The Structure and Degrees of Polynomials Computing Square Roots $\mod p$

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Abstract

For an odd prime p, we say a polynomial $f \in \mathbb{F}_p[X]$ computes square roots if $f(a)^2 = a$ for all nonzero, perfect squares $a \in \mathbb{F}_p$.

When $p \equiv 3 \mod 4$, it is easy to see that $f(X) = X^{\frac{p+1}{4}}$ is the smallest such polynomial.

For $p \equiv 1 \mod 4$, the situation is less clear. Tonelli-Shanks offers an algorithm for constructing polynomials that compute square roots, but the question of whether their degree is minimal remains. In this paper, we study the various degrees and structures of polynomials computing square roots.

1 Introduction

Let p be an odd prime and let $\mathbb{F}_p$ be the finite field with p elements. Quadratic reciprocity and the Legendre symbol give an efficient way to decide, for a given $a \in \mathbb{F}_p^\times$, whether a is a quadratic residue. On the other hand, explicitly finding a square root $b \in \mathbb{F}_p$ with $b^2 = a$ is computationally harder. One natural way to package square-root computations is via polynomials that compute square roots on the set of quadratic residues.

Let

$$S = \{a \in \mathbb{F}_p^\times : a \text{ is a square in } \mathbb{F}_p\}$$

be the set of nonzero squares, so $|S| = \frac{p-1}{2} := r$.

Definition 1.1. We say that a polynomial $f(x) \in F_p[x]$ *computes square roots* (on S), if

$$f(a)^2 = a$$

for all $a \in S$. We write $\mathcal{F} = \mathcal{F}_S$ for the set of all such polynomials.

A first observation is that if $f \in \mathcal{F}$ then

$$f(x)^2 - x$$

vanishes on S, hence is divisible by the vanishing polynomial

$$\prod_{a \in S} (x - a) = x^{\frac{p-1}{2}} - 1. \tag{1.1}$$

This immediately yields the "trivial" degree bounds

$$\frac{p-1}{4} \leq \deg f \leq \frac{p-1}{2} - 1$$

the upper bound coming from the fact that a polynomial that computes square roots on S is determined by interpolation on $|S| = \frac{p-1}{2}$ points.

When $p \equiv 3 \mod 4$, the lower bound becomes $\deg f \geq \frac{p+1}{4}$ and it is sharp. Indeed, it is easy to check that $f(x) = x^{\frac{p+1}{4}}$ is indeed a polynomial that computes square roots, and therefore, it has the minimal possible degree. In fact, we can prove it is the unique minimal polynomial. The key input is the following structural result for polynomials whose square has very few monomials.

Theorem 1.1. *Suppose $f(x) \in \mathbb{F}_p[x]$ satisfies $f^2(x) = b_0 + b_1 x + b_{2n-1} x^{2n-1} + b_{2n} x^{2n}$ for some $2 < n < \frac{p}{2}$. Then f is a monomial.*

The main focus of this paper is the more delicate case $p \equiv 1 \mod 4$.
Writing

$$p-1 = 2^s q, \qquad q \text{ odd}, \qquad r := \frac{p-1}{2}$$

we still have the trivial bounds

$$\frac{p-1}{4} \leq \deg f \leq r-1$$

Kedlaya and Kopparty [KK24] recently proved that when $p \equiv 1 \mod 4$ one in fact has

$$\frac{p-1}{3} \leq \deg(f) \leq \frac{p-1}{2} - 1 \tag{1.2}$$

although the lower bound is not sharp. This leaves two natural questions:

1. What is the minimal possible degree of a polynomial in $\mathcal{F}$?
2. More generally, what are the possible degrees of polynomials in $\mathcal{F}$, and how are they distributed?

A classical algorithm of Tonelli and Shanks produces square roots module $p \equiv 1 \mod 4$ using the factorization $p-1 = 2^s q$. Interepreted appropriately, this algorithm yields many polynomials in $\mathcal{F}$. We call these *Tonelli-Shanks (TS) polynomials*. A natural hope is that TS polynomials might have the minimal possible degree or at least be very close to minimal. One of our goals is to understand to what extent this is true.

1.1 Main results

We now summarize our main results. Throughout, we fix an odd prime $p-1 = 2^s q$ with q odd and set $r = \frac{p-1}{2}$. Via reduction modulo $x^r - 1$, every $f \in F$ can be uniquely written as

$$f(x) = \sum_{i=0}^{r-1} c_i x^i$$

(1) Tonelli–Shanks polynomials and their degrees.

We summarize some of the properties of the Tonelli-Shanks polynomials explored in this paper in the following theorem:

Theorem 1.2. *There are $2^{2^{s-1}}$ TS-polynomials. If $f_{TS}(x)$ is such a polynomial, then its possible degrees have the form*

$$\deg(f_{TS}) = \frac{(2^s-1)p+1}{2^{s+1}} - nq = r - \frac{q-1}{2} - nq$$

for $n \geq 0$. In fact, if $f_{TS}(x) = \sum c_i x^i$, the above represent the only possible degrees for the nonzero monomials $c_i x^i$, i.e. $c_i = 0$ unless $i = r - \frac{q-1}{2} - nq$ for some n. The maximum degree (with $n = 0$) is always achieved by at least one TS-polynomial (and in fact by the majority of them).

With the heuristic model we explain later, the expected minimum degree of a TS-polynomial is

$$\min \deg(f_{TS}) = r - \frac{q-1}{2} - q\lfloor \frac{2^{s-1} - s + 1}{\log_2(p)} \rfloor$$

(2) Exact counting for high-degree polynomials.

Our analysis of the polynomials in $\mathcal{F}$ uses a discrete Fourier description of the coefficients. For a generator γ of $\mathbb{F}_p^\times$, every $f \in \mathcal{F}$ is encoded by a sign vector $\epsilon = (\epsilon_0, \ldots, \epsilon_{r-1}) \in \{\pm 1\}^r$ and each coefficient c_k can be written as a *signed half sum*

$$c_k = \frac{1}{r} \sum_{n=0}^{r-1} \varepsilon_n \gamma^{n(1-2k)}.$$

We develop a symmetry on sign patterns (the *flip-shift* map) and the associated notion of alternating order $\alpha(\epsilon)$ which measures of the amount of alternating periodicicity in ϵ. For each divisor $d|r$ with $D := r/d$ odd, we show that the polynomials whose sign vectors have alternating order d form a "family" whose degrees lie in an arithmetic progression:

$$\deg f = r - \frac{D-1}{2} - nD, \qquad n \geq 0.$$

In particular, the TS polynomials correspond exactly to the family with $d = 2^{s-1}$ and $D = q$.

We next study the vanishing of individual coefficients c_k. Relating this to the signed half sums and using discrete Fourier analysis, we prove:

Theorem 1.3 (Counting solutions to a signed half sum)**.** *Let p be an odd prime, $r = \frac{p-1}{2}$, and let ℓ be an odd integer with $\gcd(\ell, p-1) = 1$. Then the number of sign vectors $\epsilon \in \{\pm 1\}^r$ such that*

$$\sum_{n=0}^{r-1} \varepsilon_n \xi^{n\ell} = 0$$

for a fixed primitive $(p-1)$-th root of unity ξ, is

$$\frac{1}{p}\left(2^r + (p-1)\left(\frac{2}{p}\right)\right)$$

where $\left(\frac{2}{p}\right)$ is the Legendre symbol.

Translating back to coefficients, this shows that as f ranges over $\mathcal{F}$ and k is such that $\gcd(1-2k, p-1) = 1$, the coefficient c_k vanishes with probability

$$\mathbb{P}(c_k = 0) = \frac{1}{p} + O(2^{-r}).$$

and we obtain an exact count of the number of polynomials in $\mathcal{F}$ with $c_k = 0$. In particular, under the mild hypothesis $3 \nmid q$, this applies to the top coefficient c_{r-1}, and yields:

Corollary 1.1. *Assume $p - 1 = 2^s q$ with q odd and $3 \nmid q$. Then*

$$\#\{f \in \mathcal{F} : \deg f = r-1\} = \frac{1}{p}\left(2^r + (p-1)\left(\frac{2}{p}\right)\right)$$

Thus we obtain a precise asymptotic bound for the "largest possible" degree among square-root polynomials.

(3) Heuristic model for degrees and minimal degree.

The exact counting result above suggests that, for a typical $f \in F$, the coefficients c_k behave almost like independent uniform elements of $\mathbb{F}_p$. Motivated by this and by extensive computations, we formulate the following heuristic:

- (H_1) For each k, $\mathbb{P}(c_k = 0) = \frac{1}{p} + O(2^{-r})$.
- (H_2) For each divisor $d|r$, the events $\{c_k = 0 : k \equiv r + td \pmod r\}$ are approximately independent as t varies.

Organizing polynomials by their alternating order $d = \alpha(\epsilon)$, we obtain for each family a set of admissible degrees (as described above) and under $(H_1) - (H_2)$, a prediction for how many polynomials of a given degree one should expect. This leads to a heuristic formula for the minimal degree in each family and hence for the global minimal degree

$$\min\{\deg f : f \in \mathcal{F}\}$$

While this expression is not a simple closed form in terms of p it is straightforward to evaluate for each fixed prime.

Specializing the heuristic to the TS family (for which $D = q$), we obtain the prediction described in Theorem 1.2.

For small primes p (up to several hundred), our computations of the full degree distribution of $\mathcal{F}$ and of the TS subfamily agree very closely with the predictions of this model.

Finally, we remark that this entire paper can be viewed as a special case of the Littlewood-Offord problem. Given an integer vector $\mathbf{a} = (a_1, .., a_n)$, its concentration probability is defined as $\rho(a) := \sup_{x \in \mathbb{Z}} \mathbb{P}(\epsilon_1 a_1 + ... + \epsilon_n a_n = x)$. The Littlewood-Offord problem asks for bounds on $\rho(a)$ under various hypotheses on $\mathbf{a}$. For an overview and recent results, see for example [TV10; Fer+21]

Organization of the paper. In Section 2 we prove Theorem 1.1. In Section 3 we define the Tonelli-Shanks polynomials formally and analyze their basic properties. Section 4 describes square roots polynomials via Lagrange interpolation and Fourier inversion on subgroups of S. Section 5 introduces signed half sums, the flip–shift symmetry, and alternating order, and uses these to analyze the possible degrees, with a detailed description of the TS family. Section 6 contains our exact counting theorem for signed half sums, the resulting heuristic model for the degree distribution, and numerical evidence. The appendix develops the algorithm that was used to compute the minimal polynomials for small primes p. It is available in [Noa25].

2 Proof of Theorem 1.1

Proof. Let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{F}_p[x],$$

and let

$$m = \min\{i : 0 \le i \le n,\ a_i \neq 0\}.$$

Then the coefficient of x^{2m} in $f(x)^2$ is

$$[x^{2m}]f(x)^2 = a_m^2 \neq 0.$$

By hypothesis all nonzero coefficients of $f(x)^2$ occur in degrees $0, 1, 2n-1, 2n$, so

$$2m \in \{0, 1, 2n-1, 2n\}.$$

Since $2m$ is even and $2n-1$ is odd, in fact $2m \in \{0, 2n\}$, hence $m = 0$ or $m = n$. If $m = n$, then f is a monomial and we are done.

Assume now that $m = 0$. Then $a_0 \neq 0$. Multiplying f by the nonzero scalar $-\frac{1}{2a_0}$ does not change the shape of $f(x)^2$ (it just scales it), so we may assume that the constant term of f is $-\frac{1}{2}$. Thus

$$f(x) = -\frac{1}{2} + a_1 x + \cdots + a_n x^n.$$

Let

$$\ell = \min\{i : 1 \le i \le n,\ a_i \neq 0\}.$$

Such an ℓ exists because $a_n \neq 0$. The coefficient of x^ℓ in $f(x)^2$ is

$$[x^\ell]f(x)^2 = 2a_0 a_\ell = 2\left(-\frac{1}{2}\right) a_\ell = -a_\ell \neq 0.$$

Again, all nonzero coefficients of $f(x)^2$ occur in degrees $0, 1, 2n-1, 2n$. Since $2 < n$, we have

$$2 \le \ell \le n \implies \ell \notin \{0, 1, 2n-1, 2n\},$$

so the only possibility is $\ell = 1$. In particular, $a_1 \neq 0$.

Claim. For every integer k with $1 \le k \le n$,

$$a_k = C_{k-1} a_1^k,$$

where C_j denotes the j-th Catalan number.

Recall that the Catalan numbers $(C_m)_{m \ge 0}$ satisfy

$$C_0 = 1, \qquad C_m = \frac{1}{m+1}\binom{2m}{m} = \sum_{i=1}^{m} C_{i-1} C_{m-i} \quad (m \ge 1).$$

We prove the claim by induction on k.

For $k = 1$ the formula is $a_1 = C_0 a_1^1$, which holds because $C_0 = 1$. Now suppose that for some k with $1 \le k \le n-1$ we have

$$a_i = C_{i-1} a_1^i \quad \text{for all } 1 \le i \le k.$$

We show that this implies $a_{k+1} = C_k a_1^{k+1}$.

