COHEN-MACAULEYNESS OF THE ZERO-DIVISOR GRAPH OF A BOOLEAN POSET

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Abstract. In this paper, we prove that the zero-divisor graph $\Gamma(P)$ of a Boolean poset P is both well-covered and Cohen–Macaulay. Furthermore, for a poset $\mathbf{P} = \prod_{i=1}^{n} P_i$ $(n \geq 3)$, where each P_i is a finite bounded poset satisfying $Z(P_i) = \{0\}$ for all i, and $2 \leq |P_1| \leq |P_2| \leq \cdots \leq |P_n|$, we show that the zero-divisor graph $\Gamma(\mathbf{P})$ is Cohen–Macaulay if and only if $\mathbf{P}$ is a Boolean lattice.

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1. Introduction

In recent years, there has been a growing interest in the study of zero-divisor graphs of ordered structures. One of the main reason is that it serve as a tool to study various other graphs associated with algebraic structures, such as the zero-divisor graph of reduced commutative rings, the intersection graph of ideals, the comaximal ideal graph, the nilpotent graph, and the annihilating ideal graph.

The concept of a zero-divisor graph was introduced by I. Beck [3] to color commutative rings with unity. He considered a simple graph $G(R)$ whose vertices are the elements of R, such that two distinct elements x and y are adjacent if and only if $xy = 0$. D. F. Anderson and P. S. Livingston [1] modified this definition of the zero-divisor graph by changing the vertex set to the set of nonzero zero-divisors and adjacency remains same.

In [7], R. Halaš and M. Jukl introduced the *zero-divisor graph* of a poset with the least element 0 by considering elements of a poset are the vertices, and there is an edge between two vertices x, y, if 0 is the only element lying below x and y. They have revealed the similar results of the zero-divisor graph of a poset as Beck did in [3].

The definition of the *zero-divisor graph* of a poset, originally introduced in [7] and later modified by Lu and Wu in [13] (see also Joshi [9]), is stated below.

Definition 1.1. Let P be a poset with 0. Define a *zero-divisor* of P to be any element of the set $Z(P) = \{a \in P \mid \text{there exists } b \in P \setminus \{0\} \text{ such that } \{a,b\}^{\ell} = \{0\}\}$. The *zero-divisor graph* of P is the graph $\Gamma(P)$ whose vertices are the elements of $Z(P) \setminus \{0\}$ such that two vertices a and b are adjacent if and only if $\{a,b\}^{\ell} = \{0\}$, where $\{a,b\}^{\ell} = \{x \in P \mid x \leq a, b\}$.

A lot of work has been done in the area of *zero-divisor graph* of ordered sets; see [4, 9, 10, 13]. In [17], R.H. Villarreal studied the ideal associated with graphs and named it as an edge ideal. He discovered the class of *Cohen-Macaulay graphs.* As in combinatorial commutative algebra, *Cohen-Macaulay* ring plays a vital role, hence it is important to find a class of simplicial complexes whose steinley-Reisnar ring (Face Ring) is Cohen-Macaulay. As for a given graph Γ, we have the simplicial complex of independent sets of the vertex set of Γ. A graph Γ is said to be *Cohen-Macaulay*, *vertex decomposable*, *Well-covered/Unmixed/Pure* and *Gorenstein* if simplicial complex of independent sets of Γ have that property. For more details; see [12, 16, 20].

In [12], A-Ming Liu and Tongsuo Wu proved that Boolean graphs are Cohen–Macaulay. In a similar direction, first part of this paper identifies a class of zero-divisor graphs of a Boolean poset possess the Cohen–Macaulay property, and thus generalizes the result of A-Ming Liu and Tongsuo Wu.

Theorem 1.2. *If P is a Boolean poset, then $\Gamma(P)$ is Cohen–Macaulay.*

In the later part, we have proved the converse of the above theorem for a special class of posets. This result generalizes Main Theorem of Asir et al. [2]

Theorem 1.3. *Let $\mathbf{P} = \prod_{i=1}^{n} P_i$, $(n \geq 3)$, where P_i's are finite bounded posets such that $Z(P_i) = \{0\}$, $\forall i$ and $2 \leq |P_1| \leq |P_2| \leq \cdots \leq |P_n|$. Then the following statements are equivalent.*

(1) $\Gamma(\mathbf{P})$ *is Cohen-Macaulay.*
(2) $\Gamma(\mathbf{P})$ *is well-covered.*
(3) $|P_i| = 2$ *for all* $i = 1, 2, \ldots, n$.
(4) $\mathbf{P}$ *is a Boolean lattice.*
(5) $\mathbf{P}$ *is a Boolean poset.*

2. Preliminaries

2.1. **Combinatorial Commutative Algebra and Graph Theory:** A *simplicial complex* Δ on a set V is a collection of subsets of V such that for every $B \in \Delta$ and every $A \subseteq B$, we also have $A \in \Delta$. The elements of Δ are called *faces*. The set V is called the *vertex set* of Δ, and its elements are called the *vertices*.

A face F of Δ is called a *facet* if there is no face $B \in \Delta$ such that $F \subset B$. The set of all facets of Δ is denoted by $\mathcal{F}(\Delta)$. The complex Δ is called a *simplex* if it has exactly one facet.

The *dimension* of a face A is defined as $\dim(A) = |A| - 1$. The *dimension* of the simplicial complex Δ is the maximum dimension of any of its faces and is denoted by $\dim(\Delta)$. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension; otherwise, it is called *nonpure*. A pure simplicial complex is also known as *well-covered*.

Hereafter, we consider the simplicial complex of independent vertex sets of a graph Γ, denoted by Δ_Γ. The faces of Δ_Γ are precisely the independent sets of Γ, and the facets of Δ_Γ are the maximal independent sets of Γ.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$, and let Δ be a simplicial complex on the vertex set $[n] = \{1, 2, \cdots, n\}$. For a subset $F \subseteq [n]$, denote $x_F = \prod_{i \in F} x_i$.

The *Stanley–Reisner ideal* of Δ over $\mathbb{K}$ is the ideal $I_\Delta \subseteq S$ generated by all squarefree monomials x_F corresponding to subsets $F \notin \Delta$. The corresponding *Stanley–Reisner ring* is defined as $\mathbb{K}[\Delta] = S/I_\Delta$.

