VARIETIES OF GRADED W-ALGEBRAS AND ASYMPTOTIC BEHAVIOR OF CODIMENSION GROWTH

GIOVANNI BUSALACCHI, FABRIZIO MARTINO, AND CARLA RIZZO

Abstract. Let W be a G-graded algebra over a field of characteristic zero, where G is a finite group. We develope a theory of generalized G-graded polynomial identities satisfied by any finite-dimensional W-algebra A, by mean of the graded multiplier algebra of A. In particular, we first prove that the graded generalized exponent exists and equals the ordinary one. Then, we explicitly compute the G-graded generalized identities of UT_2, the 2×2 upper triangular matrix algebra equipped with its canonical $\mathbb{Z}_2$-grading, under all the possible graded W-actions. Finally, we exhibit examples of varieties of graded W-algebras with almost polynomial growth of the codimensions.

1. Introduction

Let F be a field of characteristic zero and let G be a finite group. Let W be a unital G-graded algebra over F. An algebra A over F is said to be a G-graded W-algebra (or simply a *graded W-algebra* when no ambiguity may arise) if A is a graded W-bimodule satisfying certain additional conditions. For example, when $W = F$ this definition coincides with the standard notion of a graded F-algebra. In this paper, we are concerned with the polynomial identities satisfied by finite-dimensional graded W-algebras, with particular emphasis on the asymptotic growth of their W-codimension sequences.

The codimension sequence $c_n(A)$ was first introduced by Regev in [10] in the context of associative algebras over fields of characteristic zero, providing a quantitative measure of the polynomial identities satisfied by a given algebra A. Subsequently, several properties of this sequence were established. In particular, if A satisfies at least one nontrivial polynomial identity (i.e., if A is a PI-algebra), then $c_n(A)$ is exponentially bounded (see [10]). Moreover, as shown in [6], its growth is either polynomial or exponential, with no intermediate behaviour. This dichotomy motivated the definition of *varieties of almost polynomial growth*, namely varieties $\mathcal{V}$ whose codimension sequence grows exponentially but for which every proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$ has polynomially bounded codimensions. One of the consequences of the results in [6] was the complete characterization of such varieties. Finally, the exponential growth rate of any PI-algebra was determined in [4], where it was proved that the so-called PI-exponent exists and is a non-negative integer. Analogous developments were later obtained in the setting of group-graded algebras (see [1, 3, 11]).

Motivated by these results, the authors of [8] carried out a detailed study of the W-variety generated by UT_2, the trivially graded algebra of 2×2 upper triangular matrices, endowed with all possible W-actions. Furthermore, in [9] they established generalized versions of the aforementioned theorems for ordinary algebras. In particular, by employing the multiplier algebra–thus enabling them to refrain from specifying *a priori* the acting algebra W and to keep it formally defined–they classified the W-varieties of almost polynomial growth generated by finite-dimensional algebras and proved the existence of the PI-exponent when W acts as a subalgebra of A via left and right multiplication.

The aim of the present paper is to investigate the behaviour of codimension growth in the setting of G-graded W-algebras. The paper is organized as follows. Section 2 introduces the multiplier algebra $\mathcal{M}(A)$

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of a graded W-algebra A, and shows that the given G-grading on A extends naturally to $\mathcal{M}(A)$. We also establish a duality between the action of W on A and the corresponding action of $\mathcal{M}(A)$. Section 3 recalls the necessary background on generalized graded polynomial identities, including the definitions of the graded W-codimension sequence and the PI graded W-exponent. Section 4 proves the existence of the graded W-exponent and its equality with the ordinary graded PI-exponent. Sections 5 and 6 provide an extensive analysis of the graded W-variety generated by UT_2, including explicit computations of the ideals of identities, the codimension sequences, the exponents, and the graded W-cocharacters for all possible graded W-actions. Finally, Section 7 shows that the W-variety generated by UT_2 exhibits almost polynomial codimension growth whenever W acts either as F or as the subalgebra $D \subset UT_2$ of diagonal matrices via left and right multiplication.

2. W-actions and G-graded algebras

Throughout this paper, F will denote a field of characteristic 0, $G = \{g_1 = e_G, g_2, \ldots, g_s\}$ will be a finite group and all algebras will be associative algebras over F.

Let A be an F-algebra. Recall that a G-grading on A is a decomposition of A into the direct sum of subspaces $A = \oplus_{g\in G} A^g$ such that $A^g A^h \subseteq A^{gh}$ for all $g, h \in G$. The subspaces A^g are called the homogeneous components of A. Accordingly, an element $a \in A$ is homogeneous (or a homogeneous component of degree g) if $a \in A^g$ and in this case we write $|a|_A = g$ or simply $|a| = g$ if no ambiguity arises. If B is a subalgebra (ideal) of A, then we say that B is a graded subalgebra (ideal) if $B = \oplus_{g\in G} B^g$ where $B^g = B \cap A^g$ for all $g \in G$.

Given a G-graded algebra W with unit 1_W, we say that A is a G-graded W-algebra if A is a G-graded algebra and a W-bimodule, i.e., A is both a left and a right W-module with the compatibility relation, for all $w_1, w_2 \in W$, $a \in A$,

$$(w_1 a)w_2 = w_1(a w_2), \tag{2.1}$$

that satisfies the following additional relations, for any $w \in W, a_1, a_2 \in A$,

$$(a_1a_2)w = a_1(a_2w),\ w(a_1a_2) = (wa_1)a_2,\ (a_1w)a_2 = a_1(wa_2), \tag{2.2}$$

and, for any $v \in W^g$, $a \in A^h$, $h, g \in G$,

$$av \in A^{hg} \quad \text{and} \quad va \in A^{gh}. \tag{2.3}$$

Let $\mathrm{End}_F(A)$ be the algebra of the endomorphisms of A acting on the left of A. The multiplication in $\mathrm{End}_F(A)$ is given by the usual composition of endomorphisms, written by juxtaposition. Moreover, we denote by $\mathrm{End}_F(A)^{\mathrm{op}}$ the opposite algebra of $\mathrm{End}_F(A)$, which is the underlying vector space of $\mathrm{End}_F(A)$ endowed with the opposite product $\cdot^{\mathrm{op}}$ defined by $f_1 \cdot^{\mathrm{op}} f_2 := f_2 f_1$ for all $f_1, f_2 \in \mathrm{End}_F(A)$.

As in [9], we define the multipliers as follows: given $(R, L) \in \mathrm{End}_F(A)^{\mathrm{op}} \times \mathrm{End}_F(A)$, then (R, L) is a multiplier of A if for all $a, b \in A$, the following conditions hold

$$R(ab) = aR(b), \tag{2.4}$$

$$L(ab) = L(a)b, \tag{2.5}$$

$$R(a)b = aL(b). \tag{2.6}$$

For each $m \in A$, the pair (R_m, L_m), defined by $R_m(a) = am$ and $L_m(a) = ma$ for all $a \in A$, is a multiplier of A, referred to as the inner multiplier of A induced by the element m.

We denote by $\mathcal{M}(A)$ the set of all multipliers of A. This set naturally carries the structure of an F-algebra with unity, where the algebra operations are defined component-wise and the unit element is $(\mathrm{id}_A, \mathrm{id}_A)$, and it is called the multiplier algebra of A. Moreover, the set $\mathcal{IM}(A)$ of all inner multipliers of A is a two-sided ideal of $\mathcal{M}(A)$ called inner multiplier ideal of A.

In what follows, we show that $\mathcal{M}(A)$ inherits a G-grading from that of A. To this end, we introduce the following notation. For any $a \in A$, we write it in terms of its homogeneous components as $a = \sum_{g\in G} a^g$ with $a^g \in A^g$. For a fixed $h \in G$, we define the linear map $\pi_h : A \to A$ by

$$\pi_h\left(\sum_{g\in G} a^g\right) = a^h,$$

which is called h-homogeneous linear projection, and as a consequence, we can write $a = \sum_{g\in G} \pi_g(a)$. Moreover, notice that for any $h, k \in G$ and $a, b \in A$ we have that

$$\pi_k\pi_h(a) = \begin{cases} \pi_h(a) & \text{if } k = h \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

and, in case $a \in A^k$ or $b \in A^h$,

$$\pi_k(a)\pi_h(b) = \pi_{kh}(ab). \tag{2.8}$$

With this notation in place, we introduce, for each $g \in G$ and for all $(R, L) \in \mathrm{End}_F(A)^{\mathrm{op}} \times \mathrm{End}_F(A)$, the following endomorphisms of A:

$$R_g = \sum_{h\in G} \pi_{hg} R\pi_h \quad \text{and} \quad L_g = \sum_{h\in G} \pi_{gh} L\pi_h. \tag{2.9}$$

Lemma 2.1. *Let $g \in G$ and $(R, L) \in \mathcal{M}(A)$. Then $(R_g, L_g) \in \mathcal{M}(A)$.*

Proof. We prove only the condition (2.6) since the same argument works also for (2.4) and (2.5). To this end, without loss of generality, we may assume that a and b are homogeneous, so let $a = \pi_{g_1}(a) \in A^{g_1}$ and $b = \pi_{g_2}(b) \in A^{g_2}$. Then by using (2.6), (2.7) and (2.8) we have that

$$R_g(a)b = \left(\sum_{h\in G} \pi_{hg}R\pi_h\pi_{g_1}(a)\right)\pi_{g_2}(b) = \pi_{g_1g}(R(a))\pi_{g_2}(b) = \pi_{g_1gg_2}(R(a)b) = \pi_{g_1gg_2}(aL(b)).$$

On the other hand

$$aL_g(b) = \pi_{g_1}(a)\left(\sum_{h\in G} \pi_{gh}L\pi_h\pi_{g_2}(b)\right) = \pi_{g_1}(a)\pi_{gg_2}(L(b)) = \pi_{g_1gg_2}(aL(b)).$$

Then $R_g(a)b = aL_g(b)$. □

Lemma 2.2. *For all $(R, L) \in \mathcal{M}(A)$ we have that*

$$R = \sum_{g\in G} R_g \qquad \textit{and} \qquad L = \sum_{g\in G} L_g.$$

Proof. We first show that $R = \sum_{g\in G} R_g$. To this end, let $a = \sum_{h\in G} \pi_h(a) \in A$ and remark that on one side

$$R(a) = \sum_{g\in G} \pi_g R(a) = \sum_{g\in G} \pi_g R\left(\sum_{h\in G} \pi_h(a)\right) = \sum_{g,h\in G} \pi_g R\pi_h(a).$$

On the other, since $hG = G$ we have

$$\sum_{g\in G} R_g(a) = \sum_{g\in G}\left(\sum_{h\in G} \pi_{hg}R\pi_h(a)\right) = \sum_{h\in G}\left(\sum_{g\in G} \pi_{hg}R\pi_h(a)\right) = \sum_{g,h\in G} \pi_g R\pi_h(a),$$

thus $R(a) = \sum_{g\in G} R_g(a)$ as claimed. Analogously, we obtain $L = \sum_{g\in G} L_g$. □

Proposition 2.3. *Let A be a G-graded algebra. Then the multiplier algebra $\mathcal{M}(A)$ of A is graded as follows*

$$\mathcal{M}(A) = \bigoplus_{g\in G} \mathcal{M}(A)^g,$$

where, for each $g \in G$,

$$\mathcal{M}(A)^g = \{(R, L) \in \mathcal{M}(A) \mid R(A^h) \subseteq A^{hg} \textit{ and } L(A^h) \subseteq A^{gh} \text{ for all } h \in G\}.$$

Proof. Let $(R, L) \in \mathcal{M}(A)$ and R_g and L_g be the endomorphism of A defined in (2.9). Then by Lemma 2.2 and the definition of sum in $\mathcal{M}(A)$, it follows that $(R, L) = \sum_{g\in G}(R_g, L_g)$. Moreover, by Lemma 2.1 and using (2.7) we get $(R_g, L_g) \in \mathcal{M}(A)^g$ and, as a consequence, we obtain $\mathcal{M}(A) = \sum_{g\in G} \mathcal{M}(A)^g$.