Since $2 \le k+1 \le n$, the coefficient of x^{k+1} in $f(x)^2$ must vanish (because $k+1 \notin \{0, 1, 2n-1, 2n\}$):

$$[x^{k+1}]f(x)^2 = 0.$$

On the other hand,

$$\begin{aligned} 0 = [x^{k+1}]f(x)^2 &= \sum_{i=0}^{k+1} a_i a_{k+1-i} \\ &= 2a_0 a_{k+1} + \sum_{i=1}^{k} a_i a_{k+1-i} \\ &= -a_{k+1} + \sum_{i=1}^{k} a_i a_{k+1-i}, \end{aligned}$$

using $a_0 = -\frac{1}{2}$. Thus

$$a_{k+1} = \sum_{i=1}^{k} a_i a_{k+1-i}.$$

Substituting the induction hypothesis,

$$\begin{aligned} a_{k+1} &= \sum_{i=1}^{k} \bigl(C_{i-1} a_1^i\bigr)\bigl(C_{k-i} a_1^{k+1-i}\bigr) \\ &= a_1^{k+1} \sum_{i=1}^{k} C_{i-1} C_{k-i} = C_k a_1^{k+1} \end{aligned}$$

by the Catalan recurrence. This completes the induction and proves the claim.

Finally, consider the coefficient of x^{n+1} in $f(x)^2$. Since $2 < n$, we have

$$2 \le n+1 \le 2n-2,$$

so $n+1 \notin \{0, 1, 2n-1, 2n\}$ and therefore

$$[x^{n+1}] f(x)^2 = 0.$$

However, using the claim we obtain

$$\begin{aligned} [x^{n+1}] f(x)^2 &= \sum_{i=1}^{n} a_i a_{n+1-i} \\ &= \sum_{i=1}^{n} \bigl(C_{i-1} a_1^i\bigr)\bigl(C_{n-i} a_1^{n+1-i}\bigr) \\ &= a_1^{n+1} \sum_{i=1}^{n} C_{i-1} C_{n-i} = C_n a_1^{n+1} \\ &= \frac{1}{n+1}\binom{2n}{n} a_1^{n+1}. \end{aligned}$$

Because $2n < p$, neither $n+1$ nor $\binom{2n}{n}$ is divisible by p, so $C_n \not\equiv 0 \pmod p$. We already know $a_1 \neq 0$, hence $[x^{n+1}] f(x)^2 \neq 0$ in $\mathbb{F}_p$, contradicting the fact that this coefficient must be zero.

This contradiction shows that the case $m = 0$ cannot occur. Therefore $m = n$, so $f(x) = a_n x^n$ is a monomial. ■

3 Tonelli–Shanks polynomials

We now turn our attention to the case $p \equiv 1 \pmod 4$. In this section we define the Tonelli–Shanks polynomials; in later sections we see several equivalent descriptions.

Write

$$p - 1 = 2^s q, \qquad q \text{ odd},$$

and set

$$r = \frac{p-1}{2}.$$

Recall that the set of nonzero squares is

$$S = \mu_{\frac{p-1}{2}} = \mu_r \subset \mathbb{F}_p^{\times}.$$

We fix throughout a primitive 2^{s-1}-st root of unity $\zeta \in \mathbb{F}_p$. With this notation we make the following definitions and observations.

1. For each $i = 0, \ldots, 2^{s-1} - 1$ define

$$S_i = \{a \in S : a^q = \zeta^i\}.$$

Then the S_i form a partition of S,

$$S = \bigsqcup_{i=0}^{2^{s-1}-1} S_i,$$

and each set has cardinality $|S_i| = q$.

2. For $i = 0, \ldots, 2^{s-1} - 1$ define

$$F_i := F_{S_i} := \{f(x) \in \mathbb{F}_p[x] : f(a)^2 = a \text{ for all } a \in S_i\},$$

the set of polynomials that compute square roots on S_i. As before, we write F for the set of polynomials that compute square roots on the full set S.

3. If we are given polynomials $f_k \in F_k$ for all $k = 0, \ldots, 2^{s-1} - 1$, then

$$f(x) = \sum_{k=0}^{2^{s-1}-1} f_k(x) \prod_{\substack{i=0 \\ i \neq k}}^{2^{s-1}-1} \frac{x^q - \zeta^i}{\zeta^k - \zeta^i} \tag{3.1}$$

computes square roots on all of S.

The next lemma records the natural symmetry between the sets S_i and the spaces F_i.

Lemma 3.1. *Fix a prime p and $\zeta \in \mu_{2^{s-1}}$. Pick $\xi \in \mu_{2^{s-1}}$ a primitive 2^{s-1}-st root of unity, so that $\xi^q = \zeta$. Then for each $i = 0, \ldots, 2^{s-1} - 1$ we have:*

1. *$S_i = \xi^i S_0$.*
2. *There is a bijection*

$$F_0 \xrightarrow{\sim} F_i, \qquad f_0(x) \longmapsto f_i(x) := \xi^{i/2} f_0(\xi^{-i} x),$$

where we fix once and for all a choice of $\xi^{1/2} \in \mu_{2^s}$ and interpret $\xi^{i/2}$ accordingly.

Proof. The map

$$\phi_q : \mu_m \to \mu_m \qquad x \mapsto x^q$$

is an automorphism when $\gcd(q, m) = 1$, so ξ is well defined and primitive and the rest readily follows. ■

Using the same ideas as in the Introduction, we can now describe minimal-degree polynomials on each S_k.

Theorem 3.1 (Tonelli-Shanks)**.** *Let $p - 1 = 2^s q$ with q odd, and fix a primitive 2^{s-1}-st root of unity $\zeta \in \mu_{2^{s-1}}$. Fix a choice of $\zeta^{1/2} \in \mu_{2^s}$. For each $k = 0, \ldots, 2^{s-1} - 1$ the polynomial*

$$f_k(x) = \zeta^{-k/2} x^{\frac{q+1}{2}}$$

lies in F_k and has minimal possible degree among polynomials in F_k.

Moreover, if $q \geq 5$, then f_k is the unique polynomial in F_k of minimal degree, up to an overall sign.

Proof. First, for any $a \in S_k$ we have $a^q = \zeta^k$, so

$$f_k(a)^2 = \zeta^{-k} a^{q+1} = \zeta^{-k} \zeta^k a = a,$$

hence $f_k \in F_k$.

Now let $f \in F_k$ be arbitrary, and set $d = \deg f$. The vanishing polynomial on S_k is

$$\prod_{a \in S_k} (x - a) = x^q - \zeta^k,$$

and $f(x)^2 - x$ vanishes on S_k and therefore

$$x^q - \zeta^k \mid f(x)^2 - x.$$

Thus we can write

$$f(x)^2 - x = A(x)(x^q - \zeta^k)$$

for some $A(x) \in \mathbb{F}_p[x]$. If f is nonconstant then $\deg(f^2 - x) = 2d$, so

$$2d = \deg(f^2 - x) \geq \deg(x^q - \zeta^k) = q,$$

and hence

$$d \geq \frac{q+1}{2}.$$

Since f_k has degree $\frac{q+1}{2}$, this shows that f_k has minimal possible degree in F_k.

For the uniqueness statement, assume $q \geq 5$ and let $f \in F_k$ have minimal degree $d = \frac{q+1}{2}$. Then $\deg f^2 = q + 1$, and from

$$f(x)^2 - x = A(x)(x^q - \zeta^k)$$

we see that $\deg A = 1$. Write $A(x) = \alpha x + \beta$, so

$$\begin{aligned} f(x)^2 &= A(x)(x^q - \zeta^k) + x \\ &= \alpha x^{q+1} + \beta x^q - \alpha\zeta^k x - \beta\zeta^k + x \\ &= \big(-\beta\zeta^k\big) + \big(1 - \alpha\zeta^k\big)x + \beta x^q + \alpha x^{q+1}. \end{aligned}$$

Set $n = \frac{q+1}{2}$, so that $2n = q + 1$ and $2n - 1 = q$. Then

$$f(x)^2 = b_0 + b_1 x + b_{2n-1}x^{2n-1} + b_{2n}x^{2n}$$

with

$$b_0 = -\beta\zeta^k, \quad b_1 = 1 - \alpha\zeta^k, \quad b_{2n-1} = \beta, \quad b_{2n} = \alpha.$$

Since $q \geq 5$, we have $n = \frac{q+1}{2} > 2$. Moreover, $2n = q + 1 < p$, so the hypotheses of Theorem 1.1 apply and we conclude that f is a monomial:

$$f(x) = cx^n$$

for some $c \in \mathbb{F}_p^\times$.

Finally, for any $a \in S_k$ we have

$$a = f(a)^2 = c^2 a^{2n} = c^2 a^{q+1},$$

so $c^2 a^q = 1$. Using $a^q = \zeta^k$ we obtain $c^2\zeta^k = 1$, hence

$$c^2 = \zeta^{-k}.$$

Thus $c = \pm\zeta^{-k/2}$, and

$$f(x) = \pm\zeta^{-k/2}x^{\frac{q+1}{2}}.$$

This proves uniqueness up to sign when $q \geq 5$. ∎

For $q = 1$ or $q = 3$ the same degree bound shows that $\deg f \geq \frac{q+1}{2}$ and f_k still has minimal degree, but one can (and in general does) obtain additional minimal polynomials of the same degree; we do not need the uniqueness statement in these small cases in the rest of the paper.

Specializing Theorem 3.1 to $p \equiv 3 \pmod 4$ (so that $s = 1$ and $q = \frac{p-1}{2}$) recovers the well-known minimal polynomial in that case.

Corollary 3.1. *Assume $p \equiv 3 \pmod 4$. Then*

$$f(x) = x^{\frac{p+1}{4}}$$

computes square roots on S and has minimal possible degree. If $p \geq 11$ (so $\frac{p+1}{4} > 2$), then f is the unique minimal polynomial in F, up to sign.

Definition 3.1. A *Tonelli–Shanks polynomial* is any polynomial obtained by gluing together the polynomials f_k from Theorem 3.1 via the formula (3.1).

Corollary 3.2. *Let $p - 1 = 2^s q$ with q odd, and set $r = \frac{p-1}{2}$. Then:*

1. *There are $2^{2^{s-1}}$ Tonelli–Shanks polynomials that compute square roots on S.*

2. If $f_{TS}(x)$ is such a polynomial, then

$$\deg(f_{TS}) \le \frac{(2^s-1)p+1}{2^{s+1}} = \frac{q(2^s-1)+1}{2} = r - \frac{q-1}{2}.$$

Proof. For each S_i there are two choices of polynomial in F_i of minimal degree, namely $\pm f_i$ from Theorem 3.1. There are 2^{s-1} different sets S_i, so the total number of ways to choose one sign on each S_i is $2^{2^{s-1}}$. This proves (1).

For (2), fix the minimal-degree representatives $f_k(x) = \zeta^{-k/2} x^{\frac{q+1}{2}}$. In the gluing formula (3.1), the factor

$$\prod_{\substack{i=0 \\ i\neq k}}^{2^{s-1}-1} (x^q - \zeta^i)$$

has degree $(2^{s-1}-1)q$, so each summand has degree

$$(2^{s-1}-1)q + \frac{q+1}{2}.$$

Thus

$$\deg(f_{TS}) \le (2^{s-1}-1)q + \frac{q+1}{2} = \frac{(2^s-1)p+1}{2^{s+1}} = r - \frac{q-1}{2}.$$

Equality holds whenever there is no cancellation in the leading coefficient. ∎

Remark 3.1. If in each S_i we choose the "positive" polynomial $f_i(x) = \zeta^{-i/2} x^{\frac{q+1}{2}}$, then there is no cancellation in the leading coefficient of f_{TS}. Indeed, the leading coefficient is a nonzero multiple of

$$\sum_{k=0}^{2^{s-1}-1} (\zeta^{-1/2})^k = \frac{1-(\zeta^{-1/2})^{2^{s-1}}}{1-\zeta^{-1/2}} = \frac{1-\zeta^{-2^{s-2}}}{1-\zeta^{-1/2}} = \frac{2}{1-\zeta^{-1/2}} \neq 0$$

in $\mathbb{F}_p$, since $\zeta^{2^{s-2}} = -1$. Thus at least one Tonelli–Shanks polynomial has degree

$$\deg(f_{TS}) = \frac{(2^s-1)p+1}{2^{s+1}},$$

and in fact we will see later that the majority of Tonelli–Shanks polynomials attain this degree.

In particular, we obtain the upper bound

$$\min\{\deg f : f \in F\} \le r - \frac{q-1}{2}.$$

For instance, when $p \equiv 5 \pmod 8$ one has $s = 2$ and $q = \frac{p-1}{4}$, so

$$\deg f_{TS} = \frac{3p+1}{8},$$

recovering the bound of [KK24, Th. 1.3].

4 Lagrange Interpolation and Fourier Transform

In this section, we describe the coefficients of polynomials that compute square roots, using Lagrange interpolation and Fourier inversion. We start with the general case of subgroups of S and then restrict our attention to S_0 and F_0, which is sufficient by Lemma 3.1.

Let $H \le S$ be a subgroup. Then H is cyclic of the form $H = \mu_d$ for some divisor $d \mid r = \frac{p-1}{2}$. Any polynomial f of degree $\le d-1$ that computes square roots on H (we write $f \in F_H$) can be constructed by Lagrange interpolation, by interpolating the tuples

$$\{(a, \pm\sqrt{a})\}_{a\in H}.$$

This yields the expression

$$f(x) = \sum_{s\in H} (\pm\sqrt{s}) \prod_{\substack{t\in H \\ t\neq s}} \frac{x-t}{s-t}.$$

Equivalently, if we enumerate $H = \{\zeta_d^i : 0 \le i \le d-1\}$ for a primitive d-th root of unity $\zeta_d \in \mathbb{F}_p^\times$, then f is uniquely determined by a choice of signs $\epsilon_i \in \{\pm 1\}$ via

$$f(\zeta_d^i) = \epsilon_i \zeta_d^{i/2}, \qquad i = 0, 1, \dots, d-1,$$

where we implicitly fix a choice of $\zeta_d^{1/2}$.

