

b-divisorial valuations and Berkovich positivity functions

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Abstract

We prove semicontinuity properties for local positivity invariants of big and nef divisors. The usual definition of Seshadri constant and asymptotic order of vanishing along a subvariety is extended to include all seminorms in the Berkovich space, and we obtain semicontinuity of such constants as a function of the center seminorm.

We use Shokurov’s language of **b**-divisors; to each seminorm there is an associated **b**-divisor which can be used to translate questions about positivity into questions about the shape of certain cones of **b**-divisors. The theory works especially well for what we call **b**-divisorial valuations, a natural extension of the notion of divisorial valuations which encompasses e.g., all Abhyankar valuations.

1 Introduction

A profound, yet often elusive, connection links the Berkovich analytification, the Riemann-Zariski space, and the theory of **b**-divisors in birational geometry. This work aims to render this connection explicit from a geometric and birational viewpoint, tracing its origins to a recurring analytic pathology encountered in problems of a Nagata-type flavor. The unifying thread is the study of how global geometric invariants behave under variation in high-dimensional spaces of valuations, and how their discontinuities reflect the underlying structure of these spaces.

This investigation is primarily motivated by several disparate contexts where a common phenomenon—a certain discontinuity in geometric invariants—surfaces in a striking manner. A pivotal example arises in the work of [10], where a key function was constructed on a branch of the Berkovich space of the projective plane, attached to quasi-monomial valuations. The authors demonstrated that specific continuity and semi-continuity properties of this function would directly imply Nagata’s celebrated conjecture on linear systems. Surprisingly, an identical function emerged in the seemingly unrelated domain of symplectic geometry, specifically in the study of the *ball packing* problem [18], suggesting a deep, hidden correspondence between algebraic and symplectic invariants.

A third, appearance of this phenomenon occurs in the theory of Newton-Okounkov bodies. These are convex bodies associated to a line bundle on a projective variety endowed with a valuation of full rank, designed to encode asymptotic section data. In [8], a fundamental pathology was observed: while the volume of a Newton-Okounkov body remains invariant under a change of valuation, its *shape* can change *discontinuously* as the valuation moves along a branch of the Berkovich space. This *mutation* phenomenon is also observed in the symplectic setting [19] and is an indication that the standard topology

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on the space of valuations is not fine enough to control the geometric data it is meant to parameterize.

These examples, while mysterious and currently largely confined to a 2-dimensional setting, point towards a fundamental connection between the topology of valuation spaces and positivity measures in birational geometry. This work aims to provide a unifying explanation for these pathological phenomena through the lens of classical valuation theory and birational geometry. We lay a geometric groundwork that clarifies the nature of these discontinuities, paving the way for future explorations, particularly into the connections with symplectic geometry.

Let X be a normal projective variety over a closed field k . We explore the relationship between two fundamental, non-Archimedean objects:

- The *Berkovich analytification* $\mathfrak{X}^{\text{an}}$, whose points correspond to multiplicative seminorms on the function field $K(X)$ extending the base field norm, which we take to be the trivial norm. This space is endowed with a natural and well-behaved (e.g., locally path-connected) topology, making it a central object in non-Archimedean geometry.
- The *Zariski-Riemann space* $\mathcal{X}$, defined as the space of all valuation rings on $K(X)$ that are centered on X , or equivalently, as the space of all equivalence classes of valuations $\nu : K(X)^\times \rightarrow \Gamma$ on $K(X)$ centered on X .

We approach the space $\mathcal{X}$ from two complementary perspectives. The first is the classical, algebraic definition, which offers immense conceptual clarity but often obscures the underlying birational geometry. While this viewpoint yields important general results, such as the density of divisorial valuations, the non-noetherian geometric structure of $\mathcal{X}$ remains difficult to grasp.

The second perspective, championed by Shokurov (cf. [4]), provides a more geometric and birational characterization. Here, $\mathcal{X}$ is realized as the projective limit of all proper birational models of X over its base:

$$\mathcal{X} = \varprojlim_{X' \rightarrow X} X',$$

where the limit is taken over all proper birational morphisms $X' \rightarrow X$ with X' normal. This definition highlights $\mathcal{X}$ as a universal object in birational geometry. The bridge between these two definitions is furnished by the fact that the towers of birational maps are intrinsically governed by the inclusion relations of divisorial rings and valuation rings.

The novel contribution of this work is to build a bridge between these points of view by systematically analyzing the *trace* (or *support*) of valuation ideals (relative to a base model X) on the projective system of models defining the Zariski-Riemann space $\mathcal{X}$. This leads naturally to the construction of a **b**-divisor D_ξ —a concept central to modern birational geometry, representing a collection of divisors on all birational models that are compatible under pushforward—naturally associated to any seminorm ξ (D_ξ is the **anti-effective b**-divisor associated, in the sense of Boucksom–De Fernex–Favre [4], to the graded sequence of ideals of the valuation in the spirit of Jonsson–Mustață [14], see Section 3 for details). By elucidating to what extent the data of a valuation is encoded in this limiting divisor on all possible birational models, we provide a clear geometric framework for understanding the discontinuities observed in the motivating examples. This approach reveals the intricate interplay between analysis, algebra, and geometry in the study of valuation spaces,

positioning the space of $\mathbf{b}$ -divisors as a well-suited geometric setting for analyzing the continuity of positivity invariants.

In our approach, the class of valuations for which the associated $\mathbf{b}$ -divisor allows one to recover the valuation is of great importance. We propose to call these valuations *$\mathbf{b}$ -divisorial*, and they strictly include quasimonomial, or Abhyankar, valuations, and hence all divisorial valuations. Arguably, the excellent properties of Abhyankar valuations in birational geometry (see [11], [14] among many) derive from their being $\mathbf{b}$ -divisorial. We conjecture that $\mathbf{b}$ -divisorial valuations are characterized by the inequality $D_\xi \neq 0$. In the direction of characterizing $\mathbf{b}$ -divisorial valuations, we prove the following.

Theorem 1.1 (6.4, 6.5). *Let v be a valuation of $K(X)$ and ξ the associated norm. If $\dim X = 2$, then v is $\mathbf{b}$ -divisorial if and only if $D_\xi \neq 0$.*

In general, if v has sublinear log-discrepancy then v is $\mathbf{b}$ -divisorial.

Valuations with sublinear log-discrepancy include all valuations with finite log-discrepancy [14], see Definition 3.22.

For our purposes it is crucial to understand how the $\mathbf{b}$ -divisor D_ξ varies as a function of ξ . In this regard, we show the following.

Theorem 1.2 (4.4, 4.8). *The map $D : \mathfrak{X} \rightarrow \operatorname{Div}_{\mathbb{R}}(\mathcal{X})$ given by $\xi \mapsto D_\xi$ is lower semicontinuous.*

Moreover, for every model X_π above X , every point $p \in X_\pi$, and every local system of parameters z , the map D restricted to the set Δ of quasimonomial valuations with respect to z is continuous in the interior of Δ .

Then, with these results at hand, in sections 5 and 6 we address our original problem, which is to study *local positivity invariants*, such as Seshadri constants and asymptotic orders of vanishing, as functions of the associated valuation. First we extend the classical definitions of these positivity invariants to general seminorms; if D is an ample divisor on X , its Seshadri constant at a $\mathbf{b}$ -divisorial valuation v_ξ is

$$\varepsilon(D, v_\xi) := \sup \{ t \mid (D + tD_\xi) \text{ is nef} \},$$

and its asymptotic order of vanishing is

$$\omega(D, v_\xi) := \sup \{ t \mid (D + tD_\xi) \text{ is pseudoeffective} \}.$$

Finally, our main result, which sheds some light on the continuity and discontinuity phenomena that motivated us, is as follows.

Theorem 1.3 (5.11, 5.14, 6.8). *Fix D an ample divisor, and consider the functions $\mathfrak{X} \rightarrow \mathbb{R} \cup \{\infty\}$ given by $\xi \mapsto \varepsilon(D, \xi)$ and $\xi \mapsto \omega(D, \xi)$.*

- ω is lower semicontinuous, and it is continuous in the interior of every set Δ of quasimonomial valuations as above.
- ε is continuous in the interior of every set Δ of quasimonomial valuations as above.
- If X is a surface, and $p \in X$ a smooth point, then ε is lower semicontinuous over the subset $\mathfrak{X}_p$ of valuations centered at p .

Note that ε is not globally lower (nor upper) semicontinuous, see Example 5.13. Since both asymptotic order of vanishing and Seshadri constant are reflected in the shape of Newton–Okounkov bodies, our results are also significant for the understanding of the discontinuities observed in [8]. We expect to go back to the issue of semicontinuity for Newton–Okounkov bodies in later work. The use of $\mathbf{b}$ -divisorial valuations should also help in this context by unifying the analysis of bodies determined by valuations of maximal rank which are not flag valuations, especially on surfaces.

Section 2 is devoted to fixing the language and notation for both types of valuation spaces we use, including some results on the theory of $\mathbf{b}$ -divisors from [4] and on multiplicative ideals from [14] which will be used extensively. In section 3 we introduce the divisor D_ξ associated to a seminorm and we initiate its study, including the proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. In Section 5 we define and study the positivity invariants associated to seminorms, and prove most of Theorem 1.3. Finally, in Section 6 we deal with the case of surfaces, in which it is possible to give an explicit determination of D_ξ in terms of the sequence of centers of a valuation, and we prove the part of Theorem 1.3 dealing with surfaces.

2 Preliminaries on valuation spaces

Fix an algebraically closed field K of characteristic zero, endowed with the trivial valuation. Fix X a smooth irreducible projective variety over K . We will be working on two spaces whose points may be identified with valuations of $K(X)$, namely the Berkovich analytification and the Riemann–Zariski space of X . We begin by recalling the basics and setting up notation for these spaces.

We follow the usual conventions of writing valuations additively and seminorms multiplicatively.

2.1 Berkovich analytification

Let $\mathfrak{X}$ stand for the Berkovich analytification of X . It is a Berkovich analytic space (see [2], [3], [1], [23], [24] for general definitions and results) whose points are identified with real multiplicative seminorms on $\mathcal{O}_X$. More precisely, a point $\xi \in \mathfrak{X}$ is given by an affine open set $U_\xi \subset X$ and a multiplicative seminorm $\|\cdot\|_\xi : \mathcal{O}_X(U_\xi) \rightarrow \mathbb{R}_{>0}$ whose restriction to K is trivial, with the natural identifications on overlaps.

The kernel map and the reduction map. The analytification $\mathfrak{X}$ comes with a map $\ker : \mathfrak{X} \rightarrow X$ which sends each ξ to the (schematic) point associated to its kernel, the prime ideal in $\text{Spec } \mathcal{O}_X(U_\xi) \subset X$ formed by those $f \in \mathcal{O}_X(U_\xi)$ with $\|f\|_\xi = 0$. The fiber of $\ker$ over a point $p \in X$ can be identified with the set of norms on $\kappa(p)$, the field of rational functions of the closed subvariety $\bar{p}$, which have trivial restriction to K . Giving a norm $\|\cdot\|$ on $\kappa(p)$ is equivalent to giving a real valuation v on $\kappa(p)^\times$, related by $v(f) = -\log\|f\|$. Thus a point ξ in $\mathfrak{X}$ can be completely identified with a pair $(V, v) = (V_\xi, v_\xi)$ where $V \subseteq X$ is a closed subvariety and $v : K(V)^\times \rightarrow \mathbb{R}$ is a valuation. Then $\ker(V_\xi, v_\xi)$ is simply the generic point of V_ξ , and

$$\|f\|_\xi = \begin{cases} 0 & \text{if } f|_{V_\xi} = 0, \\ \exp(-v_\xi(f|_{V_\xi})) & \text{otherwise.} \end{cases}$$

The topology on $\mathfrak{X}$ is such that the map $\ker$ is continuous; in fact it can be defined as the weakest topology such that:

1. The kernel map is continuous.
2. For every open affine $U \subset X$ and every $f \in \mathcal{O}_X(U)$, the function

$$\begin{aligned} \ker^{-1}(U) &\rightarrow \mathbb{R} \\ \xi &\mapsto \|f\|_\xi \end{aligned}$$

is continuous.

There is a second map $\text{red} : \mathfrak{X} \rightarrow X$, called the ‘reduction map’ in the literature, which sends each $\xi = (V_\xi, v_\xi)$ to $\text{center}(v_\xi)$, the center of the valuation v_ξ , defined as the prime ideal in $\mathcal{O}_X(U)$ formed by those $f \in \mathcal{O}_X(U)$ with $\|f\|_\xi < 1$ (for any affine neighborhood U of $\ker \xi$). The (schematic) point $\text{red}(\xi)$ belongs to V_ξ , and there, it can be described as the prime ideal in $\mathcal{O}_{V_\xi}(U \cap V_\xi)$ formed by those f with $v_\xi(f) > 0$. The reduction map is anticontinuous. [23]

Notation 2.1. For every irreducible and reduced $V \subseteq X$, the notation $\theta_V = (X, \text{ord}_V) \in \mathfrak{X}$ will stand for the norm induced by the valuation order along V , i.e., $\ker(\theta_V) = 0$ and for every $f \in K(X)$, $\|f\|_{\theta_V} = \exp(-\text{ord}_V(f))$.

For every $\xi = (V_\xi, v_\xi) \in \mathfrak{X}$, and $\ell \in [0, +\infty)$ we use the notation ξ^ℓ for the seminorm $\xi^\ell = (V_\xi, \ell v_\xi)$; in other words,

$$\begin{aligned} \|f\|_{\xi^\ell} &= 0 \iff \|f\|_\xi = 0, \\ \|f\|_{\xi^\ell} &= \|f\|_\xi^\ell \quad \text{if } \|f\|_\xi \neq 0. \end{aligned}$$

In particular, for $\ell = 0$ we get the trivial seminorm on $V_\xi = \overline{\ker(\xi)}$, satisfying

$$\|f\|_{\xi^0} = \begin{cases} 0 & \text{if } f|_{V_\xi} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Additionally, by ξ^∞ we mean the trivial seminorm on $\text{red}(\xi)$, i.e. the seminorm which satisfies

$$\|f\|_{\xi^\infty} = \lim_{\ell \rightarrow \infty} \|f\|_{\xi^\ell} = \begin{cases} 0 & \text{if } f|_{\overline{\text{red}(\xi)}} = 0, \\ 1 & \text{otherwise,} \end{cases}$$

for every f regular in a neighborhood of $\text{red}(\xi)$. Thus, the trivial seminorm on a given subvariety V shall be denoted by θ_V^∞ . Note that for the trivial norm we have $\theta_X = \theta_X^\infty$.

Given a real valuation v on $K(X)$ with center x on X , one has a real filtration by ideal sheaves $(\mathfrak{a}_{v,m} \subset \mathcal{O}_X)_{m \in \mathbb{R}_{\geq 0}}$ defined by

$$\mathfrak{a}_{v,m}(U) = \begin{cases} \{f \in \mathcal{O}_X(U), v(f) \geq m\} & \text{if } x \in U, \\ \mathcal{O}_X(U) & \text{otherwise.} \end{cases}$$

This filtration is multiplicative, i.e., $\mathfrak{a}_{v,m} \cdot \mathfrak{a}_{v,n} \subset \mathfrak{a}_{v,m+n}$ for every $m, n \in \mathbb{R}_{\geq 0}$. This generalizes immediately to the more general setting of a seminorm $\xi \in \mathfrak{X}$; recall that ξ can

be described as a pair (kernel, valuation), $\xi = (V_\xi, v_\xi)$, with $v_\xi(f|_{V_\xi}) = -\log\|f\|_\xi$. Then the filtration is defined as

$$\mathfrak{a}_{\xi,m}(U) = \begin{cases} \{f \in \mathcal{O}_X(U), \log\|f\|_\xi \leq -m\} & \text{if } \text{red}(\xi) \in U, \\ \mathcal{O}_X(U) & \text{otherwise.} \end{cases}$$

Sometimes we need to emphasize the model X where these ideal sheaves live, and we shall use the notation $\mathfrak{a}_{\xi,m}^X$.

2.2 Riemann-Zariski space and Shokurov **b**-divisors

The Riemann-Zariski space of X was introduced by Zariski [25, Chapter VI, §17] as the set of all equivalence classes of valuations on $K(X)$ with a suitable structure of locally ringed space, which makes it isomorphic to the projective limit

$$\mathcal{X} := \varprojlim_{\pi} X_{\pi}$$

(in the category of locally ringed spaces) where $\pi: X_{\pi} \rightarrow X$ is a proper birational morphism. Such a limit is not a scheme if $\dim X > 1$. For every point $\eta \in \mathcal{X}$, the local ring $\mathcal{O}_{\mathcal{X},\eta}$ is a valuation ring of $K(X)$, namely the common valuation ring of all valuations in the equivalence class η , and every valuation ring arises this way.

The birational study of divisors on models of X is most conveniently tackled by using Shokurov's language of **b**-divisors on the Riemann-Zariski space, introduced in [21]. Let us recall some well-known facts following [4].

The group of Weil and Cartier **b**-divisors over X are respectively defined as

$$\begin{aligned} \text{Div}(\mathcal{X}) &:= \varprojlim_{\pi} \text{Div}(X_{\pi}) \quad \text{and} \\ \text{CDiv}(\mathcal{X}) &:= \varinjlim_{\pi} \text{CDiv}(X_{\pi}), \end{aligned}$$

where the first limit is taken with respect to the pushforward and the second with respect to the pullback. As is usual, $\mathbb{R}$ -Weil and $\mathbb{R}$ -Cartier **b**-divisors are obtained by extension of scalars, tensoring these abelian groups with $\otimes_{\mathbb{Z}} \mathbb{R}$.

- Given a Weil **b**-divisor D , the *trace* (or incarnation) of D on any given model X_{π} will be denoted by $D_{\pi} := D|_{X_{\pi}}$.
- We will say that a Cartier **b**-divisor C is *determined* on X_{π} for a model X_{π} over X if the equality $C_{\pi'} = \phi^* C_{\pi}$ holds for every higher model $\phi: X_{\pi'} \rightarrow X_{\pi}$. In that case we also say that C_{π} is a *determination* of C .
- Let $X_{\pi} \xrightarrow{\pi} X$ be some model and D_{π} a Cartier divisor on X_{π} . We denote by $\overline{D_{\pi}}$ the Cartier closure of D_{π} : the Cartier **b**-divisor obtained via pullback on higher models.
- Given a Weil **b**-divisor D , the fractional ideal sheaf on X whose sections on an open set $U \subset X$ are the rational functions f such that $\overline{\text{div}(f)} + W \geq 0$ is denoted by $\mathcal{O}_X(D)$. This sheaf is not coherent in general, but it is coherent for Cartier **b**-divisors and also for some important classes of Weil divisors, as will be seen below.

We will need the notion of numerical equivalence for **b**-divisors from [4]. There are two possible natural definitions:

- the 1-codimensional numerical classes over X

$$N^1(\mathcal{X}) := \varinjlim_{\pi} N^1(X_{\pi})$$

with respect to pulling-back.

