

Merging Mechanisms for Ads and Organic Items in E-commerce Platforms

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Abstract

In contemporary e-commerce platforms, search result pages display two types of items: ad items and organic items. Ad items are determined through an advertising auction system, while organic items are selected by a recommendation system. These systems have distinct optimization objectives, creating the challenge of effectively merging these two components. Recent research has explored merging mechanisms for e-commerce platforms, but none have simultaneously achieved all desirable properties: incentive compatibility, individual rationality, adaptability to multiple slots, integration of inseparable candidates, and avoidance of repeated exposure for ads and organic items. This paper addresses the design of a merging mechanism that satisfies all these properties. We first provide the necessary conditions for the optimal merging mechanisms. Next, we introduce two simple and effective mechanisms, termed the generalized fix mechanism and the generalized change mechanism. Finally, we theoretically prove that both mechanisms offer guaranteed approximation ratios compared to the optimal mechanism in both simplest and general settings.

Introduction

The rapid advancement of the Internet has established e-commerce platforms as leading online marketplaces, offering unparalleled convenience for both sellers and buyers. Sellers can swiftly set up online stores, while buyers can easily search for products using just a few keywords. However, this ease of use presents a significant challenge for platforms: how to effectively and accurately present products. Consequently, e-commerce platforms are continuously working to enhance their methods for generating relevant and suitable product listings.

Current search result pages typically feature two main components: organic items and advertising items. Organic items are selected by a *recommendation system*, which prioritizes user experience and purchase intent to boost overall sales. In contrast, advertising items are determined through a *sponsored search auction*, where advertisers bid for display slots and pay extra fees for increased exposure. This auction system has become a major revenue source for online

platforms, generating an impressive \$225 billion in 2023. The challenge lies in integrating these two types of outputs into a cohesive list, a complex multi-objective optimization problem that has attracted significant attention from both academia and industry.

Several studies have explored this problem using various approaches. Traditionally, platforms allocate the top slots in search results to advertising items, with the recommendation system and sponsored search auction operating independently. This separation makes it difficult to achieve a globally optimal arrangement. Recently, some companies have adopted merging methods, incorporating winning advertising items into the recommendation system's candidate set and using machine learning to produce a unified list. Although this approach has improved average performance, it lacks theoretical support and does not always preserve desirable auction theory properties. Theoretical studies have also aimed to design mechanisms that address both organic items and advertisements, but they often struggle with issues such as repeated exposure (where the same product appears as both an ad and an organic item) and adaptability to multi-slot scenarios (which are prevalent in search engines or e-commerce platforms where several items are displayed on a single page).

Based on existing research, we observe that current methods fall short in addressing several key questions simultaneously: 1. How can we design a mechanism that effectively incorporates both ads and organic items while optimizing multiple objectives? 2. How can we maintain essential properties of advertising auctions, such as incentive compatibility (IC) and individual rationality (IR)? 3. How can we avoid the repeated display of an item in both its ad and organic forms? 4. How can we adapt to a multi-slot setting effectively? Each of these issues is both theoretically significant and practically valuable.

Main Contribution

To address the aforementioned questions, this work tackles the mechanism design problem for e-commerce platforms in a multi-slot setting. In this scenario, several candidate items can be displayed in two forms: as ads or as organic content. Our focus is on developing IC and IR merging mechanisms that effectively balance revenue and user experience, while avoiding the repeated display of the same item in both its ad

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and organic forms.

Our work extends the problem studied by (Li et al. 2024a), which addresses a similar issue in the single-slot context. We generalize their model to the more prevalent multi-slot scenario and propose two desirable mechanisms with guaranteed approximate ratios. Our main results are summarized as follows:

- In addition to the known characterization of feasible mechanisms, we introduce additional necessary conditions for the optimal mechanisms in both general and the simplest settings (three items and two slots). Notably, in the simplest setting, we find that a partial order always exists for displaying organic items. These conditions facilitate the construction of our mechanisms and simplify the subsequent performance analysis. We propose two simple mechanisms: the generalized fix (G-FIX) mechanism and the generalized change (G-CHANGE) mechanism, both of which are IC, IR, and applicable to the multi-slot setting.
- **G-FIX mechanism:** In this mechanism, we restrict each item to participate in the mechanism in only one form, either as an ad or as an organic item, thereby avoiding overlap and simplifying the allocation process. We then select the most suitable items for display. The G-FIX mechanism is shown to be a $(4/5)^3$ -approximate mechanism in the simplest setting. For a general setting with n items and k slots, under certain assumptions, the G-FIX mechanism is nearly optimal.
- **G-CHANGE mechanism:** Unlike the G-FIX mechanism, the G-CHANGE mechanism allows a set of ordered items to participate in both forms (ad and organic). Items are evaluated in reverse order to determine their displayed form, which mitigates the effect of the other items and ensures compliance with the IC condition. In the simplest setting, we demonstrate that the G-CHANGE mechanism is optimal. In a general setting, the G-CHANGE mechanism is a $1/2$ -approximate mechanism compared to the optimal merging mechanisms, without any assumption.

Due to space limitation, complete proofs can be referred to the full version.

Related Works

Our contribution lies within the field of Internet economics, specifically in the design of sponsored search auctions. Our work is related to recent theoretical paper (Li et al. 2024a), which considers a mechanism design problem to avoid the repeated display of ads and organic items, but their models are limited to the single-slot setting. In comparison to this paper, we generalize their models to multi-slot setting and design desirable mechanisms with guaranteed approximate ratios.

In the realm of online advertising and recommendation systems, significant research has been devoted to integrating ad and organic items. Our work addresses a multi-objective auction design problem that incorporates the impact of organic items. Here, we briefly provide the related studies in online advertising. Initially, sponsored ad-

vertising was implemented through generalized first-price (GFP) and generalized second-price (GSP) auctions. Subsequent research explore equilibrium analysis (Edelman, Ostrovsky, and Schwarz 2007; Varian 2007; Leme and Tardos 2010; Han and Liu 2015) and revenue enhancement (Lahaie and Pennock 2007; Thompson and Leyton-Brown 2013) for GSP and GFP auctions. Notably, Bachrach et al. (2014) investigate multi-objective optimization in the context of sponsored advertising.

Regarding the integration of ads and organic items, two primary directions have emerged. The first direction is to utilize advanced machine learning techniques to combine ads and organic items into a unified list. For example, several studies (Wang et al. 2019; Zhao et al. 2020; Liao et al. 2022; Zhu et al. 2023) model this process as a Markov decision process and apply reinforcement learning to generate the item list. Additionally, Liu et al. (2021) and Li et al. (2024b) employ automated mechanism design to integrate one ad with multiple organic items. The second direction focuses on optimization methods, particularly Lagrangian duality. Papers in this category (Zhang et al. 2018; Yan et al. 2020; Carrion et al. 2023) typically create a virtual ranking score using Lagrangian duality and allocate slots based on this score. Particularly, Li et al. (2023) addresses the multi-slot setting and proposes a Myerson-like merge mechanism aimed at achieving the optimal objective values. However, their mechanism does not effectively address the issue of repeated exposure of items in both ad and organic forms.

Observing from the aforementioned papers, we can find that merging ads and organic items is a popular research topic in recent years. Although there have been fruitful results in both academia and industry, these works have not addressed the four questions mentioned earlier in this section simultaneously, and at least one question remains unanswered. To the best of our knowledge, our paper is the first work to design IC and IR mechanisms that directly handle and avoid the overlap of ads and organic items, while also being applicable to the multi-slot model.

Model and Preliminaries

Merging Mechanism

In this section, we formally establish our model for developing a merging mechanism. Consider a search results page with k slots for displaying either ads or organic items. Suppose that there are n candidate items, denoted by $i \in [n]$. Each item i can be displayed in one of two forms: as an ad or as an organic item. We denote these forms by Ad_i for the ad form and Org_i for the organic form. We assume that each item can be displayed at most once and in only one form, either as Ad_i or Org_i . Therefore, Ad_i and Org_i will not appear simultaneously on the same search results page.

The two forms of an item, Ad_i and Org_i , can be distinguished by the user and may have different effects. Let α_i^A represent the click-through rate (CTR) of the ad form Ad_i , and α_i^O represent the CTR of the organic form Org_i . We assume that $\alpha_i^A < \alpha_i^O$ due to users' preference for organic items. Additionally, we suppose that the CTRs for all items are public information.

Additionally, the different forms may affect user experience in varying ways. For each item i , let γ_i^A and γ_i^O denote the user experience effect of the ad form and organic form, respectively. Suppose that $0 \leq \gamma_i^A \leq \gamma_i^O$, since users prefer more on organic items.

With the above symbols in mind, we introduce the merging mechanisms used in this paper. Before the mechanism is implemented, each item $i \in [n]$ (referred to as owner i for convenience) submits a bid b_i ¹. Let $\mathbf{b} = (b_1, \dots, b_n)$ denote the bid profile. We use $\mathbf{b}_{-i}$ to refer to the bid profile excluding owner i . We assume that each bid b_i is independently drawn from a prior distribution F_i known to the platform. Each F_i is further assumed to be a regular distribution² with support $[0, B_i]$ for some $B_i \geq 0$, which is a common assumption in economics. We now define a merging mechanism as follows.

Definition 1 (Merging Mechanism). *Collecting bids from owners, a merging mechanism is denoted by $\mathcal{M} = (x, y, p)$, where:*

1. $x(\mathbf{b}) = (x_i(\mathbf{b}))_{i \in [n]}$ and $y(\mathbf{b}) = (y_i(\mathbf{b}))_{i \in [n]}$ are the allocation rules for ad items and organic items, respectively. For each item $i \in [n]$, $x_i(\mathbf{b}) = 1$ indicates that its ad form Ad_i is displayed, otherwise $x_i(\mathbf{b}) = 0$. Similarly, $y_i(\mathbf{b}) = 1$ means that its organic form Org_i is displayed, $y_i(\mathbf{b}) = 0$ otherwise.
2. $p(\mathbf{b}) = (p_i(\mathbf{b}))_{i \in [n]}$ represents the payments charged to the owners when their ads are clicked. Specifically, if Ad_i is displayed ($x_i(\mathbf{b}) = 1$), the platform charges a payment $p_i(\mathbf{b})$ if the user clicks on Ad_i (with probability α_i^A). We assume that $p_i(\mathbf{b}) = 0$ if $x_i(\mathbf{b}) = 0$.

Given the definition of the merging mechanism, we define the expected click-through rate $X_i(\mathbf{b})$ and the expected payment $P_i(\mathbf{b})$ for item i as follows:

$$X_i(\mathbf{b}) = x_i(\mathbf{b})\alpha_i^A + y_i(\mathbf{b})\alpha_i^O$$

and

$$P_i(\mathbf{b}) = p_i(\mathbf{b}) \cdot x_i(\mathbf{b})\alpha_i^A.$$

Here, $X_i(\mathbf{b})$ represents the total probability that item i is clicked in either form, and $P_i(\mathbf{b})$ is the expected payment when the item's ad form is displayed and clicked. The utility $U_i(\mathbf{b})$ of owner i under the bid profile $\mathbf{b}$ is defined as:

$$U_i(\mathbf{b}) = b_i X_i(\mathbf{b}) - P_i(\mathbf{b}).$$

In this paper, we assume that item owners are utility maximizers and do not differentiate between the displayed forms of their items.

Since at most k items can be displayed and each item can appear in only one form (either ad or organic), the merging mechanism must satisfy the following feasibility constraints:

$$\sum_{i \in [n]} (x_i(\mathbf{b}) + y_i(\mathbf{b})) \leq k.$$

¹For simplicity, we use b_i to represent both the bid and the item's value, as our focus is on incentive-compatible mechanisms.

²A regular distribution, as defined by (Myerson 1981), satisfies that $b_i - (1 - F_i(b_i))/f_i(b_i)$ is monotone non-decreasing, where f_i is the p.d.f of F_i .

And for all item $i \in [n]$,

$$x_i(\mathbf{b}) + y_i(\mathbf{b}) \leq 1.$$

We aim for the merging mechanism to satisfy two desirable properties: incentive compatibility (IC) and individual rationality (IR).

Definition 2 (Incentive Compatibility). *A merging mechanism is incentive compatible if owners are motivated to submit their true bids. Formally, for any bid profile $\mathbf{b}$ and any owner $i \in [n]$, for any misreported bid b'_i , the following condition must hold:*

$$b_i X_i(b_i, \mathbf{b}_{-i}) - P_i(b_i, \mathbf{b}_{-i}) \geq b_i X_i(b'_i, \mathbf{b}_{-i}) - P_i(b'_i, \mathbf{b}_{-i}).$$

This ensures that the utility of reporting b_i truthfully is not less than the utility of reporting any other bid b'_i .

Definition 3 (Individual Rationality). *A merging mechanism is individually rational if each owner's utility is always non-negative, meaning that payments never exceed bids. Formally, for any bid profile $\mathbf{b}$ and any owner $i \in [n]$, the following must hold:*

$$p_i(\mathbf{b}) \leq b_i.$$

This ensures that the payment $p_i(\mathbf{b})$ charged to the owner is less than or equal to their bid b_i , guaranteeing that their utility is never negative.