We can make the dependence on the signs explicit by a discrete Fourier transform on H.

Theorem 4.1 (Fourier inversion on $H \le S$)**.** *Let p be an odd prime, $S = \mu_{\frac{p-1}{2}} = \mu_r$ be the group of all squares in $\mathbb{F}_p^\times$, and let $H = \mu_d$ be a subgroup of S. Fix a primitive d-th root $\zeta_d \in \mathbb{F}_p^\times$ and a choice of $\zeta_d^{1/2}$. Let $f \in F_H$ be determined by a sign pattern $\{\epsilon_n\}_{n=0}^{d-1} \subset \{\pm 1\}$ via*

$$f(\zeta_d^n) = \epsilon_n \zeta_d^{n/2} \qquad (n = 0, 1, \dots, d-1).$$

Write

$$f(x) = \sum_{k=0}^{d-1} c_k x^k.$$

Then, for $0 \le k \le d-1$,

$$c_k = \frac{1}{d} \sum_{n=0}^{d-1} \epsilon_n \zeta_d^{n(\frac{1}{2}-k)}.$$

Proof. Write $\zeta = \zeta_d$. Evaluating $f(x) = \sum_{k=0}^{d-1} c_k x^k$ at $x = \zeta^n$ gives

$$f(\zeta^n) = \sum_{k=0}^{d-1} c_k \zeta^{nk}.$$

By hypothesis $f(\zeta^n) = \epsilon_n \zeta^{n/2}$. Using the orthogonality relations

$$\frac{1}{d} \sum_{n=0}^{d-1} \zeta^{n(k-\ell)} = \begin{cases} 1 & \text{if } k \equiv \ell \pmod d, \\ 0 & \text{otherwise,} \end{cases}$$

we invert to obtain

$$c_k = \frac{1}{d} \sum_{n=0}^{d-1} f(\zeta^n) \zeta^{-nk} = \frac{1}{d} \sum_{n=0}^{d-1} \epsilon_n \zeta^{n(\frac{1}{2}-k)}.$$

■

4.1 Lagrange interpolation and coefficients of polynomials computing S_0

We now specialize Theorem 4.1 to the case $H = S_0 \le S$ and describe the coefficients of $f_0 \in F_0$. By Lemma 3.1, the same analysis applies to any S_i.

Definition 4.1. Let $m := \frac{q+1}{2} \in \mathbb{Z}$. For $a \in S_0$ define

$$a^{1/2} := a^m \in S_0.$$

If ζ_q is a primitive q-th root of unity, this convention fixes the meaning of $\zeta_q^{n/2}$ for all n.

As before, any $f_0 \in F_0$ is uniquely determined by a choice of signs $\epsilon_i \in \{\pm 1\}$ for $i = 0, 1, \dots, q-1$ via

$$f_0(\zeta_q^i) = \epsilon_i \zeta_q^{i/2}.$$

Example 4.1. Let $f_0(x) = x^{\frac{q+1}{2}} \in F_0$ be the minimal polynomial in F_0, as in Theorem 3.1. Then the corresponding sign pattern is

$$(\epsilon_0, \dots, \epsilon_{q-1}) = (1, \dots, 1).$$

As a consequence of Theorem 4.1, we obtain the following convenient form for the coefficients of f_0.

Corollary 4.1 (Fourier inversion on F_0)**.** *Let p be an odd prime with $p-1 = 2^s q$ and q odd. Fix a primitive q-th root $\zeta_q \in \mathbb{F}_p^\times$ and set $m := \frac{q+1}{2}$. Let $f_0 \in F_0$ be determined by a sign pattern $\{\epsilon_n\}_{n\in\mathbb{Z}/q\mathbb{Z}} \subset \{\pm 1\}^q$ via*

$$f_0(\zeta_q^n) = \epsilon_n(\zeta_q^n)^{1/2} = \epsilon_n \zeta_q^{nm} \qquad (n = 0, 1, \ldots, q-1).$$

Then, for any $d \in \mathbb{Z}$,

$$\left[x^{\frac{q+1}{2}+d}\right] f_0 = \frac{1}{q}\sum_{n=0}^{q-1} \epsilon_n \zeta_q^{-nd}.$$

(Here $\left[x^r\right] f_0$ denotes the coefficient of x^r in $f_0(x)$.)

It is natural to explore various structured choices of the sign pattern and how they affect the degree of f_0.

Theorem 4.2. *Write $p - 1 = 2^s q$ as always. Fix a primitive q-th root $\zeta \in \mathbb{F}_p^\times$. Let $(\epsilon_n)_{n\in\mathbb{Z}/q\mathbb{Z}}$ be a sign pattern with period $N \mid q$, that is,*

$$\epsilon_{n+N} = \epsilon_n \quad \text{for all } n,$$

and set $Q := q/N$ (so Q is odd).

Let $f_0 \in F_0$ be the polynomial corresponding to the values ϵ_n. By Corollary 4.1, for each $k = \frac{q+1}{2} + d$ we have

$$c_k = \left[x^{\frac{q+1}{2}+d}\right] f_0 = \frac{1}{q}\sum_{n=0}^{q-1} \epsilon_n \zeta^{-dn} \qquad (0 \le k \le q-1),$$

where c_k denotes the coefficient of x^k in $f_0(x)$. Then:

1. *$c_k = 0$ if $d \not\equiv 0 \pmod{Q}$. In particular, f_0 has at most N nonzero monomials.*
2. *Consequently,*

$$\deg f_0 \le \frac{q+1}{2} + \frac{q}{2} - \frac{q}{2N} = q - \frac{Q-1}{2}.$$

Proof. Write $n = Nm + r$ with $0 \le r < N$ and $0 \le m < Q$. Using the N-periodicity of ϵ, we have

$$\begin{aligned} c_k &= \frac{1}{q}\sum_{n=0}^{q-1} \epsilon_n \zeta^{-dn} = \frac{1}{q}\sum_{r=0}^{N-1}\sum_{m=0}^{Q-1} \epsilon_{Nm+r}\zeta^{-d(Nm+r)} \\ &= \frac{1}{q}\sum_{r=0}^{N-1} \epsilon_r \zeta^{-dr} \sum_{m=0}^{Q-1} (\zeta^{-dN})^m. \end{aligned}$$

The inner sum is a geometric series. It vanishes unless $\zeta^{-dN} = 1$, which is equivalent to $q \mid dN$, hence to $Q \mid d$. This proves (1).

If $Q \mid d$ we can write $d = tQ$ for some integer t. For the corresponding exponent

$$k = \frac{q+1}{2} + d = \frac{q+1}{2} + tQ$$

to lie in the range $0 \le k \le q-1$ we need

$$\frac{q+1}{2} + tQ \le q - 1 \quad \Longleftrightarrow \quad tQ \le \frac{q-3}{2}.$$

Since $q = NQ$, this gives

$$t \le \frac{q-3}{2Q} = \frac{N}{2} - \frac{3}{2Q} < \frac{N}{2}.$$

Because t is an integer and N is odd (as a divisor of the odd number q), we have $t \le \frac{N-1}{2}$.

Thus the largest possible exponent k with $c_k \neq 0$ satisfies

$$k_{\max} = \frac{q+1}{2} + t_{\max}Q \le \frac{q+1}{2} + \frac{N-1}{2}Q = \frac{q+1}{2} + \frac{q}{2} - \frac{Q}{2} = q - \frac{Q-1}{2},$$

which gives (2) and therefore the stated degree bound. ∎

Example 4.2. Assume $p \equiv 5 \pmod 8$ and $p-1 = 2^s q$ with $3 \mid q$. Then

$$f_0(x) = \frac{x^{\frac{q+1}{2}}}{3}\left(2x^{\frac{q}{3}} - 1 + 2x^{-\frac{q}{3}}\right)$$

computes square roots on S_0 (viewed modulo $x^q - 1$).

Proof. Take $N = 3$ and the 3-periodic sign pattern $(\epsilon_0, \epsilon_1, \epsilon_2) = (1, -1, -1)$. Applying Theorem 4.2 and rewriting exponents modulo q gives the displayed formula. ∎

5 Signed half sums

This is the main section of the paper. We study signed half sums in a general setting and extend Theorem 4.2 from the subgroup setting to the full family $\mathcal{F} = \mathcal{F}_S$.

5.1 Signed half sums

Recall $r = \frac{p-1}{2}$ and $S = \mu_r \subset \mathbb{F}_p^\times$. Fix a primitive $2r$-th root of unity $\zeta \in \mathbb{F}_p^\times$; then ζ^2 generates μ_r, and we can write

$$S = \{\zeta^{2n} : 0 \le n < r\}.$$

Every polynomial $f \in F$ is determined by a sign vector $\epsilon = (\epsilon_0, \dots, \epsilon_{r-1})$, where

$$f(\zeta^{2n}) = \epsilon_n \zeta^n \qquad (0 \le n < r),$$

and, by Fourier inversion (Theorem 4.1 with $H = S$), its coefficients have the form

$$c_k = \frac{1}{r}\sum_{n=0}^{r-1} \epsilon_n \zeta^{n(1-2k)}. \tag{5.1}$$

Thus each coefficient is a signed average of roots of unity. This motivates the following abstract setup.

For each positive integer d, set

$$E_d := \{\pm 1\}^d.$$

Definition 5.1. Let p be a prime and let d be a divisor of $r = \frac{p-1}{2}$. Let $\zeta \in \mathbb{F}_p^\times$ be a primitive $2d$-th root of unity, and let k be an odd integer. For $\epsilon = (\epsilon_0, \dots, \epsilon_{d-1}) \in E_d$, the (d,k)-*signed half sum* of ϵ (with respect to ζ) is the $\{\pm 1\}$-weighted sum over half the powers of ζ:

$$H_{d,k}^{(\zeta)}(\epsilon) := \sum_{n=0}^{d-1} \epsilon_n \zeta^{nk}.$$

In this way, using (5.1) all coefficients of polynomials $f \in \mathcal{F}$ (and similarly for the subfamilies $\mathcal{F}_H$) can be seen as signed half sums, up to a constant factor and a choice of (d, k, ζ).

Definition 5.2. Let p be a prime and $d \mid r$. For an odd integer k, define

$$V_{d,k}^{(\zeta)} := \left\{\epsilon \in E_d : H_{d,k}^{(\zeta)}(\epsilon) = 0\right\}.$$

For a set K of odd integers, define

$$V_{d,K}^{(\zeta)} := \bigcap_{k \in K} V_{d,k}^{(\zeta)} = \left\{\epsilon \in E_d : H_{d,k}^{(\zeta)}(\epsilon) = 0 \text{ for all } k \in K\right\}.$$

When the choice of primitive $2d$-th root ζ is clear from context, we simply write $H_{d,k}$, $V_{d,k}$ and $V_{d,K}$.

The next lemma shows that the choice of primitive $2d$-th root is essentially irrelevant for counting solutions.

Lemma 5.1. *Let p be a prime, $d \mid r$, and let k be an odd integer. Let ζ and η be any two primitive $2d$-th roots of unity in $\mathbb{F}_p^\times$. Then there exists a permutation*

$$T : E_d \to E_d$$

such that for all $\epsilon \in E_d$,

$$H_{d,k}^{(\zeta)}(\epsilon) = H_{d,k}^{(\eta)}\big(T(\epsilon)\big).$$

Proof. Since ζ and η are both primitive $2d$-th roots, there exists an integer u with $\gcd(u, 2d) = 1$ such that $\eta = \zeta^u$. Consider the permutation

$$\tau : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}, \qquad \tau(n) = u^{-1}n \pmod{d},$$

where u^{-1} is the inverse of u in $\mathbb{Z}/d\mathbb{Z}$ (which exists since $\gcd(u, d) = 1$). Define $T : E_d \to E_d$ by

$$T(\epsilon)_n := \epsilon_{\tau(n)}.$$

Then

$$\begin{aligned} H^{(\zeta)}_{d,k}(\epsilon) &= \sum_{n=0}^{d-1} \epsilon_n \zeta^{nk} = \sum_{n=0}^{d-1} \epsilon_n (\eta^u)^{nk} = \sum_{n=0}^{d-1} \epsilon_n \eta^{(un)k} \\ &= \sum_{m=0}^{d-1} \epsilon_{\tau(m)} \eta^{mk} = H^{(\eta)}_{d,k}\big(T(\epsilon)\big). \end{aligned}$$

■

Corollary 5.1. *Let p be a prime, $d \mid r$, and let K be a set of odd integers. If ζ and η are any two primitive $2d$-th roots of unity in $\mathbb{F}_p^\times$, then*

$$\big|V^{(\zeta)}_{d,K}\big| = \big|V^{(\eta)}_{d,K}\big|.$$

We now relate these definitions back to square-root polynomials.

