NINE AND TEN LONELY RUNNERS

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Abstract. The Lonely Runner Conjecture of Wills and Cusick states that if $k+1$ runners start running at distinct constant speeds around a unit-length circular track, then for each runner there is a time when he/she is at least $1/(k+1)$ away from all other runners. Rosenfeld recently obtained a computer-assisted proof of the conjecture for 8 runners. By refining his approach with a sieve, we obtain proofs (also computer-assisted) for 9 and 10 runners.

1. Introduction

Wills [14] first posed the Lonely Runner Conjecture in the 1960s while studying Diophantine approximation. The conjecture has since become a well-studied problem in combinatorics, with interpretations in terms of geometric view obstruction [5] and colourings of distance graphs [15]. The following formulation, which gives the conjecture its name, comes from [3]:

Consider $k+1$ runners with distinct constant speeds. Suppose they simultaneously begin running around a circular track with unit circumference, starting from the same location, and continue indefinitely. We say that a runner is *lonely* if he/she is at a distance of at least $\frac{1}{k+1}$ away from all other runners. The Lonely Runner Conjecture predicts that regardless of the runners' speeds, each runner will become lonely at some time.

To prove the conjecture, it suffices to consider the case where one runner's speed is zero and show that this stationary runner is guaranteed to become lonely. A non-trivial result of [4] shows that it suffices to consider the case where all nonzero speeds are positive integers. Consequently, the Lonely Runner Conjecture for $k+1$ runners is equivalent to the following statement, where $\|x\|$ denotes the distance from a real number x to $\mathbb{Z}$.

Conjecture 1.1 (Lonely Runner Conjecture for $k+1$ runners)**.** *Let $v_1, \ldots, v_k$ be positive integers. There exists $t \in \mathbb{R}$ such that for every $i \in \{1, \ldots, k\}$,*

$$\|tv_i\| \geq \frac{1}{k+1}.$$

At present, Conjecture 1.1 is known to be true for $k \leq 7$ (i.e., for a total of at most 8 runners, including the stationary runner). The first two cases $k \in \{1, 2\}$ are easy, and $k = 3$ was resolved by Betke and Wills [2]. The case $k = 4$ was solved first by Cusick and Pomerance [6] with computer assistance, and a proof without computer assistance was given by Bienia, Goddyn, Gvozdjak, Sebő and Tarsi [3]. The first proof of $k = 5$ was due to Bohman, Holzman, and Kleitman [4], and Renault [9] later produced a simpler proof. The case $k = 6$ was settled

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by Barajas and Serra [1] in 2008. In 2025, Rosenfeld [11] proved the Lonely Runner Conjecture for $k = 7$ using a new computational approach. We prove the Lonely Runner Conjecture for $k \in \{8, 9\}$ by refining Rosenfeld's approach.

Theorem 1.2. *The Lonely Runner Conjecture holds for 9 and 10 runners.*

2. Rosenfeld's computational approach

In this preliminary section we give some context for Rosenfeld's approach and indicate the source of our improvement.

It has been known for a number of years that for each k, deciding the Lonely Runner Conjecture for up to $k+1$ runners is a finite calculation. The first result in this direction is due to Tao [12], who showed that there is an explicitly computable constant $C > 0$ such that the following holds: Suppose that the Lonely Runner Conjecture holds for k runners; then the Lonely Runner Conjecture for $k + 1$ runners holds unless there is a counterexample with all (positive integer) speeds at most k^{Ck^2}. This means that to prove or disprove the Lonely Runner Conjecture for $k + 1$ runners, it suffices to check all sets of speeds up to k^{Ck^2}. Unfortunately, the constant C is enormous, so this result is not practical for obtaining any new instances of the Lonely Runner Conjecture.

This finite-checking result was improved by Giri and Kravitz [7] and by Malikiosis, Santos and Schymura [8]. The latter authors showed that, assuming that the Lonely Runner Conjecture holds for k runners, the Lonely Runner Conjecture for $k + 1$ runners can be proven by checking that there are no positive integer counterexamples $v_1, v_2, \ldots, v_k$ with

$$v_1 v_2 \cdots v_k < \left\lceil \frac{\binom{k+1}{2}^{k-1}}{k} \right\rceil^k .$$

This quantity is much more reasonable, but even for $k = 7$ it is large enough that a naive case exhaustion is not feasible.