A simplicial complex Δ is said to be *Cohen–Macaulay* (respectively, *Gorenstein*) over $\mathbb{K}$ if its Stanley–Reisner ring $\mathbb{K}[\Delta]$ is a *Cohen–Macaulay ring* (respectively, a *Gorenstein ring*).

Let $\Gamma = (V, E)$ be a finite simple graph with the vertex set V. The *edge ideal* of Γ, denoted by $I(\Gamma)$, is the squarefree monomial ideal in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$

defined as $I(\Gamma) = \big(x_i x_j \,:\, \{i,j\} \in E\big)$. That is, each edge $\{i,j\}$ of the graph corresponds to a quadratic squarefree monomial $x_i x_j$, and the edge ideal is generated by all such monomials.

The connection with Stanley-Reisner theory arises through the independence complex of Γ. The independence complex, denoted by Δ_Γ, is the simplicial complex whose faces are the independent sets of Γ. Since the minimal nonfaces of Δ_Γ are precisely the edges of Γ, the Stanley–Reisner ideal of Δ_Γ coincides with the edge ideal of Γ, i.e., $I_{\Delta_\Gamma} = I(\Gamma)$. Consequently, the Stanley–Reisner ring of the independence complex is given by $\mathbb{K}[\Delta_\Gamma] = S/I(\Gamma)$.

A subset C of the vertex set $V(\Gamma)$ of a graph Γ, is called a *vertex cover* of a graph Γ if for every $v \in V(\Gamma)$ there exist at least one vertex $u \in C$ such that v is adjacent to u in Γ. A vertex cover C of graph Γ is called *minimal* if there does not exists any cover of Γ which is inside C. A subset I of the vertex set $V(\Gamma)$ is called an *independent set* if the induced subgraph of Γ on I is an empty graph (graph without edges). An independent set I is called *maximal* if for any $v \in V(\Gamma) \setminus I$ the induced subgraph on $I \cup \{v\}$ is not an empty graph. Note that the set of independent sets of Γ forms a simplicial complex.

Definition 2.1. A graph Γ is said to be *well-covered* if all its maximal independent sets have the same cardinality. *Well-covered* graphs are also called as *pure*.

Further, Γ is called *very well–covered* if it is well–covered, has no isolated vertices, and the number of vertices of Γ is exactly twice the size of a maximal independent set.

Let Γ be a graph and let $v \in V(\Gamma)$. A vertex $w \in V(\Gamma)$ is called a *complement* of v in Γ, if v is adjacent to w, and no vertex is adjacent to both v and w, i.e., the edge $v - w$ is not an edge of any triangle in Γ. In such a case, we write $v \perp w$. Moreover, we say that Γ is *complemented* if every vertex has a complement. An *end* is a vertex that is adjacent to precisely one other vertex. For more details, please refer, [8, 12].

2.2. **Basics of Poset Theory.** We proceed with the following basics and necessary terminologies given by Devhare, Joshi and LaGrange in [4]. Also, we mention the some results that we have used as tools to prove the main results of this paper.

Let P be a poset. For any subset $A \subseteq P$, the *upper cone* of A is defined as $A^u = \{\, b \in P \mid b \geq a \text{ for all } a \in A \,\}$, and the *lower cone* of A is given by $A^\ell = \{\, b \in P \mid b \leq a \text{ for all } a \in A \,\}$. For a single element $a \in P$, we write a^u and a^ℓ instead of $\{a\}^u$ and $\{a\}^\ell$, respectively. The notation $A^{u\ell}$ means $(A^u)^\ell$, and similarly one defines $A^{\ell u}$.

An element $a \in P$ is called an *atom* if $a > 0$ and there is no $b \in P$ satisfying $0 < b < a$. The poset P is said to be *atomic* if each nonzero element $b \in P \setminus \{0\}$ contains an atom $a \in P$ with $a \leq b$.

A poset P is *bounded* if it has both a least element 0 and a greatest element 1. In a bounded poset, an element $a' \in P$ is called a *complement* of $a \in P$ when $\{a, a'\}^\ell = \{0\}$ and $\{a, a'\}^u = \{1\}$. An element $b \in P$ is a *pseudocomplement* of a if $\{a, b\}^\ell = \{0\}$ and b is maximal with this property; that is, b is the pseudocomplement of a exactly when $a^\perp = b^\ell$, where $a^\perp = \{\, x \in P \mid \{x, a\}^\ell = \{0\} \,\}$. Pseudocomplements, when they exist, are unique; the pseudocomplement of a is denoted by a^*.

A bounded poset is called *complemented* (respectively, *pseudocomplemented*) if every element has a complement a' (respectively, a pseudocomplement a^*).

A poset P is said to be *distributive* if for all $a, b, c \in P$, the equality $\{\{a\} \cup \{b, c\}^u\}^\ell = \{\{a, b\}^\ell \cup \{a, c\}^\ell\}^{u\ell}$ holds. This notion extends the classical definition of distributivity in lattices: a lattice is distributive in the usual sense if and only if it is distributive as a poset. A bounded poset that is both distributive and complemented is called *Boolean*; see [6]. Every Boolean algebra is therefore a Boolean poset, though the converse does not necessarily hold.

It is a standard fact that in a Boolean poset, complementation coincides with pseudocomplementation (cf. [8], Lemma 2.4). Thus, if P is Boolean, it is pseudocomplemented, and each $x \in P$ has a unique complement x'. For any undefined terminology, we refer the reader to [4, 5, 19].

Next, we present several results on Cohen-Macaulay graphs and the zero-divisor graph of a Boolean poset P, which will be essential in proving our main results.

Lemma 2.2 ([14, Lemma 3.1])**.** *Let Γ be a very well-covered graph with $2h$ vertices. Then the following conditions are equivalent:*

(1) The graph Γ is Cohen-Macaulay.