Proposition 5.1. *Let $r = \frac{p-1}{2}$, and let ζ be a primitive $2r$-th root of unity. For each $\epsilon \in E_r$, let $f_\epsilon \in F$ denote the unique polynomial such that*

$$f_\epsilon(\zeta^{2n}) = \epsilon_n \zeta^n \qquad (0 \le n < r).$$

Then

$$\deg(f_\epsilon) \le d \quad \iff \quad \epsilon \in V_{r,K_d},$$

where

$$K_d := \{\, 1 - 2(d+1),\, 1 - 2(d+2),\, \ldots,\, 1 - 2(r-1) \,\}.$$

Proof. By Fourier inversion on $H = S = \mu_r$ (Theorem 4.1), the coefficients of $f_\epsilon(x) = \sum_{k=0}^{r-1} c_k x^k$ satisfy

$$c_k = \frac{1}{r} \sum_{n=0}^{r-1} \epsilon_n \zeta^{n(1-2k)} = \frac{1}{r} H_{r,\,1-2k}(\epsilon).$$

Thus, for $d + 1 \le k \le r - 1$,

$$c_k = 0 \iff \sum_{n=0}^{r-1} \epsilon_n \zeta^{n(1-2k)} = 0 \iff H_{r,\,1-2k}(\epsilon) = 0 \iff \epsilon \in V_{r,\,1-2k}.$$

Hence $c_k = 0$ for all $k \ge d + 1$ if and only if $\epsilon \in \bigcap_{k=d+1}^{r-1} V_{r,\,1-2k} = V_{r,K_d}$. This is equivalent to $\deg(f_\epsilon) \le d$. ■

5.2 Flip shifts

Fix a divisor $d \mid r$.

Definition 5.3. The *flip shift* map $\sigma : E_d \to E_d$ is defined by

$$\sigma(\epsilon)_i = \begin{cases} \epsilon_{i+1}, & 0 \le i \le d-2, \\ -\epsilon_0, & i = d-1. \end{cases}$$

Equivalently,

$$\sigma(\epsilon) = (\epsilon_1, \ldots, \epsilon_{d-1}, -\epsilon_0).$$

More generally, for $n \in \mathbb{Z}$ we define

$$\sigma^n(\epsilon)_i = (-1)^{\left\lfloor \frac{i+n}{d} \right\rfloor} \epsilon_{\,i+n \bmod d}.$$

One checks directly that this agrees with iterating the basic flip shift.

Let C_σ be the cyclic group generated by σ under composition.

Proposition 5.2. *For any $\epsilon \in E_d$,*

1. $\sigma(-\epsilon) = -\sigma(\epsilon)$.
2. $\sigma^d(\epsilon) = -\epsilon$.
3. $\sigma^{2d}(\epsilon) = \epsilon$.
4. $|C_\sigma| = 2d$.

Proof. Using the explicit formula for $\sigma^n(\epsilon)_i$ with $n = d$ and $n = 2d$ gives (2) and (3), and (1) is immediate from linearity in the signs. Since $\sigma^{2d} = \mathrm{id}$ and $\sigma^d \neq \mathrm{id}$, the order of σ is exactly $2d$, so $|C_\sigma| = 2d$. ∎

The flip shift preserves the signed half-sum equations:

Proposition 5.3. *Let p be a prime, $d \mid r$, and let k be an odd integer. Then, for all $\epsilon \in E_d$,*

$$\epsilon \in V_{d,k} \quad \Longleftrightarrow \quad \sigma(\epsilon) \in V_{d,k}.$$

Proof. Let ζ be a primitive $2d$-th root of unity. We show

$$H_{d,k}(\epsilon) = \zeta^k H_{d,k}(\sigma(\epsilon)).$$

Indeed,

$$\begin{aligned} H_{d,k}(\epsilon) &= \sum_{n=0}^{d-1} \epsilon_n \zeta^{nk} = \zeta^k \sum_{n=0}^{d-1} \epsilon_n \zeta^{(n-1)k} \\ &= \zeta^k \left(\epsilon_0 \zeta^{-k} + \sum_{m=0}^{d-2} \epsilon_{m+1} \zeta^{mk} \right). \end{aligned}$$

Since $o(\zeta) = 2d$, we have $\zeta^d = -1$, and as k is odd, $\zeta^{kd} = -1$. Thus

$$\epsilon_0 \zeta^{-k} = -\epsilon_0 \zeta^{(d-1)k}.$$

Using $\sigma(\epsilon)_m = \epsilon_{m+1}$ for $0 \leq m \leq d-2$ and $\sigma(\epsilon)_{d-1} = -\epsilon_0$, we get

$$\begin{aligned} H_{d,k}(\epsilon) &= \zeta^k \left(-\epsilon_0 \zeta^{(d-1)k} + \sum_{m=0}^{d-2} \epsilon_{m+1} \zeta^{mk} \right) \\ &= \zeta^k \sum_{m=0}^{d-1} \sigma(\epsilon)_m \zeta^{mk} = \zeta^k H_{d,k}(\sigma(\epsilon)). \end{aligned}$$

Since ζ^k is a unit, $H_{d,k}(\epsilon) = 0$ if and only if $H_{d,k}(\sigma(\epsilon)) = 0$, proving the claim. ∎

5.3 Alternating order

Consider the orbit of $\epsilon \in E_d$ under the action of C_σ:

$$\mathcal{O}_\epsilon := \{\sigma^n(\epsilon) : n \in \mathbb{Z}\}.$$

By Proposition 5.2, $\sigma^{2d}(\epsilon) = \epsilon$ and $\sigma^d(\epsilon) = -\epsilon$, so $-\epsilon \in \mathcal{O}_\epsilon$ and $(-\epsilon) \cdot (-\epsilon) = \epsilon$ in this orbit.

Definition 5.4. For $\epsilon \in E_d$, the *order* of ϵ is

$$o(\epsilon) := |\mathcal{O}_\epsilon|,$$

and the *alternating order* of ϵ is

$$a(\epsilon) := \frac{|\mathcal{O}_\epsilon|}{2}.$$

Proposition 5.4. *Let $\epsilon \in E_d$. Then:*

1. *$o(\epsilon)$ is the least positive integer n such that $\sigma^n(\epsilon) = \epsilon$, and $a(\epsilon)$ is the least positive integer m such that $\sigma^m(\epsilon) = -\epsilon$.*

2. *For every* $n \in \mathbb{Z}$, $o(\sigma^n(\epsilon)) = o(\epsilon)$.

3. $a(\epsilon) \mid d$ *and* $o(\epsilon) \mid 2d$.

Proof of (iii). The map

$$\psi : C_\sigma \to \mathcal{O}_\epsilon, \qquad \psi(\sigma^n) = \sigma^n(\epsilon)$$

is a surjective group homomorphism. Thus

$$o(\epsilon) = |\mathcal{O}_\epsilon| = |C_\sigma : \ker \psi|$$

divides $|C_\sigma| = 2d$, so $o(\epsilon) \mid 2d$. As $a(\epsilon) = \frac{o(\epsilon)}{2}$, we have $a(\epsilon) \mid d$. ■

Proposition 5.5. *Fix a set* K *of odd integers. For a positive integer* d*, define*

$$A := \big\{\epsilon \in V_{r,K} : o(\epsilon) = d\big\}.$$

Then d *divides* $|A|$.

Proof. By Proposition 5.3, for each $k \in K$,

$$\epsilon \in V_{r,k} \iff \sigma(\epsilon) \in V_{r,k},$$

and hence

$$\epsilon \in V_{r,K} \iff \sigma(\epsilon) \in V_{r,K}.$$

By Proposition 5.4(ii) and (iii), we also have

$$o(\epsilon) = d \iff o(\sigma(\epsilon)) = d.$$

Thus $\epsilon \in A$ if and only if $\sigma(\epsilon) \in A$, so C_σ acts on A. By definition of A, each orbit under this action has size d, and these orbits partition A. Therefore d divides $|A|$. ■

The next theorem explains the terminology "alternating order".

Theorem 5.1. *Let* $\epsilon \in E_r$*. Then* $a(\epsilon)$ *is the least positive integer* d *such that*

1. $\dfrac{r}{d}$ *is odd, and*

2. $$\epsilon_{dx+y} = (-1)^x \epsilon_y \qquad \text{for all } 0 \le x < \frac{r}{d},\ 0 \le y < d.$$

Proof. First suppose $d = a(\epsilon)$. Put $D := \frac{r}{d}$. By Proposition 5.4(i), we have $\sigma^d(\epsilon) = -\epsilon$. For $0 \le x \le D-2$ and $0 \le y < d$, the index $dx + y + d$ is still less than r, so using the explicit formula for σ^d on E_r,

$$-\epsilon_{dx+y} = \sigma^d(\epsilon)_{dx+y} = (-1)^{\left\lfloor \frac{dx+y+d}{r} \right\rfloor} \epsilon_{dx+y+d} = \epsilon_{d(x+1)+y},$$

hence $\epsilon_{d(x+1)+y} = -\epsilon_{dx+y}$. By induction on x, this gives

$$\epsilon_{dx+y} = (-1)^x \epsilon_y \qquad (0 \le x \le D-1,\ 0 \le y < d).$$

Taking $x = D - 1$ in this formula yields

$$\epsilon_{d(D-1)+y} = (-1)^{D-1} \epsilon_y.$$

On the other hand, evaluating $\sigma^d(\epsilon)$ at the same index,

$$\begin{aligned} -\epsilon_{d(D-1)+y} = \sigma^d(\epsilon)_{d(D-1)+y} &= (-1)^{\left\lfloor \frac{d(D-1)+y+d}{r} \right\rfloor} \epsilon_{dD+y} \\ &= (-1)^{\left\lfloor 1 + \frac{y}{r} \right\rfloor} \epsilon_y = -\epsilon_y. \end{aligned}$$

Combining these two identities gives

$$(-1)^D \epsilon_y = -\epsilon_y$$

for all y, so $(-1)^D = -1$ and hence D is odd. Thus r/d is odd and the claimed formula holds.

Conversely, suppose $m \mid r$, set $M := \frac{r}{m}$, assume M is odd, and that

$$\epsilon_{mx+y} = (-1)^x \epsilon_y \qquad (0 \le x < M,\ 0 \le y < m).$$

For $0 \le x \le M-2$ and $0 \le y < m$, we have $0 \le mx + y + m < r$, so

$$\begin{aligned}\sigma^m(\epsilon)_{mx+y} &= (-1)^{\left\lfloor \frac{mx+y+m}{r} \right\rfloor} \epsilon_{mx+y+m} \\ &= \epsilon_{m(x+1)+y} = (-1)^{x+1}\epsilon_y = -\epsilon_{mx+y},\end{aligned}$$

using the assumed relation in the last step. For $x = M-1$, we get

$$\begin{aligned}\sigma^m(\epsilon)_{m(M-1)+y} &= (-1)^{\left\lfloor \frac{m(M-1)+y+m}{r} \right\rfloor} \epsilon_{m(M-1)+y+m} \\ &= (-1)^{\left\lfloor 1+\frac{y}{r} \right\rfloor} \epsilon_y = -\epsilon_y.\end{aligned}$$

But $\epsilon_{m(M-1)+y} = (-1)^{M-1}\epsilon_y = \epsilon_y$ since M is odd, so again $\sigma^m(\epsilon)_{m(M-1)+y} = -\epsilon_{m(M-1)+y}$. Thus $\sigma^m(\epsilon) = -\epsilon$, and by Proposition 5.4(i) we must have $a(\epsilon) \le m$.

Taking $m = a(\epsilon)$ in the first part and applying this minimality in the second part shows that $a(\epsilon)$ is exactly the least positive integer d with the stated properties. ∎

5.4 Orthogonality in Signed Half Sums

In this section, we generalize Theorem 4.2 and characterize the sign patterns $\epsilon \in E_r$ for which the signed half sum

$$H_{r,k}(\epsilon) = \sum_{n=0}^{r-1} \epsilon_n \zeta^{nk}$$

exhibits orthogonality among different odd values of k. We show that the only possible form of such orthogonality comes from an alternating periodic structure in ϵ.

Definition 5.5. Let p be a prime, and let $d \mid r$. Fix a primitive $2d$-th root ζ, and set $D = \frac{r}{d}$.

We say that a vector $\epsilon \in E_r$ is *d-periodic up to sign* if there exist signs $\delta_x \in \{\pm 1\}$ such that

$$\epsilon_{dx+y} = \delta_x \epsilon_y \quad \text{for all } 0 \le x < D,\ 0 \le y < d.$$

We say that ϵ is *alternating periodic* (of order d) if

$$\epsilon_{dx+y} = (-1)^x \epsilon_y \quad \text{for all } 0 \le x < D,\ 0 \le y < d.$$

Lemma 5.2. *Let $\epsilon \in E_r$ be d-periodic up to sign, with associated signs $(\delta_x)_{0 \le x < D}$. Set*

$$B_k = \sum_{y=0}^{d-1} \epsilon_y \zeta^{yk}.$$

Then the signed half sum

$$H_{r,k}(\epsilon) = \sum_{n=0}^{r-1} \epsilon_n \zeta^{nk}$$

factors as

$$H_{r,k}(\epsilon) = B_k \sum_{x=0}^{D-1} \delta_x (\zeta^d)^{xk}.$$

Proof. Using the d-periodic structure and writing $n = dx + y$ with $0 \le x < D$, $0 \le y < d$, we have

$$\sum_{n=0}^{r-1} \epsilon_n \zeta^{nk} = \sum_{x=0}^{D-1} \sum_{y=0}^{d-1} \epsilon_{dx+y} \zeta^{(dx+y)k} = \sum_{x=0}^{D-1} \delta_x (\zeta^d)^{xk} \sum_{y=0}^{d-1} \epsilon_y \zeta^{yk}.$$

Factoring out the inner sum gives the desired factorization. ∎

Theorem 5.2 (Orthogonality Criterion)**.** *Let p be a prime, $d \mid r$, $D = \frac{r}{d}$, and let ζ be a primitive $2r$-th root of unity. Suppose $\epsilon \in E_r$ is d-periodic up to sign.*

Then the sums $H_{r,k}(\epsilon)$ vanish for all but one residue class of odd k modulo some $M \mid D$ if

$$\epsilon_{dx+y} = (-1)^x \epsilon_y \quad \textit{and} \quad D = \frac{r}{d} \textit{ is odd.}$$

In this case $M = D$, and

$$H_{r,k}(\epsilon) = \begin{cases} DB_k, & k \equiv 0 \pmod{D}, \\ 0, & \textit{otherwise,} \end{cases}$$

where $B_k = \sum_{y=0}^{d-1} \epsilon_y \zeta^{yk}$.