Rosenfeld's breakthrough idea was showing that in any putative counterexample $v_1, v_2, \ldots, v_k$ to the Lonely Runner Conjecture for $k + 1$ runners, there must be many primes dividing the product $v_1 v_2 \cdots v_k$. If one can exhibit sufficiently many primes that must divide $v_1 v_2 \cdots v_k$ in any counterexample, then one can conclude that any counterexample has $v_1 v_2 \cdots v_k$ above the threshold of Malikiosis, Santos, and Schymura. This would imply that in fact the Lonely Runner Conjecture holds for $k + 1$ runners.

Rosenfeld introduced a clever criterion for showing that a given prime p must divide $v_1 v_2 \cdots v_k$ in any counterexample. He achieved this by combining the "prejump" idea of [3] with the ansatz of checking rational times with denominator $(k + 1)p$. This computation turned out to be barely manageable for $k = 7$ with a computer; the last few primes each took tens of hours to be checked. The approach is too slow to handle larger k.

Our main innovation is introducing a "sieve lemma" that makes checking for a dividing prime p more efficient. This improvement allows us to carry out Rosenfeld's general strategy for $k \in \{8, 9\}$.

3. Reduction of possible counterexamples

For positive integers[1] ℓ, p, we define the set

$$B(\ell,p) = \{0,1,\ldots,\ell p-1\} \setminus \{0,p,\ldots,(\ell-1)p\}.$$

We call a k-tuple $\mathbf{v} = (v_1,\ldots,v_k) \in \mathbb{N}^k$ a *speed set*. A speed set $\mathbf{v}$ is said to have the *LR property* if it satisfies the Lonely Runner Conjecture.

We begin by tracing the steps of [11], but in greater generality.

Definition 3.1. Let $k \geq 2$. A speed set $\mathbf{v} \in B(\ell,p)^k$ is said to be (k,ℓ,p)-*proper* if one of the following holds:

- there exists $i \in \{1,\ldots,k\}$ such that
$$\gcd(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k,\ell p) > 1,$$
or
- there exists $t \in \{0,1,\ldots,\ell p-1\}$ such that, for all $i \in \{1,\ldots,k\}$,
$$\left\| \frac{tv_i}{\ell p} \right\| \geq \frac{1}{k+1}.$$

If $\mathbf{v} \in B(\ell,p)^k$ is not (k,ℓ,p)-proper, then we say it is (k,ℓ,p)-*improper*. Define $I(k,\ell,p)$ to be the set of all (k,ℓ,p)-improper speed sets.

The following is a fact noted in [3] and [11], which we state without proof.

Lemma 3.2 (Lemma 5 of [11])**.** *Suppose that the Lonely Runner Conjecture holds for $k-1$, where $k \geq 3$. If $\mathbf{v}$ is a speed set with $\gcd(v_1,\ldots,v_k) = 1$, and there is i such that $\gcd(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k) > 1$, then $\mathbf{v}$ has the LR property.*

We easily generalize Lemma 6 of [11], which is the special case $\ell = k+1$ of the following lemma.

Lemma 3.3. *Let $k \geq 3$, and assume that the Lonely Runner Conjecture holds for $k-1$ runners. Let p be a prime. If there is some ℓ such that $I(k,\ell,p)$ is empty, then any counterexample $\mathbf{u} = (u_1,\ldots,u_k)$ to the Lonely Runner Conjecture with k runners must have the property that p divides $u_1u_2\cdots u_k$.*

Proof. Assume that $I(k,\ell,p)$ is empty. Suppose that $\mathbf{u}$ is a speed set such that $p \nmid u_1\cdots u_k$. We may assume that $\gcd(u_1,\ldots,u_k) = 1$, because the LR property is invariant under rescaling and p divides the original $u_1u_2\cdots u_k$ if p divides the rescaled $u_1u_2\cdots u_k$.

Let $v_i = u_i \mod \ell p$. It follows that $\mathbf{v} \in B(\ell,p)^k$ because otherwise $p \mid v_i$ for some i and hence $p \mid u_i$. Thus $\mathbf{v}$ must be (k,ℓ,p)-proper.

