DISSIPATIVE SOLUTIONS TO STOCHASTIC 3D EULER EQUATIONS

UMBERTO PAPPALETTERA AND FRANCESCO TRIGGIANO

Abstract. We construct probabilistically strong solutions to the three-dimensional Euler equations perturbed by additive noise that are $\mathbb{P}$-almost surely continuous in time, Hölder in space, and satisfy the local energy inequality up to an arbitrarily large stopping time.

Contents

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1. Introduction

In this work, we consider the stochastic Euler equations on the three-dimensional torus $\mathbb{T}^3 := \mathbb{R}^3/2\pi\mathbb{Z}^3$ driven by additive noise

$$\begin{cases} du + \operatorname{div}(u \otimes u)\, dt + \nabla p\, dt = Q^{1/2} dW, \\ \operatorname{div}(u) = 0. \end{cases} \tag{SE}$$

Euler equations govern the evolution of an ideal fluid and constitute one of the most important models in fluid dynamics. The presence of a random external force in (SE) can help represent many phenomena perturbing the dynamics that are neglected by the deterministic equation, such as endogenous microscopic thermal effects or exogenous influences of the environment on the fluid. First investigations on (SE) date back to the works [MV00, Kim09], where local existence of smooth solutions is proved (see also [BF99] for other results in dimension two). Measure-valued solutions satisfying weak-strong uniqueness and a version of the energy inequality have been constructed in [BM21], building upon results by DiPerna and Majda [DM87] for the deterministic Euler equations $Q = 0$. Whether solutions are unique or not remained an open problem until the work [HZZ22] by Hofmanová, Zhu, and Zhu, adapting for the first time the convex integration scheme developed by De Lellis and Székelyhidi Jr. in [LJ09] to the stochastic case, and obtaining non-uniqueness in law for (SE). After [HZZ22] made clear that convex integration techniques can be applied to stochastic equations (let us also mention [CFF21] in this regard), a sequence of papers has appeared [HZZ25, LZ24, KK24, LLZ25] extending the works [DLS13, DLS14, BLIJ15, BDLSJ16, Ise18, BDLSV19] on the deterministic Euler equations. In particular, for every $\vartheta \in (0, 1/3)$, Lü, Lü, and Zhu proved in [LLZ25] existence and non-uniqueness of solutions of (SE) with regularity $u \in C([0,\infty), C^\vartheta(\mathbb{T}^3, \mathbb{R}^3)$ almost surely, which are strictly dissipative up to an arbitrary large stopping time $\mathfrak{t}$ in the sense that u satisfies almost surely the energy inequality

$$\frac{1}{2}\|u(t\wedge\mathfrak{t})\|^2_{L^2_x} \le \frac{1}{2}\|u(s\wedge\mathfrak{t})\|^2_{L^2_x} + \int_{s\wedge\mathfrak{t}}^{t\wedge\mathfrak{t}} \langle u(r), Q^{1/2} dW_r\rangle + \frac{1}{2}\operatorname{Tr}(Q)(t\wedge\mathfrak{t} - s\wedge\mathfrak{t}), \tag{1.1}$$

for every $s < t$, and the inequality is strict if $s < \mathfrak{t}$. In the present work we investigate solutions of (SE) that satisfy almost surely the *local energy inequality* up to time $\mathfrak{t}$:

$$\frac{1}{2} d|u|^2 + \operatorname{div}\left(\left(\frac{1}{2}|u|^2 + p\right) u\right) dt - \mathfrak{q}\, dt - u \cdot Q^{1/2} dW \le 0, \tag{LEI}$$

where $\mathfrak{q} := \frac{1}{2}\sum_k |Q^{1/2} e_k|^2$ for a CONS $\{e_k\}_{k\in\mathbb{N}}$ of the space of L^2_x solenoidal vector fields, $d|u|^2$ and dW are Itō stochastic differentials, and the equation is meant as an almost sure identity in the sense of distributions. Notice that (1.1) can be obtained from (LEI) by space integration.

The importance of a local form of energy inequality towards partial regularity results for *suitable* weak solutions of the Navier-Stokes equations is known since the works of Scheffer [Sch77] and Caffarelli, Kohn, and Niremberg [CKN82]. Similar results are available for the stochastic Navier-Stokes equations, see [Rom10, FR02]. The local energy balance (LEI) is a natural condition to impose also on the stochastic Euler equations (SE), since any physically relevant $L^3_{t,x}$ zero-viscosity limit of suitable solutions of the stochastic Navier-Stokes equations must inherit this property. It describes the local change in kinetic energy due to spatial roughness of u: indeed, the equality in (LEI) holds for smooth solutions u, and in general the inequality corresponds to no local creation of kinetic energy due to irregularities of u. We refer to the celebrated work of Duchon and Robert [DR00] for more details.

In the deterministic setting $Q = 0$, existence and non-uniqueness of solutions to the system (SE)-(LEI) have been shown in the series of papers [DLS10, Ise22, DLK22, GKN25]. The purpose of the present work is to provide similar results in the stochastic case $Q \neq 0$. More specifically, we will prove the following:

Theorem 1.1. *Let $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\ge0}, \mathbb{P})$ be a stochastic basis with right-continuous and complete filtration supporting a sufficiently smooth-in-space Q-Wiener process $Q^{1/2}W$ with zero space average and null divergence. Let be given $T > 0$, $\varkappa \in (0,1)$, $\alpha \in (0, 1/7)$ and $E \in C^1_{loc}([0,\infty), \mathbb{R})$. Then there exist two $\{\mathbb{F}_t\}_{t\ge0}$-progressively measurable stochastic processes*

$$u : \Omega \to C([0,\infty), C^\alpha(\mathbb{T}^3, \mathbb{R}^3)), \qquad p : \Omega \to C([0,\infty), C^{2\alpha}(\mathbb{T}^3, \mathbb{R})),$$

such that (u,p) *is* $\mathbb{P}$*-almost surely a solution of* (SE) *on the time interval* $[0,\infty)$ *in analytically weak sense, and a stopping time* $\mathfrak{t}$ *satisfying* $\mathbb{P}\{\mathfrak{t}\geq T\}\geq\varkappa$ *such that the local energy equality*

$$\frac{1}{2}d|u|^2+\operatorname{div}\left(\left(\frac{1}{2}|u|^2+p\right)u\right)dt-\mathfrak{q}\,dt-u\cdot Q^{1/2}dW=-E'dt \tag{LEE}$$

holds $\mathbb{P}$*-almost surely in the sense of distributions on the time interval* $[0,\mathfrak{t}]$*. In particular,* (LEI) *holds strictly if* $E'>0$*. Finally, non-uniqueness in law for* (u,p) *holds within this class.*

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2. Preliminaries and main iterative proposition

2.1. **Reformulation of the problem without local martingales.** It is convenient to rewrite (SE)-(LEE) without the local martingale terms $Q^{1/2}dW$ and $u\cdot Q^{1/2}dW$. To attain this, we reformulate the problem using the so-called Da Prato-Debussche trick, namely we decompose solutions of (SE) as $u=v+z$, where $z:=Q^{1/2}W$ and v solves the random PDE

$$\begin{cases}\partial_t v+\operatorname{div}\big((v+z)\otimes(v+z)\big)+\nabla p=0,\\ \operatorname{div}(v)=0.\end{cases}$$

A local energy inequality can be derived for v as follows. First, rewrite the stochastic differential

$$\frac{1}{2}d|v|^2=\frac{1}{2}d|u-z|^2=\frac{1}{2}d|u|^2+\frac{1}{2}d|z|^2-d(u\cdot z).$$

Since by assumption z is spatially smooth enough, Itō Formula gives

$$\frac{1}{2}d|z|^2=z\cdot dz+\mathfrak{q}\,dt,\quad d(u\cdot z)=du\cdot z+u\cdot dz+2\mathfrak{q}\,dt,$$

so that the local energy equality (LEE) is equivalent to

$$\frac{d}{dt}\frac{|v|^2}{2}+v\cdot\nabla p+v\cdot\operatorname{div}\big((v+z)\otimes(v+z)\big)=-E'.$$

We refer to [Rom10] for additional details. Therefore, solutions (u,p) of (SE)-(LEE) are equivalent to solutions of the following system:

$$\begin{cases}\partial_t v+\operatorname{div}\big((v+z)\otimes(v+z)\big)+\nabla p=0,\\ \operatorname{div}(v)=0,\\ \partial_t\frac{|v|^2}{2}+v\cdot\nabla p+v\cdot\operatorname{div}\big((v+z)\otimes(v+z)\big)=-E'.\end{cases} \tag{2.1}$$

2.2. **Notation.** Let $\mathbb{N}$ denote the set of natural numbers, including 0 in our convention, and let $n,m\in\mathbb{N}$. For a multi-index $\mathbf{r}=(r_1,r_2,\dots,r_n)\in\mathbb{N}^n$ of order $|\mathbf{r}|:=r_1+r_2+\cdots+r_n$ and a smooth function $f:\mathcal{O}\to\mathbb{R}^m$ defined on a smooth domain $\mathcal{O}\subset\mathbb{R}^n$ without boundary, we denote $D^{\mathbf{r}}f:=\partial_1^{r_1}\partial_2^{r_2}\dots\partial_n^{r_n}f$. With a slight abuse of notation, in the following we identify functions defined on the torus $\mathbb{T}^3$ as periodic functions defined on $\mathcal{O}=\mathbb{R}^3$.

Let $N\in\mathbb{N}$ and $\delta\in(0,1)$, and let us define the following Hölder seminorms:

$$[f]_N:=\max_{|\mathbf{r}|=N}\|D^{\mathbf{r}}f\|_{C^0},\quad [f]_{N+\delta}:=\max_{|\mathbf{r}|=N}\sup_{x\neq y}\frac{|D^{\mathbf{r}}f(x)-D^{\mathbf{r}}f(y)|}{|x-y|^\delta}.$$

Then, the Hölder norms are defined as

$$\|f\|_N:=\max_{j=0,\dots,N}\{[f]_j\},\quad \|f\|_{N+\delta}:=\max\{\|f\|_N,[f]_{N+\delta}\}.$$

Furthermore, in order to simplify the statement of many propositions here contained, we adopt the notational convention $\|f\|_M\equiv 0$ whenever the index $M<0$.

2.3. **Preparation of the convex integration scheme.** The proof of Theorem 1.1 relies on a convex integration scheme, mostly inspired by the works [Ise22, DLK22, LLZ25]. Let us introduce various parameters that will allow to estimate all the objects appearing in the upcoming sections. Define for $q \in \mathbb{N}$:

$$\lambda_q := \lfloor a^{b^q} \rfloor, \quad \overline{\delta}_q := \lambda_q^{-2\alpha}, \quad \delta_q := \begin{cases} \lambda_0^\gamma, & q = 0, \\ \lambda_0^\gamma \overline{\delta}_q \lambda_1^{2\alpha}, & q \geq 1, \end{cases}$$

where $a > 2$ will be very large, $b > 1$ will be chosen close to 1 and $\alpha \in (0, 1/7)$ indicates the Hölder regularity of the solution identified by the iterative procedure.

