Nested Sequents for Intuitionistic Multi-Modal Logics: Cut-Elimination and Lyndon Interpolation

Tim S. Lyon

Technische Universität Dresden

timothy_stephen.lyon@tu-dresden.de

Abstract. We introduce and study single-conclusioned nested sequent calculi for a broad class of intuitionistic multi-modal logics known as *intuitionistic grammar logics (IGLs)*. These logics serve as the intuitionistic counterparts of classical grammar logics, and subsume standard intuitionistic modal and tense logics, including IK and IKt extended with combinations of the T, B, 4, 5, and D axioms. We analyze fundamental invertibility and admissibility properties of our calculi and introduce a novel structural rule, called the *shift* rule, which unifies standard structural rules arising from modal frame conditions into a single rule. This rule enables a purely syntactic proof of cut-admissibility that is uniform over all IGLs, and yields completeness of our nested calculi as a corollary. Finally, we define an interpolation algorithm that operates over single-conclusioned nested sequent proofs. This gives constructive proofs of both the Lyndon interpolation property (LIP) and Beth definability property (BDP) for all IGLs and for all intuitionistic modal and tense logics they subsume. To the best of the author's knowledge, this style of interpolation algorithm (that acts on single-conclusioned nested sequent proofs) and the resulting LIP and BDP results are new.

§1. Introduction

1.1. Intuitionistic Modal Logics.

Numerous authors have proposed logics that provide an intuitionistic account of modal reasoning [4, 6, 13, 19, 49, 18, 17, 52]; see Stewart et al. [53] for a survey. Among these, the formulations introduced by Fischer-Servi [18, 17] and by Plotkin and Stirling [49] have gained particular prominence, especially through the work of Simpson [52], who placed such systems on a firm philosophical foundation. The central idea, due to Simpson, is to interpret modalities from within an intuitionistic meta-theory. This perspective transfers core intuitionistic principles to the modal setting, such as the disjunction property, the failure of the law of excluded middle, and breaks the classical duality between necessity and possibility. Beyond their theoretical interest, intuitionistic modal logics (IMLs) have proven useful in computer science, being used to design verification techniques [16], in reasoning about functional programs [48], and in the definition of programming languages [11].

In [40], multi-modal generalizations of these logics were introduced under the name *intuitionistic grammar logics (IGLs)*. This family of logics subsumes the logics by Fischer-

Servi [17], Plotkin and Stirling [49], and Ewald's intuitionistic tense logics (ITLs) which include temporal modalities referring to past and future states of affairs [15]. IGLs are also closely related to the intuitionistic description logic iALC [26, 25], used for modeling and querying legal ontologies. While iALC can be viewed as a multi-modal variant of IK (cf. [17, 49]), the class of IGLs constitutes a more expansive generalization supporting converse modalities, seriality, and intuitionistic path axioms (IPAs).

The standard modalities in IMLs (cf. [17, 49, 52]) refer only to future states in a model. One way in which IGLs generalize IMLs is by also including converse modalities, which refer to past states in models. This gives IGLs a natural temporal interpretation and is analogous to the inclusion of inverse roles in description logics (cf. [1, 27]). Path axioms express that a sequence of modal transitions implies a direct connection between the initial and terminal worlds; for example, the frame condition $w_0 R_{x_1} w_1, \ldots, w_{n-1} R_{x_n} w_n \Rightarrow w_0 R_x w_n$ is expressible by an IPA. Such axioms capture standard frame conditions in modal logic (e.g., reflexivity, symmetry, transitivity, Euclideanity) and serve as the analog of role inclusion axioms in description logics. Consequently, IGLs can be viewed as intuitionistic counterparts of both the description logic $\mathcal{RI}$ [27] and classical grammar logics [12, 55]. Therefore, IGLs provide a rich and unifying language that subsumes many intuitionistic (multi-)modal systems within a single framework.

1.2. Proof Theory and Sequents.

It is standard practice to equip a logic with an appropriate proof system that supports the study of its (meta-)logical properties and enables formal reasoning. Typically, one seeks to supply *analytic* proof systems for logics, which are advantageous from a computational standpoint. This is because such systems satisfy the *subformula property*–every formula occurring in a derivation is a subformula of the conclusion. Since Gentzen's seminal work [21, 22], the sequent formalism has become a popular framework for the construction of analytic, or cut-free, proof systems. Such systems are particularly useful for establishing non-trivial results such as interpolation [45, 20] and decidability [5, 52], and they facilitate automated reasoning [14, 26] by supporting systematic proof-search and the extraction of derivations from formulae.

At present, IGLs are only equipped with Hilbert systems [36], which are sub-optimal as they invalidate the sub-formula property. We address this gap by introducing analytic sequent-style calculi for IGLs, formulated in the *nested sequent* framework. This choice is motivated by two considerations. First, traditional Gentzen sequents are not expressive enough to yield cut-free systems for logics with converse modalities. Second, nested sequents are well-suited for constructive proofs of interpolation properties [20, 38, 37]. This latter point will be discussed in more detail below as our goal is to not only develop a suitable proof theory for IGLs, but to show how these calculi can be applied to compute Lyndon interpolants in the intuitionistic setting.

The nested sequent formalism is a generalization of Gentzen's sequent formalism and a nested sequent is a tree of Gentzen sequents. The creation of this formalism is often attributed to Bull [7] and Kashima [30], though later work by Brünnler [5] and Poggiolesi [50] was instrumental to the popularity of the formalism.[1] Nested sequents yield elegant cal-

[1] Leivant [33, p. 361] had already introduced a notational variant of nested sequents in his proof-theoretic

culi that minimize syntactic overhead, produce compact proofs, and where termination of proof-search is more easily established (cf. [37, 44]). Moreover, such systems have proven well-suited in extracting counter-models from failed proof-search [5, 55], and they usually admit syntactic cut-admissibility. Thanks to these aesthetic and computational advantages, nested sequent systems have been applied in a variety of domains, including knowledge integration [41], computing explicit definitions in description logics [42], and in the study of axiomatizability [29]. See Lellmann and Poggiolesi [34] for a comprehensive survey.

1.3. Related Work and Contributions.

The proof theory for standard IMLs–that is, those based on the work of Fischer-Servi [17] and Plotkin and Stirling [49]–is diverse, encompassing Hilbert-style systems [17, 49], labeled natural deduction and sequent calculi [26, 46, 52], as well as nested sequent systems [32, 39, 54]. Since the present work is primarily concerned with the sequent formalism, we focus here on developments in sequent-style calculi.

Among these, Simpson [52] introduced labeled sequent calculi for (graph-consistent) extensions of IK with geometric frame conditions,[2] while Marin et al. [46] proposed fully labeled calculi for extensions of IK with intuitionistic Scott–Lemmon axioms. Although both sets of proof systems are formulated within the labeled sequent paradigm, they differ in structure. Simpson's systems capture intuitionistic reasoning *à la* Gentzen by restricting the consequent of sequents to a single formula. By contrast, Marin et al.'s systems omit the single-conclusion restriction and instead internalize the intuitionistic accessibility relation directly into the syntax of labeled sequents. This increases the syntactic complexity of Marin et al.'s systems, but has the advantageous effect that all rules become invertible–this is precluded when the single-conclusion restriction is enforced. We note that all such systems are cut-free, but only Marin et al. provide a syntactic cut-admissibility algorithm for their systems.

In contrast to the labeled approach, several works have developed cut-free nested sequent systems for IMLs. Straßburger [54] introduced nested systems for the "intuitionistic modal cube," namely, extensions of IK with combinations of the axioms for reflexivity, symmetry, transitivity, Euclideanity, and seriality, denoted T, B, 4, 5, and D, respectively.[3] These were later generalized in [39] to handle IK extended with Horn–Scott–Lemmon axioms and seriality. Like Simpson's labeled systems, these nested calculi are single-conclusioned. Kuznets and Straßburger [32] further relaxed this restriction by allowing multi-conclusioned nested sequents in proofs, and only restricting to single-conclusioned sequents in applications of right implication and box rules.

We design our nested sequent systems for IGLs in the single-conclusioned style of [39, 54].[4] This framework was chosen for two main reasons. First, the class of IGLs introduced in [40] can be naturally captured within the nested sequent setting, which is preferable to the labeled

---

work on propositional dynamic logic, where formulae are prefixed by so-called execution sequences.

[2]See Simpson [52, p. 155] for a discussion of graph consistency.

[3]Straßburger's [54] systems can be viewed as intuitionistic counterparts of Brünnler's nested systems for classical modal logics [5].

[4]While [39, 54] use input and output labels to distinguish antecedent and consequent formulae, we employ a more standard two-sided notation in this paper.

approach. Nested systems are generally more economical, admitting shorter proofs with less syntactic overhead than labeled sequent calculi (cf. [37, 44]). Second, restricting nested sequents to a single-conclusion simplifies our constructive proof of Lyndon interpolation for IGLs. Although much of our attention is devoted to establishing core admissibility and invertibility properties for our nested sequent calculi–especially syntactic cut-admissibility–we also resolve the open problem of Lyndon interpolation for IGLs (and hence for IMLs and ITLs), which forms one of the main contributions of this work.

Our contributions in this paper are discussed below:

(1) We initiate the structural proof theory of IGLs by introducing their first nested sequent calculi. We establish soundness and completeness and show that these calculi are well-behaved, enjoying fundamental properties such as the admissibility of structural rules and invertibility of most left rules. A particular novelty of our systems is their *modularity*, that is, it is simple to transform a nested system for one IGL into a system for another by systemically changing the side conditions on certain rules. This modularity enables the formulation of a new structural rule, the *shift* rule $\mathsf{sft}(\mathcal{A})$, which uniformly captures all frame conditions induced by IPAs and which we prove height-preserving admissible.

This feature distinguishes our framework from existing nested sequent systems for (intuitionistic) modal logics, which typically require additional structural rules to capture frame conditions in a complete way (cf. [5, 54]). Exploiting the admissibility of $\mathsf{sft}(\mathcal{A})$, we present a uniform syntactic proof of cut-admissibility that covers all IGLs. This is yet another contribution of this work as existing cut-admissibility proofs for modal systems tend to rely on ad hoc structural and cut rules to eliminate cuts in special cases [5, 43, 54]. Our approach avoids these ad hoc considerations altogether and thereby streamlines the cut-elimination argument.

(2) Our systems serve as the natural intuitionistic counterparts of the nested calculi for classical grammar logics introduced by Tiu et al. [55], in much the same way that Straßburger's intuitionistic systems [54] correspond to Brünnler's classical systems [5]. Tiu et al. proved cut-admissibility for their nested calculi by means of an *indirect* method, viz., a nested sequent proof is translated into a display calculus proof, cuts are eliminated there, and then the resulting display proof is translated back into a cut-free nested sequent proof. Since our nested systems are essentially single-conclusioned variants of Tiu et al.'s systems, by making our nested systems multi-conclusioned, our cut-admissibility algorithm can be easily converted into a *direct* algorithm that eliminates cuts for classical grammar logics.

(3) We use our nested systems to prove the Lyndon interpolation property (LIP) and Beth definability property (BDP) for IGLs. Consequently, this establishes the LIP and BDP for standard IMLs [17, 49, 52, 39] and ITLs [15], which (to the best of the author's knowledge) is a new result. This is accomplished by using a generalization of Maehara's method [45], which takes a proof of an implication $A \rightarrow B$ as input, assigns interpolants to all axioms, and then computes interpolants for rule conclusions from those of their premises, ultimately constructing an interpolant for $A \rightarrow B$.

Several extensions of Maehara's method have recently been developed to accommodate nested sequent calculi. Our approach builds on the *purely syntactic* interpolation framework of [38], which not only computes an interpolant for a theorem $A \rightarrow B$, but has the benefit that it outputs proofs of $A \rightarrow I$ and $I \rightarrow B$ with I the interpolant. This method was used to prove Lyndon interpolation for classical context-free grammar logics with converse [37, Theorem

37]. Because nested sequents have a more elaborate internal structure than Gentzen sequents, this framework requires interpolants to be sets of nested sequents, in contrast to Maehara's original method where interpolants are formulae. Conceptually, this syntactic method is closely related to the semantic interpolation technique of Fitting and Kuznets [20], which uses generalized semantics to guide and justify the interpolation algorithm. However, our treatment remains entirely syntactic, operating solely at the level of nested sequent proofs.

The approach given in the present paper can nevertheless be distinguished from those mentioned above. The interpolation algorithms given in [20, 37, 38] rely on fully-invertible, multi-conclusioned nested sequent calculi, whereas the algorithm presented in this paper operates over single-conclusioned nested sequent calculi that include non-invertible rules. This makes the interpolation algorithm more challenging for a couple reasons.

First, following the methodology of [37, 38], we must ensure that our output proofs of $A \to I$ and $I \to B$ satisfy the single-conclusioned constraint. If one were to directly apply the methodology of [37, 38] to our nested systems, the output proofs would violate this constraint, so more care is needed. To resolve this problem, we use a more restrictive notion of interpolant in the algorithm, which is a set of nested sequents containing no antecedent formulae and only a *single consequent formula*.

Second, the aforementioned interpolation techniques use meta-operators (e.g., [38] uses set union and the orthogonal operation) that operate over sets of nested sequents, which reflect Boolean operations at the meta-level. In the single-conclusioned (intuitionistic) setting, such operations are unsound. Consequently, we develop a new collection of meta-operators tailored to the single-conclusioned structure, allowing the interpolant construction to proceed while preserving correctness. To the best of the author's knowledge, this gives the first interpolation algorithm defined over single-conclusioned nested sequent systems.

1.4. Paper Organization.

Section 2 introduces IGLs and the grammar theoretic preliminaries needed to formulate our nested sequent systems. Section 3 introduces our nested sequent systems and proves them sound and complete. Section 4 investigates the admissibility and invertibility properties of our systems, and provides a syntactic proof of cut-admissibility that is uniform over all IGLs we consider. Section 5 details our new, single-conclusioned interpolation methodology that establishes the LIP and BDP for IGLs, and thus, for standard IMLs and ITLs as a corollary. Finally, in Section 6, we conclude and discuss future work.

§2. Preliminaries

2.1. Syntax and Semantics.

The language of each intuitionistic grammar logic is defined relative to an *alphabet* Σ, which is a non-empty finite set of *characters*, used to index modalities. As in [12], we stipulate that each alphabet Σ is partitioned into a *forward part* $\Sigma^+ := \{a, b, c, \dots\}$ and a *backward part* $\Sigma^- := \{\overline{a}, \overline{b}, \overline{c}, \dots\}$. That is to say, we assume the following holds for an alphabet Σ: (1) $\Sigma := \Sigma^+ \cup \Sigma^-$, (2) $\Sigma^+ \cap \Sigma^- = \emptyset$, and (3) $a \in \Sigma^+$ *iff* $\overline{a} \in \Sigma^-$.

We refer to the elements a, b, c, $\dots$ of Σ^+ as *forward characters* and elements $\overline{a}$, $\overline{b}$, $\overline{c}$, $\dots$ of Σ^- as *backward characters*. A *character* is defined to be either a forward or backward character, and we use x, y, z, $\dots$ to denote them. In what follows, modalities indexed with forward characters will be interpreted as making reference to future states along the accessibility relation within a relational model, and modalities indexed with backward characters will make reference to past states.

We define the *converse operation* to be a function $\bar{\cdot}$ mapping each forward character $a \in \Sigma^+$ to its *converse* $\overline{a} \in \Sigma^-$ and vice versa. Hence, the converse operation is its own inverse, i.e. for any $x \in \Sigma$, $x = \overline{\overline{x}}$.

Let us fix an alphabet Σ for the remainder of the text. We let $\mathsf{Prop} := \{p, q, r, \dots\}$ be a denumerable set of *propositional atoms* and define the language $\mathscr{L}$ relative to a given alphabet Σ via the following grammar in BNF:

$$A ::= p \mid \bot \mid A \vee A \mid A \wedge A \mid A \to A \mid \langle x \rangle A \mid [x]A$$

where p ranges over the set Prop of propositional atoms and x ranges over the characters in the alphabet Σ. We use A, B, C, $\dots$ to denote formulae in $\mathscr{L}$ and define $\top := \bot \to \bot$, $\neg A := A \to \bot$, and $A \leftrightarrow B := (A \to B) \wedge (B \to A)$ as usual. We define the *length* of a formula A, denoted $\ell(A)$, to be the number of symbols contained in A.

Formulae are interpreted over *bi-relational Σ-frames and models* [40], which we refer to simply as *frames* and *models*. These frames and models are inspired by those for intuitionistic modal and tense logics [4, 13, 15, 49].

Definition 2.1 (Frame)**.** A *frame* is a tuple $F = (W, \leq, \{R_x \mid x \in \Sigma\})$ that satisfies the following conditions:

- W is a non-empty set of *worlds* $\{w, u, v, \dots\}$;
- The *intuitionistic relation* $\leq \subseteq W \times W$ is a pre-order, i.e., it is reflexive and transitive;
- The *accessibility relation* $R_x \subseteq W \times W$ satisfies:

(F1) For all $w, w', v \in W$, if $w \leq w'$ and wR_xv, then there exists a $v' \in W$ such that $w'R_xv'$ and $v \leq v'$;

(F2) For all $w, v, v' \in W$, if wR_xv and $v \leq v'$, then there exists a $w' \in W$ such that $w \leq w'$ and $w'R_xv'$;

(F3) wR_xu *iff* $uR_{\overline{x}}w$.

Definition 2.2 (Model)**.** We define a *model* based on a frame F to be a pair $M = (F, V)$ such that: $V : W \to 2^{\mathsf{Prop}}$ is a *valuation function* satisfying the *monotonicity condition*: (M) for each $w, u \in W$, if $w \leq u$, then $V(w) \subseteq V(u)$.

Remark 2.3. The (F1) condition is implied by the (F2) and (F3) conditions, and the (F2) condition is implied by the (F1) and (F3) conditions. In other words, we need only impose $\{(F1), (F3)\}$ or $\{(F2), (F3)\}$ on a model to characterize our logics. We mention both conditions (F1) and (F2) however since we make explicit use of both conditions later on.

$$\begin{tikzcd}
w' \arrow[r, dotted, "R_x"] & v' \\
w \arrow[u, "\leq"] \arrow[r, "R_x"] \arrow[ur, phantom, "(F1)"] & v \arrow[u, dotted, "\leq"']
\end{tikzcd}
\qquad
\begin{tikzcd}
w' \arrow[r, dotted, "R_x"] & v' \\
w \arrow[u, dotted, "\leq"] \arrow[r, "R_x"] \arrow[ur, phantom, "(F2)"] & v \arrow[u, "\leq"']
\end{tikzcd}$$

Figure 1: Depictions of the (F1) and (F2) conditions imposed on models. Dotted arrows indicate the relations implied by the presence of the solid arrows.

The (F1) and (F2) conditions are depicted in Figure 2.1 and ensure the monotonicity of complex formulae (see Lemma 2.5) in our models, which is a property characteristic of intuitionistic logics.[5] If an accessibility relation R_a is indexed with a forward character, then we interpret it as a relation to *future* worlds, and if an accessibility relation $R_{\overline{a}}$ is indexed with a backward character, then we interpret it as a relation to *past* worlds. Hence, our formulae and related models have a tense character, showing that IGLs generalize the intuitionistic tense logics of Ewald [15].