Proof. By Lemma 5.2 we have

$$H_{r,k}(\epsilon) = B_k \sum_{x=0}^{D-1} \delta_x (\zeta^d)^{xk}.$$

Thus any orthogonality in k must come from the outer sum

$$S(k) := \sum_{x=0}^{D-1} \delta_x (\zeta^d)^{xk}.$$

Assume that there exists $M \mid D$ and a residue class $\alpha \pmod{M}$ such that $S(k) = 0$ for all odd $k \not\equiv \alpha \pmod{M}$.

Since $(\zeta^d)^D = -1$ has order 2, the sequence $x \mapsto \delta_x (\zeta^d)^{xk}$ has period dividing $2D$ in x, and for fixed k the sum $S(k)$ is a linear combination of powers of $(\zeta^d)^k$. The assumption that $S(k)$ vanishes for all but one residue class of k modulo M forces $S(k)$ to be proportional to a geometric sum in k. Concretely, there must exist $\gamma \in \mathbb{F}_p^\times$ with order dividing D and some integer α such that

$$\delta_x (\zeta^d)^{xk} = \gamma^{x(k-\alpha)} \tag{5.2}$$

for all $0 \le x < D$ and all odd k. Rearranging gives

$$\delta_x = (\gamma^{-\alpha})^x (\gamma \zeta^{-d})^{xk}. \tag{5.3}$$

Taking $x = 1$, $k = 1$ in (5.3) yields

$$\delta_1 = \gamma^{-\alpha} (\gamma \zeta^{-d}),$$

so

$$\gamma^{-\alpha} = \delta_1 (\gamma \zeta^{-d})^{-1}.$$

Substituting this into (5.3) gives

$$\delta_x = \delta_1^x (\gamma \zeta^{-d})^{x(k-1)}. \tag{5.4}$$

Taking $k = 1$ in (5.4) we obtain $\delta_x = \delta_1^x$ for all x. Writing $\lambda := \delta_1 \in \{\pm 1\}$, we have $\delta_x = \lambda^x$, and since k is always odd, $\lambda^x = \lambda^{xk}$. Substituting into (5.2) gives

$$(\lambda \zeta^d)^{xk} = \gamma^{x(k-\alpha)}.$$

Rearranging,

$$(\gamma \lambda \zeta^{-d})^{xk} = \gamma^{x\alpha}. \tag{5.5}$$

Taking $x = 1$ in (5.5) we get

$$(\gamma \lambda \zeta^{-d})^k = \gamma^\alpha$$

for all odd k. This is only possible if

$$\gamma \lambda \zeta^{-d} = 1 \quad \text{and} \quad \gamma^\alpha = 1.$$

Thus $o(\gamma) \mid \alpha$, and $\gamma = \pm \zeta^d$. Without loss of generality we may take $\alpha = 0$.

If we choose $\gamma = \zeta^d$, then $o(\gamma) = 2D$, which does not divide D, so this case is excluded by our assumption that the order of γ divides D. Hence the only possibility is

$$\alpha = 0, \qquad \lambda = -1, \qquad \gamma = -\zeta^d.$$

We now check that this choice forces D to be odd. Note that $-\zeta^d = \zeta^{r+d}$, so

$$o(-\zeta^d) = \frac{2r}{\gcd(2r, r+d)} = \frac{2D}{\gcd(2D, D+1)}.$$

This divides D if and only if D is odd, and in that case $o(-\zeta^d) = D$. We also have

$$\delta_x = \lambda^x = (-1)^x,$$

so

$$S(k) = \sum_{x=0}^{D-1} \delta_x (\zeta^d)^{xk} = \sum_{x=0}^{D-1} (-\zeta^d)^{xk} = \begin{cases} D, & k \equiv 0 \pmod{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus orthogonality holds precisely when $\delta_x = (-1)^x$ and D is odd, with $M = D$, and in this case

$$H_{r,k}(\epsilon) = B_k S(k) = \begin{cases} DB_k, & k \equiv 0 \pmod{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Conversely, if ϵ is alternating periodic of order d, so that $\delta_x = (-1)^x$, and D is odd, the calculation above shows that $H_{r,k}(\epsilon)$ has exactly the stated orthogonality in k. This completes the proof. ∎

Definition 5.6. For $\epsilon \in E_r$, we say that $H_{r,k}(\epsilon)$ *reduces to a* (d, k) *signed half sum* if d is the least positive integer dividing r such that there exist $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{F}_p^\times$, and $\epsilon' \in E_d$ with

$$H_{r,k}(\epsilon) = \begin{cases} \beta\, H_{d,k}(\epsilon'), & k \equiv \alpha \pmod{\frac{r}{d}}, \\ 0, & \text{otherwise,} \end{cases}$$

for all odd k.

As an immediate consequence of Theorems 5.2 and 5.1, we obtain:

Proposition 5.6. *For every $\epsilon \in E_r$, the sums $H_{r,k}(\epsilon)$ reduce to an $(a(\epsilon), k)$ signed half sum.*

5.5 Tonelli–Shanks polynomials revisited

We now return to the Tonelli–Shanks polynomials $f_{TS}(x)$ defined in Definition 3.1 and describe their sign patterns.

Proposition 5.7. *Write $p - 1 = 2^s q$ with q odd, and set $d = 2^{s-1}$. Let $f = f_\epsilon \in \mathcal{F}$ be a square-root polynomial with sign vector $\epsilon \in E_r$. Then*

$$f \text{ is a TS-polynomial} \iff a(\epsilon) = d = 2^{s-1}.$$

Equivalently, f_ϵ is a TS-polynomial if and only if

$$\epsilon_{dx+y} = (-1)^x \epsilon_y \qquad \text{for } 0 \le x < q,\ 0 \le y < d.$$

Proof. Write $p - 1 = 2^s q$ with q odd and set $d = 2^{s-1}$, $r = \frac{p-1}{2} = dq$. Fix a generator γ of $\mathbb{F}_p^\times$ and put $\zeta_r = \gamma^2$, so $\mu_r = \langle \zeta_r \rangle$. Choose

$$\tau := \gamma^q \in \mu_{2^s},$$

so that $\tau^2 = \zeta_r^q$, and set

$$\zeta := \zeta_r^q \in \mu_d.$$

For each $y \in \{0, \dots, d-1\}$ define

$$f_y(x) = \tau^{-y} x^{\frac{q+1}{2}}.$$

Let

$$S_y := \{a \in \mu_r : a^q = \zeta^y\}.$$

Every element of S_y can be written uniquely as $a = \zeta_r^{dx+y}$ with $0 \le x < q$, and then

$$\begin{aligned} f_y(a) &= \tau^{-y} a^{\frac{q+1}{2}} = \gamma^{-qy} \gamma^{(dx+y)(q+1)} \\ &= (\gamma^{dq})^x \gamma^{dx+y} = (-1)^x \gamma^{dx+y}, \end{aligned}$$

since $\gamma^{dq} = \gamma^{(p-1)/2} = -1$.

Define

$$E_y(x) := \prod_{\substack{0 \le j < d \\ j \ne y}} \frac{x^q - \zeta^j}{\zeta^y - \zeta^j}.$$

Then $E_y(a) = 1$ for $a \in S_y$ and $E_j(a) = 0$ for $a \in S_y$, $j \ne y$.

Now let $h\colon \{0, \dots, d-1\} \to \{\pm 1\}$ be arbitrary, and define

$$f_{TS}(x) = \sum_{y=0}^{d-1} h(y)\, f_y(x)\, E_y(x).$$

For $a = \zeta_r^{dx+y} \in S_y$ we obtain

$$f_{TS}(a) = h(y)\, f_y(a) = (-1)^x h(y) \gamma^{dx+y}.$$

Thus the sign vector ϵ associated to f_{TS} (via $f_{TS}(\gamma^{2n}) = \epsilon_n \gamma^n$) satisfies

$$\epsilon_{dx+y} = (-1)^x h(y),$$

so ϵ is alternating periodic of order d and hence $a(\epsilon) = d$ by Theorem 5.1.

Conversely, suppose $f \in \mathbb{F}_p[x]$ is a square-root polynomial whose sign vector ϵ satisfies

$$\epsilon_{dx+y} = (-1)^x h(y)$$

for some function $h\colon \{0, \dots, d-1\} \to \{\pm 1\}$. Define f_y and E_y as above, and set f_{TS} accordingly. For $a = \zeta_r^{dx+y} \in S_y$ we have

$$f_{TS}(a) = (-1)^x h(y) \gamma^{dx+y} = f(a).$$

Replacing f by its remainder modulo $x^r - 1$ if necessary, we may assume $\deg f < r$; this reduction does not change the values on μ_r since $a^r = 1$ for all $a \in \mu_r$. Both f and f_{TS} now have degree $< r$ and agree on all r distinct points of μ_r, so they coincide. Hence f is a TS-polynomial. ∎

Remark 5.1. The number of TS-polynomials is $N = 2^{2^{s-1}}$: there are 2^{s-1} different subsets S_i, and in each S_i there are two choices for the Tonelli–Shanks polynomial in F_i. By contrast, the total number of square-root polynomials in $\mathcal{F}$ is

$$2^{(p-1)/2} = 2^{2^{s-1} q} = N^q.$$

In particular, when $q = 1$ every square-root polynomial is a TS-polynomial.

Corollary 5.2. *With notation as above, write $p - 1 = 2^s q$ with q odd and $d = 2^{s-1}$. Let f_{TS} be a Tonelli–Shanks polynomial over $\mathbb{F}_p$ with signs*

$$\epsilon_{dx+y} = (-1)^x h(y).$$

Then

$$f_{TS}(x) = \sum_{t=0}^{d-1} c_t x^{k_t}, \qquad k_t = \frac{q+1}{2} + tq,$$

i.e. all coefficients outside the single residue class

$$k \equiv \frac{q+1}{2} \pmod{q}$$

vanish. In particular, any cancellation of the top coefficient lowers the degree by a multiple of q each time: if $c_{d-1} = 0$ then $\deg f_{TS} \le k_{d-2}$; more generally, if the top ℓ coefficients $c_{d-1}, \dots, c_{d-\ell}$ vanish and $c_{d-\ell-1} \ne 0$, then

$$\deg f_{TS} = k_{d-1-\ell} = \frac{q+1}{2} + (d - \ell - 1)q.$$

Proof. Write $r = \frac{p-1}{2} = dq$ and let ζ be a primitive $2r$-th root of unity. For $0 \le k < r$, the coefficient of x^k in f_{TS} is

$$c_k = [x^k] f_{TS} = \frac{1}{r} \sum_{n=0}^{r-1} \epsilon_n \zeta^{n(1-2k)} = \frac{1}{r} H_{r,\,1-2k}(\epsilon),$$

where $H_{r,u}(\epsilon) = \sum_{n=0}^{r-1} \epsilon_n \zeta^{nu}$.

Since ϵ is alternating periodic of order d, Theorem 5.2 (with $D = r/d = q$ odd) implies

$$H_{r,u}(\epsilon) = 0 \quad \text{unless} \quad u \equiv 0 \pmod{q}.$$

Taking $u = 1 - 2k$ gives $c_k = 0$ unless $1 - 2k \equiv 0 \pmod{q}$, i.e.

$$k \equiv \frac{q+1}{2} \pmod{q}.$$

Thus the only possible exponents are

$$k_t = \frac{q+1}{2} + tq \qquad (0 \le t \le d-1),$$

and all other coefficients vanish. The leading admissible exponent is k_{d-1}; if $c_{d-1} = 0$ the degree drops to the next admissible exponent k_{d-2}, and in general vanishing of the top ℓ admissible coefficients moves the degree from k_{d-1} to $k_{d-1-\ell}$. Each such step lowers the degree by exactly q. ■

We close this section with some examples of TS-polynomials.

Example 5.1. Let $p = 41$ so that $p - 1 = 40 = 2^3 \cdot 5$. There are $2^{2^{s-1}} = 2^4 = 16$ TS-polynomials, each characterized by the signs $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$.

signs (ϵ)	TS-polynomial $f_\epsilon(x)$
$(-1,-1,-1,-1)$	$15x^{18} + 31x^{13} + 30x^8 + 5x^3$
$(-1,-1,-1,1)$	$37x^{18} + 24x^{13} + 8x^8 + 12x^3$
$(-1,-1,1,-1)$	$31x^{18} + 15x^{13} + 5x^8 + 30x^3$
$(-1,-1,1,1)$	$12x^{18} + 8x^{13} + 24x^8 + 37x^3$
$(-1,1,-1,-1)$	$8x^{18} + 12x^{13} + 37x^8 + 24x^3$
$(-1,1,-1,1)$	$30x^{18} + 5x^{13} + 15x^8 + 31x^3$
$(-1,1,1,-1)$	$24x^{18} + 37x^{13} + 12x^8 + 8x^3$
$(-1,1,1,1)$	$5x^{18} + 30x^{13} + 31x^8 + 15x^3$
$(1,-1,-1,-1)$	$36x^{18} + 11x^{13} + 10x^8 + 26x^3$
$(1,-1,-1,1)$	$17x^{18} + 4x^{13} + 29x^8 + 33x^3$
$(1,-1,1,-1)$	$11x^{18} + 36x^{13} + 26x^8 + 10x^3$
$(1,-1,1,1)$	$33x^{18} + 29x^{13} + 4x^8 + 17x^3$
$(1,1,-1,-1)$	$29x^{18} + 33x^{13} + 17x^8 + 4x^3$
$(1,1,-1,1)$	$10x^{18} + 26x^{13} + 36x^8 + 11x^3$
$(1,1,1,-1)$	$4x^{18} + 17x^{13} + 33x^8 + 29x^3$
$(1,1,1,1)$	$26x^{18} + 10x^{13} + 11x^8 + 36x^3$

Table 1: Sign vectors h and the corresponding TS-polynomials $f_\epsilon(x)$ for $p = 41$.