- the $(n - 1)$ -dimensional numerical classes over X

$$N_{n-1}(\mathcal{X}) := \varprojlim_{\pi} N^1(X_{\pi})$$

with respect to push-forward.

There is a natural continuous injection $N^1(\mathcal{X}) \hookrightarrow N_{n-1}(\mathcal{X})$, as well as surjections $\text{CDiv}_{\mathbb{R}}(\mathcal{X}) \rightarrow N^1(\mathcal{X})$, $\text{Div}_{\mathbb{R}}(\mathcal{X}) \rightarrow N_{n-1}(\mathcal{X})$ compatible with it, which we write simply as $D \mapsto [D]$. The latter map is not continuous in general, but one has the following, which is often sufficient in applications:

Lemma 2.2. [4, Lemma 1.12] *Let W_j be a sequence (or net) of $\mathbb{R}$ -Weil **b**-divisors which converges to an $\mathbb{R}$ -Weil **b**-divisor W coefficient-wise. If there exists a fixed finite dimensional vector space V of $\mathbb{R}$ -Weil divisors on X such that $W_{j,X} \in V$ for all j then $[W_j] \rightarrow [W]$ converges in $N_{n-1}(\mathcal{X})$.*

With numerical equivalence at hand, one can deal with the notion of *nefness* of Cartier and even Weil divisors:

- A Cartier **b**-divisor is said to be X -nef or nef over X if for every model $X_{\pi} \xrightarrow{\pi} X$ where it is determined and every curve $C \subseteq X_{\pi}$ that gets contracted to a point on X , we have $D_{\pi} \cdot C \geq 0$.
- A Weil **b**-divisor D is said to be X -nef or nef over X if its numerical class is the limit of a sequence or net of X -nef Cartier **b**-divisors. Equivalently, for every smooth model $X_{\pi} \xrightarrow{\pi} X$, D_{π} belongs to the closed movable cone [4, Lemma 2.10].
- A Cartier **b**-divisor D which is determined on the model X_{π} is said to be *relatively globally generated* over X if so is D_{π} , i.e., if $\pi^* \pi_* \mathcal{O}_{X_{\pi}}(D_{\pi}) \rightarrow \mathcal{O}_{X_{\pi}}(D_{\pi})$ is surjective.

Definition 2.3. *Let $\mathfrak{a}$ be a coherent fractional ideal sheaf on X . If $X_{\pi} \rightarrow X$ is the normalized blow-up of X along $\mathfrak{a}$, the Cartier **b**-divisor determined by $\mathfrak{a} \cdot \mathcal{O}_{X_{\pi}}$ is called $Z(\mathfrak{a})$.*

*Let $\mathfrak{a}_{\bullet}$ be a graded sequence of (coherent) fractional ideal sheaves with linearly bounded denominators (see [4, Section 2.1]). Then the limit $\lim_{m \rightarrow \infty} \frac{1}{m} Z(\mathfrak{a}_m)$ is an $\mathbb{R}$ -Weil **b**-divisor called $Z(\mathfrak{a}_{\bullet})$.*

A Cartier **b**-divisor $C \in \text{CDiv}(\mathcal{X})$ is of the form $Z(\mathfrak{a})$ for some fractional ideal $\mathfrak{a}$ if and only if C is relatively globally generated over X [4, Lemma 1.8].

Definition 2.4 ([4], Definition 2.3). *Let D be an $\mathbb{R}$ -Weil divisor on some model X_π . The nef envelope $\text{Env}_\pi(D)$ is the $\mathbb{R}$ -Weil $\mathbf{b}$ -divisor associated with the graded sequence $\mathbf{a}_\bullet = \{\pi_* \mathcal{O}_{X_\pi}(mD)\}_{m \geq 1}$, namely*

$$Z(\mathbf{a}_\bullet) = \lim_{m \rightarrow \infty} \frac{1}{m} Z(\mathbf{a}_m).$$

Lemma 2.5 ([4], Lemma 2.11 and Corollary 2.13). *We have that*

1. $Z(\mathbf{a}_\bullet)$ is X -nef for every graded sequence of fractional ideals with linearly bounded denominators, and
2. $\text{Env}_\pi(D)$ is the largest X -nef $\mathbb{R}$ -Weil $\mathbf{b}$ -divisor W such that $W_\pi \leq D$.

Nef envelopes of arbitrary $\mathbf{b}$ -divisors W are not defined, but only because the existence of X -nef $\mathbf{b}$ -divisors bounded by W is not guaranteed. Under this hypothesis they can be defined:

Definition 2.6 ([4], Proposition 2.15 and Definition 2.16). *Let W be an $\mathbb{R}$ -Weil $\mathbf{b}$ -divisor. If the set of X -nef $\mathbb{R}$ -Weil $\mathbf{b}$ -divisors Z such that $Z \leq W$ is nonempty, then it admits a largest element $\text{Env}_X(W)$, called the nef envelope of W .*

2.3 Evaluation of seminorms on Cartier $\mathbf{b}$ -divisors

Given a seminorm $\|\cdot\|_\xi$, let U_ξ be an affine open neighborhood of $\text{red}(\xi)$ such that $\|\cdot\|_\xi$ is defined on $\Gamma(\mathcal{O}_X, U)$. If D is an effective Cartier divisor on X , and $f \in \Gamma(\mathcal{O}_X, U)$ is an equation of D in U , set $\|D\|_\xi = \|f\|_\xi$. It is easy to see that such a definition does not depend on the choices, and it extends to arbitrary Cartier divisors by multiplicativity.

Since pullback by birational maps preserves *values* of any $f \in \Gamma(\mathcal{O}_X, U)$, it is clear that if $\xi \in \mathfrak{X}$ is a *norm*, i.e., it has an associated valuation v_ξ on $K(X)$ (ker ξ is the generic point of X), then $\|\cdot\|_\xi$ can be evaluated on all Cartier $\mathbf{b}$ -divisors, by evaluating on an arbitrary representative.

For seminorms which are not necessarily norms, some care is needed in evaluating them on higher models:

Lemma 2.7. *Let D be a Cartier $\mathbf{b}$ -divisor, determined on some model $X_\pi \xrightarrow{\pi} X$, and let $U_\pi \subset X$ be an open set over which π is an isomorphism.*

Consider the open set $\mathfrak{U}_\pi \subset \mathfrak{X}$ of all seminorms with $\ker(\xi) \in U_\pi$. For every $\xi \in \mathfrak{U}_\pi$, $\|D\|_\xi$ is well defined, and the map $\mathfrak{U}_\pi \rightarrow \mathbb{R}$ defined by $\xi \mapsto \|D\|_\xi$ is continuous.

Proof. We assume without loss of generality that D is effective.

Berkovich analytification is functorial, so π determines an analytic morphism $\pi^\natural : \mathfrak{X}_\pi \rightarrow \mathfrak{X}$ from the analytification of X_π to $\mathfrak{X}$, which restricts to an isomorphism over $\mathfrak{U}_\pi$. Therefore all seminorms in $\mathfrak{U}_\pi$ can be evaluated on Cartier divisors of X_π , such as D_π , i.e., $\|D\|_\xi = \|D_\pi\|_{(\pi^\natural)^{-1}(\xi)}$ is well defined for every $\xi \in \mathfrak{U}_\pi$.

Now fix $\xi_0 \in \mathfrak{U}_\pi$, and let P be the center of ξ_0 on X_π , i.e., $P = \text{red}_{X_\pi}(\xi_0)$. Since the map red_{X_π} is anticontinuous, the set $\mathfrak{U}_0$ of all $\xi \in \mathfrak{U}_\pi$ such that $\text{red}_{X_\pi}(\xi) \in \overline{\text{red}_{X_\pi}(\xi_0)}$ is an open neighborhood of ξ_0 in $\mathfrak{U}_\pi \subset \mathfrak{X}$, and we need only show continuity of $\|D\|_\xi$ for $\xi \in \mathfrak{U}_0$. Let $U_1, \dots, U_k$ be open affine neighborhoods of $\text{red}_{X_\pi}(\xi_0)$ covering $\overline{\text{red}_{X_\pi}(\xi_0)}$, and

$f_i \in \Gamma(\mathcal{O}_{X_\pi}, U_i)$ equations of D in U_i . In the (closed) subset $\mathfrak{Z}_i \subset \mathfrak{U}_0$ consisting of the seminorms with center in U_i , the function $\xi \mapsto \|D\|_\xi = \|f_i\|_\xi$ is continuous by the definition of the topology on $\mathfrak{X}$. On the other hand, on overlaps $U_i \cap U_j$ we have $\|f_i\|_\xi = \|D\|_\xi = \|f_j\|_\xi$ because the norm of D is well defined (explicitly, the quotient f_i/f_j is invertible, so it does not belong to $\text{red}_{X_\pi}(\xi)$ for any $\xi \in \mathfrak{Z}_i \cap \mathfrak{Z}_j$, i.e., its seminorm is 1). We conclude that the functions f_i patch to give a continuous function $\xi \mapsto \|D\|_\xi$ on $\mathfrak{U}_0$. $\square$

Notation 2.8. For every regular function $f \in \Gamma(\mathcal{O}_X, U)$ on some open subset $U \subset X$, write $\text{div}_+(f)$ for the effective divisor obtained as the closure of $V(f) \subset U$. Then for every seminorm ξ such that f is regular at $\text{red } \xi$, we have $\|\text{div}_+(f)\|_\xi = \|f\|_\xi \leq 1$. If f is not regular at $\text{red } \xi$, then by definition $\|\text{div}_+(f)\|_\xi = \|f'\|_\xi$ where f' is an equation for $\text{div}_+(f)$ regular at $\text{red } \xi$, and so in all cases $\|\text{div}_+(f)\|_\xi \leq 1$ (equivalently, $v_\xi(\text{div}_+(f)) \geq 0$).

2.4 Quasi-monomial valuations and seminorms

Fix a model X_π of X and a point $p \in X_\pi$ such that the local ring $\mathcal{O}_{X_\pi, p}$ is regular of dimension $c \leq \dim X$. Let $z_j \in \mathcal{O}_{X_\pi, p}$, $j = 1, \dots, c$ be a regular system of parameters, and let $D_j = \text{div}_+(z_j)$. By Cohen's theorem, $\widehat{\mathcal{O}}_{X_\pi, p} \cong \kappa(p)[[z_j]]$, where $\kappa(p)$ is the residue field of $\mathcal{O}_{X_\pi, p}$, so every $f \in \mathcal{O}_{X_\pi, p}$ has a unique expansion as $f = \sum_{\alpha \in \mathbb{N}^c} a_\alpha z^\alpha$. To every weight $w = (w_j) \in \mathbb{R}_{\geq 0}^c$ one associates the monomial valuation v_w of $\mathcal{O}_{X_\pi, p}$ determined by

$$v_w \left(\sum_{\alpha \in \mathbb{N}^c} a_\alpha z^\alpha \right) = \min \{ w \cdot \alpha \mid a_\alpha \neq 0 \},$$

which extends to a valuation of $K(X)$, and determines a norm $\|f\|_w = \exp(-v_w(f))$. We may further allow $w \in (\mathbb{R}_{\geq 0} \cup \{+\infty\})^c$, in which case the prime ideal $\mathfrak{a}_{w, \infty} = (z_j \mid w_j = +\infty)$ consists of all $f \in \mathcal{O}_{X_\pi, p}$ with $v_w(f) = +\infty$, v_w determines a valuation on the quotient $\mathcal{O}_{X_\pi, p}/\mathfrak{a}_{w, \infty}$, and $\|f\|_w = \exp(-v_w(f))$ is a seminorm. In all cases, the center of v_w is the prime ideal $(z_j)_{w_j > 0}$. The map $(\mathbb{R}_{\geq 0} \cup \{+\infty\})^c \rightarrow \mathfrak{X}$ determined by $w \mapsto \|\cdot\|_w$ is a homeomorphism onto its image $\Delta_{\pi, p, z}$, and one has $\mathfrak{X} = \overline{\bigcup_{\pi, p, z} \Delta_{\pi, p, z}}$ ([2, 24, 14]).

In fact, one can more precisely describe $\mathfrak{X}$ as an inverse limit of such subsets $\Delta_{\pi, p, z}$. The map

$$\begin{aligned} ev_z : \mathfrak{X} &\rightarrow (\mathbb{R}_{\geq 0} \cup \{+\infty\})^c \\ \xi &\mapsto (-\log \|D_1\|_\xi, \dots, -\log \|D_c\|_\xi) \end{aligned} \tag{1}$$

is a continuous retraction and satisfies $\|f\|_{ev_z(\xi)} \geq \|f\|_\xi$ for every $f \in \mathcal{O}_{X_\pi, p}$. For every log-smooth pair (X_π, D) over X , call the pair (p, z) D -adapted if each $D_j = \text{div}_+(z_j)$ is one of the irreducible components of D . Then the topology of the union

$$\Delta_{\pi, D} = \bigcup_{(p, z) \text{ } D\text{-adapted}} \Delta_{\pi, p, z} \subset \mathfrak{X}$$

is the topology of a simplicial cone complex obtained by gluing the cones $\Delta_{\pi, p, z} \simeq (\mathbb{R}_{\geq 0} \cup \{+\infty\})^c$, and $\mathfrak{X}$ is the inverse limit of all such simplicial complexes via the retraction map. This is explained in all detail for the case $w \in \mathbb{R}_{\geq 0}^c$ of valuations in [14]; the generalization to $w \in (\mathbb{R}_{\geq 0} \cup \{+\infty\})^c$ and seminorms is straightforward.

Valuations v_w and seminorms $\|\cdot\|_w$ as above which are monomial on some model X_π are generally called *quasi-monomial*. It is known by [11, Proposition 2.8] that quasi-monomial

valuations can be algebraically characterized as those valuations for which Abhyankar's inequality

$$\text{tr. deg. } v + \text{rat. rk. } v \leq \dim X$$

is in fact an equality (where tr. deg. stands for the transcendence degree of the residue field of v , and rat. rk. stands for the rational rank of the value group of v). For this reason, quasi-monomial valuations are also called Abhyankar valuations in the literature.

2.5 Noether lemma

When dealing with $\mathbf{b}$ -divisors, the role of *primes* is most naturally played by those associated to divisorial valuations. The center of a divisorial valuation on every sufficiently high model is a divisor; when passing to another model, the centers are related by strict transforms and push-forwards, and thus give rise to a Weil $\mathbf{b}$ -divisor:

Notation 2.9. Let X_π be any birational model of X . Let D be a prime divisor on X_π . We will denote by $\tilde{D}$ the $\mathbf{b}$ -divisor obtained from D via pushforward and strict-transform.

A Weil $\mathbf{b}$ -divisor $\tilde{D}$ is called *prime* if it is the $\mathbf{b}$ -divisor determined by strict transforms and pushforwards of a prime divisor D on some model $\pi : X_\pi \rightarrow X$. A prime $\mathbf{b}$ -divisor is not Cartier, but it can be easily seen as a limit of Cartier $\mathbf{b}$ -divisors.

Proposition 2.10. Let v be a rank 1 valuation of $K(X)$, and let $\tilde{D}$ be a prime Weil $\mathbf{b}$ -divisor, determined by a prime divisor D on the model X_π . If v is not equivalent to the divisorial valuation w associated to D , there exists a model $\varpi : X_\varpi \rightarrow X_\pi$ such that the center of v in X_ϖ is not contained in $\tilde{D}_\varpi$.

Proof. The valuation rings of $K(X)$ strictly contained in $\mathcal{O}_w$ are valuation rings of valuations composed with w , in particular they have rank greater than one. So, if v is not equivalent to w , $\mathcal{O}_v$ is not contained in $\mathcal{O}_w$. Let $f \in \mathcal{O}_v \setminus \mathcal{O}_w$. Then

$$\mathcal{U}_f = \{\mu \text{ valuation of } K(X) \mid \mu(f) \geq 0\}$$

is open in $\mathcal{X}$, it contains v and it does not contain w . Since $\mathcal{X}$ is the limit of all models X_ϖ , there is one such model where the center of v has a neighborhood that does not contain the center of w , which is the generic point of $\tilde{D}_\varpi$. $\square$

Suppose we are given a sequence of blowups centered at smooth irreducible subvarieties,

$$X_n \xrightarrow{bl_{V_n}} X_{n-1} \xrightarrow{bl_{V_{n-1}}} \dots \xrightarrow{bl_{V_2}} X_1 \xrightarrow{bl_{V_1}} X_0 = X.$$

For $i = 1, \dots, n$, denote E_i the exceptional divisor of the blowup bl_{V_i} above the subvariety $V_i \subset X_{i-1}$, and for every effective divisor D on X , denote $\tilde{D}_i$ the strict transform of D on X_i , and $m_i(D)$ the multiplicity of $\tilde{D}_i$ along V_i .

Lemma 2.11 (Noether equality, [20, II.2]). Let D be an effective divisor on X , v a valuation of $K(X)$. For every sequence of blowups with smooth centers as above,

$$v(D) = \sum_{i=1}^n m_i(D) v(E_i) + v(\tilde{D}_n).$$

If v is a real valuation, then there is a sequence of blowups with smooth centers such that $v(\tilde{D}_n) = 0$.

Proof. Write the pullback of D to X_n as a combination of the pullbacks of the E_i and $\tilde{D}_n$. For the second claim, use the previous proposition. $\square$

2.6 Multiplier ideals and Arnold multiplicity

The asymptotic behavior of valuation ideal sheaves is conveniently controlled by so-called asymptotic multiplier ideals. We recall their definition and main properties from [16, Part Three].

Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf, and $c > 0$ a rational number. Fix a log-resolution $\pi : X' \rightarrow X$ of $\mathfrak{a}$ with $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(Z(\mathfrak{a}))$. The *multiplier ideal* $\mathcal{J}(\mathfrak{a}^c)$ associated to c and $\mathfrak{a}$ is defined as

$$\mathcal{J}(\mathfrak{a}^c) = \pi_* \mathcal{O}_{X'}(K_{X'/X} + \lceil c \cdot Z(\mathfrak{a}) \rceil)$$

where $K_{X'/X}$ stands for the relative canonical divisor and $\lceil c \cdot Z(\mathfrak{a}) \rceil$ is the least integral divisor $D \geq c \cdot Z(\mathfrak{a})$ ¹. The resulting ideal is independent of the choice of log-resolution π , it contains $\mathfrak{a}$, and the subadditivity relation

$$\mathcal{J}(\mathfrak{a}^{\ell c}) \subseteq \mathcal{J}(\mathfrak{a}^c)^\ell \quad (2)$$

holds for every $\ell \in \mathbb{N}$ [16, Theorem 9.5.20].

The *Arnold multiplicity* of an ideal $\mathfrak{a}$ at $p \in V(\mathfrak{a})$ can be defined as

$$\text{Arn}_p(\mathfrak{a}) = (\inf\{c > 0 \mid \mathcal{J}(\mathfrak{a}^c) \subset \mathfrak{m}_p\})^{-1},$$

i.e., it is the inverse of the *log-canonical threshold* (in addition to [16], see [15], [14]).