(2) There is a relabeling of vertices $V(\Gamma) = \{x_1, \dots, x_h, y_1, \dots, y_h\}$ such that the following five conditions hold:

(a) $\{x_1, \dots, x_h\}$ is a minimal vertex cover of Γ and $\{y_1, \dots, y_h\}$ is a maximal independent set of Γ,

(b) $\{x_1, y_1\}, \dots, \{x_h, y_h\} \in E(\Gamma)$,

(c) if $\{z_i, x_j\}, \{y_j, x_k\} \in E(\Gamma)$, then $\{z_i, x_k\} \in E(\Gamma)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$,

(d) if $\{x_i, y_j\} \in E(\Gamma)$, then $\{x_i, x_j\} \notin E(\Gamma)$,

(e) if $\{x_i, y_j\} \in E(\Gamma)$, then $i \leq j$.

Theorem 2.3 ([8, Theorem 2.4])**.** *If P be a Boolean poset, then every vertex of $\Gamma(P)$ has the unique complement in $\Gamma(P)$.*

Lemma 2.4 ([8, Lemma 2.2])**.** *Let P be a Boolean poset. Then $(1 \neq)b$ is an atom if and only if its complement b' is the unique end adjacent to b in $\Gamma(P)$.*

3. Cohen-Macaulayness of the zero-divisor graph of a Boolean Poset

It is well known that every *Cohen–Macaulay graph* is necessarily *unmixed*; see, for example, [15, 16]. Consequently, in order to investigate the Cohen–Macaulayness of the zero-divisor graph $\Gamma(P)$, it is essential first to determine whether $\Gamma(P)$ is *well-covered*. The following theorem provides the well-coveredness of $\Gamma(P)$ in the case P is a Boolean poset.

Throughout the paper, unless otherwise specified, P denotes a finite Boolean poset with at least two atoms.

Theorem 3.1. *The zero-divisor graph of a Boolean poset is* well-covered*.*

Proof. Let $\Gamma(P)$ be a zero-divisor graph of a Boolean poset P. We prove that all maximal independent vertex sets of $\Gamma(P)$ have the same cardinality, namely $\frac{|V(\Gamma(P))|}{2}$. Consequently, $\Gamma(P)$ is *well-covered*.

Since P is a Boolean poset, every element has a unique complement. Hence, $|V(\Gamma(P))| = 2t$, where t is the number of complementary pairs in P. Moreover, for each vertex $a \in V(\Gamma(P))$, its complement a' also belongs to $V(\Gamma(P))$. Thus, the vertex set can be decomposed as

$$V(\Gamma(P)) = B \cup B', \quad B = \{a_1, a_2, \dots, a_t\}, \quad B' = \{a'_1, a'_2, \dots, a'_t\},$$

with $B \cap B' = \varnothing$.

It follows that the cardinality of any independent set cannot exceed the number of complementary pairs. In particular, if $S = \{a_1, a_2, \dots, a_s\}$ is an independent set of $\Gamma(P)$, then

$$|S| \leq t.$$

We, now, show that if $|S| < t$, then S can be extended to a larger independent set until $|S| = t$. Since $|S| < t$, there exists at least one complementary pair $\{a_{s+1}, a'_{s+1}\}$ such that

$$\{a_{s+1}, a'_{s+1}\} \cap S = \varnothing.$$

We claim that at least one of $S \cup \{a_{s+1}\}$ or $S \cup \{a'_{s+1}\}$ remains an independent set.

Suppose on the contrary that neither is independent. Then there exist distinct vertices $a_i, a_j \in S$ such that

$$\{a_i, a_{s+1}\}^\ell = \{0\} \quad \text{and} \quad \{a_j, a'_{s+1}\}^\ell = \{0\},$$

for some $i, j \in \{1, 2, \dots, s\}$. If $a_i = a_j$, then $\Gamma(P)$ would contain a triangle

$$a_{s+1} - a_i(= a_j) - a'_{s+1} - a_{s+1},$$

contradicting Theorem 2.3, since a'_{s+1} is the complement of a_{s+1} in $\Gamma(P)$.

Thus, $a_i \neq a_j$. As P is pseudo-complemented, the relations $\{a_i, a_{s+1}\}^\ell = \{0\}$ and $\{a_j, a'_{s+1}\}^\ell = \{0\}$ imply that $a_i \leq a'_{s+1}$ and $a_j \leq a_{s+1}$. However, since $\{a_{s+1}, a'_{s+1}\}^\ell = \{0\}$, we obtain $\{a_i, a_j\}^\ell = \{0\}$, which contradicts the assumption that a_i and a_j are both in the independent set S.

Hence, at least one of a_{s+1} or a'_{s+1} can be added to S, enlarging it. By repeating this process, S can always be extended until $|S| = t$.

Therefore, every maximal independent set of $\Gamma(P)$ has size $t = \frac{|V(\Gamma(P))|}{2}$, and hence $\Gamma(P)$ is *well-covered.* □

Corollary 3.2. *The zero-divisor graph of a Boolean poset is very well-covered.*

Proof. Let P be a Boolean poset. By Theorem 3.1, all maximal independent sets of $\Gamma(P)$ have the same cardinality, namely $\frac{|V(\Gamma(P))|}{2}$. Hence, $\Gamma(P)$ is very well-covered. □

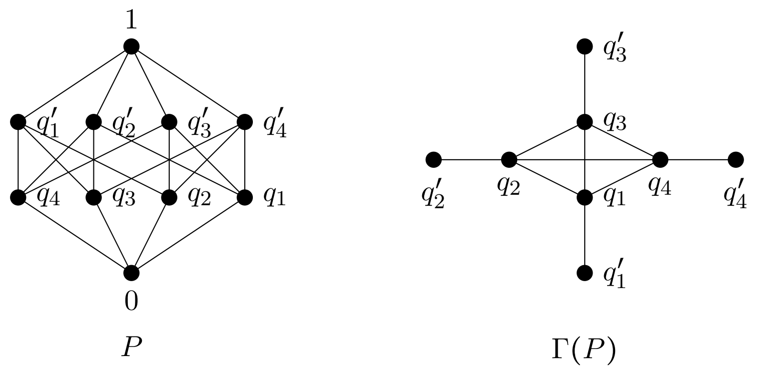

FIGURE 1. The Boolean poset P and its zero-divisor graph $\Gamma(P)$.