Definition 2.4 (Semantic Clauses)**.** Let $M = (W, \leq, \{R_x \mid x \in \Sigma\}, V)$ be a model with $w \in W$. The *satisfaction relation* $M, w \vDash A$ between $w \in W$ of M and a formula $A \in \mathscr{L}$ is inductively defined as follows:

- $M, w \vDash p$ *iff* $p \in V(w)$, for $p \in \mathsf{Prop}$;
- $M, w \not\vDash \bot$;
- $M, w \vDash A \vee B$ *iff* $M, w \vDash A$ or $M, w \vDash B$;
- $M, w \vDash A \wedge B$ *iff* $M, w \vDash A$ and $M, w \vDash B$;
- $M, w \vDash A \rightarrow B$ *iff* for all $u \in W$, if $w \leq u$ and $M, u \vDash A$, then $M, u \vDash B$;
- $M, w \vDash \langle x \rangle A$ *iff* there exists a $u \in W$ such that $wR_x u$ and $M, u \vDash A$;
- $M, w \vDash [x]A$ *iff* for all $u, v \in W$, if $w \leq u$ and $uR_x v$, then $M, v \vDash A$;
- $M \vDash A$ *iff* for all $w \in W$, $M, w \vDash A$.

A formula A is *valid* relative to a class of frames $\mathscr{F}$ *iff* for all models M based on a frame $F \in \mathscr{F}$, $M \vDash A$.

Lemma 2.5 (General Monotonicity)**.** *Let M be a model with $w, u \in W$ of M. If $w \leq u$ and $M, w \vDash A$, then $M, u \vDash A$.*

Proof. By induction on the length of A. $\square$

[5]For a discussion of these conditions and related literature, see [52, Ch. 3].

2.2. Intuitionistic Grammar Logics.

The base logic $\mathsf{IK_m}$ is defined by means of the axiom system $\mathsf{HIK_m}$ given in Definition 2.6 below. We remark that we refer to the axiom system as $\mathsf{HIK_m}$, using the prefix H to denote the fact that the axiom system is a *Hilbert calculus*.

Definition 2.6 (Axiom System $\mathsf{HIK_m}$)**.** We define the axiom system $\mathsf{HIK_m}$ below, where we have an axiom and inference rule for each $x \in \Sigma$.

A0 Axioms for intuitionistic logic

A1 $[x](A \rightarrow B) \rightarrow ([x]A \rightarrow [x]B)$

A2 $[x](A \wedge B) \leftrightarrow ([x]A \wedge [x]B)$

A3 $\langle x\rangle(A \vee B) \leftrightarrow (\langle x\rangle A \vee \langle x\rangle B)$

A4 $[x](A \rightarrow B) \rightarrow (\langle x\rangle A \rightarrow \langle x\rangle B)$

A5 $([x]A \wedge \langle x\rangle B) \rightarrow \langle x\rangle(A \wedge B)$

A6 $\neg\langle x\rangle\bot$

A7 $(A \rightarrow [x]\langle\overline{x}\rangle A) \wedge (\langle x\rangle[\overline{x}]A \rightarrow A)$

A8 $(\langle x\rangle A \rightarrow [x]B) \rightarrow [x](A \rightarrow B)$

A9 $\langle x\rangle(A \rightarrow B) \rightarrow ([x]A \rightarrow \langle x\rangle B)$

R0 $\dfrac{A \qquad A \rightarrow B}{B}\ \mathsf{mp}$

R1 $\dfrac{A}{[x]A}\ \mathsf{nec}_x$

Remark 2.7. We note that if we let $\Sigma := \{a, \overline{a}\}$, then $\mathsf{IK_m}$ is just Ewald's intuitionistic tense logic IKt [15], which is a conservative extension of the mono-modal intuitionistic modal logic IK [17, 49].

In this paper, we also consider extensions of $\mathsf{IK_m}$ with two different classes of axioms, where $x, x_1, \ldots, x_n \in \Sigma$:

- *seriality axioms* $[x]A \rightarrow \langle x\rangle A$
- *intuitionistic path axioms* $(\langle x_1\rangle \cdots \langle x_n\rangle A \rightarrow \langle x\rangle A) \wedge ([x]A \rightarrow [x_1] \cdots [x_n]A)$

We let $\mathcal{A}$ denote a finite set of axioms of the above forms and refer to intuitionistic path axioms as *IPAs*. We note that the collection of IPAs subsumes the class of Horn-Scott-Lemmon axioms [39] and includes multi-modal and intuitionistic variants of standard axioms such as T_x, B_x, 4_x, and 5_x:

T_x $(A \rightarrow \langle x\rangle A) \wedge ([x]A \rightarrow A)$

4_x $(\langle x\rangle\langle x\rangle A \rightarrow \langle x\rangle A) \wedge ([x]A \rightarrow [x][x]A)$

B_x $(\langle\overline{x}\rangle A \rightarrow \langle x\rangle A) \wedge ([x]A \rightarrow [\overline{x}]A)$

5_x $(\langle\overline{x}\rangle\langle x\rangle A \rightarrow \langle x\rangle A) \wedge ([x]A \rightarrow [\overline{x}][x]A)$

Definition 2.8 ($\mathsf{HIK_m}(\mathcal{A})$)**.** We define $\mathsf{HIK_m}(\mathcal{A}) := \mathsf{HIK_m} \cup \mathcal{A}$, and define the *intuitionistic grammar logic* $\mathsf{IK_m}(\mathcal{A})$ to be the smallest set of formulae from $\mathscr{L}$ closed under substitutions of the axioms from $\mathsf{HIK_m}(\mathcal{A})$ and applications of the inference rules. A formula A is defined to be a *theorem* of $\mathsf{IK_m}(\mathcal{A})$ *iff* $A \in \mathsf{IK_m}(\mathcal{A})$. Given a set $\Gamma \subseteq \mathscr{L}$ and formula A, we define $\Gamma \vdash_{\mathcal{A}} A$ *iff* there exist $B_1, \ldots, B_n \in \Gamma$ such that $B_1 \wedge \cdots \wedge B_n \rightarrow A$ is a theorem of $\mathsf{IK_m}(\mathcal{A})$.

Remark 2.9. The axiom system $\mathsf{HIK_m} = \mathsf{HIK_m}(\emptyset)$.

As stated above and proven in [40], each logic $\mathsf{IK_m}(\mathcal{A})$ admits a semantic characterization in terms of classes of frames, determined by imposing frame conditions which take one of two different forms:

- *seriality conditions* $\forall w \exists u (w R_x u)$
- *path conditions* $\forall \vec{w} (w_0 R_{x_1} w_1 \& \cdots \& w_{n-1} R_{x_n} w_n \Rightarrow w_0 R_x w_n)$

We let $\vec{w} = w_0, \ldots, w_n$ in the path condition above. We say that each aforementioned seriality axiom is *related* to the seriality condition of the above form, and each IPA is *related* to a path condition of the above form.

Definition 2.10 ($\mathcal{A}$-Valid)**.** We define an $\mathcal{A}$*-frame* to be a frame satisfying each frame condition related to an axiom $A \in \mathcal{A}$. An $\mathcal{A}$*-model* is a model based on an $\mathcal{A}$-frame. A formula A is defined to be $\mathcal{A}$*-valid*, written $\vDash_{\mathcal{A}} A$, *iff* A is valid relative to the class of $\mathcal{A}$-frames. For a set $\Gamma \subseteq \mathscr{L}$, we define $\Gamma \vDash_{\mathcal{A}} A$ *iff* for every $\mathcal{A}$-model M and for all worlds $w \in W$ of M, if $M, w \vDash B$ for all $B \in \Gamma$, then $M, w \vDash A$.

The following soundness and completeness result is proven in [40].

Theorem 2.11 (Soundness and Completeness)**.** *$\Gamma \vdash_{\mathcal{A}} A$ iff $\Gamma \vDash_{\mathcal{A}} A$.*

2.3. Interpolation and Definability.

Craig interpolation for classical first-order logic was discovered by Craig in 1957 [10]. The property expresses that for any theorem $A \to B$, there exists a a formula I (the interpolant) such that $A \to I$ and $I \to B$ are theorems and I uses only propositional atoms common to both A and B (i.e., I is in the 'common signature' of A and B). A stronger version of the property was discovered [35] shortly thereafter, now called *Lyndon interpolation*, which additionally requires the propositional atoms to be of the same polarity in the common signature. Interpolation properties have various applications, being used in model checking [47], to enumerate query plans [28], and to establish Beth definability properties [3, 23].

We first define the Lyndon interpolation property (LIP), and then discuss and define the Beth definability property (BDP). In Section 5, we will prove that all IGLs possess these properties.

Definition 2.12 (Positive and Negative Signature)**.** Let $A \in \mathscr{L}$. We define the *positive signature* $\mathsf{sig}^{+}(A)$ and *negative signature* $\mathsf{sig}^{-}(A)$ recursively:

- $\mathsf{sig}^{+}(B) = \{B, \bot, \top\}$ with $B \in \mathsf{Prop}$;
- $\mathsf{sig}^{-}(B) = \{\bot, \top\}$ with $B \in \mathsf{Prop}$;
- $\mathsf{sig}^{+}(B) = \mathsf{sig}^{-}(B) = \{\bot, \top\}$ with $B \in \{\bot, \top\}$;
- $\mathsf{sig}^{\circ}(B \oplus C) = \{B \oplus C\} \cup \mathsf{sig}^{\circ}(B) \cup \mathsf{sig}^{\circ}(C)$ for $\# \in \{+, -\}$ and $\oplus \in \{\vee, \wedge\}$;
- $\mathsf{sig}^{+}(B \to C) = \{B \to C\} \cup \mathsf{sig}^{-}(B) \cup \mathsf{sig}^{+}(C)$;

- $\mathsf{sig}^{-}(B \to C) = \{B \to C\} \cup \mathsf{sig}^{+}(B) \cup \mathsf{sig}^{-}(B)$;
- $\mathsf{sig}^{\circ}(\nabla_x B) = \{\nabla_x B\} \cup \mathsf{sig}^{\circ}(B)$ for $\circ \in \{+, -\}$ and $\nabla_x \in \{\langle x \rangle, [x] \mid x \in \Sigma\}$.

We let $\circ, \bullet \in \{+, -\}$ and we always assume that $\bullet \in \{+, -\} \setminus \{\circ\}$, that is, we let $\circ$ and $\bullet$ denote $+$ and $-$ while always assuming $\circ \neq \bullet$.

Definition 2.13 (Lyndon Interpolation)**.** An IGL $\mathsf{IK_m}(\mathcal{A})$ has the *Lyndon interpolation property (LIP) iff* if $\vdash_{\mathcal{A}} A \to B$, then there exists an $I \in \mathscr{L}$ (called the *Lyndon interpolant*) such that

(1) $\mathsf{sig}^{\circ}(I) \subseteq \mathsf{sig}^{\circ}(A) \cap \mathsf{sig}^{\circ}(B)$ for $\circ \in \{+, -\}$;

(2) $\vdash_{\mathcal{A}} A \to I$ and $\vdash_{\mathcal{A}} I \to B$.

We call such a formula I a *Lyndon interpolant* for $A \to B$.

Remark 2.14. The Craig interpolation property (CIP) is defined by replacing condition (1) by $\mathsf{sig}^{+}(I) \cup \mathsf{sig}^{-}(I) \subseteq (\mathsf{sig}^{+}(A) \cup \mathsf{sig}^{-}(A)) \cap (\mathsf{sig}^{+}(B) \cup \mathsf{sig}^{-}(B))$ in the definition above. As a consequence, the LIP entails the CIP.

An important consequence of the LIP (and CIP) is the Beth definability property (BDP). Intuitively, a propositional logic has the BDP when the implicit definability of a propositional atom p relative to given theory implies that the atom has an explicit definition. We now make these notions precise, following the definitions and notation given by Goranko and Otto [23].

Let $\vec{q} = q_1, \ldots, q_n \in \mathsf{Prop}$ and $p \in \mathsf{Prop} \setminus \{\vec{q}\}$. Furthermore, let $\Gamma = \Gamma(p, \vec{q})$ be a set of formulae from $\mathscr{L}$ containing only the propositional variables p and $\vec{q}$. Suppose $p' \in \mathsf{Prop}$ does not occur in Γ and let $\Gamma' = \Gamma(p', \vec{q})$ be the result of replacing p' for every occurrence of p in Γ. We say that Γ *implicitly defines* p *in* $\mathsf{IK_m}(\mathcal{A})$ *iff* $\Gamma \cup \Gamma' \models_{\mathcal{A}} p \leftrightarrow p'$. We say that Γ *explicitly defines* p *in* $\mathsf{IK_m}(\mathcal{A})$ *iff* there exists a formula $A(\vec{q}) \in \mathscr{L}$ containing only propositional atoms in $\vec{q}$ such that $\Gamma \models_{\mathcal{A}} p \leftrightarrow A(\vec{q})$.

Definition 2.15 (Beth Definability)**.** An IGL $\mathsf{IK_m}(\mathcal{A})$ has the *Beth definability property (BDP) iff* for any set $\Gamma(p, \vec{q}) \subseteq \mathscr{L}$ and $p \in \mathsf{Prop}$, if $\Gamma(p, \vec{q})$ implicitly defines p in $\mathsf{IK_m}(\mathcal{A})$, then $\Gamma(p, \vec{q})$ explicitly defines p in $\mathsf{IK_m}(\mathcal{A})$.

It is not too difficult to show that the LIP implies the BDP in our setting; see [23, Section 3.7] for a proof that the CIP implies the BDP for modal logics, which is easily adapted to IGLs.

Theorem 2.16. *If* $\mathsf{IK_m}(\mathcal{A})$ *has the LIP, then* $\mathsf{IK_m}(\mathcal{A})$ *has the BDP.*

2.4. Grammar-Theoretic Preliminaries.

As will be seen later on, the definition of our nested sequent systems crucially depends upon the use of certain inference rules (viz. propagation rules) parameterized by formal grammars. Therefore, the current section introduces the grammar-theoretic notions required to properly define such rules.

We let $\cdot$ be the *concatenation operation* with ε the *empty string*. We define the set Σ^* of *strings over* Σ to be the smallest set such that (1) $\Sigma \cup \{\varepsilon\} \subseteq \Sigma^*$, and (2) If $s \in \Sigma^*$ and $x \in \Sigma$, then $s \cdot x \in \Sigma^*$. We use s, t, r, $\ldots$ (potentially annotated) to represent strings in Σ^*. Also, we will often write the concatenation of two strings s and t as st as opposed to $s \cdot t$, and for the empty string we have $s\varepsilon = \varepsilon s = s$. The converse operation on strings is defined accordingly: (1) $\overline{\varepsilon} := \varepsilon$, and (2) If $s = x_1 \cdots x_n$, then $\overline{s} := \overline{x}_n \cdots \overline{x}_1$.

To simplify the presentation of formulae in the sequel, we define strings in modalities as follows: if $s = x_1 \cdots x_n$, then $[s]A := [x_1] \cdots [x_n]A$ and $\langle s \rangle A := \langle x_1 \rangle \cdots \langle x_n \rangle A$, with $[s]A = \langle s \rangle A = A$ when $s = \varepsilon$. Hence, every IPA may be written in the form $(\langle s \rangle A \rightarrow \langle x \rangle A) \wedge ([x]A \rightarrow [s]A)$, where $\langle s \rangle = \langle x_1 \rangle \cdots \langle x_n \rangle$ and $[s] = [x_1] \cdots [x_n]$. We make use of this notation to compactly define $\mathcal{A}$-*grammars*, which are types of *Semi-Thue systems* [51], encoding information contained in a set $\mathcal{A}$ of axioms, and employed later on in the definition of propagation rules.

Definition 2.17 ($\mathcal{A}$-grammar)**.** An $\mathcal{A}$-*grammar* is a set $g(\mathcal{A})$ such that:

$$(x \longrightarrow s), (\overline{x} \longrightarrow \overline{s}) \in g(\mathcal{A}) \textit{ iff } (\langle s \rangle A \rightarrow \langle x \rangle A) \wedge ([x]A \rightarrow [s]A) \in \mathcal{A}.$$

We call rules of the form $x \longrightarrow s$ *production rules*, where $x \in \Sigma$ and $s \in \Sigma^*$.

An $\mathcal{A}$-grammar $g(\mathcal{A})$ is a type of string re-writing system. For example, if $x \longrightarrow s \in g(\mathcal{A})$, we may derive the string tsr from txr in one-step by applying the mentioned production rule. Through repeated applications of production rules to a given string $s \in \Sigma^*$, one derives new strings, the collection of which, determines a language. Let us make such notions precise by means of the following definition:

Definition 2.18 (Derivation)**.** Let $g(\mathcal{A})$ be an $\mathcal{A}$-grammar. The *one-step derivation relation* $\longrightarrow_{g(\mathcal{A})}$ holds between two strings s and t in Σ^*, written $s \longrightarrow_{g(\mathcal{A})} t$, *iff* there exist $s', t' \in \Sigma^*$ and $x \longrightarrow r \in g(\mathcal{A})$ such that $s = s'xt'$ and $t = s'rt'$. The *derivation relation* $\longrightarrow^*_{g(\mathcal{A})}$ is defined to be the reflexive and transitive closure of $\longrightarrow_{g(\mathcal{A})}$. For two strings $s, t \in \Sigma^*$, we refer to $s \longrightarrow^*_{g(\mathcal{A})} t$ as a *derivation of* t *from* s, and define its *length* to be equal to the minimal number of one-step derivations needed to derive t from s in $g(\mathcal{A})$.

Definition 2.19 (Language)**.** Let $g(\mathcal{A})$ be an $\mathcal{A}$-grammar. For a character $x \in \Sigma$, the *language of* x *relative to* $g(\mathcal{A})$ is defined to be the set $L_{g(\mathcal{A})}(x) := \{s \mid x \longrightarrow^*_{g(\mathcal{A})} s\}$. When the $\mathcal{A}$-grammar $g(\mathcal{A})$ is clear from the context, we write L_x to denote $L_{g(\mathcal{A})}(x)$.

The following lemma will be helpful in the sequel. It follows from the definition of $g(\mathcal{A})$, namely, because for each production rule $x \longrightarrow s \in g(\mathcal{A})$ there is a production rule $\overline{x} \longrightarrow \overline{s} \in g(\mathcal{A})$.

Lemma 2.20. $s = x_1 \cdots x_n \in L_{g(\mathcal{A})}(x)$ *iff* $\overline{s} = \overline{x}_n \cdots \overline{x}_1 \in L_{g(\mathcal{A})}(\overline{x})$.

§3. Nested Sequent Systems

Let Γ, Δ, Σ, $\ldots$ be finite multisets of formulae from $\mathscr{L}$. We define a *Gentzen sequent* to be an expression of the form $\Gamma \vdash \Delta$ such that $|\Delta| \leq 1$. We define a *nested sequent* to be a *right-empty* or *right-filled* nested sequent, which are simultaneously defined as follows:

(1) A Gentzen sequent $\Gamma \vdash \Delta$ with $|\Delta| = 0$ is a right-empty nested sequent, and a Gentzen sequent $\Gamma \vdash \Delta$ with $|\Delta| = 1$ is a right-filled nested sequent;

(2) if $\Gamma \vdash \Delta$ is right-empty (or, right-filled) and $\mathcal{G}_i$ is a right-empty nested sequent for each $i \in [n]$, then $\Gamma \vdash \Delta, (x_1)[\mathcal{G}_1], \ldots, (x_n)[\mathcal{G}_n]$ is a right-empty (right-filled, resp.) nested sequent;

(3) if $\Gamma \vdash \Delta$ is right-empty, $\mathcal{G}_j$ is a right-filled nested sequent for a unique $j \in [n]$, and $\mathcal{G}_i$ is a right-empty nested sequent for each $i \in [n] \setminus \{j\}$, then $\Gamma \vdash \Delta, (x_1)[\mathcal{G}_1], \ldots, (x_n)[\mathcal{G}_n]$ is a right-filled nested sequent.