If there is i such that $\gcd(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k,\ell p) = d > 1$, then as $d \mid \ell p$, we have

$$d \mid \gcd(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_k).$$

By Lemma 3.2, $\mathbf{u}$ has the LR property.

Alternatively, $\mathbf{v}$ is proper in the second sense, i.e., there exists $t \in \{0,1,\ldots,\ell p-1\}$ such that, for all $i \in \{1,\ldots,k\}$, $\|\frac{tv_i}{\ell p}\| \geq \frac{1}{k+1}$. Then for all i, $\|\frac{tu_i}{\ell p}\| \geq \frac{1}{k+1}$, so $\mathbf{u}$ also has the LR property. Therefore, $\mathbf{u}$ is not a counterexample to the Lonely Runner Conjecture. □

[1] In particular, p does not need to be a prime, although it will be in our applications.

In theory, nothing prevents us from picking any ℓ to check if $I(k,\ell,p)$ is empty. The significance of $\ell = k+1$ is that there is no hope of having $I(k,\ell,p) = \emptyset$ for smaller ℓ. In fact, for $p > k+1$, $I(k,\ell,p) = \emptyset$ only if ℓ is a multiple of $k+1$. To see this, note that speed set $\mathbf{u} = (1,2,\ldots,k)$ achieves its maximum loneliness of $\frac{1}{k+1}$ exactly at the times $t = \frac{a}{k+1}$ for a coprime to $k+1$.

Even if it may be impossible to exclude everything from $I(k,\ell,p)$ for small ℓ, we can still obtain valuable information regarding $I(k,m,p)$ from $I(k,\ell,p)$, provided that $m = c\ell$ for some integer c. The following sieve lemma tells us that if we have already checked that $(v_1,\ldots,v_k)$ is (k,ℓ,p)-proper, we can discard from $I(k,m,p)$ all $\mathbf{v}$ of the form

$$(v_1 + a_1\ell p, \ldots, v_k + a_k\ell p)$$

where $a_1,\ldots,a_k \in \mathbb{Z}_c$. This observation will later make it easier to show computationally that $I(k,k+1,p)$ is empty.

Definition 3.4. Let $m = c\ell$ for some positive integer c. The *shadow* of $I(k,\ell,p)$ at level m, denoted $\delta_m I(k,\ell,p)$, is defined as

$$\big\{(v_1 + a_1\ell p, \ldots, v_k + a_k\ell p) : \mathbf{v} \in I(k,\ell,p), a_i \in \{0,1,\ldots,c-1\}\big\},$$

i.e., all $\mathbf{v} \in B(m,p)^k$ whose images modulo ℓp are in $I(k,\ell,p)$.

Lemma 3.5. *For any p, if ℓ divides m, then*

$$I(k,m,p) \subseteq \delta_m I(k,\ell,p).$$

Proof. Let $m = c\ell$. We must show that

$$B(m,p)^k \setminus \delta_m I(k,\ell,p) \subseteq B(m,p)^k \setminus I(k,m,p).$$

Let $\mathbf{u} \in B(m,p)^k \setminus \delta_m I(k,\ell,p)$. Then, we can write $u_i = v_i + a_i\ell p$ for some $a_i \in \{0,1,\ldots,c-1\}$ and (k,ℓ,p)-proper speed set $\mathbf{v}$.

If there is $i \in \{1,\ldots,k\}$ such that $\gcd(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k,\ell p) = d > 1$, then $d \mid v_j$ for all $j \neq i$ and $d \mid \ell p$, so in particular $d \mid u_j$ for all $j \neq i$ and $d \mid mp$. Hence $d \mid \gcd(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_k,mp)$ and so $\mathbf{u}$ is (k,m,p)-proper.