For $\mathfrak{a}_\bullet$ a graded sequence of (coherent) ideal sheaves, such as the valuation ideals we are mostly concerned with, one has $\mathcal{J}(\mathfrak{a}_m) \subseteq \mathcal{J}(\mathfrak{a}_{pm}^{1/p})$ for every natural p [16, Lemma 11.1.4]. By noetherianity then, there is a unique maximal element in the set of ideals $\{\mathcal{J}(\mathfrak{a}_{pm}^{1/p})\}_{p \in \mathbb{N}}$, which is called the m -th asymptotic multiplier ideal $\mathfrak{j}_m = \mathfrak{j}_m(\mathfrak{a}_\bullet)$. Its essential property is that for every m and every natural number ℓ ,

$$\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m} \subseteq \mathfrak{j}_m^\ell, \quad (3)$$

which is a consequence of subadditivity [16, Theorem 11.2.3]: $\mathfrak{j}_{s+t} \subseteq \mathfrak{j}_s \cdot \mathfrak{j}_t$. The Arnold multiplicity of the graded sequence is defined as

$$\text{Arn}_p(\mathfrak{a}_\bullet) := \inf_{m \in \mathbb{N}} \frac{1}{m} \text{Arn}_p(\mathfrak{a}_m) = \lim_{m \rightarrow \infty} \frac{1}{m} \text{Arn}_p(\mathfrak{a}_m).$$

The divisor $K_{X'/X} + \lceil c \cdot Z(\mathfrak{a}) \rceil$ is not necessarily X -nef, hence we will be interested in the “multiplier ideal divisor”

$$Z(\mathfrak{a}^c) := Z(\mathcal{J}(\mathfrak{a}^c)) = \text{Env}_\pi(K_{X'/X} + \lceil c \cdot Z(\mathfrak{a}) \rceil)$$

seen as a Cartier $\mathbf{b}$ -divisor. Again, for our purposes, we are mainly interested in the asymptotic counterparts; given a graded sequence of ideals $\mathfrak{a}_\bullet$, we define the m -th asymptotic multiplier ideal divisor as the Cartier $\mathbf{b}$ -divisor

$$\mathring{Z}_m(\mathfrak{a}_\bullet) := Z(\mathfrak{j}_m(\mathfrak{a}_\bullet)).$$

¹We depart from the notational convention used in [16, Definition 9.2.3] which uses the effective divisor $F = -Z(\mathfrak{a})$ and round-down; our choice looks for consistency with the usual convention when working with Nef envelopes: $Z(\mathfrak{a})$ is X -nef whereas F is anti- X -nef.

By (3), these $\mathbf{b}$ -divisors satisfy

$$Z(\mathbf{a}_m) \leq \frac{Z(\mathbf{a}_{\ell m})}{\ell} \leq \dot{Z}_m(\mathbf{a}_\bullet) \quad (4)$$

for every $m > 0$ and every natural ℓ .

Recall that a *complete* ideal is an ideal determined by the values $\mathbf{m} = (m_i)_{i \in I}$ of its elements at some set of valuations v_i

$$\mathbf{a}_\mathbf{m} = \{f \in \mathcal{O}_X \mid v_i(f) \geq m_i \forall i \in I\}.$$

This is equivalent to the ideal being integrally closed [13, Chapter 6].

Proposition 2.12. *If $\phi : X \rightarrow Y$ is a birational morphism, and $\mathbf{a} = \mathbf{a}_\mathbf{m}$ is a complete ideal, then $Z(\phi_*(\mathbf{a})) = \text{Env}_Y(Z(\mathbf{a}))$.*

Note that, since $Z(\mathbf{a})$ is a Cartier $\mathbf{b}$ -divisor determined on the normalized blowup $\pi : X_\pi \rightarrow X$ of X along $\mathbf{a}$, one has $\text{Env}_Y(Z(\mathbf{a})) = \text{Env}_{\phi \circ \pi}((Z(\mathbf{a}))_\pi)$.

Proof. Note first that $\phi_*(\mathbf{a})$ is also a valuation ideal (given by the same set of valuations on $K(Y) = K(X)$), and $\phi^*\phi_*(\mathbf{a}) \subseteq \mathbf{a}$.

Let $\pi_Y : Y_\mathbf{m} \rightarrow Y$ be the normalized blowup of Y along $\phi_*(\mathbf{a})$, and let $\pi' : X_\mathbf{m} \rightarrow X_\pi$ be a birational model such that the induced map $\phi_\mathbf{m} : X_\mathbf{m} \rightarrow Y_\mathbf{m}$ is a morphism. For simplicity write also $\pi_X = \pi \circ \pi'$.

By the definitions then, the Cartier $\mathbf{b}$ -divisors $Z(\phi_*(\mathbf{a}))$ and $Z(\mathbf{a})$ are defined on $Y_\mathbf{m}$ and $X_\mathbf{m}$ respectively, and

$$\begin{aligned} \mathbf{a} &= \pi_{X*}(\mathcal{O}_{X_\mathbf{m}}(Z(\mathbf{a}))_{X_\mathbf{m}}), \\ \phi_*(\mathbf{a}) &= \pi_{Y*}(\mathcal{O}_{Y_\mathbf{m}}(Z(\phi_*(\mathbf{a})))_{Y_\mathbf{m}}). \end{aligned}$$

We will prove that $\phi_\mathbf{m}^*(Z(\phi_*(\mathbf{a}))_{Y_\mathbf{m}}) = \text{Env}_{\pi_Y \circ \phi_\mathbf{m}}(Z(\mathbf{a})_{X_\mathbf{m}})$. This will show that the two $\mathbf{b}$ -divisors in the claim are equal, as they are defined on $X_\mathbf{m}$ and agree there, by commutativity of the diagram

$$\begin{array}{ccc} X_\mathbf{m} & \xrightarrow{\phi_\mathbf{m}} & Y_\mathbf{m} \\ \pi' \downarrow & & \downarrow \pi_Y \\ X_\pi & & \\ \pi \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & Y \end{array}$$

Since $\phi^*\phi_*(\mathbf{a}) \subseteq \mathbf{a}$, we deduce that $\phi_\mathbf{m}^*(Z(\phi_*(\mathbf{a}))_{Y_\mathbf{m}}) \leq Z(\mathbf{a})_{X_\mathbf{m}}$. Since $Z(\phi_*(\mathbf{a}))_{Y_\mathbf{m}}$ is Y -nef, it follows by [4, Corollary 2.13] that $\phi_\mathbf{m}^*(Z(\phi_*(\mathbf{a}))_{Y_\mathbf{m}}) \leq \text{Env}_{\pi_Y \circ \phi_\mathbf{m}}(Z(\mathbf{a})_{X_\mathbf{m}})$.

To see the converse inequality $\phi_\mathbf{m}^*(Z(\phi_*(\mathbf{a}))_{Y_\mathbf{m}}) \geq \text{Env}_{\pi_Y \circ \phi_\mathbf{m}}(Z(\mathbf{a})_{X_\mathbf{m}})$, it will be enough to see that, for every Cartier divisor D on Y such that $(\phi \circ \pi_X)^*(D) + \text{Env}_{\pi_Y \circ \phi_\mathbf{m}}(Z(\mathbf{a})_{X_\mathbf{m}}) \geq 0$, the inequality $(\phi \circ \pi_X)^*(D) + \phi_\mathbf{m}^*(Z(\phi_*(\mathbf{a}))_{Y_\mathbf{m}}) \geq 0$ is also satisfied.

Now for every $f \in \mathcal{O}_Y$ such that $(\phi \circ \pi_X)^*(\text{div}(f)) + \text{Env}_{\pi_Y \circ \phi_\mathbf{m}}(Z(\mathbf{a})_{X_\mathbf{m}}) \geq 0$ we get $(\phi \circ \pi_X)^*(\text{div}(f)) + Z(\mathbf{a})_{X_\mathbf{m}} \geq 0$, so $f \in \phi_*\mathbf{a}$ which implies $v_i(\phi^*f) \geq m_i$ for all $i \in I$ and therefore $v_i(f) \geq m_i$. Thus $(\pi_Y)^*(\text{div}(f)) + Z(\phi_*(\mathbf{a}))_{Y_\mathbf{m}} \geq 0$ and $(\phi \circ \pi_X)^*(\text{div}(f)) + \phi_\mathbf{m}^*(Z(\phi_*(\mathbf{a}))_{Y_\mathbf{m}}) \geq 0$ as claimed. $\square$

Corollary 2.13. *If $X \rightarrow Y$ is a birational morphism between smooth varieties, and $\mathbf{a}_\bullet$ is a graded sequence of valuation ideal sheaves on X , then $Z(\phi_*(\mathbf{a}_m)) = \text{Env}_Y(Z(\mathbf{a}_m))$ and $\check{Z}_m(\phi_*(\mathbf{a}_\bullet)) \geq \text{Env}_Y(\check{Z}_m(\mathbf{a}_\bullet))$ for every m .*

Proof. The equality $Z(\phi_*(\mathbf{a}_m)) = \text{Env}_Y(Z(\mathbf{a}_m))$ is immediate from the proposition, as $\mathbf{a}_m$ is a complete ideal for every m . For multiplier ideal divisors, we have

$$\begin{aligned} \check{Z}_m(\phi_*(\mathbf{a}_\bullet)) &= Z(\mathbf{j}_m(\phi_*(\mathbf{a}_\bullet))) = \text{Env}_{\phi \circ \pi}(K_{X_m/Y} + \lceil (1/p) \cdot Z(\phi_*(\mathbf{a}_{pm}))_{X_m} \rceil) = \\ &= \text{Env}_{\phi \circ \pi}(Z(\mathbf{j}_m(\mathbf{a}_\bullet)) + \pi^*(K_{X/Y})). \end{aligned}$$

Since $K_{X/Y} \geq 0$, it follows that

$$\check{Z}_m(\phi_*(\mathbf{a}_\bullet)) \geq \text{Env}_{\phi \circ \pi}(Z(\mathbf{j}_m(\mathbf{a}_\bullet))) = \text{Env}_Y(\check{Z}_m(\mathbf{a}_\bullet))$$

as claimed. $\square$

3 The $\mathbf{b}$ -divisor associated to a seminorm

In this section we define the map $D : \mathfrak{X} \rightarrow \text{Div } \mathcal{X}$ which is central in our work and we study its formal properties. In later sections it will be used to connect valuative properties of divisors with positivity properties.

3.1 X -nef $\mathbf{b}$ -divisor associated to an analytic point $\xi \in \mathfrak{X}$

Recall from section 2.1 that we associate to every seminorm $\xi \in \mathfrak{X}$ a real filtration by ideal sheaves $(\mathbf{a}_{\xi,m} \subset \mathcal{O}_X)_{m \in \mathbb{R}}$ defined by

$$\mathbf{a}_{\xi,m}(U) = \begin{cases} \{f \in \mathcal{O}_X(U), \log \|f\|_\xi \leq -m\} & \text{if } \text{red}(\xi) \in U, \\ \mathcal{O}_X(U) & \text{otherwise.} \end{cases}$$

Consider the corresponding Cartier divisors $Z(\mathbf{a}_{\xi,m})$ as defined by Boucksom–de-Fernex–Favre [4] (see section 2.2).

Proposition 3.1. *The limit $\lim_{m \rightarrow \infty} (1/m)Z(\mathbf{a}_{\xi,m})$ exists as an $\mathbb{R}$ -Weil $\mathbf{b}$ -divisor.*

Proof. This proposition is a slight generalization of [4, Proposition 2.1], which deals with filtrations indexed by $\mathbb{N}$, to our setting in which the filtration is indexed by $m \in \mathbb{R}_{>0}$. In this setting, Example 2.7 of [14] proves that for every divisorial valuation ord_E , the limit $\lim_{m \rightarrow \infty} (1/m) \text{ord}_E Z(\mathbf{a}_{\xi,m})$ exists; this is enough to show the result (for the topology of pointwise convergence). $\square$

Notation 3.2. *In the sequel, we keep the notation D_ξ (or D_ξ^X if need be) for the Weil $\mathbf{b}$ -divisor $\lim_{m \rightarrow \infty} (1/m)Z(\mathbf{a}_{\xi,m})$.*

Remark 3.3. *Note that, even though valuations and filtrations by ideals indexed by ordered groups larger than $\mathbb{R}$ have been studied and are relevant in other contexts, the result does not generalize to them. In this sense, Proposition 3.1 is the most general version possible.*

To the best of our knowledge, the $\mathbf{b}$ -divisor D_ξ has not been explicitly introduced in the literature. However, Jonsson and Mustařa do use the notation $\text{ord}_E(\mathbf{a}_\bullet)$ for what we call

$\text{ord}_E(D_\xi)$ and so they implicitly deal with D_ξ , as the data of all these numbers is equivalent to the datum of D_ξ . Therefore several results in [14] useful to us can be aptly expressed in the language of $\mathbf{b}$ -divisors. We shall give such rephrasings in the sequel.

Proposition 3.4 ([14, Proposition 2.13]). *For every ξ ,*

$$D_\xi = \lim_{m \rightarrow \infty} \frac{\mathring{Z}_m(\mathbf{a}_{\xi, \bullet})}{m}.$$

Proposition 3.5. *For every $m \geq 0$, the following inequalities of $\mathbf{b}$ -divisors hold:*

$$Z(\mathbf{a}_{\xi, m}) \leq mD_\xi \leq \mathring{Z}_m(\mathbf{a}_{\xi, \bullet}) \leq 0.$$

Proof. Immediate from (4) taking limits for $\ell \rightarrow \infty$. □

Remark 3.6. *If $\xi = \theta_X$ is the trivial norm on X , then the filtration $\mathbf{a}_{\xi, \bullet}$ is also trivial, i.e., $\mathbf{a}_{\xi, m} = 0 \ \forall m$, and D_ξ is not defined. We agree to write $D_{\theta_X} = -\infty$, meaning that $D_{\theta_X} < D$ for every $\mathbf{b}$ -divisor D on X .*

Remark 3.7. *By definition, for every $\ell \in \mathbb{R}_{>0}$ we have $\mathbf{a}_{\xi^\ell, m} = \mathbf{a}_{\xi, m/\ell}$, so*

$$D_{\xi^\ell} = \lim_{m \rightarrow \infty} \frac{1}{m} Z(\mathbf{a}_{\xi, m/\ell}) = \frac{1}{\ell} D_\xi.$$

Example 3.8. *Let $\tilde{P}$ be a prime Weil $\mathbf{b}$ -divisor on X , determined by a prime divisor $P_\pi \subset X_\pi$. Let v_P, θ_P stand for the corresponding divisorial valuation and norm. Then the associated divisor is a nef envelope: $D_{\theta_P} = \text{Env}_\pi(-P_\pi) = \text{Env}_X(-\tilde{P})$ (see Definition 2.4). Thus the $\mathbf{b}$ -divisor associated to a seminorm generalizes the nef envelope in the case of prime divisors.*

Proposition 3.9. *Let P be a prime $\mathbf{b}$ -divisor on X , determined by an irreducible divisor $P_\pi \subset X_\pi$ on a suitable model. Then for every $\xi \in \mathfrak{X}$,*

1. *If $\xi \in \mathfrak{X}$ is not a norm, or equivalently, if $\xi = (V_\xi, v_\xi)$ with V_ξ a proper subvariety of X , then $D_\xi = 0$.*
2. $\text{ord}_P(D_\xi) = -\inf_{f \in \mathcal{O}_{X, \text{red}(\xi)}} \frac{\text{ord}_{P_\pi}(\pi^*(f))}{-\log \|f\|_\xi}.$
3. *If $\text{ord}_{P_\pi}(D_{\xi, \pi}) \neq 0$, then $\pi(P_\pi) \subseteq \overline{\text{red}(\xi)} \subset X$.*
4. $\text{ord}_{\tilde{P}_\pi}(D_{\xi, X}) \geq -\frac{1}{v_\xi(P_\pi)}.$

Proof. To prove the first claim, assume $\xi = (V_\xi, v_\xi)$ with V_ξ a proper subvariety of X and let U be an affine open neighborhood of the center $\text{red}(\xi)$. Note that, since the center of $\xi = (V_\xi, v_\xi)$ belongs to V_ξ , we have $U \cap V_\xi \neq \emptyset$. Hence there exist nonzero f in $\mathcal{I}_V(U) \subset \mathcal{O}_X(U)$, the ideal of V in U , and such an f satisfies $\|f\|_\xi = 0$. Denote D the effective Cartier divisor on X obtained as the closure of $Z(f)$. Now for every m , $f \in \mathbf{a}_{\xi, m}(U)$ and hence $\pi_m^*(D) + Z(\mathbf{a}_{\xi, m}) \geq 0$. Taking limits we obtain $\overline{D} + mD_\xi \geq 0$, or $0 \geq D_\xi \geq -(1/m)\overline{D}$ for every m , and hence $D_\xi = 0$.

The second claim, in the case that ξ is not a norm, follows immediately from the first one, as the infimum in the formula is computed as zero by those f with $\|f\|_\xi = 0$. On

the other hand, if ξ is a norm, then because of multiplicativity of the filtration, the second claim is just a rewriting of the definition.

For every natural number m and any f in the ideal

$$\pi_* \mathcal{O}_{X_\pi, P_\pi} \left(-\frac{m}{v_\xi(P_\pi)} P_\pi \right) \subset \mathcal{O}_{X, \text{red}(\xi)},$$

$v_\xi(f) \geq m$, so this ideal is contained in $\mathfrak{a}_{\xi, m}$; the two last claims follow. $\square$

3.2 Characterization of nonvanishing D_ξ

The $\mathbf{b}$ -divisor D_ξ can be trivial for non-trivial seminorms, as we saw in part 1 of 3.9; in this subsection we show further examples of the phenomenon and give a characterization of the underlying seminorms in terms of the Arnold multiplicity, inspired in the work of Jonsson-Mustață [14]. In section 3.3 we will show that in many cases of interest, ξ can be recovered from D_ξ , and we conjecture that this happens whenever $D_\xi \neq 0$.

Remark 3.10. *For every $\xi \in \mathfrak{X}$ distinct from the trivial norm, ξ^∞ is the trivial seminorm on a strict subvariety of X , and so it is not a norm; hence Proposition 3.9, part 1 implies that the equality $D_{\xi^\ell} = \frac{1}{\ell} D_\xi$ (Remark 3.7) holds for $\ell = \infty$ as well. On the other hand, for $\ell = 0$, recall that the power ξ^0 is the trivial seminorm on $\ker(\xi)$. If ξ is not a norm, all involved divisors are trivial by the same proposition; if it is a norm then by definition $D_{\xi^0} = -\infty$, which can be seen as the limit of $\frac{1}{\ell} D_\xi$ when ℓ tends to zero by virtue of D_ξ being anti-effective.*

Remark 3.11. *The converse to Proposition 3.9, part 1 is not true, even on surfaces. To see an example, let (x, y) be a system of parameters of the local ring $\mathcal{O}_{X, p}$ where X is a smooth projective algebraic surface. The formal power series ring $k[[x, y]]$ can be identified with the completion of $\mathcal{O}_{X, p}$. Let $s(x) \in k[[x]]$ be a non-algebraic series, so that there is no polynomial $P(x, y)$ with $P(x, s(x)) = 0$ (since $K(X)$ is a finite algebraic extension of $k(x, y)$, there is no $f \in \mathcal{O}_{X, p}$ with $f(x, s(x)) = 0$ either). Consider the valuation on $\mathcal{O}_{X, p}$ determined by $v(f) = \text{ord}_x(f(x, s(x)))$, which extends to a valuation of $K(X)$ and determines a multiplicative norm $\|f\|_\xi = \exp(-v(f))$ and a point ξ of $\mathfrak{X}$. It is not hard to check that $\ker(\xi)$ is the generic point of X , $\text{red}(\xi) = p$, and $D_\xi = 0$.*

Indeed, consider every truncation f_k of $y - s(x)$ at order k ; it is obvious that $\lim \|f_k\|_\xi = 0$ whereas it is not hard to see that for every fixed P , $\text{ord}_P(f_k)$ stabilizes for $k \gg 0$. Thus for each ε , there exists $k \gg 0$ such that $\|f_k\|_\xi \leq \exp(-\text{ord}_P(f_k)/\varepsilon)$. Therefore, for every prime $\mathbf{b}$ -divisor P , $\text{ord}_P(D_\xi) = 0$ by Proposition 3.9.