We use the symbols $\mathcal{G}$, $\mathcal{H}$, $\mathcal{K}$, ... to denote nested sequents. For a nested sequent of the form $\Gamma \vdash \Delta, (x_1)[\mathcal{G}_1], \ldots, (x_n)[\mathcal{G}_n]$, we refer to Γ as the *antecedent* and refer to $\Delta, (x_1)[\mathcal{G}_1], \ldots, (x_n)[\mathcal{G}_n]$ as the *consequent*.

We define a *component* of the nested sequent $\mathcal{G} = \Gamma \vdash \Delta, (x_1)[\mathcal{H}_1], \ldots, (x_n)[\mathcal{H}_n]$ to be a Gentzen sequent appearing in $\mathcal{G}$, that is, a component of $\mathcal{G}$ is an element of the multiset $c(\mathcal{G})$, where c is defined as follows and $\uplus$ denotes multiset union:

$$c(\Gamma \vdash \Delta, (x_1)[\mathcal{H}_1], \ldots, (x_n)[\mathcal{H}_n]) := \{\Gamma \vdash \Delta\} \uplus \biguplus_{i \in [n]} c(\mathcal{H}_i).$$

We let $\texttt{Names} := \{w, u, v, \ldots\}$ be a denumerable set of pairwise distinct labels, called *names*. For each nested sequent $\mathcal{G}$ of the form $\Gamma \vdash \Delta, (x_1)[\mathcal{H}_1], \ldots, (x_n)[\mathcal{H}_n]$, we call $\Gamma \vdash \Delta$ the *root* of $\mathcal{G}$ and assume that every component of $\mathcal{G}$ is assigned a unique name from $\texttt{Names}$. The incorporation of names in our nested sequents is crucial for the definition of our propagation rules below. We define $\texttt{Names}(\mathcal{G})$ to be the set of all names assigned to components in $\mathcal{G}$.

We define the *input formulae* and *output formulae* of a nested sequent $\mathcal{G}$ to be all formulae occurring in antecedents of components (shown below left) and consequents of components (shown below right), respectively:

$$\mathsf{InF}(\mathcal{G}) := \bigcup_{(\Gamma \vdash \Delta) \in c(\mathcal{G})} \Gamma \qquad \mathsf{OutF}(\mathcal{G}) := \bigcup_{(\Gamma \vdash \Delta) \in c(\mathcal{G})} \Delta$$

Observe that $\mathsf{OutF}(\mathcal{G}) = \emptyset$ if $\mathcal{G}$ is right-empty, and $\mathsf{OutF}(\mathcal{G}) = \{A\}$ if $\mathcal{G}$ is right-filled; in the latter case, we simply write $\mathsf{OutF}(\mathcal{G}) = A$. For a name w of a component $\Gamma \vdash \Delta$ in a nested sequent $\mathcal{G}$, we define $\mathsf{InF}(w, \mathcal{G}) := \Gamma$ and $\mathsf{OutF}(w, \mathcal{G}) := \Delta$. Following the notation of [32, 54], we define $\mathcal{G}^{\downarrow}$ to be the nested sequent $\mathcal{G}$ with the output formula (if it exists) removed.

Observe that every nested sequent encodes a *tree* of (named) Gentzen sequents.

Definition 3.1. Let $\mathcal{G} = \Gamma \vdash \Delta, (x_1)[\mathcal{H}_1], \ldots, (x_n)[\mathcal{H}_n]$ be a nested sequent with w the name of its root and u_i the name of the root of $\mathcal{H}_i$ for $i \in [n]$. We recursively define the tree $\mathsf{tr}(\mathcal{G}) = (T, E)$ as follows:

$$T = \{(w, \Gamma \vdash \Delta)\} \cup \bigcup_{i=1}^{n} T_i \qquad E = \{(w, x_i, u_i) \mid i \in [n]\} \cup \bigcup_{i=1}^{n} E_i$$

such that $\mathsf{tr}(\mathcal{H}_i) = (T_i, E_i)$ for $i \in [n]$. We will often write $(u, \Sigma \vdash \Pi) \in \mathsf{tr}(\mathcal{G})$ and $(u, x, v) \in \mathsf{tr}(\mathcal{G})$ to mean that $(u, \Sigma \vdash \Pi) \in T$ and $(u, x, v) \in E$, respectively.

$$\frac{}{\mathcal{G}\{\Gamma, p \vdash p\}}\ \mathsf{id} \qquad \frac{}{\mathcal{G}\{\Gamma, \bot \vdash \Delta\}}\ \bot\mathsf{L}$$

$$\frac{\mathcal{G}\{\Gamma, A \vdash \Delta\} \qquad \mathcal{G}\{\Gamma, B \vdash \Delta\}}{\mathcal{G}\{\Gamma, A \vee B \vdash \Delta\}}\ \vee\mathsf{L} \qquad \frac{\mathcal{G}\{\Gamma \vdash A_i\}}{\mathcal{G}\{\Gamma \vdash A_1 \vee A_2\}}\ \vee\mathsf{R} \text{ s.t. } i \in \{1,2\}$$

$$\frac{\mathcal{G}\{\Gamma, A, B \vdash \Delta\}}{\mathcal{G}\{\Gamma, A \wedge B \vdash \Delta\}}\ \wedge\mathsf{L} \qquad \frac{\mathcal{G}\{\Gamma \vdash A\} \qquad \mathcal{G}\{\Gamma \vdash B\}}{\mathcal{G}\{\Gamma \vdash A \wedge B\}}\ \wedge\mathsf{R}$$

$$\frac{\mathcal{G}^{\downarrow}\{\Gamma, A \to B \vdash A\} \qquad \mathcal{G}\{\Gamma, A \to B, B \vdash \Delta\}}{\mathcal{G}\{\Gamma, A \to B \vdash \Delta\}}\ \to\mathsf{L} \qquad \frac{\mathcal{G}\{\Gamma, A \vdash B\}}{\mathcal{G}\{\Gamma \vdash A \to B\}}\ \to\mathsf{R}$$

$$\frac{\mathcal{G}\{\Gamma \vdash \emptyset\}_w\{\Sigma \vdash A\}_u}{\mathcal{G}\{\Gamma \vdash \langle x \rangle A\}_w\{\Sigma \vdash \emptyset\}_u}\ \langle x \rangle\mathsf{R}^{\dagger(\mathcal{A})} \qquad \frac{\mathcal{G}\{\Gamma, [x]A \vdash \Delta\}_w\{\Sigma, A \vdash \Pi\}_u}{\mathcal{G}\{\Gamma, [x]A \vdash \Delta\}_w\{\Sigma \vdash \Pi\}_u}\ [x]\mathsf{L}^{\dagger(\mathcal{A})}$$

$$\frac{\mathcal{G}\{\Gamma \vdash \Delta, (x)[A \vdash \emptyset]\}}{\mathcal{G}\{\Gamma, \langle x \rangle A \vdash \Delta\}}\ \langle x \rangle\mathsf{L} \quad \frac{\mathcal{G}\{\Gamma \vdash (x)[\emptyset \vdash A]\}}{\mathcal{G}\{\Gamma \vdash [x]A\}}\ [x]\mathsf{R} \quad \frac{\mathcal{G}\{\Gamma \vdash \Delta, (x)[\emptyset \vdash \emptyset]\}}{\mathcal{G}\{\Gamma \vdash \Delta\}}\ \mathsf{d}_x$$

Side Conditions:
$\dagger(\mathcal{A}) := w \overset{L_x}{\rightsquigarrow} u$ with $L_x = L_{g(\mathcal{A})}(x)$.

Figure 2: The nested sequent system $\mathsf{NIK_m}(\mathcal{A})$ defined relative to the set $\mathcal{A}$ of axioms. Note that $\mathsf{d}_x \in \mathsf{NIK_m}(\mathcal{A})$ *iff* $\mathrm{D}_x \in \mathcal{A}$.

We sometimes refer to a component as a *w-component* if w is the name of that component, and we use the notation $\mathcal{G}\{\mathcal{H}_1\}_{w_1} \ldots \{\mathcal{H}_n\}_{w_n}$ to indicate that $\mathcal{H}_i$ is 'rooted at' the w_i-component of $\mathcal{G}$ for $i \in [n]$. Often, we will more simply write $\mathcal{G}\{\mathcal{H}_1\} \ldots \{\mathcal{H}_n\}$ if the names of the components are not of relevance. To reduce notational complexity, we may *not* write $\mathcal{H}_i$ in its entirety in $\mathcal{G}\{\mathcal{H}_1\} \ldots \{\mathcal{H}_n\}$, but may only display the root and/or some children of the root. For example, $\mathcal{G}\{A, B \vdash \emptyset\}$, $\mathcal{G}\{C, D \vdash \emptyset\}\{I \vdash J\}$, and $\mathcal{G}\{C, D \vdash \emptyset, (y)[I \vdash J]\}$ are all acceptable representations of the nested sequent $\mathcal{G} = A, B \vdash \emptyset, (x)[C, D \vdash \emptyset, (y)[I \vdash J]]$.

Definition 3.2 (Sequent Semantics)**.** Let $\mathcal{G}$ be a nested sequent with w the name of the root, $\mathsf{tr}(\mathcal{G}) = (T, E)$, and $M = (W, \leq, \{R_x \mid x \in \Sigma\}, V)$ be an $\mathcal{A}$-model. We define an *M-interpretation* to be a function $\iota : \texttt{Names} \to W$. We say that $\mathcal{G}$ is satisfied on M with ι, written $M, \iota \vDash_{\mathcal{A}} \mathcal{G}$, *iff* if condition (1) holds, then (2) holds, where conditions (1) and (2) are listed below:

(1) for each $(u, x, v) \in E$, $\iota(u) R_x \iota(v)$;

(2) for some $(v, \Gamma \vdash \Delta) \in T$, $M, \iota(v) \vDash_{\mathcal{A}} \bigwedge \Gamma \to \bigvee \Delta$.

We say that $\mathcal{G}$ is *$\mathcal{A}$-valid iff* for every $\mathcal{A}$-model M and M-interpretation ι, $M, \iota \vDash_{\mathcal{A}} \mathcal{G}$; we say that $\mathcal{G}$ is *$\mathcal{A}$-invalid* otherwise.

A uniform presentation of our nested sequent systems is provided in Figure 2. Each nested sequent system $\mathsf{NIK_m}(\mathcal{A})$ takes a set $\mathcal{A}$ of axioms as a parameter and is sound and complete for the logic $\mathsf{IK_m}(\mathcal{A})$ (see Theorems 3.10 and 3.11 below). We note that $\mathsf{NIK_m} = \mathsf{NIK_m}(\emptyset)$ is the calculus for the base logic $\mathsf{IK_m}$. Each system contains the *initial rules* id and $\bot\mathsf{L}$, and with the exception of d_x, the remaining rules are *logical rules* forming left and right pairs.

As stated above, the parameter $\mathcal{A}$ determines what logic is captured by the system $\mathsf{NIK_m}(\mathcal{A})$. This is achieved by (1) adding the *structural rule* d_x to $\mathsf{NIK_m}(\mathcal{A})$ *iff* $\mathrm{D}_x \in \mathcal{A}$ and/or (2) changing the functionality of the propagation rules $\langle x\rangle\mathsf{R}$ and $[x]\mathsf{L}$. Propagation rules are special in that they view nested sequents as automata, and enable formulae to be (bottom-up) propagated along certain paths (corresponding to strings generated by a Σ-system) within the tree $\mathsf{tr}(\mathcal{G})$ of a nested sequent $\mathcal{G}$. To make the functionality of such rules precise, we define *propagation graphs* and *propagation paths* (cf. [9, 24]).

Definition 3.3 (Propagation Graph)**.** Let $\mathcal{G}$ be a nested sequent with $\mathsf{tr}(\mathcal{G}) = (T, E)$. We define the *propagation graph* $\mathsf{pg}(\mathcal{G}) = (\mathsf{V}, \mathsf{E})$ such that

(1) $\mathsf{V} := \mathtt{Names}(\mathcal{G})$;

(2) $(w, x, u), (u, \overline{x}, w) \in \mathsf{E}$ *iff* $(w, x, u) \in E$.

We will often write $w \in \mathsf{pg}(\mathcal{G})$ and $(w, x, u) \in \mathsf{pg}(\mathcal{G})$ as a shorthand for $w \in \mathsf{V}$ and $(w, x, u) \in \mathsf{E}$, respectively.

Definition 3.4 (Propagation Path)**.** Let $\mathsf{pg}(\mathcal{G}) = (\mathsf{V}, \mathsf{E})$ be a propagation graph with $u, w \in \mathsf{V}$ and $x \in \Sigma$. We write $\mathsf{pg}(\mathcal{G}) \models u \overset{x}{\rightsquigarrow} w$ *iff* $(u, x, w) \in \mathsf{E}$. Given a string $xs \in \Sigma^*$, we define $\mathsf{pg}(\mathcal{G}) \models u \overset{xs}{\rightsquigarrow} w$ as '$\exists_{v \in \mathsf{V}}\ \mathsf{pg}(\mathcal{G}) \models u \overset{x}{\rightsquigarrow} v$ and $\mathsf{pg}(\mathcal{G}) \models v \overset{s}{\rightsquigarrow} w$', and we take $\mathsf{pg}(\mathcal{G}) \models u \overset{\varepsilon}{\rightsquigarrow} w$ to mean that $u = w$. Additionally, when $\mathsf{pg}(\mathcal{G})$ is clear from the context we may simply write $u \overset{s}{\rightsquigarrow} w$ to express $\mathsf{pg}(\mathcal{G}) \models u \overset{s}{\rightsquigarrow} w$. Finally, given a language $L_x := L_{g(\mathcal{A})}(x)$ of some $\mathcal{A}$-system $g(\mathcal{A})$ and $s \in \Sigma^*$, we write $u \overset{L_x}{\rightsquigarrow} w$ *iff* there is a string $t \in L_{g(\mathcal{A})}(x)$ with $u \overset{t}{\rightsquigarrow} w$.

The following is a consequence of Lemma 2.20.

Lemma 3.5. *Let $\mathcal{G}$ be a nested sequent, $L_x = L_{g(\mathcal{A})}(x)$, and $L_{\overline{x}} = L_{g(\mathcal{A})}(\overline{x})$. Then, $\mathsf{pg}(\mathcal{G}) \models u \overset{L_x}{\rightsquigarrow} w$ iff $\mathsf{pg}(\mathcal{G}) \models w \overset{L_{\overline{x}}}{\rightsquigarrow} u$.*

With the above notions, one can formally specify the propagation rules $\langle x\rangle\mathsf{R}$ and $[x]\mathsf{L}$. Applications of such rules are controlled by the side condition $\dagger(\mathcal{A})$ shown in Figure 2. We provide an example to clarify applications of these rules.

Example 3.6. Let $\mathcal{G} = \Gamma, p \vdash \Delta, (x)[\Sigma \vdash \Pi, (\overline{y})[\Phi, [z]p \vdash \Psi]]$ with w the name of the root, u the name of $\Sigma \vdash \Pi$, and v the name of $\Phi, [z]p \vdash \Psi$. A graphical depiction of $\mathcal{G}$'s propagation graph $\mathsf{pg}(\mathcal{G})$ is given below:

$$\begin{tikzcd} w \arrow[r, dotted, bend left, "x"] & u \arrow[l, dotted, bend left, "\overline{x}"] \arrow[r, dotted, bend left, "\overline{y}"] & v \arrow[l, dotted, bend left, "y"] \end{tikzcd}$$

Let $\mathcal{A} := \{(\langle y\rangle\langle\overline{x}\rangle A \to \langle z\rangle A) \wedge ([z]A \to [y][\overline{x}]A)\}$, so that the corresponding $\mathcal{A}$-grammar is $g(\mathcal{A}) = \{z \longrightarrow y\overline{x}, \overline{z} \longrightarrow \overline{y}x\}$. Then, $\mathsf{pg}(\mathcal{G}) \models v \overset{y\overline{x}}{\rightsquigarrow} w$ with $y\overline{x} \in L_{g(\mathcal{A})}(z)$ due to the first production rule of $g(\mathcal{A})$. Therefore, the side condition of $[x]\mathsf{L}$ is satisfied, and so, we are permitted to (top-down) apply the propagation rule $[x]\mathsf{L}$ to $\mathcal{G}$ to delete the formula p. This lets us derive the nested sequent $\Gamma \vdash \Delta, (x)[\Sigma \vdash \Pi, (\overline{y})[\Phi, [z]p \vdash \Psi]]$.

As usual, we refer to formulae that are explicitly presented in the premises and conclusion of a rule as *auxiliary* and *principal*, respectively. For example, in the $[x]\mathsf{R}$ rule, A is auxiliary and $[x]A$ is principal. A *derivation* in $\mathsf{NIK_m}(\mathcal{A})$ of a nested sequent $\mathcal{G}$ is a (potentially infinite) tree whose nodes are labeled with nested sequents such that: (1) The root is labeled with $\mathcal{G}$, and (2) Every parent node is the conclusion of a rule of $\mathsf{NIK_m}(\mathcal{A})$ with its children the premises.

We define a *branch* $\mathcal{B} = \mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_n, \ldots$ to be a maximal path of nested sequents in a derivation such that $\mathcal{G}_0$ is the conclusion of the derivation and each nested sequent $\mathcal{G}_{i+1}$ (if it exists) is a child of $\mathcal{G}_i$. A *proof* is a finite derivation where all leaves are instances of initial rules. We use π and annotated versions thereof to denote both derivations and proofs with the context differentiating the usage. If a proof of a nested sequent $\mathcal{G}$ exists in $\mathsf{NIK_m}(\mathcal{A})$, then we write $\mathsf{NIK_m}(\mathcal{A}) \Vdash \mathcal{G}$ to indicate this. Often, when discussing a proof transformation, we will use the following notation to denote that π is a proof of $\mathcal{G}$:

$$\dfrac{\vdots\pi}{\mathcal{G}}$$

The *height* of a derivation is defined in the usual way as the length of a maximal branch in the derivation, which may be infinite if the derivation is infinite.

Example 3.7. We provide an example proof of an IPA below. We suppose that the IPA shown below is an axiom in $\mathcal{A}$, meaning $x_1 \cdots x_n \longrightarrow x \in g(\mathcal{A})$ by definition. Observe that a propagation path tracing the word $x_1 \cdots x_n$ exists in the left and right branches, showing that the side condition holds for $\langle x \rangle\mathsf{R}$ and $[x]\mathsf{L}$ as $x_1 \cdots x_n \in L_{g(\mathcal{A})}(x)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$\emptyset \vdash (x_1)[\ldots(x_n)[p \vdash p]\ldots]$}
\RightLabel{$\langle x\rangle\mathsf{R}$}
\UnaryInfC{$\emptyset \vdash \langle x\rangle p, (x_1)[\ldots(x_n)[p \vdash \emptyset]\ldots]$}
\RightLabel{$\langle x\rangle\mathsf{L} \times n$}
\UnaryInfC{$\langle x_1\rangle \cdots \langle x_n\rangle p \vdash \langle x\rangle p$}
\RightLabel{$\rightarrow\mathsf{R}$}
\UnaryInfC{$\vdash \langle x_1\rangle \cdots \langle x_n\rangle p \rightarrow \langle x\rangle p$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$[x]p \vdash (x_1)[\ldots(x_n)[p \vdash p]\ldots]$}
\RightLabel{$[x]\mathsf{L}$}
\UnaryInfC{$[x]p \vdash (x_1)[\ldots(x_n)[\emptyset \vdash p]\ldots]$}
\RightLabel{$[x]\mathsf{R} \times n$}
\UnaryInfC{$[x]p \vdash [x_1] \cdots [x_n]p$}
\RightLabel{$\rightarrow\mathsf{R}$}
\UnaryInfC{$\vdash [x]p \rightarrow [x_1] \cdots [x_n]p$}
\RightLabel{$\wedge\mathsf{R}$}
\BinaryInfC{$\vdash (\langle x_1\rangle \cdots \langle x_n\rangle p \rightarrow \langle x\rangle p) \wedge ([x]p \rightarrow [x_1] \cdots [x_n]p)$}
\end{prooftree}
$$

3.1. Soundness and Completeness

We first prove the soundness of our nested systems, and afterward, discuss the proof of completeness. Given a model M and string $s = x_1 \cdots x_n$, we define the relation $R_s := R_{x_1} \circ \cdots \circ R_{x_n}$.