We now prove that this sum is direct. Suppose, by contradiction, that there exists $0 \neq (R, L) \in \mathcal{M}(A)^g \cap \sum_{h\neq g} \mathcal{M}(A)^h$ for some $g \in G$. Then there must exist $h_0 \neq g$ such that $R(A^k) \subseteq A^{kg} \cap A^{kh_0} = 0$ for all $k \in G$, which implies that $R = 0$. By a similar argument, it follows that $L = 0$, and thus $(R, L) = 0$, contradicting our assumption. Therefore, we have proved that $\mathcal{M}(A)$ is the direct sum of the subspaces $\mathcal{M}(A)^g$ for all $g \in G$.

Finally, it is clear that by construction

$$\mathcal{M}(A)^g \mathcal{M}(A)^h \subseteq \mathcal{M}(A)^{gh},$$

for all $g, h \in G$, and the statement is proved. $\square$

Corollary 2.4. *If A is a G-graded algebra, then $\mathcal{IM}(A)$ is a graded ideal of $\mathcal{M}(A)$. In particular, $\mathcal{IM}(A) = \bigoplus_{g\in G} \mathcal{IM}(A)^g$ with $\mathcal{IM}(A)^g = \{(R_m, L_m) \mid m \in A^g\}$ for all $g \in G$.*

As in [9], now we define the acting homomorphism. The right and left actions of the algebra W on A naturally define the following homomorphisms of F-algebras:

$$\rho\colon W \to \mathrm{End}_F(A)^{\mathrm{op}} \qquad \text{and} \qquad \lambda\colon W \to \mathrm{End}_F(A)$$

such that $\rho(w)(a) = aw$ and $\lambda(w)(a) = wa$ for all $w \in W$ and $a \in A$.

Assume now that A is a G-graded W-algebra. In this setting, A admits both left and right graded W-module structures, and one may then consider the corresponding representations λ and ρ. From identity (2.2), it immediately follows that, for any $w \in W$,

$$(\rho(w), \lambda(w)) \in \mathcal{M}(A).$$

This leads to the definition of the following homomorphism of F-algebras

$$\begin{aligned} \Phi : W &\longrightarrow \mathcal{M}(A) \\ w &\longmapsto (\rho(w), \lambda(w)) \end{aligned}$$

which is called the acting homomorphism of W on A. Notice that, since (2.3) holds, Φ is a graded homomorphism; therefore, in particular, $\Phi(W)$ is a graded subalgebra of $\mathcal{M}(A)$. Moreover, for all $w \in W$ by (2.1), $\Phi(W)$ satisfies the following compatibility condition

$$\rho(w_2)\lambda(w_1) = \lambda(w_1)\rho(w_2)$$

for all $w_1, w_2 \in W$. As a consequence, A becomes a G-graded $\Phi(W)$-algebra with the action given by:

$$(R, L) \cdot a = L(a) \qquad a \cdot (R, L) = R(a), \tag{2.10}$$

for any $a \in A$ and for any $(R, L) \in \Phi(W)$.

Proposition 2.5. *Let W be a G-graded F-algebra. If A is a unital graded W-algebra, then the action of W on A is equivalent to the action of a suitable unital graded subalgebra B of A by left and right multiplication.*

Proof. If A has unity, then by [9, Corollary 2.5] follows that $\mathcal{M}(A) = \mathcal{IM}(A)$ and $A \cong \mathcal{IM}(A)$ as F-algebras using the isomorphism of F-algebras $\mu : A \to \mathcal{IM}(A)$ such that $\mu(m) = (R_m, L_m)$ for any $m \in A$. Notice that if $m \in A^g$, then for any $a \in A^h$, $R_m(a) = am \in A^{hg}$ and $L_m(a) = ma \in A^{gh}$, so $\mu(m) \in \mathcal{IM}(A)^g$. Then $\mu(A^g) \subseteq \mathcal{IM}(A)^g$ for all $g \in G$, and μ is a G-graded isomorphism. Thus, $A \cong \mathcal{IM}(A)$ as G-graded algebras.

Let Φ be the acting homomorphism of W on A. Since both W and A are unital, we have that $\Phi(1_W) = (R_{1_A}, L_{1_A})$. Now, if we consider $\Phi(W) \subseteq \mathcal{M}(A) = \mathcal{IM}(A)$, then the action of W on A is equivalent to the action of $\Phi(W)$ on A via (2.10). Therefore, since Φ is also a graded homomorphism, it follows that the unital graded algebra $\Phi(W)$ is, through μ, isomorphic to a unital graded subalgebra B of A that acts on A by right and left multiplication. $\square$

3. On the free G-graded W-algebra

In this section, we first construct the free W-algebra. Then, we define the generalized graded codimension sequence and present some basic results that will be useful in the next section, where we will prove the first main result of the paper.

Let $X = \{x_1, x_2, \ldots\}$ be an infinite countable set and $W\langle X\rangle$ be the free W-algebra. Recall that we set

$$1_W x 1_W = x \tag{3.1}$$

for any $x \in X$. Also, if we fix a a vector spaces basis $\mathcal{B}_W = \{w_i\}_{i\in\mathcal{I}}$ of W, then a vector space basis of $W\langle X\rangle$ consists of the following monomials

$$w_{i_0}x_{j_1}w_{i_1}\cdots w_{i_{n-1}}x_{j_n}w_{i_n}, \quad n \geq 1,$$

where $w_{i_0}, \dots, w_{i_n}$ are from the basis $\mathcal{B}_W$. The product of two monomials is given first by juxtaposition and then by multiplication in W, and the W-action is given by the multiplication of W.

Next, we introduce a grading on $W\langle X\rangle$. We write $X = \bigcup_{g\in G} X^g$, where $X^g = \{x_1^g, x_2^g, \dots\}$ are disjoint sets and we are requiring that the variables in X^g have homogeneous degree g, for all $g \in G$. If $w_{i_0}, w_{i_1}, \dots, w_{i_t}$ are homogeneous elements of W, the homogeneous degree of a monomial $w_{i_0}x_{j_1}^{g_{k_1}}w_{i_1}\cdots w_{i_{t-1}}x_{j_n}^{g_{k_n}}w_{i_n}$ is

$$|w_{i_0}|\cdot g_{k_1}\cdot|w_{i_1}|\cdots|w_{i_{n-1}}|\cdot g_{k_n}\cdot|w_{i_n}|.$$

If we denote by $W\langle X\rangle^g$ the subspace of $W\langle X\rangle$ generated by all the monomials having homogeneous degree g, then $W\langle X\rangle^g W\langle X\rangle^h \subseteq W\langle X\rangle^{gh}$, for every g, h in G. Therefore, it follows that

$$W\langle X\rangle = \bigoplus_{g\in G} W\langle X\rangle^g$$

is a G-grading on $W\langle X\rangle$. We denote such graded W-algebra as $W\langle X\rangle^G$ and we call it free G-graded W-algebra of countable rank over F. As free algebra, $W\langle X\rangle^G$ has the following universal property: given any G-graded W-algebra A, then any set theoretical map $\psi : X \to A$ such that $\psi(X^g) \subseteq A^g$ extends uniquely to a G-graded W-homomorphism $\overline{\psi} : W\langle X\rangle^G \to A$.

An element $f \in W\langle X\rangle^G$ is called G-graded W-polynomial, or graded generalized polynomial if the roles of W and G are clear. We say that $f \in W\langle X\rangle^G$ is W-trivial if $f = 0$; otherwise, we say f is W-nontrivial. By naturally extending the ordinary case (when G is trivial and $W = F$), the total degree of a monomial M in a variable $x \in X$, is defined as the number of times the variable x appears in M (regardless of the exponent in G and the elements of W).

Given a G-graded W-algebra A, an element $f = f(x_1^{g_1}, \dots, x_{t_1}^{g_1}, \dots, x_1^{g_s}, \dots, x_{t_k}^{g_s})$ of $W\langle X\rangle^G$ is called G-graded W-identity of A, or graded generalized identity (if the roles W and G are clear), and we write $f \equiv 0$, if f vanishes under all evaluations $x_i^{g_j} \to a_{g_j,i} \in A^{g_j}, 1 \leq j \leq s$. The set of all graded generalized identities of A is an ideal denoted by

$$\mathrm{Id}^{\mathrm{G,W}}(A) = \{f \in W\langle X\rangle^G \mid f \equiv 0 \text{ on } A\}$$

which is invariant under all G-graded W-endomorphisms of $W\langle X\rangle^G$. In this case, we say that $\mathrm{Id}^{\mathrm{G,W}}(A)$ is a $T_{G,W}$-ideal. For any $n \geq 1$ we define

$$P_n^{G,W} = \mathrm{span}_F\{w_{i_0}x_{\sigma(1)}^{g_{j_1}}w_{i_1}\cdots w_{i_{n-1}}x_{\sigma(n)}^{g_{j_n}}w_{i_n} \mid \sigma \in S_n,\ g_{j_1}, \dots, g_{j_n} \in G,\ w_{i_0}, \dots, w_{i_n} \in \mathcal{B}_W\}$$

the space of multilinear G-graded generalized polynomials in the graded variables $x_1^{g_{j_1}}, \dots, x_n^{g_{j_n}}, g_{i_l} \in G$. Here S_n stands for the symmetric group of order n. Since the characteristic of the base field is zero, by standard arguments, we have that the G-graded W-identities follow from the multilinear ones, and we study

$$P_n^{G,W}(A) = \frac{P_n^{G,W}}{P_n^{G,W} \cap \mathrm{Id}^{\mathrm{G,W}}(A)}.$$

The dimension

$$c_n^{G,W}(A) = \dim_F P_n^{G,W}(A), \quad n \geq 1,$$

is called the nth G-graded W-codimension, or graded generalized codimension, of A and, roughly speaking, it gives an idea of how many generalized polynomial identities are satisfied by A.

Recall that the graded codimension sequence, in the case of ordinary polynomial identities, is defined similarly. More in detail, let $F\langle X\rangle^G$ be the ordinary free G-graded algebra and $\mathrm{Id}^{\mathrm{G}}(A) = \{f \in F\langle X\rangle^G | f \equiv 0 \text{ on } A\}$ be the T_G-ideal of ordinary G-graded polynomial identities of A. If P_n^G is the space of the ordinary multilinear G-graded polynomials in the variables $x_1^{g_{j_1}}, \dots, x_n^{g_{j_n}}$, $g_{j_l} \in G$, then $c_n^G(A) = \dim_F \frac{P_n^G}{P_n^G \cap \mathrm{Id}^{\mathrm{G}}(A)}$ is the nth ordinary graded codimension of A.