Otherwise, there exists $t \in \{0,1,\ldots,\ell p-1\}$ such that, for all $i \in \{1,\ldots,k\}$, $\|\frac{tv_i}{\ell p}\| \geq \frac{1}{k+1}$. Let $s = ct \in \{0,1,\ldots,mp-1\}$. Then, for all i,

$$\left\|\frac{su_i}{mp}\right\| = \left\|\frac{ct(v_i + a_i\ell p)}{mp}\right\| = \left\|\frac{ctv_i}{mp} + a_i t\right\| = \left\|\frac{ctv_i}{mp}\right\| = \left\|\frac{tv_i}{\ell p}\right\| \geq \frac{1}{k+1}.$$

Therefore, $\mathbf{u}$ is (k,m,p)-proper. In both cases, $\mathbf{u} \in B(m,p)^k \setminus I(k,m,p)$. □

Lemma 3.5 can be interpreted probabilistically. If k,p are fixed and $\ell \mid m$, then the proportion of improper speed sets at level m is at most the proportion of improper speed sets at level ℓ. Moreover, if we have a sequence (ℓ_n) where $\ell_n \mid \ell_{n+1}$, then this proportion at level ℓ_n is non-increasing.

4. Verification of dividing primes

From Lemma 3.3, if we can show that $I(k,k+1,p)$ is empty, then we have that $p \mid v_1v_2\ldots v_k$ whenever $\mathbf{v}$ does not have the LR property. Following [11], we wish to fins many primes p for which $I(k,k+1,p)$ is empty. Consider a fixed prime p.

However, unlike [11], instead of directly checking all possible $(p-1)^k(k+1)^k$ speed sets in $B(k+1,p)^k$, we first use Lemma 3.5 to "sieve out" most candidates. More precisely, we compute $I(k,\ell,p)$ for some smaller $\ell \mid (k+1)$ first, and the compute $I(k,k+1,p)$ inside the shadow of $I(k,\ell,p)$. Furthermore, we can choose $\ell_1,\dots,\ell_n$ such that $\ell_1 = 1, \ell_n = k+1$ and $\ell_{r+1} = c_r\ell_r$ for some $c_r \in \mathbb{N}$ for $r \in \{1,\dots,n-1\}$. At the end, we have to check only

$$(p-1)^k\left(\ell_1^k + s_1\ell_2^k + s_1s_2\ell_3^k + \cdots + s_1\cdots s_{n-1}\ell_n^k\right)$$

tuples, where $0 \le s_r \le 1$ is the proportion of tuples that survive the sieve at level ℓ_r. This number can be made even smaller if we use two distinct sequences $\ell_1,\dots,\ell_n$ and $\ell'_1,\dots,\ell'_{n'}$ that merge (by intersection) at the end, as illustrated by the case $k+1 = 10$.

In the worst case where $s_1 = \cdots = s_{n-1} = 1$, this number is actually slightly larger than $(k+1)^k(p-1)^k$ (because we wasted time checking prior to the last step). In practice, however, s_r tends to be very small and so this improves on the direct method.

The method we described applies to any value of k, but it is most effective when $k+1$ is highly composite. For the sake of illustration and completeness, we explicitly state the algorithms for the cases $k = 8, 9$, which are relevant to proving the Lonely Runner Conjecture for 9 and 10 runners.

4.1. **Case $k = 8$.**

(1) Compute $I(8,1,p)$. Then find its shadow $\delta_3 I(8,1,p)$.
(2) Compute $I(8,3,p)$ within $\delta_3 I(8,1,p)$. Again, find the shadow $\delta_9 I(8,3,p)$.
(3) Compute $I(8,9,p)$ within $\delta_9 I(8,3,p)$.

4.2. **Case $k = 9$.**

(1) Compute $I(9,1,p)$. Then find its shadows $\delta_2 I(9,1,p)$ and $\delta_5 I(9,1,p)$.
(2) (a) Compute $I(9,2,p)$ within $\delta_2 I(9,1,p)$, and find $\delta_{10} I(9,2,p)$.
    (b) Compute $I(9,5,p)$ within $\delta_5 I(9,1,p)$, and find $\delta_{10} I(9,5,p)$.
(3) Compute $I(9,10,p)$ within $\delta_{10} I(9,2,p) \cap \delta_{10} I(9,5,p)$.

5. Proof of Theorem 1.2

Let $C_k = \left\lceil \frac{\binom{k+1}{2}^{k-1}}{k} \right\rceil^k$ be the quantity from the result of Malikiosis, Santos and Schymura [8]. Notice that $C_8 < 10^{80}$ and $C_9 < 10^{111}$.