Example 3.3. *In the above figure, P denotes a Boolean poset, and $\Gamma(P)$ is the corresponding zero-divisor graph. Consider the simplicial complex of independent sets of $\Gamma(P)$, denoted by $\Delta_{\Gamma(P)}$. We have*

$$\Delta_{\Gamma(P)} = \big\langle \{q_1, q'_2, q'_3, q'_4\}, \{q_2, q'_1, q'_3, q'_4\}, \{q_3, q'_1, q'_2, q'_4\}, \{q_4, q'_1, q'_2, q'_3\}, \{q'_1, q'_2, q'_3, q'_4\} \big\rangle.$$

It is evident that all facets of $\Delta_{\Gamma(P)}$ have the same cardinality. Therefore, $\Delta_{\Gamma(P)}$ is a well-covered simplicial complex. Hence, $\Gamma(P)$ is well-covered.

In order to prove that $\Gamma(P)$ is a Cohen-Macaulay graph, we introduce the following definition for elements of a Boolean poset P.

Definition 3.4. The *weight* (wt) of an element x in a finite bounded poset P is defined as the number of atoms lying below x.

The weight of an element x will be denoted by $\mathrm{wt}(x)$. Clearly, every atom has weight 1, and the *weight of a poset* P, denoted by $wt(P)$, is defined as the weight of the greatest element $1 \in P$. Furthermore, the *weight of a vertex* $v \in \Gamma(P)$ is defined as the weight of the element v in P.

Let B_i be the set of all elements in P with weight $k - i$, where k is the number of all atoms of a Boolean poset P. One can verify that B_1 and B_3 are nonempty and B_2 is empty in the case of a Boolean poset depicted in Figure 1.

Lemma 3.5. *Let P be a Boolean poset and let $x \in P$. Then x' is the complement of x if and only if* $\mathrm{wt}(x') = \mathrm{wt}(P) - \mathrm{wt}(x)$.

Proof. Let P be a Boolean poset with atoms $\{p_1, p_2, \dots, p_k\}$, and let $x \in P$ with $\mathrm{wt}(x) = t$. Since every element in P has a unique complement, say x'. As $\mathrm{wt}(x) = t$, consider the set $\{p_{i_1}, p_{i_2}, \dots, p_{i_t}\}$ of t atoms lying below x, where $\{i_1, i_2, \dots, i_t\} \subseteq \{1, 2, \dots, k\}$. Similarly, let $\{p_{j_1}, p_{j_2}, \dots, p_{j_{\mathrm{wt}(x')}}\}$ be the set of atoms lying below x'.

Now, suppose that $\mathrm{wt}(x') > k - t$. Then there exists at least one atom, say p_s, that is common to both sets $\{p_{i_1}, p_{i_2}, \dots, p_{i_t}\}$ and $\{p_{j_1}, p_{j_2}, \dots, p_{j_{\mathrm{wt}(x')}}\}$. Hence $p_s \leq x$ and $p_s \leq x'$, which implies that $\{p_s\} \subseteq \{x, x'\}^{\ell} \neq \{0\}$, contradicting the fact that x' is the complement of x. Therefore, $\mathrm{wt}(x') \leq k - t$.

Next, suppose $\mathrm{wt}(x') < k - t$. Then there exists an atom p_q that is not below either x or x'. Consequently, $\{x, p_q\}^{\ell} = \{0\} = \{x', p_q\}^{\ell}$, implying that the edge x–x' forms an edge of a triangle, contradicting Theorem 2.3. Thus, $\mathrm{wt}(x') = k - t = \mathrm{wt}(P) - \mathrm{wt}(x)$.

Conversely, suppose $\mathrm{wt}(x') = \mathrm{wt}(P) - \mathrm{wt}(x) = k - t$. If $\{x, x'\}^{\ell} \neq \{0\}$, then there exists an atom lying below both x and x', implying that $\mathrm{wt}(x) + \mathrm{wt}(x') > k$, a contradiction. Hence $\{x, x'\}^{\ell} = \{0\}$.

Finally, let $y \in \{x, x'\}^u$. Then y must lie above all $t + (k - t) = k$ atoms of P, and hence $y = 1$. Therefore, $\{x, x'\}^u = \{1\}$, showing that x' is a complement of x. Since P is Boolean, this complement is unique. ☐

Let P be a poset with 0. The poset P is called *weakly section semi-complemented* if, for $a, b \in P$ and $a < b$, there exists $c \in P$ such that $0 < c \leq b$ and $\{a, c\}^{\ell} = \{0\}$. Further, P is said be *section semi-complemented* or $a, b \in P$ and $b \nleq a$, there exists $c \in P$ such that $0 < c \leq b$ and $\{a, c\}^{\ell} = \{0\}$. Clearly, every section semi-complemented poset is weak section semi-complemented. A finite bounded section semi-complemented poset is called *atomistic.*

Lemma 3.6. *Every uniquely complemented poset is weakly section semi-complemented.*

Proof. Let P be a uniquely complemented poset and let $a < b$ for $a, b \in P$. Consider the complement a' of a in P. If $\{a', b\}^{\ell} = \{0\}$, then a will have two complements namely a' and b, as we have $\{1\} = \{a, a'\}^u \subseteq \{b, a'\}^u$. Hence $\{a', b\}^{\ell} \neq \{0\}$. Choose any $c \in \{a', b\}^{\ell} = \{0\}$. Then we have $0 < c \leq b$ and $\{c, a\}^{\ell} = \{0\}$. This shows that every uniquely complemented poset is weakly section semi-complemented. ☐

However, the converse is not true; see Waphare and Joshi [18, Figure 1].

As an immediate consequence of the above lemma, we have

Corollary 3.7. *Let P be a Boolean poset and $a, b \in P$ such that $a < b$. Then* $\mathrm{wt}(a) < \mathrm{wt}(b)$.

Proof. In the proof of Lemma 3.6, take any atom below c, say p. Then p will be below b but not a. Thus $\mathrm{wt}(a) < \mathrm{wt}(b)$. ☐

Figure 1 given in Waphare and Joshi [18] shows that a uniquely complemented poset need not be section semi-complemented. However, if we consider a Boolean poset then the assertion is true.