If we consider $W \cong F$ then $P_n^{G,W} = P_n^G$. Hence, in general, since $F \cong F1_W$ is a subalgebra of W, we have that $P_n^G \subseteq P_n^{G,W}$. Therefore, if A is a G-graded W-algebra, by following the argument of [5, Lemma 10.1.2] with the necessary modifications, we obtain the following relation between the ordinary graded codimensions and the generalized graded codimensions:

$$c_n^G(A) \le c_n^{G,W}(A), \quad n \ge 1. \tag{3.2}$$

In [1,3] the authors proved that if A is a G-graded PI-algebra, i.e., if A is a G-graded algebra that satisfies a non-trivial G-graded identity, then the limit

$$\exp^{\mathrm{G}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$$

exists and is an integer called the G-graded exponent of A. Similarly, we define

$$\exp^{\mathrm{G,W}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{G,W}(A)}.$$

One of our main goals will be to prove that this limit exists and coincides with $\exp^{\mathrm{G}}(A)$ when A is a finite-dimensional G-graded W-algebra.

In the last part of this section we find a formula that simplifies the computation of the generalized graded codimension sequence. To this end, for $n \ge 1$, consider $0 \le n_1, \dots, n_s \le n$ such that $n_1 + n_2 + \cdots + n_s = n$ and

$$I_1, \dots, I_s \subseteq \{1, \dots, n\}, \quad I_i \cap I_j = \emptyset \text{ for all } i \ne j, \quad |I_i| = n_i. \tag{3.3}$$

If $I_1 = \{i_1, \dots, i_{n_1}\}$, $I_2 = \{i_{n_1+1}, \dots, i_{n_1+n_2}\}, \dots,$ $I_s = \{i_{n_1+\cdots+n_{s-1}+1}, \dots, i_{n_1+\cdots+n_s}\}$, we denote by $P^{G,W}_{I_1,\dots,I_s}$ the vector subspace of $P_n^{G,W}$ of multilinear graded generalized polynomials in the variables $x_{i_1}^{g_1}, \dots, x_{i_{n_1}}^{g_1}$, $x_{i_{n_1+1}}^{g_2}, \dots, x_{i_{n_1+n_2}}^{g_2}, \dots, x_{i_{n_1+\cdots+n_{s-1}+1}}^{g_s}, \dots, x_{i_{n_1+\cdots+n_s}}^{g_s}$. Then it is clear that $P_n^{G,W}$ has the decomposition

$$P_n^{G,W} = \bigoplus_{I_1,\dots,I_s} P^{G,W}_{I_1,\dots,I_s}, \tag{3.4}$$

with $I_1, \dots, I_s$ as in (3.3).

Given a G-graded W-algebra A, if we consider $P^{G,W}_{n_1,\dots,n_s} := P^{G,W}_{\{1,\dots,n_1\},\dots,\{n_1+\cdots+n_{s-1}+1,\dots,n_1+\cdots+n_s\}}$, i.e., the vector subspace of $P_n^{G,W}$ of multilinear generalized graded polynomials in the variables $x_1^{g_1}, \dots, x_{n_1}^{g_1}, \dots,$ $x^{g_s}_{n_1+\cdots+n_{s-1}+1}, \dots, x^{g_s}_{n_1+\cdots+n_s}$, then, clearly,

$$\frac{P^{G,W}_{I_1,\dots,I_s}}{P^{G,W}_{I_1,\dots,I_s} \cap \mathrm{Id}^{\mathrm{G,W}}(A)} \cong \frac{P^{G,W}_{n_1,\dots,n_s}}{P^{G,W}_{n_1,\dots,n_s} \cap \mathrm{Id}^{\mathrm{G,W}}(A)}, \tag{3.5}$$

for all $I_1, \dots, I_s$ as in (3.3).

For $n = n_1 + \cdots + n_s$, let us define

$$c^{G,W}_{n_1,\dots,n_s}(A) := \dim_F \frac{P^{G,W}_{n_1,\dots,n_s}}{P^{G,W}_{n_1,\dots,n_s} \cap \mathrm{Id}^{\mathrm{G,W}}(A)}$$

and denote by $\binom{n}{n_1,\dots,n_s} = \frac{n!}{n_1!\cdots n_s!}$ the corresponding multinomial coefficient. Then we have the following result.

Proposition 3.1. *Let A be a W-algebra over a field F of characteristic zero. Then, for any $n \ge 1$,*

$$c_n^{G,W}(A) = \sum_{\substack{n_1,\dots,n_s \\ n_1+\cdots+n_s=n}} \binom{n}{n_1, \dots, n_s} c^{G,W}_{n_1,\dots,n_s}(A).$$

Proof. Let $f \in P_n^{G,W} \cap \mathrm{Id}^{\mathrm{G,W}}(A)$. By (3.4) we can write $f = \sum_{I_1,\dots,I_s} f_{I_1,\dots,I_s}$, with $I_1, \dots, I_s$ as in (3.3) and $f_{I_1,\dots,I_s} \in P^{G,W}_{I_1,\dots,I_s}$. Then by standard arguments it is clear that $f_{I_1,\dots,I_s} \in \mathrm{Id}^{\mathrm{G,W}}(A)$. Hence

$$P_n^{G,W} \cap \mathrm{Id}^{\mathrm{G,W}}(A) = \bigoplus_{I_1,\dots,I_s} \Big(P^{G,W}_{I_1,\dots,I_s} \cap \mathrm{Id}^{\mathrm{G,W}}(A)\Big), \tag{3.6}$$

with $I_1, \dots, I_s$ as in (3.3). Combining (3.4) and (3.6) we have that

$$\frac{P_n^{G,W}}{P_n^{G,W} \cap \mathrm{Id}^{\mathrm{G,W}}(A)} \cong \bigoplus_{I_1,\dots,I_s} \frac{P^{G,W}_{I_1,\dots,I_s}}{P^{G,W}_{I_1,\dots,I_s} \cap \mathrm{Id}^{\mathrm{G,W}}(A)}. \tag{3.7}$$

Since for any partition $n_1, \dots, n_s$ of n, we have $\binom{n}{n_1,\dots,n_s}$ possibilities of $I_1, \dots, I_s$, the claim is proved using (3.5) and (3.7). $\square$

4. Generalized G-graded W-exponent

Throughout this section, A will be a finite dimensional G-graded W-algebra. In what follows, it will also be useful the following notation: if $B \subseteq A$, then

$$WBW = \left\{ \sum w_i b_k w_j \mid b_k \in B,\ w_i, w_j \in W \right\}.$$

Thus, an ideal (subalgebra) I of A is called W-ideal (W-subalgebra) if it is W-invariant, i.e., $WIW \subseteq I$. Moreover, we say that I is a G-graded W-ideal (G-graded W-subalgebra) if it is a G-graded ideal (subalgebra) W-invariant.

By the Wedderburn-Malcev Theorem for finite dimensional G-graded algebras (see [7]), we write

$$A = B + J$$

where B is a maximal semisimple subalgebra of A, which we may assume to be G-graded, and $J = J(A)$ is its Jacobson radical, which is a graded ideal of A. Moreover, if F is algebraically closed, we can write

$$B = B_1 \oplus \cdots \oplus B_t$$

where $B_1, \dots, B_t$ are simple G-graded algebras, i.e., they have no proper graded ideals. Notice that in such decomposition, the semisimple part is not in general W-invariant. In fact, for instance, if $W \cong A$ and the action is given by left and right multiplication, since J is an ideal, we have that $JB,\ BJ \subseteq J$. Nevertheless, the Jacobson radical is a W-invariant, as proved in [9, Theorem 3.3] which statement, in the setting of graded algebras, is the following.

Proposition 4.1. *Let W be a G-graded algebra over a field F of characteristic zero and let A be a G-graded finite dimensional W-algebra. Then the Jacobson radical J of A is a G-graded W-ideal of A. Moreover, if F is algebraically closed, then $A = B_1 \oplus \cdots \oplus B_t + J$, where $B_1, \dots, B_t$ are G-graded simple algebras and $WB_iW \subseteq B_i + J$ for all $1 \le i \le t$.*

Since the main goal of this section is to prove the existence and the integrality of the generalized graded exponent, we give the following definition that one can find in [3, Definition 2].

Definition 4.2. Let A be a finite dimensional G-graded W-algebra such that $A = B_1 \oplus \cdots \oplus B_t + J$ where $B_1, \dots, B_t$ are G-graded simple algebras. We say that a G-graded subalgebra $B_{i_1} \oplus \cdots \oplus B_{i_k}$, where $B_{i_1}, \dots, B_{i_k}$ are distinct simple G-graded algebras, is admissible if for some permutation $(l_1, \dots, l_k)$ of $(i_1, \dots, i_k)$ we have that $B_{l_1} J B_{l_2} J \cdots J B_{l_k} \neq 0$.

In the same paper the authors showed that $\exp^{\mathrm{G}}(A)$ exists and equals the maximal dimension of an admissible subalgebra (see [3, Theorem 2]). Here, we want to prove that $\exp^{\mathrm{G,W}}(A) = \exp^{\mathrm{G}}(A)$. To this end, let us compute an upper bound for $c_n^{G,W}(A)$.

Recall that $G = \{g_1, \dots, g_s\}$. Moreover, from now on, we let $\mathcal{B}_A = \mathcal{B}_{g_1} \cup \cdots \cup \mathcal{B}_{g_s}$ be a basis of A such that the elements of $\mathcal{B}_{g_l}$ have homogeneous degree g_l, $1 \le l \le s$, and $\mathcal{B}_{g_1} = \{a_1, \dots, a_{r_1}\}, \dots, \mathcal{B}_{g_s} = \{a_{r_1+\cdots+r_{s-1}+1}, \dots, a_{r_1+\cdots+r_s}\}$. Moreover, let $n \ge 1$ and $\xi_{i,j}$, $1 \le i \le n,\ 1 \le j \le r_1 + \cdots + r_s = \dim_F A$, be commutative variables. Then we define the generic elements

$$\xi_i^{g_l} = \sum_{j=r_1+\cdots+r_{l-1}+1}^{r_1+\cdots+r_l} a_j \otimes \xi_{i,j},\ 1 \le i \le n,\ 1 \le l \le s. \tag{4.1}$$

Let $\mathcal{H}$ be the algebra generated by the elements in (4.1). Then $\mathcal{H}$ is a G-graded W-algebra, where $\xi_i^{g_l}$ has homogeneous degree g_l and, for any $w \in W$,

$$w\xi_i^{g_l} = \sum_{j=r_1+\cdots+r_{l-1}+1}^{r_1+\cdots+r_l} wa_j \otimes \xi_{i,j},\ 1 \le i \le n,\ 1 \le l \le s.$$

$$\xi_i^{g_l} w = \sum_{j=r_1+\cdots+r_{l-1}+1}^{r_1+\cdots+r_l} a_j w \otimes \xi_{i,j},\ 1 \le i \le n,\ 1 \le l \le s.$$

Notice that $\mathcal{H} \subseteq A \otimes F[\xi_{i,j}]$.

Proposition 4.3. *The G-graded W-algebra $\mathcal{H}$ is isomorphic to the a relatively free G-graded W-algebra of A in $n \cdot s$ graded generators.*

Proof. Let
$$\psi:\ W\langle x_1^{g_1},\dots,x_n^{g_1},\dots,x_1^{g_s},\dots,x_n^{g_s}\rangle^G \to \mathcal{H}$$
be the G-graded W-homomorphism induced by mapping $x_i^{g_j} \to \xi_i^{g_j}$, $g_j \in G$, $1 \le i \le n$. We will prove that $\ker\psi = \mathrm{Id}^{\mathrm{G,W}}(A)$.