Let

$$\begin{aligned} S_8 = \{&47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109,\\ &113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179,\\ &181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241\} \end{aligned}$$

and

$$\begin{aligned} S_9 = \{&137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193,\\ &197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263,\\ &269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337,\\ &347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401\}. \end{aligned}$$

Using the algorithms presented in Section 4, we found that every $p \in S_k$, $I(k, k+1, p)$ is the empty set, for $k \in \{8, 9\}$.

As the products satisfy

$$\prod_{p \in S_8} p > 10^{82}$$

and

$$\prod_{p \in S_9} p > 10^{112},$$

it follows that the Lonely Runner Conjecture holds for $k = 8$ and (then) $k = 9$.

6. Implementation details

We carry out the verification algorithms in C++. Our code is developed based on that of Rosenfeld [10], but we optimize for parallelization across cores. We employed OpenAI GPT-5 to assist with code generation, especially in low-level optimization. The code used, along with the result receipts, is uploaded to the author's GitHub [13]. The computations were performed on a 14-core Apple M3 Max processor. On this machine, the verification for $k = 8$ takes 15 minutes, and for $k = 9$ takes under 23 hours.

We remark that we work with sets instead of k-tuples (to save a factor of $k!$).

For $k = 8$, we tested every prime from[2] 11 to 241. For all $11 \le p \le 43$, the set $I(8, 9, p)$ was non-empty. On the other hand, for $k = 9$, we only checked primes from 137 to 401. It should not be inferred from our prior discussion that $I(9, 10, p)$ is non-empty for $11 \le p \le 131$.

There are two reasons why we start p from a large value. One is that larger p contributes more to the product. But more importantly, here it is actually faster to verify large p (in the range of low hundreds). The running time forms a U-shaped curve with run time of 1-2 hours from the start[3], going as low as 5 minutes for $227 \le p \le 277$, and going back up again to 40 minutes for the last p.

The surprising behavior of running time reflects the trade-off between the size of $B(1, p)^9$ and the size of $I(9, 1, p)$ in the first step. When p is small, there are not many $\mathbf{v} \in B(1, p)^9$ to check, but it is much more likely (because the ansatz in Lemma 3.3 allows fewer witness times) that $\mathbf{v}$ belongs in $I(9, 1, p)$, so we need more time in steps 2 and 3 to check $\delta_m I(9, 1, p)$. The opposite is true for p large. In the middle, there exists a "Goldilocks zone" of primes such that both $I(9, 1, p)$ and $B(1, p)^9$ are not too large, and in this case it happens to be around 250.

7. Further work

Since 11 is prime, our sieve algorithm in the case $k = 10$ has only two steps: find $I(10, 1, p)$ first, then $I(10, 11, p)$. In contrast to when $k + 1$ is composite, we do not have an intermediate sieve. Each $\mathbf{v} \in I(10, 1, p)$ corresponds to 11^{10} speed sets in $\delta_{11} I(10, 1, p)$, so this quickly becomes the bottleneck that makes verifying p impossible in a reasonable time.

[2]For $p \le k + 1$, in Lemma 3.3, $t = 1$ serves as a witness time for every $\mathbf{v} \in B(k + 1, p)^k$, so trivially there is no $(k, k + 1, p)$-improper speed set. Had we exploited this fact in Section 5, we would have needed to check a couple fewer primes.

[3]$p = 151$ is a far outlier here, taking 7.5 hours to verify.

However, since we only want to conclude that $I(10, \ell, p)$ is empty, we can also insert an additional filter, say 2, and then compute $I(10, 22, p)$ within $\delta_{11} I(10, 2, p)$. This additional filter may sufficiently reduce $I(10, 2, p)$ to a manageable number of cases. Still, within this framework we cannot avoid extending from ℓ to 11ℓ at some point because, as we previously discussed, for $I(10, \ell, p)$ to be empty, we must have $11 \mid \ell$.

Ultimately, our algorithm, like the original proposed by Rosenfeld, still needs to perform $\Theta(p^k)$ checks to verify that $I(k, \ell, p)$ is empty. Our reduction makes the constant of the scaling much smaller, but we see no easy way to escape this scaling law.

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