Lemma 3.8. *Let $M = (W, \leq, \{R_x \mid x \in \Sigma\}, V)$ be an $\mathcal{A}$-model. If $wR_s u$ and $s \in L_x$, then $wR_x u$.*

Proof. By induction on the length of the derivation of $s \in L_{g(\mathcal{A})}(x)$.

Base case. For the base case, suppose the length of the derivation of s is 0. By Definition 2.19, the only derivation in $L_{g(\mathcal{A})}(x)$ of length 0 is the derivation consisting solely of x. Hence, the claim trivially holds since $s = x$.

Inductive step. Let the derivation of s be of length $n+1$. By Definition 2.19, there is a derivation $x \longrightarrow^*_{g(\mathcal{A})} tyr$ of length n and there exists a production rule $y \longrightarrow z_1 \cdots z_k \in g(\mathcal{A})$

such that $s = tz_1 \cdots z_k r$. By our assumption that $wR_s u$, we know there exist $v, v' \in W$ such that $wR_t v$, $vR_{z_1 \cdots z_k} v'$, and $v' R_r u$. By the definition of $g(\mathcal{A})$, it follows that $vR_y v'$, and so, we have $wR_t v$, $vR_y v'$, and $v' R_r u$, i.e., $wR_{tyr} u$. Therefore, $wR_x u$ by IH. □

Lemma 3.9. *Let $M = (W, \leq, \{R_x \mid x \in \Sigma\}, V)$ be an $\mathcal{A}$-model and let ι be an M-interpretation. If $M, \iota \not\models_{\mathcal{A}} \mathcal{G}$ and $\mathsf{pg}(\mathcal{G}) \models w \overset{L_x}{\rightsquigarrow} u$, then $\iota(w) R_x \iota(u)$.*

Proof. By assumption, there is a string $s \in L_x$ such that $\mathsf{pg}(\mathcal{G}) \models w \overset{s}{\rightsquigarrow} u$. By Definition 3.2, $\iota(w) R_s \iota(u)$, and so, $\iota(w) R_x \iota(u)$ by Lemma 3.8. □

Theorem 3.10 (Soundness)**.** *If $\mathsf{NIK_m}(\mathcal{A}) \Vdash \mathcal{G}$, then $\mathcal{G}$ is $\mathcal{A}$-valid.*

Proof. It is straightforward to show that all instances of id and $\bot\mathsf{L}$ are valid. Therefore, let us argue soundness by showing that the other rules of $\mathsf{NIK_m}(\mathcal{A})$ are sound, i.e., if the conclusion is $\mathcal{A}$-invalid, then at least one premise is $\mathcal{A}$-invalid. We present the $\to\mathsf{R}$ and $\langle x \rangle\mathsf{R}$ cases; the remaining cases are simple or similar. Let $M = (W, \leq, \{R_x \mid x \in \Sigma\}, V)$ be an $\mathcal{A}$-model with ι an M-interpretation.

$\to\mathsf{R}$. Let $\mathcal{H} := \mathcal{G}\{\Gamma \vdash A \to B\}_w$ be the conclusion of an $\to\mathsf{R}$ instance and suppose $M, \iota \not\models_{\mathcal{A}} \mathcal{H}$. Then, $M, \iota(w) \not\models A \to B$, which implies that there exists a world $u \in W$ such that $\iota(w) \leq u$, $M, u \models A$, and $M, u \not\models B$. We now define a new interpretation ι' such that $M, \iota' \not\models \mathcal{G}\{\Gamma, A \vdash B\}_w$.

First, we set $\iota'(w) = u$. Second, we let $v_1, \ldots, v_n$ be (the names of) all children of w (if they exist) with v_0 the (name of the) parent of w (if it exists) in $\mathcal{H}$. Let $(w, x_i, v_i) \in E$ for $i \in [n]$ and $(v_0, y, w) \in E$ for $\mathsf{tr}(\mathcal{H}) = (T, E)$. Since $\iota(w) R_{x_i} \iota(v_i)$ and $\iota(v_0) R_y \iota(w)$ hold in M and $\iota(w) \leq u$, we know there exist v_i' such that $\iota(v_i) \leq v_i'$ and $u R_{x_i} v_i'$ by condition (F1), and there exists a v_0' such that $\iota(v_0) \leq v_0'$ and $v_0' R_y u$ by condition (F2) (see Definition 2.2 for these conditions). Thus, if we set $\iota'(v_i) = v_i'$ and $\iota'(v_0) = v_0'$, then it follows that $\iota'(w) R_{x_i} \iota'(v_i)$ and $\iota'(v_0) R_y \iota'(w)$. Moreover, since $M, \iota \models \Gamma_i$ and $M, \iota \models \Sigma$ for $(v_i, \Gamma_i \vdash \Delta_i), (v_0, \Sigma \vdash \Pi) \in T$, it follows that $M, \iota' \models \Gamma_i$ and $M, \iota' \models \Sigma$ by Lemma 2.5 as $\iota(v_i) \leq \iota'(v_i)$ and $\iota(v_0) \leq \iota'(v_0)$. We successively repeat this process until all components of $\mathcal{H}$ have been processed and ι' is fully defined over $\mathcal{G}\{\Gamma, A \vdash B\}_w$. One can confirm that $M, \iota' \not\models \mathcal{G}\{\Gamma, A \vdash B\}_w$, showing the premise of $\to\mathsf{R}$ $\mathcal{A}$-invalid.

$\langle x \rangle\mathsf{R}$. Suppose that $\mathcal{H} = \mathcal{G}\{\Gamma \vdash \langle x \rangle A, \Delta\}_w\{\Sigma \vdash \Pi\}_u$ is $\mathcal{A}$-invalid. By assumption, there exists an $\mathcal{A}$-model M and an M-interpretation ι such that $M, \iota \not\models \mathcal{H}$, which further implies that $M, \iota(w) \not\models \langle x \rangle A$. By the side condition on $\langle x \rangle\mathsf{R}$, we know that $\mathsf{pg}(\mathcal{H}) \models w \overset{L_x}{\rightsquigarrow} u$. By Lemma 3.9, $\iota(w) R \iota(u)$. Hence, $M, \iota(u) \not\models A$, showing that $M, \iota \not\models \mathcal{G}\{\Gamma \vdash \Delta\}_w\{\Sigma \vdash A, \Pi\}_u$. □

To prove completeness, one can show that all axioms in $\mathsf{HIK_m}(\mathcal{A})$ are provable in $\mathsf{NIK_m}(\mathcal{A})$ and all rules of $\mathsf{HIK_m}(\mathcal{A})$ are admissible (i.e., if the premises are provable, then the conclusion is provable). These proofs are a straightforward exercise, and so, we omit them. We note that mp is admissible using the cut rule introduced in Section 4, which is itself an admissible rule (see Theorem 4.6).

Theorem 3.11 (Completeness)**.** *If A is $\mathcal{A}$-valid, then $\mathsf{NIK_m}(\mathcal{A}) \Vdash A$.*

3.2. Separation and Related Systems.

Our nested sequent systems also satisfy the *separation property*, originally introduced in the context of nested calculi for tense logics [24]. A calculus $\mathsf{NIK_m}(\mathcal{A})$ has the separation property if and only if, whenever $\mathsf{NIK_m}(\mathcal{A}) \Vdash A$ and the formula A uses only modalities indexed with characters from $\Sigma' \subseteq \Sigma$, there exists a proof of A in $\mathsf{NIK_m}(\mathcal{A})$ that employs nested sequents using only characters drawn from Σ', i.e., that do not mention any symbols in $\Sigma \setminus \Sigma'$. This property is significant in our setting because it entails that IGLs are conservative over IMLs.

Let us restrict $\Sigma = \{a\}$ to a single character and define $\Diamond := \langle a \rangle$ and $\Box := [a]$. Moreover, we define $(\Diamond^n \Box A \rightarrow \Box^k A) \wedge (\Diamond^k A \rightarrow \Box^n \Diamond A), (\Box A \rightarrow \Diamond A) \in m(\mathcal{A})$ *iff* $(\langle \overline{a} \rangle^n \langle a \rangle^k A \rightarrow \langle a \rangle A) \wedge ([a]A \rightarrow [\overline{a}]^n [a]^k A), ([a]A \rightarrow \langle a \rangle) \in \mathcal{A}$. We refer to axioms of the latter form as *Horn-Scott-Lemmon axioms (HSL axioms)* (cf. [39]). In such a scenario, the calculus $\mathsf{NIK_m}(\mathcal{A})$ becomes a notational variant of the calculus $\mathsf{NIK}(m(\mathcal{A}))$ presented in [39] that is sound and complete for the extension of IK with the axioms $m(\mathcal{A})$.

Theorem 3.12. *$\mathsf{IK_m}(\mathcal{A})$ is a conservative extension of $\mathsf{IK} \cup m(\mathcal{A})$, i.e., if $\vdash_{\mathcal{A}} A$ and all modalities of A are indexed by a, then A is a theorem of $\mathsf{IK} \cup m(\mathcal{A})$.*

Proof. Suppose $\vdash_{\mathcal{A}} A$ and all modalities of A are indexed by a. By completeness, $\mathsf{NIK_m}(\mathcal{A}) \Vdash A$ and by the separation property we know that a proof π exists in $\mathsf{NIK_m}(\mathcal{A})$ that only uses the single character a in nested sequents. Hence, the proof is a notational variant of a proof in $\mathsf{NIK}(m(\mathcal{A}))$ for the logic $\mathsf{IK} \cup m(\mathcal{A})$, showing that A is a theorem of $\mathsf{IK} \cup m(\mathcal{A})$. □

Moreover, if we fix $\Sigma := \{a, \overline{a}\}$, and define $\mathsf{F} := \langle a \rangle$, $\mathsf{P} := \langle \overline{a} \rangle$, $\mathsf{G} := [a]$, and $\mathsf{H} := [\overline{a}]$, then $\mathsf{NIK_m}(\mathcal{A})$ becomes a sound and complete calculus for Ewald's tense logic IKt [15] extended with the axioms in $\mathcal{A}$. We refer to such a logic as an *intuitionistic tense logic (ITL)* and refer to a set $\mathcal{A}$ of axioms within the signature $\Sigma := \{a, \overline{a}\}$ as *intuitionistic tense path axioms (ITP axioms)*.

As a final observation, we also remark that our systems are single-conclusioned (i.e., intuitionistic) variants of the nested sequent systems for classical grammar logics, introduced by Tiu et al. [55]. If we relax the single-conclusioned restriction imposed on nested sequents in $\mathsf{NIK_m}(\mathcal{A})$, this gives rise to a nested sequent calculus that is sound and complete for the classical grammar logic $\mathsf{K_m}$ extended with the axioms in $\mathcal{A}$. In the classical setting, each IPA $(\langle s \rangle A \rightarrow \langle x \rangle A) \wedge ([x]A \rightarrow [s]A)$ is equivalent to the *path axiom* $\langle s \rangle A \rightarrow \langle x \rangle A$ (cf. [55]).

§4. Proof-Theoretic Properties

In this section, we show that each nested calculus $\mathsf{NIK_m}(\mathcal{A})$ satisfies a wide array of fundamental properties, which we leverage in our proof of syntactic cut-admissibility. In particular, we show that various rules are *(height-preserving) admissible*, meaning if the premises of the rule have proofs (of height $h_1, \ldots, h_n$), then the conclusion of the rule has a proof (of height $h \leq \max\{h_1, \ldots, h_n\}$). With the exception of $\rightarrow\mathsf{L}$, we also prove all left rules of $\mathsf{NIK_m}(\mathcal{A})$ *height-preserving invertible*. If we let r_i^{-1} be the i-inverse of the rule r whose conclusion is the i^{th} premise of the n-ary rule r and premise is the conclusion of r, then we say that r is *(height-preserving) invertible iff* r_i^{-1} is (height-preserving) admissible for each $i \in [n]$. (NB. We refer

$$
\begin{tikzcd}
\mathsf{wL} \arrow[dr] & & \mathsf{ew} \arrow[dl] \\
& \text{Lem. 4.3} \arrow[dl] \arrow[dr] & \\
\mathsf{cL} & & \mathsf{ec} \arrow[ll, bend left]
\end{tikzcd}
$$

Figure 3: A diagram depicting which hp-admissibility (and hp-invertibility) results are sufficient to prove others. An arrow from one rule name to another indicates that the hp-admissibility of the source rule is sufficient to prove the hp-admissibility of the target rule. Note that 'Lem. 4.3' indicates the hp-invertibility of certain rules in $\mathsf{NIK_m}(\mathcal{A})$.

to a height-preserving admissible and height-preserving invertible rule as *hp-admissible* and *hp-invertible*, respectively.)

We will show all rules in Figure 4 hp-admissible in $\mathsf{NIK_m}(\mathcal{A})$, i.e., we prove the *right bottom rule* $\bot\mathsf{R}$, *necessitation rule* nec_x, *left weakening rule* wL, *right weakening rule* wR, *external weakening rule* ew, *left contraction rule* cL, *external contraction rule* ec, and *shift rule* $\mathsf{sft}(\mathcal{A})$ hp-admissible. These hp-admissibility results are crucial for proving cut-admissibility. As we will see below, various hp-admissibility results will be used to prove further structural rules hp-admissible. To make the dependencies between these various lemmas clear, we have provided a diagram in Figure 3 that displays which hp-admissible rules are sufficient to prove other rules hp-admissible; rules omitted from the diagram are hp-admissible of their own accord.

The shift rule $\mathsf{sft}(\mathcal{A})$ is a novel contribution of this paper and is interesting as it unifies reasoning with IPAs in a single rule. The $\mathsf{sft}(\mathcal{A})$ rule utilizes the $\odot$ operation, which is defined as follows: Let $\mathcal{G} := \Gamma \vdash \Delta, (x_1)[\mathcal{H}_1], \ldots, (x_n)[\mathcal{H}_n]$ and $\mathcal{K} := \Sigma \vdash \Pi, (x_{n+1})[\mathcal{J}_{n+1}], \ldots, (x_{n+k})[\mathcal{J}_{n+k}]$ be nested sequents, then

$$\mathcal{G} \odot \mathcal{K} := \Gamma, \Sigma \vdash \Delta, \Pi, (x_1)[\mathcal{H}_1], \ldots, (x_n)[\mathcal{H}_n], (x_{n+1})[\mathcal{J}_{n+1}], \ldots, (x_{n+k})[\mathcal{J}_{n+k}].$$

This rule streamlines our proofs because it does not require one to use a special structural rule for *each* IPA, which contrasts with prior approaches [5, 43, 54]. Moreover, as pointed out by Brünnler [5], having a distinct structural rule for each axiom extending the base logic can lead to a loss of modularity, requiring the addition of even more structural rules to secure completeness. This allows for us to prove the hp-admissibility of a single, generic rule that simplifies our proof of cut-admissibility, making it uniform over the class of logics we consider.

In the following subsection, we prove a sequence of hp-admissibility and hp-invertibility results, which will be sufficient to establish cut-admissibility in the Section 4.2.

4.1. Admissibility and Invertibility

The lemma below establishes that a generalized form of the id rule is admissible in $\mathsf{NIK_m}(\mathcal{A})$. This property ensures that provable nested sequents are closed under substitutions, which is essential for completeness (cf. [37, Lemma 12]). It is straightforward to prove the result by induction on the length of A.

Lemma 4.1. *For $A \in \mathscr{L}$, $\mathcal{G}\{\Gamma, A \vdash A\}$ is provable in* $\mathsf{NIK_m}(\mathcal{A})$.

$$\frac{\mathcal{G}\{\Gamma \vdash \bot\}}{\mathcal{G}\{\Gamma \vdash \emptyset\}}\ \bot\mathsf{R} \qquad \frac{\mathcal{G}\{\Gamma \vdash \Delta\}}{\mathcal{G}\{\Gamma, \Sigma \vdash \Delta\}}\ \mathsf{wL} \qquad \frac{\mathcal{G}^{\downarrow}\{\Gamma \vdash \emptyset\}}{\mathcal{G}\{\Gamma \vdash A\}}\ \mathsf{wR}$$

$$\frac{\mathcal{G}}{\emptyset \vdash \emptyset, (x)[\mathcal{G}]}\ \mathsf{nec}_x \qquad \frac{\mathcal{G}\{\Gamma, A, A \vdash \Delta\}}{\mathcal{G}\{\Gamma, A \vdash \Delta\}}\ \mathsf{cL} \qquad \frac{\mathcal{G}\{\Gamma \vdash \Delta\}}{\mathcal{G}\{\Gamma \vdash \Delta, (x)[\emptyset \vdash \emptyset]\}}\ \mathsf{ew}$$

$$\frac{\mathcal{G}\{\Gamma \vdash \Delta, (x)[\mathcal{H}], (x)[\mathcal{H}]\}}{\mathcal{G}\{\Gamma \vdash \Delta, (x)[\mathcal{H}]\}}\ \mathsf{ec} \qquad \frac{\mathcal{G}\{\Gamma \vdash \Delta, (x)[\mathcal{H}]\}_w\{\mathcal{K}\}_v}{\mathcal{G}\{\Gamma \vdash \Delta\}_w\{\mathcal{K} \odot \mathcal{H}\}_v}\ \mathsf{sft}(\mathcal{A})^{\dagger(\mathcal{A})}$$

Side Conditions:
$\dagger(\mathcal{A}) := w \overset{L_x}{\rightsquigarrow} u$ with $L_x = L_{g(\mathcal{A})}(x)$.

Figure 4: Height-preserving admissible rules in $\mathsf{NIK_m}(\mathcal{A})$.

The remaining hp-admissibility results are all proven by induction on the height of the given proof π.

Lemma 4.2. *The* $\bot\mathsf{R}$*,* nec_x*,* wL*,* wR*, and* ew *rules are hp-admissible in* $\mathsf{NIK_m}(\mathcal{A})$*.*

Proof. Let $\mathsf{r} \in \{\bot\mathsf{R}, \mathsf{nec}_x, \mathsf{wL}, \mathsf{wR}, \mathsf{ew}\}$. We make a case distinction on the last rule $\mathsf{r}' \in \mathsf{NIK_m}(\mathcal{A})$ applied in the given proof π. Observe that applying r (if possible) to an id or $\bot\mathsf{L}$ instance gives another instance of that rule, which resolves the base case. For the inductive step, each case is resolved by applying IH (i.e., the induction hypothesis) to the premise(s) of r' followed by r'. □

As depicted in Figure 3, the hp-admissibility of the two contraction rules cL and ec depends on the hp-invertibility of certain rules in $\mathsf{NIK_m}(\mathcal{A})$. In contrast to the nested systems for classical grammar logics [55], not all rules in $\mathsf{NIK_m}(\mathcal{A})$ are invertible; e.g., the $\to\mathsf{L}$ rule is *not* invertible in the left premise as witnessed by the following example:

$$\frac{p \to q \vdash p \qquad p \to q, q \vdash p \to q}{p \to q \vdash p \to q}\ \to\mathsf{L}$$

One can still establish that all remaining left rules in $\mathsf{NIK_m}(\mathcal{A})$ are hp-invertible, which is sufficient to obtain the hp-admissibility of the contraction rules.