Remark that if C is a commutative F-algebra, then $A \otimes_F C$ is also a G-graded W-algebra where the W-action is defined by
$$w\Big(\sum_{i\in\mathcal{I}}(b_i\otimes c_i)\Big) = \sum_{i\in\mathcal{I}} wb_i \otimes c_i \qquad \text{and} \qquad \Big(\sum_{i\in\mathcal{I}}(b_i\otimes c_i)\Big)w = \sum_{i\in\mathcal{I}} b_i w \otimes c_i,$$
with $b_i \in A, c_i \in C$, for all $i \in \mathcal{I}$, and the grading is given by setting $(A\otimes_F C)^g = A^g \otimes_F C$, for all $g \in G$. Since the characteristic of F is zero, following step by step [5, Lemma 1.4.2], with the necessary changes, we have that if f is a graded generalized identity for A, then f is also a graded generalized identity for $A\otimes_F C$. As a consequence, $\mathcal{H} \subseteq A\otimes F[\xi_{i,j}]$ implies that $\mathrm{Id}^{\mathrm{G,W}}(A) = \mathrm{Id}^{\mathrm{G,W}}(A\otimes F[\xi_{i,j}]) \subseteq \mathrm{Id}^{\mathrm{G,W}}(\mathcal{H})$. Thus $\mathrm{Id}^{\mathrm{G,W}}(A) \subseteq \ker\psi$.

Suppose now that $g = g(x_1^{g_1},\dots,x_n^{g_1},\dots,x_1^{g_s},\dots,x_n^{g_s}) \in \ker\psi$, i.e., $g(\xi_1^{g_1},\dots,\xi_n^{g_1},\dots,\xi_1^{g_s},\dots,\xi_n^{g_s}) = 0$ in $\mathcal{H}$, and let $b_1^{g_1},\dots,b_n^{g_1},\dots,b_1^{g_s},\dots,b_n^{g_s}$ be arbitrary elements of A such that $b_i^{g_l} \in A^{g_l}$, $1\le i\le n,\ 1\le l\le s$. Write each element $b_i^{g_l}$ as a linear combination of the basis $\mathcal{B}_{g_l} = \{a_{r_1+\cdots+r_{l-1}+1},\dots,a_{r_1+\cdots+r_l}\}$ of A^{g_l} and consider
$$b_i^{g_l} = \sum_{j=r_1+\cdots+r_{l-1}+1}^{r_1+\cdots+r_l} \mu_{i,j}a_j, \quad 1\le i\le n,\ 1\le l\le s,$$
with $\mu_{i,j} \in F$. Since $F[\xi_{i,j}]$ is the free commutative algebra of countable rank, any set-theorical map $\xi_{i,r_1+\cdots+r_{l-1}+q} \to \mu_{i,r_1+\cdots+r_{l-1}+q}$ extends to a homomophism $F[\xi_{i,j}] \to F$. Hence, due to the universal property of the tensor product, the map
$$a\to a,\ a\in A, \quad \xi_{i,r_1+\cdots+r_{l-1}+q} \to \mu_{i,r_1+\cdots+r_{l-1}+q},\ 1\le i\le n,\ 1\le l\le s,\ 1\le q\le r_l,$$
extends to a homomorphism $\phi: A\otimes F[\xi_{i,q}] \to A$ such that $\phi(\xi_i^{g_l}) = b_i^{g_l},\ 1\le i\le n$. Hence
$$\begin{aligned}0 &= \phi(g(\xi_1^{g_1},\dots,\xi_n^{g_1},\dots,\xi_1^{g_s},\dots,\xi_n^{g_s})) = g(\phi(\xi_1^{g_1}),\dots,\phi(\xi_n^{g_1}),\dots,\phi(\xi_1^{g_s}),\dots,\phi(\xi_n^{g_s}))\\ &= g(b_1^{g_1},\dots,b_n^{g_1},\dots,b_1^{g_s},\dots,b_n^{g_s}).\end{aligned}$$
Since $b_1^{g_1},\dots,b_n^{g_s}$ are homogeneous arbitrary elements of A, $g(x_1^{g_1},\dots,x_n^{g_1},\dots,x_1^{g_s},\dots,x_n^{g_s}) \equiv 0$ is a graded generalized identity of A and $\ker\psi \subseteq \mathrm{Id}^{\mathrm{G,W}}(A)$ follows. □

As a consequence, we have that a graded generalized polynomial f is a graded generalized identity of A if and only if it vanishes on the generic elements ξ_i^g, $g\in G$. Next, we will use this fact to determine an upper bound for the graded generalized codimensions.

Lemma 4.4. *Let A be a finite dimensional G-graded W-algebra over an algebraically closed field F. Then there exist constants C, u such that*
$$c_n^{G,W}(A) \le Cn^u d^n,$$
where d is the maximal dimension of an admissible G-graded subalgebra.

Proof. Let ψ be the homomorphism defined in Proposition 4.3. Since $\ker\psi = \mathrm{Id}^{\mathrm{G,W}}(A)$, it is clear that, for any $n_1,\dots,n_s \ge 0$ such that $n_1+\cdots+n_s = n$,
$$c_{n_1,\dots,n_s}^{G,W}(A) = \dim_F \mathcal{P}_{n_1,\dots,n_s}^{G,W},$$
where
$$\begin{aligned}\mathcal{P}_{n_1,\dots,n_s}^{G,W} = \mathrm{span}_F\{w_{q_0}\eta_{\sigma(1)}w_{q_1}\cdots w_{q_{n-1}}\eta_{\sigma(n)}w_{q_n} \mid \sigma\in S_n,\ w_{q_0},\dots,w_{q_n}\in\mathcal{B}_W,\\ \eta_i = \xi_i^{g_l} \quad \text{if } n_1+\cdots+n_{l-1}+1 \le i \le n_1+\cdots+n_l\},\end{aligned}$$

i.e, $\mathcal{P}^{G,W}_{n_1,\dots,n_s}$ is the vector space of multilinear graded generalized polynomials in variables $\xi^{g_1}_1,\dots,\xi^{g_1}_{n_1}$, $\xi^{g_2}_{n_1+1},\dots,\xi^{g_2}_{n_1+n_2},\dots,\xi^{g_s}_{n_1+\cdots+n_{s-1}+1},\dots,\xi^{g_s}_{n_1+\cdots+n_s}$. Thus, our first aim is to compute an upper bound of $\dim_F \mathcal{P}^{G,W}_{n_1,\dots,n_s}$.

Taking a monomial $w_{q_0}\eta_{\sigma(1)}w_{q_1}\cdots w_{q_{n-1}}\eta_{\sigma(n)}w_{q_n}$ in $\mathcal{P}^{G,W}_{n_1,\dots,n_s}$, then, by means of the definition (4.1), we can write it as a linear combination of elements of the type $c\otimes\xi_{1,j_1}\cdots\xi_{n,j_n}$, where c is an element of the above basis $\mathcal{B}_A$ of A. Thus we shall estimate $c^{G,W}_{n_1,\dots,n_s}(A)$ through an estimate of the number of possible monomials $\xi_{1,j_1}\cdots\xi_{n,j_n}$ which can appear as coefficients of $w_{q_0}\eta_{\sigma(1)}w_{q_1}\cdots w_{q_{n-1}}\eta_{\sigma(n)}w_{q_n}$ in the chosen basis.

Write $A = B + J$ where B is a maximal semisimple G-graded subalgebra of A and $J = J(A)$ is its Jacobson radical. Since F is algebraically closed, the semisimple part decomposes as a direct sum of simple G-graded algebras: $B = B_1\oplus\cdots\oplus B_t$. Suppose, as we may, that the above basis $\mathcal{B}_A$ of A is a basis of J and a basis for each of the B_i. Then, by abuse of notation, since each variable $\xi_{i,j}$ in (4.1) is attached to a basis element of some homogeneous degree, we will say that $\xi_{i,j}$ is a radical variable or a semisimple variable of some homogeneous degree.

Take a non-zero monomial $\Xi := \xi_{1,j_1}\cdots\xi_{n,j_n}$ of degree n that contains i radical variables. Write $i = i_1+\cdots+i_s$ where $i_l \le n_l$ is the number of radical variables of homogeneous degree g_l, for $1\le l\le s$. If $J^u \neq 0$ and $J^{u+1} = 0$, then Ξ is non-zero only if $i\le u$ since $WJW\subseteq J$. Thus the number of possible distributions of radical i variables in a non-zero monomial is at most $\binom{n_1}{i_1}\cdots\binom{n_s}{i_s}(\dim J)^i \le C_1 n^u$. Now each such monomial involves $n-i$ semisimple variables with $n_l - i_l$ variables of homogeneous degree g_l, $1\le l\le s$, which come from some distinct G-graded simple components $B_{m_1},\dots,B_{m_k}$. Since $WB_kW\subseteq B_k + J$, this means that there exists distincts $p_1,\dots,p_q\in\{m_1,\dots,m_k\}$ with $1\le q\le k$ such that the subalgebra $D = B_{p_1}\oplus\cdots\oplus B_{p_q}\subseteq B$ is admissible.

Now let D be an admissible G-graded subalgebra of B and let $d_1 = |\mathcal{B}_{g_1}\cap D|, d_2 = |\mathcal{B}_{g_2}\cap D|,\dots,d_s = |\mathcal{B}_{g_s}\cap D|$. If there are $i = i_1+\cdots+i_s$ radical variables, the number of non-zero monomials with semisimple variables coming from D is at most $\binom{n_1}{i_1}\cdots\binom{n_s}{i_s}(\dim J)^i d_1^{n_1-i_i}\cdots d_s^{n_s-i_s}$. Hence such admissible subalgebra D may contribute with at most $C_2 n^u d_1^{n_1}d_2^{n_2}\cdots d_s^{n_s}$ possible monomials, with C_2 a constant.

Now, if M is the number of admissible G-graded subalgebras of B and E is an admissible G-graded subalgebra of maximal dimension such that $e_1 = |\mathcal{B}_1\cap E|,\ e_2 = |\mathcal{B}_2\cap E|,\dots,e_s = |\mathcal{B}_s\cap E|$, then we have that $e_1+\cdots+e_s = d$ and an upper bound for the number of possible non-zero monomials is $MC_2n^u e_1^{n_1}\cdots\ e_s^{n_s}$. Taking into account that we rewrote any product of n basis elements of A as a linear combination of basis elements, we get that

$$c^{G,W}_{n_1,\dots,n_s}(A)\le C_3 n^u e_1^{n_1}\cdots e_s^{n_s},$$

where C_3 is a constant. Thus, by Proposition 3.1 we have that

$$c^{G,W}_n(A) = \sum_{\substack{n_1,\dots,n_s\\ n_1+\cdots+n_s=n}}\binom{n}{n_1,\dots,n_s}c^{G,W}_{n_1,\dots,n_s}(A)\le C_3n^u\sum_{\substack{n_1,\dots,n_s\\ n_1+\cdots+n_s=n}}\binom{n}{n_1,\dots,n_s}e_1^{n_1}\cdots e_s^{n_s} = C_3n^ud^n,$$

and the proof is complete. □

As in the ordinary case, following step by step [5, Theorem 4.1.9] with the necessary changes, we have that the codimensions do not change upon extension of the base field F provided F is infinite. So, combining (3.2) and Lemma 4.4 with [1, Theorem 2.3], we have the following.

Theorem 4.5. *Let A be a finite dimensional G-graded W-algebra over a field F of characteristic zero. Then there exists constants C_1, C_2, r_1, r_2 such that $C_1\neq 0$ and*

$$C_1n^{r_1}d^n\le c^{G,W}_n(A)\le C_2n^{r_2}d^n,$$

where $d = \exp^G(A)$ is the maximal dimension of an admisible G-graded subalgbera of A. As a consequence the G-graded W-exponent $\exp^{\mathrm{G,W}}(A)$ exists and is equal to $\exp^G(A)$.