6 Heuristics

6.1 A count we know

Theorem 6.1. *Let p be an odd prime, let $r = \frac{p-1}{2}$, and consider an (r, ℓ) signed half sum with $\gcd(\ell, p-1) = 1$. Then*

$$\big|V_{r,\ell}\big| = \frac{1}{p}\left(2^r + (p-1)\prod_{k=1}^{r}(\zeta_p^k + \zeta_p^{-k})\right) = \frac{1}{p}\left(2^r + (p-1)\Big(\frac{2}{p}\Big)\right),$$

where ζ_p is a primitive p-th root of unity and $\left(\frac{2}{p}\right)$ is the Legendre symbol.

Proof. Define the additive character $\chi\colon \mathbb{F}_p \to \mathbb{C}^\times$ by

$$\chi(t) = \zeta_p^t \text{ for } t \in \mathbb{F}_p^\times.$$

For any $\epsilon \in \mathcal{E}^r$ we have the standard identity

$$\mathbf{1}_{\{H_{r,\ell}(\epsilon)=0\}} = \frac{1}{p}\sum_{t\in\mathbb{F}_p}\chi\big(tH_{r,\ell}(\epsilon)\big).$$

Fix a primitive $(p-1)$-st root of unity ξ in $\mathbb{F}_p^\times$, so that

$$H_{r,\ell}(\epsilon) = \sum_{n=0}^{r-1}\epsilon_n\xi^{n\ell}.$$

Then

$$\begin{aligned}
\big|V_{r,\ell}\big| &= \sum_{\epsilon\in\mathcal{E}^r}\mathbf{1}_{\{H_{r,\ell}(\epsilon)=0\}} \\
&= \sum_{\epsilon\in\mathcal{E}^r}\frac{1}{p}\sum_{t\in\mathbb{F}_p}\chi(tH_{r,\ell}(\epsilon)) \\
&= \frac{1}{p}\sum_{\epsilon\in\mathcal{E}^r}\left(1 + \sum_{t\in\mathbb{F}_p^\times}\chi(tH_{r,\ell}(\epsilon))\right) \\
&= \frac{1}{p}\left(2^r + \sum_{\epsilon\in\mathcal{E}^r}\sum_{t\in\mathbb{F}_p^\times}\chi\left(t\sum_{n=0}^{r-1}\epsilon_n\xi^{n\ell}\right)\right) \\
&= \frac{1}{p}\left(2^r + \sum_{t\in\mathbb{F}_p^\times}\sum_{\epsilon\in\mathcal{E}^r}\chi\left(\sum_{n=0}^{r-1}\epsilon_n t\xi^{n\ell}\right)\right).
\end{aligned}$$

Writing $a_n = t\xi^{n\ell} \in \mathbb{F}_p^\times$ and using the fact that the ϵ_n are independent signs,

$$\begin{aligned}
\sum_{\epsilon\in\mathcal{E}^r}\chi\left(\sum_{n=0}^{r-1}\epsilon_n a_n\right) &= \sum_{\epsilon\in\mathcal{E}^r}\prod_{n=0}^{r-1}\chi(\epsilon_n a_n) \\
&= \prod_{n=0}^{r-1}\sum_{\epsilon_n\in\{\pm1\}}\chi(\epsilon_n a_n) \\
&= \prod_{n=0}^{r-1}\big(\chi(a_n) + \chi(-a_n)\big).
\end{aligned}$$

Thus

$$\big|V_{r,\ell}\big| = \frac{1}{p}\left(2^r + \sum_{t\in\mathbb{F}_p^\times}\prod_{n=0}^{r-1}\big(\chi(t\xi^{n\ell}) + \chi(-t\xi^{n\ell})\big)\right). \tag{6.1}$$

For each $t \in \mathbb{F}_p^\times$ set

$$A_t = \{t\xi^{n\ell} : 0 \le n \le r-1\} \subset \mathbb{F}_p^\times.$$

Since $\chi(u) = \zeta_p^u$ for $u \in \mathbb{F}_p^\times$, the inner product in (6.1) can be written as

$$\prod_{a \in A_t} (\zeta_p^a + \zeta_p^{-a}).$$

We claim that the pairs $\{\pm a\}$ with $a \in A_t$ are all distinct. Indeed, if $\xi^{n\ell} = \xi^{m\ell}$ then $(p-1) \mid (n-m)\ell$; since $\gcd(\ell, p-1) = 1$ we have $(p-1) \mid (n-m)$, so $n \equiv m \pmod{p-1}$. As $0 \le n, m \le r-1$ and $r = \frac{p-1}{2}$, this forces $n = m$. Hence there are exactly r distinct pairs $\{\pm a\}$, so they must be all pairs $\{\pm k\}$ with $1 \le k \le r$. Thus, for each t,

$$\prod_{a \in A_t} (\zeta_p^a + \zeta_p^{-a}) = \prod_{k=1}^{r} (\zeta_p^k + \zeta_p^{-k}),$$

independent of t. Plugging this into (6.1) gives

$$\left|V_{r,\ell}\right| = \frac{1}{p}\left(2^r + (p-1)\prod_{k=1}^{r}(\zeta_p^k + \zeta_p^{-k})\right).$$

Finally, the well-known identity

$$\prod_{k=1}^{\frac{p-1}{2}} (\zeta_p^k + \zeta_p^{-k}) = \left(\frac{2}{p}\right)$$

(see Lemma 6.1) yields the desired formula. ∎

Lemma 6.1. *Let $\zeta_p = e^{2\pi i/p}$ be a primitive p-th root of unity. Then*

$$\prod_{k=1}^{\frac{p-1}{2}} (\zeta_p^k + \zeta_p^{-k}) = \left(\frac{2}{p}\right).$$

Proof. This is Exercise 8.3 in [Was82]. ∎

Remark 6.1. If ℓ is not coprime to $p-1$, the formula (6.1) still holds, but the pairs $\{\pm a\}$ with $a \in A_t$ are no longer distinct. As a result, the product

$$P(t) := \prod_{a \in A_t} (\zeta_p^a + \zeta_p^{-a})$$

is no longer independent of t. For each $u \in (\mathbb{Z}/p\mathbb{Z})^\times$, let $\sigma_u \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ be the automorphism with $\sigma_u(\zeta_p) = \zeta_p^u$. Then one checks that

$$P(t) = \sigma_t\big(P(1)\big).$$

Thus $P(t)$ is independent of t if and only if $P(1) \in \mathbb{Q}$. When $\gcd(\ell, p-1) = 1$, the square $P(1)^2$ is the norm of $\zeta_p + \zeta_p^{-1}$, hence $P(1) = \pm 1$. In general this need not be the case. On the other hand,

$$\sum_{t \in \mathbb{F}_p^\times} P(t) = \mathrm{Tr}_{K/\mathbb{Q}}\big(P(1)\big),$$

where $K = \mathbb{Q}(\zeta_p)$, so the sum is always an integer. We do not currently have a closed form for $\left|V_{r,\ell}\right|$ or even a sharp general upper bound in the case $\gcd(\ell, p-1) \neq 1$.

We finish this subsection with the following corollary.

Corollary 6.1. *Assume $p - 1 = 2^s q$ with q odd and $3 \nmid q$, and let $r = \frac{p-1}{2}$. Then*

$$\#\{f \in F : \deg(f) = r-1\} = 2^r - \frac{1}{p}\left(2^r + (p-1)\left(\frac{2}{p}\right)\right).$$

Proof. The top coefficient c_{r-1} of $f \in \mathcal{F}$ corresponds to the signed half sum with parameter $\ell = 1 - 2(r-1) = 4 - p$. Since

$$\gcd(4-p, p-1) = \gcd(p-1, 3) = \gcd(q, 3),$$

the hypothesis $3 \nmid q$ implies $\gcd(\ell, p-1) = 1$, so Theorem 6.1 applies. The number of $f \in \mathcal{F}$ with $\deg f = r-1$ is exactly the number of sign vectors with $c_{r-1} = 0$, which is $\left|V_{r,\ell}\right|$ for this choice of ℓ. ∎

6.2 Heuristic and computations

Let $f(x) = \sum_k c_k x^k$ be a polynomial that computes square roots. Motivated by the exact counting result above and by extensive computations, we make the following heuristic assumptions.

Conjecture 6.1. (H_1) *For each k,*

$$\mathbb{P}(c_k = 0) = \frac{1}{p} + O(2^{-r}).$$

(H_2) *For each divisor $d \mid r$, put $D = \frac{r}{d}$. Inside the family of polynomials whose sign vectors have alternating order d, the events that the admissible coefficients vanish (that is, the coefficients c_k with $k \equiv r - \frac{D-1}{2} \pmod{D}$) are approximately independent as k ranges over this arithmetic progression.*

Remark 6.2.

1. As observed above, (H_1) is proved (with an exact count) for all indices k such that $\gcd(1-2k, p-1) = 1$ by Theorem 6.1. The second heuristic seems out of reach at present, but it is strongly supported by numerical data.
2. Strictly speaking, (H_2) is not true: the events $\{c_k = 0\}$ along a given arithmetic progression are not genuinely independent. Our working hypothesis is that the dependence is weak enough not to affect the asymptotics of the expected minimal degree.
3. A simpler but numerically very similar model is to assume that the coefficients c_k are independent and uniformly distributed in $\mathbb{F}_p$. This would give the naive heuristic

$$(H') \qquad \mathbb{P}\big(c_i = 0 \text{ for all } i \in I\big) = p^{-|I|}$$

for any finite index set I.

With these heuristics, we can estimate the expected number of square-root polynomials. We group polynomials according to their alternating order $a(\epsilon)$ and set

$$\mathcal{F}_d = \{f_\epsilon \in F : a(\epsilon) = d\}, \qquad N(d) = \#\mathcal{F}_d.$$

Every sign vector of length d has alternating order dividing d, so

$$\sum_{\ell \mid d} N(\ell) = 2^d.$$

Hence

$$N(d) = 2^d - \sum_{\substack{\ell \mid d \\ \ell < d}} N(\ell).$$

Using the trivial bound $N(\ell) \le 2^\ell \le 2^{d/2}$ for $\ell \le d/2$, we obtain

$$N(d) \ge 2^d - \tau(d) 2^{d/2},$$

where $\tau(d)$ is the number of positive divisors of d. In particular,

$$\log N(d) = d \log 2 + O(2^{-d/2}).$$

For large d it is therefore reasonable to approximate

$$N(d) \approx 2^d.$$

Notice that when $d = 2^{s-1}$ (so that $D = \frac{r}{d} = q$), we are in the Tonelli–Shanks case, and in fact $N(d) = 2^d$ exactly: every choice of the 2^{s-1} signs on the blocks yields a TS-polynomial.

We will need the following structural fact.

Proposition 6.1. *Let $d \mid r$ and set $D = \frac{r}{d}$. Suppose D is odd. Then the possible degrees of polynomials $f_\epsilon \in \mathcal{F}_d$ are*

$$\deg(f_\epsilon) = r - \frac{D-1}{2} - nD, \qquad n = 0, 1, 2, \ldots.$$

Proof. This follows directly from Theorem 5.2, applied to sign vectors of alternating order d: within such a family, all coefficients c_k vanish except for $k \equiv r - \frac{D-1}{2} \pmod{D}$, and the largest admissible index gives the degree. ∎

For example, when $D = q$ (the TS family), the possible degrees are

$$\deg(f_{TS}) = r - \frac{q-1}{2} - nq, \qquad n \geq 0.$$

Remark 6.3. If $f_\epsilon \in \mathcal{F}_d$, then there are $2d$ polynomials in the same orbit under the flip–shift map σ from Section 5.2. These polynomials have the same degree and belong to the same family $\mathcal{F}_d$. Thus, for counting purposes we may quotient by this action and only keep one representative from each orbit. In other words, we may replace $N(d)$ by

$$N'(d) = \frac{N(d)}{2d}$$

as the relevant number of polynomials in $\mathcal{F}_d$.