Let also also denote $N_1 = n_0 + 5$, where the parameter $n_0 \in \mathbb{N}$ will be chosen in the following sections and will depend on the value b. In particular, n_0 gets larger as b is closer to 1. We assume that the Q-Wiener process z takes values $\mathbb{P}$-almost surely in the Sobolev space $H^{2+N_1}(\mathbb{T}^3, \mathbb{R}^3)$ with zero mean and divergence.

Let us also introduce $0 < \delta \ll 1$ sufficiently small such that

$$\frac{1/2 + \delta}{1/2 - \delta} < \frac{1 - 4\alpha}{3\alpha} \wedge \frac{4}{3},$$

and $L > 1$ such that the following stopping time

$$\mathfrak{t} := \inf \left\{ t \geq 0 : \|z\|_{C^{1/2-\delta}([0,t], C_x^{N_1})} \geq L \right\} \wedge 2T$$

is larger than T with probability greater than $\varkappa \in (0, 1)$ given by the statement of Theorem 1.1. Notice that is always possible to find such L since $H^{2+N_1} \subset C^{N_1}$ by Sobolev embedding. In this way, we have a $\mathbb{P}$-almost sure control over the size of $\|z_t\|_{N_1}$ on the time interval $t \in [0, \mathfrak{t}]$. However, since z is temporally rough, we need to introduce a time mollification

$$z_q(t) := \int_{\mathbb{R}} r_{i_q}(t - s) z(s)\, ds, \quad t \in \mathbb{R} \tag{2.2}$$

where $z(-s) := z(0)$ for all $s > 0$, r is a smooth time mollifier with support contained in $[0, 1]$, and the parameter i_q is defined as

$$i_q := \begin{cases} \lambda_0^{m_5} \lambda_1^{m_6}, & q = 0, \\ \lambda_q^{m_1} \lambda_{q+1}^{m_2} \delta_q^{m_3} \delta_{q+1}^{m_4}, & q \geq 1, \end{cases}$$

where $(m_1, m_2, m_3, m_4) = (1/2 - \delta)^{-1} (0, 0, -2, 7/2)$ and $(m_5, m_6) = (7\alpha/(1 - 2\delta), -m_5)$. The choice of the asymmetric mollifier guarantees that z_q remains adapted to the filtration $\{\mathbb{F}_t\}_{t \in \mathbb{R}}$, extended to negative times identically equal to $\mathbb{F}_0$. Hereafter, we will always adopt this convention when referring to the filtration $\{\mathbb{F}_t\}_{t \in \mathbb{R}}$.

2.4. **Induction scheme.** The main idea of [Ise22] is to obtain a solution of Euler equations satisfying the local energy inequality as a limit of smooth solutions of approximating systems, which include a Reynolds stress tensor R in the momentum equation and a flux current φ in the local energy inequality, and boil down to the Euler system when $R = 0$ and $\phi = 0$. In this sense, the size of the *errors* R and ϕ give an indication of "how far" a solution of the approximating system is from being an exact solution of the Euler system.

In this work, the approximating system to (2.1) is slightly more involved, since we need to keep track of the random process z affecting the dynamics. Throughout this work we shall deal with *probabilistically strong* solutions, i.e. our objects will be defined on the same stochastic basis $(\Omega, \mathbb{F}, (\mathbb{F}_t), \mathbb{P})$ supporting the Wiener process z.

Definition 2.1. A tuple of $\{\mathbb{F}_t\}$-progressively measurable smooth tensors (v, p, R, ϕ, z) is a dissipative Euler-Reynolds flow, with energy loss $E \in C_b^1$, if it satisfies the following system

$$\begin{cases} \partial_t v + \operatorname{div}\big((v + z) \otimes (v + z)\big) + \nabla p = \operatorname{div}(R), \\ \operatorname{div}(v) = 0, \\ \partial_t \frac{|v|^2}{2} + v \cdot \nabla p + v \cdot \operatorname{div}\big((v + z) \otimes (v + z)\big) = -E' + \frac{1}{2} D_t(\operatorname{Tr}(R)) + \operatorname{div}(Rv) + R : \nabla z^T + \operatorname{div}(\phi), \end{cases} \tag{2.3}$$

where $D_t := \partial_t + (v + z) \cdot \nabla$ denotes the advective derivative.

We point out that our approximating Euler-Reynolds flow will solve (2.3) (in an analytically strong sense) also for negative times $t < 0$, as we work with one-sided time mollifications in order to preserve adaptedness to the filtration $\{\mathbb{F}_t\}_{t\in\mathbb{R}}$. We shall impose zero external noise for negative times, so that the process (v, p, R, ϕ, z) will be deterministic for negative times (and thus progressively measurable with respect to the filtration $\{\mathbb{F}_t\}_{t\in\mathbb{R}}$).

As usual in convex integration schemes, we set up an iterative procedure to modify a given dissipative Euler-Reynolds flow $(v_q, p_q, R_q, \phi_q, z_q)$, $q \in \mathbb{N}$, in such a way to identify a new dissipative Euler-Reynolds flow $(v_{q+1}, p_{q+1}, R_{q+1}, \phi_{q+1}, z_{q+1})$ whose errors satisfy $\|R_{q+1}\|_0 \ll \|R_q\|_0$ and $\|\phi_{q+1}\|_0 \ll \|\phi_q\|_0$.

Lemma 2.2 (Iteration Lemma)**.** *Let $\alpha \in (0, 1/7)$ and $E \in C^1_b((-\infty, 2T], \mathbb{R})$. Then there exist parameters a, b as above, $C_v, \overline{M} \in (1, \infty)$ and $n_0 \in \mathbb{N}$ such that the following holds.*

Assume that $(v_q, p_q, R_q, \phi_q, z_q)$, $q \in \mathbb{N}$, is a $\{\mathbb{F}_t\}$-progressively measurable dissipative Euler-Reynolds flow solving (2.3) on the time interval $(-\infty, \mathfrak{t}]$ and such that $\mathbb{P}$-almost surely the following estimates hold for all $t \le \mathfrak{t}$:

$$\|v_q(t)\|_N \le C_v \overline{M} \lambda_q^N \delta_q^{1/2}, \qquad N \in \{1, 2, \dots, n_0 + 3\}, \tag{2.4}$$

$$\|p_q(t)\|_N \le \overline{M} \lambda_q^N \delta_q, \qquad \|D_{t,q} p_q(t)\|_{N-2} \le \overline{M} \lambda_q^{N-1} \delta_q^{3/2}, \qquad N \in \{1, 2, \dots, n_0 + 3\}, \tag{2.5}$$

$$\|R_q(t)\|_N \le \overline{M} \lambda_q^{N-\gamma} \delta_{q+1}, \qquad \|D_{t,q} R_q(t)\|_{N-2} \le \overline{M} \lambda_q^{N-1-\gamma} \delta_{q+1} \delta_q^{1/2}, \qquad N \in \{0, 1, \dots, n_0 + 3\}, \tag{2.6}$$

$$\|\phi_q(t)\|_N \le \overline{M} \lambda_q^{N-3\gamma/2} \delta_{q+1}^{\frac{3}{2}}, \qquad \|D_{t,q} \phi_q(t)\|_{N-1} \le \overline{M} \lambda_q^{N-3\gamma/2} \delta_{q+1}^{\frac{3}{2}} \delta_q^{1/2}, \qquad N \in \{0, 1, 2\}, \tag{2.7}$$

where $\gamma := (b-1)^2$ and $D_{t,q} := \partial_t + (v_q + z_q) \cdot \nabla$.

Then, there exists a new tuple $(v_{q+1}, p_{q+1}, R_{q+1}, \phi_{q+1}, z_{q+1})$ of $\{\mathbb{F}_t\}$-progressively measurable tensors which solves (2.3), such that the estimates (2.4), (2.5), (2.6), (2.7) hold with q replaced by $q+1$ and, in addition, $\mathbb{P}$-almost surely for every $t \le \mathfrak{t}$:

$$\|v_{q+1} - v_q\|_0 + \frac{1}{\lambda_{q+1}} \|v_{q+1} - v_q\|_1 \le C_v \overline{M} \delta_{q+1}^{1/2} \lambda_q^{-\gamma/2}, \tag{2.8}$$

$$\|p_{q+1} - p_q\|_0 + \frac{1}{\lambda_{q+1}} \|p_{q+1} - p_q\|_1 \le \overline{M} \delta_{q+1}. \tag{2.9}$$

We point out that, by definition of $\|\cdot\|_{-1}$ and $\|\cdot\|_{-2}$, estimates (2.5), (2.6) and (2.7) on the advective derivatives are void in these specific cases. Notice also that $\mathfrak{t} \le 2T$ $\mathbb{P}$-almost surely by definition, so that the system (2.3) is meaningful on the time interval $(-\infty, \mathfrak{t}]$.

Since $\delta_q \to 0$ exponentially fast, the estimates above imply $\|R_q\|_0, \|\phi_q\|_0 \to 0$ as $q \to \infty$. Moreover, (2.8) and (2.9) on the increments and interpolation guarantee convergence $v_q \to v$ and $p_q \to p$ in a space of continuous-in-time, Hölder-in-space functions. In fact, given the previous lemma, we can prove Theorem 1.1 arguing as follows.