Lemma 3.8. *Every Boolean poset P is section semi-complemented. In particular, if P is finite, then it is atomistic.*

Proof. Suppose $a \nleq b$. We claim that $\{a, b'\}^{\ell} \smallsetminus \{0\} \neq \varnothing$. Suppose on the contrary that $\{a, b'\}^{\ell} = \{0\}$. Then we have $a \leq b'' = b$, a contradiction, as P is a Boolean poset and hence it is pseudocomplemented. Hence $\{a, b'\}^{\ell} \smallsetminus \{0\} \neq \varnothing$. Hence there exists a nonzero element $c \in P$ such that $c \leq a$ and $c \leq b'$; hence $c \nleq b$.

Since P is finite, there exists an *atom* p with $0 < p \leq c$. Then $p \leq a$ and $p \leq b'$, that is, $\{p, b\}^{\ell} = \{0\}$. □

With the above preparation, we are ready to prove the following main theorem.

Theorem 3.9. *If P is a Boolean poset, then $\Gamma(P)$ is Cohen–Macaulay.*

Proof. Let P be a Boolean poset and $\mathrm{wt}(P) = k$. By Theorem 3.1, every maximal independent set have the cardinality equals to the number of complementary pairs, say, t in P. Also, note that $t = \frac{V(\Gamma(P))}{2}$. If $k = 2$, then $\Gamma(P)$ is K_2 and in this case the result is true. Hence assume that $k > 2$.

First, let us construct a maximal independent set B of $\Gamma(P)$.

Suppose that the weight of the poset k is odd. Then for $1 \leq i \leq \frac{k-1}{2}$, define B_i be the set of all elements in P with weight $k - i$.

Let $B_1 = \{y_1, y_2, \dots, y_k\}$ be the set elements of weight $k-1$. Clearly, $B_1 \neq \varnothing$, as P has at least two atoms and contains all the dual atoms of P.

As there is no guaranty that P contains an elements of weight $k - i$ for $2 \leq i \leq k-2$, B_i may or may not be empty for $2 \leq i \leq k-2$.

Now, suppose that k is an even. For $i = \frac{k}{2}$, define the set $B_i = \widehat{B}$.

If there are complementary pairs of weight $\frac{k}{2}$, then the set $B_i = \widehat{B}$ contains any one element from the each complementary pairs. Now,

$$B = \begin{cases} B_1 \cup B_2 \cup \cdots \cup B_{\frac{k-1}{2}}, & \text{if } k \text{ is odd,} \\ B_1 \cup B_2 \cup \cdots \cup B_{\frac{k-2}{2}} \cup \widehat{B}, & \text{if } k \text{ is even.} \end{cases}$$

Now, we show that the set B is a maximal independent set.

Suppose that k is even. Observe that B contains all the elements of weight greater than or equal to $\frac{k}{2}$. Let $x, y \in B$. Then $\mathrm{wt}(x) + \mathrm{wt}(y) \geq \frac{k}{2} + \frac{k}{2} = k$. If equality hols, then y is the complement of x, by Lemma 3.5. However, by the construction, B contains one element from each complementary pair having weight $\frac{k}{2}$, a contradiction. Thus equality does not hold.

If $\mathrm{wt}(x) + \mathrm{wt}(y) > k$, then $\{x, y\}^{\ell}$ contains at least one atom. So $\{x, y\}^{\ell} \neq \{0\}$. Hence x and y are not adjacent.

Now, assume that k is odd. Then all the elements of B are of weight greater than or equal to $\frac{k+1}{2}$. Thus for any two elements $x, y \in B$ satisfy $\mathrm{wt}(x) + \mathrm{wt}(y) > k$. Therefore, there exists at least one atom below x and y that implies that x and y are not adjacent.

Thus, B is an independent set.

Now, we prove B is a maximal independent set. For that, let $y \in V(\Gamma(P)) \smallsetminus B$, Clearly, $0 < \mathrm{wt}(y) \leq \frac{k}{2}$, if k is even. If $\mathrm{wt}(y) = \frac{k}{2}$, then by Lemma 3.5 and the construction of B, $\mathrm{wt}(y') = \frac{k}{2}$ and $y' \in B$. So, there is an edge between y and y'.

If $\mathrm{wt}(y) = l < \frac{k}{2}$. Clearly, y' is in P with $\mathrm{wt}(y') = k - l$. As $l < \frac{k}{2}$, $\mathrm{wt}(y') = k - l > \frac{k}{2}$, by the construction, $y' \in B$. So, we can not extend the independent set B, when k is even.

Now, assume that k is odd. Then B contains all elements of weight greater than or equal to $\frac{k+1}{2}$. Thus the element $y \in V(\Gamma(P)) \smallsetminus B$ has weight at most $\frac{k-1}{2}$. Thus by Lemma 3.5, every element in $y \in V(\Gamma(P)) \smallsetminus B$ has the unique complement in B that is adjacent to y.

Thus B is the maximal independent set of $\Gamma(P)$.

Now, lets give relabeling to the elements of B.

Start labeling the elements of the set B_1 by $x'_1, x'_2, \dots, x'_k$ as there are exactly k elements of weight $k-1$. If $B_2 \neq \varnothing$, then label the elements of B_2 by $x'_{k+1}, x'_{k+2}, \dots, x'_{k+m_1}$ where, m_1 is the number of elements of weight $k-2$. Continue in this way, label the last element of B by x'_t, because every maximal independent set have the cardinality $t = \frac{V(\Gamma(P))}{2}$. The set B can be written as $B = \{x'_1, x'_2, \dots, x'_k, x'_{k+1}, x'_{k+2}, \dots, x'_{k+m_1}, \dots, x'_t\}$.

Now, let the set $A = V(\Gamma(P)) \smallsetminus B = \{x_1, x_2, \dots, x_t\}$ such that x'_i is the complement of x_i for all $i = 1, 2, \dots, t$ and $V(\Gamma(P)) = A \cup B$

Finally, to prove that $\Gamma(P)$ is Cohen-Macaulay, the above labeling must satisfy the conditions given in Lemma 2.2.

Condition (a): Since the complement of any maximal independent set is a minimal vertex cover, the set $A = \{x_1, x_2, \dots, x_t\}$ is a minimal vertex cover of $\Gamma(P)$.

Condition (b): Since $\{x, x'_i\}^\ell = \{0\}$, we have x_i is adjacent to x'_i in $\Gamma(P)$ for each $i = 1, 2, \dots, t$.