5. Generalized graded identities of $UT_2(F)$

From now on, we carry out an exhaustive study of the generalized graded polynomial identities of $UT_2 := UT_2(F)$, the algebra of 2×2 upper triangular matrices over a field F of characteristic zero, together with the variety it generates. In particular, in this section we begin by classifying all possible graded W-actions on UT_2 and by computing the corresponding ideal of identities. As a consequence, we obtain the generalized graded codimension sequences and the generalized graded PI-exponents for each case under consideration.

Recall that, given a finite group G, the canonical grading on UT_2 is defined as follows: let $g \in G$, $g \neq 1_G$, and set $UT_2 = UT_2^1 \oplus UT_2^g$ where $UT_2^1 = Fe_{11} \oplus Fe_{22}$ and $UT_2^g = Fe_{12}$. According to [12, Theorem 1], up to isomorphism, UT_2 admits only two distinct gradings, the trivial grading and the canonical one. This readily implies that we can regard UT_2 as a superalgebra, i.e., graded by $\mathbb{Z}_2$, the cyclic group of order two.

Hence, since in what follows we will work exclusively with superalgebras, we adopt the convention that variables y represent elements of X of homogeneous degree 0, while the variables z correspond to those of homogeneous degree 1.

As is well-known, studying UT_2 with the trivial grading is equivalent to study the ungraded case, which has been extensively discussed in [8, 9]. So we assume that the algebra UT_2 is endowed with the canonical grading.

Since the unital subalgebras of UT_2 are isomorphic to either $F1_{UT_2}$ or $D = Fe_{11} \oplus Fe_{22}$ or $C = F1_{UT_2} + Fe_{12}$ or UT_2 itself (see [9, Remark 5.1]), and each of these are also $\mathbb{Z}_2$-graded subalgebras of UT_2, it follows from Proposition 2.5 that the graded algebra $\Phi(W)$ must be isomorphic to one of F, D, C, or UT_2. Consequently, we can define four non-equivalent structures of $\mathbb{Z}_2$-graded W-algebras on UT_2, each determined by the choice of the graded subalgebra of UT_2 acting by left and right multiplication. We will denote them by UT_2^F, UT_2^D, UT_2^C and simply UT_2 depending on whether $\Phi(W)$ is isomorphich to F, D, C or the full algebra UT_2, respectively.

We start by considering $\Phi(W) \cong UT_2$ and, for simplicity of notation, we let $W = UT_2$. In other words, we are considering UT_2 with the action of the whole algebra UT_2 on itself by left and right multiplication.

We consider as a basis the set $\mathcal{B}_{UT_2} = \{1 := e_{11} + e_{22}, e_{22}, e_{12}\}$ and we recall that

$$z_1 z_2 \equiv 0 \quad \text{and} \quad [y_1, y_2] \equiv 0$$

are ordinary $\mathbb{Z}_2$-graded identities of UT_2 and by (3.1) are also $\mathbb{Z}_2$-graded generalized identities.

Now, following the same reasoning as in [8, Proposition 2.2], we determine under which condition a polynomial in 1 variable is UT_2-trivial.

In particular, if $\sum_{i=1}^m L_{v_i} R_{u_i} = 0$ where L_{v_i} and R_{u_i} denote the left and right multiplication by $v_i, u_i \in W$, then $\sum_{i=1}^m v_i y u_i$ and $\sum_{i=1}^m v_i z u_i$ are W-trivial graded generalized polynomials. Conversely, if both $\sum_{i=1}^m v_i y u_i$ and $\sum_{i=1}^m v_i z u_i$ are W-trivial for some $v_i, u_i \in W$, then $\sum_{i=1}^m L_{v_i} R_{u_i} = 0$.

According to this, the polynomials $e_{22}y - ye_{22}$, $e_{22}z$ and $ze_{22} - z$ are UT_2-nontrivial while $e_{12}ye_{12}$ and $e_{12}ze_{12}$ are UT_2-trivial. Moreover, a straightforward computation shows that $e_{22}y - ye_{22},\ e_{22}z,\ ze_{22} - z \in \mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2)$.

Lemma 5.1. *The generalized graded polynomials z_1z_2, $e_{12}yz$, zye_{12}, $e_{12}z$ and ze_{12} are consequences of $e_{22}z$ and $ze_{22} - z$.*

Proof. From $ze_{22} - z$ it follows that $z_1e_{22}z_2 - z_1z_2$, then by $e_{22}z$ we obtain z_1z_2. Now, by multiplying $e_{22}z$ by e_{12} on the left we get $e_{12}z$. By multiplying $ze_{22} - z$ by $(-e_{12})$ on the left we get ze_{12}. Finally, the $\mathbb{Z}_2$-graded UT_2-endomorphisms φ_1 and φ_2 of the free graded algebra $UT_2\langle Y, Z\rangle$ such that $\varphi_1(z) = yz$ and $\varphi_2(z) = zy$ transform the $\mathbb{Z}_2$-graded generalized polynomials $e_{12}z$ and ze_{12} into $e_{12}yz$ and zye_{12}, respectively. $\square$

Theorem 5.2. *Let UT_2 be the 2×2-upper triangular matrix UT_2-superalgebra with canonical grading, i.e., $UT_2^0 = Fe_{11} \oplus Fe_{22}$, $UT_2^1 = Fe_{12}$, and UT_2 acts on itself by left and right multiplication. Then*

$$\mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2) = \langle [y_1, y_2],\ e_{22}y - ye_{22},\ ze_{22} - z,\ e_{22}z\rangle_{T_{\mathbb{Z}_2,UT_2}}$$

and $c_n^{\mathbb{Z}_2,UT_2}(UT_2) = 2^{n-1}(n+2) + 2$.

Proof. Let $T = \langle [y_1, y_2],\ e_{22}y - ye_{22},\ e_{22}z,\ ze_{22} - z\rangle_{T_{\mathbb{Z}_2,UT_2}}$. It is clear that $T \subseteq \mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)$, thus in order to reach our goal, suppose by contradiction that there exists a multilinear polynomial $f \in P_n^{\mathbb{Z}_2,UT_2} \cap \mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)$ such that $f \notin T$, i.e., f is a non-zero polynomial modulo T.

By Lemma 5.1, $z_1z_2 \in T$ so we can assume that

$$f \equiv f_1(y_1, \ldots, y_n) + f_2(y_1, \ldots, y_{n-1}, z) \pmod{T}.$$

Since char $F = 0$, every multihomogeneous component of f is still a generalized graded polynomial identity for UT_2, therefore f_1 and f_2 are both graded generalized identities of UT_2. Since $[y_1, y_2] \equiv 0$, $e_{22}y - ye_{22} \equiv 0$ and $e_{12}ye_{12} = 0$, it follows that

$$f_1(y_1, \dots, y_n) \equiv \alpha y_1 \cdots y_n + \beta e_{22} y_1 \cdots y_n + \sum_{\mathcal{I},\mathcal{J}} \gamma_{\mathcal{I},\mathcal{J}} y_{i_1} \cdots y_{i_r} e_{12} y_{j_1} \cdots y_{j_{n-r}} (\text{mod } T),$$

where $\mathcal{I} = \{i_1, \dots, i_r\}$ and $\mathcal{J} = \{j_1, \dots, j_{n-r}\}$ are disjoint subsets of $\{1, \dots, n\}$ such that $i_1 < i_2 < \dots < i_r$, $j_1 < j_2 < \dots < j_{n-r}$ and $0 \leq r \leq n$.

If we substitute $y_1, \dots, y_n$ with $1 = e_{11} + e_{22}$ we get $\alpha 1 + \beta e_{22} + \sum_{\mathcal{I},\mathcal{J}} \gamma_{\mathcal{I},\mathcal{J}} e_{12} = 0$, then $\alpha = \beta = 0$. So

$$f_1(y_1, \dots, y_n) \equiv \sum_{\mathcal{I},\mathcal{J}} \gamma_{\mathcal{I},\mathcal{J}} y_{i_1} \cdots y_{i_r} e_{12} y_{j_1} \cdots y_{j_{n-r}} \ (\text{mod } T).$$

For any fixed $\mathcal{I}$ and $\mathcal{J}$, if we replace $y_{i_1}, \dots, y_{i_r}$ with e_{11} and $y_{j_1}, \dots, y_{j_{n-r}}$ with e_{22}, we get $\gamma_{\mathcal{I},\mathcal{J}} e_{12} = 0$. Therefore $\gamma_{\mathcal{I},\mathcal{J}} = 0$ for all $\mathcal{I}$ and $\mathcal{J}$, thus $f_1 = 0$ modulo T.

Now let focus our attention on f_2. By Lemma 5.1 $e_{12}z$, ze_{12}, $e_{12}yz$, $zye_{12} \in T$. Then from $e_{22}z \equiv 0$, $e_{22}y - ye_{22} \equiv 0$, $e_{12}z \equiv 0$, $e_{12}yz \equiv 0$, $ze_{22} - z \equiv 0$, $ze_{12} \equiv 0$ and $zye_{12} \equiv 0$ we can suppose that e_{22} and e_{12} do not appear in f_2. Moreover $[y_1, y_2] \equiv 0$, thus

$$f_2 \equiv \sum_{\mathcal{I},\mathcal{J}} \alpha_{\mathcal{I},\mathcal{J}} y_{i_1} \cdots y_{i_r} z y_{j_1} \cdots y_{j_{n-1-r}} \ (\text{mod } T),$$

where as before $\mathcal{I} = \{i_1, \dots, i_r\}$ and $\mathcal{J} = \{j_1, \dots, j_{n-1-r}\}$ are disjoint subsets of $\{1, \dots, n-1\}$ such that $i_1 < i_2 < \dots < i_r$, $j_1 < j_2 < \dots < j_{n-1-r}$ and $0 \leq r \leq n-1$.

For any fixed $\mathcal{I}$ and $\mathcal{J}$, if we substitute $y_{i_1}, \dots, y_{i_r}$ with e_{11}, $y_{j_1}, \dots, y_{j_{n-1-r}}$ with e_{22} and z with e_{12}, we get $\alpha_{\mathcal{I},\mathcal{J}} e_{12} = 0$. Thus $\alpha_{\mathcal{I},\mathcal{J}} = 0$ for all $\mathcal{I}$ and $\mathcal{J}$ and $f_2 = 0$ modulo T.

This implies that $f \equiv 0 \ (\text{mod } T)$, a contradiction. Therefore $T = \mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)$.

We have also proved that the following monomials form a basis of $P_n^{\mathbb{Z}_2,UT_2}$ modulo $P_n^{\mathbb{Z}_2,UT_2} \cap \mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)$:

$$\begin{aligned} & y_1 \cdots y_n, \\ & e_{22} y_1 \cdots y_n, \\ & y_{i_1} \cdots y_{i_r} e_{12} y_{j_1} \cdots y_{j_{n-r}}, \\ & y_{l_1} \cdots y_{l_u} z y_{q_1} \cdots y_{q_{n-u-1}}, \end{aligned}$$

where $0 \leq r \leq n$ and $0 \leq u \leq n-1$ such that $i_1 < \dots < i_r$, $j_1 < \dots < j_{n-r}$, $l_1 < \dots < l_u$, $q_1 < \dots < q_{n-u-1}$. Hence, by counting them, we get

$$c_{n,0}^{\mathbb{Z}_2,UT_2}(UT_2) = 1 + 1 + \sum_{r=0}^{n} \binom{n}{r} = 2^n + 2, \tag{5.1}$$

and

$$c_{n-1,1}^{\mathbb{Z}_2,UT_2}(UT_2) = \sum_{r=0}^{n-1} \binom{n-1}{r} = 2^{n-1}. \tag{5.2}$$

As a cosequence, by Proposition 3.1, we have that $c_n^{\mathbb{Z}_2,UT_2}(UT_2) = 2^{n-1}(n+2) + 2$. □

We now translate all the previous results in terms of the $\mathbb{Z}_2$-graded W-algebra UT_2 where W acts on it as the algebra UT_2 by left and right multiplication. To do so, we establish the following notation.