Proof of Theorem 1.1. Step 1: Initialization. Let $E \in C^1_{loc}([0, \infty), \mathbb{R})$ be as in the statement of the theorem. Then we can find an extension of E to negative times, still denoted E with a little abuse of notation, such that $C^1_b((-\infty, 2T], \mathbb{R})$. Consider the following dissipative Euler-Reynolds flow at step $q = 0$:

$$(z_0, v_0, p_0, R_0, \phi_0) := \left(z_0, 0, \frac{|z_0|^2}{3}, z_0 \mathring{\otimes} z_0 + \frac{2E}{3}\mathrm{Id}, -\frac{|z_0|^2}{2} z_0 \right),$$

where $\mathring{\otimes}$ denotes the traceless tensor product. Using $\mathrm{div}(z_0) = 0$ and the algebraic identity $z_0 \otimes z_0 : \nabla z_0^T = \mathrm{div}(z_0 |z_0|^2/2)$, it is immediate to verify that $(z_0, v_0, p_0, R_0, \phi_0)$ solves the system (2.3) for every $t \in \mathbb{R}$. Moreover, on the time interval $t \in (-\infty, \mathfrak{t}]$, the following estimates on the Reynolds stress are valid for every $N \le n_0 + 3$:

$$\|R_0(t)\|_N \le L^2 + |E(t)| \le \lambda_0^{N-\gamma} (L^2 + |E(t)|) \lambda_0^{\gamma},$$
$$\|D_{t,0} R_0(t)\|_{N-2} \le i_0^{-1/2-\delta} L^2 + L^3 + |E'(t)| \le \lambda_0^{N-1-\gamma} (L^3 + |E'(t)|) \lambda_0^{3\gamma/2},$$

where we have used that $i_0^{-1/2-\delta} \le \lambda_0^{1+\gamma/2}$, as well as the bounds on the current

$$\|\phi_0(t)\|_N \le L^3 \le L^3 \lambda_0^{N-3\gamma/2} \lambda_0^{3\gamma/2}, \quad \|D_{t,0} \phi_0(t)\|_{N-1} \le i_0^{-1/2-\delta} L^3 + L^4 \le L^4 \lambda_0^{N-3\gamma/2} \lambda_0^{2\gamma},$$

and the pressure

$$\|p_0(t)\|_N \le L^2 \le L^2 \lambda_0^N, \quad \|D_{t,0} p_0(t)\|_{N-2} \le i_0^{-1/2-\delta} L^2 + L^3 \le L^3 \lambda_0^{N-1} \lambda_0^{\gamma/2}.$$

Then, recalling that by definition $\delta_0 = \delta_1 = \lambda_0^\gamma > 1$, the estimates above imply the existence of a constant $\overline{M} \in (1,\infty)$, depending only on L and E, such that (2.4), (2.5), (2.6), (2.7) hold true $\mathbb{P}$-almost surely on the time interval $(-\infty, \mathfrak{t}]$.

Step 2: Construction of $(u_{t\wedge \mathfrak{t}}, p_{t\wedge\mathfrak{t}})$. We iteratively apply Lemma 2.2, starting from the initial tuple $(z_0, v_0, p_0, R_0, \phi_0)$ defined in Step 1. We obtain a sequence of dissipative Euler-Reynolds flows, $(v_q, p_q, R_q, \phi_q, z_q)$, and by (2.8) we have that v_q converges $\mathbb{P}$-almost surely to some v in $C^0_t C^\beta_x$ for any $\beta < \alpha$. Similarly, by (2.9), p_q converges $\mathbb{P}$-almost surely to some p in $C^0_t C^{2\beta}_x$. Moreover, by construction z_q converges $\mathbb{P}$-almost surely to z uniformly on the time interval $[0,\mathfrak{t}]$, while $\|R_q\|_0$ and $\|\phi_q\|_0$ vanish by (2.6) and (2.7). Notice that the tuple (v,p,z) is $\{\mathbb{F}_t\}$-progressively measurable since each (v_q,p_q,z_q) is so, by Lemma 2.2. Thus, we have built a solution of the system (2.1) on the time interval $[0,\mathfrak{t}]$. By the discussion in Section 2.1, we have that the process (u,p) with $u := v + z$ solves (SE) and (LEE) on the same time interval, and has the regularity stated in Theorem 1.1.

Step 2: Continuation of (u,p). The proof is similar to [LLZ25, Corollary 1.8]. Let $\tilde{W}_t := W_{t+\mathfrak{t}} - W_{\mathfrak{t}}$ be a cylindrical Wiener process on the filtration $\tilde{\mathbb{F}}_t := \sigma(\{\tilde{W}_s\}_{s\le t}, u_{\mathfrak{t}}, p_{\mathfrak{t}})$. We intend to apply [LLZ25, Theorem 1.7] with initial condtion $u^{in} := u_{\mathfrak{t}} \in C^\beta_x$ $\mathbb{P}$-almost surely. This produces at least two distinct solutions $(\tilde{u}^1, \tilde{p}^1)$ and $(\tilde{u}^2, \tilde{p}^2)$ to (SE) with noise $Q^{1/2} d\tilde{W}$, $\mathbb{P}$-almost surely of class $\tilde{u}^i \in C^0_t C^\vartheta_x$ with arbitrary $\vartheta < \beta$. The pressure fields $\tilde{p}^i$ can be recovered from $\tilde{u}^i$ given by the aforementioned theorem by solving the equations $-\Delta \tilde{p}^i = \operatorname{divdiv}(\tilde{u}^i \otimes \tilde{u}^i)$ subject to the boundary conditions $\int_{\mathbb{T}^3} \tilde{p}^i(t,x)dx = \int_{\mathbb{T}^3} p_{\mathfrak{t}}(x)dx$ for every $t \ge 0$, and are of class $\tilde{p}^i \in C_t C^{2\vartheta}_x$ $\mathbb{P}$-almost surely by [CDR20, Corollary 3.3].

Then the glued processes $(u^i_t, p^i_t) := (u_t, p_t)\mathbf{1}_{\{t\le \mathfrak{t}\}} + (\tilde{u}^i_{t-\mathfrak{t}}, \tilde{p}^i_{t-\mathfrak{t}})\mathbf{1}_{\{t>\mathfrak{t}\}}$, $i = 1,2$, are solutions to (SE) that satisfy (LEE) by construction on the time interval $t \in [0,\mathfrak{t}]$. Since $\vartheta < \beta < \alpha$ is arbitrary, each (u^i, p^i) has the space regularity stated in Theorem 1.1, concluding the proof. □

Remark 2.3. Let us stress that (2.4) and (2.8) imply that, for any $q \in \mathbb{N}$ and $N \le n_0 + 3$,

$$\|u_q(t)\|_N + \|v_q(t)\|_N \lesssim \begin{cases} \lambda_q^N \delta_q^{1/2}, & N > 0 \\ 1\,, & N = 0 \end{cases}, \tag{2.10}$$

where the implicit constant does not depend on the parameter a. This independence of a will be crucial in the following sections. Indeed, we will be able to absorb various implicit constants emerging from the convex integration scheme by picking a large enough. Since this remark is not trivial just for $N = 0$, let us focus on that case. Indeed, since $a > 2$ it holds for every $q \in \mathbb{N}$

$$\begin{aligned} \|v_q(t)\|_0 &\le \sum_{j=1}^q \|v_j(t) - v_{j-1}(t)\|_0 \le C_v \overline{M} \sum_{j=1}^q \delta_j^{1/2} \lambda_{j-1}^{-\gamma/2} \\ &\le C_v \overline{M} \sum_{j=1}^q \left(\frac{\lambda_1}{\lambda_j}\right)^\alpha \left(\frac{\lambda_0}{\lambda_{j-1}}\right)^{\gamma/2} \le C_v \overline{M} \left(1 + \sum_{j=2}^q a^{\alpha(b-b^j)}\right) \le C_v \overline{M} \left(1 + \sum_{j=2}^q 2^{\alpha(b-b^j)}\right). \end{aligned}$$

3. Construction of the velocity perturbation

This section explains the construction of the perturbation $w := v_{q+1} - v_q$, where $(v_q, p_q, R_q, \phi_q, z_q)$ is a solution to (2.3). In particular, w is obtained by adapting the construction introduced in [DLK22] to our stochastic framework.

3.1. Mikado flows.

The perturbation w is built by combining highly oscillating vector fields modulated by scalar amplitudes, that will be chosen to cancel out R_q and ϕ_q through the following two geometric lemmas.

Lemma 3.1. *Let $\mathcal{S}^3$ denote the space of 3×3 symmetric matrices, and consider a family of vectors $\mathcal{F}_{\mathcal{R}} := \{k_i\}_{i=1}^6 \subset \mathbb{Z}^3$ and a constant $C > 0$ such that*

$$\sum_{i=1}^6 k_i \otimes k_i = C\mathrm{Id}, \quad \textit{and } \{k_i \otimes k_i\} \textit{ is a basis of } \mathcal{S}^3.$$

Then, there exists a positive constants $N_0 = N_0(\mathcal{F}_{\mathcal{R}})$ such that for any $N \le N_0$ we can find functions $\{\Gamma_{k_i}\}_{i=1}^6 \subset \mathscr{C}^\infty(S_N, (0,\infty))$, where $S_N := \{K \in \mathcal{S}^3 : |K - Id|_\infty \le N\}$, such that for every $K \in S_N$ it *holds*

$$K=\sum_{i=1}^{6}\Gamma_{k_i}^2(K)k_i\otimes k_i.$$

Lemma 3.2. *Suppose that* $\mathcal{F}_\phi:=\{k_1,k_2,k_3\}\subset\mathbb{Z}^3$ *is an orthogonal frame and define*

$$k_4:=-\sum_{i=1}^{3}k_i.$$

Then, for any $N_0>0$*, there exist real-valued affine functions* $\{\Lambda_{k_i}\}_{i=1}^4$ *on* $\mathbb{R}^3$ *such that their restrictions to* $\mathcal{V}_{N_0}:=\{u\in\mathbb{R}^3:|u|\le N_0\}$ *are of class* $\mathscr{C}^\infty(\mathcal{V}_{N_0},[N_0,\infty))$ *and for every* $u\in\mathcal{V}_{N_0}$ *it holds*

$$u=\sum_{i=1}^{4}\Lambda_{k_i}(u)k_i.$$

As in [DLK22], these two lemmas allow to find 27 disjoint families of directions $\{\mathcal{F}^j\}_{j\in\mathbb{Z}_3^3}$, indexed by equivalence classes $j\in\mathbb{Z}_3^3$ (triples of integers modulo 3), such that any $\mathcal{F}^j$ can be decomposed as $\mathcal{F}^j=\mathcal{F}_R^j\cup\mathcal{F}_\phi^j$, where $\mathcal{F}_R^j$ and $\mathcal{F}_\phi^j$ satisfy the assumptions of Lemmas 3.1 and 3.2, respectively. For any direction $f\in\mathcal{F}:=\bigcup_{j\in\mathbb{Z}_3^3}\mathcal{F}^j$ let us define

$$l_f:=\{x\in\mathbb{T}^3:x-\sigma f\in 2\pi\mathbb{Z}^3\text{ for some }\sigma\in\mathbb{R}\},$$

which is nothing but the the line in direction f passing through the origin, but seen as subset of $\mathbb{T}^3$. Moreover, le η be a positive constant and $\psi_f\in C^\infty(\mathbb{T}^3,\mathbb{R})$ be a function such that:

$$\operatorname{Supp}(\psi_f)\subset B\left(l_f,\frac{\eta}{10}\right),\quad f\cdot\nabla\psi_f=0,\tag{3.1}$$

namely with support concentrated in the pipe of radius $\eta/10$ around the line l_f, and invariant by translations in the direction f. We assume the following renormalization properties:

$$\int_{\mathbb{T}^3}\psi_f dx=\int_{\mathbb{T}^3}\psi_f^3 dx=0,\quad \fint_{\mathbb{T}^3}\psi_f^2dx=1,\qquad\text{for all } f\in\mathcal{F}_R;\tag{3.2}$$

$$\int_{\mathbb{T}^3}\psi_f dx=0,\quad \fint_{\mathbb{T}^3}\psi_f^3dx=1,\qquad\text{for all } f\in\mathcal{F}_\phi.\tag{3.3}$$

3.2. **Partition of unity and shifts.** In the following we denote $Q(x,r):=\{y\in\mathbb{T}^3:|y-x|_\infty\le r\}$ the hypercube in $\mathbb{T}^3$ of points whose box distance from x is at most r. Let $\chi_0:\mathbb{T}^3\to\mathbb{R}_+$ and $\theta_0:\mathbb{R}\to\mathbb{R}_+$ be nonnegative smooth functions and suppose $\operatorname{Supp}(\chi_0)\subset Q(0,9\pi/8)$ with $\chi_0\equiv 1$ on $Q(0,7\pi/8)$, and similarly $\operatorname{Supp}(\theta_0)\subset[-1/8,9/8]$ with $\theta_0\equiv 1$ on $[1/8,7/8]$.