Condition (c): Assume that $\{z_i, x_j\}, \{x'_j, x_k\} \in E(\Gamma(P))$ for $z_i \in \{x_i, x'_i\}$. Then $\{z_i, x_j\}^\ell = \{0\} = \{x_k, x'_j\}^\ell$. Since, x'_j is pseudocomplement of x_j, $z_i \leq x'_j$ and $x_k \leq x''_j = x_j$. This implies that that $\{z_i, x_k\}^\ell = \{0\}$. Therefore, $\{z_i, x_k\}$ is an edge in $\Gamma(P)$.

Condition (d): Suppose $\{x_i, x'_j\} \in E(\Gamma(P))$. Then $\{x_i, x'_j\}^\ell = \{0\}$. This gives $x_i \leq x''_j = x_j$. Hence $\{x_i, x_j\} \notin E(\Gamma(P))$.

Condition (e): Assume that $\{x_i, x'_j\} \in E(\Gamma(P))$. We claim that $i \leq j$.

Now, if possible let $i > j$. By the relabeling $\mathrm{wt}(x'_i) \leq \mathrm{wt}(x'_j)$. As x'_i is the unique complement of x_i in P. Thus, $\mathrm{wt}(x_i) + \mathrm{wt}(x'_i) = k$. By the above inequality, we have $\mathrm{wt}(x_i) + \mathrm{wt}(x'_j) \geq k$. If the equality holds, then by Lemma 3.5 x'_j is the complement of x_i. This contradicts the uniquely complementation in P. If $\mathrm{wt}(x_i) + \mathrm{wt}(x'_j) > k$, then there exists at least one atom below x_i and x'_j and hence $\{x_i, x'_j\}^\ell \neq \{0\}$. This contradicts to the assumption that $\{x_i, x'_j\} \in E(\Gamma(P))$. Hence, $i \leq j$.

Thus all the conditions of Lemma 2.2 are satisfied, $\Gamma(P)$ is Cohen-Macaulay. □

As an immediate consequence of the above result, we have the following results proved by A-Ming Liu and Tongsuo Wu in [12].

Corollary 3.10. *The zero-divisor graph of a Boolean algebra (equivalently Boolean ring) is Cohen–Macaulay.*

Since the simplicial complex of independent sets of a graph G coincides with the simplicial complex of cliques in its complement G^c, we further deduce the following.

Corollary 3.11. *The complement $\Gamma^c(P)$ of the zero-divisor graph of a Boolean poset P is Cohen–Macaulay with its clique complex.*

In general, the converses of Theorems 3.1, and 3.9 do not hold. The following example serves as a counterexample for the same.

Example 3.12. *Consider a bounded poset P, having three atoms say a, b, c covered by 1, which is not a Boolean poset, but the corresponding zero-divisor graph is the complete graph K_3. Hence, it is well-covered and Cohen-Macaulay.*

Now, we prove the converse of Theorem 3.9 for a special class of posets. For this purpose, we introduce the notations and definitions given in [11], which will be necessary for proving the forthcoming results.

The direct product of posets $P_1, \dots, P_n$ is the poset $\mathbf{P} = \prod_{i=1}^{n} P_i$ with $\le$ defined such that $a \le b$ in $\mathbf{P}$ if and only if $a_i \le b_i$ (in P_i) for every $i \in \{1, \dots, n\}$. For any $\varnothing \ne A \subseteq \prod_{i=1}^{n} P_i$, note that $A^u = \{b \in \prod_{i=1}^{n} P_i \mid b_i \ge a_i$ for every $a \in A$ and $i \in \{1, \dots, n\}\}$. Similarly, $A^\ell = \{b \in \prod_{i=1}^{n} P_i \mid b_i \le a_i$ for every $a \in A$ and $i \in \{1, \dots, n\}\}$.

Throughout this section, $\mathbf{P} = \prod_{i=1}^{n} P_i$, $(n \ge 2)$, where P_i's are finite bounded posets such that $Z(P_i) = \{0\}$, $\forall i$ and $2 \le |P_1| \le |P_2| \le \cdots \le |P_n|$. Since P_i's are finite bounded poset, P_i's are atomic. Note that $Z(P_i) = 0$ if and only if P_i has the unique atom. Further, assume that $q_i \in P_i$ is the unique atom of P_i for every i.

All the elements of $\mathbf{P}$ *are denoted by bold letters.*

Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbf{P}$, where $x_i \in P_i$. In particular, $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ are the only atoms of $\mathbf{P}$. That is, $\mathbf{q}_i = (0, \dots, 0, q_i, 0, \dots, 0)$. We observe that $(0, 0, \dots, 0), (1, 1, \dots, 1) \in \mathbf{P}$ are the least and the greatest elements of $\mathbf{P}$ denoted by $\mathbf{0}$ and $\mathbf{1}$ respectively. We set $D = \mathbf{P} \setminus Z(\mathbf{P})$. The elements $d \in D$ are the dense elements of $\mathbf{P}$.

Theorem 3.13. *Let* $\mathbf{P} = \prod_{i=1}^{n} P_i$*,* $(n = 2)$*, where* P_i*'s are finite bounded posets such that* $Z(P_i) = \{0\}$*,* $\forall i$ *and* $2 \le |P_1| \le |P_2|$*. Then the following statements are equivalent.*

(1) $\Gamma(\mathbf{P})$ *is Cohen-Macaulay.*

(2) $|P_1| = |P_2| = 2$*, therefore,* $\mathbf{P}$ *is a Boolean lattice*

Proof. (1) $\implies$ (2): Observe that, $\Gamma(\mathbf{P})$ is a complete bipartite graph $K_{|P_1|,|P_2|}$ which is *Cohen-Macaulay* only if $|P_1| = |P_2| = 2$.