For an e-commerce platform, the objective includes two main components: revenue and user experience. The revenue target is defined as the expected total payment. Formally, it is given by:

$$\text{Rev}(\mathcal{M}) := \mathbb{E}_{\mathbf{b} \sim F} \left[\sum_{i \in [n]} p_i(\mathbf{b}) x_i(\mathbf{b}) \alpha_i^A \right],$$

where F is the joint distribution on $\mathbf{b}$. The user experience target is defined as the total expected user experience, i.e.,

$$\text{UE}(\mathcal{M}) := \mathbb{E}_{\mathbf{b} \sim F} \left[\sum_{i \in [n]} (x_i(\mathbf{b})\gamma_i^A + y_i(\mathbf{b})\gamma_i^O) \right].$$

Note that the user experience of an item is independent of whether it is clicked. Once an item is displayed, its user experience is realized. Therefore, the platform's objective combines both revenue and user experience:

$$\text{OBJ}(\mathcal{M}) := \text{Rev}(\mathcal{M}) + \text{UE}(\mathcal{M}).$$

This formulation can be generalized to any weighted combination of revenue and user experience by incorporating weight coefficients into γ_i^A and γ_i^O . Thus, the merging mechanism design problem can be expressed as:

$$\begin{aligned} \max_{\mathcal{M}} \quad & \text{OBJ}(\mathcal{M}) \\ \text{s.t.} \quad & \text{IC, IR, Feasibility.} \end{aligned} \tag{1}$$

Let $\mathcal{M}^*$ denote the optimal mechanism for the problem. We say that a merging mechanism $\mathcal{M}$ is τ -approximate if the ratio of $\text{OBJ}(\mathcal{M})$ to $\text{OBJ}(\mathcal{M}^*)$ is at least τ . Formally, $\mathcal{M}$ is τ -approximate if $\text{OBJ}(\mathcal{M})/\text{OBJ}(\mathcal{M}^*) \geq \tau$.

Characterization of IC, IR, and Feasibility

In the rest of this section, we summarize key results from previous work that are crucial for characterizing IC, IR, and feasible merging mechanisms. These results will also aid in the construction and analysis of mechanisms in the subsequent sections. Notably, Myerson’s seminal work provides a fundamental framework for understanding IC and IR mechanisms.

Lemma 1 (Myerson (1981)). *A merging mechanism $\mathcal{M}$ satisfies IC and IR if and only if the following conditions are met:*

1. **Allocation Monotonicity:** *For any item $i \in [n]$, given the bids $\mathbf{b}_{-i}$ of the other items, its expected click-through rate is non-decreasing in b_i . Formally, for any b_i and b'_i with $b_i < b'_i$, it holds that*

$$X_i(b_i, \mathbf{b}_{-i}) \leq X_i(b'_i, \mathbf{b}_{-i}).$$

2. **Payment Identity:** *For any item $i \in [n]$ and any bid profile $\mathbf{b}$, the payment $P_i(\mathbf{b})$ is given by*

$$P_i(\mathbf{b}) = b_i X_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} X_i(t, \mathbf{b}_{-i}) dt.$$

Recent work by Li et al. (2024a) addresses a similar problem in the single-slot setting and provides necessary conditions for an IC, IR, and feasible merging mechanism. These conditions also apply to the multi-slot case. One important condition is form stability, which states that the allocation of an organic item depends solely on the bids of other owners and is unaffected by its own bid.

Lemma 2 (Form Stability (Li et al. 2024a)). *Suppose $\mathcal{M}$ satisfies IC, IR, and feasibility. Then, for any item $i \in [n]$, the allocation $y_i(\mathbf{b})$ of an organic item depends only on the bids $\mathbf{b}_{-i}$ of other owners and is unaffected by the bid b_i of the owner i . Formally, for any $\mathbf{b}_{-i}$ and for any $b_i \neq b'_i$, we have:*

$$y_i(b_i, \mathbf{b}_{-i}) = y_i(b'_i, \mathbf{b}_{-i}).$$

Leveraging Myerson’s lemma (Lemma 1) and the condition of form stability (Lemma 2), we can express the payment $p_i(\mathbf{b})$ for each item concisely and simplify the total payment into an elegant form.

Lemma 3 (Li et al. (2024a)). *Suppose that $\mathcal{M}$ satisfies IC, IR, and feasibility, and consequently form stability. For any bid profile $\mathbf{b}$, if Ad_i is displayed, then the payment $p_i(\mathbf{b})$ is given by*

$$p_i(\mathbf{b}) = \inf\{b'_i \geq 0 : x_i(b'_i, \mathbf{b}_{-i}) = 1\},$$

which is the critical bid required for Ad_i to be displayed.

Lemma 4 (Li et al. (2024a)). *If $\mathcal{M}$ satisfies IC, IR, feasibility and form stability, the revenue can be expressed as*

$$\text{Rev}(\mathcal{M}) = \mathbb{E}_{\mathbf{b} \sim F} \left[\sum_{i \in [n]} \phi_i(b_i) x_i(\mathbf{b}) \alpha_i^A \right],$$

where $\phi_i(b_i) = b_i - (1 - F_i(b_i))/f_i(b_i)$ represents Myerson’s virtual value function.

Note that, although the conclusions in Lemma 3 and 4 resemble the original results from Myerson (1981), they cannot be directly derived from Myerson’s lemma due to the presence of two possible displayed forms for each item in our model. Therefore, form stability (Lemma 2) is essential to establish these results.

Given the above lemmas, a merging mechanism $\mathcal{M}$ is fully characterized by the allocation rules $x(\mathbf{b})$ and $y(\mathbf{b})$ and can be denoted by $\mathcal{M}^{x,y} = (x, y)$. An ad item contributes to both revenue and user experience. Specifically, we define $a_i = \phi_i(b_i) \alpha_i^A + \gamma_i^A$ as the contribution of Ad_i , and $o_i = \gamma_i^O$ as the contribution of Org_i to the objective function. By using $\mathbf{a} = (a_1, \dots, a_n)$ as a proxy for $\mathbf{b}$, the optimization problem (1) can be reformulated as:

$$\begin{aligned} \max_{x,y} \quad & \text{OBJ}(\mathcal{M}^{x,y}) = \mathbb{E}_{\mathbf{a} \sim G} \left[\sum_{i \in [n]} (a_i x_i(\mathbf{a}) + o_i y_i(\mathbf{a})) \right] \\ \text{s.t.} \quad & \sum_{i \in [n]} (x_i(\mathbf{a}) + y_i(\mathbf{a})) \leq k, \quad \forall \mathbf{a}, \\ & x_i(\mathbf{a}) + y_i(\mathbf{a}) \leq 1, \quad \forall i \in [n], \mathbf{a}, \\ & y_i(a_i, \mathbf{a}_{-i}) = y_i(a'_i, \mathbf{a}_{-i}), \quad \forall i \in [n], \mathbf{a}_{-i}, a_i, a'_i, \\ & x_i(a_i, \mathbf{a}_{-i}) \leq x_i(a'_i, \mathbf{a}_{-i}), \quad \forall i \in [n], \mathbf{a}_{-i}, a_i < a'_i, \end{aligned} \quad (2)$$

where G_i is the (regular) distribution of a_i , induced by F_i , and $G = \times_{i \in [n]} G_i$ is the joint distribution on $\mathbf{a}$. For convenience, we will refer to this optimization problem as the merging problem.

Finally, we introduce a mathematical notation that will be frequently used in the following sections. Given a set S of finite real numbers and $k \leq |S|$, we define $\max^{(k)}$ as an operator that computes the sum of the largest k numbers in S . For example, if $S = \{1, 2, 3\}$, then $\max^{(2)} S = 3 + 2 = 5$.

Form Stability in the Optimal Mechanisms

Before introducing our merging mechanisms, we first establish necessary conditions, specifically form stability, that the optimal mechanisms must satisfy for the merging problem. These conditions guide the construction of our mechanisms and facilitate performance analysis. In this section, we examine form stability in two scenarios: the general case with n items and k slots, and the simplest multi-slot case with 3 items and 2 slots. To avoid ambiguity, we refer to the mechanism in the former scenario as the k -out-of- n mechanism, and in the latter as the 2-out-of-3 mechanism. Additionally, for any owner $i \in [n]$, we denote the allocation rules in the optimal mechanism $\mathcal{M}^*$ by $x_i^*(\mathbf{a})$, $y_i^*(\mathbf{a})$, and $X_i^*(\mathbf{a})$.

We begin by characterizing the optimal k -out-of- n mechanism. According to form stability of any merging mechanism (Lemma 2), an owner’s change in bid does not affect the allocation of her organic item. This implies that if an item is displayed in its organic form in the optimal mechanism, it will continue to be displayed in that form regardless of changes in its own bid. Intuitively, knowing that an item must be displayed in its organic form reduces the problem to designing a $(k-1)$ -out-of- $(n-1)$ mechanism. Therefore, by the optimality, the displayed forms of the remaining $n-1$ items in the optimal mechanism will also remain unchanged. Similarly, if an item is displayed in its ad form, as long as

its displayed form remains unchanged, bid changes will not affect the others. We formally summarize this discussion in Proposition 1.

Proposition 1. *In the optimal k -out-of- n mechanism satisfying IC and IR, for any $i \in [n]$, $\mathbf{a}_{-i}$, and a_i , the following three statements hold:*

1. *If $y_i^*(a_i, \mathbf{a}_{-i}) = 1$, it holds that for any j and any a'_i ,*

$$y_j^*(a'_i, \mathbf{a}_{-i}) = y_j^*(a_i, \mathbf{a}_{-i}) \text{ and } x_j^*(a'_i, \mathbf{a}_{-i}) = x_j^*(a_i, \mathbf{a}_{-i}).$$
2. *If $x_i^*(a_i, \mathbf{a}_{-i}) = 0$, it holds that for any j and $a'_i \leq a_i$,*

$$y_j^*(a'_i, \mathbf{a}_{-i}) = y_j^*(a_i, \mathbf{a}_{-i}) \text{ and } x_j^*(a'_i, \mathbf{a}_{-i}) = x_j^*(a_i, \mathbf{a}_{-i}).$$
3. *If $x_i^*(a_i, \mathbf{a}_{-i}) = 1$, it holds that for any j and $a'_i > a_i$,*

$$y_j^*(a'_i, \mathbf{a}_{-i}) = y_j^*(a_i, \mathbf{a}_{-i}) \text{ and } x_j^*(a'_i, \mathbf{a}_{-i}) = x_j^*(a_i, \mathbf{a}_{-i}).$$

As a corollary, in the optimal mechanism, if an owner's bid adjustment does not alter her own displayed form, it will not affect the displayed forms of others.

Corollary 1. *In the optimal k -out-of- n mechanism satisfying IC and IR, given any $\mathbf{a}_{-i}$, if owner i changes from a_i to a'_i such that $X_i^*(a_i, \mathbf{a}_{-i}) = X_i^*(a'_i, \mathbf{a}_{-i})$, then for any j , X_j^* remains unchanged.*

From the characterization of the optimal k -out-of- n mechanism, it is clear that form stability holds when only one owner changes her bid. However, the outcome becomes uncertain when more than two owners change their bids simultaneously. In the rest of this section, we examine the simplest multi-slot scenario with three items and two slots, further exploring form stability when all owners' bids can change simultaneously. Generally, maintaining a fixed allocation of ad items, $x(\mathbf{a})$, is challenging, so we focus on the allocation of organic items, $y(\mathbf{a})$. Intuitively, if under a given bid profile, the optimal mechanism favors owner i over owner j (i.e., $y_i(\mathbf{a}) = 1$ and $y_j(\mathbf{a}) = 0$) in terms of organic items, it indicates that owner i contributes more to the objective than owner j , regardless of the bids. The following lemma supports this intuition by demonstrating that in the simplest setting, if there exists a bid profile where the optimal mechanism displays owner i 's organic form but not owner j 's, then it is impossible for the optimal mechanism to display owner j 's organic form without also displaying owner i 's in any bid profile.

Lemma 5. *In the optimal 2-out-of-3 mechanism, if there exists some $\mathbf{a}$ such that $y_i^*(\mathbf{a}) = 1$ and $y_j^*(\mathbf{a}) = 0$, then for any $\mathbf{a}'$, it is not possible that $y_i^*(\mathbf{a}') = 0$ but $y_j^*(\mathbf{a}') = 1$.*

Using Lemma 5, we can compare the priority of any two items' organic forms in the optimal mechanism. Consequently, with three items, the optimal mechanism induces a partial order among the organic items, as outlined in Theorem 1. Moreover, we can demonstrate that the bids of items lower in this order can influence the allocation of items higher in the order, while the reverse is not true.

Theorem 1. *In the optimal 2-out-of-3 mechanism, there exists a partial order (i_1, i_2, i_3) such that $y_{i_1}^* \geq y_{i_2}^* \geq y_{i_3}^*$, and $y_{i_1}^* = y_{i_1}^*(a_{i_2}, a_{i_3})$, $y_{i_2}^* = y_{i_2}^*(a_{i_3})$, $y_{i_3}^* = 0$.*

Generalized Fix Mechanism

In the last section, we provide a more detailed characterization of the optimal merging mechanisms, but we find that designing the optimal mechanism is not trivial. Intuitively, one might think that ranking all a_i and o_i in non-increasing order and selecting the k items with the highest values, while avoiding conflicts in form, would be sufficient based on the objective function in the merging problem (2). Yet, due to a key feature of our model—where displaying an item in its organic form yields greater utility—this approach may not be incentive compatible. Some items could manipulate their displayed form from ad to organic by misreporting, aiming for better utility. Thus, a Myerson-like mechanism cannot be directly applied to our problem.