For every $n\in\mathbb{Z}^3$ and $m\in\mathbb{Z}$, define $\chi_n(x):=\chi_0(x-2\pi n)$ and $\theta_m(t)=\theta_0(t-m)$. The functions χ_0,θ_0 can be chosen in such a way that

$$\sum_{n\in\mathbb{Z}^3}\chi_n^6(x)=1,\quad\forall x\in\mathbb{T}^3;\qquad\sum_{m\in\mathbb{Z}}\theta_m^6(t)=1,\quad\forall t\in\mathbb{R}.$$

Next, let $\nu:=\frac{1-7\alpha}{3\alpha}$ such that $(7+\nu)\alpha<1$ and $(4+2\nu)\alpha-1<-3\alpha$. Then, we introduce the parameters

$$\epsilon:=\begin{cases}C_\epsilon\lambda_0^{h_5}\lambda_1^{h_6}, & q=0,\\ C_\epsilon\lambda_q^{h_1}\lambda_{q+1}^{h_2}\delta_q^{h_3}\delta_{q+1}^{h_4}, & q\ge 1,\end{cases}\qquad C_\epsilon:=(C_v\overline{M}+L)^{-1}\min\left(\frac{1}{2},\frac{\inf_j N_0(\mathcal{F}_R^j)}{6}\right),\tag{3.4}$$

which represents the size of the temporale partition, and

$$\mu:=\begin{cases}\lambda_0^{u_5}\lambda_1^{u_6}, & q=0\\ \lambda_q^{u_1}\lambda_{q+1}^{u_2}\delta_q^{u_3}\delta_{q+1}^{u_4}, & q\ge 1\end{cases},\tag{3.5}$$

which represents instead the size of the spatial partition, where the exponents h_i,u_i are given by:

$$h_1=u_1=-1,\quad h_2=u_2=0,\quad h_3=u_3=-(2+\nu),\quad u_4=-u_3,\quad h_4=\frac{3+2\nu}{2},$$
$$h_5=u_5=-1/2,\quad h_6=u_6=-1/2.$$

Indeed, given index sets

$$\begin{aligned}\mathcal{I} &:= \{(m,n,f) \in \mathbb{Z} \times \mathbb{Z}^3 \times \mathcal{F} : f \in \mathcal{F}^n\},\\ \mathcal{I}_R &:= \{I \in \mathcal{I} : f \in \mathcal{F}^n_R\},\\ \mathcal{I}_\phi &:= \{I \in \mathcal{I} : f \in \mathcal{F}^n_\phi\},\end{aligned}$$

then let us introduce

$$\chi_I(x) := \begin{cases} \chi^2_n(\frac{x}{\mu}), & I \in \mathcal{I}_\phi,\\ \chi^3_n(\frac{x}{\mu}), & I \in \mathcal{I}_R,\end{cases} \qquad \text{and} \qquad \theta_I(t) := \begin{cases} \theta^2_m(\frac{t}{\epsilon}), & I \in \mathcal{I}_\phi,\\ \theta^3_m(\frac{t}{\epsilon}), & I \in \mathcal{I}_R.\end{cases}$$

Given a radial smooth function $\mathfrak{m} : \mathbb{R}^3 \to \mathbb{R}$ supported in $B(0,2)$ and such that $\mathfrak{m} \equiv 1$ on $\overline{B(0,1)}$, consider the standard Littlewood-Paley operators acting on Fourier transforms as follows:

$$\mathcal{F}[\mathbf{P}_{\leq 2^j} f](\xi) := \mathfrak{m}(\xi\, 2^{-j})\mathcal{F}[f](\xi), \quad \mathcal{F}[\mathbf{P}_{>2^j} f](\xi) := [1-\mathfrak{m}(\xi\, 2^{-j})]\mathcal{F}[f](\xi),$$

The symbol $\mathcal{F}$ here indicated the Fourier transform and should not be confused with the set of directions introduced in Section 3.1. Given the vector $(e_1,e_2,e_3,e_4) := (-1,0,-\frac{4+\nu}{2},\frac{4+\nu}{2}), (e_5,e_6) := (-1/2,-1/2)$, let us introduce the frequency cutoff parameter

$$l := \begin{cases} \lambda_0^{e_5}\lambda_1^{e_6}, & q = 0\\ \lambda_q^{e_1}\lambda_{q+1}^{e_2}\delta_q^{e_3}\delta_{q+1}^{e_4}, & q \geq 1\end{cases},$$

and denote

$$\mathbf{P}_{\leq l^{-1}} := \mathbf{P}_{\leq 2^S}, \quad S := \sup\{j \in \mathbb{Z} \,:\, 2^j \leq l^{-1}\}.$$

Incidentally, we also define here the Littlewood-Paley operator projecting on the frequency shell of size approximately equal to 2^j, namely:

$$\mathbf{P}_{2^j} f := \mathbf{P}_{>2^{j-1}} - \mathbf{P}_{>2^j},$$

that will be useful later. Then we define the spatial regularization of our velocity fields in the following way:

$$v_l := \mathbf{P}_{\leq l^{-1}}[v_q], \quad z_l := \mathbf{P}_{\leq l^{-1}}[z_q], \quad u_l := v_l + z_l. \tag{3.6}$$

Denote $D_{t,l} := \partial_t + u_l \cdot \nabla$ the advective derivative with respect to the velocity field u_l. Finally, for any $I \in \mathcal{I}_m := \{(m,n,f) : n \in \mathbb{Z}^3, f \in \mathcal{F}^n\}$, let us consider the flow ξ_I solving

$$\begin{cases} D_{t,l}\xi_I(t,x) = 0,\\ \xi_I(\epsilon(m-1/8),x) = x.\end{cases} \tag{3.7}$$

3.3. **Perturbation of the velocity.** The principal part of the perturbation $w = v_{q+1} - v_q$ is defined as follows

$$w_o(t,x) := \sum_{I\in\mathcal{I}} \theta_I(t)\chi_I(\xi_I(t,x))a_I(t,x)\nabla\xi_I(t,x)^{-1} f_I \psi_I(\lambda_{q+1}\xi_I(t,x)) =: \sum_{I\in\mathcal{I}} w^o_I(t,x),$$

where the index o stands for *oscillatory*. The amplitudes a_I will be defined in the following Section 3.4, and ψ_I is given by

$$\psi_I(x) = \psi_{f_I}(x - s_I),$$

where f_I is the direction appearing in the index I (i.e. $I = (m,n,f_I)$ for some m,n), and the set of shifts $\{s_I\}_{I\in\mathcal{I}} \subset \mathbb{R}^3$ is chosen such that the following conditions hold:

- $\operatorname{Supp} w^o_I \cap \operatorname{Supp} w^o_J = \emptyset$ for all $I \neq J$;
- For every $I = (m,n,f)$, the shift s_I satisfies $s_I = s_{m,n} + p_{n,f}$ for some $s_{m,n}$ depending only on $m,n \in \mathbb{Z} \times \mathbb{Z}^3$ and $p_{n,f}$ depending only on $f \in \mathcal{F}^n$;
- $s_{m,n} = s_{m,n'}$ for $(n'-n)\mu \in \mathbb{Z}^3$.

The proof of the existence of such a family of shifts is a simple adaptation of [DLK22, Proposition 3.5], with only minor modifications needed to take the randomness of u_l into account, and is omitted for the sake of brevity. As in [DLK22], we can expand $\psi_I, \psi_I^2, \psi_I^3$ in Fourier series as follows:

$$\psi_I = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \mathring{b}_{I,k}e^{ik\cdot x}, \quad \psi_I^2 = \mathring{c}_{I,0} + \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \mathring{c}_{I,k}e^{ik\cdot x}, \quad \psi_I^3 = \mathring{d}_{I,0} + \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \mathring{d}_{I,k}e^{ik\cdot x},$$

where (3.1) implies that $\mathring{b}_{I,k}(f_I \cdot k) = \mathring{c}_{I,k}(f_I \cdot k) = \mathring{d}_{I,k}(f_I \cdot k) = 0$. Then:

$$w_o = \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} b_{m,k} e^{i\lambda_{q+1}\xi_m(x,t)\cdot k}, \tag{3.8}$$

$$w_o \otimes w_o = c_0 + \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} c_{m,k} e^{i\lambda_{q+1}\xi_m(x,t)\cdot k}, \tag{3.9}$$

$$\frac{|w_o|^2}{2} w_o = d_0 + \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} d_{m,k} e^{i\lambda_{q+1}\xi_m(x,t)\cdot k}, \tag{3.10}$$

where the coefficients $b_{m,k}, c_{m,k}, d_{m,k}$ are given by

$$\begin{aligned} b_{m,k} &= \theta_m(t) \sum_{I\in\mathcal{I}_m} \chi_I(\xi_I(t,x)) a_I(x,t) \overline{f_I}(x,t)\, \mathring{b}_{I,k}, \\ c_{m,k} &= \theta_m^2(t) \sum_{I\in\mathcal{I}_m} \chi_I^2(\xi_I(x,t)) a_I^2(t,x) \left(\overline{f_I} \otimes \overline{f_I}\right)(t,x)\, \mathring{c}_{m,k}, \\ d_{m,k} &= \theta_m^3(t) \sum_{I\in\mathcal{I}_m} \chi_I^3(\xi_I(x,t)) a_I^3(t,x) \left(\frac{|\overline{f_I}|^2}{2}\overline{f_I}\right)(t,x)\, \mathring{d}_{m,k}, \end{aligned} \tag{3.11}$$

with

$$\overline{f_I}(x,t) := (\nabla\xi_m)^{-1}(x,t) f_I, \tag{3.12}$$

whereas the zero modes c_0, d_0 are given by

$$\begin{aligned} c_0 &= \sum_{I\in\mathcal{I}} \theta_I^2(t)\chi_I^2(\xi(x,t)) a_I^2(t,x) \left(\overline{f_I} \otimes \overline{f_I}\right)(t,x) \fint_{\mathbb{T}^3} \psi_I(t,y)^2 dy, \\ d_0 &= \sum_{I\in\mathcal{I}} \theta_I^3(t)\chi_I^3(\xi(x,t)) a_I^3(t,x) \left(\frac{|\overline{f_I}|^2}{2}\overline{f_I}\right)(t,x) \fint_{\mathbb{T}^3} \psi_I(t,y)^3 dy. \end{aligned} \tag{3.13}$$