A complete characterization of the valuations which lead to zero or nonzero D_ξ can be given using the Arnold multiplicity.

Proposition 3.12. *If $\xi \in \mathfrak{X}$ is a norm, or equivalently, if $\xi = (X, v_\xi)$, then $D_\xi \neq 0$ if and only if $\text{Arn}_p(\mathfrak{a}_{\xi, \bullet}) > 0$ for $p = \text{red}_X(\xi)$.*

Proof. By definition, for each prime divisor P on some model X_π above p , $\text{ord}_P(D_\xi)$ equals Jonsson-Mustață's asymptotic order $\text{ord}_P(\mathfrak{a}_{\xi, \bullet})$. Then the result is a particular case of [14, Corollary 6.10]. $\square$

Corollary 3.13. *If v is a valuation of $K(X)$ such that the log-discrepancy $A_X(v)$ is finite, then the associated norm $\xi = (X, v_\xi)$ satisfies $D_\xi \neq 0$.*

Proof. Let $p = \text{red}_X(\xi)$. By [14, Proposition 5.10], for every $m > 0$ it holds

$$\mathfrak{a}_{\xi, m} \subseteq \mathfrak{m}_p^{\lceil m/A(v) \rceil}$$

so $\text{ord}_{\tilde{E}_p}(D_\xi) \leq -1/A(v) < 0$ where $\tilde{E}_p$ is the prime divisor of the exceptional divisor of blowing up p . $\square$

Note that this applies to Abhyankar (or quasi-monomial) valuations [14, Proposition 5.1], so we get the following:

Corollary 3.14. *If v is an Abhyankar valuation of $K(X)$ and $\xi = (X, v_\xi)$ is the associated norm, then $D_\xi \neq 0$.*

3.3 Nonvanishing and $\mathbf{b}$ -divisorial valuations

If v is a valuation on X , and $\xi = (X, v)$ is the associated norm, the valuation of a rational function, or more generally of a Cartier divisor D , can often be replaced by the “order of vanishing” along D_ξ , i.e., the number

$$\text{ord}_{D_\xi}(D) := \sup\{t \in \mathbb{R} \mid \overline{D} + tD_\xi \geq 0\}.$$

This will provide a translation of valuative properties of Cartier divisors into positivity properties, generalizing the behavior of divisorial valuations.

Proposition 3.15. *Let D be an effective Cartier divisor on X , and $\xi \in \mathfrak{X}$. Then*

$$\overline{D} - \log\|D\|_\xi \cdot D_\xi \geq 0.$$

In particular, $-\log\|D\|_\xi \leq \text{ord}_{D_\xi}(D)$.

Proof. The claim is empty if $D_\xi = 0$, so by the previous proposition, we can assume that $\xi = (X, v_\xi)$ is given by a valuation v_ξ of $K(X)$; write $m = -\log\|D\|_\xi = v_\xi(D) \in \mathbb{R}$.

For every affine $U \subset X$, and every equation $f \in \mathcal{O}_X(U)$ of $D \cap U$, we have $f \in \mathfrak{a}_{\xi, m}(U)$, and therefore on the normalized blowup X_{π_m} , $\overline{D}_{\pi_m} + Z(\mathfrak{a}_{\xi, m})_{\pi_m}$ is effective. Thus $\overline{D} + Z(\mathfrak{a}_{\xi, m}) \geq 0$ as a $\mathbf{b}$ -divisor, and since $Z(\mathfrak{a}_{\xi, m}) \leq mD_\xi$, this implies that $\overline{D} + mD_\xi \geq 0$. $\square$

Definition 3.16. *Let v be a valuation on X , and $\xi = (X, v)$ the corresponding norm. We say that the valuation v is $\mathbf{b}$ -divisorial if $v(D) = \text{ord}_{D_\xi}(D)$ for every divisor D on X . Equivalently, v is $\mathbf{b}$ -divisorial if $v(f) = \text{ord}_{D_\xi}(\text{div}(f))$ for every $f \in K(X)$.*

Proposition 3.17. *If v is a divisorial valuation, then it is $\mathbf{b}$ -divisorial.*

Proof. We can assume that $v = \text{ord}_P$ where P is a prime divisor in some model $X_\pi \xrightarrow{\pi} X$, and then $D_{\theta_P} = \text{Env}_\pi(-P_\pi)$ as noted above. Therefore, the inequality of $\mathbb{R}$ - $\mathbf{b}$ -divisors $\overline{D} + tD_{\theta_P} \geq 0$ implies $\pi^*(D) - tP_\pi \geq \pi^*(D) + t(D_{\theta_P})_\pi \geq 0$ and $v(D) \geq t$. Therefore $v(D) \geq \text{ord}_{D_\xi}(D)$, and by 3.15 $v(D) = \text{ord}_{D_\xi}(D)$ for every D . Thus v is $\mathbf{b}$ -divisorial. $\square$

Proposition 3.18. *D_ξ is a nonzero Cartier $\mathbf{b}$ -divisor if and only if ξ is a norm and v_ξ is divisorial.*

Proof. If ξ is a norm and v_ξ is divisorial then D_ξ is Cartier and nonzero as shown in Example 3.8.

Let us now prove the converse, so assume D_ξ is Cartier and nonzero. By Proposition 3.9, part 1, since D_ξ is nonzero, ξ is a norm associated to a valuation $v = v_\xi$ of $K(X)$, and since D_ξ is Cartier, it is meaningful to evaluate $v(D_\xi)$. We claim that $v(D_\xi) = -1$. If D is an effective divisor on X , the inequality of $\mathbb{R}$ -**b**-divisors $\bar{D} + tD_\xi \geq 0$ implies $v(D) \geq tv(-D_\xi)$. Therefore $v(D) \geq \text{ord}_{D_\xi}(D)v(-D_\xi)$, and since $\text{ord}_{D_\xi}(D) \geq v(D)$ by Proposition 3.15, we conclude that $v(D_\xi) \geq -1$, with an equality implying that $v = \text{ord}_{D_\xi}$ (so ord_{D_ξ} is a valuation on $\mathcal{O}_{X, \text{red}(\xi)}$). Let X_π be a model where the Cartier divisor D_ξ is determined, and write $T = (D_\xi)_{X_\pi}$. If $v(D_\xi) = -1$ then we have shown that ord_{D_ξ} , the order of vanishing along $-T$, is a valuation on $\mathcal{O}_{X, \text{red}(\xi)}$; this implies that there is a prime component P of T with $\text{ord}_{D_\xi}(\pi^*f) = \text{ord}_P(\pi^*f)/\text{ord}_P(-T)$ for every $f \in \mathcal{O}_{X, \text{red}(\xi)}$, and in particular v is a divisorial valuation.

It remains to prove that indeed $v(D_\xi) = -1$, so fix a model X_π where D_ξ is determined, and let $T = (D_\xi)_{X_\pi}$, $V = \text{red}_{X_\pi}(\xi)$, and let ξ_π the seminorm in $\mathfrak{X}_\pi$ determined by the valuation v of $K(X_\pi) = K(X)$. By the birational nature of D_ξ (4.1) $D_\xi = \text{Env}_{\mathcal{X}}(D_{\xi_\pi})$, and in particular $D_{\xi_\pi} \neq 0$.

If V is a divisor we are done, so assume it is not and let E_V be the exceptional divisor of blowing up V . We know that D_{ξ_π} is supported on divisors above $\text{red}_{X_\pi}(\xi)$, so in particular $\alpha := \text{ord}_{E_V}(D_{\xi_\pi}) < 0$, and by definition

$$\alpha = \lim_{m \rightarrow \infty} \frac{\text{ord}_{E_V} Z(\mathbf{a}_{\xi_\pi, m})}{m}.$$

The fact that D_ξ is Cartier determined on X_π means that

$$\lim_{m \rightarrow \infty} \frac{\text{ord}_{E_V} Z(\mathbf{a}_{\xi, m})}{m} = \text{ord}_{E_V} D_\xi = \text{ord}_{E_V} \bar{T}.$$

Since $\mathbf{a}_{\xi, m} = \pi_* \mathbf{a}_{\xi_\pi, m}$, we get $Z(\mathbf{a}_{\xi, m}) \leq Z(\mathbf{a}_{\xi_\pi, m})$ and it follows that

$$\lim_{m \rightarrow \infty} \frac{\text{ord}_{E_V}(Z(\mathbf{a}_{\xi_\pi, m}) - m\bar{T})}{m} \geq \lim_{m \rightarrow \infty} \frac{\text{ord}_{E_V}(Z(\mathbf{a}_{\xi, m}) - m\bar{T})}{m} = 0$$

Let $\pi' : X_{\pi'} \rightarrow X_\pi$ be the normalized blowup of $\mathbf{a}_{\xi_\pi, m}$. On that variety, we have

$$\mathcal{O}_{X_{\pi'}}(Z(\mathbf{a}_{\xi_\pi, m}) - \lfloor m\pi'^*(T) \rfloor) = (\mathbf{a}_{\xi_\pi, m} : \mathcal{O}_{X_\pi}(\lfloor mT \rfloor)) \cdot \mathcal{O}_{X_{\pi'}}$$

and the definition of valuation ideals gives

$$\mathbf{a}_{\xi_\pi, m} : \mathcal{O}_{X_\pi}(\lfloor mT \rfloor) = \mathbf{a}_{\xi_\pi, m+v(\lfloor mT \rfloor)}.$$

Therefore

$$\begin{aligned} 0 &\leq \lim_{m \rightarrow \infty} \frac{\text{ord}_{E_V}(Z(\mathbf{a}_{\xi_\pi, m}) - m\bar{T})}{m} = \lim_{m \rightarrow \infty} \frac{\text{ord}_{E_V} Z(\mathbf{a}_{\xi_\pi, m+v(\lfloor mD_\xi \rfloor)})}{m} = \\ &\lim_{m \rightarrow \infty} \frac{Z(\mathbf{a}_{\xi_\pi, m(1+v(D_\xi))})}{m} = (1 + v(D_\xi)) \lim_{m \rightarrow \infty} \frac{Z(\mathbf{a}_{\xi_\pi, m})}{m} = (1 + v(D_\xi))\alpha. \end{aligned}$$

Since $\alpha < 0$ and $1 + v(D_\xi) \geq 0$, it follows that $1 + v(D_\xi) = 0$ as claimed. $\square$

Conjecture 3.19. ξ is $\mathbf{b}$ -divisorial if and only if $D_\xi \neq 0$.

In all cases with $D_\xi \neq 0$ that we are able to compute, there is actually an equality $v_\xi = \text{ord}_{D_\xi}$ (see section 4 below).

Example 3.20. Let $X = \mathbb{A}^2$ with coordinates (x, y) and maximal ideal $\mathfrak{m} = (x, y)$. It is not hard to construct a sequence of quasi-monomial valuations whose log discrepancies diverge.

- For $t \geq 1$, define v_t by:

$$v_t(x) = 1, \quad v_t(y) = t, \quad (5)$$

extended to $\mathcal{O}_{X,0}$ via:

$$v_t\left(\sum c_{a,b}x^a y^b\right) = \min\{a + tb \mid c_{a,b} \neq 0\}. \quad (6)$$

These satisfy:

- $v_t(\mathfrak{m}) = \min(1, t) = 1$ for all t ,
- $A(v_t) = 1 + t \xrightarrow{t \rightarrow \infty} +\infty$,
- The pointwise limit $v_\infty := \lim_{t \rightarrow \infty} v_t$ satisfies:

$$v_\infty(f) = \text{ord}_x(f) = \max\{k \mid x^k \text{ divides } f\}. \quad (7)$$

This has:

- $A(v_\infty) = +\infty$ (non-quasi-monomial), but
- $v_\infty(\mathfrak{m}) = 0 \neq 1$.

This shows that to have a well defined limit in the Berkovich space it is necessary to normalize the valuation.

- For example:

$$v'_t := \frac{v_t}{t}, \quad \text{i.e., } v'_t(x) = t^{-1}, \quad v'_t(y) = 1. \quad (8)$$

Now:

- $v'_t(\mathfrak{m}) = t^{-1} \xrightarrow{t \rightarrow \infty} 0$,
- $A(v'_t) = t^{-1}(1 + t) \xrightarrow{t \rightarrow \infty} 1$,
- So that in this case $\lim_{t \rightarrow \infty} \frac{A(v'_t)}{t} = 0$.

Proposition 3.21. Let v be a valuation of $K(X)$ and $\xi = (X, v)$ the associated norm. Suppose that there exists a sequence of models X_m and regular systems of parameters $z_j^m \in \mathcal{O}_{X_m, \text{red}_{X_m}(v)}$, $j = 1, \dots, c_m$

1. $Z(\mathfrak{a}_{\xi, m})$ and $\hat{Z}_m(\mathfrak{a}_\bullet)$ are determined by SNC divisors on X_m for every m ,
2. $-Z(\mathfrak{a}_{\xi, m})$ and $-\hat{Z}_m(\mathfrak{a}_\bullet)$ are locally given by monomial equations in z^m at $\text{red}_{X_m}(v)$,

$$3. \lim_{m \rightarrow \infty} \frac{A(ev_{z^m}(v))}{m} = 0,$$

where $ev_{z^m} : \mathfrak{X} \rightarrow (\mathbb{R}_{\geq 0} \cup \{+\infty\})^c$ is the retraction associated to the system of parameters z^m and the model X_m .

Then $v = \text{ord}_{D_\xi}$, i.e., v is **b**-divisorial.

Proof. Let $p = \text{red}_X(\xi)$ and let (X_m, z^m) be a sequence of models and regular sequences satisfying the hypotheses. Due to Proposition 3.15, we only need to prove that for every $f \in \mathcal{O}_{X,p}$, $v(f) \geq \text{ord}_{D_\xi}(f)$. By Proposition 3.5, for every $m \geq 0$ the inequality $\text{ord}_{\dot{Z}_m(\mathfrak{a}_{\xi,\bullet})}(f) \geq \text{ord}_{D_\xi}(f)/m$ is satisfied, and hence

$$v(f) \geq \frac{\text{ord}_{D_\xi}(f)}{m} v(-\dot{Z}_m(\mathfrak{a}_{\xi,\bullet})).$$

So the result will follow by proving that $\liminf v(-\dot{Z}_m(\mathfrak{a}_{\xi,\bullet}))/m \geq 1$.

Now we use the sequence of retractions ev_{z^m} ; since for every m the divisors $Z(\mathfrak{a}_{\xi,m})$ and $\dot{Z}_m(\mathfrak{a}_\bullet)$ are determined by monomials on X_m , the definition of the retraction $ev_{z^m}(v)$ gives that

$$\frac{v(-\dot{Z}_m(\mathfrak{a}_{\xi,\bullet}))}{m} = \frac{ev_{z^m}(v)(-\dot{Z}_m(\mathfrak{a}_{\xi,\bullet}))}{m}$$

which due to [14, 6.2] is bounded below by

$$\frac{ev_{z^m}(v)(-Z(\mathfrak{a}_{\xi,m})) - A(ev_{z^m}(v))}{m} = \frac{v(-Z(\mathfrak{a}_{\xi,m})) - A(ev_{z^m}(v))}{m} = 1 - \frac{A(ev_{z^m}(v))}{m},$$

and the claim follows. $\square$

Definition 3.22. *The hypothesis in the preceding Proposition will be useful later on, hence we give a name to it. A valuation v has sublinear log-discrepancy if there is a sequence of models X_m satisfying properties 1-3 in Proposition 3.21. Naturally, if v has bounded log-discrepancy then it has sublinear log-discrepancy:*

Theorem 3.23. *If $A(v) < \infty$ then v has sublinear log-discrepancy. In particular v is **b**-divisorial.*

Proof. We only need to show that there exists a sequence of models X_m and regular systems of parameters z^m as in the previous proposition. Let X_m be any model where $Z(\mathfrak{a}_{\xi,m})$ and $\dot{Z}_m(\mathfrak{a}_\bullet)$ are determined by SNC Cartier divisors, and z^m any regular system of parameters at $\text{red}_{X_m}(v)$ such that $-Z(\mathfrak{a}_{\xi,m})$ and $-\dot{Z}_m(\mathfrak{a}_\bullet)$ are given by monomials (which can be done as they are effective SNC divisors). By definition of the log discrepancy, $A(ev_{z^m}(v)) \leq A(v)$ for each model, so $\lim A(ev_{z^m}(v))/m = 0$ as needed. $\square$

Corollary 3.24. *If v is an Abhyankar valuation, then it is **b**-divisorial.*

Proof. This can be proved directly using the fact proved by Ein–Lazarsfeld–Smith in [11] that if $v = v_\xi$ is Abhyankar, then there exists k (fixed for a given v) such that

$$\mathfrak{a}_{\xi,m}^\ell \subseteq \mathfrak{a}_{\xi,\ell m} \subseteq \mathfrak{a}_{\xi,m-k}^\ell$$

for every $m \in \mathbb{R}_{\geq 0}$ and every $\ell \in \mathbb{N}$.

Alternatively, since by [14, Proposition 5.1], Abhyankar valuations have finite log-discrepancy, it follows from the previous Theorem. $\square$

Having sublinear log-discrepancy is a technical hypothesis which simplifies the proofs of some statements which we conjecture to hold for all $\mathbf{b}$ -divisorial valuations.

Example 3.25. *Let v be a valuation with sublinear log-discrepancy, and let $\mathfrak{j}_m(\mathbf{a}_\bullet)$ be the subadditive sequence of asymptotic multiplier ideals. Then $\lim v(\mathfrak{j}_m(\mathbf{a}_\bullet))/m = 1$. If v has bounded log-discrepancy, this was proved by Jonsson and Mustař in [14, 6.2]. As seen in the proof of Proposition 3.21, their computation can be adapted to bound*

$$\frac{v(\mathfrak{j}_m(\mathbf{a}_\bullet))}{m} = \frac{v(-\mathring{Z}_m(\mathbf{a}_{\xi,\bullet}))}{m}$$

in the sublinear case as well, giving the desired limit.