Starting from this section, we introduce two simple merging mechanisms and analyze their performance relative to the optimal mechanism. In this section, we first introduce the generalized fix (G-FIX) mechanism and then present the generalized change mechanism in next section. The primary challenge in designing a merging mechanism lies in determining the displayed form of each item. To address this, we restrict some items to be displayed only in their organic form, while others are restricted to ad form. We then select the k items with the highest contribution, leading to the G-FIX- I mechanism.

Definition 4 (G-FIX- I Mechanism). *Define a set $I \subseteq [n]$ such that items in I can only be displayed in organic form, while items in $[n] \setminus I$ can only be displayed in ad form. The G-FIX- I mechanism, denoted by $\mathcal{M}_I^F$, is defined as follows: Let L be*

$$L = \{o_i : i \in I\} \cup \{a_i : i \in [n] \setminus I\}.$$

Then, for each item $i \in [n]$:

$$x_i(\mathbf{a}) = \begin{cases} 1 & \text{if } i \notin I \text{ and } a_i > \max^{(k+1)} L - \max^{(k)} L, \\ 0 & \text{otherwise.} \end{cases}$$

$$y_i(\mathbf{a}) = \begin{cases} 1 & \text{if } i \in I \text{ and } o_i > \max^{(k+1)} L - \max^{(k)} L, \\ 0 & \text{otherwise.} \end{cases}$$

The objective value of $\mathcal{M}_I^F$ is then calculated by:

$$\text{OBJ}(\mathcal{M}_I^F) = \mathbb{E}_{\mathbf{a} \sim G} \left[\max^{(k)} \{o_i : i \in I\} \cup \{a_i : i \in [n] \setminus I\} \right].$$

It is easy to verify that the constraints in the merging problem (2) are satisfied by the allocation rules of the G-FIX- I mechanism, ensuring that $\mathcal{M}_I^F$ is IC, IR, and feasible. By evaluating all $\mathcal{M}_I^F$ mechanisms with $|I| \leq k$ and selecting the one with the highest objective value, we obtain our first merging mechanism, the G-FIX mechanism.

Definition 5 (G-FIX Mechanism). *The G-FIX mechanism, denoted by $\mathcal{M}^F$, selects the mechanism with the highest objective value among all $\mathcal{M}_I^F$ mechanisms, where $I \subseteq [n]$ and $|I| \leq k$.*

We next compare the performance of the G-FIX mechanism to the optimal merging mechanism, starting with the simplest setting of 3 items and 2 slots.

Theorem 2. For content merging problem with 3 items and 2 slots, $\mathcal{M}^F$ is $(4/5)^3$ -approximate relative to the optimal mechanism.

Returning to the general setting with n items and k slots, we also demonstrate that $\mathcal{M}^F$ has a guaranteed approximate ratio under certain assumptions.

Theorem 3. For content merging problem with n items and k slots, when $2k \leq n$ and the bid distributions are identical, $\mathcal{M}^F$ is C_{n-k}^k/C_n^k -approximate mechanism.

Proof Sketch. Without loss of generality, we assume that $o_1 \geq o_2 \geq \dots \geq o_n$ in this proof. Before giving the proof of this theorem, we first introduce a mathematical lemma to aid us analyze the approximate ratio.

Lemma 6. When $1 \leq l \leq k$, we have

$$\frac{1 - \sum_{i=1}^l C_{n-k}^{l-i} x^{(n-k)-(l-i)} (1-x)^{(l-i)}}{1 - \sum_{i=1}^l C_n^{l-i} x^{n-(l-i)} (1-x)^{(l-i)}} \geq \frac{C_{n-k}^l}{C_n^l}.$$

Firstly, observe the objective of the merging problem (2). We know that for the optimal mechanism $\mathcal{M}^*$, its objective is always bounded by $\max^{(k)}\{o_1 \dots o_k, a_1 \dots a_n\}$. Denote by λ_i the i -th largest number in the set $\{o_1 \dots o_k, a_1 \dots a_n\}$. We have

$$\begin{aligned} \text{OBJ}(\mathcal{M}^*) &\leq \mathbb{E}_{\mathbf{a} \sim G} \left[\max^{(k)}\{o_1 \dots o_k, a_1 \dots a_n\} \right] \\ &= \sum_{i=1}^k \int_0^\infty \mathbb{P}_{\mathbf{a} \sim G}(\lambda_i \geq t) dt. \end{aligned} \quad (3)$$

On the other hand, we want to find a lower bound on the objective of $\mathcal{M}^F$. Since $\mathcal{M}^F$ selects the best mechanism among $\mathcal{M}_I^F$, herein, we focus on a specific $\mathcal{M}_I^F$ with $I = \{1, 2, \dots, k\}$, whose objective is $\max^{(k)}\{o_1, \dots, o_k, a_{k+1} \dots a_n\}$. Denote by θ_i the i -th largest number in the set $\{o_1 \dots o_k, a_{k+1} \dots a_n\}$. We have

$$\begin{aligned} \text{OBJ}(\mathcal{M}^F) &\geq \mathbb{E}_{\mathbf{a} \sim G} \left[\max^{(k)}\{o_1 \dots o_k, a_{k+1} \dots a_n\} \right] \\ &= \sum_{i=1}^k \int_0^\infty \mathbb{P}_{\mathbf{a} \sim G}(\theta_i \geq t) dt. \end{aligned} \quad (4)$$

With a calculation on the probability of order statistics, and by Lemma 6 and the following inequality

$$1 = \frac{C_{n-k}^0}{C_n^0} > \dots > \frac{C_{n-k}^l}{C_n^l} > \frac{C_{n-k}^{l+1}}{C_n^{l+1}} > \dots > \frac{C_{n-k}^k}{C_n^k},$$

we can replace the simplified terms in inequality (4) by the corresponding simplified terms in inequality (3) with C_{n-k}^k/C_n^k times and we get

$$\text{OBJ}(\mathcal{M}^F) \geq \frac{C_{n-k}^k}{C_n^k} \text{OBJ}(\mathcal{M}^*).$$

In conclusion, we show that $\mathcal{M}^F$ is C_{n-k}^k/C_n^k -approximate relative to the optimal mechanism $\mathcal{M}^*$. $\square$

Based on Theorem 3, if $n \geq k^2/\epsilon + k$ for any given $0 < \epsilon < 1$ —that is, if the number of candidate items n is sufficiently large compared to the slots k —we can further show that $\mathcal{M}^F$ is a near-optimal mechanism.

Theorem 4. For content merging problem with n items and k slots, given any $0 < \epsilon < 1$, when $n \geq k^2/\epsilon + k$ and the bid distributions are identical, $\mathcal{M}^F$ is a nearly-optimal mechanism with approximate ratio $1 - \epsilon$.

Proof. We directly analyze the approximate ratio of the mechanism $\mathcal{M}^F$, presented in Theorem 3, i.e.,

$$\begin{aligned} \frac{C_{n-k}^k}{C_n^k} &= \frac{n-k}{n} \cdot \frac{n-k-1}{n-1} \dots \frac{n-2k+1}{n-k+1} \\ &> \left(\frac{n-2k}{n-k} \right)^k = \left(1 - \frac{k}{n-k} \right)^k. \end{aligned}$$

Applying the general inequality $(1+x)^a \geq 1+ax$ where $a > 1$, we obtain:

$$\left(1 - \frac{k}{n-k} \right)^k \geq 1 - \frac{k^2}{n-k}.$$

Therefore, to ensure that the approximate ratio is greater than $1 - \epsilon$, it is sufficient to require:

$$\frac{k^2}{n-k} \leq \epsilon \implies n \geq \frac{k^2}{\epsilon} + k.$$

This completes the proof. $\square$

Generalized Change Mechanism

In the last section, we introduce the G-FIX mechanism, which restricts each owner to participate in the mechanism with a specific form. However, it lacks flexibility and results in a loss of objective value. In this section, we relax this constraint by incorporating an order factor, leading to the establishment of the generalized change (G-CHANGE) mechanism.

We begin by introducing the generalized change- I (G-CHANGE- I) mechanism. The G-CHANGE- I mechanism defines an ordered set I in which items can be displayed either as ads or in their organic form, while all other items are restricted to ads only. The final display forms are determined by a recursive rule, which we will detail later. Compared to the G-FIX mechanism, the G-CHANGE mechanism offers more flexibility by allowing some items to have alternative forms and introducing a criterion to guide the decision-making process. The detailed definition is as follows.

Definition 6 (Generalized Change- I Mechanism). Let $I = \{i_1, i_2, \dots, i_k\}$ be a set of ordered items, where $k \leq n$. Given the bid profile $\mathbf{a}$, define $w_{i_t}^{\mathbf{a}}, t \in [k]$, recursively as follows:

$$\begin{cases} w_{i_1}^{\mathbf{a}} = \mathbb{E}_{a_{i_1}} [\max^{(k)}\{a_1, \dots, a_n\}] - R_1 & t = 1, \\ w_{i_t}^{\mathbf{a}} = \mathbb{E}_{a_{i_t}} [\max\{o_{t-1} + R_{t-1}, w_{i_{t-1}}^{\mathbf{a}} + R_{t-1}\}] - R_t & t > 1, \end{cases}$$

where

$$R_t = \sum_{j=1}^{t-1} o_{i_j} + \max^{(k-t)} \{a_i \mid i \notin \{i_1, \dots, i_t\}\}.$$

Define a threshold s^* as

$$s^* := \max \{s \in [k] \mid \forall t > s, o_{i_t} < w_{i_t}^a \text{ and } o_{i_s} \geq w_{i_s}^a\}.$$

Having the above symbols, generalized change-I mechanism is given by

$$x_i(\mathbf{a}) = \begin{cases} 1 & \text{if } i \notin \{i_1, \dots, i_{s^*}\} \\ & \text{and } a_i > \max^{(k-s^*+1)} \{a_j \mid j \notin \{i_1, \dots, i_{s^*}\}\} \\ & - \max^{(k-s^*)} \{a_j \mid j \notin \{i_1, \dots, i_{s^*}\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$y_i(\mathbf{a}) = \begin{cases} 1 & \text{if } i \in \{i_1, \dots, i_{s^*}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Herein, we elaborate on the decision rule in the G-CHANGE-I mechanism. For the items in I , we evaluate them in reverse order. Take item i_t as an example. R_t represents the total contribution when the first $t-1$ items in I are displayed in organic form, and the remaining $k-t$ items with the highest a_i values are selected in ad form. w_{i_t} can be interpreted as the expected marginal contribution of item i_t 's ad form, relative to the local optimal value. Thus, the display form of item i_t is determined by comparing w_{i_t} with o_{i_t} in reverse order, identifying the first item for which $o_{i_t} \geq w_{i_t}$, denoted by s^* . Finally, the mechanism displays the first s^* items in I in organic form and selects $k-s^*$ ads from the remaining items based on their a_i values.

It is not hard to find that the decision rule is independent of the items' bids. Additionally, the reverse decision also guarantees that flexible items in I cannot affect bilaterally. These two features indicates that the G-CHANGE-I mechanism satisfies the IC condition. Furthermore, the objective function of the mechanism $\mathcal{M}_I^C$ can be expressed as:

$$\text{OBJ}(\mathcal{M}_I^C) = \mathbb{E}_{\mathbf{a} \sim G} \left[\max \left\{ \sum_{j=1}^k o_{i_j}, w_{i_k}^a + \sum_{j=1}^{k-1} o_{i_j} \right\} \right].$$

We can evaluate all possible ordered subset I and select the one with the highest objective value as the generalized change mechanism:

Definition 7 (Generalized Change Mechanism). *The generalized change mechanism, denoted by $\mathcal{M}^C$, executes the mechanism with the highest objective value among $\mathcal{M}_I^C$, where I is any ordered subset (permutation) satisfying $I \subseteq [n]$ and $|I| = k$.*

Next, we examine the performance of the G-CHANGE mechanism in the simplest setting.

Theorem 5. *For content merging problem with 3 items and 2 slots, $\mathcal{M}^C$ is optimal.*

Proof. By Theorem 1, we know that there exists a partial order (i_1, i_2, i_3) , which is a permutation of $\{1, 2, 3\}$, such that $y_{i_1}^* \geq y_{i_2}^* \geq y_{i_3}^* = 0$, and $y_{i_1}^* = y_{i_1}^*(a_{i_2}, a_{i_3})$, $y_{i_2}^* = y_{i_2}^*(a_{i_3})$, $y_{i_3}^* = 0$. Therefore, the objective value of the optimal mechanism $\mathcal{M}^*$ is given by:

$$\text{OBJ}(\mathcal{M}^*) = \mathbb{E}_{\mathbf{a} \sim G} \left[\sum_{i=1}^3 (x_i^*(\mathbf{a})a_i + y_i^*(\mathbf{a})o_i) \right]$$

$$= \mathbb{E}_{a_{i_3}} \left[\max \left\{ o_{i_1} + o_{i_2}, \mathbb{E}_{a_{i_2}} \left[\max \left\{ o_{i_1} + \max \{a_{i_2}, a_{i_3}\}, \right. \right. \right. \right. \\ \left. \left. \left. \mathbb{E}_{a_{i_1}} [\max^{(2)} \{a_{i_1}, a_{i_2}, a_{i_3}\}] \right\} \right] \right\} \right].$$

Observing from the objective of G-CHANGE-I, we can find that if we let $I = \{i_1, i_2\}$, it holds that $\text{OBJ}(\mathcal{M}^*) = \text{OBJ}(\mathcal{M}_{\{i_1, i_2\}}^C)$. On the other hand, we know that $\text{OBJ}(\mathcal{M}_{\{i_1, i_2\}}^C) \leq \text{OBJ}(\mathcal{M}^C)$. Thus, the G-CHANGE mechanism $\mathcal{M}^C$ is optimal in the simplest setting. $\square$

Finally, in the general case with non-identical distributions, the G-CHANGE mechanism still achieves guaranteed performance relative to the optimal mechanism.

