mathbb{N}$ with $n\equiv 1 \pmod 2$, then

$$\{l_{a,b,m}(n):\ m\in 2\mathbb{Z}+1\} \subseteq Q(L_2)\cap(4\mathbb{Z}+1)\neq\emptyset. \tag{3.14}$$

- Suppose $n\in\mathbb{N}$ with $\mathrm{ord}_2(n)=1$, i.e., $n\equiv 2 \pmod 4$. Then

$$\{l_{a,b,m}(n):\ m\in 2\mathbb{Z}+1\} \subseteq Q(L_2)\cap(8\mathbb{Z}+3)\neq\emptyset. \tag{3.15}$$

- Suppose $n \in \mathbb{N}$ with $\mathrm{ord}_2(n) = 2$, i.e., $n \equiv 4 \pmod 8$. Then

$$\{l_{a,b,m}(n) : \ m \in 4\mathbb{Z}\} \subseteq Q(L_2) \cap (16\mathbb{Z} + 8) \neq \emptyset. \tag{3.16}$$

- Suppose $n \in \mathbb{N}$ with $\mathrm{ord}_2(n) = 3$, i.e., $n \equiv 8 \pmod{16}$. Then

$$\begin{aligned} &\left\{ l_{a,b,m}(n) : \ m \in 4\mathbb{Z} \text{ and } \frac{m}{4} \equiv \frac{(a^2+b^2)n - 16}{32} \pmod 2 \right\} \\ &\subseteq Q(L_2) \cap ((64\mathbb{Z} + 16) \cup (64\mathbb{Z} + 32)) \neq \emptyset. \end{aligned} \tag{3.17}$$

In addition, by Lemma 2.4(ii),

$$\mathbb{Z}_2^{\times}\mathbb{Q}_2^{\times 2} \subseteq \mathbb{Q}_2^{\times} = \theta_{\mathbb{Q}_2}(L_2). \tag{3.18}$$

After the discussions of the above two cases, combining (3.13) and (3.18) with (3.3) and (3.5), by Lemma 2.3 we obtain that there is a unique spinor genus in $\mathrm{gen}(L)$, i.e.,

$$\mathrm{spn}(L) = \mathrm{gen}(L),$$

and

$$\{v \in \Omega_{\mathbb{Q}} \setminus \{\infty\} : \text{the associated quadratic space of } L_v \text{ is anisotropic}\} = \{2\}. \tag{3.19}$$

Now for any sufficiently large integer n with $\mathrm{ord}_2(n) \le C$, we write $n = 16^{\delta} \cdot n_1$, where

$$\delta = \max\{s \in \mathbb{N} : \ n \equiv 0 \pmod{16^s}\}.$$

As $\mathrm{ord}_2(n) \le C$, we see that n_1 is also a sufficiently large integer. Thus, we may assume that

$$\left((a^2+b^2)n_1\right)^{1/4} - \left(b^2 n_1\right)^{1/4} > 8.$$

This implies that there is an integer α such that

$$(\alpha + i)^2 \in \left[\left(b^2 n_1\right)^{1/2}, \left((a^2+b^2)n_1\right)^{1/2}\right] \tag{3.20}$$

for $i = 0, 1, 2, \cdots, 7$, that is, there are at least eight consecutive squares in the above interval. Using (3.20) and by (3.1), (3.2), (3.4), (3.8)–(3.12) and (3.14)–(3.17), we can choose a positive square $m^2 \in [(b^2 n_1)^{1/2}, ((a^2+b^2)n_1)^{1/2}]$ satisfying the following three conditions:

- $l_{a,b,m^2}(n_1) \in Q(\mathrm{gen}(L))$,
- $\mathrm{ord}_2(l_{a,b,m^2}(n_1)) \le 5$,
- $\gcd(m, p) = 1$.

This, together with $\mathrm{spn}(L) = \mathrm{gen}(L)$, (3.19) and Lemma 2.5, implies that

$$l_{a,b,m^2}(n_1) \in Q(L).$$

Thus, there exist $u \in \mathbb{Z}$ and $z, w \in \mathbb{N}$ such that

$$(a^2+b^2)n_1 - m^4 = u^2 + (a^2+b^2)z^2 + (a^2+b^2)w^2. \tag{3.21}$$

As $\gcd(a, b) = 1$, there exist $s, t \in \mathbb{Z}$ such that

$$m^2 = as + bt.$$

Also, by this we obtain

$$(\pm(at - bs))^2 \equiv -(as + bt)^2 \equiv -m^4 \pmod{a^2 + b^2}. \tag{3.22}$$

We first consider the case $a^2 + b^2 > 2$. Recall that the multiplicative group

$$\big(\mathbb{Z}/(a^2 + b^2)\mathbb{Z}\big)^\times = \big\{c \bmod (a^2 + b^2)\mathbb{Z} : \ \gcd(c, a^2 + b^2) = 1\big\}$$

is a cyclic group of even order.