Notation 5.3. We fix an ordered basis $\mathcal{B}_W = \{w_i\}_{i \in \mathcal{I}}$ of the graded algebra W over the field F, choosing it so that the first element is $w_0 = 1_W$, the unity of W. Whenerver we deal with a finite-dimensional unital G-graded W-algebra A, where the action of W is given by left and right multiplication by elements of a unital graded subalgebra $B \subseteq A$, we also fix an ordered basis $\mathcal{B}_B = \{b_0 = 1_A, b_1, \dots, b_n\}$ of B over F. Under these assumptions, we assume, as we may, that the action homomorphism Φ maps each w_i to the pair (R_{b_i}, L_{b_i}) in the multiplier algebra $\mathcal{M}(A)$, for all $0 \leq i \leq n$. Furthermore, for all $i \geq n+1$, we assume that w_i lies in the kernel of Φ, meaning $\Phi(w_i) = 0$.

Accordingly, fixing the ordered basis $\mathcal{B}_{UT_2} = \{1 := e_{11} + e_{22}, e_{22}, e_{12}\}$ for UT_2, the following result holds.

Corollary 5.4. *Let UT_2 be the $\mathbb{Z}_2$-graded W-algebra UT_2 endowed with the canonical grading where W acts on it as UT_2 itself by left and right multiplication. Then $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2)$ is generated, as $T_{\mathbb{Z}_2,W}$-ideal, by the following polynomials:*

$$[y_1, y_2], \quad w_1y - yw_1, \quad zw_1 - z, \quad w_1z, \quad yw_i, \quad w_iy, \quad zw_i, \quad w_iz$$

for any $i \geq 3$. Moreover, $c_n^{\mathbb{Z}_2,W}(UT_2) = 2^{n-1}(n+2) + 2$.

Now analyze the case of $\Phi(W) \cong D = Fe_{11} \oplus Fe_{22}$ and we choose the ordered basis of D as the following

$$\mathcal{B}_D = \{1_{UT_2}, e_{22}\}.$$

With this assumption and similar argumentation as in Theorem 5.2, the following theorem can be proven.

Theorem 5.5. *Let UT_2^D be the $\mathbb{Z}_2$-graded W-algebra UT_2 endowed with the canonical grading where W acts on it as the algebra D by left and right multiplication. Then $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)$ is generated, as $T_{\mathbb{Z}_2,W}$-ideal, by the following polynomials:*

$$[y_1, y_2], \quad w_1y - yw_1, \quad zw_1 - z, \quad w_iz, \quad zw_j, \quad w_jy, \quad yw_j$$

for all $i \geq 1$, $j \geq 2$. Moreover, $c_n^{\mathbb{Z}_2,W}(UT_2^D) = n2^{n-1} + 2$.

We now study the algebra UT_2^C, i.e. the W-algebra UT_2 with action given by left and right multiplication by elements of $C = F1_{UT_2} + Fe_{12}$. We highlight that the proof of the following result is easily obtained from the one of Theorem 5.2, with some mild and intuitive modifications.

To this end, let

$$\mathcal{B}_C = \{1_{UT_2}, e_{12}\}$$

a fixed ordered basis of C.

Theorem 5.6. *Let UT_2^C be the $\mathbb{Z}_2$-graded W-algebra UT_2 endowed with the canonical grading where W acts as the algebra C by left and right multiplication. Then $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^C)$ is generated, as $T_{\mathbb{Z}_2,W}$-ideal, by the following polynomials:*

$$[y_1, y_2], \quad z_1z_2, \quad yw_i, \quad w_iy, \quad zw_j, \quad w_jz$$

for all $i \geq 2$ and $j \geq 1$. Moreover, $c_n^{\mathbb{Z}_2,W}(UT_2^C) = 2^{n-1}(n+2) + 1$.

As a final case, for the sake of completeness, we analyze UT_2^F. Recall that in this case $\Phi(W) \cong F$, therefore we are dealing with the ordinary graded identities of UT_2 and we can state the following results from [12].

Theorem 5.7. *Let UT_2^F be the $\mathbb{Z}_2$-graded W-algebra UT_2 endowed with the canonical grading where W acts on it as the algebra F by left and right multiplication. Then $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^F)$ is generated, as $T_{\mathbb{Z}_2,W}$-ideal, by the following polynomials:*

$$[y_1, y_2], \quad z_1z_2, \quad yw_i, \quad w_iy, \quad zw_i, \quad w_iz$$

for all $i \geq 1$. Moreover, $c_n^{\mathbb{Z}_2,W}(UT_2^F) = n2^{n-1} + 1$.

6. Graded generalized cocharacter sequence of $UT_2(F)$

In this section, we aim to provide a description of the space of generalized multilinear polynomial identities of UT_2, for each possible action of W, in the language of Young diagrams through the representation theory of a Young subgroup of S_n.

To this end, let A be a $\mathbb{Z}_2$-graded W-algebra. For $n \geq 1$ and $0 \leq r \leq n$ we consider the left permutation action of the group $S_r \times S_{n-r}$ on the space $P_{r,n-r}^{\mathbb{Z}_2,W}$ by letting S_r act on $y_1, \ldots, y_r$ and S_{n-r} on $z_1, \ldots, z_{n-r}$. Since $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)$ is invariant under this $S_r \times S_{n-r}$ action, we have that

$$P_{r,n-r}^{\mathbb{Z}_2,W}(A) := \frac{P_{r,n-r}^{\mathbb{Z}_2,W}}{P_{r,n-r}^{\mathbb{Z}_2,W} \cap \mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)}$$

is a $S_r \times S_{n-r}$ left module. The corresponding character, denoted by $\chi_{r,n-r}^{\mathbb{Z}_2,W}(A)$, is called $(r, n-r)$th generalized graded cocharacter of A.

On one hand, it is well-known that the irreducible characters of the symmetric group S_n are in a one-to-one correspondence with the partitions of the integer n, on the other the irreducible $S_r \times S_{n-r}$ characters are obtained by taking the outer product of S_r and S_{n-r} irreducible characters, respectively. Then, by complete reducibility we can write

$$\chi^{\mathbb{Z}_2,W}_{r,n-r}(A) = \sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu)$$

where χ_λ (respectively, χ_μ) denotes the irreducible S_r-character (respectively S_{n-r}-character) and $m_{\lambda,\mu} \geq 0$ are the corresponding multiplicities.

Recall that the multiplicities in the cocharacter sequence are equal to the maximal number of linearly independent highest weight vectors, according to the representation theory of GL_n. We also recall that a highest weight vector is obtained from the polynomial corresponding to an essential idempotent by identifying the variables whose indices lie in the same row and alternating the ones whose indices lie in the same column of the corresponding Young tableau (see [2] for more details). Furthermore

$$c^{\mathbb{Z}_2,W}_{r,n-r}(A) = \sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu} d_\lambda d_\mu \tag{6.1}$$

where d_λ and d_μ are the degrees of χ_λ and χ_μ, i.e., the number of standard tableaux of shape λ and μ, respectively.

In what follows, given a partition $\lambda \vdash n$, we define height of λ, and we write $h(\lambda)$, as the number of rows of λ. We can immediately remark that if $\dim A^0 = k$ and $\dim A^1 = t$, then every generalized polynomial alternating in k or more even variables or t or more odd variables is a polynomial identity, thus $m_{\lambda,\mu} = 0$ whenever $h(\lambda) > k$ or $h(\mu) > t$.

Next, we compute the decomposition into irreducible components of the generalized graded cocharacter of the superalgebra UT_2 endowed with the canonical grading.

We start, as in the previous section, considering $\Phi(W) \cong UT_2$ and for the simplicity of the notation $W = UT_2$. Then we prove the following lemmas for the multplicities $m_{\lambda,\mu}$ of the $(r, n-r)$th generalized graded cocharacter of UT_2

$$\chi^{\mathbb{Z}_2,UT_2}_{r,n-r}(UT_2) = \sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu). \tag{6.2}$$

Lemma 6.1. *If $r = n$, then in (6.2) we have*

$$m_{\lambda,\mu} = \begin{cases} n+3 & \text{if } \lambda = (n),\ \mu = \emptyset \\ q+1 & \text{if } \lambda = (p+q,p),\ \mu = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all $p \geq 1$ and $q \geq 0$ such that $2p + q = n$.

Proof. Since $\dim UT_2^0 = 2$, then $m_{\lambda,\mu} = 0$ whenever $h(\lambda) > 2$. If $\lambda = (n)$ and $\mu = \emptyset$, we consider the following tableau

1	2	$\cdots$	n

and we associate to it the polynomials

$$y^n,$$
$$e_{22}y^n,$$
$$y^i e_{12} y^{n-i},$$

for all $0 \leq i \leq n$. It is clear that such polynomials are not generalized identities of UT_2 and moreover they are linearly independent modulo $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)$. Indeed, if

$$f(y) = \alpha y^n + \beta e_{22} y^n + \sum_{i=0}^{n} \gamma_i y^i e_{12} y^{n-i} \in \ \mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)$$

and we evaluate y in 1, we have

$$\alpha + \beta e_{22} + \sum_{i=0}^{n} \gamma_i e_{12} = 0$$

so $\alpha = \beta = 0$. Therefore, if we consider $0 \neq \delta \in F$ and $y = \delta e_{11} + e_{22}$ then

$$f(\delta e_{11} + e_{22}) = \sum_{i=0}^{n} \gamma_i (\delta e_{11} + e_{22})^i e_{12} (\delta e_{11} + e_{22})^{n-i} = \sum_{i=0}^{n} \gamma_i \delta^i e_{12} = 0$$

Since the matrix associated to the above linear system is a Vandermonde matrix and the field F is infinite, it follows that $\gamma_i = 0$ for all $0 \leq i \leq n$. This implies that $m_{\lambda,\mu} \geq n+3$.

Now let $\mu = \emptyset$ and $\lambda = (p+q, p)$ with $p \geq 1, q \geq 0$ and $2p + q = n$. For all $0 \leq i \leq q$, we consider the following tableau:

$$T_\lambda^i = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline i+1 & i+2 & \cdots & i+p & 1 & \cdots & i & i+2p+1 & \cdots & n \\ \hline i+p+1 & i+p+2 & \cdots & i+2p \\ \cline{1-4} \end{array}$$

and its associate polynomials

$$c_{p,q}^{(i)}(y_1, y_2) = y_1^i \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p-1} (y_1 e_{12} y_2 - y_2 e_{12} y_1) \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p-1} y_1^{q-i} \tag{6.3}$$

where the symbol $\tilde{}$ or $\bar{}$ means alternation on the corresponding variables. Moreover, the polynomials in (6.3) are linearly independent modulo $\mathrm{Id}^{\mathbb{Z}_2, \mathrm{UT}_2}(UT_2)$. Indeed if

$$\sum_{i=0}^{q} \alpha_i c_{p,q}^{(i)}(y_1, y_2) \in \mathrm{Id}^{\mathbb{Z}_2, \mathrm{UT}_2}(UT_2),$$

then by substituting $y_1 = \delta e_{11} + e_{22}$, where $\delta \in F$, $\delta \neq 0$, and $y_2 = e_{11}$, we get

$$\sum_{i=0}^{q} (-1)^p \alpha_i \delta^i = 0.$$

Again, by a Vandermonde argument, we have that $\alpha_i = 0$ for any $0 \leq i \leq q$. It follows that the polynomials in (6.3) are linearly independent (mod $\mathrm{Id}^{\mathbb{Z}_2, \mathrm{W}}(A)$) and this implies that $m_{\lambda,\mu} \geq q+1$.