Theorem 6. *For content merging problem with n items and k slots, $\mathcal{M}^C$ is $1/2$ -approximate relative to the optimal mechanism $\mathcal{M}^*$.*

Proof. Without loss of generality, let $o_1 \geq o_2 \geq \dots \geq o_n$. Firstly, the objective value of the optimal mechanism is upper bounded by

$$\begin{aligned} \text{OBJ}(\mathcal{M}^*) &\leq \mathbb{E}_{\mathbf{a} \sim G} [\max^{(k)} \{o_1 \dots o_k, a_1 \dots a_n\}] \\ &\leq \sum_{i=1}^k o_i + \mathbb{E}_{\mathbf{a} \sim G} [\max^{(k)} \{a_1, \dots, a_n\}]. \end{aligned}$$

Let $I = \{1, \dots, k\}$ and we can get a low bound of $\mathcal{M}^C$

$$\begin{aligned} \text{OBJ}(\mathcal{M}^C) &\geq \text{OBJ}(\mathcal{M}_{\{1, \dots, k\}}^C) \\ &\geq \max \left\{ \sum_{i=1}^k o_i, \mathbb{E}_{\mathbf{a} \sim G} [\max^{(k)} \{a_1, \dots, a_n\}] \right\}. \end{aligned}$$

Thus, by triangle inequality, we obtain

$$\begin{aligned} \frac{\text{OBJ}(\mathcal{M}^C)}{\text{OBJ}(\mathcal{M}^*)} &\geq \frac{\max \left\{ \sum_{i=1}^k o_i, \mathbb{E}_{\mathbf{a} \sim G} [\max^{(k)} \{a_1, \dots, a_n\}] \right\}}{\sum_{i=1}^k o_i + \mathbb{E}_{\mathbf{a} \sim G} [\max^{(k)} \{a_1, \dots, a_n\}]} \\ &\geq \frac{1}{2}. \end{aligned}$$

$\square$

Conclusion

In this paper, we tackle the merging mechanism design problem in multiple slots, addressing an open question from (Li et al. 2024a), where items can be displayed as either ads or in organic form. These different forms impact platform performance outcomes, such as click-through rates and user experience. We propose two straightforward mechanisms that satisfy the incentive compatibility, individual rationality, and feasibility conditions, and analyze their approximation ratio in both simple and general settings. Future research directions include: first, designing the optimal merging mechanisms or improving approximation ratio within this framework; and second, exploring the merging problem in heterogeneous slots, which presents additional challenges.

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Appendix

Missing Proofs in Section

Proof of Proposition 1

Proof. To analyze how changing the bid of a single item affects the objective function, for any $i \in [n]$, a_i and $\mathbf{a}_{-i}$, we can decompose the objective function in problem (2) as follows:

$$\begin{aligned} \max_{x,y} \text{OBJ}(\mathcal{M}^{x,y}) &= \mathbb{E}_{\mathbf{a} \sim G} \left[\sum_{j=1}^n (a_j x_j(\mathbf{a}) + o_j y_j(\mathbf{a})) \right] \\ &= \mathbb{E}_{\mathbf{a}} \left[\mathbb{I}[\mathbf{a}_{-i}] \mathbb{I}[a'_i \leq a_i] \left(\sum_{j=1}^n a_j x_j(\mathbf{a}) + o_j y_j(\mathbf{a}) \right) \right] \quad (\text{s}_1) \\ &+ \mathbb{E}_{\mathbf{a}} \left[\mathbb{I}[\mathbf{a}_{-i}] \mathbb{I}[a'_i > a_i] \left(\sum_{j=1}^n a_j x_j(\mathbf{a}) + o_j y_j(\mathbf{a}) \right) \right] \quad (\text{s}_2) \\ &+ \mathbb{E}_{\mathbf{a}} \left[(1 - \mathbb{I}[\mathbf{a}_{-i}]) \left(\sum_{j=1}^n a_j x_j(\mathbf{a}) + o_j y_j(\mathbf{a}) \right) \right]. \quad (\text{s}_3) \end{aligned}$$

Here, s_1 represents the scenario where the values of all Ads except for item i , are given as $\mathbf{a}_{-i}$ and the value of Ad_i is $a'_i \leq a_i$. Therefore, we can rewrite this as:

$$\mathbb{P}(\mathbf{a}_{-i}) \mathbb{E}_{a'_i} \left[\mathbb{I}[a'_i \leq a_i] \sum_{j=1}^n (a_j x_j(a'_i, \mathbf{a}_{-i}) + o_j y_j(a'_i, \mathbf{a}_{-i})) \right].$$

Similarly, s_2 represents the scenario where the values of all Ads except for item i , are given as $\mathbf{a}_{-i}$ and the value of Ad_i is $a'_i > a_i$. Therefore, we can rewrite this as:

$$\mathbb{P}(\mathbf{a}_{-i}) \mathbb{E}_{a'_i} \left[\mathbb{I}[a'_i > a_i] \sum_{j=1}^n (a_j x_j(a'_i, \mathbf{a}_{-i}) + o_j y_j(a'_i, \mathbf{a}_{-i})) \right].$$

Part s_3 represents the case where the values of the remaining items are not equal to $\mathbf{a}_{-i}$.

Statement 1: Recall Statement 1 in Proposition 1, and we only need to focus on s_1 and s_2 and optimize the corresponding objective:

$$\mathbb{E}_{a'_i} \left[\sum_{j=1}^n (a_j x_j(a'_i, \mathbf{a}_{-i}) + o_j y_j(a'_i, \mathbf{a}_{-i})) \right].$$

Since $y_i^*(a_i, \mathbf{a}_{-i}) = 1$, by form stability from Lemma 2, we know that for any a'_i , $y_i(a'_i, \mathbf{a}_{-i}) = 1$ and $x_i(a'_i, \mathbf{a}_{-i}) = 0$. Therefore, the equation above can be rewritten as:

$$o_i + \mathbb{E}_{a'_i} \left[\sum_{j \neq i} (a_j x_j(a'_i, \mathbf{a}_{-i}) + o_j y_j(a'_i, \mathbf{a}_{-i})) \right].$$

Since a_j and o_j ($j \neq i$) remain unchanged in this scenario, the optimal mechanism's allocation for the remaining items also remains the same. That is, for any j and any a'_i , we have $y_j^*(a'_i, \mathbf{a}_{-i}) = y_j^*(a_i, \mathbf{a}_{-i})$ and $x_j^*(a'_i, \mathbf{a}_{-i}) = x_j^*(a_i, \mathbf{a}_{-i})$.

Statement 2: To demonstrate Statement 2 in the proposition, we focus on part s_1 and optimize the following objective:

$$\mathbb{E}_{a'_i} \left[\mathbb{I}[a'_i \leq a_i] \left(\sum_{j=1}^n (a_j x_j(a'_i, \mathbf{a}_{-i}) + o_j y_j(a'_i, \mathbf{a}_{-i})) \right) \right].$$

By form stability from Lemma 2 and allocation monotonicity from Lemma 1, we know that for any $a'_i \leq a_i$, we have $y_i(a'_i, \mathbf{a}_{-i}) = y_i(a_i, \mathbf{a}_{-i})$ and $x_i(a'_i, \mathbf{a}_{-i}) = 0$. Therefore, the equation above can be rewritten as:

$$o_i y_i(a'_i, \mathbf{a}_{-i}) + \mathbb{E}_{a'_i} \left[\mathbb{I}[a'_i < a_i] \left(\sum_{j \neq i} (a_j x_j(a'_i, \mathbf{a}_{-i}) + o_j y_j(a'_i, \mathbf{a}_{-i})) \right) \right].$$

Since a_j and o_j ($j \neq i$) remain unchanged in this scenario, the optimal mechanism's allocation for the remaining items does not change. That is, for any j and any a'_i , we have $y_j^*(a'_i, \mathbf{a}_{-i}) = y_j^*(a_i, \mathbf{a}_{-i})$ and $x_j^*(a'_i, \mathbf{a}_{-i}) = x_j^*(a_i, \mathbf{a}_{-i})$.

Statement 3: Similar to the argument in Statement 2, we concentrate on part s_2 and optimize the objective

$$\mathbb{E}_{a'_i} [\mathbb{I}[a'_i > a_i] a'_i] + \mathbb{E}_{a'_i} \left[\mathbb{I}[a'_i > a_i] \left(\sum_{j \neq i} (a_j x_j(a'_i, \mathbf{a}_{-i}) + o_j y_j(a'_i, \mathbf{a}_{-i})) \right) \right].$$

Due to form stability, allocation monotonicity, and the optimality, we conclude that for any j and any $a'_i > a_i$, it holds that $y_j^*(a'_i, \mathbf{a}_{-i}) = y_j^*(a_i, \mathbf{a}_{-i})$ and $x_j^*(a'_i, \mathbf{a}_{-i}) = x_j^*(a_i, \mathbf{a}_{-i})$.

In summary, we have shown this proposition. $\square$

Proof of Corollary 1

Proof. In an optimal mechanism that satisfies IC, IR, and feasibility, if the display form of item i remains unchanged, i.e., $X_i^*(a'_i, \mathbf{a}_{-i}) = X_i^*(a_i, \mathbf{a}_{-i})$, this corresponds precisely to one of the three cases outlined in Proposition 1. Consequently, Proposition 1 directly implies that the display forms of the other items also remain unchanged. $\square$

Proof of Lemma 5

Proof. Without loss of generality, let $i = 1$ and $j = 2$. Suppose there exist $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{a}' = (a'_1, a'_2, a'_3)$ such that $y_1^*(a_1, a_2, a_3) = 1$, $y_2^*(a_1, a_2, a_3) = 0$, but $y_1^*(a'_1, a'_2, a'_3) = 0$, $y_2^*(a'_1, a'_2, a'_3) = 1$.

According to Statement 1 in Proposition 1, which states that when $y_i^* = 1$, a change in a_i alone does not affect the optimal allocation of all items, we can derive that changing a_1 to a'_1 in profile $\mathbf{a}$ leads to $y_1^*(a'_1, a_2, a_3) = 1$ and $y_2^*(a'_1, a_2, a_3) = 0$. On the other hand, but changing a'_2 to a_2 in profile $\mathbf{a}'$ leads to $y_1^*(a'_1, a'_2, a'_3) = 0$ and $y_2^*(a'_1, a'_2, a'_3) = 1$.

This implies that changing the bid of item 3 alone causes the allocations of items 1 and 2 to change. According to

Corollary 1, this change also indicates that the allocation form of item 3 must itself change.

Without loss of generality, let $a'_3 > a_3$. Since there are only two slots, we observe that before and after the change, the corresponding allocations must be $y_1^*(a'_1, a_2, a_3) = 1$, $x_2^*(a'_1, a_2, a_3) = 1$, $x_3^*(a'_1, a_2, a_3) = 0$ and $y_2^*(a'_1, a_2, a'_3) = 1$, $x_3^*(a'_1, a_2, a'_3) = 1$.

Since item 2 is displayed in the ad form under the profile (a'_1, a_2, a_3) , we know

$$\mathbb{E}_{a_2}[\max^{(2)}\{o_1, a_2, a_3\}] > \max^{(2)}\{o_1, o_2, a_3\}.$$

Because Org_1 is ultimately selected, i.e., $o_1 > a_3$, it follows that

$$o_1 + \mathbb{E}_{a_2}[\max\{a_2, a_3\}] > o_1 + \max\{o_2, a_3\},$$

and consequently

$$\mathbb{E}_{a_2}[\max\{a_2, a_3\}] > \max\{o_2, a_3\} \geq o_2.$$

On the other hand, since item 2 is displayed in the organic form under the profile (a'_1, a_2, a'_3) , we know

$$\begin{aligned} o_2 + a'_3 &> \mathbb{E}_{a_2}\left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i + y_1^*(\mathbf{a})o_1 + y_3^*(\mathbf{a})o_3\right] \\ &\geq \mathbb{E}_{a_2}[\max^{(2)}\{o_1, a_2, a'_3\}] \\ &\geq a'_3 + \mathbb{E}_{a_2}[\max\{o_1, a_2\}], \end{aligned}$$

which implies a contradiction:

$$o_2 > \mathbb{E}_{a_2}[\max\{o_1, a_2\}] > \mathbb{E}_{a_2}[\max\{a_2, a_3\}] > o_2.$$

Therefore, in an optimal 2-out-of-3 mechanism that satisfies IC, IR, and feasibility, if there exists some $\mathbf{a}$ such that $y_i^*(\mathbf{a}) = 1$ and $y_j^*(\mathbf{a}) = 0$, then for any $\mathbf{a}'$, it is not possible for $y_i^*(\mathbf{a}') = 0$ and $y_j^*(\mathbf{a}') = 1$. $\square$

Proof of Theorem 1

Proof. Firstly, based on Lemma 5, for any two items i and j , either $y_i^* \geq y_j^*$ or $y_j^* \geq y_i^*$. This implies that there is a total ordering among the three items, resulting in a partial order such that $y_{i_1}^* \geq y_{i_2}^* \geq y_{i_3}^*$, where i_1, i_2, i_3 is a permutation of the three items.