At last, we need to define a *compressibility* corrector term w_c such that $\operatorname{div}(w_c + w_o) = 0$. Let

$$\begin{aligned} w_c &:= \frac{1}{\lambda_{q+1}} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} e_{m,k} e^{i\lambda_{q+1}\xi_m(x,t)\cdot k}, \\ e_{m,k} &= \sum_{I\in\mathcal{I}_m} \nabla\left(\theta_m(t)\chi_I(\xi_I(t,x)) a_I(t,x)\mathring{b}_{I,k}\right) \times \left(\nabla\xi_I^T(t,x)\frac{ik\times f_I}{|k|^2}\right). \end{aligned} \tag{3.14}$$

3.4. **Choice of the amplitudes.** The amplitudes a_I are built differently based on I being in $\mathcal{I}_R$ or in $\mathcal{I}_\phi$. First of all, consider a temporal mollifier η with support in $[-1,0]$, and a time mollification parameter

$$l_{\text{temp}} := \begin{cases} \lambda_0^{s_5}\lambda_1^{s_6}, & q = 0 \\ \lambda_q^{s_1}\lambda_{q+1}^{s_2}\delta_q^{s_3}\delta_{q+1}^{s_4}, & q \geq 1 \end{cases},$$

with $(s_1, s_2, s_3, s_4) = (-1, 0, -\frac{4+\nu}{2}, \frac{3+\nu}{2})$, $(s_5, s_6) = (-1/2, -1/2)$. The asymmetric choice for the support of η guarantees that convolutions with η preserve $\{\mathbb{F}_t\}_{t\geq 0}$ adaptedness. Then, define

$$\begin{aligned} R_l(t,x) &= \int_{\mathbb{R}} \left(\mathbf{P}_{\leq l^{-1}} R_q\right)(t+s, \mu(t;t+s,x))\, \eta_{l_{\text{temp}}}(s) ds, \\ \phi_l(t,x) &= \int_{\mathbb{R}} \left(\mathbf{P}_{\leq l^{-1}} \phi_q\right)(t+s, \mu(t;t+s,x))\, \eta_{l_{\text{temp}}}(s) ds, \end{aligned} \tag{3.15}$$

where $\mu(t;\cdot,\cdot)$ is the unique solution to the transport equation

$$\begin{cases} \partial_r \mu(t;r,x) = u_l(r, \mu(t;r,x)), & r \geq t, \\ \mu(t;t,x) = x. \end{cases}$$

Notice that this mollification along the Lagrangian flow implies that

$$\begin{aligned} D_{t,l} R_l(t,x) &= -\int_{\mathbb{R}} \left(\mathbf{P}_{\leq l^{-1}} R_q\right)(t+s, \mu(t;t+s,x))\, \eta'_{l_{\text{temp}}}(s) ds, \\ D_{t,l} \phi_l(t,x) &= -\int_{\mathbb{R}} \left(\mathbf{P}_{\leq l^{-1}} \phi_q\right)(t+s, \mu(t;t+s,x))\, \eta'_{l_{\text{temp}}}(s) ds. \end{aligned}$$

First case: $I \in \mathcal{I}_\phi$. We define a_I to cancel out ϕ_l through d_0. Indeed, consider

$$a_I(t,x) := |\overline{f_I}|^{-2/3} 2^{1/3} \Lambda_{f_I}^{1/3} \left(- \frac{\nabla \xi_I(t,x) \phi_l(t,x)}{\overline{M}^3 \lambda_q^{-3\gamma/2} \delta_{q+1}^{3/2}} \right) \overline{M} (\lambda_q^{-3\gamma/2} \delta_{q+1}^{3/2})^{1/3}.$$

Notice that $\epsilon \|u_l(t)\|_1 < 1/2$ implies $|\overline{f}_I| > 1/2$ by (3.12). Then, (3.3) and (3.13) plus the disjoint supports property gives

$$\begin{aligned}
d_0 &= \sum_{I \in \mathcal{I}_\phi} \theta_I^3(t) \chi_I^3(\xi(x,t)) a_I^3(t,x) \frac{|\overline{f_I}|^2}{2} \overline{f_I}(t,x) \\
&= \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}^3} \theta_m^6(t) \chi_n^6(\xi_m(x,t)) \nabla \xi_m^{-1}(t,x) \sum_{\substack{I \in \mathcal{I}_\phi \\ I=(m,n,f_I)}} \overline{M}^3 \lambda_q^{-\gamma} \delta_{q+1}^{3/2} \Lambda_{f_I} \left(- \frac{\nabla \xi_I(t,x) \phi_l(t,x)}{\overline{M}^3 \lambda_q^{-\gamma} \delta_{q+1}^{3/2}} \right) f_I \\
&= \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}^3} \theta_m^6(t) \chi_n^6(\xi_m(x,t)) (-\phi_l(t,x)) = -\phi_l(t,x),
\end{aligned}$$

where the second-to-last equality is due to Lemma 3.2 applied with $N_0 = e$, while the last one is due to the space and time partitions of unity.

Second case: $I \in \mathcal{I}_R$. We define a_I to cancel out R_l through c_0. Indeed, let $I = (m,n,f)$ and define

$$a_I(t,x) := \Gamma_f \left(\mathrm{Id} + (\nabla \xi_I \nabla \xi_I^T - \mathrm{Id}) - \frac{\nabla \xi_I (R_l - M_{m,n}) \nabla \xi_I^T}{\rho} \right) \rho^{1/2}, \tag{3.16}$$

$$\rho := \frac{C \, \overline{M}^2 \lambda_q^{-\gamma} \delta_{q+1}}{\inf_j N_0(\mathcal{F}_R^j)}, \tag{3.17}$$

where Γ_f and $N_0(\mathcal{F}_R^j)$ are given by Lemma 3.1, $C > 0$ is a constant depending on the Γ_f's functions and $\sup_{f \in \mathcal{F}} |f|_\infty$, and

$$M_{m,n} := \sum_{\substack{J \in I_\phi, \\ J=(m,n,f_J), \\ |m-m'|_\infty \le 1, \\ |n-n'|_\infty \le 1}} a_J^2(t,x) \chi_J^2(\xi_J(t,x)) \theta_J^2(t) \left(\overline{f_J} \otimes \overline{f_J} \right)(t,x). \tag{3.18}$$

Then, (3.2) and (3.9) imply that

$$\begin{aligned}
c_0 &= \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}^3} \theta_m^6(t) \chi_n^6(\xi_m(x,t)) \left[\sum_{I \in \mathcal{I}_{m,n,R}} a_I^2(t,x) \nabla \xi_I^{-1}(t,x) (f_I \otimes f_I) \nabla \xi_I^{-T}(t,x) + M_{m,n} \right] \\
&= \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}^3} \theta_m^6(t) \chi_n^6(\xi_m(x,t)) \left[\rho \nabla \xi_I^{-1} \left(\nabla \xi_I \nabla \xi_I^T - \frac{\nabla \xi_I (R_l - M_{m,n}) \nabla \xi_I^T}{\rho} \right) \nabla \xi_I^{-T} + M_{m,n} \right] \\
&= \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}^3} \theta_m^6(t) \chi_n^6(\xi_m(x,t)) \left[\rho \mathrm{Id} - R_l \right] = \rho \mathrm{Id} - R_l,
\end{aligned} \tag{3.19}$$

where the second to last equality comes from Lemma 3.1 under the additional assumption

$$\epsilon \|u_l(t)\|_1 \le \frac{\inf_j N_0(\mathcal{F}_R^j)}{6}.$$

3.5. **Estimates on the velocity perturbation.** With the construction above, we are able to control the size of the perturbation w, as stated in the following key result. Since its proof is highly technical and involved, the reader could skip it at a first reading without losing much insight on the remainder of this work. In brief, the main takeaway of this result is that taking the advective derivative $D_{t,l}$ effectively "costs" a factor ϵ^{-1}, whereas a space derivative "costs" a factor λ_{q+1}.

Proposition 3.3. *For $s \in \{0,1,2\}$, $N \le n_0 + 3$ and for every $t \le \mathsf{t}$, it holds*

$$\epsilon^s \|D_{t,l}^s w_o(t)\|_N \lesssim \lambda_{q+1}^N \rho^{1/2}, \tag{3.20}$$

$$\epsilon^s \|D_{t,l}^s w_c(t)\|_N \lesssim (\mu \lambda_{q+1})^{-1} \lambda_{q+1}^N \rho^{1/2}, \tag{3.21}$$

and moreover there exists a finite constant C_2, independent on C_v, $\overline{M}$ and L, such that

$$\|w(t)\|_N \le C_2\, \lambda_{q+1}^N \rho^{1/2}.$$

In addition, for all $N \le n_0 + 3$ and $t \le \mathfrak{t}$, we have

$$\|u_{q+1}(t) - u_q(t)\|_N \lesssim \lambda_{q+1}^N \rho^{1/2}, \tag{3.22}$$

$$\|u_{q+1}(t) - u_l(t)\|_N \lesssim \lambda_{q+1}^N \rho^{1/2}. \tag{3.23}$$

3.5.1. *Proof of Proposition 3.3: Preliminary estimates.* We report some estimates that will be helpful in the proof of Proposition 3.3, as well as in the remainder of the paper.

Lemma 3.4. *For any $N \le n_0 + 4$ and $t \le \mathfrak{t}$, we have*

$$\|z_l(t) - z_{q+1}(t)\|_N \vee \|z_q(t) - z_{q+1}(t)\|_N \lesssim L\, i_q^{1/2-\delta}; \tag{3.24}$$

$$\|\partial_t z_q(t)\|_N \lesssim L\, i_q^{-1/2-\delta}. \tag{3.25}$$

Proof. The estimate for $\|z_q(t) - z_{q+1}(t)\|_N$ is standard, while the one for $\|z_l(t) - z_{q+1}(t)\|_N$ is due to

$$\|z_l(t) - z_q(t)\|_N \lesssim l \|z(t)\|_{N+1}, \quad l \le i_q^{1/2-\delta}.$$

Instead, (3.25) follows from noticing that the time mollifier α used in (2.2) has compact support, implying

$$\int_{\mathbb{R}} r'(s) z(t) ds = 0,$$

and giving in particular the bound

$$\begin{aligned} |\partial_t z_q(t)| &= \left| \partial_t z_q(t) - i_q^{-2} \int_{\mathbb{R}} r'(s\, i_q^{-1}) z(t)\, ds \right| \\ &= i_q^{-2} \left| \int_{\mathbb{R}} r'(s\, i_q^{-1}) [z(t-s) - z(t)]\, ds \right| \\ &= i_q^{-1} \left| \int_{\mathbb{R}} r'(s) [z(t - i_q s) - z(t)]\, ds \right| \lesssim L i_q^{-1/2-\delta}. \end{aligned}$$

Lemma 3.4 and (2.10) imply the following estimates, that will be useful later on.