(2) $\implies$ (1): If $|P_1| = |P_2| = 2$, then $\Gamma(\mathbf{P}) = K_2$. Hence, it is *Cohen-Macaulay.* □

Lemma 3.14. *Let* $\mathbf{P} = \prod_{i=1}^{n} P_i$*,* $(n \ge 3)$*, where* P_i*'s are finite bounded posets such that* $Z(P_i) = \{0\}$*,* $\forall i$ *and* $2 \le |P_1| \le |P_2| \le \cdots \le |P_n|$*. Let* $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ *be the all atoms of* $\mathbf{P}$*. For each* $1 \le i < j < k \le n$*, define*

$$\mathfrak{J}_{i,j,k} = \left\{ \mathbf{x} \in \mathbf{P} \;\middle|\; \mathbf{x} \in \{\mathbf{q}_i, \mathbf{q}_j\}^u \text{ or } \mathbf{x} \in \{\mathbf{q}_j, \mathbf{q}_k\}^u \text{ or } \mathbf{x} \in \{\mathbf{q}_i, \mathbf{q}_k\}^u \right\} \setminus D.$$

Then $\mathfrak{J}_{i,j,k}$ *is a maximal independent set in* $\Gamma(\mathbf{P})$*.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathfrak{J}_{i,j,k}$. We prove the result by using following cases.

Case I. Without loss of generality, assume that $\mathbf{x}, \mathbf{y} \in \{\mathbf{q}_i, \mathbf{q}_j\}^u$. Then $\{\mathbf{q}_i, \mathbf{q}_j\} \subseteq \{\mathbf{x}, \mathbf{y}\}^\ell \ne \{\mathbf{0}\}$.

Case II. Without loss of generality, assume that $\mathbf{x} \in \{\mathbf{q}_i, \mathbf{q}_j\}^u$ and $\mathbf{y} \in \{\mathbf{q}_j, \mathbf{q}_k\}^u$, then $\{\mathbf{q}_j\} \subseteq \{\mathbf{x}, \mathbf{y}\}^\ell \ne \{\mathbf{0}\}$.

Thus, in every case, $\{\mathbf{x}, \mathbf{y}\}^\ell \ne \{\mathbf{0}\}$, which means no two vertices of $J_{i,j,k}$ are adjacent in $\Gamma(\mathbf{P})$. Hence $\mathfrak{J}_{i,j,l}$ is an independent set.

Now, to prove maximality of $\mathfrak{J}_{i,j,k}$, let $\mathbf{w} \in V(\Gamma(\mathbf{P})) \setminus \mathfrak{J}_{i,j,k}$. Then $\mathbf{w} \notin \{\mathbf{q}_i, \mathbf{q}_j\}^u$, $\mathbf{w} \notin \{\mathbf{q}_j, \mathbf{q}_k\}^u$, and $\mathbf{w} \notin \{\mathbf{q}_i, \mathbf{q}_k\}^u$. Without loss of generality, assume that $\mathbf{q}_i \nleq \mathbf{w}$ and $\mathbf{q}_j \nleq \mathbf{w}$. So, $x_i = 0 = x_j$ for $\mathbf{w} = (x_1, x_2, \dots, x_n)$. Choose an element $\mathbf{x} = (0, \dots, 0, \mathbf{q}_i, \dots, \mathbf{q}_j, 0, \dots, 0)$. Then $\mathbf{x} \in \{\mathbf{q}_i, \mathbf{q}_j\}^u$. Hence $\mathbf{x} \in \mathfrak{J}_{i,j,k}$. Further, $\{\mathbf{x}, \mathbf{w}\}^\ell = \{0\}$. This means $\mathbf{x}$ and $\mathbf{w}$ are adjacent in $\Gamma(\mathbf{P})$. Therefore, $\mathbf{w}$ is adjacent to at least one vertex of $\mathfrak{J}_{i,j,k}$, and hence, $\mathfrak{J}_{i,j,k} \cup \{\mathbf{w}\}$ is not an independent set. Hence, $\mathfrak{J}_{i,j,k}$ is a maximal independent set in $\Gamma(\mathbf{P})$. □

The proof of the next result follows the same ideas as the proof of Theorem 3.5 presented by Asir et al. [2], but rewritten in the language of poset theory. For the sake of completeness, we include the full argument here.

Theorem 3.15. *Let* $\mathbf{P} = \prod_{i=1}^{n} P_i$, $(n \geq 3)$, *where* P_i*'s are finite bounded posets such that* $Z(P_i) = \{0\}$, $\forall i$ *and* $2 \leq |P_1| \leq |P_2| \leq \cdots \leq |P_n|$. *Then the zero-divisor graph* $\Gamma(\mathbf{P})$ *is well-covered if and only if* $|P_i| = 2$ *for all* $i = 1, 2, \ldots, n$.

Proof. If $|P_i| = 2$ for all $i = 1, 2, \ldots, n$, then P is a Boolean poset. By Theorem 3.1, it follows that $\Gamma(P)$ is well-covered.

Conversely, suppose that $\Gamma(\mathbf{P})$ is well-covered, i.e., all its maximal independent sets have the same cardinality.

Let $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ be the atoms of $\mathbf{P}$. For each atom $\mathbf{q}_i$, consider the maximal independent set

$$\mathfrak{J}_i = \{\mathbf{q}_i\}^u \smallsetminus D = \{(x_1, \ldots, x_k) \in \mathbf{P} \mid x_i \neq 0\} \smallsetminus D,$$

where D denotes the set of dense elements of $\mathbf{P}$. Since $|D| = (|P_1| - 1)(|P_2| - 1)\cdots(|P_n| - 1)$, we obtain $|\mathfrak{J}_i| = \big(|P_1||P_2|\cdots(|P_i| - 1)\cdots|P_n|\big) - \big((|P_1| - 1)(|P_2| - 1)\cdots(|P_n| - 1)\big)$. As $\Gamma(\mathbf{P})$ is well-covered, $|\mathfrak{J}_i| = |\mathfrak{J}_j|$ for all $i \neq j$. This equality simplifies to

$$(|P_i| - 1)|P_j| = (|P_j| - 1)|P_i| \quad \Longrightarrow \quad |P_i| = |P_j|.$$

Hence $|P_1| = |P_2| = \cdots = |P_n| = \alpha$ (say).