Suppose $\gcd(m, a^2 + b^2) = 1$. Then by the property of even cyclic group and (3.22),

$$u \equiv \pm(at - bs) \pmod{a^2 + b^2}$$

are exactly all the solutions of the equation

$$u^2 \equiv -m^4 \pmod{a^2 + b^2}$$

in the even cyclic group $(\mathbb{Z}/(a^2 + b^2)\mathbb{Z})^\times$. By this, (3.21) and replacing u by $-u$ if necessary, we may set

$$u = (a^2 + b^2)u' + (at - bs)$$

for some $u' \in \mathbb{Z}$. Using this and (3.21) again, we deduce that

$$n_1 = (s - bu')^2 + (t + au')^2 + z^2 + w^2.$$

Letting $x = s - bu'$, $y = t + au'$ and noting that $m^2 = as + bt$, we obtain

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = n_1, \\ ax + by = m^2. \end{cases}$$

Suppose $\gcd(m, a^2 + b^2) > 1$. Then by the choice of m, in this case, we have $2 \nmid ab$, $\mathrm{ord}_2(a^2+b^2) = 1$ and $\gcd(m, a^2+b^2) = 2$. As $a^2+b^2 > 2$, as mentioned before, the multiplicative group

$$\big(\mathbb{Z}/p^k\mathbb{Z}\big)^\times = \big\{c \bmod p^k\mathbb{Z} : \ \gcd(c, p^k) = 1\big\}$$

is also a cyclic group of even order, where

$$p^k = (a^2 + b^2)/2.$$

As $\gcd(m, p) = 1$, we see that

$$u \equiv \pm(at - bs) \pmod{p^k}$$

are exactly all the solutions of the equation

$$u^2 \equiv -m^4 \pmod{p^k}$$

in the even cyclic group $(\mathbb{Z}/p^k\mathbb{Z})^\times$. Thus, by (3.21) and replacing u by $-u$ if necessary, we may assume

$$u = \frac{a^2 + b^2}{2}u' + (at - bs)$$

for some $u' \in \mathbb{Z}$. Note that $at - bs$ and u are both even and $(a^2 + b^2)/2$ is odd. We therefore obtain that $u' = 2u''$ is an even integer, that is,

$$u = (a^2 + b^2)u'' + (at - bs).$$

This, together with (3.21), implies that

$$n_1 = (s - bu'')^2 + (t + au'')^2 + z^2 + w^2.$$

Letting $x = s - bu''$ and $y = t + au''$ and noting that $as + bt = m^2$ again, we also have

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = n_1, \\ ax + by = m^2. \end{cases}$$

Now we consider the case $a = b = 1$. As $s + t = m^2$, (3.21) implies

$$u \equiv m^2 \equiv s + t \equiv s - t \pmod 2.$$

Thus, by replacing u by $-u$ if necessary, we may set

$$u = 2u''' + (t - s)$$

for some $u''' \in \mathbb{Z}$. By this and (3.21) again, we obtain

$$2n_1 - m^4(2u''' + (t - s))^2 + 2z^2 + 2w^2,$$

which implies that

$$n_1 = (s - u''')^2 + (t + u''')^2 + z^2 + w^2.$$

Setting $x = s - u''', y = t + u'''$, we obtain

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = n_1, \\ x + y = m^2. \end{cases}$$

By the above discussions, for n_1, we can always find a square $m^2 \in [(b^2n_1)^{1/2}, ((a^2+b^2)n_1)^{1/2}]$, $x, y \in \mathbb{Z}$ and $z, w \in \mathbb{N}$ such that

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = n_1, \\ ax + by = m^2. \end{cases}$$

Now applying Lemma 2.6, we immediately obtain that $x, y \in \mathbb{N}$. Recall that $n = 16^\delta \cdot n_1$. We thus directly obtain

$$\begin{cases} (4^\delta x)^2 + (4^\delta y)^2 + (4^\delta z)^2 + (4^\delta w)^2 = n, \\ a(4^\delta x) + b(4^\delta y) = (2^\delta m)^2 \in \mathcal{S}. \end{cases}$$

In view of the above, we have completed the proof. $\square$

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Declaration of competing interest

The authors declare that they have no conflict of interest.

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No data was used for the research described in the article.

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