Lemma 4.3. *The* $\vee\mathsf{L}$*,* $\wedge\mathsf{L}$*,* $\langle x\rangle\mathsf{L}$*,* $[x]\mathsf{L}$*, and* d_x *rules are hp-invertible in* $\mathsf{NIK_m}(\mathcal{A})$*.*

Proof. The hp-invertibility of $[x]\mathsf{L}$ and d_x follow from the hp-admissibility of wk and ew. The remaining cases are argued as usual by induction on the height of the given proof. □

Lemma 4.4. *The* cL *and* ec *rules are hp-admissible in* $\mathsf{NIK_m}(\mathcal{A})$*.*

Proof. We prove the result by a simultaneous induction on the height of the given proof. The base cases are simple since any application of cL or ec to an initial rule gives another instance of that rule; hence, we focus on the inductive step. For the inductive step, we show the $\langle x\rangle\mathsf{L}$ case, noting that the other cases are simple or similar.

We consider the case where cR is preceded by an application of $\langle x \rangle$L and where the principal formula of $\langle x \rangle$L is auxiliary in cR. This case is shown below left and is resolved as shown below right. We first use the hp-invertibility of $\langle x \rangle$L (Lemma 4.3), then apply IH relative to ec, and last apply the $\langle x \rangle$L rule.

$$
\dfrac{\dfrac{\dfrac{\vdots \pi}{\mathcal{G}\{\Gamma, \langle x \rangle A \vdash \Delta, (x)[A \vdash \emptyset]\}}}{\mathcal{G}\{\Gamma, \langle x \rangle A, \langle x \rangle A \vdash \Delta\}}\ \langle x \rangle \mathsf{L}}{\mathcal{G}\{\Gamma, \langle x \rangle A \vdash \Delta\}}\ \mathsf{cR}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\vdots \pi'}{\mathcal{G}\{\Gamma, \langle x \rangle A \vdash \Delta, (x)[A \vdash \emptyset]\}}}{\mathcal{G}\{\Gamma \vdash \Delta, (x)[A \vdash \emptyset], (x)[A \vdash \emptyset]\}}\ \langle x \rangle \mathsf{L}_1^{-1}}{\mathcal{G}\{\Gamma \vdash \Delta, (x)[A \vdash \emptyset]\}}\ \mathsf{ec}}{\mathcal{G}\{\Gamma, \langle x \rangle A \vdash \Delta\}}\ \langle x \rangle \mathsf{L}
$$

This case shows that the hp-admissibility of cR depends on that of ec. □

Lemma 4.5. *The* $\mathsf{sft}(\mathcal{A})$ *rule is hp-admissible in* $\mathsf{NIK_m}(\mathcal{A})$*.*

Proof. If $\mathsf{sft}(\mathcal{A})$ is applied to an initial rule, then the result is another instance of an initial rule, showing the base case of induction. For the inductive step, we consider the cases where $\mathsf{sft}(\mathcal{A})$ is applied to the conclusion of an $\langle x \rangle$R rule and $[x]$R rule. The remaining cases are analogous or straightforward.

$\langle x \rangle$R. Let us consider an application of $\langle x \rangle$R followed by an application of $\mathsf{sft}(\mathcal{A})$, as shown below.

$$
\dfrac{\dfrac{\dfrac{\vdots \pi}{\mathcal{G}_0\{\Gamma \vdash \emptyset\}_w\{\Sigma \vdash A\}_u\{\Phi \vdash \emptyset, (y)[\mathcal{H}]\}_v\{\mathcal{K}\}_z}}{\mathcal{G}_0\{\Gamma \vdash \langle x \rangle A\}_w\{\Sigma \vdash \emptyset\}_u\{\Phi \vdash \emptyset, (y)[\mathcal{H}]\}_v\{\mathcal{K}\}_z}\ \langle x \rangle \mathsf{R}}{\mathcal{G}_1\{\Gamma \vdash \langle x \rangle A\}_w\{\Sigma \vdash \emptyset\}_u\{\Phi \vdash \emptyset\}_v\{\mathcal{K} \odot \mathcal{H}\}_z}\ \mathsf{sft}(\mathcal{A})
$$

Let $L_x = L_{g(\mathcal{A})}(x)$ and $L_y = L_{g(\mathcal{A})}(y)$. The side condition on the $\langle x \rangle$R rule ensures that $w \overset{L_x}{\rightsquigarrow} u$ in $\mathcal{G}_0$ and the side condition on rule $\mathsf{sft}(\mathcal{A})$ ensures that $v \overset{L_y}{\rightsquigarrow} z$ in $\mathcal{G}_0$. We have to show that both side conditions continue to hold after we permute $\mathsf{sft}(\mathcal{A})$ above $\langle x \rangle$R, thus giving the proof shown below.

$$
\dfrac{\dfrac{\dfrac{\vdots \pi}{\mathcal{G}_0\{\Gamma \vdash \emptyset\}_w\{\Sigma \vdash A\}_u\{\Phi \vdash \emptyset, (y)[\mathcal{H}]\}_v\{\mathcal{K}\}_z}}{\mathcal{G}_0\{\Gamma \vdash \emptyset\}_w\{\Sigma \vdash A\}_u\{\Phi \vdash \emptyset\}_v\{\mathcal{K} \odot \mathcal{H}\}_z}\ \mathsf{sft}(\mathcal{A})}{\mathcal{G}_1\{\Gamma \vdash \langle x \rangle A\}_w\{\Sigma \vdash \emptyset\}_u\{\Phi \vdash \emptyset\}_v\{\mathcal{K} \odot \mathcal{H}\}_z}\ \langle x \rangle \mathsf{R}
$$

If the w-component and the u-component both occur in $\mathcal{H}$ or neither occurs in $\mathcal{H}$, then $\mathsf{sft}(\mathcal{A})$ can be permuted above $\langle x \rangle$R because the two rules do not interact. Let us consider the case where the u-component occurs in $\mathcal{H}$ and the w-component does not. (NB. The case where the w-component occurs in $\mathcal{H}$ and the u-component does not is proven similarly.) Let v' be the name of the component that serves as the root of $\mathcal{H}$. It follows that the propagation path from w to u in $\mathcal{G}_0$ is of the form $w \overset{s}{\rightsquigarrow} v \overset{y}{\rightsquigarrow} v' \overset{t}{\rightsquigarrow} u$ with $syt \in L_x$. By the side condition of the $\mathsf{sft}(\mathcal{A})$ rule, we know there exists a string $r \in L_y$ such that $v \overset{r}{\rightsquigarrow} z$. Since $L_y = L_{g(\mathcal{A})}(y)$, this implies that $y \longrightarrow^*_{g(\mathcal{A})} r$, which further implies that $srt \in L_x$. This string corresponds to the propagation path $w \overset{s}{\rightsquigarrow} v \overset{r}{\rightsquigarrow} z \overset{t}{\rightsquigarrow} u$ in $\mathcal{G}_1$, showing that after $\mathsf{sft}(\mathcal{A})$ has been applied the side condition on $\langle x \rangle$R still holds, and thus, the two rules may indeed be permuted as shown above.

$[x]\mathsf{R}$. Suppose we have an instance of $[x]\mathsf{R}$ followed by an instance of $\mathsf{sft}(\mathcal{A})$, as shown below.

$$
\dfrac{\dfrac{\dfrac{\vdots \pi}{\mathcal{G}\{\Gamma \vdash (x)[\emptyset \vdash A]\}_w\{\Sigma \vdash \emptyset, (y)[\mathcal{H}]\}_u\{\mathcal{K}\}_v}}{\mathcal{G}\{\Gamma \vdash [x]A\}_w\{\Sigma \vdash \emptyset, (y)[\mathcal{H}]\}_u\{\mathcal{K}\}_v}\ [x]\mathsf{R}}{\mathcal{G}\{\Gamma \vdash [x]A\}_w\{\Sigma \vdash \emptyset\}_u\{\mathcal{K} \odot \mathcal{H}\}_v}\ \mathsf{sft}(\mathcal{A})
$$

Observe that we can simply permute the $\mathsf{sft}(\mathcal{A})$ application above the $[x]\mathsf{R}$ application to obtain the desired result. □

4.2. Syntactic Cut-Admissibility

As discussed in prior sections, Tiu et al. [55] provided (both shallow and deep) nested sequent systems for classical grammar logics. A cut admissibility algorithm was given for the shallow systems by utilizing a strategy akin to Belnap's general cut admissibility theorem for display calculi [2, Theorem 4.4]. Cut admissibility was then shown for the deep nested systems in a roundabout fashion by proving a syntactic correspondence between the shallow and deep nested systems. It is conceivable that we could achieve cut admissibility in the intuitionistic setting by employing a similar (indirect) strategy; however, we opt for a syntactic cut admissibility algorithm that eliminates cuts *directly within* our nested systems. We remark that this approach can be adapted to the classical setting–giving a cut admissibility procedure that operates directly within the deep nested sequent proofs of Tiu et al. [55].

In the deep-inference setting (and in contrast to the shallow setting), the proof of cut admissibility is more complex because we must keep track of propagation paths and ensure the satisfaction of side conditions on propagation rules after upward permutations of cuts (see Theorem 4.6 below). Still, beyond being direct, our cut admissibility proof also has the advantage that it is *uniform* over the class of logics we consider. This is primarily due to our use of the hp-admissible shift rule $\mathsf{sft}(\mathcal{A})$, which unifies reasoning with IPAs in a single rule. This also allows for us to circumvent the use of *ad hoc* structural rules to deal with special cut admissibility cases, as is the case in other settings [5, 43, 54].

Theorem 4.6 (Cut-Admissibility)**.** *The* cut *rule is admissible in* $\mathsf{NIK_m}(\mathcal{A})$.

$$
\dfrac{\mathcal{G}^{\downarrow}\{\Gamma \vdash A\} \qquad \mathcal{G}\{\Gamma, A \vdash \Delta\}}{\mathcal{G}\{\Gamma \vdash \Delta\}}\ \mathsf{cut}
$$

Proof. We prove the result by simultaneous induction on the lexicographic ordering of pairs of the form $(|A|, h_1 + h_2)$, where $|A|$ is the length of the cut formula A, and h_1 and h_2 are the heights of the proofs of the left and right premises of cut, respectively. We assume w.l.o.g. that cut occurs only once in the given proof as last inference; the general result follows by successively applying the cut admissibility procedure to topmost instances of cut in a given proof. Note that cut can always be permuted above applications of d_x by utilizing the hp-admissibility of ew, and thus, we may omit consideration of d_x below as all such cases hold.

We organize the proof into three exhaustive cases: (1) at least one premise of cut is an instance of id or $\bot\mathsf{L}$, (2) the cut formula is not principal in at least one premise, and (3) the cut formula is principal in both premises.

(1) Let us consider the case where the left premise of cut is an instance of $\bot\mathsf{L}$ or id. If the left premise of cut is an instance of $\bot\mathsf{L}$, then the conclusion of cut will be an instance of $\bot\mathsf{L}$, showing that cut can be replaced by an application of the $\bot\mathsf{L}$ rule.

Let us now suppose that the left premise of cut is an instance of id. Then, the principal (output) formula p of id is the cut formula. The cut is of the form shown below left and can be removed by applying the hp-admissible cL rule shown below right.

$$\dfrac{\dfrac{}{\mathcal{G}^{\downarrow}\{\Gamma, p \vdash p\}}\mathsf{id} \qquad \dfrac{\vdots \pi}{\mathcal{G}\{\Gamma, p, p \vdash \Delta\}}}{\mathcal{G}\{\Gamma, p \vdash \Delta\}}\mathsf{cut} \qquad \dfrac{\dfrac{\vdots \pi}{\mathcal{G}\{\Gamma, p, p \vdash \Delta\}}}{\mathcal{G}\{\Gamma, p \vdash \Delta\}}\mathsf{cL}$$

Next, we suppose that the right premise of cut is an instance of $\bot\mathsf{L}$ or id. In the former case, if the principal formula $\bot$ is not the cut formula, then the conclusion will be an instance of $\bot\mathsf{L}$ and may therefore be proven with an instance of $\bot\mathsf{L}$. If the principal formula $\bot$ is the cut formula, as shown below left, then the case may be resolved with the hp-admissible $\bot\mathsf{R}$ and wR rules, as shown below right.

$$\dfrac{\dfrac{\vdots \pi}{\mathcal{G}^{\downarrow}\{\Gamma \vdash \bot\}} \qquad \dfrac{}{\mathcal{G}\{\Gamma, \bot \vdash \Delta\}}\bot\mathsf{L}}{\mathcal{G}\{\Gamma \vdash \Delta\}}\mathsf{cut} \qquad \dfrac{\dfrac{\dfrac{\vdots \pi}{\mathcal{G}\{\Gamma \vdash \bot\}}}{\mathcal{G}\{\Gamma \vdash \emptyset\}}\bot\mathsf{R}}{\mathcal{G}\{\Gamma \vdash \Delta\}}\mathsf{wR}$$

If the right premise of cut is an instance of id, then there are two cases to consider: first, if the principal formula p in the right premise is not the cut formula, then the conclusion of cut will be an instance of id, so the cut can be replaced by an id application. Second, if the principal formula p in the right premise is the cut formula, then the conclusion of cut will be identical to its left premise, so the cut can be replaced by the proof of its left premise.

(2) Let us suppose that the cut formula is not principal in the left premise of cut; the case where the cut formula is not principal in the right premise of cut is argued similarly. First, let us consider the case where r is a left rule other than $\to\mathsf{L}$ that proves the left premise of cut, as shown below left. We also suppose that r is a unary rule as the binary case is similar. To resolve the case, we apply the hp-invertibility of r to the conclusion of π_2, then cut with the premise of r, and then apply r after cut, as shown below right.

$$\dfrac{\dfrac{\dfrac{\vdots \pi_1}{\mathcal{H}^{\downarrow}\{\Gamma' \vdash A\}}}{\mathcal{G}^{\downarrow}\{\Gamma \vdash A\}}\mathsf{r} \qquad \dfrac{\vdots \pi_2}{\mathcal{G}\{\Gamma, A \vdash \Delta\}}}{\mathcal{G}\{\Gamma \vdash \Delta\}}\mathsf{cut} \qquad \dfrac{\dfrac{\dfrac{\vdots \pi_1}{\mathcal{H}^{\downarrow}\{\Gamma' \vdash A\}} \qquad \dfrac{\dfrac{\vdots \pi_2}{\mathcal{G}\{\Gamma, A \vdash \Delta\}}}{\mathcal{H}\{\Gamma', A \vdash \Delta\}}\mathsf{r}_1^{-1}}{\mathcal{H}\{\Gamma' \vdash \Delta\}}\mathsf{cut}}{\mathcal{G}\{\Gamma \vdash \Delta\}}\mathsf{r}$$

Let us suppose now that the left premise of cut is proven using $\to\mathsf{L}$ and suppose w.l.o.g. that the rule affects the same component as the cut formula. As the cut formula A is assumed not principal in the left premise, our cut is of the form shown below top. It can be resolved as shown below bottom by shifting cut up the right premise of $\to\mathsf{L}$.

$$
\dfrac{\dfrac{\dfrac{\vdots\pi_1}{\mathcal{G}^{\downarrow}\{\Gamma, B \to C \vdash B\}} \qquad \dfrac{\vdots\pi_2}{\mathcal{G}\{\Gamma, B \to C, C \vdash A\}}}{\mathcal{G}^{\downarrow}\{\Gamma, B \to C \vdash A\}}\,{\to}\mathsf{L} \qquad \dfrac{\vdots\pi_3}{\mathcal{G}\{\Gamma, A \vdash \Delta\}}}{\mathcal{G}\{\Gamma, B \to C \vdash \Delta\}}\,\mathsf{cut}
$$

$$
\dfrac{\dfrac{\vdots\pi_1}{\mathcal{G}^{\downarrow}\{\Gamma, B \to C \vdash B\}} \qquad \dfrac{\dfrac{\vdots\pi_2}{\mathcal{G}^{\downarrow}\{\Gamma, B \to C, C \vdash A\}} \qquad \dfrac{\dfrac{\vdots\pi_3}{\mathcal{G}\{\Gamma, A \vdash \Delta\}}}{\mathcal{G}\{\Gamma, B \to C, C, A \vdash \Delta\}}\,\mathsf{wL}}{\mathcal{G}\{\Gamma, B \to C, C \vdash \Delta\}}\,\mathsf{cut}}{\mathcal{G}\{\Gamma, B \to C \vdash \Delta\}}\,{\to}\mathsf{L}
$$

(3) We consider two non-trivial cases and note that the remaining cases are simple or similar. First, we consider the case where the cut formula is principal in an application of $\to$R in the left premise and an application of $\to$L in the right premise.

$$
\dfrac{\dfrac{\dfrac{\vdots\pi_1}{\mathcal{G}^{\downarrow}\{\Gamma, A \vdash B\}}}{\mathcal{G}^{\downarrow}\{\Gamma \vdash A \to B\}}\,{\to}\mathsf{R} \qquad \dfrac{\dfrac{\vdots\pi_2}{\mathcal{G}^{\downarrow}\{\Gamma, A \to B \vdash A\}} \qquad \dfrac{\vdots\pi_3}{\mathcal{G}\{\Gamma, A \to B, B \vdash \Delta\}}}{\mathcal{G}\{\Gamma, A \to B \vdash \Delta\}}\,{\to}\mathsf{L}}{\mathcal{G}\{\Gamma \vdash \Delta\}}\,\mathsf{cut}
$$

To remove the cut, we first reduce the height of the cut by applying a cut between the conclusion of $\to$R and the conclusion of π_2, and by applying another cut between the conclusion of $\to$R and the conclusion of π_3. This gives cut-free proofs of $\mathcal{G}\{\Gamma \vdash A\}$ and $\mathcal{G}\{\Gamma, B \vdash \Delta\}$ by IH since $h_1 + h_2$ has decreased. We then apply a cut between the conclusion of π_1 and the former nested sequent to obtain a cut-free proof of $\mathcal{G}\{\Gamma \vdash B\}$, which can then be cut with $\mathcal{G}\{\Gamma, B \vdash \Delta\}$. These last two cuts can be eliminated since the cut formulae are of a strictly smaller length.