Lemma 3.26. *Let $\xi \in \mathfrak{X}$ be a seminorm, and let $\tilde{P}$ be a prime $\mathbf{b}$ -divisor. Then*

$$\text{ord}_{\tilde{P}}(D_\xi) = \sup\{\text{ord}_{\tilde{P}}(D_{ev_z(\xi)})\}$$

where the supremum runs over all regular systems of parameters z in all $\mathcal{O}_{X_\pi,p}$ for all models X_π .

Proof. First observe that if ξ is not a norm, i.e., $\ker(\xi)$ is (the generic point of) a proper subvariety V , then for every smooth point p_0 of V and every system of parameters z with respect to which V is locally monomial, $\ker(ev_z(\xi)) = V$, so

$$\text{ord}_{\tilde{P}}(D_\xi) = 0 = D_{ev_z(\xi)} = \sup_z \{\text{ord}_{\tilde{P}}(D_{ev_z(\xi)})\}$$

by Proposition 3.9, part 1. Thus in the rest of the proof we may assume that ξ is a norm, corresponding to a valuation v_ξ on $K(X)$.

The inequality $\|f\|_\xi \leq \|f\|_{ev_z(\xi)}$ holds for all $f \in \mathcal{O}_{X,p}$ by [14, Lemma 4.7], therefore $\mathfrak{a}_{\xi,m} \supseteq \mathfrak{a}_{ev_z(\xi),m}$ for every m and so $D_\xi \geq D_{ev_z(\xi)}$. Therefore $\text{ord}_{\tilde{P}}(D_\xi) \geq \sup\{\text{ord}_{\tilde{P}}(D_{ev_z(\xi)})\}$.

We will show that a strict inequality

$$\text{ord}_{\tilde{P}}(D_\xi) > \sup_z \{\text{ord}_{\tilde{P}}(D_{ev_z(\xi)})\} \tag{9}$$

would lead to a contradiction. By Proposition 3.9,

$$\text{ord}_P(D_\xi) = - \inf_{f \in \mathcal{O}_{X,\text{red}(\xi)}} \frac{\text{ord}_{P_\pi}(\pi^*(f))}{v_\xi(f)} = \sup_{f \in \mathcal{O}_{X,\text{red}(\xi)}} \frac{-\text{ord}_{P_\pi}(\pi^*(f))}{v_\xi(f)}.$$

Thus, the strict inequality (9) means that there exist $f \in \mathcal{O}_{X,\text{red}_X(\xi)}$ with

$$\frac{-\text{ord}_{P_\pi}(\pi^*(f))}{v_\xi(f)} > \sup_z \{\text{ord}_{\tilde{P}}(D_{ev_z(\xi)})\}$$

Now, since $v_\xi(f) = \lim v_{ev_z(\xi)}(f)$, it follows that there exists some z_0 with

$$\frac{-\text{ord}_{P_\pi}(\pi^*(f))}{v_{ev_{z_0}(\xi)}(f)} > \sup_z \{\text{ord}_{\tilde{P}}(D_{ev_z(\xi)})\} = \sup_z \sup_{g \in \mathcal{O}_{X,\text{red}(\xi)}} \frac{-\text{ord}_{P_\pi}(\pi^*(g))}{v_{ev_z(\xi)}(g)},$$

a contradiction. □

4 Continuity properties of the valuative divisor D_ξ

4.1 Birational nature of the valuative divisor D_ξ

If $\phi : X \rightarrow Y$ is a generically finite morphism and $\xi \in \mathfrak{X}$, it is then natural to consider $\phi_*(\xi)$, the seminorm defined by pulling back functions, where $\|f\|_{\phi_*\xi} = \|\phi^*(f)\|_\xi$.

Proposition 4.1. *Let $\phi : X \rightarrow Y$ be a birational morphism and let $\xi \in \mathfrak{X}$, then*

$$D_{\phi_*\xi} = \text{Env}_Y(\phi_*D_\xi).$$

Remark 4.2. *Note that, since we are assuming that ϕ is birational, it induces a natural isomorphism of Riemann-Zariski spaces $\mathcal{X} \cong \mathcal{Y}$, and $\mathbf{b}$ -divisors in $\text{Div } \mathcal{Y}$ and $\text{Div } \mathcal{X}$ are also naturally identified via the isomorphism ϕ^* which “forgets” the traces of a $\mathbf{b}$ -divisor on all models that do not dominate X . The inverse of ϕ^* consists in taking pushforward to those models, and so it coincides with the pushforward ϕ_* of $\mathbf{b}$ -divisors, defined in [4, Section 1.5] (see especially Proposition 1.15). Because of these identifications, in the sequel we will write $\text{Env}_Y(D_\xi) := \text{Env}_Y(\phi_*D_\xi)$ for brevity, as it can not possibly lead to any confusion. The nef envelope operators Env_X and Env_Y do differ however, because the conditions of being X -nef and Y -nef differ; thus, D_ξ is X -nef by definition but need not be Y -nef, and so will generally differ from $\text{Env}_Y D_\xi$.*

Proof. Let $\xi \in \mathfrak{X}$ be any seminorm. By definition, $\ker \phi_*(\xi) = \phi(\ker \xi)$, so if ξ is not a norm then $\phi_*\xi$ is not a norm either, both divisors D_ξ and $D_{\phi_*\xi}$ are zero, and the equality holds trivially. Thus we may assume that $\xi = (X, v_\xi)$ is the norm associated to a valuation v_ξ .

Note that $\text{Env}_Y(D_\xi)$ is well defined. Indeed, for every m , the $\mathbf{b}$ -divisor $Z(\mathfrak{a}_{\xi,m})$ is Cartier, so $\text{Env}_Y(Z(\mathfrak{a}_{\xi,m}))$ is defined, is X -nef, and satisfies

$$\text{Env}_Y(Z(\mathfrak{a}_{\xi,m})) \leq Z(\mathfrak{a}_{\xi,m}) \leq D_\xi.$$

Now for a given $m \in \mathbb{R}_{>0}$, we know by Corollary 2.13 that

$$\text{Env}_Y(Z(\mathfrak{a}_{\xi,m})) = \text{Env}_{\phi \circ \pi_m}(Z(\mathfrak{a}_{\xi,m})_{X_m}) = Z(\phi_*(\mathfrak{a}_m)) = Z(\mathfrak{a}_{\phi_*(\xi),m})$$

where $\pi_m : X_m \rightarrow X$ is the normalized blowup of $\mathfrak{a}_m$, and

$$\mathring{Z}_m(\mathfrak{a}_{\phi_*\xi,\bullet}) = \mathring{Z}_m(\phi_*(\mathfrak{a}_{\xi,\bullet})) \geq \text{Env}_Y(\mathring{Z}_m(\mathfrak{a}_\bullet)).$$

Therefore by Proposition 3.5 we have

$$Z(\mathfrak{a}_{\phi_*(\xi),m}) = \text{Env}_Y(Z(\mathfrak{a}_{\xi,m})) \leq \text{Env}_Y(D_\xi) \leq \text{Env}_Y(\mathring{Z}_m(\mathfrak{a}_\bullet)) \leq \mathring{Z}_m(\mathfrak{a}_{\phi_*\xi,\bullet}).$$

Both the leftmost and rightmost terms in these inequalities converge to $D_{\phi_*\xi}$ as $m \rightarrow \infty$ so the claim follows. $\square$

Remark 4.3. *It should be possible, via an appropriate study of nef envelopes of pushforward Weil divisors, to extend Proposition 4.1 to finite proper morphisms. We do not need such a general functoriality result in this work.*

4.2 Global semicontinuity

Theorem 4.4. *The map $D : \mathfrak{X} \rightarrow \text{Div}_{\mathbb{R}}(\mathcal{X})$ given by $\xi \mapsto D_{\xi}$ is lower semicontinuous.*

Proof. We only have to see that, for every prime $\mathbf{b}$ -divisor $\tilde{P}$, the map given by $\xi \mapsto \text{ord}_{\tilde{P}}(D_{\xi})$ is lower semicontinuous. So fix a prime $\mathbf{b}$ -divisor $\tilde{P}$ and a sequence or net $\{\xi_i\}_{i \in I}$ converging to $\xi = \lim_i \xi_i$; we need to show that $\text{ord}_{\tilde{P}}(D_{\xi}) \leq \liminf_i \text{ord}_{\tilde{P}}(D_{\xi_i})$. We argue by contradiction, assuming that

$$0 \geq e := \text{ord}_{\tilde{P}}(D_{\xi}) > \liminf_{i \in I} \text{ord}_{\tilde{P}}(D_{\xi_i}) =: \ell. \quad (10)$$

Let $\Delta = e - \ell > 0$, and let $p = \text{red}_X(\xi)$. By Proposition 3.9, for every $e' < e$ there exists $f \in \mathcal{O}_{X,p}$ with

$$-\frac{\text{ord}_{\tilde{P}}(f)}{-\log\|f\|_{\xi}} > e'.$$

Choosing $e' = e - \Delta/4$ we get $0 \leq \text{ord}_{\tilde{P}}(f) < (e - \Delta/4) \log\|f\|_{\xi}$ (note that in this case $\log\|f\|_{\xi} < 0$). Now since $\lim_i \xi_i = \xi$, we have that $\lim_i \log\|f\|_{\xi_i} = \log\|f\|_{\xi}$, so for every $\varepsilon > 0$ there exists $i_0 \in I$ such that for $i \geq i_0$, the inequality $\log\|f\|_{\xi_i} < \log\|f\|_{\xi}(1 - \varepsilon) < 0$ holds. But then, choosing $\varepsilon = \Delta/(2\Delta - 4e)$ (which is positive since $e < 0$ and $\Delta > 0$), we have

$$\text{ord}_{\tilde{P}}(f) < \left(e - \frac{\Delta}{4}\right) \log\|f\|_{\xi} < \left(e - \frac{\Delta}{4}\right) \frac{\log\|f\|_{\xi_i}}{1 - \varepsilon} = \left(e - \frac{\Delta}{2}\right) \log\|f\|_{\xi_i},$$

which again by 3.9 implies that $\text{ord}_{\tilde{P}} D_{\xi_i} \geq e - \Delta/2 = \ell + \Delta/2$ for every $i \geq i_0$, in contradiction with (10). This finishes the proof. $\square$

Remark 4.5. *Note that Theorem 4.4 applies in particular to cases with $D_{\xi} = 0$. For them we have that, if $\lim_i \xi_i = \xi$ with $D_{\xi} = 0$, then*

$$0 = \text{ord}_{\tilde{P}}(D_{\xi}) \leq \liminf_{i \in I} \text{ord}_{\tilde{P}}(D_{\xi_i}) \leq 0$$

for every $\tilde{P}$ by the anti-effectivity of all D_{ξ_i} . Therefore $\liminf_i \text{ord}_{\tilde{P}}(D_{\xi_i}) = 0$ which, again since $\text{ord}_{\tilde{P}}(D_{\xi_i}) \leq 0$ for all n , means that the sequence (or net) $\text{ord}_{\tilde{P}}(D_{\xi_i})$ actually converges to 0. So the map D is in fact continuous at every ξ with $D_{\xi} = 0$.

4.3 Continuity on quasimonomial families

Our next goal is to show that the map D is in fact *continuous* over large sections of $\mathfrak{X}$, notably over the sets $\Delta_{\pi,p,z}$ of quasimonomial (semi)norms described in section 2.4. To this end, we will consider first the case of monomial seminorms, and then we will apply the results of section 4.1 about the dependency of the valuative divisor on the base model relatively to which it is defined to be nef.

If $X_{\pi} \rightarrow X$ is some birational model above X , we will use the notation D_{ξ}^{π} for the $\mathbf{b}$ -divisor associated to ξ with respect to the model X_{π} , i.e., $D_{\xi}^{\pi} = \lim_{m \rightarrow \infty} Z(\mathfrak{a}_{\xi,m}^{X_{\pi}})$, where, as above, $\mathfrak{a}_{\xi,m}^{X_{\pi}} \subset \mathcal{O}_{X_{\pi}}$ is the valuative ideal sheaf of ξ on X_{π} . Recall from section 2.4 that, if $p \in X_{\pi}$ is a point such that $\mathcal{O}_{X_{\pi},p}$ is regular of dimension c , and $z_1, \dots, z_c$ is a regular system of parameters, there is a subset $\Delta_{\pi,p,z}$ parametrizing all seminorms that are monomial with respect to z , which is homeomorphic to $(\mathbb{R}_{\geq 0} \cup \infty)^c$ and is a retraction of $\mathfrak{X}$.

Proposition 4.6. *Fix a model X_π of X , a point $p \in X_\pi$ such that the local ring $\mathcal{O}_{X_\pi, p}$ is regular of dimension $c \leq \dim X$, and $z_j \in \mathcal{O}_{X_\pi, p}$, $j = 1, \dots, c$ a regular system of parameters. For every prime $\mathbf{b}$ -divisor $\tilde{P}$, the function $(\mathbb{R}_{>0} \cup \{\infty\})^c \rightarrow \mathbb{R}$ defined by the composition of $w \mapsto \|\cdot\|_w$ and $\xi \mapsto \text{ord}_{\tilde{P}}(D_\xi^\pi)$ is continuous in $(\mathbb{R}_{>0} \cup \infty)^c \subsetneq \Delta_{\pi, p, z}$.*

Proof. This result is a consequence of [14, Proposition 3.1]. For every m the ideal $\mathfrak{a}_{\xi_w, m}$ is monomial, so we only need to consider toric divisors P at p or on toric blowups above p and the prime $\mathbf{b}$ -divisors $\tilde{P}$ determined by them.

In particular, if all $w_i \neq 0$ this allows to bound the P -valuation of $\mathfrak{a}_{\xi_w, m}$ by the minimum of the toric pairings of P with each “monomial” z_i^{m/w_i} , the actual value being computed by a monomial with integer exponents close to z_i^{m/w_i} . Then the limit as $m \rightarrow \infty$ of $v_P(\mathfrak{a}_{\xi_w, m})/m$, which is the opposite of the coefficient of P in $D_{\xi_w}^\pi$, is the minimum of the formal toric pairings of P with z_i^{1/w_i} . This is clearly continuous as a function of $w \in (0, \infty]^c$ independently of P . $\square$

Observe that the function above is *never* continuous at any point of the boundary. Indeed, let $I \subseteq \{1, \dots, c\}$ be the nonempty set of indices with $w_i = 0$. If $I = \{1, \dots, c\}$ then ξ_w is the trivial norm, $D_\xi^\pi = -\infty$, and clearly the map is not continuous at w , so we can assume that $\{1, \dots, c\} \setminus I$ is nonempty. Let P be the toric divisor associated to the blowup centered at the toric ideal $(z_i)_{i \notin I}$; in this case the toric pairing is given by the “indicatrix” vector $\sum_{i \notin I} \mathbf{e}_i$. For every $i \in I$ and every w' near w with $w'_i \neq 0$, $z_i^{\lceil m/w_i \rceil} \in \mathfrak{a}_{\xi_w, m}$ has $\text{ord}_P(z_i^{\lceil m/w_i \rceil}) = 0$, so $\text{ord}_{\tilde{P}} D_{\xi_{w'}}^\pi = 0$. On the other hand,

$$\text{ord}_{\tilde{P}}(D_{\xi_w}^\pi) = -\min_{i \notin I} 1/w_i < 0$$

(since $w_i \neq 0$ for $i \notin I$). In other words, since at the limit, the ideals $\mathfrak{a}_{\xi_w, m}$ contain no monomial involving only the $z_i, i \in I$, the toric pairing with $\mathbf{v}$ “jumps” to a strictly negative coefficient of P in $D_{\xi_w}^\pi$.

Notice that if $c < n = \dim X$, then at each point $p' \in \bar{p}$ it is possible to extend the regular system of parameters $z = (z_1, \dots, z_c)$ to a regular system $z' = (z_1, \dots, z_c, \dots, z_n)$ for $\mathcal{O}_{X_\pi, p'}$. Each such extension provides a larger set $\Delta_{\pi, p', z'}$ of seminorms, containing $\Delta_{\pi, p, z}$ on its boundary, and the collection of all such extensions cover all “branches” of the Berkovich space $\mathfrak{X}$ near $\Delta_{\pi, p, z}$.

Example 4.7. *We will now describe an example of discontinuity and its connection with the non-continuity of the Nakayama–Zariski decomposition (also known as σ -decomposition).*

Let us fix a model X_π of X , a point $p \in X_\pi$ such that the local ring $\mathcal{O}_{X_\pi, p}$ is regular of dimension $c \leq \dim X$, and $z_j \in \mathcal{O}_{X_\pi, p}$, $j = 1, \dots, c$ a regular system of parameters. Let $D_1 = \text{div}_+(z_1), \dots, D_c = \text{div}_+(z_c)$ be the divisors on X_π associated to these parameters, and let $\Gamma_{D_1} = \text{ord}_{D_1}$ be the divisorial valuation associated to the first divisor D_1 , with $\theta_{D_1} = (X, \text{ord}_{D_1}) \in \mathfrak{X}$ the associated norm, i.e., for $f \in K(X)$,

$$\|f\|_{\theta_{D_1}} = \exp(-\text{ord}_{D_1}(f)).$$

Recall that, to consider the σ -decomposition, we need to work with effective divisors. Let us denote $\tilde{D}_1$ the $\mathbf{b}$ -divisor obtained from D_1 via pushforward and strict transform, and $\overline{D}_1$ the Cartier closure of D_1 . In particular, $\tilde{D}_1 \leq \overline{D}_1$.

We now construct explicitly a sequence $\xi_k \in \mathfrak{X}$ such that:

- $\text{red}(\xi_k) = p$ for all k ,
- In local coordinates, $\xi_k \in \mathbb{R}_{>0}^c$ and $\lim_{k \rightarrow \infty} \xi_k = \theta_{D_1}$,
- $\bar{D}_1 + D_{\xi_k} \geq 0$ for all k .

For example, consider the sequence of quasimonomial valuations ξ_k defined in coordinates $(z_1, \dots, z_c)$ by the weight vector $(1 + \frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})$.

Then:

- $\text{red}(\xi_k) = p$ since all ξ_k are centered at p ,
- $\xi_k \rightarrow \theta_{D_1}$ as $k \rightarrow \infty$ because the weights converge to $(1, 0, \dots, 0)$,
- $\bar{D}_1 + D_{\xi_k} \geq 0$ since D_{ξ_k} is effective and $\bar{D}_1$ is Cartier.