By taking into account (5.1) and (6.1), we have proved that

$$2^n + 2 = c_{n,0}^{\mathbb{Z}_2, UT_2}(UT_2) \geq (n+3) d_{(n)} d_\emptyset + \sum_{\substack{1 \leq p \leq \lfloor \frac{n}{2} \rfloor \\ 0 \leq q \leq n-2p}} (q+1) d_{(p+q,p)} d_\emptyset,$$

so far.

Now, in the proof of [8, Theorem 4.4], it was showed that

$$\sum_{\substack{1 \leq p \leq \lfloor \frac{n}{2} \rfloor \\ 0 \leq q \leq n-2p}} (q+1) d_{(p+q,p)} = 2^n - n - 1, \tag{6.4}$$

therefore, recalling that $d_\emptyset = 1$, we get

$$(n+3) d_{(n)} d_\emptyset + \sum_{\substack{1 \leq p \leq \lfloor \frac{n}{2} \rfloor \\ 0 \leq q \leq n-2p}} (q+1) d_{(p+q,p)} d_\emptyset = n + 3 + 2^n - n - 1 = 2^n + 2.$$

This readily implies that $m_{(n),\emptyset} = n+3$ and $m_{(p+q,p),\emptyset} = q+1$, as required. □

Lemma 6.2. *If $r = n-1$, then in (6.2) we have*

$$m_{\lambda,\mu} = \begin{cases} n & \lambda = (n-1), \ \mu = (1) \\ q+1 & \lambda = (p+q, p), \ \mu = (1) \\ 0 & \textit{otherwise} \end{cases}$$

for all $p \geq 1, q \geq 0$ such that $2p + q = n-1$

Proof. As in Lemma 6.1, we have $m_{\lambda,\mu} = 0$ as long as $h(\lambda) > 2$. Now, let $\lambda = (n-1)$. Then for any $0 \le i \le n-1$ we consider the following two tableaux

$$T_\lambda^{(i)} = \boxed{1 \mid \cdots \mid i \mid i+2 \mid \cdots \mid n}$$

$$T_\mu^{(i)} = \boxed{i+1}$$

and the polynomial associated to them

$$a^{(i)}(y,z) = y^i z y^{n-i-1}.$$

The above polynomials are not generalized identities, moreover they are linearly independent modulo the generalized graded identities of UT_2, in fact if we suppose the contrary, i.e. $\sum_{i=0}^{n-1} \alpha_i a^{(i)} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)}$ with some non-zero scalar α_i, then we let t be the greatest integer such that $\alpha_t \neq 0$ and we substitute y with $y_1 + y_2$. We get

$$\alpha_t(y_1+y_2)^t z(y_1+y_2)^{n-t-1} + \sum_{i<t} \alpha_i (y_1+y_2)^i z (y_1+y_2)^{n-i-1} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)}.$$

Let us consider the homogeneous component of degree t in y_1 and $n-t-1$ in y_2. If we make the substitution $y_1 = e_{11}$, $y_2 = e_{22}$ and $z = e_{12}$, we obtain $\alpha_t e_{12} = \alpha_t = 0$, a contradiction. Hence the polynomials $a^{(i)}, 0 \le i \le n-1$ are linearly independent $\pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)}$. Therefore $m_{\lambda,\mu} \ge n$.

In the case of $\lambda = (p+q,p)$ and $\mu = (1)$ with $p \ge 1$, for any $0 \le i \le q$, we consider the following two tableaux:

$$T_\lambda^{(i)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline i+1 & i+2 & \cdots & i+p & 1 & \cdots & i & i+2p+2 & \cdots & n \\ \hline i+p+2 & i+p+3 & \cdots & i+2p+1 & \multicolumn{6}{c}{} \\ \cline{1-4}\end{array}$$

$$T_\mu^{(i)} = \boxed{i+p+1}$$

and the polynomials associated with them:

$$b_{p,q}^{(i)}(y_1,y_2,z) = y_1^i \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p} z \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p} y_1^{q-i}$$

Since the previous highest weight vectors are exactly the same that arise in the computation of the ordinary $\mathbb{Z}_2$-graded cocharacter of UT_2, following word by word the lines of [12, Theorem 3], it can be proven that these polynomials are linearly independent modulo $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{UT}_2}(UT_2)$, so $m_{\lambda,\mu} \ge q+1$.

By (5.2), (6.1) and (6.4), we have proved that

$$2^{n-1} = c_{n-1,1}^{\mathbb{Z}_2,UT_2}(UT_2) \ge n d_{(n-1)} d_{(1)} + \sum_{\substack{1 \le p \le \lfloor \frac{n-1}{2} \rfloor \\ 0 \le q \le n-2p-1}} (q+1) d_{(p+q,p)} d_{(1)} = n + 2^{n-1} - n + 1 - 1 = 2^{n-1},$$

thus $m_{(n-1),(1)} = n$ and $m_{(p+q,p),(1)} = q+1$. The proof is now complete. □

Combining the previous lemmas, we obtain the following result.

Theorem 6.3. *Let UT_2 be the $\mathbb{Z}_2$-graded W-algebra of upper 2×2 triangular matrices over a field F of characteristic zero endowed with the canonical grading where W acts as the full algebra UT_2 by left and right multiplication. Let $0 \le r \le n$ and let*

$$\chi_{r,n-r}^{\mathbb{Z}_2,W}(UT_2) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} (\chi_\lambda \otimes \chi_\mu)$$

be the $(r,n-r)$th generalized graded cocharacter of UT_2. Then either $r = n$ and

$$m_{\lambda,\mu} = \begin{cases} n+3 & \lambda = (n),\ \mu = \emptyset \\ q+1 & \lambda = (p+q,p),\ \mu = \emptyset \\ 0 & \textit{otherwise} \end{cases},$$

or $r = n-1$ and

$$m_{\lambda,\mu} = \begin{cases} n & \lambda = (n-1),\ \mu = (1) \\ q+1 & \lambda = (p+q,p),\ \mu = (1) \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. Notice that $z_1z_2 \equiv 0$, therefore $m_{\lambda,\mu} = 0$ as long as $\mu \vdash k$ with $k \geq 2$. Now the proof follows directly from Lemmas 6.1 and 6.2. $\square$

Now with similar techniques, we can also compute the $(r, n-r)$th graded generalized cocharacter of UT_2^D, i.e., UT_2 with canonical grading when $\Phi(W) \cong D = Fe_{11} \oplus Fe_{22}$.

Theorem 6.4. *Let UT_2^D be the $\mathbb{Z}_2$-graded W-algebra of upper 2×2 triangular matrices over a field F of characteristic zero endowed with the canonical grading where W acts as the algebra D by left and right multiplication. Let $0 \leq r \leq n$ and let*

$$\chi^{\mathbb{Z}_2,W}_{r,n-r}(UT_2^D) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu)$$

be the $(r, n-r)$th generalized graded cocharacter of UT_2^D. Then either $r = n$ and

$$m_{\lambda,\mu} = \begin{cases} 2 & \lambda = (n),\ \mu = \emptyset \\ 0 & otherwise \end{cases},$$

or $r = n-1$ and

$$m_{\lambda,\mu} = \begin{cases} n & \lambda = (n-1),\ \mu = (1) \\ q+1 & \lambda = (p+q, p),\ \mu = (1) \\ 0 & otherwise \end{cases}.$$

Analogously, we can also compute the $(r, n-r)$th graded generalized cocharacter of $\mathbb{Z}_2$-graded W-algebra UT_2^C, i.e., UT_2 with canonical grading when $\Phi(W) \cong C = F1_{UT_2} + Fe_{12}$.

Theorem 6.5. *Let UT_2^C be the $\mathbb{Z}_2$-graded W-algebra of upper 2×2 triangular matrices over a field F of characteristic zero endowed with the canonical grading where W acts as the algebra C by left and right multiplication. Let $0 \leq r \leq n$ and let*

$$\chi^{\mathbb{Z}_2,W}_{r,n-r}(UT_2^C) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu)$$

be the $(r, n-r)$th generalized graded cocharacter of UT_2^C. Then either $r = n$ and

$$m_{\lambda,\mu} = \begin{cases} n+2 & \lambda = (n),\ \mu = \emptyset \\ 0 & otherwise \end{cases},$$

or $r = n-1$ and

$$m_{\lambda,\mu} = \begin{cases} n & \lambda = (n-1),\ \mu = (1) \\ q+1 & \lambda = (p+q, p),\ \mu = (1) \\ 0 & otherwise \end{cases}.$$

Finally, for the sake of completeness, we also state the result for UT_2^F, i.e., UT_2 with the canonical grading and with $\Phi(W) \cong F$, whose proof can be found in [12].

Theorem 6.6. *Let UT_2^F be the $\mathbb{Z}_2$-graded W-algebra of upper 2×2 triangular matrices over a field F of characteristic zero endowed with the canonical grading where W acts as the algebra F by left and right multiplication. Let $0 \leq r \leq n$ and let*

$$\chi^{\mathbb{Z}_2,W}_{r,n-r}(UT_2^F) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu)$$

be the $(r, n-r)$th generalized graded cocharacter of UT_2^F. Then either $r = n$ and

$$m_{\lambda,\mu} = \begin{cases} 1 & \lambda = (n),\ \mu = \emptyset \\ 0 & otherwise \end{cases},$$

or $r = n-1$ *and*

$$m_{\lambda,\mu} = \begin{cases} n & \lambda = (n-1),\ \mu = (1) \\ q+1 & \lambda = (p+q,p),\ \mu = (1) \\ 0 & \textit{otherwise} \end{cases}.$$

7. On almost polynomial growth

In this section we deal with generalized varieties of superalgebras of almost polynomial growth (APG). To this end, recall that a variety of superalgebras $\mathcal{V}$ is said to be of almost polynomial growth of the codimensions, if $c_n^{G,W}(\mathcal{V})$ grows exponentially, but for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$, $c_n^{G,W}(\mathcal{U})$ grows polynomially.

Concerning this topic, given a class of algebras (associative, non-associative, with additional structure, etc), a central problem in the theory is twofold: on one hand, to construct algebras that are not PI-equivalent but still generate APG varieties; on the other hand, to classify, up to equivalence, all varieties exhibiting almost polynomial growth. For instance, a celebrated theorem by Kemer states that in characteristic zero up to equivalence, the only APG varieties of ordinary algebras are the ones generated by UT_2 and by the infinite dimensional Grassmann algebra E (see [6]). Following this approach, in [8] it was proved that the generalized varieties of ordinary algebras generated by UT_2^F and UT_2^D are APG, while in [9] the authors showed that up to equivalence, these are the only generalized APG varieties generated by finite dimensional algebras.