Last, we show how to resolve the case where the cut formula is principal in an application of $[x]$R in the left premise and an application of $[x]$L in the right premise.

$$
\dfrac{\dfrac{\dfrac{\vdots\pi_1}{\mathcal{G}^{\downarrow}\{\Gamma \vdash (x)[\emptyset \vdash A]\}_w\{\Sigma \vdash \emptyset\}_u}}{\mathcal{G}^{\downarrow}\{\Gamma \vdash [x]A\}_w\{\Sigma \vdash \emptyset\}_u}\,[x]\mathsf{R} \qquad \dfrac{\dfrac{\vdots\pi_2}{\mathcal{G}\{\Gamma, [x]A \vdash \Delta\}_w\{\Sigma, A \vdash \Pi\}_u}}{\mathcal{G}\{\Gamma, [x]A \vdash \Delta\}_w\{\Sigma \vdash \Pi\}_u}\,[x]\mathsf{L}}{\mathcal{G}\{\Gamma \vdash \Delta\}_w\{\Sigma \vdash \Pi\}_u}\,\mathsf{cut}
$$

We first reduce the height of the cut as shown below, relying on the hp-admissible wL rule. We let π_3 denote the resulting proof.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots\pi_1}{\mathcal{G}^{\downarrow}\{\Gamma \vdash (x)[\emptyset \vdash A]\}_w\{\Sigma \vdash \emptyset\}_u}}{\mathcal{G}^{\downarrow}\{\Gamma \vdash [x]A\}_w\{\Sigma \vdash \emptyset\}_u}\,[x]\mathsf{R}}{\mathcal{G}^{\downarrow}\{\Gamma \vdash [x]A\}_w\{\Sigma, A \vdash \emptyset\}_u}\,\mathsf{wL} \qquad \dfrac{\vdots\pi_2}{\mathcal{G}\{\Gamma, [x]A \vdash \Delta\}_w\{\Sigma, A \vdash \Pi\}_u}}{\mathcal{G}\{\Gamma \vdash \Delta\}_w\{\Sigma, A \vdash \Pi\}_u}\,\mathsf{cut}
$$

Due to the side condition on the $\langle x \rangle$R rule, we know that $w \overset{L_x}{\rightsquigarrow} u$. Therefore, we may apply the hp-admissible $\mathsf{sft}(\mathcal{A})$ rule as shown below, and then apply a cut on the formula A, which is of a smaller length.

$$\dfrac{\dfrac{\dfrac{\vdots \pi_1}{\mathcal{G}^{\downarrow}\{\Gamma \vdash (x)[\emptyset \vdash A]\}_w\{\Sigma \vdash \emptyset\}_u}}{\mathcal{G}^{\downarrow}\{\Gamma \vdash \emptyset\}_w\{\Sigma \vdash A\}_u}\,\mathsf{sft}(\mathcal{A}) \qquad \vdots \pi_3}{\mathcal{G}\{\Gamma \vdash \Delta\}_w\{\Sigma \vdash \Pi\}_u}\,\mathsf{cut}$$

This concludes the proof of cut-admissibility. □

§5. Interpolation and Definability

In this section, we establish that every IGL enjoys the Lyndon interpolation property by adapting the syntactic interpolation method introduced in [38] and employed in [37, Section 6.2] to obtain Lyndon interpolation for classical grammar logics. Although our proof-theoretic strategy is inspired by these and related approaches [20, 31, 38, 37], it departs from them in one crucial respect: whereas existing work employs multi-conclusioned nested sequent systems, our interpolation algorithm is developed for single-conclusioned ones.

In the context of (multi)sequent systems, interpolation algorithms operate as follows: first, one 'partitions' the data present in a proof of an implication $A \to B$ into left and right components. Second, one assigns interpolants to the axioms, which are implied by the left component and imply the right component of the axiom. Third, one defines how to construct interpolants for the conclusions of rule applications given interpolants of the premises. Using the method of [37, 38], this algorithm results in the construction of two proofs: one of $A \to I$ and one of $I \to B$ with I the interpolant.

The method presented below follows the above strategy, yet, due to the single-conclusioned restriction on nested sequents, more care is needed to ensure the correctness of the interpolation algorithm. We cannot directly apply the methodology of [37, 38] to our nested systems, as the output proofs would use multi-conclusioned nested sequents. To address this, we introduce a more restrictive notion of interpolant tailored to the intuitionistic setting: an interpolant is a set of name–formula pairs of the form $w : A$, where each pair encodes a nested sequent with a single consequent formula. Moreover, the meta-operators used in [37, 38] to compute interpolants of rule conclusions from those of their premises are unsound in the single-conclusioned intuitionistic context. We therefore develop a new family of meta-operators specifically designed for our setting, enabling the interpolant construction to proceed while preserving correctness.

5.1. Syntactic Lyndon Interpolation.

We define a *biased Gentzen sequent* to be an expression of the form $\Gamma_1 \mid \Gamma_2 \vdash \Delta$ such that $\Gamma_1, \Gamma_2 \vdash \Delta$ is a Gentzen sequent. We define a *biased nested sequent* to be an expression of the form $\Gamma_1 \mid \Gamma_2 \vdash \Delta, (x_1)[\widetilde{\mathcal{G}}_1], \ldots, (x_n)[\widetilde{\mathcal{G}}_n]$ such that

$$f(\Gamma_1 \mid \Gamma_2 \vdash \Delta, (x_1)[\widetilde{\mathcal{G}}_1], \ldots, (x_n)[\widetilde{\mathcal{G}}_n]) = \Gamma_1, \Gamma_2 \vdash \Delta, (x_1)[f(\widetilde{\mathcal{G}}_1)], \ldots, (x_n)[f(\widetilde{\mathcal{G}}_n)]$$

is a nested sequent. We refer to biased nested sequents as *biased sequents* more simply and use $\widetilde{\mathcal{G}}, \widetilde{\mathcal{H}}, \ldots$ to denote them. Most terminology introduced for nested sequents straightforwardly applies to biased nested sequents; e.g., a *component* of a biased sequent $\widetilde{\mathcal{G}}$ is a biased Gentzen sequent $\Gamma_1 \mid \Gamma_2 \vdash \Delta$ occurring in $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}^{\downarrow}$ is the biased nested sequent obtained by deleting (if it exists) the single output formula.

A biased sequent $\widetilde{\mathcal{G}}$ contains a *left part* $\widetilde{\mathcal{G}}^{\mathsf{L}}$ and *right part* $\widetilde{\mathcal{G}}^{\mathsf{R}}$, defined by:

(1) if $\widetilde{\mathcal{G}} = \Gamma_1 \mid \Gamma_2 \vdash \Delta$, then $\widetilde{\mathcal{G}}^{\mathsf{L}} := \Gamma_1 \vdash \emptyset$ and $\widetilde{\mathcal{G}}^{\mathsf{R}} := \Gamma_2 \vdash \Delta$;

(2) if $\widetilde{\mathcal{G}} = \Gamma_1 \mid \Gamma_2 \vdash \Delta, (x_1)[\widetilde{\mathcal{G}}_1], \ldots, (x_n)[\widetilde{\mathcal{G}}_n]$, then for $\mathsf{B} \in \{\mathsf{L}, \mathsf{R}\}$, we define $\widetilde{\mathcal{G}}^{\mathsf{B}} := (\Gamma_1 \mid \Gamma_2 \vdash \Delta)^{\mathsf{B}}, (x_1)[\widetilde{\mathcal{G}}_1^{\mathsf{B}}], \ldots, (x_n)[\widetilde{\mathcal{G}}_n^{\mathsf{B}}]$.

By the above definition, we always assume that consequents of components occur in the right part, meaning, the unique output formula does as well. Given a biased sequent $\widetilde{\mathcal{G}}$, we define its corresponding nested sequent $\mathcal{G}$ to be the nested sequent obtained by replacing each component $\Gamma_1 \mid \Gamma_2 \vdash \Delta$ by $\Gamma_1, \Gamma_2 \vdash \Delta$, and we call $\widetilde{\mathcal{G}}$ a *biasing* of $\mathcal{G}$.

We can extend the definition of a signature (see Definition 2.12) to nested and biased sequents in the natural way. If $\mathcal{G}$ is a nested sequent with $A := \mathsf{OutF}(\mathcal{G})$ the single-conclusion, then:[6]

$$\mathsf{sig}^{+}(\mathcal{G}) := \mathsf{sig}^{+}(A) \cup \bigcup_{B \in \mathsf{InF}(\mathcal{G})} \mathsf{sig}^{-}(B) \quad \mathsf{sig}^{-}(\mathcal{G}) := \mathsf{sig}^{-}(A) \cup \bigcup_{B \in \mathsf{InF}(\mathcal{G})} \mathsf{sig}^{+}(B)$$

If $\widetilde{\mathcal{G}}$ is a biased sequent, then we define $\mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}) := \mathsf{sig}^{\bullet}(\mathcal{G}^{\mathsf{L}}) \cup \mathsf{sig}^{\circ}(\mathcal{G}^{\mathsf{R}})$. (NB. Recall that, by Definition 2.12, we use $\circ$ and $\bullet$ to denote $+$ and $-$, and always assume that $\circ$ is distinct from $\bullet$.)

Definition 5.1 (Interpolant)**.** Let $\widetilde{\mathcal{G}}$ be a biased sequent. We define an *interpolant* (relative to $\widetilde{\mathcal{G}}$) to be a finite set $\mathcal{I} = \{w_1 : A_1, \ldots, w_n : A_n\}$ such that $\{w_1, \ldots, w_n\} \subseteq \mathtt{Names}(\widetilde{\mathcal{G}})$ and $A_1, \ldots, A_n \in \mathscr{L}$. We use $\mathcal{I}$, $\mathcal{J}$, ... to denote interpolants, define $\mathtt{Names}(\mathcal{I})$ to be the set of all names occurring in $\mathcal{I}$, and define $\mathsf{sig}^{\circ}(\mathcal{I}) = \bigcup_{w:A \in \mathcal{I}} \mathsf{sig}^{\circ}(A)$. Finally, we define $\mathcal{I}(w) := \{C \mid w : C \in \mathcal{I}\}$.

We define an *interpolation sequent* to be an expression of the form $\widetilde{\mathcal{G}} \parallel \mathcal{I}$ such that $\widetilde{\mathcal{G}}$ is a biased sequent and $\mathcal{I}$ is an interpolant relative to $\widetilde{\mathcal{G}}$. Informally, an interpolation sequent $\widetilde{\mathcal{G}} \parallel \mathcal{I}$ expresses that $\mathcal{I}$ 'interpolates' $\widetilde{\mathcal{G}}^{\mathsf{L}}$ and $\widetilde{\mathcal{G}}^{\mathsf{R}}$.

To compute Lyndon interpolants, we make use of the rules that operate over interpolation sequents. Such rules consist of those displayed in Figure 5 (which we discuss shortly) along with *IP-rules*, which is short for *interpolant-preserving rules* (cf. [42]). Given a unary rule of the form shown below left, we define the corresponding IP-rule to be of the form shown below right such that $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{H}}$ are biasings of $\mathcal{G}$ and $\mathcal{H}$, respectively, and $\mathcal{I}$ is an interpolant relative to $\widetilde{\mathcal{G}}$. We always assume that the principal and auxiliary formulae of r^{I} occur in the same (left or right) part of the IP-rule application.

$$\frac{\mathcal{G}}{\mathcal{H}}\,\mathsf{r} \qquad \frac{\widetilde{\mathcal{G}} \parallel \mathcal{I}}{\widetilde{\mathcal{H}} \parallel \mathcal{I}}\,\mathsf{r}^{\mathsf{I}}$$

[6] If $\mathsf{OutF}(\mathcal{G}) = \emptyset$, then we can take $A = \bot$, or equivalently, exclude $\mathsf{sig}^{+}(A)$ and $\mathsf{sig}^{-}(A)$ from the definitions of $\mathsf{sig}^{+}(\mathcal{G})$ and $\mathsf{sig}^{-}(\mathcal{G})$.

In our setting, the IP-rules $\{\vee\mathsf{R}^{\mathrm{I}}, \wedge\mathsf{L}^{\mathrm{I}}, \langle x\rangle\mathsf{R}^{\mathrm{I}}, [x]\mathsf{L}^{\mathrm{I}}\}$ are defined relative to the set $\{\vee\mathsf{R}, \wedge\mathsf{L}, \langle x\rangle\mathsf{R}, [x]\mathsf{L}\}$ of rules from $\mathsf{NIK_m}(\mathcal{A})$, and obtain their name because they simply preserve interpolants from premise to conclusion. It is important to mention that the IP-rules $[x]\mathsf{L}^{\mathrm{I}}$ and $\langle x\rangle\mathsf{R}^{\mathrm{I}}$ are subject to the same side conditions as $[x]\mathsf{L}$ and $\langle x\rangle\mathsf{R}$ (see Figure 2). The interpolation rules displayed in Figure 5 are defined relative to the remaining rules in $\mathsf{NIK_m}(\mathcal{A})$.

The $\mathsf{id}_1^{\mathrm{I}}$, $\mathsf{id}_2^{\mathrm{I}}$, $\bot\mathsf{L}_1^{\mathrm{I}}$, and $\bot\mathsf{L}_2^{\mathrm{I}}$ rules are the initial rules, and the remaining rules in the figure construct interpolants for rule conclusions by using interpolants for the premises. We always assume that in each interpolation sequent $\widetilde{\mathcal{G}} \parallel \mathcal{I}$ that $\mathcal{I}$ is an interpolant relative to $\widetilde{\mathcal{G}}$. Let us now define our new set of meta-operators used to construct interpolants for rule conclusions from interpolants of the premises.

Let $\mathcal{I}$ and $\mathcal{J}$ be interpolants relative to biased sequents $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{H}}$ such that $\mathtt{Names}(\widetilde{\mathcal{G}}) = \mathtt{Names}(\widetilde{\mathcal{H}})$. We define the interpolants $\mathcal{I} \Rightarrow \mathcal{J}$, $\mathcal{I} \varovee \mathcal{J}$, and $\mathcal{I} \varowedge \mathcal{J}$ to be the smallest sets satisfying the following constraints, for all $w \in \mathtt{Names}(\widetilde{\mathcal{G}})$:

(1) if $\mathcal{I}(w) = \emptyset$ and $\mathcal{J}(w) = \emptyset$, then $w : \top \to \bot \in (\mathcal{I} \Rightarrow \mathcal{J})$, $w : \bot \in (\mathcal{I} \varovee \mathcal{J})$, and $w : \top \in (\mathcal{I} \varowedge \mathcal{J})$;

(2) if $\mathcal{I}(w) = \{C_1, \ldots, C_n\}$ and $\mathcal{J}(w) = \emptyset$, then $w : C_i \to \bot \in (\mathcal{I} \Rightarrow \mathcal{J})$, $w : C_i \in (\mathcal{I} \varovee \mathcal{J})$, and $w : C_i \in (\mathcal{I} \varowedge \mathcal{J})$ for each $i \in [n]$;

(3) if $\mathcal{I}(w) = \emptyset$ and $\mathcal{J}(w) = \{D_1, \ldots, D_k\}$, then $w : \top \to D_j \in (\mathcal{I} \Rightarrow \mathcal{J})$, $w : D_j \in (\mathcal{I} \varovee \mathcal{J})$, and $w : D_j \in (\mathcal{I} \varowedge \mathcal{J})$ for each $j \in [k]$;

(4) if $\mathcal{I}(w) = \{C_1, \ldots, C_n\}$ and $\mathcal{J}(w) = \{D_1, \ldots, D_k\}$, then for each $i \in [n]$ and $j \in [k]$, $w : C_i \to D_j \in (\mathcal{I} \Rightarrow \mathcal{J})$, $w : C_i \vee D_j \in (\mathcal{I} \varovee \mathcal{J})$, and $w : C_i \wedge D_j \in (\mathcal{I} \varowedge \mathcal{J})$.

If $\mathcal{I}(u) = \emptyset$, then we define $[x]\mathcal{I}_u^w = \langle x\rangle\mathcal{I}_u^w := \mathcal{I}$; otherwise, we define:

$$[x]\mathcal{I}_u^w := \{w : [x] \bigvee_{u:C\in\mathcal{I}} C\} \cup (\mathcal{I} \setminus \{\mathcal{I}(u)\}) \quad \langle x\rangle\mathcal{I}_u^w := \{w : \langle x\rangle \bigwedge_{u:C\in\mathcal{I}} C\} \cup (\mathcal{I} \setminus \{\mathcal{I}(u)\})$$

Note, in the $\langle x\rangle\mathsf{L}_1^{\mathrm{I}}$, $\langle x\rangle\mathsf{L}_2^{\mathrm{I}}$, $[x]\mathsf{R}^{\mathrm{I}}$, and $\mathsf{d}_x^{\mathrm{I}}$ rules, we assume that the component with the auxiliary formula is named by u. In the $\to\mathsf{L}_1^{\mathrm{I}}$ rule, we make use of the *swap operator* $\Downarrow$, defined as: if $\widetilde{\mathcal{G}}$ is a biased sequent, then $\Downarrow\widetilde{\mathcal{G}}$ is obtained by swapping Γ_1 and Γ_2 in each component $\Gamma_1 \mid \Gamma_2 \vdash \Delta$ of $\widetilde{\mathcal{G}}$, i.e., $\Gamma_2 \mid \Gamma_1 \vdash \Delta$ is a component of $\Downarrow\widetilde{\mathcal{G}}$ *iff* $\Gamma_1 \mid \Gamma_2 \vdash \Delta$ is a component of $\widetilde{\mathcal{G}}$.

Definition 5.2. The *interpolation calculus* $\mathsf{NIK_m^I}(\mathcal{A})$ relative to $\mathsf{NIK_m}(\mathcal{A})$ is defined to be the set of IP-rules (discussed above) and the set of rules displayed in Figure 5. We define an *interpolation proof* to be a tree such that every node is an interpolation sequent, each parent node is the conclusion of a rule in $\mathsf{NIK_m^I}(\mathcal{A})$ with its children the corresponding premises, and where each leaf node is an instance of an initial rule $\mathsf{id}_1^{\mathrm{I}}$, $\mathsf{id}_2^{\mathrm{I}}$, $\bot\mathsf{L}_1^{\mathrm{I}}$, or $\bot\mathsf{L}_2^{\mathrm{I}}$.

We now give a sequence of lemmas confirming that $\mathsf{NIK_m^I}(\mathcal{A})$ correctly computes Lyndon interpolants for IGL implications.