We obtain that $\Gamma_{D_1}(\bar{D}_1 + D_{\xi_k}) = 1$ for all k , but

$$\Gamma_{D_1}(\bar{D}_1 + D_{\Gamma_{D_1}}) = \Gamma_{D_1}(\bar{D}_1 - \tilde{D}_1) = 0.$$

Corollary 4.8. *The map $\mathbb{R}_{>0}^c \rightarrow \text{Div}_{\mathbb{R}}(\mathcal{X})$ defined by the composition of $w \mapsto \|\cdot\|_w$ and $\xi \mapsto D_{\xi}^{\pi}$ is continuous for the topology of coefficient-wise convergence on $\text{Div}_{\mathbb{R}}(\mathcal{X})$.*

Proof. Follows from Proposition 4.6 by the definition of the topology on $\text{Div}_{\mathbb{R}}(\mathcal{X})$. $\square$

Recall from section 2.4 that for every smooth model X_{π} , every point $p \in X_{\pi}$ and every regular system of parameters $z = (z_j)$ in $\mathcal{O}_{X_{\pi}, p}$ the map

$$\text{ev}_z : \mathfrak{X} \rightarrow \Delta_{\pi, p, z} \cong (\mathbb{R}_{>0} \cup \{+\infty\})^c$$

defined in (1) is a retraction and $\mathfrak{X}$ is an inverse limit of such $\Delta_{\pi, p, z}$'s.

Lemma 4.9. *Let W_i be a sequence (or net) of $\mathbb{R}$ -Weil $\mathbf{b}$ -divisors which converges to an $\mathbb{R}$ -Weil $\mathbf{b}$ -divisor W coefficient-wise. Assume that each W_i admits a nef envelope. Then W admits a nef envelope which is bounded below by*

$$W' := \liminf \text{Env}_{\mathcal{X}}(W_i)$$

Proof. Since for every i we have an inequality $\text{Env}_{\mathcal{X}}(W_i) \leq W_i$, the limit inferior satisfies $W' \leq W$. On the other hand, by closedness of the X -nef cone, W' is X -nef. Thus W admits a nef envelope, bounded below by W' . $\square$

Theorem 4.10. *Fix a model X_{π} of X , a point $p \in X_{\pi}$ such that the local ring $\mathcal{O}_{X_{\pi}, p}$ is regular of dimension $c \leq \dim X$, and $z_j \in \mathcal{O}_{X_{\pi}, p}$, $j = 1, \dots, c$ a regular system of parameters. For every prime $\mathbf{b}$ -divisor $\tilde{P}$, the function $(\mathbb{R}_{>0} \cup \{\infty\})^c \rightarrow \mathbb{R}$ defined by the composition of $w \mapsto \|\cdot\|_w$ and $\xi \mapsto \text{ord}_{\tilde{P}}(D_{\xi})$ is continuous.*

Proof. By Theorem 4.4, the map is lower semicontinuous. We will show that it is also upper semicontinuous.

We saw in Proposition 4.6 that the function $w \mapsto \text{ord}_{\tilde{P}}(D_{\|\cdot\|_w}^{\pi})$ is continuous, where (D_{ξ}^{π}) denotes the $\mathbf{b}$ -divisor associated to ξ with respect to the model X_{π} .

By the birational nature of D_{ξ} 4.1, $D_{\|\cdot\|_w} = \text{Env}_{\mathcal{X}}(D_{\|\cdot\|_w}^{\pi})$ for every w , and so the needed upper semicontinuity follows from Lemma 4.9. $\square$

5 Positivity functions and their continuity

To define Seshadri constants for $\mathbf{b}$ -divisors, we rely on the intersection products from [4] explained above.

Note that, by lemma 2.2, the numerical class $[D_\xi]$ is the limit of all $[Z(\mathfrak{a}_{\xi,m})/m]$. Indeed, by submultiplicativity all components of $Z(\mathfrak{a}_{\xi,m})_{X_\pi}$, $m > 1$ are among the (finitely many) components of $Z(\mathfrak{a}_{\xi,1})_{X_\pi}$, for each model X_π over X , so we can apply Lemma 2.2 to $W_j = Z(\mathfrak{a}_{\xi,j})/j$, with V_i the span of the components of $Z(\mathfrak{a}_{\xi,1})_{X_\pi}$, to conclude that $[D_\xi]$ is the limit of $[Z(\mathfrak{a}_{\xi,m})/m]$.

Theorem 4.4 implies the following:

Corollary 5.1. *The map $D : \mathfrak{X} \rightarrow N_{n-1}(\mathcal{X})$ given by $\xi \mapsto [D_\xi]$ is lower semicontinuous.*

Note that the natural map $\text{Div}_{\mathbb{R}}(\mathcal{X}) \rightarrow N_{n-1}(\mathcal{X})$ (“taking numerical equivalence class”) is not continuous (see [4, section 1.4]) so this is not immediate from Theorem 4.4, but it needs Lemma 2.2.

Proof. Let ξ_j be a sequence (or net) of seminorms in $\mathfrak{X}$ converging to a seminorm $\xi \in \mathfrak{X}$. We distinguish two cases.

1. The sequence (or net) ξ_j has a subsequence (or net) consisting of seminorms *divisorial* on X , i.e., such that the center $\text{red}(\xi_j)$ is a divisor D_j on X .

If the divisor D_j is fixed, say $D_j = D$, for j large enough, then $\|f\|_{\xi_j} = \exp(-w_j) \text{ord}_D(f)$ with $w_j \in [0, \infty]$ converging to $w \in [0, \infty]$, and $\|f\|_\xi = \exp(-w) \text{ord}_D(f)$. In that case $[D_{\xi_j}] = -(1/w_j)[D]$ converge to $[D_\xi] = (-1/w)[D]$.

Alternatively, there is no fixed divisor D in $\text{center}(\xi_j)$ for $j \gg 0$. Therefore, $\lim \|D\|_{\xi_j} = 0$ for every D (because $\text{red}(\xi_j)$ is a divisor D_j and for almost all j , D does not contain D_j). Thus ξ is the trivial norm, $D_\xi = -\infty$, and semicontinuity holds.

2. For all j large enough, the center $\text{red}(\xi_j)$ is not divisorial on X . Then by Proposition 3.9, $(D_{\xi_j})_X = 0$ for all j large enough, and the result follows from Theorem 4.4 and Lemma 2.2. $\square$

Definition 5.2. [4, Lemma 2.10]

$$\text{Nef}(\mathcal{X}) = \lim_{\leftarrow \pi} \overline{\text{Mov}(X_\pi)}$$

where the limit is taken over all smooth (or $\mathbb{Q}$ -factorial) models X_π , i.e. an $\mathbb{R}$ -Weil $\mathbf{b}$ -divisor W is nef iff W_π is movable on each smooth (or $\mathbb{Q}$ -factorial) model X_π .

We leave to the reader to check the details of the goodness of the definition as explained in [4]. Next we introduce Seshadri constants in the context of $\mathbf{b}$ -divisorial valuations.

Definition 5.3 (Seshadri constant and asymptotic order of vanishing of a $\mathbf{b}$ -divisorial valuation). *Let $v = v_\xi$ be a $\mathbf{b}$ -divisorial valuation on X and D a nef Cartier $\mathbf{b}$ -divisor on $\mathcal{X}$, then its Seshadri constant is*

$$\varepsilon(D, v) := \sup \{ t \mid (D + tD_\xi) \text{ is nef} \}.$$

Dually, we also define an asymptotic order of vanishing if D is a big divisor:

$$\omega(D, v) := \sup \{ t \mid (D + tD_\xi) \text{ is pseudoeffective} \}.$$

Here by pseudoeffective Weil $\mathbf{b}$ -divisor we mean a limit of effective Cartier $\mathbf{b}$ -divisors. By definition pseudoeffective $\mathbf{b}$ -divisors form a closed convex cone.

The asymptotic order of vanishing [10, 6] is also called *pseudoeffective threshold* in the literature, see e.g. [5]. Its reciprocal is commonly called *Waldschmidt constant*, especially on projective spaces.

For seminorms that are not $\mathbf{b}$ -divisorial we expect D_ξ to vanish (Conjecture 3.19), so we suggest alternative definitions, explicitly constructing the constant as a limit.

Definition 5.4 (Seshadri constant and asymptotic order of vanishing of a seminorm). *Let $\xi \in \mathfrak{X}$ and D a nef (resp. big) Cartier $\mathbf{b}$ -divisor on $\mathcal{X}$, then*

$$\varepsilon(D, \xi) := \limsup_m (\sup \{t \mid mD + tZ(\mathbf{a}_{\xi, m}) \text{ is nef}\}).$$

$$\omega(D, \xi) := \limsup_m (\sup \{t \mid mD + tZ(\mathbf{a}_{\xi, m}) \text{ is pseudoeffective}\}).$$

Some remarks on these definitions are in order.

Remark 5.5. *Definitions 5.3 and 5.4 mimic and extend the classical definition of Seshadri (resp. Waldschmidt) constant at a subvariety in terms of nefness (resp. bigness) on the blowup, which therefore coincide with our definition when the valuation or seminorm is divisorial.*

Remark 5.6. *The asymptotic order of vanishing of a seminorm satisfies*

$$\begin{aligned} \omega(D, \xi) &= \limsup_m (\sup \{t \mid mD + tZ(\mathbf{a}_{\xi, m}) \text{ is big}\}) \\ &= \limsup_m (\sup \{t \mid mD + tZ(\mathbf{a}_{\xi, m}) \text{ is effective}\}), \end{aligned}$$

as in fact for every m we have

$$\begin{aligned} \sup \{t \mid mD + tZ(\mathbf{a}_{\xi, m}) \text{ is pseudoeffective}\} &= \sup \{t \mid mD + tZ(\mathbf{a}_{\xi, m}) \text{ is big}\} \\ &= \sup \{t \mid mD + tZ(\mathbf{a}_{\xi, m}) \text{ is effective}\}. \end{aligned}$$

Remark 5.7. *We expect that for $\mathbf{b}$ -divisorial valuations, both definitions 5.3 and 5.4 of the Seshadri constant should agree. In Lemma 5.10 below we prove the consistency of both definitions for valuations with sublinear log-discrepancy.*

Lemma 5.8. *Let $D' \geq D$ be two $\mathcal{X}$ -nef $\mathbb{R}$ -Weil $\mathbf{b}$ -divisors with $(D')_X = D_X$. If D is nef then D' is nef.*

Proof. By closedness of the nef cones, it is not restrictive to assume that D and D' are $\mathbb{Q}$ -Weil $\mathbf{b}$ -divisors, and taking a suitable multiple, that they are integral Weil $\mathbf{b}$ -divisors.

Assume D is nef, which by [4, Lemma 2.10] means that on every model X_π , D_π is movable. We need to prove that the same holds for D' . Write $\Delta_\pi = D'_\pi - D_\pi$ which is an effective exceptional divisor. As D_π is movable, every fixed component of $|D'_\pi| = |D_\pi + \Delta_\pi|$ must be a component of Δ_π . But D' is $\mathcal{X}$ -nef, so D'_π is X -movable, in particular it has no fixed exceptional components. $\square$

Remark 5.9. In Definition 5.4, both limits superior are in fact limits. For the asymptotic order of vanishing this is essentially [6, 2.6]; let us give some details on the proof in the present setting. Set $t_m = \sup \{t \mid mD + tZ(\mathbf{a}_{\xi,m}) \text{ is pseudoeffective}\}$ and consider the auxiliary number $a_m = m/t_m$ which is the least real number such that $a_mD + Z(\mathbf{a}_{\xi,m})$ is pseudoeffective. Then, for every positive m and n we have

$$(a_m + a_n)D + Z_{m+n} \geq (a_m + a_n)D + Z_m + Z_n = (a_mD + Z_m) + (a_nD + Z_n)$$

which is pseudoeffective, as the sum of two pseudoeffective divisors. Therefore $a_{m+n} \leq a_m + a_n$, i.e., a_m is subadditive and hence

$$\lim_{m \rightarrow \infty} t_m^{-1} = \lim_{m \rightarrow \infty} \frac{a_m}{m} = \inf \frac{a_m}{m} = \inf t_m^{-1}$$

by Fekete's lemma. Taking inverses, the claim follows.

For the Seshadri constant, the same proof works, observing that the difference between the two sides in the inequality

$$(a_m + a_n)D + Z_{m+n} \geq (a_m + a_n)D + Z_m + Z_n$$

is exceptional, so Lemma 5.8 applies and nefness of the right hand side implies nefness for the left hand side. $\square$

Lemma 5.10. For every $\mathbf{b}$ -divisorial valuation $v = v_\xi$, $\varepsilon(D, \xi) \leq \varepsilon(D, v)$ and $\omega(D, \xi) \leq \omega(D, v)$. If v has sublinear log-discrepancy, then $\varepsilon(D, \xi) = \varepsilon(D, v)$ and $\omega(D, \xi) = \omega(D, v)$.

Proof. The definition of $\varepsilon(D, \xi)$ implies that $D + \varepsilon(D, \xi)D_\xi$ is a limit of nef $\mathbb{R}$ -Cartier $\mathbf{b}$ -divisors $D + (t_i/m_i)Z(\mathbf{a}_{\xi,m_i})$ for a suitable sequence m_i with $\lim m_i = \infty$ and $\lim t_i = \varepsilon(D, \xi)$. In particular, $D + \varepsilon(D, \xi)D_\xi$ is nef and $\varepsilon(D, \xi) \leq \varepsilon(D, v)$.

On the other hand, as explained in Example 3.25,

$$\lim \frac{v(j_m(\mathbf{a}_\bullet))}{m} = \lim \frac{v(-\mathring{Z}_m(\mathbf{a}_{\xi,\bullet}))}{m} = 1$$

hence there exist rational numbers $c_m = \frac{a_m}{b_m} > 1$ such that $\lim_{m \rightarrow \infty} c_m = 1$ and $j_m(\mathbf{a}_\bullet)^{a_m} \subset \mathbf{a}_{\xi,m}^{b_m}$. For these numbers it holds

$$a_m \text{Env}_{\mathcal{X}}(\mathring{Z}_m(\mathbf{a}_{\xi,\bullet})) < b_m Z(\mathbf{a}_{\xi,m}),$$

and since $\mathring{Z}_m(\mathbf{a}_{\xi,\bullet}) \geq mD_\xi$ and D_ξ is X -nef, we deduce that

$$c_m D_\xi \leq c_m \text{Env}_{\mathcal{X}}(\mathring{Z}_m(\mathbf{a}_{\xi,\bullet}))/m < Z(\mathbf{a}_{\xi,m})/m.$$

We claim that for every $t < \varepsilon(D, v)$ there exist $m \gg 0$ such that $mD + tZ(\mathbf{a}_{\xi,m})$ is nef. Since the c_m converge to 1, for m large enough the inequality $c_m \leq \varepsilon(D, v)/t$ holds, and hence $D + tc_m D_\xi$ is nef. Since $mD + tZ(\mathbf{a}_{\xi,m})$ is $\mathcal{X}$ -nef and

$$mD + tZ(\mathbf{a}_{\xi,m}) \geq mD + tmc_m(D_\xi)_\pi \geq m(D + tc_m D_\xi),$$

we can apply Lemma 5.8 and conclude that $mD + tZ(\mathbf{a}_{\xi,m})$ is nef, as needed.

The proof for the asymptotic order of vanishing is similar, but simpler, as Lemma 5.8 is not needed. We leave the details to the interested reader. $\square$

5.1 Semicontinuity of the asymptotic order of vanishing

Theorem 5.11. *For every big divisor D , the map $\omega : \mathfrak{X} \rightarrow \mathbb{R}$ defined by $\xi \mapsto \omega(D, \xi)$ is lower semicontinuous.*

Proof. Let $v_\xi = -\log \|\cdot\|_\xi$. If ξ is a norm then v_ξ is the associated valuation; if it is not a norm, then v_ξ is merely a semivaluation taking values in $\mathbb{R} \cup \{\infty\}$. By Remark 5.6,

$$\begin{aligned}\omega(D, \xi) &= \sup\{m/k \mid H^0(X_\pi, \mathfrak{a}_{\xi, m}(\pi^*(kD))) \neq 0\} \\ &= \sup\{v_\xi(s)/k \mid s \in H^0(X_\pi, \mathcal{O}_{X_\pi}(\pi^*(kD)))\}.\end{aligned}$$

For every $s \in H^0(X_\pi, \mathcal{O}_{X_\pi}(\pi^*(kD)))$, the map $\xi \mapsto v_\xi(s)/k$ is continuous. Therefore $\omega(D, \cdot)$, being the supremum of continuous functions, is lower semicontinuous. $\square$

Proposition 5.12. *Let $\mathfrak{Y} \subset \mathfrak{X}$ be any subset of $\mathbf{b}$ -divisorial valuations with sublinear log-discrepancy over which the map $\xi \mapsto D_\xi$ is continuous. Then the map $\omega : \mathfrak{Y} \rightarrow \mathbb{R}$ defined by $\xi \mapsto \omega(D, \xi)$ is upper semicontinuous.*

In particular, for every model X_π of X , every point $p \in X_\pi$ such that the local ring $\mathcal{O}_{X_\pi, p}$ is regular of dimension $c \leq \dim X$, and every regular system of parameters $z_j \in \mathcal{O}_{X_\pi, p}$, $j = 1, \dots, c$, the map $\omega : \text{interior}(\Delta_{\pi, p, z}) \rightarrow \mathbb{R}$ defined by $w \mapsto \omega(D, \xi)$ is continuous.

Proof. Let us consider a sequence or net ξ_i converging to ξ in $\mathfrak{Y}$. Since $D_\xi = \lim D_{\xi_i}$, we have

$$D + \lim_{w_i \rightarrow w} \omega(D, \xi_i) D_\xi = \lim_{w_i \rightarrow w} (D + \omega(D, \xi_i) D_{\xi_i})$$

which is pseudoeffective (as the limit of pseudoeffective divisors), so $\omega(D, \xi) \geq \lim_i \omega(D, \xi_i)$. $\square$

5.2 Semicontinuity of the Seshadri constant

In the remaining part of the section we will partially prove semicontinuity of Seshadri constants with respect to the topology of the Berkovich space. Here we will give an example of why there cannot be a global semicontinuity of such constants and gives a strong motivation for the reason why in the literature the center of the considered valuations is generally fixed and most likely a point.

Example 5.13. *Let us consider a monomial valuation on $\mathbb{P}^2$ centered at the intersection of two lines that we can locally think as $x = 0$, $y = 0$. We can then consider the sequence of monomial valuations of the form*

$$v_{w_m}(\sum a_{ij} x^i y^j) = \min\{i + j/m \mid a_{ij} \neq 0\}.$$

The limit of such valuations is clearly the divisorial valuation centered at the line $x = 0$.

On $\mathbb{P}^2$ the class of a line $D \sim l$ is ample and it is then natural to consider $\varepsilon(D, \xi_{w_m})$. It is obvious to compute $\varepsilon(D, \xi_l) = 1$, while $\varepsilon(D, \xi_{w_m}) = 1/m \rightarrow 0$.

In this subsection all seminorms (or valuations) under consideration have bounded log-discrepancy, and hence the distinction between the two definitions of Seshadri constants is irrelevant.