It is trivial to see that $y_{i_3}^* = 0$; otherwise, we would have $y_{i_1}^* = y_{i_2}^* = y_{i_3}^* = 1$, which violates the feasibility constraint of selecting only two items.

Next, we consider the effect of changing a_{i_1} on $y_{i_2}^*$. When $y_{i_1}^* = 1$, Proposition 1 tells us that changes in a_{i_1} do not affect $y_{i_2}^*$. When $y_{i_1}^* = 0$, the partial order constraint implies that $y_{i_2}^* = 0$. Therefore, $y_{i_2}^*$ is not affected by changes in a_{i_1} .

Additionally, by form stability from Lemma 2, changes in a_i alone do not affect y_i . Combining these results, we conclude that:

$$\begin{aligned} y_{i_1}^* &= y_{i_1}^*(a_{i_2}, a_{i_3}), \\ y_{i_2}^* &= y_{i_2}^*(a_{i_3}), \\ y_{i_3}^* &= 0. \end{aligned}$$

Thus, we have shown that there exists a partial order (i_1, i_2, i_3) such that $y_{i_1}^* \geq y_{i_2}^* \geq y_{i_3}^*$, with $y_{i_1}^* = y_{i_1}^*(a_{i_2}, a_{i_3})$, $y_{i_2}^* = y_{i_2}^*(a_{i_3})$, and $y_{i_3}^* = 0$. $\square$

Missing Proofs in Section

Proof of Theorem 2

Proof. When $n = 3$ and $k = 2$, the mechanism $\mathcal{M}^F$ performs best among the following: $\mathcal{M}_0$, $\mathcal{M}_{\{1\}}^F$, $\mathcal{M}_{\{2\}}^F$, $\mathcal{M}_{\{3\}}^F$, $\mathcal{M}_{\{1,2\}}^F$, $\mathcal{M}_{\{1,3\}}^F$ and $\mathcal{M}_{\{2,3\}}^F$. We have the following result:

$$\begin{aligned} \text{OBJ}(\mathcal{M}_0) &= \mathbb{E}_{\mathbf{a} \sim G}[\max^{(2)}\{a_1, a_2, a_3\}], \\ \text{OBJ}(\mathcal{M}_{\{1\}}^F) &= \mathbb{E}_{a_2, a_3}[\max^{(2)}\{o_1, a_2, a_3\}], \\ \text{OBJ}(\mathcal{M}_{\{2\}}^F) &= \mathbb{E}_{a_1, a_3}[\max^{(2)}\{a_1, o_2, a_3\}], \\ \text{OBJ}(\mathcal{M}_{\{3\}}^F) &= \mathbb{E}_{a_1, a_2}[\max^{(2)}\{a_1, a_2, o_3\}], \\ \text{OBJ}(\mathcal{M}_{\{1,2\}}^F) &= \mathbb{E}_{a_3}[\max^{(2)}\{o_1, o_2, a_3\}], \\ \text{OBJ}(\mathcal{M}_{\{1,3\}}^F) &= \mathbb{E}_{a_2}[\max^{(2)}\{o_1, a_2, o_3\}], \\ \text{OBJ}(\mathcal{M}_{\{2,3\}}^F) &= \mathbb{E}_{a_1}[\max^{(2)}\{a_1, o_2, o_3\}]. \end{aligned}$$

To analyze the performance of $\mathcal{M}^F$, we first establish an upper bound for the optimal mechanism $\mathcal{M}^*$. Based on Theorem 1, there exists a partial order relationship in the 2-out-of-3 optimal mechanism. Without loss of generality, assume $y_1^* \geq y_2^* \geq y_3^*$. Let $y_1^* = y_1^*(a_2, a_3)$, $y_2^* = y_2^*(a_3)$, and $y_3^* \equiv 0$. Consequently, the objective function can be rewritten as follows:

$$\begin{aligned} \text{OBJ}(\mathcal{M}^*) &= \mathbb{E}_{\mathbf{a}}\left[\sum_{i=1}^3 y_i^*(\mathbf{a})o_i + \sum_{i=1}^3 x_i^*(\mathbf{a})a_i\right] \\ &= \mathbb{E}_{a_3}\left[y_2^*(a_3)o_2 + \mathbb{E}_{a_2}\left[y_1^*(a_2, a_3)o_1 + \mathbb{E}_{a_1}\left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i\right]\right]\right]. \end{aligned}$$

We simplify this objective function by examining it from the outside in. Fixing any a_3 , we consider two cases:

- (1) If $y_2^*(a_3) = 1$ for all a_1 and a_2 , then by the partial order relationship, we have $y_1^* \geq y_2^* = 1$. Consequently, the mechanism selects two organic items. Thus, we have:

$$\begin{aligned} &y_2^*(a_3)o_2 + \mathbb{E}_{a_2}\left[y_1^*(a_2, a_3)o_1 + \mathbb{E}_{a_1}\left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i\right]\right] \\ &= o_1 + o_2 \leq \max^{(2)}\{o_1, o_2, a_3\}. \end{aligned}$$

- (2) If $y_2^*(a_3) = 0$ for all a_1 and a_2 , then in this case, we have:

$$\begin{aligned} &y_2^*(a_3)o_2 + \mathbb{E}_{a_2}\left[y_1^*(a_2, a_3)o_1 + \mathbb{E}_{a_1}\left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i\right]\right] \\ &= \mathbb{E}_{a_2}\left[y_1^*(a_2, a_3)o_1 + \mathbb{E}_{a_1}\left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i\right]\right]. \end{aligned}$$

We then fix a_2 along with the previously fixed a_3 and consider the following two subcases:

(2.1) If $y_1^*(a_2, a_3) = 1$ for all a_1 , then we have

$$\begin{aligned} & y_1^*(a_2, a_3)o_1 + \mathbb{E}_{a_1} \left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i \right] \\ &= o_1 + \max\{a_2, a_3\} \\ &\leq \max^{(2)}\{o_1, a_2, a_3\}. \end{aligned}$$

(2.2) If $y_1^*(a_2, a_3) = 0$ for all a_1 , then we get

$$\begin{aligned} & y_1^*(a_2, a_3)o_1 + \mathbb{E}_{a_1} \left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i \right] \\ &= \mathbb{E}_{a_1} \left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i \right] \\ &\leq \mathbb{E}_{a_1} \left[\max^{(2)}\{a_1, a_2, a_3\} \right]. \end{aligned}$$

Combining above subcases, we obtain

$$\begin{aligned} & \mathbb{E}_{a_2} \left[y_1^*(a_2, a_3)o_1 + \mathbb{E}_{a_1} \left[\sum_{i=1}^3 x_i^*(\mathbf{a})a_i \right] \right] \leq \\ & \mathbb{E}_{a_2} \left[\max \left\{ \max^{(2)}\{o_1, a_2, a_3\}, \mathbb{E}_{a_1} \left[\max^{(2)}\{a_1, a_2, a_3\} \right] \right\} \right]. \end{aligned}$$

To simplify the mathematical expression, we introduce the following notation:

$$\begin{aligned} m_{\{1,2\}}^F &= \max^{(2)}\{o_1, o_2, a_3\}, \\ m_{\{1\}}^F &= \max^{(2)}\{o_1, a_2, a_3\}, \\ m_0 &= \max^{(2)}\{a_1, a_2, a_3\}. \end{aligned} \quad (5)$$

Therefore, by taking the expectation over a_3 , we obtain an upper bound for the optimal mechanism:

$$\text{OBJ}(\mathcal{M}^*) \leq \mathbb{E}_{a_3} \left[\max \left\{ m_{\{1,2\}}^F, \mathbb{E}_{a_2} \left[\max \{ m_{\{1\}}^F, \mathbb{E}_{a_1} [m_0] \} \right] \right\} \right].$$

Using the same notation, we can also derive a lower bound for $\mathcal{M}^F$:

$$\begin{aligned} \text{OBJ}(\mathcal{M}^F) &\geq \max \left\{ \text{OBJ}(\mathcal{M}_{\{1,2\}}^F), \text{OBJ}(\mathcal{M}_{\{1\}}^F), \text{OBJ}(\mathcal{M}_0) \right\} \\ &= \max \left\{ \mathbb{E}_{a_3} [m_{\{1,2\}}^F], \mathbb{E}_{a_2, a_3} [m_{\{1\}}^F], \mathbb{E}_{\mathbf{a}} [m_0] \right\}. \end{aligned}$$

Thus, we can determine the approximate ratio of the G-FIX mechanism relative to the optimal mechanism:

$$\frac{\text{OBJ}(\mathcal{M}^F)}{\text{OBJ}(\mathcal{M}^*)} \geq \frac{\max \left\{ \mathbb{E}_{a_3} [m_{\{1,2\}}^F], \mathbb{E}_{a_2, a_3} [m_{\{1\}}^F], \mathbb{E}_{\mathbf{a}} [m_0] \right\}}{\mathbb{E}_{a_3} \left[\max \left\{ m_{\{1,2\}}^F, \mathbb{E}_{a_2} \left[\max \{ m_{\{1\}}^F, \mathbb{E}_{a_1} [m_0] \} \right] \right\} \right]}.$$

Before proceeding with the analysis, it is crucial to grasp the expression intuitively: Firstly, $m_{\{1\}}^F$ represents the outcome of running the corresponding $\mathcal{M}_{\{1\}}^F$ mechanism with a given set of bids. For example:

- $m_{\{1,2\}}^F$ represents the outcome of $\mathcal{M}_{\{1,2\}}^F$ given a_3 .

- $m_{\{1\}}^F$ represents the outcome of $\mathcal{M}_{\{1\}}^F$ given a_2 and a_3 .
- m_0 represents the outcome of $\mathcal{M}_0$ given a_1, a_2 , and a_3 .

Thus, the numerator can be interpreted as running one of the mechanisms $\mathcal{M}_{\{1,2\}}^F$, $\mathcal{M}_{\{1\}}^F$, or $\mathcal{M}_0$ for all possible $\mathbf{a}$ and choosing one with the best expected outcome. The denominator, on the other hand, can be understood as follows:

- Given a_3 , the mechanism can decide whether to run $\mathcal{M}_{\{1,2\}}^F$ or incorporate more information based on the values of $m_{\{1,2\}}^F$ and $\mathbb{E}_{a_2} \left[\max \left\{ m_{\{1\}}^F, \mathbb{E}_{a_1} [m_0] \right\} \right]$.
 - If $m_{\{1,2\}}^F$ is larger, $\mathcal{M}_{\{1,2\}}^F$ is executed.
 - Otherwise, the mechanism introduces a_2 and, based on the values of a_2 and a_3 , decides whether to run $\mathcal{M}_{\{1\}}^F$ or $\mathcal{M}_0$.
 - * If $m_{\{1\}}^F$ is larger, $\mathcal{M}_{\{1\}}^F$ is executed.
 - * If $\mathbb{E}_{a_1} [m_0]$ is larger, $\mathcal{M}_0$ is executed.

Intuitively, the mechanism in the numerator pre-selects the one with the highest expected outcome across all possible bids, while the mechanism in the denominator is capable of making decisions based on the actual situation step by step.

We will now decompose this fraction based on the step-by-step intuition, breaking it down into a more manageable form.

$$\frac{\text{OBJ}(\mathcal{M}^F)}{\text{OBJ}(\mathcal{M}^*)} \geq \frac{\max \left\{ \mathbb{E}_{a_3} [m_{\{1,2\}}^F], \mathbb{E}_{a_2, a_3} [m_{\{1\}}^F], \mathbb{E}_{\mathbf{a}} [m_0] \right\}}{\max \left\{ \mathbb{E}_{a_3} [m_{\{1,2\}}^F], \mathbb{E}_{a_3} [\max \left\{ \mathbb{E}_{a_2} [m_{\{1\}}^F], \mathbb{E}_{a_1, a_2} [m_0] \right\}] \right\}} \quad (6)$$

$$\cdot \frac{\max \left\{ \mathbb{E}_{a_3} [m_{\{1,2\}}^F], \mathbb{E}_{a_3} [\max \left\{ \mathbb{E}_{a_2} [m_{\{1\}}^F], \mathbb{E}_{a_1, a_2} [m_0] \right\}] \right\}}{\max \left\{ \mathbb{E}_{a_3} [m_{\{1,2\}}^F], \mathbb{E}_{a_3, a_2} [\max \left\{ m_{\{1\}}^F, \mathbb{E}_{a_1} [m_0] \right\}] \right\}} \quad (7)$$

$$\cdot \frac{\max \left\{ \mathbb{E}_{a_3} [m_{\{1,2\}}^F], \mathbb{E}_{a_3, a_2} [\max \left\{ m_{\{1\}}^F, \mathbb{E}_{a_1} [m_0] \right\}] \right\}}{\mathbb{E}_{a_3} [\max \left\{ m_{\{1,2\}}^F, \mathbb{E}_{a_2} [\max \left\{ m_{\{1\}}^F, \mathbb{E}_{a_1} [m_0] \right\}] \right\}]} \quad (8)$$

In the three decomposed fractions, each denominator adds some flexibility compared to the numerator. For example, in fraction (6), the denominator adds the flexibility of a_3 , allowing the choice of running $\mathcal{M}_{\{1\}}^F$ or $\mathcal{M}_0$ based on the value of a_3 . In fraction (7), given a_3 , the denominator adds the flexibility of a_2 , enabling the choice of running $\mathcal{M}_{\{1\}}^F$ or $\mathcal{M}_0$ based on the value of a_2 . Fraction (8) adds the flexibility of a_1 .