Under (H_1)–(H_2), the expected number of polynomials in $\mathcal{F}_d$ whose degree is at most

$$r - \frac{D-1}{2} - nD$$

is

$$\lambda_n^{(d)} = \frac{N'(d)}{p^n} \approx \frac{2^d}{2d\,p^n}.$$

We look for the largest n such that $\lambda_n^{(d)} \geq \frac{1}{2}$. This condition is

$$\frac{2^d}{2d\,p^n} \geq \frac{1}{2} \quad \Longleftrightarrow \quad p^n \leq \frac{2^d}{d}.$$

Equivalently,

$$n \leq \frac{\log_2(2^d/d)}{\log_2 p} = \frac{d - \log_2 d}{\log_2 p},$$

so we expect

$$n_{\max}^{(d)} \approx \left\lfloor \frac{d - \log_2 d}{\log_2 p} \right\rfloor.$$

Thus the expected minimal degree within the family $\mathcal{F}_d$ is

$$\min \deg(\mathcal{F}_d) \approx r - \frac{D-1}{2} - D \left\lfloor \frac{d - \log_2 d}{\log_2 p} \right\rfloor.$$

Globally,

$$\min\{\deg f : f \in F\} = \min_{d \mid r} \min \deg(\mathcal{F}_d).$$

This expression does not simplify to a closed form in p, but for each fixed prime it is straightforward to evaluate numerically.

We now compare these predictions with numerical data. The algorithm and code used to compute the minimal degrees are described in the appendix and can be found in [Noa25]. The "expected" counts in the tables are computed using the naive heuristic (H').

Table 2: ts long

p	s	q	degree	ALL_Counts	ALL_expected	TS_counts	TS_expected
29	2	7	10	0	1	0	0
29	2	7	11	4	19	4	4
29	2	7	12	560	545	0	0
29	2	7	13	15820	15819	0	0
37	2	9	14	4	5	4	4
37	2	9	15	180	186	0	0
37	2	9	16	6984	6893	0	0
37	2	9	17	254976	255059	0	0
41	3	5	16	0	15	0	0
41	3	5	17	640	609	0	0
41	3	5	18	24936	24951	16	16
41	3	5	19	1023000	1023001	0	0
53	2	13	20	4	0	4	4
53	2	13	21	0	8	0	0
53	2	13	22	416	442	0	0
53	2	13	23	23660	23440	0	0
53	2	13	24	1242124	1242314	0	0
53	2	13	25	65842660	65842659	0	0
61	2	15	23	4	0	4	4
61	2	15	24	0	1	0	0
61	2	15	25	120	76	0	0
61	2	15	26	5100	4653	0	0
61	2	15	27	283500	283832	0	0
61	2	15	28	17292540	17313762	0	0
61	2	15	29	1056160560	1056139499	0	0
73	3	9	30	0	33	0	0
73	3	9	31	2088	2387	0	0
73	3	9	32	172480	174229	16	16
73	3	9	33	12716784	12718730	0	0
73	3	9	34	928438416	928467316	0	0
73	3	9	35	67778146968	67778114041	0	0
89	3	11	37	nan	35	0	0
89	3	11	38	2904	3115	0	0
89	3	11	39	273784	277238	16	16
89	3	11	40	24706616	24674164	0	0
89	3	11	41	2195987728	2196000635	0	0
89	3	11	42	195444040704	195444056547	0	0
89	3	11	43	17394521032680	17394521032681	0	0
97	5	3	40	nan	3	0	0
97	5	3	41	608	334	32	7
97	5	3	42	33024	32440	0	0
97	5	3	43	3238272	3146678	0	0
97	5	3	44	305133760	305227801	640	669
97	5	3	45	29607090144	29607096698	0	0
97	5	3	46	2871888387456	2871888379660	0	0
97	5	3	47	278573172827392	278573172827041	64864	64860
101	2	25	38	4	0	4	4
101	2	25	42	0	10	0	0
101	2	25	43	600	1050	0	0
101	2	25	44	115700	106065	0	0
101	2	25	45	10713100	10712551	0	0
101	2	25	46	1081898900	1081967680	0	0
101	2	25	47	109278676200	109278735713	0	0

Continued on next page

Table 2: ts long

p	s	q	degree	ALL_Counts	ALL_expected	TS_counts	TS_expected
101	2	25	48	11037152425620	11037152307054	0	0
101	2	25	49	1114752383012500	1114752383012490	0	0
109	2	27	41	4	0	4	4
109	2	27	45	0	1	0	0
109	2	27	46	0	98	0	0
109	2	27	47	10728	10643	0	0
109	2	27	48	1174176	1160071	0	0
109	2	27	49	126427068	126447728	0	0
109	2	27	50	13781571648	13782802392	0	0
109	2	27	51	1502326744776	1502325460684	0	0
109	2	27	52	163753475167764	163753475214549	0	0
109	2	27	53	17849128798385820	17849128798385800	0	0
113	4	7	46	0	0	0	2
113	4	7	47	0	3	0	0
113	4	7	48	672	304	0	0
113	4	7	49	35168	34304	0	0
113	4	7	50	3844064	3876387	0	0
113	4	7	51	437844064	438031721	0	0
113	4	7	52	49497868448	49497584522	0	0
113	4	7	53	5593227289920	5593227051029	256	254
113	4	7	54	632034656462160	632034656766225	0	0
113	4	7	55	71419916214583400	71419916214583400	0	0

6.3 Tonelli-Shanks heuristics

We end by specializing the above heuristics to the case of TS-polynomials.

Recall that the TS-polynomials correspond to the family $d = 2^{s-1}$ and $D = q$. The discussion in the previous subsection gives us the following:

Corollary 6.2. *Let p be an odd prime such that $p - 1 = 2^s q$. Set $r = \frac{p-1}{2}, d = 2^{s-1}, D = q$. Then, the TS-polynomials are of degrees*

$$r - \frac{q-1}{2} - nq$$

for $n \geq 0$.

The expected number of TS-polynomials of that given degree is

$$\lambda_n^{(d)} \approx \left(1 - \frac{1}{p}\right) \frac{N(d)}{p^n} = \left(1 - \frac{1}{p}\right) \frac{2^d}{p^n}$$

The expected minimum degree of a TS-polynomial is

$$\min \deg(f_{TS}) = r - \frac{q-1}{2} - q\lfloor \frac{2^{s-1} - s + 1}{\log_2(p)} \rfloor$$

Computationally, this heuristic appears accurate for small primes p:

p	s	q	degree counts	expected counts rounded
7	1	3	{ deg 2: 2 }	{ deg 2: 2 }
11	1	5	{ deg 3: 2 }	{ deg 3: 2 }
13	2	3	{ deg 5: 4 }	{ deg 5: 4 }
17	4	1	{ deg 7: 16, deg 8: 240 }	{ deg 6: 1, deg 7: 14, deg 8: 241 }
19	1	9	{ deg 5: 2 }	{ deg 5: 2 }
23	1	11	{ deg 6: 2 }	{ deg 6: 2 }
29	2	7	{ deg 11: 4 }	{ deg 11: 4 }
31	1	15	{ deg 8: 2 }	{ deg 8: 2 }
37	2	9	{ deg 14: 4 }	{ deg 14: 4 }
41	3	5	{ deg 18: 16 }	{ deg 18: 16 }
43	1	21	{ deg 11: 2 }	{ deg 11: 2 }
47	1	23	{ deg 12: 2 }	{ deg 12: 2 }
53	2	13	{ deg 20: 4 }	{ deg 20: 4 }
59	1	29	{ deg 15: 2 }	{ deg 15: 2 }
61	2	15	{ deg 23: 4 }	{ deg 23: 4 }
67	1	33	{ deg 17: 2 }	{ deg 17: 2 }
71	1	35	{ deg 18: 2 }	{ deg 18: 2 }
73	3	9	{ deg 32: 16 }	{ deg 32: 16 }
79	1	39	{ deg 20: 2 }	{ deg 20: 2 }
83	1	41	{ deg 21: 2 }	{ deg 21: 2 }
89	3	11	{ deg 39: 16 }	{ deg 39: 16 }
97	5	3	{ deg 41: 32, deg 44: 640, deg 47: 64864 }	{ deg 41: 7, deg 44: 669, deg 47: 64860 }
101	2	25	{ deg 38: 4 }	{ deg 38: 4 }
103	1	51	{ deg 26: 2 }	{ deg 26: 2 }
107	1	53	{ deg 27: 2 }	{ deg 27: 2 }
109	2	27	{ deg 41: 4 }	{ deg 41: 4 }
113	4	7	{ deg 53: 256 }	{ deg 46: 2, deg 53: 254 }
127	1	63	{ deg 32: 2 }	{ deg 32: 2 }
131	1	65	{ deg 33: 2 }	{ deg 33: 2 }
137	3	17	{ deg 60: 16 }	{ deg 60: 16 }
139	1	69	{ deg 35: 2 }	{ deg 35: 2 }
149	2	37	{ deg 56: 4 }	{ deg 56: 4 }
151	1	75	{ deg 38: 2 }	{ deg 38: 2 }
157	2	39	{ deg 59: 4 }	{ deg 59: 4 }
163	1	81	{ deg 41: 2 }	{ deg 41: 2 }
167	1	83	{ deg 42: 2 }	{ deg 42: 2 }
173	2	43	{ deg 65: 4 }	{ deg 65: 4 }
179	1	89	{ deg 45: 2 }	{ deg 45: 2 }
181	2	45	{ deg 68: 4 }	{ deg 68: 4 }
191	1	95	{ deg 48: 2 }	{ deg 48: 2 }
353	5	11	{ deg 160: 96, deg: 171: 65440 }	{ deg 149: 1, deg 160: 185, deg 171: 65536 }

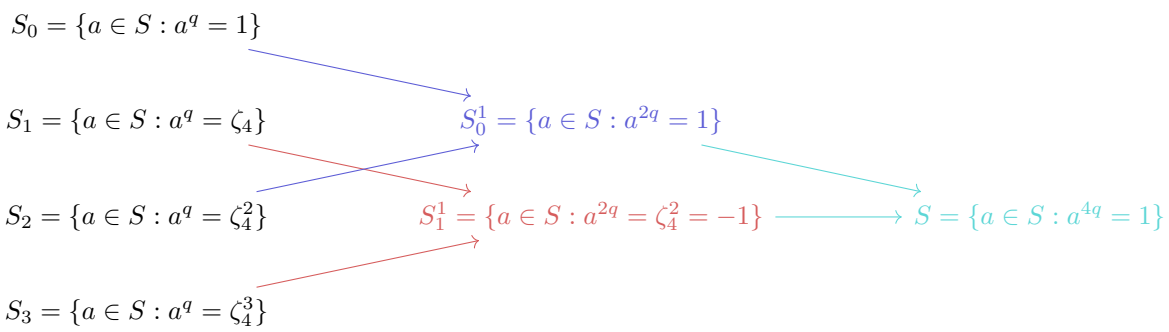
Table 3: TS degree distributions and heuristic expected counts (rounded to nearest integer). For each p, $p-1 = 2^s q$ with q odd.

Appendix

In this appendix, we give a more detailed discussion of the tree construction of root computing polynomials, and how it was used to build a fast algorithm for computing the minimal degree root computing polynomial for a given prime p.

A Tree Construction Of Root Computing Polynomials

The following discussion is best motivated through an example. Assume $p \equiv 9 \mod 16$. Therefore, $s = 3$ and one creates S_0, S_1, S_2, S_3 and $f_i \in F_i$. Instead of immediately gluing the f_i's together using Equation 3.1, to get an $f \in \mathcal{F}$, one can perform the process in steps, as in the following tree diagram:



where $\zeta_4 \in \mathbb{F}_p$ is a primitive 4-th root of unity.

For a general prime p, one can generalize this by setting

$$S_i^k = \{a \in S_i : a^{2^k q} = \zeta_{2^{s-1}}^{2^k i}\}, \qquad i = 0, ..., 2^{s-k-1} - 1$$

and $S_i^0 := S_i$. Notice that as a variety, $S_i^k = Z(x^{2^k q} - \zeta_{2^{s-1}}^{2^k i})$. In more generality, given a subvariety $A \subseteq S$ with characterizing equation $g_A(x) = 0$, let F_A be the set of polynomials in the coordinate ring $\mathbb{F}_p[x]/(g_A(x))$ that compute square roots on A, i.e.

$$F_A = \{f \in \mathbb{F}_p[x]/(g_A(x)) : f^2(x) = x\}$$

Theorem A.1. *Let A be an algebraic subvariety of S and $A_1 \cup A_2 \cup \ldots \cup A_n = A \subset S$, with A_i's pairwise disjoint. Then,*

$$F_A \cong F_{A_1} \times F_{A_2} \times \ldots \times F_{A_n}$$

Corollary A.1. *Write $\zeta = \zeta_{2^{s-1}}$. For $A = S_i^k$, denote $F_i^k = F_A$. Set*

$$M_k := 2^k q, \qquad \alpha := \zeta^{2^k i}.$$

Then

$$F_i^k \times F_{i+2^{s-2-k}}^k \simeq F_i^{k+1}$$

is given by

$$(f_i(x), f_j(x)) \mapsto f(x) = \frac{1}{2\alpha}\Big[x^{M_k}\big(f_i(x) - f_j(x)\big) + \alpha\big(f_i(x) + f_j(x)\big)\Big]$$

Consequently,

$$\deg(f) = M^k + \deg(f_i - f_j) = 2^k q + \deg(f_i - f_j) \tag{A.1}$$

Proof. It follows immediately from the Chinese Remainder Theorem that

$$\mathbb{F}_p[x]/(g_A) \cong \mathbb{F}_p[x]/(g_{A_1}) \times \mathbb{F}_p[x]/(g_{A_2}) \times \ldots \times \mathbb{F}_p[x]/(g_{A_n})$$

It remains to show that this isomorphism preserves computing square roots. It is clear that if $f \in F_A$ then $f \pmod{g_{A_i}}$ computes square roots on A_i. Now, suppose $(f_1, \ldots f_2) \in F_{A_1} \times F_{A_2} \times \ldots \times F_{A_n}$, and let f be their composition using the Chinese Remainder Theorem. Then, $\forall a \in A$, $\exists 1 \leq i \leq n : a \in A_i$, so writing $f(x) = h(x)g_{A_i}(x) + f_i(x)$, we see that $f(a) = h(a)g_{A_i}(a) + f_i(a) = 0 \pm \sqrt{a} = \pm\sqrt{a}$. ∎

A.1 Lower Bounds on $\deg f$ for all odd primes p

We start by recalling a couple of known facts on the lower bound of the minimum degree of polynomials that compute square roots.