Corollary 3.5. *For $N \le n_0 + 3$ and $t \le \mathfrak{t}$, we have that*

$$\begin{aligned} \|D_{t,q+1}(z_q - z_l)(t)\|_N &\lesssim l \left(i_q^{-1/2-\delta} + (\lambda_{q+1}^N \delta_{q+1}^{1/2} \vee 1) \right), \\ \|D_{t,q+1}(z_{q+1} - z_q)(t)\|_N &\lesssim i_{q+1}^{-1/2-\delta} + i_q^{1/2-\delta} (\lambda_{q+1}^N \delta_{q+1}^{1/2} \vee 1). \end{aligned} \tag{3.26}$$

□

Lemma 3.6. *For $N \le n_0 + 1$ and $t \le \mathfrak{t}$, we have*

$$\|D_{t,l}(u_q - u_l)(t)\|_N + \|D_{t,l}(v_q - v_l)(t)\|_N \lesssim l^{1-N} \lambda_q^2 \delta_q. \tag{3.27}$$

Proof. First of all, we have that

$$D_{t,q} u_q = \partial_t z_q + \operatorname{div}(R_q) + \nabla p_q.$$

Therefore, the bounds (2.5), (2.6), [DLK22, Lemma A.3], $\lambda_q \delta_q^{1/2} \ge L$, $i_q^{-1/2-\delta} \le \lambda_q^2 \delta_q$ and $\lambda_q < l^{-1}$ imply

$$\begin{aligned} \|D_{t,l}(u_q - u_l)\|_N &\le \|\mathbf{P}_{>l^{-1}} D_{t,l} u_q\|_N + \|[\mathbf{P}_{>l^{-1}}, u_l \cdot \nabla] u_q\|_N \\ &\lesssim l \|D_{t,q} u_q\|_{N+1} + l \|(u_l - u_q) \cdot \nabla u_q\|_{N+1} + l^{1-N} \|u_q\|_1^2 \lesssim l^{1-N} \lambda_q^2 \delta_q. \end{aligned}$$

The same computations and Lemma 3.4 yield the result for $\|D_{t,l}(v_q - v_l)(t)\|_N$. □

Lemma 3.7. *For any $N \le n_0 + 3$ and $t \le \mathfrak{t}$, we have*

$$\|D_{t,l} \nabla v_l(t)\|_N + \|D_{t,l} \nabla u_l(t)\|_N \lesssim l^{-N} \lambda_q^2 \delta_q. \tag{3.28}$$

Proof. Since it holds

$$D_{t,l}(\nabla u_l) = \nabla(\partial_t u_l + u_l \cdot \nabla u_l) - (\nabla u_l)^2 = \nabla\left(\operatorname{div}(R_l + R_{comm}) - \nabla p_l + \partial_t z_l\right) - (\nabla u_l)^2,$$

where $(\nabla u_l)^2$ denotes the standard product between matrices $[(\nabla u_l)^2]^{i,j} = \sum_k \partial_i u^k \partial_k u^j$, then by [DLK22, Lemma A.3],Lemma 3.4, and the bound $L\, i_q^{-1/2-\delta} \le \lambda_q^2 \delta_q$ we have:

$$\|D_{t,l}\nabla u_l(t)\|_N \lesssim l^{-N}\lambda_q^2\delta_q + L\, i_q^{-1/2-\delta} \lesssim l^{-N}\lambda_q^2\delta_q.$$

Then, the estimate (3.28) on $\|D_{t,l}\nabla v_l(t)\|_N$ follows from the identity $D_{t,l}\nabla v_l(t) = D_{t,l}\nabla u_l(t) - D_{t,l}\nabla z_l(t)$ and computations similar to those above. □

3.5.2. *Proof of Proposition 3.3: Estimates on the oscillatory and compressible Fourier coefficients.* Recall the definition of the parameters ϵ and μ, given respectively in (3.4) and (3.5).

Let us start with estimates aimed to control the Fourier coefficients appearing w and in the products $w_o \otimes w_o$ and $|w_o|^2 w_o$, cf. Equations (3.8), (3.9), (3.10), (3.14).

The key takeaway of the following lemma is that taking the advective derivative of the building blocks costs a factor ϵ^{-1}, whereas taking a space derivative costs a factor μ^{-1}.

Lemma 3.8. *For $s \in \{0,1,2\}$ and $N \le n_0 + 3$, it holds for all $t \le \mathfrak{t}$ that*

$$\epsilon^s \|D^s_{t,l} b_{m,k}(t)\|_N \lesssim \sup_{I\in\mathcal{I}} |\mathring{b}_{I,k}|\rho^{1/2}\mu^{-N},$$
$$\epsilon^s \|D^s_{t,l} c_{m,k}(t)\|_N \lesssim \sup_{I\in\mathcal{I}} |\mathring{c}_{I,k}|\rho\,\mu^{-N},$$
$$\epsilon^s \|D^s_{t,l} d_{m,k}(t)\|_N \lesssim \sup_{I\in\mathcal{I}} |\mathring{d}_{I,k}|\rho^{3/2}\mu^{-N},$$
$$\epsilon^s \|D^s_{t,l} e_{m,k}(t)\|_N \lesssim \sup_{I\in\mathcal{I}} |\mathring{b}_{I,k}|\rho^{1/2}\mu^{-N-1}.$$

Proof. Let us show the claim for $b_{m,k}$. The estimates on the other coefficients follow from similar inequalities. Due to the expression (3.11) and the spatial partition of unity, we can focus on a single addend corresponding to $I \in \mathcal{I}_m$ and $k \in \mathbb{Z}^3 \setminus \{0\}$:

$$X_I(t,x) := \theta_m(t)\chi_I(\xi_I(t,x))a_I(x,t)\overline{f}_I(x,t)\mathring{b}_{I,k}. \tag{3.29}$$

First, we have that

$$|D^s_{t,l}\theta_m(t)| \lesssim \epsilon^{-s}. \tag{3.30}$$

Second, by Proposition A.1 and (3.28), together with the assumed bound $\epsilon\|u_l(t)\|_1 < 1$ and the algebraic identities

$$D_{t,l}(\nabla\xi_I)^{-1} = \nabla u_l(\nabla\xi_I)^{-1}, \quad D^2_{t,l}(\nabla\xi_I)^{-1} = D_{t,l}(\nabla u_l)(\nabla\xi_I)^{-1} + (\nabla u_l)^2(\nabla\xi_I)^{-1}, \tag{3.31}$$

imply that for every $K \in \{0,1\ldots,N\}$ we have

$$\|D^s_{t,l}\overline{f_I}(t)\|_K \lesssim l^{-K}(\lambda_q\delta_q^{1/2})^s, \quad \text{for } s \in \{0,1,2\}. \tag{3.32}$$

Third, by Proposition A.1, (3.7) and [BDLIS15, Proposition C.1], we have

$$\|D^s_{t,l}\chi_I(\xi_I)\|_K \lesssim \mu^{-K}\mathbf{1}_{\{s=0\}}. \tag{3.33}$$

Now, we need to estimate $\|D^s_{t,l}a_I\|_K$ for $K \in \{0,1\ldots,N\}$, either for $I \in \mathcal{I}_\phi$ or $I \in \mathcal{I}_R$. Before treating each case separately, let us recall some useful inequalities valid for general C^K maps $g : \mathbb{T}^3 \to \mathbb{R}^k$ and $\Lambda : \mathbb{R}^k \to \mathbb{R}^n$, for general k,n. First, it is easy to check that the advective derivative of $\Lambda(g) : \mathbb{T}^3 \to \mathbb{R}^n$ is given by $D_{t,l}\Lambda(g) = (\nabla\Lambda)(g)\cdot D_{t,l}g$. In particular, by Faà di Bruno's formula we obtain for every $K \in \{0,1\ldots,N\}$:

$$\|D_{t,l}\Lambda(g)\|_K \lesssim \sum_{K_1+K_2\le K} \|D_{t,l}g\|_{K_1}\|(\nabla\Lambda)(g)\|_{K_2}; \tag{3.34}$$

$$\|D^2_{t,l}\Lambda(g)\|_K \lesssim \sum_{K_1+K_2\le K} \|D^2_{t,l}g\|_{K_1}\|(\nabla\Lambda)(g)\|_{K_2} + \|D_{t,l}g\otimes D_{t,l}g\|_{K_1}\|(\nabla^2\Lambda)(g)\|_{K_2}. \tag{3.35}$$

First case: $I \in \mathcal{I}_\phi$. Let $\rho_1 := \lambda_q^{-\gamma}\delta_{q+1}^{3/2}$. Recalling the definition of ρ given in (3.17), one immediately has $\rho_1 \lesssim \rho^{3/2}$ up to multiplicative constants independent of ϵ, μ. For $s = 0$, [BDLIS15, Proposition C.1] implies that

$$\|a_I\|_K \lesssim l^{-K}\rho_1^{1/3} \lesssim l^{-K}\rho^{1/2}. \tag{3.36}$$

For $s = 1$ instead, we have by (3.31), Proposition A.1 and Eq. (3.15):

$$\|D_{t,l}(\nabla\xi_I\phi_l)\rho_1^{-1/3}\|_K \lesssim l_{\text{temp}}^{-1}\, l^{-K}. \tag{3.37}$$

On the other hand, [BDLIS15, Proposition C.1] implies that for $k \in \{1, 2\}$

$$\left\|(\nabla^k\Lambda_{f_I}^{1/3})\left((\nabla\xi_I\phi_l)\rho_1^{-1/3}\right)\right\|_K \lesssim l^{-K}, \tag{3.38}$$

where the implicit constant depends on $\|\nabla^k\Lambda_{f_I}^{1/3}\|_K$. Therefore, (3.34), (3.37), (3.38) imply that

$$\begin{aligned}\|D_{t,l}a_I\|_K &\lesssim \sum_{K_1+K_2\le K}\rho_1^{1/3}\|D_{t,l}(\nabla\xi_I\phi_l)\rho_1^{-1/3}\|_{K_1}\left\|(\nabla\Lambda_{f_I}^{1/3})\left((\nabla\xi_I\phi_l)\rho_1^{-1/3}\right)\right\|_{K_2}\\ &\lesssim \rho_1^{1/3}l_{\text{temp}}^{-1}l^{-K} \lesssim \rho^{1/2}l_{\text{temp}}^{-1}l^{-K}.\end{aligned} \tag{3.39}$$