Next, we will separately prove the lower bound for each of these three fractions. We will start by introducing a mathematical result from the previous paper.

Lemma 7 ((Li et al. 2024a)). *If $\frac{h_2(t)}{h_1(t)} \in [r, r+1]$ for all t , where $0 < r < 1$, then for any distribution G_1 ,*

$$\frac{\max \left\{ \mathbb{E}_{a_1 \sim G_1} [h_1(a_1)], \mathbb{E}_{a_1 \sim G_1} [h_2(a_1)] \right\}}{\mathbb{E}_{a_1 \sim G_1} [\max \{ h_1(a_1), h_2(a_1) \}]} \geq \frac{4}{5}.$$

Next, we will prove the following three claims based on this claim.

Claim 1. With $m_{\{1,2\}}^F$, $m_{\{1\}}^F$ and m_0 defined in notation (5), we have

$$\frac{\max\{\mathbb{E}_{a_3}[m_{\{1,2\}}^F], \mathbb{E}_{a_2, a_3}[m_{\{1\}}^F], \mathbb{E}_a[m_0]\}}{\max\{\mathbb{E}_{a_3}[m_{\{1,2\}}^F], \mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[m_{\{1\}}^F], \mathbb{E}_{a_1, a_2}[m_0]\}]\}} \geq \frac{4}{5}.$$

Proof. Since

$$\max\{\mathbb{E}_{a_2, a_3}[m_{\{1\}}^F], \mathbb{E}_a[m_0]\} \leq \mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[m_{\{1\}}^F], \mathbb{E}_{a_1, a_2}[m_0]\}],$$

deleting $\mathbb{E}_{a_3}[m_{\{1,2\}}^F]$ on numerator and denominator, we get

$$\begin{aligned} & \frac{\max\{\mathbb{E}_{a_3}[m_{\{1,2\}}^F], \mathbb{E}_{a_2, a_3}[m_{\{1\}}^F], \mathbb{E}_a[m_0]\}}{\max\{\mathbb{E}_{a_3}[m_{\{1,2\}}^F], \mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[m_{\{1\}}^F], \mathbb{E}_{a_1, a_2}[m_0]\}]\}} \\ & \geq \frac{\max\{\mathbb{E}_{a_2, a_3}[m_{\{1\}}^F], \mathbb{E}_a[m_0]\}}{\mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[m_{\{1\}}^F], \mathbb{E}_{a_1, a_2}[m_0]\}]} \end{aligned} \quad (9)$$

Define

$$h_1(t) = \mathbb{E}_{a_2}[\max^{(2)}\{o_1, a_2, t\}]$$

and

$$h_2(t) = \mathbb{E}_{a_1 a_2}[\max^{(2)}\{a_1, a_2, t\}],$$

we can rewrite right hand side of inequality (9) as follow:

$$\frac{\max\{\mathbb{E}_{a_3}[h_1(a_3)], \mathbb{E}_{a_3}[h_2(a_3)]\}}{\mathbb{E}_{a_3}[\max\{h_1(a_3), h_2(a_3)\}]}.$$

In order to analyze the value of this fraction, we will calculate the derivatives of $h_1(t)$ and $h_2(t)$ and their values at critical points. First, for $h_1(t)$, we can obtain:

1. The derivative of $h_1(t)$:

$$h_1'(t) = \begin{cases} \mathbb{P}(t \geq a_2) & t < o_1 \\ 1 & t > o_1, \end{cases}$$

2. $h_1(t)$'s values at critical points:

$$h_1(0) = o_1 + \mathbb{E}_{a_2}[a_2],$$

$$h_1(o_1) = 2o_1 + \mathbb{E}_{a_2}[\mathbb{I}[a_2 > o_1](a_2 - o_1)],$$

$$\lim_{t \rightarrow \infty} h_1(t) = \lim_{t \rightarrow \infty} t + o_1 + \mathbb{E}_{a_2}[\mathbb{I}[a_2 > o_1](a_2 - o_1)].$$

Similarly for $h_2(t)$, we can obtain:

1. The derivative of $h_2(t)$:

$$h_2'(t) = 1 - \mathbb{P}(t < a_1)\mathbb{P}(t < a_2),$$

2. $h_2(t)$'s values at critical points:

$$h_2(0) = \mathbb{E}_{a_1}[a_1] + \mathbb{E}_{a_2}[a_2],$$

$$h_2(o_1) = a_1 + a_2 + \mathbb{E}_{a_1 a_2}[\mathbb{I}[o_1 > \min\{a_1, a_2\}](o_1 - \min\{a_1, a_2\})],$$

$$\begin{aligned} \lim_{t \rightarrow \infty} h_2(t) &= \lim_{t \rightarrow \infty} t + \mathbb{E}_{a_1 a_2}[\max\{a_1, a_2\}] \\ &\geq \lim_{t \rightarrow \infty} t + \mathbb{E}_{a_2}[\max\{\mathbb{E}_{a_1}[a_1], a_2\}]. \end{aligned}$$

From the characterization of the functions above, we can observe that $h_1(t)$ grows more slowly than $h_2(t)$ in the interval $(0, o_1)$, and grows faster than $h_2(t)$ in the interval $(o_1, +\infty)$. Additionally, the relationship between $h_1(0)$ and $h_2(0)$, as well as $h_1(+\infty)$ and $h_2(+\infty)$, depend on the relationship between o_1 and $\mathbb{E}_{a_1}[a_1]$.

It is evident from Figure 1 that when $o_1 \leq \mathbb{E}_{a_1}[a_1]$, $h_1(t)$ is always less than or equal to $h_2(t)$, and when $o_1 > \mathbb{E}_{a_1}[a_1]$ and $h_1(o_1) \geq h_2(o_1)$, as shown in Figure 2, $h_1(t)$ is always greater than or equal to $h_2(t)$. In both cases, we have:

$$\frac{\max\{\mathbb{E}_{a_3}[h_1(a_3)], \mathbb{E}_{a_3}[h_2(a_3)]\}}{\mathbb{E}_{a_3}[\max\{h_1(a_3), h_2(a_3)\}]} = 1.$$

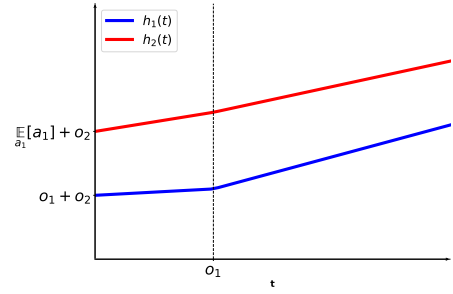


Figure 1: The relationship between $h_1(t)$ and $h_2(t)$ when $o_1 \leq \mathbb{E}_{a_1}[a_1]$.

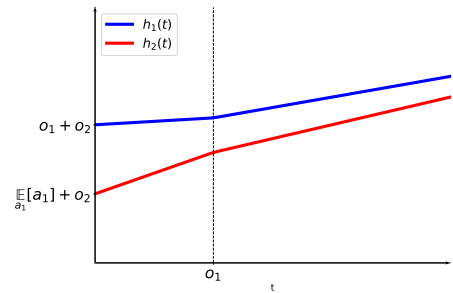


Figure 2: The relationship between $h_1(t)$ and $h_2(t)$ when $o_1 > \mathbb{E}_{a_1}[a_1]$ and $h_1(o_1) \geq h_2(o_1)$.

Therefore we only need to consider the case that $o_1 > \mathbb{E}_{a_1}[a_1]$ and $h_1(o_1) < h_2(o_1)$. See Figure 3 to help us understand.

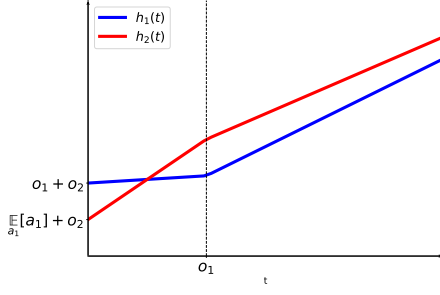


Figure 3: The relationship between $h_1(t)$ and $h_2(t)$ when $o_1 > \mathbb{E}[a_1]$ and $h_1(o_1) < h_2(o_1)$.

Since $\frac{h_2(t)}{h_1(t)}$ is monotonically increasing in the interval $(0, o_1)$ and monotonically decreasing in the interval $(o_1, +\infty)$, and $\lim_{t \rightarrow \infty} \frac{h_2(t)}{h_1(t)} = 1$, let

$$\frac{h_2(0)}{h_1(0)} = \frac{\mathbb{E}[a_1] + \mathbb{E}[a_2]}{o_1 + \mathbb{E}[a_2]} = r \in (0, 1),$$

and we have:

$$r = \frac{h_2(0)}{h_1(0)} \leq \frac{h_2(t)}{h_1(t)} \leq \frac{h_2(o_1)}{h_1(o_1)} \leq \frac{\mathbb{E}[a_1] + \mathbb{E}[a_2] + o_1}{\mathbb{E}[a_2] + o_1} \leq r + 1.$$

Thus, according to Lemma 7, we have:

$$\frac{\max\{\mathbb{E}_{a_3}[h_1(a_3)], \mathbb{E}_{a_3}[h_2(a_3)]\}}{\mathbb{E}_{a_3}[\max\{h_1(a_3), h_2(a_3)\}]} \geq \frac{4}{5}.$$

By inequality (9), we prove this claim. $\square$

Claim 2. With $m_{\{1,2\}}^F$, $m_{\{1\}}^F$ and m_0 defined in notation (5), we have

$$\frac{\max\{\mathbb{E}_{a_3}[m_{\{1,2\}}^F], \mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[m_{\{1\}}^F], \mathbb{E}_{a_1, a_2}[m_0]\}]\}}{\max\{\mathbb{E}_{a_3}[m_{\{1,2\}}^F], \mathbb{E}_{a_3}[\mathbb{E}_{a_2}[\max\{m_{\{1\}}^F, \mathbb{E}_{a_1}[m_0]\}]]\}} \geq \frac{4}{5}.$$

Proof. Since

$$\mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[m_{\{1\}}^F], \mathbb{E}_{a_1, a_2}[m_0]\}] \leq \mathbb{E}_{a_2, a_3}[\max\{m_{\{1\}}^F, \mathbb{E}_{a_1}[m_0]\}],$$

with the similar argument in proof of Claim 1, we know

$$\begin{aligned} & \frac{\max\{\mathbb{E}_{a_3}[m_{\{1,2\}}^F], \mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[m_{\{1\}}^F], \mathbb{E}_{a_1, a_2}[m_0]\}]\}}{\max\{\mathbb{E}_{a_3}[m_{\{1,2\}}^F], \mathbb{E}_{a_3}[\mathbb{E}_{a_2}[\max\{m_{\{1\}}^F, \mathbb{E}_{a_1}[m_0]\}]]\}} \\ & \geq \frac{\mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[m_{\{1\}}^F], \mathbb{E}_{a_1, a_2}[m_0]\}]}{\mathbb{E}_{a_3}[\mathbb{E}_{a_2}[\max\{m_{\{1\}}^F, \mathbb{E}_{a_1}[m_0]\}]]}. \end{aligned} \quad (10)$$

Define

$$h_3(t, a_3) = \max^{(2)}\{o_1, t, a_3\}$$

and

$$h_4(t, a_3) = \mathbb{E}_{a_1}[\max^{(2)}\{a_1, t, a_3\}],$$

we can rewrite fraction (10) as follow:

$$\frac{\mathbb{E}_{a_3}[\max\{\mathbb{E}_{a_2}[h_3(a_2, a_3)], \mathbb{E}_{a_2}[h_4(a_2, a_3)]\}]}{\mathbb{E}_{a_3}[\mathbb{E}_{a_2}[\max\{h_3(a_2, a_3), h_4(a_2, a_3)\}]]}.$$

To ease our analysis, we will consider two cases: $a_3 \leq o_1$ and $a_3 > o_1$. In the first case, for $h_3(t, a_3)$, we can obtain:

1. The derivative of $h_3(t, a_3)$:

$$h_3'(t, a_3) = \begin{cases} 0 & t < a_3, \\ 1 & t > a_3. \end{cases}$$

2. $h_3(t, a_3)$'s values at critical points:

$$\begin{aligned} h_3(0, a_3) &= o_1 + a_3, \\ h_3(a_3, a_3) &= o_1 + a_3. \end{aligned}$$