1. When $p \equiv 3 \mod 4$, then $\min \deg(f) = \frac{p+1}{4}$
2. When $p \equiv 1 \mod 4$, we have $\min \deg(f) \geq \frac{p-1}{3}$ by [KK24, Theoroem 1.1]

This second statement can be easily generalized to any $f \in F_i^k$ as follows:

Theorem A.2. *Assume $s \geq 2$, $1 \leq k \leq s-1$ and let $f \in F_i^k$. Set $M_k = 2^k q$. Then*

$$\frac{2M_k}{3} = \frac{p-1}{3 \cdot 2^{s-k-1}} \leq \deg f \leq M_k = \frac{p-1}{2^{s-k}}$$

For $k = 0$ and $f \in F_i^0$, instead we have

$$\deg(f) \geq \frac{q+1}{2}$$

Proof. The proof is almost identical to [KK24, Theorem 1.1], simply adjusting it to make sure we are working over F_i^k. For convenience, we present it here again.

The upper bound is because $f(x)$ interpolates M_k points.

When $k = 0$, this is Theorem 3.1. So assume $k > 0$.

Without loss of generality, assume $f(x) \in F_0^k$. Set $M_k = 2^k q = \frac{p-1}{2^{s-k}}$. Recall that the vanishing polynomial on S_0^k is $X^{M_k} - 1$.

Assume by contradiction that $f(X)$ is of degree $d < \frac{2M_k}{3}$. Then, we have

$$f^2(X) - X = (X^{M_k} - 1)A(X)$$

Rewrite this as

$$f^2(X) = A(X)X^{M_k} + B(X), \qquad B(X) = X - A(X) \tag{A.2}$$

We remark that:

- $\deg(f(X)) = d$
- $\deg(A(X)), \deg(B(X)) \leq 2d - M_k$
- $A(X) \neq 0$ as that would imply $f^2(X) = X$.
- $B(X) \neq 0$. Otherwise, $A(X) = X$, and $f^2(X) = X^{M_k+1}$. But that would imply M is odd and therefore $k = 0$.

Taking (formal) derivatives of both sides of A.2, we get:

$$2f(X)f'(X) = A'(X)X^{M_k} + A(X)M_k X^{M_k-1} + B'(X) \tag{A.3}$$

Computing $2f(X)^2 f'(X)$ in two ways using A.2 and A.3, we get:

$$\underbrace{(2f'A - fA')X^{M_k} - fAMX^{M_k-1}}_{\equiv 0 \pmod{X^{M_k-1}}} + (2f'B - fB') = 0$$

Hence

$$2f'(X)B(X) - f(X)B'(X) \equiv 0 \mod X^{M_k-1}$$

By the degree bounds,

$$\deg(2f'B), \deg(fB') \leq (d-1) + (2d - M_k) = 3d - M_k - 1 < M - 1$$

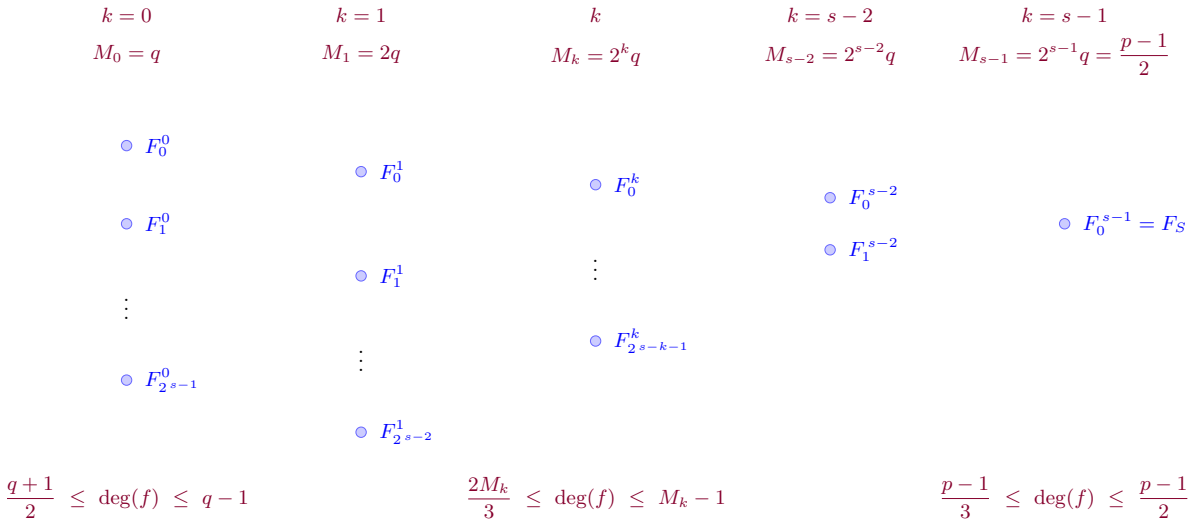
so the congruence forces the polynomial identity $2f'B = fB'$

Since $B(X) \neq 0$, we get $\frac{2f'(X)}{f(X)} = \frac{B'(X)}{B(X)}$. Since $\deg(f), deg(B) < p$, by a basic property of logarithmic derivatives, this implies $f(X)^2 = \lambda B(X)$ for some nonzero λ, contradicting the fact that $A(X) \neq 0$. ■

We summarize our discussion so far in the following diagram:

In particular, if $f \in \mathcal{F}$ is minimal, then

$$\frac{p-1}{3} = \frac{2M_{s-1}}{3} \leq \deg(f) \leq \deg(f_{TS}) = M_{s-1} - \frac{q-1}{2} \leq M_{s-1} - 1$$



A.2 Algorithm for finding minimal root computing polynomials using tree construction

We now restrict to primes $p \equiv 1 \pmod 4$. A naïve strategy would enumerate all $2^{\frac{p-1}{2}}$ root–computing polynomials on $S = \mu_{\frac{p-1}{2}}$ and then select those of minimal degree. Since each polynomial can be represented modulo $X^{\frac{p-1}{2}} - 1$ with $O(p)$ coefficients, this costs

$$O\Big(p \cdot 2^{\frac{p-1}{2}}\Big).$$

Using the tree structure from §4, we reduce this to

$$O\Big(p \cdot 2^{\frac{p-1}{4}}\Big).$$

Write $p - 1 = 2^s q$ with q odd (so $s \ge 2$). At level $k = s - 2$ we have $M_{s-2} = 2^{s-2} q = \frac{p-1}{4}$ and a partition

$$F_0^{s-1} \rightsquigarrow F_0^{s-2} \sqcup F_1^{s-2},$$

with $|F_0^{s-2}| = |F_1^{s-2}| = 2^{\frac{p-1}{4}}$. By Lemma 3.1 there is a bijection $F_0^{s-2} \cong F_1^{s-2}$.

Step 1 — Build the penultimate families. Compute all polynomials in F_0^{s-2} (e.g. via the Lagrange/DFT formula of §3.1), then obtain F_1^{s-2} from the bijection. Since each representative has degree $< M_{s-2} = O(p)$, the work here is

$$O\Big(p \cdot 2^{\frac{p-1}{4}}\Big).$$

Step 2 — Order by leading–coefficient profiles. Represent each $f \in F_i^{s-2}$ by its coefficient vector modulo $X^{M_{s-2}} - 1$,

$$f(x) = \sum_{j=0}^{M_{s-2}-1} c_j \, x^j,$$

and sort F_0^{s-2} and F_1^{s-2} lexicographically from highest degree to lowest (MSD radix over $\mathbb{F}_p$). With $M_{s-2} = O(p)$ and $|F_i^{s-2}| = 2^{\frac{p-1}{4}}$, this costs

$$O\Big(M_{s-2} \cdot 2^{\frac{p-1}{4}}\Big) = O\Big(p \cdot 2^{\frac{p-1}{4}}\Big).$$

Step 3 — Glue with maximal prefix match. For $f_0 \in F_0^{s-2}$ and $f_1 \in F_1^{s-2}$, the one–level gluing formula produces $f \in F_0^{s-1}$ with

$$\deg f \;=\; M_{s-2} \;+\; \deg(f_0 - f_1) \qquad (\text{provided } f_0 \neq f_1).$$

Thus minimizing $\deg f$ is equivalent to minimizing $\deg(f_0 - f_1)$, i.e. maximizing the length of the common high–degree prefix of the two coefficient vectors. After the MSD sort, perform a bucketed two–pointer sweep: group entries with the same leading coefficient, then recurse to the next coefficient within each bucket. Each pass is linear in the list sizes, so the total here remains

$$O\left(p \cdot 2^{\frac{p-1}{4}}\right).$$

(Edge case: the cancellation $f_0 = f_1$ would give $\deg f < M_{s-2} = \frac{p-1}{4}$, which is ruled out by the lower bound $\deg f \geq \frac{2}{3} M_{s-1} = \frac{p-1}{3}$ for $s \geq 2$; see Theorem 4.2.)

Conclusion. Combining the three steps, we enumerate all minimal–degree root–computing polynomials in time

$$O\left(p \cdot 2^{\frac{p-1}{4}}\right).$$

(A parallel implementation simply shards the construction of F_0^{s-2} and the bucketed matching across threads; memory usage is linear in the number of polynomials stored.)

A multi-threaded C++ implementation of this can be found here [Noa25]

A.3 An example

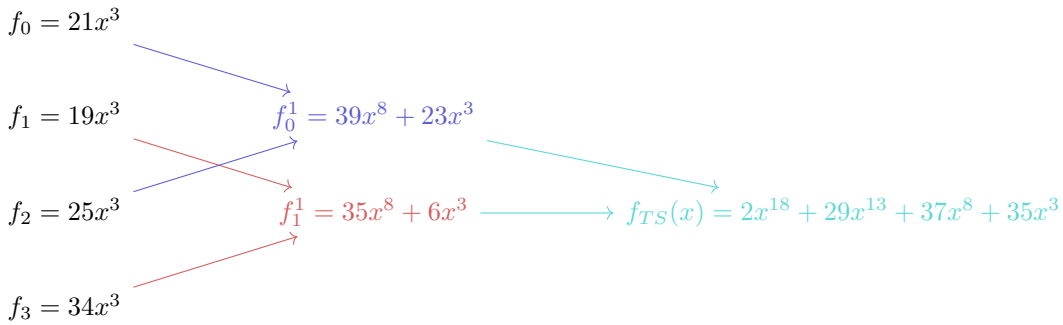
For $p = 41$, a minimal polynomial in $\mathcal{F}$ is of degree 17, while the Tonelli-Shanks polynomial is of degree 18. An example of a minimal polynomial is

$$\begin{aligned} f(x) =& 15x^{17} + 32x^{16} + 4x^{15} + 12x^{14} + 37x^{13} + 25x^{12} + 5x^{11} + x^{10} + x^9 \\ &+ x^8 + 9x^6 + 29x^5 + 6x^4 + 36x^3 + 7x^2 + 33x + 35 \end{aligned}$$

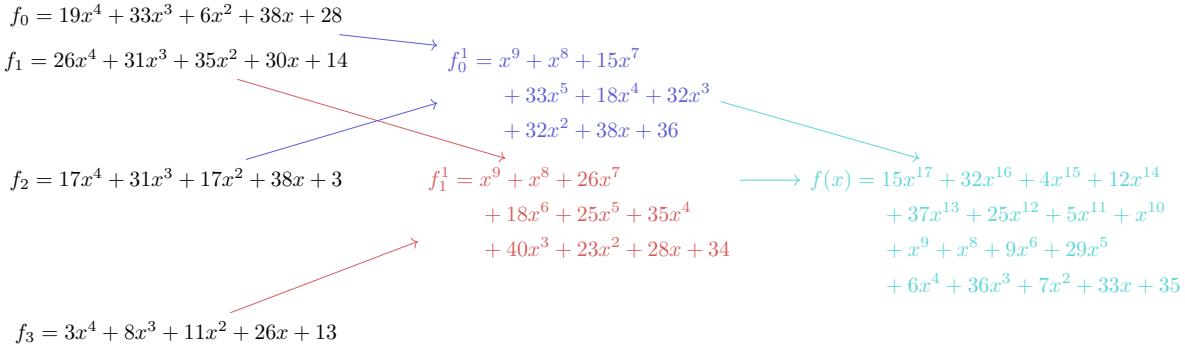
while the Tonelli-Shanks polynomial is

$$f_{TS}(x) = 26x^{18} + 10x^{13} + 11x^8 + 36x^3$$

Below are the tree decompositions of these polynomials, as in Section 4.
First is the tree decomposition of $f_{TS}(x)$:



And second is the tree decomposition of $f(x)$:



We observe that for the Tonelli-Shanks polynomial, there is no degree reduction at any level of the tree, while for the minimal polynomial $f(x)$, there is a degree reduction in the last step. This pattern seems to hold for every polynomial which we have tested: If $f(x)$ is a *minimal* polynomial, different from the Tonelli-Shanks, then the degree cancellation happens only at the last step of the tree. We leave this as an open question for future research.

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