Theorem 5.14. *Fix a model X_π of X , a smooth closed point $p \in X_\pi$, and $z_j \in \mathcal{O}_{X_\pi, p}$, $j = 1, \dots, n$ a regular system of parameters. Then the map $\varepsilon : \text{interior}(\Delta_{\pi, p, z}) \rightarrow \mathbb{R}$ defined by $\xi \mapsto \varepsilon(D, \xi)$ is continuous.*

Conjecture 5.15. *We believe that Theorem 5.14 should also hold for p any point such that the local ring $\mathcal{O}_{X_\pi, p}$ is regular of dimension $c \leq \dim X$. However, our proof below using toric pairings only works if all relevant curves can be assumed to have image in X_π meeting p properly, which is not assured in the general case.*

Proof. Consider, for each given curve C on X_π , the value

$$\varepsilon_C(D, \xi_w) := \limsup_m \frac{mD \cdot C}{-Z(\mathfrak{a}_{\xi_w, m}) \cdot C} \in \mathbb{R}_\infty$$

where $Z(\mathfrak{a}_{\xi_w, m}) \cdot C$ stands for the intersection product, on any model where $Z(\mathfrak{a}_{\xi_w, m})$ is defined, of this divisor with the strict transform of C (by the projection formula, this intersection number is independent on the model chosen). The obvious equality

$$\varepsilon_C(D, \xi_w) = \limsup_m \left(\sup_t (t \mid (mD + tZ(\mathfrak{a}_{\xi_w, m})) \cdot C \geq 0) \right)$$

and the definition of the Seshadri constant above show that

$$\varepsilon(D, \xi_w) = \inf_C \varepsilon_C(D, \xi_w),$$

where the infimum is taken over all curves C on X_π . Note that curves in higher models whose image in X_π is a point are not relevant for the computation of ε , because $Z(\mathfrak{a}_{\xi_w, m})$ is X_π -nef.

We next analyze the dependence of $\varepsilon_C(D, \xi_w)$ on w , and we shall prove that there exist a neighborhood U_w and a constant k_w such that for every curve C relevant on U_w for the computation of the Seshadri constant, the map $w \mapsto \varepsilon_C(D, \xi_w)$ is k_w -Lipschitz continuous on U_w . This will imply that $\varepsilon(D, \xi)$ is also k_w -Lipschitz continuous on U_w , i.e., it is locally Lipschitz continuous.

To make this argument precise we introduce a technical definition:

Definition 5.16. *Let $U \subset \Delta_{\pi, p, z}$. We say that a collection of curves $\mathcal{C}$ on X_π is Seshadri U -sufficient if $\varepsilon(D, \xi_w) = \inf_{C \in \mathcal{C}} \varepsilon_C(D, \xi)$ for every $w \in U$.*

The statement we need to prove is then the following:

Proposition 5.17. *There exists a neighborhood U_w , a constant k_w , and a Seshadri U_w -sufficient collection of curves $\mathcal{C}_w$ such that for every curve $C \in \mathcal{C}_w$, the map*

$$w \mapsto \varepsilon_C(D, \xi_w)$$

is k_w -Lipschitz continuous.

As soon as Proposition 5.17 is proved, since the infimum of k -Lipschitz continuous functions is k -Lipschitz continuous the claimed continuity will follow. $\square$

The proof of Proposition 5.17 is quite simpler in the case $X = X_\pi$; we prove first this case as Lemma 5.18. The difference with the general case is due to the presence of components of the divisors $Z_{\mathfrak{a}_{\xi_w, m}}$ contracted by π , and the continuity of these components' coefficients is dealt with in Lemma 5.19. With the help of these preliminary lemmas the proof of 5.17 will be significantly streamlined.

Lemma 5.18. *For every $w \in \text{interior}(\Delta_{\pi, p, z})$ there exists a neighborhood U_w , a constant k_w , and a Seshadri U_w -sufficient collection of curves $\mathcal{C}_w$ such that for every curve $C \in \mathcal{C}_w$, the map*

$$w' \mapsto \varepsilon_C^{X_\pi}(D, \xi_{w'})$$

is k_w -Lipschitz continuous on U_w .

Lemma 5.19. *For every $w \in \text{interior}(\Delta_{\pi, p, z})$ there exist a neighborhood U_w and a constant k_w such that for every prime divisor E on X_π contracted by π , the coefficient of E in $D_{\xi_{w'}}$ is k_w -Lipschitz continuous as a function of $w' \in U_w$.*

Proof of Lemma 5.18. For the purposes of this lemma, all Seshadri constants and Z -divisors refer to the base model X_π , so we simply write $\varepsilon(D, \xi) = \varepsilon^{X_\pi}(D, \xi)$ and $Z(\mathbf{a}) = Z^{X_\pi}(\mathbf{a})$.

Note first that if C is a curve on X_π which does not pass through p $Z(\mathfrak{a}_{\xi_w, m}) \cdot C = 0$ for every w and m , so $\varepsilon_C(D, \xi_w)$ is infinite and we may disregard it for the computation of Seshadri constants on a neighborhood of w (i.e., C will not be included in our Seshadri U_w -sufficient collection). Now let C be any curve that is potentially relevant, i.e., assume that C has $k > 0$ branches through p . The k branches can be parameterised as $(f_1^{(i)}(t), \dots, f_n^{(i)}(t))$, where each f is a holomorphic function on a disc $t \in U \subset \mathbb{C}$ around 0. Let $a_j = \sum_i \text{ord}_t(f_j^{(i)})$ for $j = 1, \dots, n$ and $\mathbf{a} = (a_1, \dots, a_c)$.

Next observe as in the proof of 4.6 that, for every m , the ideal $\mathfrak{a}_{\xi_w, m}$ is monomial, generated by those monomials z^α with $\langle w, \alpha \rangle \geq m$. Then, $Z(\mathfrak{a}_{\xi_w, m})$ is supported on toric divisors P on toric blowups above p , and its intersection with C is in fact equal to the opposite of the colength of $\mathfrak{a}_{\xi_w, m}|_C$, which can be computed torically as

$$-\min\{\langle \mathbf{a}, \alpha \rangle \mid \langle w, \alpha \rangle \geq m\}.$$

For large m , this minimum will be attained by a monomial x^α with α/m close to $1/w_j \mathbf{e}_j$ for some $j \in \{1, \dots, c\}$ (where we use $\mathbf{e}_j$ for the j -th basis vector in $\mathbb{R}^c$). In any event,

$$\varepsilon_C(D, \xi_w) = \frac{D \cdot C}{\min_{j=1, \dots, c} a_j / w_j} = D \cdot C \max_{j=1, \dots, c} w_j / a_j.$$

Now fix a curve C_0 through p and such that $\max\{w_j / a_j(C_0)\}$ is attained for a single $j = j_0$. Then

$$\varepsilon_{C_0}(D, \xi_{w'}) = \frac{D \cdot C_0}{a_{j_0}(C_0)} \cdot w'_{j_0}$$

for every w' in a neighborhood U_w of w .

Since w belongs to the interior of $\Delta_{\pi, p, z}$, $w_i > 0 \forall i$, so we may pick an arbitrary neighborhood U_w and there exist constants $M_1, \dots, M_c > 0$ such that $w'_i \geq M_i w'_{j_0}$ for every $w' \in U$ and every i . Then, the following set of curves is Seshadri U_w -sufficient:

$$\left\{ C \mid \frac{D \cdot C}{a_j(C)} w'_j \leq \frac{D \cdot C_0}{a_{j_0}(C_0)} w'_{j_0} \forall j \text{ s.t. } \frac{w'_j}{a_j(C)} = \max_i \frac{w'_i}{a_i(C)} \right\}.$$

We can enlarge the set somewhat (so that it stays Seshadri U_w -sufficient) using a condition which is independent on $w' \in U_w$:

$$\mathcal{C} := \left\{ C \mid \frac{D \cdot C}{a_j(C)} \leq \frac{D \cdot C_0}{M_j a_{j_0}(C_0)} \forall j \right\}.$$

In this way it is clear that the function

$$\varepsilon_C(D, \xi_w) = D \cdot C \max_j \frac{w'_j}{a_j(C)}$$

is k_w -Lipschitz continuous on U_w where $k_w = \frac{D \cdot C_0}{\min(M_j) a_{j_0}(C_0)}$ for every $C \in \mathcal{C}$. $\square$

Proof of Lemma 5.19. Given two vectors w, w' in the interior of Δ , define

$$\alpha(w, w') = \max_{i=1, \dots, c} w_i / w'_i.$$

The description above of the monomial ideals $\mathfrak{a}_{\xi_w, m}^{X_\pi}$ shows that, for every m , the following inclusions hold:

$$\mathfrak{a}_{\xi_w, m\alpha(w, w')}^{X_\pi} \subseteq \mathfrak{a}_{\xi_{w'}, m}^{X_\pi}, \quad \text{and hence } \mathfrak{a}_{\xi_w, m\alpha(w, w')} \subseteq \mathfrak{a}_{\xi_{w'}, m}.$$

These in turn imply inequalities

$$Z_{\xi_w, m\alpha(w, w')} \leq Z_{\xi_{w'}, m}$$

valid for all m , and it follows that

$$\alpha(w, w') D_{\xi_w} \leq D_{\xi_{w'}}. \quad (11)$$

For a given w in the interior of Δ , choose a neighborhood U_w and constants $A, B > 0$ such that for every $w' = (w'_1, \dots, w'_n) \in U_w$, $A < w'_i < B$ for every i .

Again, the description above shows that $(z_1, \dots, z_n)^{\lceil m/A \rceil} \subset \mathfrak{a}_{\xi_w, m}$ for every $w \in U_w$ (where $\lceil x \rceil$ stands for the least integer greater or equal to x).

Fix $m_0 \geq A/B$ and let $Z = Z((z_1, \dots, z_n)^{\lceil m_0/A \rceil})_{X_\pi}$, so that for every $w' \in U_w$, $Z \leq Z_{\xi_{w'}, m_0}$ and hence $Z \leq m_0 D_{\xi_{w'}}$. For every a, b positive real numbers, there are inequalities

$$aZ + Z_{\xi_{w'}, b} \leq Z_{\xi_{w'}, am_0 + b}. \quad (12)$$

and therefore

$$aZ + bD_{\xi_{w'}} \leq (am_0 + b)D_{\xi_{w'}}. \quad (13)$$

Now let $\gamma(w', w'') = \max(0, \alpha(w', w'') - 1)$, and note that $\gamma(w', w'') \leq \|w' - w''\|_1 / A$ in U_w . For every $w', w'' \in U_w$ we have

$$D_{\xi''} \geq \alpha(w', w'') D_{\xi_{w'}} \geq D_{\xi_{w'}} + \frac{\gamma(w', w'')}{m_0} Z \geq D_{\xi_{w'}, m} + \frac{\|w' - w''\|_1}{Am_0} Z$$

Symmetrically, we also have $D_{\xi_{w'}} \geq D_{\xi_{w''}, m} + \frac{\|w' - w''\|_1}{Am_0} Z$ and so, all components of Z (which are the components of the $D_{\xi_{w'}}$ contracted by π) appear in $D_{\xi_{w'}}$ with coefficients that are Lipschitz continuous functions of $w' \in U_w$ with Lipschitz constant at most $1/Am_0$ times the largest coefficient of Z . $\square$

Proof of Proposition 5.17. Fix, as in lemma 5.19, a contracted divisor $Z = Z((z_1, \dots, z_n)^{\lceil m_0/A \rceil})_{X_\pi}$. Let C be any curve on X_π not contained in $|Z|$, the support of Z . C has $k \geq 0$ branches through p . If $k = 0$, i.e., C does not go through p , then $Z(\mathfrak{a}_{\xi_w, m}) \cdot C = 0$ for every w and m , so $\varepsilon_C(D, \xi_w)$ is infinite and we may disregard it for the computation of Seshadri constants on a neighborhood of w (i.e., C will not be included in our Seshadri U_w -sufficient collection).

As in the proof of lemma 5.18, the intersection of C with $Z_{\xi_w, m}$ can be computed on X_π as the opposite of the colength of $(\mathfrak{a}_{\xi_w, m} \cdot \mathcal{O}_{X_\pi})|_C$. Now, the ideal $(\mathfrak{a}_{\xi_w, m} \cdot \mathcal{O}_{X_\pi})$ is the intersection of a divisorial part supported on $|Z|$ and the monomial ideal described in 5.18; equivalently, it is the product of the divisorial part and the monomial part $\mathfrak{a}_{\xi_w, m'_w}^{X_\pi}$ where $m - m'_w$ is the ξ_w -valuation of the divisorial part. When m tends to ∞ , $\lim \frac{m - m'_w}{m} = v_{\xi_w}((D_{\xi_w})X_\pi)$. Therefore,

$$C \cdot D_{\xi_w} = - \min_{j=1, \dots, c} \frac{a_j}{w_j}$$

The colength of the restricted ideal can be computed as the sum of the colengths of the factors, which are continuous as functions of w , and we are done. $\square$

Lower semicontinuity extends to larger subsets of the space of seminorms, due to the following proposition, but it is not easy to characterize such subsets.

Proposition 5.20. *Let $\xi \in \mathfrak{X}$ be a seminorm, and let D be a nef Cartier $\mathbf{b}$ -divisor. Then*

$$\varepsilon(D, \xi) = \sup\{\varepsilon(D, ev_z(\xi))\}$$

where the supremum runs over all regular systems of parameters z in all $\mathcal{O}_{X_\pi, p}$ for all models X_π .

In particular, if $\xi \in \mathfrak{X}$ is such that $ev_z(\xi)$ belongs to the interior of $\Delta_{\pi, p, z}$ for every z , then $\varepsilon(D, \cdot)$ is lower semicontinuous at ξ .

Note that there do exist norms ξ satisfying the latter condition, and that the statement for such ξ follows as there $\varepsilon(D, \cdot)$ is the supremum of continuous functions.

Proof. By Lemma 3.26, D_ξ is the limit of $D_{ev_z(\xi)}$. For every t such that $mD + tZ(\mathfrak{a}_{ev_z(\xi), m})$ is nef, by Lemma 5.8, $mD + tZ(\mathfrak{a}_{\xi, m})$ is nef as well. Therefore $\varepsilon(D, \xi) \geq \sup\{\varepsilon(D, ev_z(\xi))\}$. However, a strict inequality leads to contradiction, by the same argument as in lemma 5.10. $\square$

6 The case of Surfaces

In this subsection, we determine D_ξ for every ξ when X is a smooth surface. Because of Proposition 3.9,1, we need only consider the case of a real valuation v on $K(X)$ and its associated norm $\xi = (X, v)$. If the center $P \subset X$ of v on X is divisorial, then $v = a \cdot \text{ord}_P$ for some $a > 0$ and one immediately checks that $D_\xi = -(1/a)\overline{P}$; so we assume in the rest of the section that the center of v in X is a point $p = \text{red}_X(\xi)$.

We recall, following [7], the classification of valuations centered at a point of a smooth surface using its sequence of centers (see also [22]). The valuation v determines a sequence of point blowing ups

$$\dots \xrightarrow{b_{p_i}} X_{p_{i-1}} \xrightarrow{b_{p_{i-1}}} \dots \xrightarrow{b_{p_2}} X_{p_1} \xrightarrow{b_{p_1}} X$$

where $p_1 = p$ and in general $p_i = \text{red}_{X_{p_{i-1}}}(\xi)$ is the center of v on $X_{p_{i-1}}$; b_q denotes the blowing up centered at q . If v is a divisorial valuation, then eventually $\text{red}_{X_{p_i}}$ is a divisor, so the sequence of centers and blowing ups is finite; otherwise it is infinite. As customary, we use the notation E_i for the exceptional divisor of the i -th blowing up b_{p_i} and $\tilde{E}_i$ for the corresponding prime $\mathbf{b}$ -divisor. We also fix the notation $\pi_i : X_{p_i} \rightarrow X$ for the composition of the first i blowing ups. There is a natural relation, called *proximity*, between the centers: p_i is proximate to p_j if $p_i \in (\tilde{E}_j)_{X_{p_{i-1}}}$. Every center p_i , $i > 1$, is clearly proximate to its predecessor p_{i-1} , and it can be proximate to at most another p_j with $j < i - 1$; in that case, p_i is said to be a *satellite*, otherwise it is *free*.

There exist valuations for which there is a center p_j with infinitely many centers p_i , $i > j$ proximate to it; however these are not real valuations by [7, 8.1.14], so we assume henceforth that the set of p_i proximate to each given p_j is finite. Consider the numbers $v_i = v(E_i)$. They are determined by v_1 and the so-called *proximity equalities* [7, 8.1.7]

$$v_j = \sum_{p_i \text{ prox. to } p_j} v_i. \quad (14)$$

The fact that v is a real valuation similarly implies that, for each given divisor D , only finitely many centers belong to the strict transform of D , and hence $v(D)$ can be evaluated using Noether's equality 2.11:

$$v(D) = \sum_{i \geq 1} v_i \text{mult}_{p_i}(\tilde{D})_{X_{p_{i-1}}},$$

in particular v is determined by v_1 and the sequence of centers.

Now we can describe explicitly the ideals $\mathfrak{a}_{\xi, v}$ and hence the divisor D_ξ .

Lemma 6.1. *Let v be a real valuation on X and $\xi = (X, v)$ the associated norm. For every $m \in \mathbb{R}$, there is an n such that X_{p_n} is the minimal resolution of X_{π_m} .*

Proof. Since $\mathfrak{a}_{\xi, m}$ is a complete ideal, it is determined by a (finite) cluster of infinitely near points, based at p_1 , and the blowup X_{π_m} is a sandwiched singularity whose minimal resolution is obtained from X by blowing up all points in the cluster of base points of $\mathfrak{a}_{\xi, m}$ [17, 12]. We proceed by induction on the number n of base points of $\mathfrak{a}_{\xi, m}$.

Let $e_1 = \text{ord}_{p_1}(\mathfrak{a}_{\xi, m})$ be the minimal multiplicity at p_1 of an element in $\mathfrak{a}_{\xi, m}$.

The ideal $\mathfrak{a}_{\xi, m}$ has a unique base point (i.e., $n = 1$, in which case the claim is obvious) exactly when it is a power of the maximal ideal, $\mathfrak{a}_{\xi, m} = \mathfrak{m}_{p_1}^{e_1}$. This happens if and only if every element in $\mathcal{O}_{X_{p_1}}$ with multiplicity at least e_1 belongs to $\mathfrak{a}_{\xi, m}$, i.e., $e_1 v_1 \geq m$.

So assume $e_1 v_1 < m$. There exist elements f in $\mathfrak{a}_{\xi, m}$ with multiplicity exactly e_1 , and in this case, by Noether's equality 2.11, $b_{p_1}^*(f) - e_1 E_1$ has positive valuation, i.e., it goes through p_2 , for every such f . More precisely, if we let $\mathfrak{a}'_{\xi, x}$ be the ideal in $\mathcal{O}_{X_{p_1}, p_2}$ formed by functions with valuation at least x , Noether's equality gives

$$\mathfrak{a}_{\xi, m} = (b_{p_1})_*(\mathfrak{a}'_{\xi, m-e_1 v_1}(-e_1 E_{p_1})). \quad (15)$$

It follows that the base points of $\mathfrak{a}_{\xi, m}$ are exactly p_1 and the base points of $\mathfrak{a}'_{\xi, m-e_1 v_1}$. Now by the induction hypothesis, these base points are centers of v_ξ , and X_{p_n} is the resolution of the normalized blowup of X_{p_1} centered at $\mathfrak{a}'_{\xi, m-e_1 v_1}$. Moreover, the equality (15) tells us that the normalized blowup of X_{p_1} centered at $\mathfrak{a}'_{\xi, m-e_1 v_1}$ factors through X_{π_m} (and they coincide if E_{p_1} is not contracted in X_{π_m}). $\square$

Next we determine the decomposition in prime divisors of D_ξ for a divisorial valuation v . In this case there is a finite number of centers, which we denote by n , and for the last 0-dimensional center p_n , it holds $v = v_n \cdot \text{ord}_{E_{p_n}}$.