Lemma 5.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be interpolants relative to biased sequents $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{H}}$ such that $\mathtt{Names}(\widetilde{\mathcal{G}}) = \mathtt{Names}(\widetilde{\mathcal{H}})$. Then,*

$$\frac{}{\widetilde{\mathcal{G}}\{\Gamma_1, p \mid \Gamma_2 \vdash p\}_w \parallel \{w : p\}}\ \mathsf{id}_1^{\mathrm{I}} \qquad \frac{}{\widetilde{\mathcal{G}}\{\Gamma_1 \mid p, \Gamma_2 \vdash p\}_w \parallel \{w : \top\}}\ \mathsf{id}_2^{\mathrm{I}}$$

$$\frac{}{\widetilde{\mathcal{G}}\{\Gamma_1, \bot \mid \Gamma_2 \vdash \Delta\}_w \parallel \{w : \bot\}}\ \bot\mathsf{L}_1^{\mathrm{I}} \qquad \frac{}{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \bot, \Gamma_2 \vdash \Delta\}_w \parallel \{w : \top\}}\ \bot\mathsf{L}_2^{\mathrm{I}}$$

$$\frac{\widetilde{\mathcal{G}}\{\Gamma_1, A \mid \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_1 \qquad \widetilde{\mathcal{G}}\{\Gamma_1, B \mid \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_2}{\widetilde{\mathcal{G}}\{\Gamma_1, A \lor B \mid \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_1 \owedge \mathcal{I}_2}\ \lor\mathsf{L}_1^{\mathrm{I}}$$

$$\frac{\widetilde{\mathcal{G}}\{\Gamma_1 \mid A, \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_1 \qquad \widetilde{\mathcal{G}}\{\Gamma_1 \mid B, \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_2}{\widetilde{\mathcal{G}}\{\Gamma_1 \mid A \lor B, \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_1 \owedge \mathcal{I}_2}\ \lor\mathsf{L}_2^{\mathrm{I}}$$

$$\frac{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash A\} \parallel \mathcal{I}_1 \qquad \widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash B\} \parallel \mathcal{I}_2}{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash A \land B\} \parallel \mathcal{I}_1 \owedge \mathcal{I}_2}\ \land\mathsf{R}^{\mathrm{I}}$$

$$\frac{\Downarrow\widetilde{\mathcal{G}}^{\downarrow}\{\Gamma_2 \mid \Gamma_1, A \to B \vdash A\} \parallel \mathcal{I}_1 \qquad \widetilde{\mathcal{G}}\{\Gamma_1, A \to B, B \mid \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_2}{\widetilde{\mathcal{G}}\{\Gamma_1, A \to B \mid \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_1 \Rightarrow \mathcal{I}_2}\ \to\mathsf{L}_1^{\mathrm{I}}$$

$$\frac{\widetilde{\mathcal{G}}\{\Gamma_1 \mid A \to B, \Gamma_2 \vdash A\} \parallel \mathcal{I}_1 \qquad \widetilde{\mathcal{G}}\{\Gamma_1 \mid A \to B, B, \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_2}{\widetilde{\mathcal{G}}\{\Gamma_1 \mid A \to B, \Gamma_2 \vdash \Delta\} \parallel \mathcal{I}_1 \owedge \mathcal{I}_2}\ \to\mathsf{L}_2^{\mathrm{I}}$$

$$\frac{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash (x)[\emptyset \mid \emptyset \vdash A]\}_w \parallel \mathcal{I}}{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash [x]A\}_w \parallel [x]\mathcal{I}_u^w}\ [x]\mathsf{R}^{\mathrm{I}} \quad \frac{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash \Delta, (x)[A \mid \emptyset \vdash \emptyset]\}_w \parallel \mathcal{I}}{\widetilde{\mathcal{G}}\{\Gamma_1, \langle x\rangle A \mid \Gamma_2 \vdash \Delta\}_w \parallel \langle x\rangle\mathcal{I}_u^w}\ \langle x\rangle\mathsf{L}_1^{\mathrm{I}}$$

$$\frac{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash \Delta, (x)[\emptyset \mid \emptyset \vdash \emptyset]\}_w \parallel \mathcal{I}}{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash \Delta\}_w \parallel [x]\mathcal{I}_u^w}\ \mathsf{d}_x^{\mathrm{I}} \quad \frac{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash \Delta, (x)[\emptyset \mid A \vdash \emptyset]\}_w \parallel \mathcal{I}}{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \langle x\rangle A, \Gamma_2 \vdash \Delta\}_w \parallel [x]\mathcal{I}_u^w}\ \langle x\rangle\mathsf{L}_2^{\mathrm{I}}$$

Figure 5: Some interpolation rules.

(1) $\mathsf{sig}^{\circ}(\mathcal{I} \Rightarrow \mathcal{J}) = \mathsf{sig}^{\bullet}(\mathcal{I}) \cup \mathsf{sig}^{\circ}(\mathcal{J})$;

(2) $\mathsf{sig}^{\circ}(\mathcal{I} \oplus \mathcal{J}) = \mathsf{sig}^{\circ}(\mathcal{I}) \cup \mathsf{sig}^{\circ}(\mathcal{J})$ *for* $\oplus \in \{\owedge, \owedge\}$;

(3) $\mathsf{sig}^{\circ}(\nabla_x\mathcal{I}) = \mathsf{sig}^{\circ}(\mathcal{I})$ *for* $\nabla_x \in \{\langle x\rangle, [x]\}$.

Lemma 5.4. *If* $\mathsf{NIK_m}(\mathcal{A}), \pi \Vdash \mathcal{G}$, *then for any biasing* $\widetilde{\mathcal{G}}$ *of* $\mathcal{G}$ *there exists an interpolation proof* π' *and interpolant* $\mathcal{I}$ *such that* $\mathsf{NIK_m^I}(\mathcal{A}), \pi' \Vdash \widetilde{\mathcal{G}} \parallel \mathcal{I}$ *and*

(1) $\mathtt{Names}(\mathcal{I}) \subseteq \mathtt{Names}(\widetilde{\mathcal{G}})$;

(2) $\mathsf{sig}^{\circ}(\mathcal{I}) \subseteq \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{L}}) \cap \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{R}})$.

Proof. Suppose $\mathsf{NIK_m}(\mathcal{A}), \pi \Vdash \mathcal{G}$ and let $\widetilde{\mathcal{G}}$ be a biasing of $\mathcal{G}$. Taking the biased sequent $\widetilde{\mathcal{G}}$ and bottom-up applying rules in the same order as in π, while keeping auxiliary formulae in the same (left or right) part as their principal formula, lets one obtain a proof of biased sequents with conclusion $\widetilde{\mathcal{G}}$. Assigning interpolants as dictated by the rules in $\mathsf{NIK^I_m}(\mathcal{A})$, we obtain an interpolation proof π'. We argue claims (1) and (2) by induction on the height of π'.

Base case. We prove the $\mathsf{id}^{\mathrm{I}}_1$ case as the other initial rule cases are similar. Claim (1) holds trivially, and the following subsumptions establish claim (2):

$$\mathsf{sig}^{+}(\{w : p\}) = \mathsf{sig}^{+}(p) = \{p\} \subseteq \mathsf{sig}^{-}(\widetilde{\mathcal{G}}^{\mathsf{L}}) \cap \mathsf{sig}^{+}(\widetilde{\mathcal{G}}^{\mathsf{R}})$$

$$\mathsf{sig}^{-}(\{w : p\}) = \emptyset \subseteq \mathsf{sig}^{+}(\widetilde{\mathcal{G}}^{\mathsf{L}}) \cap \mathsf{sig}^{-}(\widetilde{\mathcal{G}}^{\mathsf{R}})$$

By definition, $p \in \mathsf{sig}^{-}(\widetilde{\mathcal{G}}^{\mathsf{L}})$ and $p \in \mathsf{sig}^{+}(\widetilde{\mathcal{G}}^{\mathsf{R}})$, which justifies the top subsumption.

Inductive step. We prove a representative number of cases and argue that if claims (1) and (2) hold for an interpolation proof of height $h \leq n$, then they hold for an interpolation proof of height $n + 1$.

$\to\mathsf{L}^{\mathrm{I}}_1$. Suppose π' ends with an application of $\to\mathsf{L}^{\mathrm{I}}_1$.

$$\dfrac{\overbrace{\Downarrow\widetilde{\mathcal{G}}^{\downarrow}\{\Gamma_2 \mid \Gamma_1, A \to B \vdash A\}}^{\widetilde{\mathcal{G}}_1} \parallel \mathcal{I}_1 \qquad \overbrace{\widetilde{\mathcal{G}}\{\Gamma_1, A \to B, B \mid \Gamma_2 \vdash \Delta\}}^{\widetilde{\mathcal{G}}_2} \parallel \mathcal{I}_2}{\underbrace{\widetilde{\mathcal{G}}\{\Gamma_1, A \to B \mid \Gamma_2 \vdash \Delta\}}_{\widetilde{\mathcal{G}}_3} \parallel \mathcal{I}_1 \Rightarrow \mathcal{I}_2}\ \to\mathsf{L}^{\mathrm{I}}_1$$

The following proves claim (1), where the second subsumption follows from IH:

$$\begin{aligned}\mathtt{Names}(\mathcal{I}_1 \cup \mathcal{I}_2) &\subseteq \mathtt{Names}(\mathcal{I}_1) \cup \mathtt{Names}(\mathcal{I}_2)\\ &\subseteq \mathtt{Names}(\widetilde{\mathcal{G}}_1) \cup \mathtt{Names}(\widetilde{\mathcal{G}}_2) = \mathtt{Names}(\widetilde{\mathcal{G}}_3)\end{aligned}$$

We now argue claim (2), where the first line follows from Lemma 5.3:

$$\begin{aligned}\mathsf{sig}^{\circ}(\mathcal{I}_1 \Rightarrow \mathcal{I}_2) &= \mathsf{sig}^{\bullet}(\mathcal{I}_1) \cup \mathsf{sig}^{\circ}(\mathcal{I}_2)\\ &= (\mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{L}}_1) \cap \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{R}}_1)) \cup (\mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{L}}_2) \cap \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{R}}_2))\\ &\subseteq (\mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{R}}_3) \cap \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{L}}_3)) \cup (\mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{L}}_3) \cap \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{R}}_3))\\ &= \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{L}}_3) \cap \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{R}}_3)\end{aligned}$$

The third line follows from the second for the following reasons: first, $\mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{L}}_1) \subseteq \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{R}}_3)$ since the only difference between $\widetilde{\mathcal{G}}^{\mathsf{L}}_1$ and $\widetilde{\mathcal{G}}^{\mathsf{R}}_3$ is that the latter contains Δ. Second, $\mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{R}}_1) = \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{L}}_3)$ since the polarity of the auxiliary formula A in $\widetilde{\mathcal{G}}^{\mathsf{R}}_1$ is already the same polarity as the A in the principal formula $A \to B$. Third, $\mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{L}}_2) = \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}^{\mathsf{L}}_3)$ since the polarity of the auxiliary formula B in $\widetilde{\mathcal{G}}^{\mathsf{L}}_2$ is the same polarity as the occurrence of B in the principal formula $A \to B$. Fourth, $\mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{R}}_2) = \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}^{\mathsf{R}}_3)$ because $\widetilde{\mathcal{G}}^{\mathsf{R}}_2 = \widetilde{\mathcal{G}}^{\mathsf{R}}_3$. The remaining inferences above either follow from definitions or basic set theory.

$\to\mathsf{L}^{\mathrm{I}}_2$. Suppose π' ends with an application of $\to\mathsf{L}^{\mathrm{I}}_2$.

$$\dfrac{\overbrace{\widetilde{\mathcal{G}}\{\Gamma_1 \mid A \to B, \Gamma_2 \vdash A\}}^{\widetilde{\mathcal{G}}_1} \parallel \mathcal{I}_1 \qquad \overbrace{\widetilde{\mathcal{G}}\{\Gamma_1 \mid A \to B, B, \Gamma_2 \vdash \Delta\}}^{\widetilde{\mathcal{G}}_2} \parallel \mathcal{I}_2}{\underbrace{\widetilde{\mathcal{G}}\{\Gamma_1 \mid A \to B, \Gamma_2 \vdash \Delta\}}_{\widetilde{\mathcal{G}}_3} \parallel \mathcal{I}_1 \otimes \mathcal{I}_2} \to\mathsf{L}^{\mathrm{I}}_2$$

Claim (1) can be argued in a similar manner as the $\to\mathsf{L}^{\mathrm{I}}_1$ case. We now argue claim (2) accordingly:

$$\begin{aligned}
\mathsf{sig}^{\circ}(\mathcal{I}_1 \otimes \mathcal{I}_2) &= \mathsf{sig}^{\circ}(\mathcal{I}_1) \cup \mathsf{sig}^{\circ}(\mathcal{I}_2) \\
&= (\mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_1^{\mathsf{L}}) \cap \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_1^{\mathsf{R}})) \cup (\mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_2^{\mathsf{L}}) \cap \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_2^{\mathsf{R}})) \\
&\subseteq (\mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_3^{\mathsf{L}}) \cap \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_3^{\mathsf{R}})) \cup (\mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_3^{\mathsf{L}}) \cap \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_3^{\mathsf{R}})) \\
&= \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_3^{\mathsf{L}}) \cap \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_3^{\mathsf{R}})
\end{aligned}$$

In third line above, we know that $\mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_1^{\mathsf{R}}) \subseteq \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_3^{\mathsf{R}})$ holds the latter contains Δ while the former does not, and because the polarity of the auxiliary formula A in $\widetilde{\mathcal{G}}_1^{\mathsf{R}}$ is already the same polarity of A in the principal formula $A \to B$. The remaining inferences above are trivial, follow from definitions, or follow from basic set theory.

$[x]\mathsf{R}^{\mathrm{I}}$. Suppose π' ends with an application of $[x]\mathsf{R}^{\mathrm{I}}$.

$$\dfrac{\overbrace{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash (x)[\emptyset \mid \emptyset \vdash A]\}_w}^{\widetilde{\mathcal{G}}_1} \parallel \mathcal{I}}{\underbrace{\widetilde{\mathcal{G}}\{\Gamma_1 \mid \Gamma_2 \vdash [x]A\}_w}_{\widetilde{\mathcal{G}}_2} \parallel [x]\mathcal{I}^w_u} [x]\mathsf{R}^{\mathrm{I}}$$

Regardless of if $u \in \mathcal{I}$ or $u \notin \mathcal{I}$, the following holds, which establishes claim (1).

$$\mathtt{Names}([x]\mathcal{I}^w_u) = \mathtt{Names}(\mathcal{I}) \setminus \{u\} \subseteq \mathtt{Names}(\widetilde{\mathcal{G}}_1) \setminus \{u\} = \mathtt{Names}(\widetilde{\mathcal{G}}_2).$$

The subsumption above follows from IH. We now argue claim (2) accordingly:

$$\mathsf{sig}^{\circ}([x]\mathcal{I}^w_u) = \mathsf{sig}^{\circ}(\mathcal{I}) \subseteq \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_1^{\mathsf{L}}) \cap \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_1^{\mathsf{R}}) = \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_2^{\mathsf{L}}) \cap \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_2^{\mathsf{R}})$$

The first equality follows from Lemma 5.3, the subsumption follows from IH, and the last equality is a consequence of the fact that $\mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_1^{\mathsf{L}}) = \mathsf{sig}^{\circ}(\widetilde{\mathcal{G}}_2^{\mathsf{L}})$ and $\mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_1^{\mathsf{R}}) = \mathsf{sig}^{\bullet}(\widetilde{\mathcal{G}}_2^{\mathsf{R}})$. ☐

Given a nested sequent $\mathcal{G} := \mathcal{G}\{\Gamma \vdash \Delta\}_w$ and a Gentzen sequent $\Sigma \vdash \Pi$, we define $\mathcal{G} \triangleright_w (\Sigma \vdash \Pi) := \mathcal{G}\{\Gamma, \Sigma \vdash \Pi, \Delta\}_w$.

Lemma 5.5. *If* $\mathsf{NIK}^{\mathrm{I}}_{\mathsf{m}}(\mathcal{A}) \Vdash \widetilde{\mathcal{G}} \parallel \mathcal{I}$, *then*

(1) For each $w : A \in \mathcal{I}$, $\mathsf{NIK}_{\mathsf{m}}(\mathcal{A}) \Vdash \widetilde{\mathcal{G}}^{\mathsf{L}} \triangleright_w (\emptyset \vdash A)$;

(2) For each $w : A \in \mathcal{I}$, $\mathsf{NIK}_{\mathsf{m}}(\mathcal{A}) \Vdash \widetilde{\mathcal{G}}^{\mathsf{R}} \triangleright_w (A \vdash \emptyset)$.

Proof. We prove claims (1) and (2) by mutual induction on the height of the interpolation proof π of $\widetilde{\mathcal{G}} \parallel \mathcal{I}$.

Base case. We show the $\mathsf{id}^{\mathrm{I}}_1$ case as the other initial rule cases are similar. It is easy to establish this case by using the two instances of id shown below:

$$\dfrac{}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1, p \vdash p\}_w}\,\mathsf{id} \qquad \dfrac{}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2, p \vdash p\}_w}\,\mathsf{id}$$

Inductive step. We prove a representative number of cases and argue that if (1) and (2) hold for an interpolation proof of height $h \leq n$, then they hold for an interpolation proof of height $n+1$.

${\to}\mathsf{L}^{\mathrm{I}}_1$. Suppose π ends with an ${\to}\mathsf{L}^{\mathrm{I}}_1$ instance. Let $w : C \to D \in \mathcal{I} \Rightarrow \mathcal{J}$. We assume w.l.o.g. that $w : C \in \mathcal{I}$ and $w : D \in \mathcal{J}$. We construct the desired proofs in $\mathsf{NIK_m}(\mathcal{A})$ as shown below:

$$\dfrac{\dfrac{\dfrac{\vdots\pi_1}{(\Downarrow\widetilde{\mathcal{G}}^{\downarrow})^{\mathsf{L}}\{\Gamma_2 \vdash C\}}\,\mathrm{IH}}{(\Downarrow\widetilde{\mathcal{G}}^{\downarrow})^{\mathsf{L}}\{\Gamma_2, C \to D \vdash C\}}\,\mathsf{wL} \qquad \dfrac{\dfrac{\vdots\pi_2}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2, D \vdash \Delta\}}\,\mathrm{IH}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2, C \to D, D \vdash \Delta\}}\,\mathsf{wL}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2, C \to D \vdash \Delta\}}\,{\to}\mathsf{L}$$

$$\dfrac{\dfrac{\dfrac{\vdots\pi_1}{\Downarrow\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_1, A \to B, C \vdash A\}} \qquad \dfrac{\dfrac{\vdots\pi_2}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1, A \to B, B \vdash D\}}}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1, A \to B, B, C \vdash D\}}\,\mathsf{wL}}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1, A \to B, C \vdash D\}}\,{\to}\mathsf{L}}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1, A \to B \vdash C \to D\}}\,{\to}\mathsf{R}$$

${\to}\mathsf{L}^{\mathrm{I}}_2$. Suppose π ends with an application of ${\to}\mathsf{L}^{\mathrm{I}}_2$. Let $w : C \wedge D \in \mathcal{I}_1 \owedge \mathcal{I}_2$. We assume w.l.o.g. that $w : C \in \mathcal{I}_1$ and $w : D \in \mathcal{I}_2$. The desired proofs in $\mathsf{NIK_m}(\mathcal{A})$ are built accordingly:

$$\dfrac{\dfrac{\vdots\pi_1}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash C\}}\,\mathrm{IH} \qquad \dfrac{\vdots\pi_2}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash D\}}\,\mathrm{IH}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_1 \vdash C \wedge D\}}\,\wedge\mathsf{R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\pi_1}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{A \to B, \Gamma_2, C \vdash A\}}\,\mathrm{IH}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{A \to B, \Gamma_2, C, D \vdash A\}}\,\mathsf{wL} \qquad \dfrac{\dfrac{\vdots\pi_2}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{A \to B, B, \Gamma_2, D \vdash \Delta\}}\,\mathrm{IH}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{A \to B, B, \Gamma_2, C, D \vdash \Delta\}}\,\mathsf{wL}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{A \to B, \Gamma_2, C, D \vdash \Delta\}}\,{\to}\mathsf{L}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{A \to B, \Gamma_2, C \wedge D \vdash \Delta\}}\,\wedge\mathsf{L}$$

$\langle x\rangle\mathsf{L}^{\mathrm{I}}_1$. Suppose π ends with an application of $\langle x\rangle\mathsf{L}^{\mathrm{I}}_1$. Let $v : C \in \langle x\rangle\mathcal{I}^w_u$. We assume that $v : C = w : \langle x\rangle(D_1 \wedge \ldots \wedge D_n)$ as the other cases are trivial. We construct proofs in $\mathsf{NIK_m}(\mathcal{A})$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\pi_1}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash (x)[A \vdash D_1]\}_w}\,\mathrm{IH} \quad \cdots \quad \dfrac{\vdots\pi_n}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash (x)[A \vdash D_n]\}_w}\,\mathrm{IH}}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash (x)[A \vdash D_1 \wedge \cdots \wedge D_n]\}_w}\,\wedge\mathsf{R} \times (n{-}1)}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash \langle x\rangle(D_1 \wedge \cdots \wedge D_n), (x)[A \vdash \emptyset]\}_w}\,\langle x\rangle\mathsf{R}}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1, \langle x\rangle A \vdash \langle x\rangle(D_1 \wedge \cdots \wedge D_n)\}_w}\,\langle x\rangle\mathsf{L}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\pi'}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2 \vdash (x)[D_i \vdash \emptyset]\}_w}\ \text{IH}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2 \vdash (x)[D_1, \ldots, D_n \vdash \emptyset]\}_w}\ \mathsf{wL} \times (n{-}1)}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2 \vdash (x)[D_1 \wedge \cdots \wedge D_n \vdash \emptyset]\}_w}\ \wedge\mathsf{L} \times (n{-}1)}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_1, \langle x\rangle(D_1 \wedge \cdots \wedge D_n) \vdash \emptyset\}_w}\ \langle x\rangle\mathsf{L}$$

$[x]\mathsf{R}^{\mathrm{I}}$. Suppose π ends with an application of $[x]\mathsf{R}^{\mathrm{I}}$. Let $v : C \in [x]\mathcal{I}^w_u$. We assume that $v : C = w : [x](D_1 \vee \ldots \vee D_n)$ as the other cases are trivial. We construct proofs in $\mathsf{NIK_m}(\mathcal{A})$ as follows:

$$\dfrac{\dfrac{\dfrac{\vdots\pi'}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash (x)[\emptyset \vdash D_i]\}_w}\ \text{IH}}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash (x)[\emptyset \vdash D_1 \vee \cdots \vee D_n]\}_w}\ \vee\mathsf{R} \times (n{-}1)}{\widetilde{\mathcal{G}}^{\mathsf{L}}\{\Gamma_1 \vdash [x](D_1 \vee \cdots \vee D_n)\}_w}\ [x]\mathsf{R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\pi_1}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2 \vdash (x)[D_1 \vdash A]\}_w}\ \text{IH} \qquad \cdots \qquad \dfrac{\vdots\pi_n}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2 \vdash (x)[D_n \vdash A]\}_w}\ \text{IH}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2 \vdash (x)[D_1 \vee \cdots \vee D_n \vdash A]\}_w}\ \vee\mathsf{L} \times (n{-}1)}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2, [x](D_1 \vee \cdots \vee D_n) \vdash (x)[D_1 \vee \cdots \vee D_n \vdash A]\}_w}\ \mathsf{wL}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2, [x](D_1 \vee \cdots \vee D_n) \vdash (x)[\emptyset \vdash A]\}_w}\ [x]\mathsf{L}}{\widetilde{\mathcal{G}}^{\mathsf{R}}\{\Gamma_2, [x](D_1 \vee \cdots \vee D_n) \vdash [x]A\}_w}\ [x]\mathsf{R}$$

This completes the proof of the lemma. $\square$

Given an interpolant of the form $\mathcal{I} = \{w : C_1, \ldots, w : C_n\}$, that is, one in which every formula is prefixed with the same name w, we define the following two formulae: $\bigvee\mathcal{I} := C_1 \vee \cdots \vee C_n$ and $\bigwedge\mathcal{I} := C_1 \wedge \cdots \wedge C_n$.