For $h_4(t, a_3)$, we can obtain:

1. The derivative of $h_4(t, a_3)$:

$$h_4'(t, a_3) = \begin{cases} \mathbb{P}(t \geq a_1) & t < a_3, \\ 1 & t > a_3. \end{cases}$$

2. $h_4(t, a_3)$'s values at critical points:

$$\begin{aligned} h_4(0, a_3) &= \mathbb{E}_{a_1}[a_1] + a_3, \\ h_4(a_3, a_3) &= 2a_3 + \mathbb{E}_{a_1}[\mathbb{I}[a_1 \geq a_3](a_1 - a_3)]. \end{aligned}$$

We observe that $h_4(t, a_3)$ grows faster than $h_3(t, a_3)$ in the interval $(0, a_3)$ (since $h_3(t, a_3)$ does not grow in this interval), and both grow at the same rate in the interval $(a_3, +\infty)$. Therefore, when $h_4(0, a_3) \geq h_3(0, a_3)$, it follows that $h_4(t, a_3) \geq h_3(t, a_3)$ for all t . Besides, when $h_4(0, a_3) < h_3(0, a_3)$ and $h_4(a_3, a_3) \leq h_3(a_3, a_3)$, it follows that $h_4(t, a_3) \leq h_3(t, a_3)$ for all $t \geq 0$. In both cases, we have:

$$\frac{\max\{\mathbb{E}_{a_2}[h_3(a_2, a_3)], \mathbb{E}_{a_2}[h_4(a_2, a_3)]\}}{\mathbb{E}_{a_2}[\max\{h_3(a_2, a_3), h_4(a_2, a_3)\}]} = 1.$$

Therefore, we only need to consider the case that $h_4(0, a_3) < h_3(0, a_3)$ and $h_4(a_3, a_3) > h_3(a_3, a_3)$. And since $\frac{h_4(t, a_3)}{h_3(t, a_3)}$ is monotonically increasing in the interval $(0, a_3)$ and monotonically decreasing in the interval $(a_3, +\infty)$, and $\lim_{t \rightarrow \infty} [h_4(t)/h_3(t)] = 1$, let $\frac{h_4(0, a_3)}{h_3(0, a_3)} = \frac{\mathbb{E}_{a_1}[a_1] + a_3}{o_1 + a_3} = r \in (0, 1)$, we have:

$$\begin{aligned} r &= \frac{h_4(0, a_3)}{h_3(0, a_3)} \leq \frac{h_4(t, a_3)}{h_3(t, a_3)} \leq \frac{h_4(a_3, a_3)}{h_3(a_3, a_3)} \\ &\leq \frac{\mathbb{E}[a_1] + 2a_3}{o_1 + a_3} \leq r + 1. \end{aligned}$$

Thus, according to claim 7, we have:

$$\frac{\max_{a_2}\{\mathbb{E}[h_3(a_2, a_3)], \mathbb{E}[h_4(a_2, a_3)]\}}{\mathbb{E}[\max_{a_2}\{h_3(a_2, a_3), h_4(a_2, a_3)\}]} \geq \frac{4}{5}.$$

Next, let us consider the case when $a_3 > o_1$. Similarly, we can obtain:

1. The derivative of $h_3(t, a_3)$

$$h'_3(t, a_3) = \begin{cases} 0 & t < o_1, \\ 1 & t > o_1. \end{cases}$$

2. $h_3(t, a_3)$'s values at critical points

$$\begin{aligned} h_3(0, a_3) &= o_1 + a_3, \\ h_3(o_1, a_3) &= o_1 + a_3, \\ h_3(a_3, a_3) &= 2a_3. \end{aligned}$$

For $h_4(t, a_3)$, we can obtain:

1. the derivative of $h_4(t, a_3)$

$$h'_4(t, a_3) = \begin{cases} \mathbb{P}(t \geq a_1) & t < o_1, \\ \mathbb{P}(t \geq a_1) & o_1 \leq t < a_3, \\ 1 & t > a_3. \end{cases}$$

2. $h_4(t, a_3)$'s values at critical points

$$\begin{aligned} h_4(0, a_3) &= \mathbb{E}[a_1] + a_3, \\ h_4(o_1, a_3) &= o_1 + a_3 + \mathbb{E}[\mathbb{I}[a_1 \geq o_1](a_1 - o_1)], \\ h_4(a_3, a_3) &= 2a_3 + \mathbb{E}[\mathbb{I}[a_1 \geq a_3](a_1 - a_3)]. \end{aligned}$$

We observe that $h_4(t, a_3)$ grows faster than $h_3(t, a_3)$ in the interval $(0, o_1)$ (since $h_3(t, a_3)$ has a derivative of 0 in this interval). In the interval (o_1, a_3) , $h_4(t, a_3)$ grows more slowly than $h_3(t, a_3)$, and in the interval $(a_3, +\infty)$, both functions grow at the same rate. Additionally, we can easily see that $h_4(o_1, a_3) \geq h_3(o_1, a_3)$ and $h_4(a_3, a_3) \geq h_3(a_3, a_3)$. Therefore, in the interval $(o_1, +\infty)$, we always have $h_4(t, a_3) \geq h_3(t, a_3)$. Consequently, when $h_4(0, a_3) \geq h_3(0, a_3)$, we have $h_4(t, a_3) \geq h_3(t, a_3)$ for all $t \in (0, +\infty)$. That means, in this case, we have:

$$\frac{\max_{a_2}\{\mathbb{E}[h_3(a_2, a_3)], \mathbb{E}[h_4(a_2, a_3)]\}}{\mathbb{E}[\max_{a_2}\{h_3(a_2, a_3), h_4(a_2, a_3)\}]} = 1.$$

Therefore, we only need to consider the case when $h_4(0, a_3) < h_3(0, a_3)$. In this situation, as can be seen from Figure 4, we can easily observe that $\frac{h_4(t, a_3)}{h_3(t, a_3)}$ is monotonically increasing in the interval $(0, o_1)$, and monotonically decreasing in the interval $(o_1, +\infty)$. Furthermore, $\lim_{t \rightarrow +\infty} \frac{h_4(t, a_3)}{h_3(t, a_3)} = 1$. So, let $\frac{h_4(0, a_3)}{h_3(0, a_3)} = r \in (0, 1)$, we have:

$$\begin{aligned} r &= \frac{h_4(0, a_3)}{h_3(0, a_3)} \leq \frac{h_4(t, a_3)}{h_3(t, a_3)} \leq \frac{h_4(o_1, a_3)}{h_3(o_1, a_3)} \\ &\leq \frac{\mathbb{E}[a_1] + a_3 + o_1}{o_1 + a_3} \leq r + 1. \end{aligned}$$

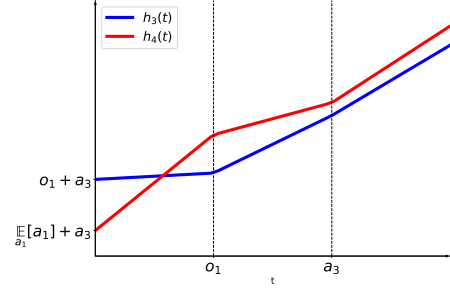


Figure 4: The relationship between $h_3(t)$ and $h_4(t)$ when $a_3 > o_1$ and $h_4(0, a_3) < h_3(0, a_3)$.

Thus, according to claim 7, we have:

$$\frac{\max_{a_2}\{\mathbb{E}[h_3(a_2, a_3)], \mathbb{E}[h_4(a_2, a_3)]\}}{\mathbb{E}[\max_{a_2}\{h_3(a_2, a_3), h_4(a_2, a_3)\}]} \geq \frac{4}{5}.$$

Finally, taking expectation of a_3 we can get:

$$\frac{\mathbb{E}_{a_3}[\max_{a_2}\{\mathbb{E}[h_3(a_2, a_3)], \mathbb{E}[h_4(a_2, a_3)]\}]}{\mathbb{E}_{a_3}[\mathbb{E}_{a_2}[\max_{a_2}\{h_3(a_2, a_3), h_4(a_2, a_3)\}]]} \geq \frac{4}{5}.$$

By inequality (10), we show this claim. $\square$

Claim 3. With $m_{\{1,2\}}^F$, $m_{\{1\}}^F$ and m_0 defined in notation (5), we have

$$\frac{\max_{a_3}\{\mathbb{E}[m_{\{1,2\}}^F], \mathbb{E}_{a_3}[\mathbb{E}_{a_2}[\max_{a_1}\{m_{\{1\}}^F, \mathbb{E}[m_0]\}]]\}}{\mathbb{E}_{a_3}[\max_{a_2}\{m_{\{1,2\}}^F, \mathbb{E}_{a_2}[\max_{a_1}\{m_{\{1\}}^F, \mathbb{E}[m_0]\}]]\}} \geq \frac{4}{5}.$$

Proof. Define

$$\begin{aligned} h_5(t) &= \max^{(2)}\{o_1, o_2, t\}, \\ h_6(t) &= \mathbb{E}_{a_2}[\max\{\max^{(2)}\{o_1, a_2, t\}, \mathbb{E}_{a_1}[\max^{(2)}\{a_1, a_2, t\}]\}]. \end{aligned}$$

We can rewrite fraction in the claim as follow:

$$\frac{\max_{a_3}\{\mathbb{E}[h_5(a_3)], \mathbb{E}_{a_3}[h_6(a_3)]\}}{\mathbb{E}_{a_3}[\max_{a_2}\{h_5(a_3), h_6(a_3)\}]} \quad (11)$$

To facilitate the analysis, we will consider two cases: $o_1 \leq \mathbb{E}[a_1]$ and $o_1 > \mathbb{E}[a_1]$.

Firstly, we analyze the simpler case where $o_1 \leq \mathbb{E}[a_1]$. In this situation, we have:

$$\begin{aligned} h_6(t) &= \mathbb{E}_{a_2} \left[\max \left\{ \max^{(2)}\{o_1, a_2, t\}, \mathbb{E}_{a_1}[\max^{(2)}\{a_1, a_2, t\}] \right\} \right] \\ &= \mathbb{E}_{a_1, a_2} [\max^{(2)}\{a_1, a_2, t\}]. \end{aligned}$$

For $h_5(t)$, we have:

1. The derivative of $h_5(t)$:

$$h'_5(t) = \begin{cases} 0 & t < \min\{o_1, o_2\}, \\ 1 & t > \min\{o_1, o_2\}. \end{cases}$$

2. $h_5(t)$'s values at critical points:

$$\begin{aligned} h_5(0) &= o_1 + o_2, \\ h_5(\min\{o_1, o_2\}) &= o_1 + o_2, \\ \lim_{t \rightarrow \infty} h_5(t) &= \lim_{t \rightarrow \infty} t + \max\{o_1, o_2\}. \end{aligned}$$

For $h_6(t)$, we have:

1. The derivative of $h_6(t)$

$$h_6'(t) = 1 - \mathbb{P}(t < a_1)\mathbb{P}(t < a_2).$$

2. $h_6(t)$'s values at critical points

$$\begin{aligned} h_6(0) &= \mathbb{E}_{a_1}[a_1] + \mathbb{E}_{a_2}[a_2], \\ \lim_{t \rightarrow \infty} h_6(t) &= \lim_{t \rightarrow \infty} t + \mathbb{E}_{a_1, a_2}[\max\{a_1, a_2\}]. \end{aligned}$$

We observe that in the interval $(0, \min\{o_1, o_2\})$, $h_6(t)$ grows faster than $h_5(t)$ (since $h_5(t)$ has a growth rate of 0 in this interval). In the interval $(\min\{o_1, o_2\}, +\infty)$, $h_6(t)$ grows more slowly than $h_5(t)$, and $\lim_{t \rightarrow \infty} \frac{h_6(t)}{h_5(t)} = 1$. It is straightforward to see that if $h_6(0) \geq h_5(0)$, then $h_6(t) \geq h_5(t)$ holds for all t . Conversely, if $h_6(0) < h_5(0)$ and $h_6(\min\{o_1, o_2\}) \leq h_5(\min\{o_1, o_2\})$, then $h_6(t) \leq h_5(t)$ holds for all t . Therefore, we only need to consider the case where $h_6(0) < h_5(0)$ and $h_6(\min\{o_1, o_2\}) > h_5(\min\{o_1, o_2\})$. In this case, we can see that $\frac{h_6(t)}{h_5(t)}$ is increasing in $(0, \min\{o_1, o_2\})$ and decreasing in $(\min\{o_1, o_2\}, +\infty)$. Thus, denote $\frac{h_6(0)}{h_5(0)} = r \in (0, 1)$ we have:

$$\begin{aligned} r &= \frac{h_6(0)}{h_5(0)} \leq \frac{h_6(t)}{h_5(t)} \leq \frac{h_6(\min\{o_1, o_2\})}{h_5(\min\{o_1, o_2\})} \\ &= \frac{\mathbb{E}_{a_1}[a_1] + \mathbb{E}_{a_2}[a_2] + \min\{o_1, o_2\}}{o_1 + o_2} \leq r + 1. \end{aligned}$$

Thus, according to claim 7, we have fraction (11) is at least $\frac{4}{5}$ when $o_1 \leq \mathbb{E}_{a_1}[a_1]$.