It remains to show that $\alpha = 2$. By Lemma 3.14, consider the maximal independent set

$$\mathfrak{J}_{1,2,3} = \Big\{\mathbf{x} \in \mathbf{P} \;\Big|\; \mathbf{x} \in \{\mathbf{q}_1, \mathbf{q}_2\}^u \text{ or } \mathbf{x} \in \{\mathbf{q}_2, \mathbf{q}_3\}^u \text{ or } \mathbf{x} \in \{\mathbf{q}_1, \mathbf{q}_3\}^u\Big\} \smallsetminus D.$$

Writing the set $\mathfrak{J}_{1,2,3} = A \cup B \cup C$, where $A = \{\mathbf{x} \in \mathbf{P} \mid \mathbf{x} \in \{\mathbf{q}_1, \mathbf{q}_2\}^u\} \smallsetminus D$, $B = \{\mathbf{x} \in \mathbf{P} \mid \mathbf{x} \in \{\mathbf{q}_2, \mathbf{q}_3\}^u\} \smallsetminus D$, and $C = \{\mathbf{x} \in \mathbf{P} \mid \mathbf{x} \in \{\mathbf{q}_1, \mathbf{q}_3\}^u\} \smallsetminus D$. By the principle of Inclusion-Exclusion,

$$|\mathfrak{J}_{1,2,3}| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Note that

$$|A| = |B| = |C| = (\alpha - 1)^2 \alpha^{n-2} - (\alpha - 1)^n,$$

and

$$|A \cap B| = |A \cap C| = |B \cap C| = |A \cap B \cap C| = (\alpha - 1)^3 \alpha^{n-3} - (\alpha - 1)^n.$$

Thus

$$|\mathfrak{J}_{1,2,3}| = 3\big((\alpha - 1)^2 \alpha^{n-2} - (\alpha - 1)^n\big) - 2\big((\alpha - 1)^3 \alpha^{n-3} - (\alpha - 1)^n\big).$$

On the other hand, we know that

$$|\mathfrak{J}_1| = (\alpha - 1)\alpha^{n-1} - (\alpha - 1)^n.$$

Since, $\Gamma(\mathbf{P})$ is unmixed, $|\mathfrak{J}_{1,2,3}| = |\mathfrak{J}_1|$. Equating the two expressions, we have

$$3(\alpha - 1)^2 \alpha^{n-2} - 2(\alpha - 1)^3 \alpha^{n-3} = (\alpha - 1)\alpha^{n-1}.$$

Dividing through by $\alpha^{n-3} > 0$, we obtain

$$3(\alpha - 1)\alpha - 2(\alpha - 1)^2 - \alpha^2 = 0,$$

which simplifies to

$$\alpha - 2 = 0.$$

Thus, $\alpha = 2$, and therefore, $|P_1| = |P_2| = \cdots = |P_n| = \alpha = 2$, completing the proof. $\square$

With this, we give a proof of Theorem, 1.3 that characterizes the Cohen-Macaulayness of the zero-divisor graph $\Gamma(\mathbf{P})$.

Proof of Theorem 1.3 :

Proof. (1) $\implies$ (2): It is well-known that, every Cohen-Macaulay graph is well-covered.

(2) $\implies$ (3): follows from Theorem 3.15.

(3) $\implies$ (4): Let $\mathbf{P} = \prod_{i=1}^{n} P_i$, $(n \geq 3)$, where $|P_i| = 2$ for all $i = 1, 2, \ldots, n$. Hence each P_i is a 2-element chain. Thus, $\mathbf{P}$ is a Boolean lattice.

(4) $\implies$ (5): Every Boolean lattice is Boolean poset.

(5) $\implies$ (1): From Theorem 3.9, $\Gamma(\mathbf{P})$ is Cohen-Macaulay. $\square$

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References

[1] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217(2)** (1999), 434-447.
[2] T. Asir, P. V. Cheri, M. R. Pournaki, and M. Poursoltani, *Cohen–Macaulay and Gorenstein zero-divisor graphs of divisor posets*, Period. Math. Hungar. https://doi.org/10.1007/s10998-025-00690-w.
[3] I. Beck, *Coloring of a commutative ring*, J. Algebra **116** (1988), 208-226.
[4] S. Devhare, V. Joshi and J. D. LaGrange, *Eulerianand Hamiltonian complements of zero-divisor graphs of pseudocomplemented posets*, Palestine Math. **6(1)** (2017), 1-10.
[5] G. Grätzer, *Lattice Theory: Foundation*, Springer Science and Business Media.
[6] R. Halaš, Some properties of Boolean ordered sets, Czechoslovak Math. J. **46** (1996), 93-98.
[7] R. Halaš and M. Jukl, *On Beck's coloring of posets*, Discrete Math. **309** (2009), 4584-4589.
[8] V. Joshi and A. Khiste, *The zero-divisor graph of Boolean posets*, Math. Slovaca **64 (2)** (2014), 511-519.
[9] V. Joshi, *Zero divisor graph of a poset with respect to an ideal*, Order **29** (2012), 499-506.
[10] V. Joshi, H. Y. Pourali and B. N. Waphare, *The graph of equivalence classes of zero-divisors*, ISRN Discrete Math., Volume 2014, Article ID 896270, 7 pages.
[11] N. Khandekar and V. Joshi, *Zero-divisor graphs and total coloring conjecture*, Soft Comput., **24** (2020), 18273–18285.
[12] A-M. Liu and T. Wu, *Boolean graphs are Cohen-Macaulay*, Comm. Algebra **46** (2018), 4498-4510.
[13] D. Lu and T. Wu, *The zero-divisor graphs of partially ordered sets and an application to semigroups*, Graphs Combin. **26** (2010), 793-804.
[14] M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai, S. Yassemi, *Vertex decomposability and regularity of very well-covered graphs*, J. Pure Appl. Algebra **215** (2011), no. 10, 2473–2480.
[15] R. P. Stanley, *Combinatorics and Commutative Algebra*, second edition, Birkhäuser Boston, 1996.
[16] R.H. Villarreal, *Monomial Algebras*, Boca Raton- London-New York: Taylor & Francis Group, LLC. (First Edition (2005)) New York: Marcel Dekker, Inc. (2015).
[17] R.H. Villarreal, *Cohen-Macaulay Graphs*, Manuscripta Math. **66** (1990), 277 - 293.
[18] B.N. Waphare, V. V. Joshi, *On Uniquely Complemented Posets*, Order **22**(2005), 11–20.
[19] D. B. West, *Introduction to Graph Theory*, Prentice Hall, 1996.
[20] R. Woodroofe, *Vertex decomposable graphs and obstructions to shellability*, Proc. American Math. Soc. **137**(2009), 3235-3246.