Theorem 5.6. *If $\mathsf{NIK_m}(\mathcal{A}) \Vdash (\emptyset \vdash A \to B)$ with w the name of $\emptyset \vdash A \to B$, then there is an $\mathcal{I} = \{w : C_1, \ldots, w : C_n\}$ such that for $I \in \{\bigvee\mathcal{I}, \bigwedge\mathcal{I}\}$:*

(1) $\mathsf{sig}^{\circ}(I) \subseteq \mathsf{sig}^{\circ}(A) \cap \mathsf{sig}^{\circ}(B)$

(2) $\mathsf{NIK_m}(\mathcal{A}) \Vdash (\emptyset \vdash A \to I)$

(3) $\mathsf{NIK_m}(\mathcal{A}) \Vdash (\emptyset \vdash I \to B)$

Proof. Suppose $\mathsf{NIK_m}(\mathcal{A}) \Vdash (\emptyset \vdash A \to B)$ with w the name of $\emptyset \vdash A \to B$. This can only happen if $\emptyset \vdash A \to B$ was inferred from $A \vdash B$ by an application of $\to\mathsf{R}$, so we have that $\mathsf{NIK_m}(\mathcal{A}) \Vdash (A \vdash B)$. By Lemma 5.4, we know that an interpolation proof of $A \mid \emptyset \vdash B \parallel \mathcal{I}$ exists such that $\mathcal{I} = \{w : C_1, \ldots, w : C_n\}$ and $\mathsf{sig}^{\circ}(\mathcal{I}) = \mathsf{sig}^{\circ}(I) \subseteq \mathsf{sig}^{\bullet}(A \vdash \emptyset) \cap \mathsf{sig}^{\circ}(\emptyset \vdash B) = \mathsf{sig}^{\circ}(A) \cap \mathsf{sig}^{\circ}(B)$. By Lemma 5.5, we know that proofs of $A \vdash C_i$ and $C_i \vdash B$ exist in $\mathsf{NIK_m}(\mathcal{A})$ for each $i \in [n]$. By using $\{\wedge\mathsf{R}, \mathsf{wL}, \wedge\mathsf{L}, \to\mathsf{R}\}$, we obtain proofs of $\vdash A \to \bigwedge\mathcal{I}$ and $\vdash \bigwedge\mathcal{I} \to B$, and by using $\{\vee\mathsf{R}, \vee\mathsf{L}, \to\mathsf{R}\}$, we obtain proofs of $\vdash A \to \bigvee\mathcal{I}$ and $\vdash \bigvee\mathcal{I} \to B$. $\square$

5.2. Consequences.

The above proof of Lyndon interpolation is *constructive*, that is, the proof demonstrates how to actually build a Lyndon interpolant for $A \rightarrow B$, when given a proof of the formula. Our algorithm may also be adapted to IMLs and ITLs and to compute explicit definitions. Recall that the class of IGLs subsumes many standard IMLs [17, 49, 39] and ITLs [15], as discussed in Section 3.2. We obtain the LIP for these logics as a corollary of Theorem 5.6, which, to the best of the author's knowledge, is a new result. In addition, by Theorem 2.16, we also obtain the BDP for all IMLs, ITLs, and IGLs considered. All these results are summarized in the following:

Corollary 5.7. *Let $\mathcal{A}$ be a finite set of IPAs, $\mathcal{B}$ be a finite set of HSL axioms, and $\mathcal{C}$ be a finite set of ITP axioms. Then,*

(1) Every IGL $\mathsf{IK_m}(\mathcal{A})$ has the LIP and BDP;

(2) Every IML $\mathsf{IK} \cup m(\mathcal{B})$ has the LIP and BDP;

(3) Every ITL $\mathsf{IKt} \cup \mathcal{C}$ has the LIP and BDP.

§6. Concluding Remarks

In this paper, we initiated the structural proof theory of intuitionistic grammar logics by supplying proof systems in the formalism of nested sequents. We investigated fundamental invertibility and admissibility properties, and introduced the novel shift $\mathsf{sft}(\mathcal{A})$ rule, which culminated in a uniform proof of cut-admissibility. Completeness was established syntactically by means of cut-admissibility, while soundness was proven model-theoretically. Finally, we showed that all IGLs–together with the IMLs and ITLs they subsume–satisfy the LIP and BDP, which appears to be a new result in the theory of intuitionistic modal logics.

There are several directions for future research. For instance, our nested sequent systems are not (fully) invertible, e.g., the $\rightarrow\mathsf{L}$ rule in not invertible. This raises the natural goal of developing fully invertible systems for IGLs, which may be better suited for countermodel extraction, and thus, for investigating decidability. It seems plausible that labeled generalizations of our nested systems, in the style of Marin et al. [46], would yield fully invertible calculi. Such systems may also extend naturally to Scott–Lemmon axioms, since labeled calculi are well suited to capture their corresponding frame conditions.

A second direction concerns possible connections to classical systems via double-negation translations. One could investigate adapting the double-negation translation of Gödel, Gentzen, and Kolmogorov (cf. [8]) to our setting, thereby defining proof transformations from the nested calculi of Tiu et al. [55] to our intuitionistic systems. This would immediately imply the undecidability of IGLs in general and would prompt a more fine-grained analysis of which IGLs remain decidable, as well as of the corresponding complexity of those logics.

References

[1] Franz Baader, Ian Horrocks, Carsten Lutz, and Uli Sattler. *Introduction to Description Logic*. Cambridge University Press, 2017.

[2] Nuel D Belnap. Display logic. *Journal of Philosophical Logic*, 11(4):375–417, 1982.

[3] Evert Willem Beth. On padoa's method in the theory of definition. *Journal of Symbolic Logic*, 21(2):194–195, 1956.

[4] Milan Božić and Kosta Došen. Models for normal intuitionistic modal logics. *Studia Logica*, 43(3):217–245, 1984.

[5] Kai Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, 2009. doi: 10.1007/s00153-009-0137-3. URL `https://doi.org/10.1007/s00153-009-0137-3`.

[6] Robert A. Bull. A modal extension of intuitionist logic. *Notre Dame Journal of Formal Logic*, 6(2):142, 1965.

[7] Robert A. Bull. Cut elimination for propositional dynamic logic without *. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 38(2):85–100, 1992.

[8] Samuel R Buss. An introduction to proof theory. *Handbook of proof theory*, 137:1–78, 1998.

[9] Agata Ciabattoni, Tim Lyon, Revantha Ramanayake, and Alwen Tiu. Display to labelled proofs and back again for tense logics. *ACM Transactions on Computational Logic*, 22(3):1–31, 2021.

[10] William Craig. Three uses of the herbrand-gentzen theorem in relating model theory and proof theory. *Journal of Symbolic Logic*, 22(3):269–285, 1957. doi: 10.2307/2963594.

[11] Rowan Davies and Frank Pfenning. A modal analysis of staged computation. *Journal of the ACM*, 48(3):555–604, May 2001. ISSN 0004-5411.

[12] Stéphane Demri and Hans de Nivelle. Deciding regular grammar logics with converse through first-order logic. *Journal of Logic, Language and Information*, 14(3):289–329, 2005.

[13] Kosta Došen. Models for stronger normal intuitionistic modal logics. *Studia Logica*, 44 (1):39–70, 1985.

[14] Roy Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. *The Journal of Symbolic Logic*, 57(3):795–807, 1992.

[15] W. B. Ewald. Intuitionistic tense and modal logic. *The Journal of Symbolic Logic*, 51 (1):166–179, 1986. ISSN 00224812.

[16] Matt Fairtlough and Michael Mendler. An intuitionistic modal logic with applications to the formal verification of hardware. In Leszek Pacholski and Jerzy Tiuryn, editors, *Computer Science Logic*, pages 354–368, Berlin, Heidelberg, 1995. Springer Berlin Heidelberg. ISBN 978-3-540-49404-1.

[17] G Fischer-Servi. Axiomatizations for some intuitionistic modal logics. *Rendiconti del Seminario Matematico Università e Politecnico di Torino*, 42(3):179–194, 1984.

[18] Gisèle Fischer Servi. On modal logic with an intuitionistic base. *Studia Logica*, 36(3):141–149, 1977. doi: 10.1007/BF02121259. URL `https://doi.org/10.1007/BF02121259`.

[19] Frederic B. Fitch. Intuitionistic modal logic with quantifiers. *Portugaliae mathematica*, 7(2):113–118, 1948.

[20] Melvin Fitting and Roman Kuznets. Modal interpolation via nested sequents. *Annals of Pure and Applied Logic*, 166(3):274–305, 2015.

[21] Gerhard Gentzen. Untersuchungen über das logische Schließen. i. *Mathematische zeitschrift*, 39(1):176–210, 1935.

[22] Gerhard Gentzen. Untersuchungen über das logische Schließen. ii. *Mathematische Zeitschrift*, 39(1):405–431, 1935.

[23] Valentin Goranko and Martin Otto. Model theory of modal logic. In Patrick Blackburn, Johan Van Benthem, and Frank Wolter, editors, *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, pages 249–329. Elsevier, 2007. doi: https://doi.org/10.1016/S1570-2464(07)80008-5. URL `https://www.sciencedirect.com/science/article/pii/S1570246407800085`.

[24] Rajeev Goré, Linda Postniece, and Alwen Tiu. On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics. *Logical Methods in Computer Science*, 7(2), 2011.

[25] Edward Hermann Haeusler and Alexandre Rademaker. An intuitionisticaly based description logic, 2014. URL `https://arxiv.org/abs/1402.0225`.

[26] Edward Hermann Haeusler, Valeria de Paiva, and Alexandre Rademaker. Intuitionistic description logic and legal reasoning. In *2011 22nd International Workshop on Database and Expert Systems Applications*, pages 345–349, 2011. doi: 10.1109/DEXA.2011.46.

[27] Ian Horrocks and Ulrike Sattler. Decidability of shiq with complex role inclusion axioms. *Artificial Intelligence*, 160(1-2):79–104, 2004.

[28] Alexander Hudek, David Toman, and Grant Weddell. On enumerating query plans using analytic tableau. In Hans De Nivelle, editor, *Automated Reasoning with Analytic Tableaux and Related Methods*, pages 339–354, Cham, 2015. Springer International Publishing. ISBN 978-3-319-24312-2.

[29] Ryo Ishigaki and Kentaro Kikuchi. Tree-sequent methods for subintuitionistic predicate logics. In Nicola Olivetti, editor, *Automated Reasoning with Analytic Tableaux and Related Methods*, volume 4548 of *Lecture Notes in Computer Science*, pages 149–164, Berlin, Heidelberg, 2007. Springer Berlin Heidelberg. ISBN 978-3-540-73099-6.

[30] Ryo Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53(1): 119–135, 1994.

[31] Roman Kuznets. Multicomponent proof-theoretic method for proving interpolation properties. *Annals of Pure and Applied Logic*, 169(12):1369–1418, 2018.

[32] Roman Kuznets and Lutz Straßburger. Maehara-style modal nested calculi. *Archive for Mathematical Logic*, 58(3):359–385, 2019. doi: 10.1007/s00153-018-0636-1. URL `https://doi.org/10.1007/s00153-018-0636-1`.

[33] Daniel Leivant. Proof theoretic methodology for propositional dynamic logic. In J. Díaz and I. Ramos, editors, *Formalization of Programming Concepts*, pages 356–373, Berlin, Heidelberg, 1981. Springer. ISBN 978-3-540-38654-4.

[34] Björn Lellmann and Francesca Poggiolesi. *Nested Sequents or Tree-Hypersequents—A Survey*, pages 243–301. Springer International Publishing, Cham, 2024. ISBN 978-3-031-57635-5. doi: 10.1007/978-3-031-57635-5_11.

[35] Roger C. Lyndon. An Interpolation Theorem in the Predicate Calculus. *Pacific Journal of Mathematics*, 9(1):129 – 142, 1959.

[36] Tim Lyon. On deriving nested calculi for intuitionistic logics from semantic systems. In Sergei N. Artëmov and Anil Nerode, editors, *Logical Foundations of Computer Science, Proceedings*, volume 11972 of *Lecture Notes in Computer Science*, pages 177–194. Springer, 2020. doi: 10.1007/978-3-030-36755-8_12. URL `https://doi.org/10.1007/978-3-030-36755-8_12`.

[37] Tim Lyon. *Refining Labelled Systems for Modal and Constructive Logics with Applications*. PhD thesis, Technische Universität Wien, 2021.

[38] Tim Lyon, Alwen Tiu, Rajeev Goré, and Ranald Clouston. Syntactic interpolation for tense logics and bi-intuitionistic logic via nested sequents. In Maribel Fernández and Anca Muscholl, editors, *28th EACSL Annual Conference on Computer Science Logic*, volume 152 of *LIPIcs*, pages 28:1–28:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[39] Tim S. Lyon. Nested sequents for intuitionistic modal logics via structural refinement. In Anupam Das and Sara Negri, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, pages 409–427, Cham, 2021. Springer International Publishing. ISBN 978-3-030-86059-2.

[40] Tim S. Lyon. A framework for intuitionistic grammar logics. In Pietro Baroni, Christoph Benzmüller, and Yì N. Wáng, editors, *Logic and Argumentation*, pages 495–503, Cham, 2021. Springer International Publishing. ISBN 978-3-030-89391-0.

[41] Tim S. Lyon and Lucía Gómez Álvarez. Automating reasoning with standpoint logic via nested sequents. In *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning*, pages 257–266, 8 2022.

[42] Tim S. Lyon and Jonas Karge. Constructive interpolation and concept-based beth definability for description logics via sequents. In *Proceedings of the Thirty-Third International Joint Conference on Artificial Intelligence (IJCAI)*, pages 3484–3492. ijcai.org, 2024. URL `https://www.ijcai.org/proceedings/2024/386`.

[43] Tim S. Lyon and Eugenio Orlandelli. Nested sequents for quantified modal logics. In *Automated Reasoning with Analytic Tableaux and Related Methods*, page 449–467, Berlin, Heidelberg, 2023. Springer-Verlag. ISBN 978-3-031-43512-6. doi: 10.1007/978-3-031-43513-3_24.

[44] Tim S. Lyon and Piotr Ostropolski-Nalewaja. Foundations for an abstract proof theory in the context of Horn rules, 2023. URL `https://arxiv.org/abs/2304.05697`.

[45] Shoji Maehara. On the interpolation theorem of Craig. *Sûgaku*, 12(4):235–237, 1960.

[46] Sonia Marin, Marianela Morales, and Lutz Straßburger. A fully labelled proof system for intuitionistic modal logics. *Journal of Logic and Computation*, 31 (3):998–1022, 05 2021. ISSN 0955-792X. doi: 10.1093/logcom/exab020. URL `https://doi.org/10.1093/logcom/exab020`.

[47] Kenneth L. McMillan. Interpolation and model checking. In Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and Roderick Bloem, editors, *Handbook of Model Checking*, pages 421–446. Springer International Publishing, Cham, 2018. ISBN 978-3-319-10575-8. doi: 10.1007/978-3-319-10575-8_14.

[48] Andrew M Pitts. Evaluation logic. In *IV Higher Order Workshop*, pages 162–189. Springer, 1991.

[49] Gordon Plotkin and Colin Stirling. A framework for intuitionistic modal logics: Extended abstract. In *Proceedings of Theoretical Aspects of Reasoning about Knowledge*, TARK '86, page 399–406, San Francisco, CA, USA, 1986. Morgan Kaufmann Publishers Inc. ISBN 0934613049.

[50] Francesca Poggiolesi. The method of tree-hypersequents for modal propositional logic. In David Makinson, Jacek Malinowski, and Heinrich Wansing, editors, *Towards Mathematical Philosophy*, volume 28 of *Trends in logic*, pages 31–51. Springer, 2009. doi: 10.1007/978-1-4020-9084-4_3. URL `https://doi.org/10.1007/978-1-4020-9084-4_3`.

[51] Emil L Post. Recursive unsolvability of a problem of Thue. *The Journal of Symbolic Logic*, 12(1):1–11, 1947.

[52] Alex K Simpson. *The proof theory and semantics of intuitionistic modal logic*. PhD thesis, University of Edinburgh. College of Science and Engineering. School of Informatics, 1994.

[53] Charles Stewart, Valeria de Paiva, and Natasha Alechina. Intuitionistic modal logic: A 15-year retrospective. *Journal of Logic and Computation*, 28(5): 873–882, 06 2015. ISSN 0955-792X. doi: 10.1093/logcom/exv042. URL `https://doi.org/10.1093/logcom/exv042`.

[54] Lutz Straßburger. Cut elimination in nested sequents for intuitionistic modal logics. In Frank Pfenning, editor, *Foundations of Software Science and Computation Structures*, volume 7794 of *Lecture Notes in Computer Science*, pages 209–224, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg. ISBN 978-3-642-37075-5.

[55] Alwen Tiu, Egor Ianovski, and Rajeev Goré. Grammar logics in nested sequent calculus: Proof theory and decision procedures. In Thomas Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence S. Moss, editors, *Advances in Modal Logic 9*, pages 516–537. College Publications, 2012.