Lemma 6.2. *Let v be a divisorial valuation with n 0-dimensional centers with values $v_1, \dots, v_n$, and let $\xi = (X, v)$ be the associated norm. Then for every $m \in \mathbb{R}$ such that $mv_n/\mathbf{v}^2$ is an integer, we have:*

$$Z(\mathfrak{a}_{\xi, m})_{\pi_n} = -\frac{m}{\mathbf{v}^2} \sum_{i \leq n} v_i \overline{E_i}_{X_{p_n}}.$$

where $\mathbf{v}^2 = \sum v_i^2$.

Note that certainly there exist $m \in \mathbb{R}$ such that $mv_n/\mathbf{v}^2$ is an integer, and due to 14, this implies that for every $i = 1, \dots, n$, $mv_i/\mathbf{v}^2$ is an integer.

Proof. Since $v = v_n \cdot \text{ord}_{E_{p_n}}$, by definition we have

$$\mathfrak{a}_{\xi, m} = (\pi_n)_* \left(\mathcal{O}_{X_{p_n}} \left(-\frac{m}{v_n} E_n \right) \right).$$

Let D be the divisor $-\sum_{i \leq n} \frac{mv_i}{\mathbf{v}^2} \overline{E_i}_{X_{p_n}}$ and $\mathfrak{a}_m = (\pi_n)_*(\mathcal{O}_{X_{p_n}}(D))$. We need to show that $\mathfrak{a}_{\xi, m} = \mathfrak{a}_m$.

It is immediate to check that $D \cdot (\tilde{E}_i)_{X_n} = 0$ for every $i < n$. By [9, Lemma 3] this implies that

$$-\text{ord}_{E_n}(D) \cdot E_n = (-\text{ord}_{E_n}(D) \cdot E_n - D) + D$$

is the relative Zariski decomposition of $-\text{ord}_{E_n}(D) \cdot E_n$ and therefore

$$\mathfrak{a}_m = (\pi_n)_*(\mathcal{O}_{X_{p_n}}(\text{ord}_{E_n}(D) \cdot E_n)).$$

On the other hand, since $\text{ord}_{E_n}(\overline{E_i}) = v_i/v_n$, one sees that

$$\text{ord}_{E_{p_n}}(D) = -\sum \frac{mv_i}{\mathbf{v}^2} \frac{v_i}{v_n} = -\frac{m}{v_n}$$

so $\mathfrak{a}_m = \mathfrak{a}_{\xi, m}$ and $D = Z(\mathfrak{a}_{\xi, m})$ as claimed. $\square$

Proposition 6.3. *Let v be a divisorial valuation and let $\xi = (X, v)$ be the associated norm. Then $\text{vol}_X(v) = 1/\mathbf{v}^2$ and*

$$D_\xi = -\text{vol}_X(v) \sum v_i \overline{E_i}.$$

Later we will show that this formula holds in all generality.

Recall from [11] that the volume of v is

$$\text{vol}_X(v) = \lim_{m \rightarrow \infty} \frac{\text{length } \mathcal{O}_{X, p}/\mathfrak{a}_m}{m^2/2}.$$

Proof. By the previous lemma, for every m such that $mv_n/\mathbf{v}^2$ is an integer the Hoskin–Deligne–Casas-Alvero formula [7, 6.1] gives

$$\text{length } \mathcal{O}_{X,p}/\mathfrak{a}_m = \sum_{i=1}^n \frac{\frac{mv_i}{\mathbf{v}^2} \left(\frac{mv_i}{\mathbf{v}^2} + 1 \right)}{2} = \frac{m^2}{2\mathbf{v}^2} + \frac{m \sum v_i}{2(\mathbf{v}^2)^2}.$$

Dividing by $m^2/2$ and taking the limit we get $\text{vol}_X(v) = 1/\mathbf{v}^2$ as claimed. On the other hand the previous lemma also gives

$$D_\xi = \lim_{m \rightarrow \infty} Z(\mathfrak{a}_{\xi,m})/m = -\frac{1}{\mathbf{v}^2} \sum_{i \leq n} v_i \overline{E_{iX_{p_n}}} = -\text{vol}(v) \sum_{i \leq n} v_i \overline{E_{iX_{p_n}}}.$$

□

Theorem 6.4. *Let v be a valuation on X with zero-dimensional center, and let $\xi = (X, v)$ be the associated norm. Then*

$$D_\xi = -\text{vol}_X(v) \sum v_i \overline{E_i},$$

where $v_1, v_2, \dots$ are the values of v at its sequence of centers.

Proof. Because of the Proposition above, we need only consider the case that v is not divisorial. Our proof proceeds in two steps:

1. We construct a sequence of divisorial valuations $v^{(k)}$ which is nondecreasing (i.e., $v^{(k)}(f) \leq v^{(k+1)}(f)$ for every $f \in \mathcal{O}_{X,p_1}$ and every $k \geq 1$), converges to v , and satisfies

$$\lim_{k \rightarrow \infty} D_{\xi^{(k)}} = -\text{vol}_X(v) \sum v_i \overline{E_i}.$$

2. We show that for every i , the order of the $\mathbf{b}$ -divisor D_ξ along E_{p_i} is $\lim_k \text{ord}_{E_{p_i}} D_{\xi^{(k)}}$.

By lemma 6.1 then, we obtain an equality $D_\xi = \lim_{k \rightarrow \infty} D_{\xi^{(k)}}$ and the two steps combined finish the proof.

Proof of step 1. Since v is not divisorial, the sequence of centers is infinite, and we begin the construction by considering for each $k \in \mathbb{N}$ the divisorial valuation

$$w^{(k)} = \frac{v_1}{\text{ord}_{p_k}(\overline{E_{p_1}})} \cdot \text{ord}_{p_k}.$$

The normalizing factor $v_1/\text{ord}_{p_k}(\overline{E_{p_1}})$ is chosen so that $w_1^{(k)} = w^{(k)}(E_1) = v_1$ for every k . The proximity equalities then imply that, for every i , there is a k_i such that $w_i^{(k)} = v_i$ for all $k \geq k_i$. Then by Noether's formula, for every f there is a k_f such that $w^{(k)}(f) = v(f)$ for all $k \geq k_f$. Therefore, $w^{(k)}$ converge to v .

The sequence $w^{(k)}$ is in general not nondecreasing, but we will find a suitable subsequence $v^{(k)} = w^{(\ell_k)}$ with this property, which is equivalent to the inequalities $v_i^{(k)} \leq v_i^{(k+1)}$ for all $i, k \geq 1$. If p_{k+1} is free, the proximity equalities involving v_i and $w_i^{(k')}$ are the same for every $i \leq k \leq k'$, so $w_i^{(k)} = v_i = w_i^{(k')}, i = 1, \dots, k$ and therefore $w^{(k)} \leq w^{(k')}$. Thus, if

v has infinitely many free centers, the subsequence formed by those $v^{(k)} = w^{(\ell_k)}$ such that p_{ℓ_k+1} is nondecreasing.

Assume now v has only finitely many free centers, and let p_{ℓ_1} be the last free center. Then we can find a sequence ℓ_k such that $p_{\ell_k+1}, \dots, p_{\ell_{k+1}+1}$ are all the centers p_i proximate to p_{ℓ_k} (recall that no center has infinitely many centers proximate to it). The valuation v is Type 5 in the classification of valuations of [7, Chapter 8]: an irrational valuation associated to an irrational characteristic exponent α , and the valuations $v^{(\ell_k)}$ approach v as the approximants of the continued fraction of α approach it: alternatively from above and from below. Then we choose the subsequence of the lower approximants, obtaining a nondecreasing sequence of valuations whose centers are contained in the sequence of centers of v and converging to v .

Now let ℓ_k be the number of centers of $v^{(k)}$ and consider

$$m_k = \frac{\sum_{i=1}^{\ell_k} v_i^2}{v_{\ell_k}}.$$

The real numbers m_k form a divergent increasing sequence. For any fixed j and for every k we claim the equality $\mathfrak{a}_{\xi, m_j} = \mathfrak{a}_{\xi^{(k)}, m_j}$ holds. The inclusion $\mathfrak{a}_{\xi, m_j} \supset \mathfrak{a}_{\xi^{(k)}, m_j}$ holds because $v^{(k)} \geq v$. On the other hand, if $f \in \mathfrak{a}_{\xi, m_j}$, let e_i be the multiplicity at p_i of the strict transform of $f = 0$. Noether's equality 2.11

$$v(f) = \sum_{i \geq 1} v_i e_i$$

implies that either $\sum_{i=1}^{\ell_k} v_i e_i \geq m$ (in which case $f \in \mathfrak{a}_{\xi^{(k)}, m_j}$) or $e_{\ell_k+1} > 0$. In the latter case, the proximity inequalities give $e_{\ell_k} \geq 1 = v_{\ell_k}/v_{\ell_k}$ and for every $i < \ell_k$,

$$e_i \geq \sum_{p_{i'} \text{ prox. to } p_i} v_{i'},$$

so by recurrence we get $e_i \geq v_i/v_{\ell_k}$ for all $i \leq \ell_k$ and therefore

$$v^{(k)}(f) = \sum_{i=1}^{\ell_k} v_i e_i \geq \sum_{i=1}^{\ell_k} v_i^2/v_{\ell_k} = m$$

i.e., $f \in \mathfrak{a}_{\xi^{(k)}, m_j}$ as claimed.

The choice of m_k guarantees that Lemma 6.2 applies and so

$$\text{length } \mathcal{O}_{X,p}/\mathfrak{a}_{m_k} = \text{length } \mathcal{O}_{X,p}/\mathfrak{a}_{\xi^{(k)}, m_k} = \sum_{i=1}^{\ell_j} \frac{\frac{mv_i}{\sum_{i=1}^{\ell_j} v_i^2} \left(\frac{mv_i}{\sum_{i=1}^{\ell_j} v_i^2} + 1 \right)}{2}$$

and in particular

$$\text{vol}_X(v) = \frac{1}{\sum_{i \geq 1} v_i^2} = \lim \text{vol}_X(v^{(k)}) \in [0, \infty),$$

so that

$$\lim_{k \rightarrow \infty} D_{\xi^{(k)}} = -\text{vol}_X(v) \sum v_i \overline{E_i}$$

which finishes the proof of step 1.

Proof of step 2. By Proposition 3.9, for every i we have equalities

$$\begin{aligned}\mathrm{ord}_{E_i}(D_\xi) &= - \inf_{f \in \mathcal{O}_{X,p_1}} \frac{\mathrm{ord}_{E_i}(\pi_i^*(f))}{v(f)}, \\ \mathrm{ord}_{E_i}(D_{\xi^{(k)}}) &= - \inf_{f \in \mathcal{O}_{X,p_1}} \frac{\mathrm{ord}_{E_i}(\pi_i^*(f))}{v^{(k)}(f)}.\end{aligned}$$

Since $v^{(k)} \leq v$ for every k , this immediately gives that $\mathrm{ord}_{E_i}(D_\xi) \geq \lim_{k \rightarrow \infty} \mathrm{ord}_{E_i}(D_{\xi^{(k)}})$. The opposite weak inequality can be justified directly, for each $i \geq 1$: for every $\varepsilon > 0$ we will provide an $f \in \mathcal{O}_{X,p_1}$ and a k_0 such that, for every $k \geq k_0$,

$$\frac{\mathrm{ord}_{E_i}(\pi_i^*(f))}{v^{(k)}(f)} \leq -\mathrm{ord}_{E_i}(D_\xi) + \varepsilon.$$

Indeed, let $f \in \mathcal{O}_{X,p_1}$ be such that

$$\frac{\mathrm{ord}_{E_i}(\pi_i^*(f))}{v(f)} \leq -\mathrm{ord}_{E_i}(D_\xi) + \frac{\varepsilon}{2},$$

and assume (replacing f by a suitable power f if necessary) that $\mathrm{ord}_{E_i}(\pi_i^*(f)) \geq 2$. Let $k \geq 1$ be such that $v^{(k)}(f) \geq v(f)(1 - \frac{\varepsilon}{\mathrm{ord}_{E_i}(\pi_i^*(f)) + \varepsilon})$. Then

$$\frac{\mathrm{ord}_{E_i}(\pi_i^*(f))}{v^{(k)}(f)} \leq \frac{\mathrm{ord}_{E_i}(\pi_i^*(f))}{v(f)} + \frac{\varepsilon}{\mathrm{ord}_{E_i}(\pi_i^*(f))} \leq -\mathrm{ord}_{E_i}(D_\xi) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

as needed. □

Corollary 6.5. *If X is a surface, then given a nontrivial norm $\xi \in \mathfrak{X}$, the following are equivalent:*

1. ξ is **b**-divisorial.
2. $D_\xi \neq 0$.
3. $\mathrm{red}_X(\xi)$ is a curve or $\mathrm{vol}_X(v_\xi) > 0$.

Proof. We know that (1) implies (2) always, and by the preceding Theorem it is clear that (2) implies (3). Let us prove that (3) implies (1).

Of course if $\mathrm{red}_X(\xi)$ is a curve then ξ is divisorial, so assume $\mathrm{vol}_X(v_\xi) > 0$, and keep the notations of the previous proof.

We need to show that if $\mathrm{div}(f) + tD_\xi \geq 0$ then $v(f) \geq t$ (the converse implication has been proved before). We have, for every k ,

$$\mathrm{div}(f) + t \frac{\mathrm{vol}_X(v)}{\mathrm{vol}_X(v^{(k)})} D_{\xi^{(k)}} = \mathrm{div}(f) - t \mathrm{vol}_X(v) \sum_{i=1}^{\ell_k} v_i \overline{E_i} \geq \mathrm{div}(f) + tD_\xi \geq 0,$$

so, because $v^{(k)}$ is divisorial, $v^{(k)}(f) \geq t \mathrm{vol}_X(v) / \mathrm{vol}_X(v^{(k)})$. Now, for k large enough,

$$v(f) = v^{(k)}(f) \geq t \frac{\mathrm{vol}_X(v)}{\mathrm{vol}_X(v^{(k)})}$$

and taking the limit for $k \rightarrow \infty$, we get the desired inequality. □

Remark 6.6. *Let us highlight the special properties of valuations on surfaces which are leveraged in the proof of Corollary 6.5 but fail to prove Conjecture 3.19 in higher dimension.*

In dimension 2 the sequence of centers of a valuation v is in ordered bijection with the regular local rings dominated by $\mathcal{O}_v$ and dominating $\mathcal{O}_{X, \text{center}_X(v)}$. In higher dimension, the analogous set of local rings is not totally ordered, and the sequence of centers need not be cofinal with it, and so it does not capture the equivalence class of the valuation (i.e., nonequivalent valuations with the same sequence of centers exist, see [20]). An attempt to generalize the proof of Corollary 6.5 to prove Conjecture 3.19 would begin by selecting a suitable nondecreasing sequence of divisorial valuations $v^{(k)}$ converging to v , and decompositions of the corresponding birational maps in blowups of smooth centers. If this is feasible, then one would still need to determine coefficients $c(k)$ converging to 1 and such that

$$c(k)D_{\xi^{(k)}} \geq D_\xi.$$

This seems totally out of reach.

In the case of surfaces, X -nefness is controlled by the proximity inequalities, which boil down to controlling nefness on the $\mathbb{P}^1$'s which arise as exceptional divisors of smooth point blowups. On the other hand, the values of centers satisfy proximity equalities. It is the combination of proximity equalities and inequalities that allowed our computation in terms of the volume.

In higher dimension, X -nefness becomes a much more challenging question, where a similar approach would require the explicit determination of the nef cone on a sequence of blown up $\mathbb{P}^r$ -bundles. This is in general a hard problem, and it is known that in many cases the nef cone is not polyhedral, so the nefness conditions involved in determining the coefficients of $Z(\mathbf{a}_{\xi, m})$ will not be linear nor related to proximity equalities. The sequence of values will continue to satisfy proximity equalities by 2.11, but these will also be harder to control as there is no natural choice for a sequence of centers of the blowups.

In conclusion, we do not expect an explicit formula like

$$\frac{\text{vol}_X(v_\xi)}{\text{vol}_X(v^{(k)})} D_{\xi^{(k)}} \geq D_\xi$$

can hold with the needed generality, at least not with a similar proof.

We end by showing that for every smooth point $p \in X$ on a surfaces, the Seshadri function is lower semicontinuous over the subset of $\mathfrak{X}$ consisting of seminorms centered at p .

Lemma 6.7. *Assume p_1 is a smooth point of the surface X . Let*

$$X_{p_i} \xrightarrow{b_{p_i}} X_{p_{i-1}} \xrightarrow{b_{p_{i-1}}} \dots \xrightarrow{b_{p_1}} X$$

be a sequence of blowing-ups, write $\pi_i = b_{p_i} \circ b_{p_{i-1}} \circ \dots \circ b_{p_1}$ and let $z = (z_1, z_2)$ be a system of parameters at $p_i \in X_{p_i}$ such that $z_1 = 0$ is a local equation of the last exceptional divisor E_{i-1} . Then for every nef divisor D on X , the map $\varepsilon_i(D, \cdot) : \Delta_{\pi_i, p_i, z} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is continuous in the interior and on the half-ray $(w_1, 0)$ with $w_1 > 0$.

Proof. Continuity in the interior is Theorem 5.14. Now since (z_1, z_2) is a local system of parameters with $z_1 = 0$ an equation of E_{i-1} , $z_2 = 0$ is transverse to E_{i-1} and so there exist

local parameters $u = (u_1, u_2)$ at p_{i-1} such that the map b_i is given locally as $(z_1, z_1 z_2)$. This implies that the retraction map $\Delta_{\pi_i, p_i, z} \rightarrow \Delta_{\pi_{i-1}, p_{i-1}, u}$ is an inclusion, given in coordinates as $(w_1, w_2) \mapsto (w_1, w_1 + w_2)$. Since ε_{i-1} is continuous in the interior of $\Delta_{\pi_{i-1}, p_{i-1}, u}$, and the ray $(w_1, 0)$ is mapped to the interior, the claim follows. $\square$

Theorem 6.8. *Assume p is a smooth point of the surface X . Let $\mathfrak{X}_p = \text{red}^{-1}(p)$. The map $\varepsilon(D, \cdot) : \mathfrak{X}_p \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is lower semicontinuous.*

Proof. For every $\xi \in \mathfrak{X}_p$, every model X_π , point $p \in X_\pi$ and system of parameters z , by the previous lemma, the map $\varepsilon(D, \cdot) : \Delta_{\pi, p, z} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is continuous at $ev_z(\xi)$. Then the claim follows by Proposition 5.20, as $\varepsilon(D, \cdot)$ is the supremum of continuous functions at a neighborhood of ξ . $\square$

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