Analogously for $s = 2$ we have by (3.31) and (3.15)

$$\|D_{t,l}^2(\nabla\xi_I\phi_j)\rho_1^{-1/3}\|_K \lesssim l_{\text{temp}}^{-2}\, l^{-K}. \tag{3.40}$$

Then, (3.35), (3.40) and (3.38) imply that

$$\|D_{t,l}^2 a_I\|_N \lesssim \rho_1^{1/3}l_{\text{temp}}^{-2}\, l^{-N} \lesssim \rho^{1/2}l_{\text{temp}}^{-2}\, l^{-N}. \tag{3.41}$$

To summarize, recalling the decomposition (3.29), we can combining all the estimates given in (3.30), (3.33), (3.32), (3.36), (3.39), and (3.41) above and finally obtain

$$\|D_{t,l}^s X_I\|_N \lesssim \epsilon^{-s}\sup_I|\mathring{b}_{I,k}|\rho^{1/2}\mu^{-N}. \tag{3.42}$$

Second case: $I \in \mathcal{I}_R$. The desired estimates follow from the same set of inequalities used in the case $I \in \mathcal{I}_\phi$. The main difference, due to (3.16), is that we now need to estimate the quantity

$$\left\|D_{t,l}^s\left((\nabla\xi_I\nabla\xi_I^T - \mathrm{Id}) - \frac{\nabla\xi_I(R_l - M_{m,n})\nabla\xi_I^T}{\rho}\right)\right\|_N,$$

where $M_{m,n}$ is given by Eq. (3.18). Then, (3.31), (3.15), (3.42) imply that

$$\|D_{t,l}^s\left((\nabla\xi_I\nabla\xi_I^T - \mathrm{Id}) - \frac{\nabla\xi_I(R_l - M_{m,n})\nabla\xi_I^T}{\rho}\right)\|_N \lesssim \epsilon^{-s}\mu^{-N}.$$

Let us stress that we have exploited the bound (3.42) since $M_{m,n}(t,x)$ is given by a finite summation of X_J for $J \in \mathcal{I}_\phi$. Therefore,

$$\|D_{t,l}^s X_I\|_N \lesssim \epsilon^{-s}\sup_I|\mathring{b}_{I,k}|\rho^{1/2}\mu^{-N}.$$

□

Remark 3.9. In particular, the choice of C_ϵ implies that there exists C_1 independent on C_v, $\overline{M}$ and L such that

$$\|b_{m,k}(t)\|_N + \mu^{-1}\|e_{m,k}(t)\|_N \le C_1\rho^{1/2}\mu^{-N}.$$

We point out that the estimate above depends on ϵ because of the construction: whenever one has to bound ξ_m, terms of the form $\epsilon\|u_q\|_1 l^{-j}$ appear; therefore the constant C_ϵ permits to compensate for the dependence of these terms on the parameters C_v and $\overline{M}$ implicit in the quantity $\|v_q\|_1 \le C_v\overline{M}\lambda_q\delta_q^{1/2}$.

3.5.3. *Proof of Proposition 3.3: Conclusion.* Let us start from the estimates on w. It is enough to show the result for w_o, as the claim for w_c follows by similar arguments. Notice that

$$D^s_{t,l} w_o = \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} D^s_{t,l} b_{m,k} e^{i\lambda_{q+1}\xi_m(x,t)\cdot k},$$

where the sum over $m \in \mathbb{Z}$ is effectively a finite sum because the coefficients $b_{m,k}$ have disjoint supports for $|m-m'|>1$. Furthermore, for every fixed $m\in\mathbb{Z}$ we have

$$\|e^{i\lambda_{q+1}\xi_m(t)\cdot k}\|_N \lesssim |k|^N \lambda_{q+1}^N, \tag{3.43}$$

and thus the desired inequality follows from Lemma 3.8 and the analog of [DLK22, Equation (3.24)].

Moreover, the estimates (3.20), (3.21), (3.24), (2.4), $i_q^{1/2-\delta} \le \rho^{1/2}$ and $l\lambda_q\delta_q^{1/2} \le \rho^{1/2}$ imply the bounds (3.22) and (3.23) on u.

4. Construction of the new Reynolds stress and pressure

Based on the previous section, we have defined $v_{q+1} = v_q + w_o + w_c$. Now, we focus on identifying the new Reynolds stress R_{q+1} and pressure p_{q+1}. Using $v_{q+1} = v_q + w_o + w_c$ and the equation $\partial_t v_q + \operatorname{div}\big(u_q \otimes u_q\big) + \nabla p_q = \operatorname{div}\big(R_q\big)$, we obtain that any putative R_{q+1} and p_{q+1} satisfying the momentum equation in (2.3) must necessarily satisfy also

$$\begin{aligned} -\nabla p_{q+1} + \operatorname{div}(R_{q+1}) &= \partial_t v_{q+1} + \operatorname{div}\big(u_{q+1}\otimes u_{q+1}\big) \\ &= \partial_t w + \operatorname{div}(w_o\otimes w_o + R_l) - \nabla p_q + \operatorname{div}(w_c \otimes w + w_o\otimes w_c) \\ &\quad + u_l\cdot\nabla w + w\cdot\nabla u_l + \operatorname{div}\big((v_q - v_l)\odot w + R_q - R_l\big) \\ &\quad + \operatorname{div}\big(w\odot(z_{q+1}-z_l) + u_q\odot\overline{z} + \overline{z}\otimes\overline{z}\big), \end{aligned}$$

where $\odot$ is the symmetric tensor product, $\overline{z} := z_{q+1} - z_q$ and z_l is given by (3.6). Now, we can exploit (3.9) and (3.19) to get (hereafter the symbol $\mathring{\odot}$ stands for the traceless symmetric product)

$$\begin{aligned} -\nabla p_{q+1} + \operatorname{div}(R_{q+1}) &= \partial_t w + u_l\cdot\nabla w + \operatorname{div}\Big(w_o\otimes w_o - c_0\Big) + w\cdot\nabla u_l \\ &\quad + \operatorname{div}\Big(w_c\otimes w + w_o\otimes w_c + w\odot(v_q - v_l) + R_q - R_l\Big) \\ &\quad + \operatorname{div}\Big(w\,\mathring{\odot}\,(z_{q+1}-z_l) + u_q\,\mathring{\odot}\,\overline{z} + \overline{z}\mathring{\otimes}\overline{z}\Big) \\ &\quad + \nabla\Big(\frac{2}{3}w\cdot(z_{q+1}-z_l) + \frac{2}{3}u_q\cdot\overline{z} + \frac{|\overline{z}|^2}{3}\Big) - \nabla p_q. \end{aligned} \tag{4.1}$$

Notice that the last line in the expression above is a gradient; hence, we define:

$$\overline{p} := p_{q+1} - p_q := -\Big(\frac{2}{3}w\cdot(z_{q+1}-z_l) + \frac{2}{3}u_q\cdot\overline{z} + \frac{|\overline{z}|^2}{3}\Big). \tag{4.2}$$

Therefore, in order to enforce (4.1) we can define R_{q+1} as follows:

$$R_{q+1} := R_A + R_T + R_S + \frac{2}{3}f(t)\mathrm{Id},$$

where f is an arbitrary space-independent scalar function and

$$R_A := \mathcal{R}\Big[\partial_t w + u_l\cdot\nabla w + w\cdot\nabla u_l + \operatorname{div}\Big(w_o\otimes w_o - c_0\Big)\Big]; \tag{4.3}$$

$$R_T := w_c\otimes w + w_o\otimes w_c + w\odot(v_q - v_l) + R_q - R_l; \tag{4.4}$$

$$R_S := w\,\mathring{\odot}\,(z_{q+1}-z_l) + u_q\,\mathring{\odot}\,\overline{z} + \overline{z}\mathring{\otimes}\overline{z}. \tag{4.5}$$

In (4.3) above, the operator $\mathcal{R}$ denotes the anti-divergence operator (cf. for instance [DLS13, Definition 4.2]), which to a given smooth periodic vector field v associates a field $\mathcal{R}v$ taking values in the space of symmetric and traceless matrices and satisfying $\operatorname{div}\mathcal{R}v = v - \fint_{\mathbb{T}^3} v$. In fact, the *antidivergence stress* R_A denotes the part of the Reynolds stress R_{q+1} containing all the terms that we will estimate through properties of the the antidivergence operator $\mathcal{R}$. Similarly, the *trace stress* R_T contains all the contributions of terms having trace (except for the term $\frac{2}{3}f(t)\mathrm{Id}$), and the *stochastic stress* R_S contains all the terms with an explicit dependence of the stochastic process z. Our particular f entering in the convex integration scheme will be defined by (5.3) below. It will depend on the flux current ϕ and it will serve the purpose of canceling out the space average of several terms appearing in the equation (5.2) for the new current ϕ_{q+1}. In passing, we record the following expression for the trace of R_{q+1}, that we shall use later:

$$\operatorname{Tr}\left(R_{q+1}\right) = \operatorname{Tr}\left(R_q - R_l\right) + |w_c|^2 + 2w_c \cdot w_o + 2(v_q - v_l)\cdot w + 2f(t). \tag{4.6}$$

4.1. **Estimates on the Reynolds stress.** We denote $D_{t,q+1} := \partial_t + u_{q+1}\cdot\nabla$. Our main proposition on the Reynolds stress is the following: morally speaking, taking either a space derivative or and advective derivative of the new Reynolds stress R_{q+1} "costs" a factor λ_{q+1}.