Next, we analyze the case when $o_1 > \mathbb{E}_{a_1}[a_1]$. Denote

$$\begin{aligned} A &= \max^{(2)}\{o_1, a_2, t\}, \\ B &= \mathbb{E}_{a_1}[\max^{(2)}\{a_1, a_2, t\}]. \end{aligned}$$

For $h_6(t)$, we have

1. The derivative of $h_6(t)$:

when $t < o_1$:

$$\mathbb{I}[A \geq B]\mathbb{P}(t > a_2) + \mathbb{I}[A < B](1 - \mathbb{P}(t < a_1)\mathbb{P}(t < a_2)),$$

when $t > o_1$:

$$\mathbb{I}[A \geq B] + \mathbb{I}[A < B](1 - \mathbb{P}(t < a_1)\mathbb{P}(t < a_2)).$$

2. $h_6(t)$'s values at critical points:

$$h_6(0) = o_1 + \mathbb{E}_{a_2}[a_2].$$

From the above analysis, we can observe that the derivative of $h_6(t)$ always lies within the range $(0, 1]$. Therefore, in the interval $(0, \min\{o_1, o_2\})$, $h_6(t)$ grows faster than $h_5(t)$ (since $h_5'(t) = 0$), while in the interval $(\min\{o_1, o_2\}, +\infty)$, $h_6(t)$ always grows slower than $h_5(t)$, and $\lim_{t \rightarrow +\infty} \frac{h_6(t)}{h_5(t)} = 1$. Consequently, it is straightforward to deduce that when $h_6(0) \geq h_5(0)$, we always have $h_6(t) \geq h_5(t)$; and when $h_6(0) < h_5(0)$ and $h_6(\min\{o_1, o_2\}) \leq h_5(\min\{o_1, o_2\})$, we always have $h_6(t) \leq h_5(t)$. Therefore, the only case we need to consider is when $h_6(0) < h_5(0)$ and $h_6(\min\{o_1, o_2\}) > h_5(\min\{o_1, o_2\})$. In this scenario, as shown in Figure 5, $\frac{h_6(t)}{h_5(t)}$ is monotonically increasing in the interval $(0, \min\{o_1, o_2\})$ and monotonically decreasing in the interval $(\min\{o_1, o_2\}, +\infty)$. Let $\frac{h_6(0)}{h_5(0)} = r \in (0, 1)$, we have:

$$\begin{aligned} r &= \frac{h_6(0)}{h_5(0)} \leq \frac{h_6(t)}{h_5(t)} \leq \frac{h_6(\min\{o_1, o_2\})}{h_5(\min\{o_1, o_2\})} \\ &= \frac{\mathbb{E}_{a_1}[a_1] + \mathbb{E}_{a_2}[a_2] + \min\{o_1, o_2\}}{o_1 + o_2} \leq r + 1. \end{aligned}$$

Thus, according to claim 7, we have fraction (11) is at least $\frac{4}{5}$ when $o_1 > \mathbb{E}_{a_1}[a_1]$.

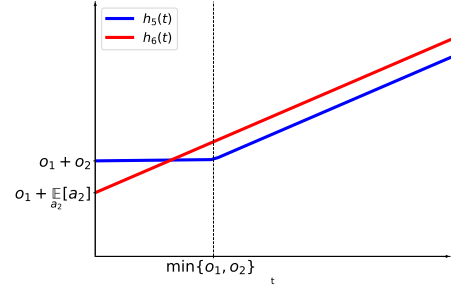


Figure 5: The relationship between $h_5(t)$ and $h_6(t)$ when $o_1 > \mathbb{E}_{a_1}[a_1]$, $h_6(0) < h_5(0)$ and $h_6(\min\{o_1, o_2\}) > h_5(\min\{o_1, o_2\})$.

Therefore, we have:

$$\frac{\max\{\mathbb{E}_{a_3}[\mathbf{m}_{\{1,2\}}^F], \mathbb{E}_{a_3, a_2}[\mathbb{E}[\max\{\mathbf{m}_{\{1\}}^F, \mathbb{E}_{a_1}[\mathbf{m}_0]\}]]\}}{\mathbb{E}_{a_3}[\max\{\mathbf{m}_{\{1,2\}}^F, \mathbb{E}_{a_2}[\max\{\mathbf{m}_{\{1\}}^F, \mathbb{E}_{a_1}[\mathbf{m}_0]\}]]]} \geq \frac{4}{5}.$$

Hence, the claim is proved. $\square$

Therefore, combining the conclusions of Claims 1, 2, and 3, we can obtain

$$\begin{aligned} &\frac{\text{OBJ}(\mathcal{M}^F)}{\text{OBJ}(\mathcal{M}^*)} \\ &\geq \frac{\max\{\mathbb{E}_{a_3}[\mathbf{m}_{\{1,2\}}^F], \mathbb{E}_{a_2, a_3}[\mathbf{m}_{\{1\}}^F], \mathbb{E}_{a_1}[\mathbf{m}_0]\}}{\mathbb{E}_{a_3}[\max\{\mathbf{m}_{\{1,2\}}^F, \mathbb{E}_{a_2}[\max\{\mathbf{m}_{\{1\}}^F, \mathbb{E}_{a_1}[\mathbf{m}_0]\}]]]} \\ &\geq \left(\frac{4}{5}\right)^3. \end{aligned} \quad \square$$

Proof of Lemma 6

Proof. Firstly, define

$$f(x) = C_n^l \left[1 - \sum_{i=1}^l C_{n-k}^{l-i} x^{(n-k)-(l-i)} (1-x)^{l-i} \right] - C_{n-k}^l \left[1 - \sum_{i=1}^l C_n^{l-i} x^{n-(l-i)} (1-x)^{l-i} \right].$$

We can find that:

$$\begin{aligned} f(0) &= C_n^l - C_{n-k}^l \geq 0, \\ f(1) &= 0, \\ f'(x) &= -g(x) + h(x), \end{aligned}$$

where

$$g(x) = C_n^l C_{n-k}^{l-1} [(n-k) - (l-1)] x^{(n-k)-(l-1)-1} (1-x)^{l-1}$$

and

$$h(x) = C_{n-k}^l C_n^{l-1} [n - (l-1)] x^{n-(l-1)-1} (1-x)^{l-1}.$$

Since

$$C_n^l C_{n-k}^{l-1} [(n-k) - (l-1)] = C_{n-k}^l C_n^{l-1} [n - (l-1)],$$

we have

$$\begin{aligned} \frac{g(x)}{h(x)} &= x^{-k} > 1, x \in (0, 1), \\ f'(x) &= -g(x) + h(x) < 0, x \in (0, 1). \end{aligned}$$

Thus, $f(x)$ is monotonically decreasing on the interval $(0, 1)$, and we have:

$$\frac{1 - \sum_{i=1}^l C_{n-k}^{l-i} x^{(n-k)-(l-i)} (1-x)^{l-i}}{1 - \sum_{i=1}^l C_n^{l-i} x^{n-(l-i)} (1-x)^{l-i}} \geq \frac{C_{n-k}^l}{C_n^l}. \quad \square$$

Full Proof of Theorem 3

Proof. Without loss of generality, we assume that $o_1 \geq o_2 \geq \dots \geq o_n$ in this proof. Firstly, observe the objective of merging problem (2). We know that for the optimal mechanism $\mathcal{M}^*$, its objective is always bounded by $\max^{(k)}\{o_1 \dots o_k, a_1 \dots a_n\}$, i.e.,

$$\text{OBJ}(\mathcal{M}^*) \leq \mathbb{E}_{\mathbf{a} \sim G} \left[\max^{(k)}\{o_1 \dots o_k, a_1 \dots a_n\} \right].$$

Let λ_i be the i -th largest number in the set $\{o_1, \dots, o_k, a_1, \dots, a_n\}$. We can rewrite the above inequality as:

$$\begin{aligned} \text{OBJ}(\mathcal{M}^*) &\leq \mathbb{E}_{\mathbf{a} \sim G} \left[\max^{(k)}\{o_1 \dots o_k, a_1 \dots a_n\} \right] \\ &= \sum_{i=1}^k \int_0^\infty \mathbb{P}_{\mathbf{a} \sim G}(\lambda_i \geq t) dt. \end{aligned} \quad (12)$$

Since all a_i are identical, for $i \in [k]$, we can calculate $\mathbb{P}_{\mathbf{a} \sim G}(\lambda_i \geq t)$ as

$$\begin{aligned} \mathbb{P}_{\mathbf{a} \sim G}(\lambda_1 \geq t) &= \begin{cases} 1 & 0 \leq t \leq o_1, \\ 1 - G^n(t) & t > o_1. \end{cases} \\ \mathbb{P}_{\mathbf{a} \sim G}(\lambda_2 \geq t) &= \begin{cases} 1 & 0 \leq t \leq o_2, \\ 1 - G^n(t) & o_2 < t \leq o_1, \\ 1 - \sum_{i=1}^2 C_n^{2-i} G^{n-(2-i)}(t) (1 - G(t))^{2-i} & t > o_1. \end{cases} \\ &\dots \\ \mathbb{P}_{\mathbf{a} \sim G}(\lambda_k \geq t) &= \begin{cases} 1 & 0 \leq t \leq o_k, \\ 1 - \sum_{i=j}^k C_n^{k-i} G^{n-(k-i)}(t) (1 - G(t))^{k-i} & o_j < t \leq o_{j-1}, \\ 1 - \sum_{i=1}^k C_n^{k-i} G^{n-(k-i)}(t) (1 - G(t))^{k-i} & j \in [2, \dots, k], \\ & t > o_1. \end{cases} \end{aligned}$$

Plugging all $\mathbb{P}_{\mathbf{a} \sim G}(\lambda_i \geq t)$ into inequality (12), we get an upper bound of the optimal objective.

$$\begin{aligned} \text{OBJ}(\mathcal{M}^*) &\leq \sum_{i=1}^k \int_0^\infty \mathbb{P}_{\mathbf{a} \sim G}(\lambda_i \geq t) dt \\ &= \sum_{i=1}^k o_i + \int_{o_k}^\infty (1 - G^n(t)) dt + \dots \\ &\quad + \int_{o_1}^{+\infty} \left[1 - \sum_{i=1}^k C_n^{k-i} G^{n-(k-i)}(t) (1 - G(t))^{k-i} \right] dt. \end{aligned} \quad (13)$$

On the other hand, we want to find a lower bound on the objective of $\mathcal{M}^F$. Since $\mathcal{M}^F$ selects the best mechanism among $\mathcal{M}_I^F$, herein, we focus on a specific $\mathcal{M}_I^F$ with $I = \{1, 2, \dots, k\}$, whose objective is $\max^{(k)}\{o_1 \dots o_k, a_{k+1} \dots a_n\}$. Denote θ_i by the i -th largest number in the set $\{o_1 \dots o_k, a_{k+1} \dots a_n\}$. We have

$$\begin{aligned} \text{OBJ}(\mathcal{M}^F) &\geq \mathbb{E}_{\mathbf{a} \sim G} \left[\max^{(k)}\{o_1 \dots o_k, a_{k+1} \dots a_n\} \right] \\ &= \sum_{i=1}^k \int_0^\infty \mathbb{P}_{\mathbf{a} \sim G}(\theta_i \geq t) dt \end{aligned}$$

Note that there are only $n-k$ identical $a_i, i \in \{k+1, \dots, n\}$. With the similar calculation, we get a lower bound of $\text{OBJ}(\mathcal{M}^F)$, i.e.,

$$\begin{aligned} \text{OBJ}(\mathcal{M}^F) &\geq \sum_{i=1}^k o_i + \int_{o_k}^\infty (1 - G^{n-k}(t)) dt + \dots \\ &\quad + \int_{o_1}^{+\infty} \left[1 - \sum_{i=1}^k C_{n-k}^{k-i} G^{(n-k)-(k-i)}(t) (1 - G(t))^{k-i} \right] dt. \end{aligned} \quad (14)$$

It is not hard to find that inequalities (13) and (14) have the same number of terms. By Lemma 6 and the following inequality

$$1 = \frac{C_{n-k}^0}{C_n^0} > \dots > \frac{C_{n-k}^l}{C_n^l} > \frac{C_{n-k}^{l+1}}{C_n^{l+1}} > \dots > \frac{C_{n-k}^k}{C_n^k},$$

we can replace the terms in inequality (14) by the corresponding terms in inequality (13) with C_{n-k}^k/C_n^k times and we get

$$\begin{aligned} \text{OBJ}(\mathcal{M}^F) &\geq \frac{C_{n-k}^k}{C_n^k} \left[\sum_{i=1}^k o_i + \int_{o_k}^{\infty} (1 - G^n(t)) dt + \dots \right. \\ &\quad \left. + \int_{o_1}^{+\infty} \left[1 - \sum_{i=1}^k C_n^{k-i} G^{n-(k-i)}(t) (1 - G(t))^{k-i} \right] dt \right] \\ &= \frac{C_{n-k}^k}{C_n^k} \text{OBJ}(\mathcal{M}^*). \end{aligned}$$

In conclusion, we show that $\mathcal{M}^F$ is C_{n-k}^k/C_n^k competitive relative to the optimal mechanism $\mathcal{M}^*$. $\square$