Motivated by the above results, in this section we aim to prove that in case of superalgebras, the generalized varieties generated by UT_2^F and by UT_2^D, both endowed with the canonical grading, are APG.

Remark 7.1. Recall that by Theorem 6.4, if $\lambda = (n)$ and $\mu = \emptyset$ then the linearly independent graded generalized highest weight vectors corresponding to λ and μ are

$$y^n \quad \text{and} \quad w_1 y^n,$$

where w_1 acted on UT_2 as the left and right multiplication by e_{22}. Moreover, if $\lambda = (n-1)$ and $\mu = (1)$ then the linearly independent graded generalized highest weight vectors are

$$y^i z y^{n-i-1}, \quad 0 \le i \le n-1.$$

Finally, for all $p \ge 1$, $q \ge 0$, $2p+q = n-1$, if $\lambda = (p+q,p)$ and $\mu = (1)$, the linearly independent graded generalized highest weight vectors corresponding to λ and μ are

$$y_1^i \underbrace{\overline{y}_1 \cdots \tilde{y}_1}_{p} z \underbrace{\overline{y}_2 \cdots \tilde{y}_2}_{p} y_1^{q-i}, \quad 0 \le i \le q.$$

Theorem 7.2. $\mathrm{var}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)$ *has almost polynomial growth of the codimensions.*

Proof. Let A be a $\mathbb{Z}_2$-graded W-algebra such that $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D) \subsetneq \mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)$. For $n \ge 1$ and $0 \le r \le n$, denote by $m_{\lambda,\mu}(A)$ and $m_{\lambda,\mu}(UT_2^D)$ the multiplicity of the generalized graded $(r, n-r)$th cocharacter of A and UT_2^D, respectively, corresponding to the partitions $\lambda \vdash r$ and $\mu \vdash n-r$. We claim that there exists $N \ge 0$ such that $m_{\lambda,\mu}(A) \le N$ for all $\lambda \vdash r$ and $\mu \vdash n-r$. Moreover, if $\lambda = (p+q,p)$ with $p \ge 1$ and $q \ge 0$, then there exists $M \ge 0$ such that $m_{\lambda,\mu}(A) = 0$ when $p \ge M$.

Since $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D) \subsetneq \mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)$, there exist $n_0 \ge 1$, $0 \le r_0 \le n_0$, and partitions $\lambda' \vdash r_0$ and $\mu' \vdash n_0 - r_0$ such that $m_{\lambda',\mu'}(A) < m_{\lambda',\mu'}(UT_2^D)$. By Remark 7.1, It follows that either

$$\alpha_1 y^{n_0} + \alpha_2 w_1 y^{n_0} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)} \tag{7.1}$$

with α_1, α_2 not both zero, or

$$\sum_{i=0}^{n_0-1} \beta_i y^i z y^{n_0-i-1} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)} \tag{7.2}$$

with β_i not all zero, or

$$\sum_{i=0}^{q_0} \gamma_i y_1^i \underbrace{\overline{y}_1 \cdots \tilde{y}_1}_{p_0} z \underbrace{\overline{y}_2 \cdots \tilde{y}_2}_{p_0} y_1^{q_0-i} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)} \tag{7.3}$$

with γ_i not all zero, and where $p_0 \geq 1$, $q_0 \geq 0$, $2p_0 + q_0 = n_0 - 1$.

If (7.2) or (7.3) hold, then since such highest weight vectors are exactly the same of the superalgebra UT_2 in the ordinary case, the claim follows directly from the proof of [12, Theorem 4].

Suppose now that (7.1) holds. Since $zw_1 \equiv z \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)}$, for $\alpha_1 \neq -\alpha_2$ from (7.1) we get that

$$zy^{n_0} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)}. \tag{7.4}$$

Moreover, since $w_1 y \equiv y w_1 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)}$ and $w_1 z \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)}$, for $\alpha_1 \neq 0$ from (7.1) we have that

$$y^{n_0} z \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)}. \tag{7.5}$$

Notice that by Theorem 6.4, to reach our goal it is enough to consider only the cases when $\lambda = (n-1)$ and $\mu = (1)$ or $\lambda = (p+q, p)$ and $\mu = (1)$.

First suppose that $\lambda = (n-1)$ and $\mu = (1)$. If $n \leq n_0$, then again by Theorem 6.4, $m_{\lambda,\mu}(A) \leq n \leq n_0$ and we have nothing to prove.

Now, assume that $n > n_0$. If $\alpha_1 \neq -\alpha_2$, then due to the identity (7.4), it follows that $y^i z y^{n-i-1} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)}$ as long as $n - i - 1 \geq n_0$. Therefore for all $0 \leq i \leq n - n_0 - 1$ the polynomials $y^i z y^{n-i-1}$ are $\mathbb{Z}_2$-graded W-identities of A and, since the remaining highest weight vectors correspond to the index i running in the range from $n - n_0$ to $n-1$, the maximal number of linearly independent highest weight vectors is at most n_0, i.e., $m_{\lambda,\mu}(A) \leq n_0$ for all $\lambda = (n-1)$, $\mu = (1)$ and $n \geq 2$.

Now, if $\alpha_1 = -\alpha_2$, then $\alpha_1 \neq 0$ because α_1, α_2 are not both zero. Thus by the identity (7.5), we get that $y^i z y^{n-i-1} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)}$ for all $i \geq n_0$. Since the remaining highest weight vectors correspond to the index i running in the range from 0 to $n_0 - 1$, again as before, the maximal number of linearly independent highest weight vectors is at most n_0. Hence $m_{\lambda,\mu}(A) \leq n_0$ for all $\lambda = (n-1)$, $\mu = (1)$ and $n \geq 2$.

We now focus our attention on the polynomials associated to partitions of the type $\lambda = (p+q, p)$ and $\mu = (1)$. First notice that, since $[y_1, y_2] \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)}$, if $p \geq 2n_0$, then from either (7.4) or (7.5) it follows that

$$\underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p} \, z \, \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)}$$

because, either before or after the variable z, there are at least n_0 copies of either y_1 or y_2. Therefore $m_{\lambda,\mu}(A) = 0$ when $p \geq 2n_0$. Thus, we can assume that $p < 2n_0$.

Now, if $q < n_0$, then by Theorem 6.4, $m_{\lambda,\mu}(A) \leq q + 1 \leq n_0$ and we have nothing to prove. Then assume that $q \geq n_0$. If $\alpha_1 \neq -\alpha_2$, then from (7.4) and since $[y_1, y_2] \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)}$, it follows that

$$y_1^i \, \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p} \, z \, \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p} \, y_1^{q-i} \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(A)}$$

for all $0 \leq i \leq q - n_0$. Since the remaining highest weight vectors correspond to the index i running in the range from $q - n_0 + 1$ to q, the maximal number of linearly independent highest weight vectors is at most n_0, i.e., $m_{\lambda,\mu}(A) \leq n_0$ for all $\lambda = (p+q, p)$, $\mu = (1)$ and $p \geq 1$, $q \geq 0$.

Now, if $\alpha_1 = -\alpha_2$, then $\alpha_1 \neq 0$ because α_1, α_2 are not both zero. Thus by using (7.5) and $[y_1, y_2] \equiv 0 \pmod{\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)}$, we get as above that $m_{\lambda,\mu}(A) \leq n_0$ for all $\lambda = (p+q, p)$, $\mu = (1)$ and $p \geq 1$, $q \geq 0$.

In conclusion, in all cases we have proved that $m_{\lambda,\mu}(A) \leq N = n_0$ for all partitions λ, μ, and $m_{\lambda,\mu}(A) = 0$ for all λ with more than $M = 2n_0$ boxes below the first row, as claimed. Therefore

$$\chi_{r,n-r}^{\mathbb{Z}_2,W}(A) = \sum_{\substack{\lambda \vdash r,\, \mu \vdash n-r \\ r - \lambda_1 \leq M-1 \\ n-r \leq 1}} m_{\lambda,\mu}(A) \, (\chi_\lambda \otimes \chi_\mu)$$

where λ_1 stands for the number of boxes of the first row of λ.

Since $r - \lambda_1 \leq M - 1$, then $\lambda_1 \geq r - (M-1)$ and by the hook formula we get

$$d_\lambda = \chi_\lambda(1) \leq \frac{n!}{(r - (M-1))!} \leq n^{M-1}.$$

Thus

$$
\begin{aligned}
c_n^{\mathbb{Z}_2,W}(A) &= \sum_{r=0}^{n}\binom{n}{r}\chi^{\mathbb{Z}_2,W}_{r,n-r}(A)(1)=\sum_{r=0}^{n}\binom{n}{r}\sum_{\substack{\lambda\vdash r,\,\mu\vdash n-r\\ r-\lambda_1\le M-1\\ n-r\le 1}} m_{\lambda,\mu}d_\lambda d_\mu\\
&= n\sum_{\substack{\lambda\vdash n-1,\\ n-\lambda_1\le M\\ \mu=(1)}} m_{\lambda,\mu}d_\lambda d_\mu+2\le n\sum_{\substack{\lambda\vdash n-1,\\ n-\lambda_1\le M\\ \mu=(1)}} Nn^{M-1}+2\le 2(M-1)^2Nn^M+2.
\end{aligned}
$$

The latter inequality holds since the number of partitions $\lambda\vdash n-1$ such that $n-\lambda_1\le M$ is bounded by $(M-1)^2$. Therefore $c_n^{\mathbb{Z}_2,W}(A)$ is polynomially bounded and $\mathrm{var}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)$ has almost polynomial growth of the codimensions, as required. □

With respect to the superalgebra UT_2^F, recall that in this case $\Phi(W)\cong F$, therefore we are dealing with the ordinary graded identities of UT_2 and by [12, Theorem 4] we have the following.

Theorem 7.3. *The variety of $\mathbb{Z}_2$-graded W-algebras generated by UT_2^F has almost polynomial growth.*

As consequence of Theorems 5.5 and 5.7, we have that $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^F)\not\subseteq \mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)$ and $\mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)\not\subseteq \mathrm{Id}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^F)$. Thus, by Theorems 7.2 and 7.3 we have the following.

Corollary 7.4. *The superalgebras UT_2^F and UT_2^D generate two distinct varieties of $\mathbb{Z}_2$-graded W-algebras of almost polynomial growth.*

Finally, notice that by Theorems 5.2 and 5.5, $UT_2^D\in \mathrm{var}^{\mathbb{Z}_2,\mathrm{W}}(UT_2)$ and $\mathrm{var}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^D)$ grows exponentially. Moreover, by Theorems 5.6 and 5.7, $UT_2^F\in \mathrm{var}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^C)$ and $\mathrm{var}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^F)$ grows exponentially. Then it immediately follows that

Corollary 7.5. $\mathrm{var}^{\mathbb{Z}_2,\mathrm{W}}(UT_2)$ *and* $\mathrm{var}^{\mathbb{Z}_2,\mathrm{W}}(UT_2^C)$ *do not have almost polynomial growth of the codimensions.*

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Giovanni Busalacchi, Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, via Archirafi 34, 90123, Palermo, Italy. *E-mail address*: giovanni.busalacchi@unipa.it

Fabrizio Martino, Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, via Archirafi 34, 90123, Palermo, Italy. *E-mail address*: fabrizio.martino@unipa.it

Carla Rizzo, CMUC, Departamento de Matemática, Universidade de Coimbra, Largo D. Dinis, 3001-501, Coimbra, Portugal. *E-mail address*: carlarizzo@mat.uc.pt