Proposition 4.1. *There exists $b_0 := b_0(\alpha) > 1$ such that, for any $b \in (1, b_0)$, there are parameters $\gamma := (b-1)^2$, $a_0 > 1$ and $n_0, h_0 \in \mathbb{N}$ sufficiently large with the following property. For every $a > a_0$, $t \le \mathfrak{t}$, and $N \le n_0 + 3$ the following estimates hold:*

$$\left\|\left(R_{q+1}(t) - \frac{2}{3}f(t)\mathrm{Id}\right)w(t)\right\|_N \le \lambda_{q+1}^{N-2\gamma}\delta_{q+2}^{3/2}, \tag{4.7}$$

$$\left\|D_{t,q+1}\left(R_{q+1}(t) - \frac{2}{3}f(t)\mathrm{Id}\right)w(t)\right\|_{N-1} \le \lambda_{q+1}^{N-2\gamma}\delta_{q+2}^{3/2}\delta_{q+1}^{1/2}, \tag{4.8}$$

$$\left\|R_{q+1}(t) - \frac{2}{3}f(t)\mathrm{Id}\right\|_N \le \lambda_{q+1}^{N-2\gamma}\delta_{q+2}, \tag{4.9}$$

$$\left\|D_{t,q+1}\left(R_{q+1}(t) - \frac{2}{3}f(t)\mathrm{Id}\right)\right\|_{N-1} \le \lambda_{q+1}^{N-2\gamma}\delta_{q+2}\delta_{q+1}^{1/2}. \tag{4.10}$$

Before proving the previous proposition, let us introduce the following parameters:

$$n_0 := \left\lceil \frac{24b}{b-1}\right\rceil - 1, \quad h_0 := \left\lceil \frac{8b}{b-1}\right\rceil.$$

They will play a crucial role when perfoming estimates involving quantities in which we need to apply the antidivergence operator $\mathcal{R}$, more specifically in the bounds (A.7), (A.18), (A.24) from the Appendix. The values of n_0, h_0 ensure that

$$\begin{aligned}\frac{\rho}{\mu}\lambda_q\delta_q^{1/2}(\mu\lambda_{q+1})^{-n_0-1} &\lesssim \lambda_{q+1}^{-2\gamma}\delta_{q+2}^{3/2},\\ \frac{\rho}{\epsilon}\lambda_q\delta_q^{1/2}(l\lambda_{q+1})^{-h_0} &\lesssim \lambda_{q+1}^{-1-2\gamma}\delta_{q+2}^{3/2}\delta_{q+1}^{1/2}.\end{aligned}$$

4.1.1. *Proof of Proposition 4.1: bounds on R_T, R_A, R_S.* As a first step towards proving Proposition 4.1, we separately give bounds on the stress tensors R_T, R_A, R_S appearing in the decomposition of the new Reynolds stress R_{q+1}.

Lemma 4.2. *Let R_T be defined by (4.4). For $N \le n_0 + 3$, it holds for any $t \le \mathfrak{t}$ that*

$$\begin{aligned}\|R_T(t)\|_N &\lesssim l^{1-N}\lambda_q^{1-\gamma}\delta_{q+1}(l^{-1}\delta_q^{1/2}l_{\mathrm{temp}} + 1)\\ &\quad + \lambda_{q+1}^N\rho^{1/2}\left(\frac{\rho^{1/2}\mu^{-1}}{\lambda_{q+1}} + l\lambda_q\delta_q^{1/2}\right);\\ \|D_{t,q+1}R_T(t)\|_{N-2} &\lesssim \lambda_q^{N-1-\gamma}\delta_{q+1}\delta_q^{1/2} + \lambda_{q+1}^{N-2}\lambda_q^{1-\gamma}\delta_{q+1}\rho^{1/2} + l^{1-N}\lambda_q^{-\gamma}\delta_{q+1}l_{\mathrm{temp}}^{-1}\\ &\quad + \lambda_{q+1}^{N-2}\rho^{1/2}\left[\lambda_{q+1}\rho^{1/2}\left(\frac{\rho^{1/2}\mu^{-1}}{\lambda_{q+1}} + l\lambda_q\delta_q^{1/2}\right)\right].\end{aligned}$$

Proof. We start by estimating $R_q - R_l = (R_q - \mathbf{P}_{\le l^{-1}}R_q) + (\mathbf{P}_{\le l^{-1}}R_q - R_l)$. Notice that (3.15) implies that

$$\mathbf{P}_{\le l^{-1}}R_q - R_l(t,x) = \int_{-l_{\mathrm{temp}}}^0 ds\, \eta_{l_{\mathrm{temp}}}(s)\int_s^0 (D_{t,l}\mathbf{P}_{\le l^{-1}}R_q)(t+r, \mu(t; t+r, x))dr. \tag{4.11}$$

Moreover, due to (2.4), (2.6), and [DLK22, Lemma A.4], we have the following inequality:

$$\begin{aligned}\|D_{t,l}\mathbf{P}_{\le l^{-1}}R_q\|_N &\le \|\mathbf{P}_{\le l^{-1}}(D_{t,q}R_q)\|_N + \|[u_l\cdot\nabla, \mathbf{P}_{\le l^{-1}}]R_q\|_N + \|\mathbf{P}_{\le l^{-1}}((u_q - u_l)\cdot\nabla R_q)\|_N\\ &\lesssim l^{-N}\lambda_q^{1-\gamma}\delta_{q+1}\delta_q^{1/2} + l^{1-N}\|u_q\|_1\lambda_q^{1-\gamma}\delta_{q+1} \lesssim l^{-N}\lambda_q^{1-\gamma}\delta_{q+1}\delta_q^{1/2}.\end{aligned} \tag{4.12}$$

Therefore, by (4.11), (4.12), [BDLIS15, Proposition C.1], and (A.1) we obtain the bound

$$\|\mathbf{P}_{\le l^{-1}}R_q(t) - R_l(t)\|_N \lesssim l^{-N}\lambda_q^{1-\gamma}\delta_{q+1}\delta_q^{1/2}l_{\mathrm{temp}}.$$

On the other hand, Bernstein inequality and $\lambda_q < l^{-1}$ imply that

$$\|R_q(t) - \mathbf{P}_{\leq l^{-1}} R_q(t)\|_N \lesssim \begin{cases} l\|R_q(t)\|_1, & \text{if } N = 0 \\ \lambda_q^{N-\gamma}\delta_{q+1}, & \text{if } N \geq 1 \end{cases}$$
$$\lesssim \begin{cases} l\lambda_q^{1-\gamma}\delta_{q+1}, & \text{if } N = 0 \\ l^{1-N}\lambda_q^{1-\gamma}\delta_{q+1}, & \text{if } N \geq 1 \end{cases}$$
$$\lesssim l^{1-N}\lambda_q^{1-\gamma}\delta_{q+1}.$$

As for the material derivative

$$\begin{aligned}\|D_{t,q+1}(R_q - R_l)(t)\|_{N-2} &\leq \|D_{t,q+1}R_q(t)\|_{N-2} + \|D_{t,q+1}R_l(t)\|_{N-2} \\ &\leq \|D_{t,q}R_q(t)\|_{N-2} + \|(u_{q+1} - u_q)\cdot\nabla R_q(t)\|_{N-2} \\ &+ \|D_{t,l}R_l(t)\|_{N-2} + \|(u_{q+1} - u_l)\cdot\nabla R_l(t)\|_{N-2} \\ &\lesssim \lambda_q^{N-1-\gamma}\delta_{q+1}\delta_q^{1/2} + \lambda_{q+1}^{N-2}\lambda_q^{1-\gamma}\delta_{q+1}\rho^{1/2} + l^{2-N}\lambda_q^{-\gamma}\delta_{q+1}l_{\text{temp}}^{-1},\end{aligned}$$

where the last inequality is due to (2.6), (3.22), (3.23), the bounds $l < l_{\text{temp}}$ and $l_{\text{temp}}\|u_q\|_1 \leq 1$, as well as the inequality $\|R_l(t)\|_N \lesssim l^{-N}\lambda_q^{-\gamma}\delta_{q+1}$ valid for $N \geq 0$.

Now we estimate $R_T' := R_T - (R_q - R_l) = w_c \otimes w + w_o \otimes w_c + w \odot (v_q - v_l)$. The bounds (3.20), (3.21), and (2.4) imply

$$\|R_T'(t)\|_N \lesssim \lambda_{q+1}^N \rho^{1/2}\left(\frac{\rho^{1/2}\mu^{-1}}{\lambda_{q+1}} + l\lambda_q\delta_q^{1/2}\right).$$

Instead, the material derivative is controlled thanks to bounds (3.20), (3.21), (3.27), (3.23), as well as $l\lambda_q\delta_q^{1/2} \leq \rho^{1/2}$ and $\epsilon^{-1} \lesssim \lambda_{q+1}\rho^{1/2}$, giving

$$\begin{aligned}\|D_{t,q+1}R_T'\|_{N-2} &\leq \|D_{t,l}R_T'\|_{N-1} + \|(u_{q+1} - u_l)\cdot\nabla R_T'\|_{N-2} \\ &\lesssim \lambda_{q+1}^{N-2}\rho^{1/2}\left[\lambda_{q+1}\rho^{1/2}\left(\frac{\rho^{1/2}\mu^{-1}}{\lambda_{q+1}} + l\lambda_q\delta_q^{1/2}\right)\right].\end{aligned}$$

□

Lemma 4.3. *Let R_A be defined by (4.3). For $N \leq n_0 + 3$ and $t \leq \mathfrak{t}$ we have*

$$\|R_A(t)\|_N \lesssim \lambda_{q+1}^N\left[\lambda_{q+1}^{-1} + (\lambda_{q+1}\mu)^{-n_0-1}\right]\frac{\rho}{\mu},$$
$$\|D_{t,q+1}R_A(t)\|_{N-2} \lesssim \lambda_{q+1}^{N-2}\left[(\epsilon\lambda_{q+1})^{-1} + \lambda_{q+1}^{-n_0}\mu^{-n_0-1} + (l\lambda_{q+1})^{-h_0}\lambda_{q+1}^2\lambda_q\delta_q^{1/2}\right]\frac{\rho}{\mu}$$

Proof. Let us preliminarily observe that

$$R_A = \mathcal{R}\left(\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\sum_{m\in\mathbb{Z}} a_{m,k}e^{i\lambda_{q+1}\xi_m\cdot k}\right),$$
$$a_{m,k} := \operatorname{div}(c_{m,k}) + D_{t,l}(b_{m,k} + \lambda_{q+1}^{-1}e_{m,k}) + (b_{m,k} + \lambda_{q+1}^{-1}e_{m,k})\cdot\nabla u_l.$$

For the estimate on $\|R_A(t)\|_N$ we intend to apply Proposition A.3 with $F = R_A$. Indeed, thanks to Lemma 3.8 and the bound $\epsilon\|u_q\| \leq 1$ one can check that the assumptions of the proposition are satisfied with $\lambda = \lambda_{q+1}$, $s_j = \frac{\rho}{\mu}\mu^{-j}$, and with numbers $\mathring{a}_k$ obtained from the constants in the inequalities stated in Lemma 3.8 and a_F depending only on finitely many derivatives of the Mikado flows ψ_I, in particular independent of λ_{q+1}, μ, ρ et cetera. Similarly, the estimate on $\|D_{t,q+1}R_A(t)\|_{N-2}$ follows from Proposition A.4, where the assumptions therein hold true with $c = \epsilon$ because of Lemma 3.8 and

$$\|D_{t,l}\operatorname{div}(c_{m,k})\|_j \leq \|D_{t,l}c_{m,k}\|_{j+1} + \|\nabla u_l : \nabla c_{m,k}^T\|_j \lesssim \mu^{-j}\frac{\rho}{\mu\,\epsilon}, \tag{4.13}$$
$$\left\|D_{t,l}\left[(b_{m,k} + \lambda_{q+1}^{-1}e_{m,k})\cdot\nabla u_l\right]\right\|_j \lesssim \mu^{-j}\rho^{1/2}(\epsilon^{-1}\lambda_q\delta_q^{1/2} + \lambda_q^2\delta_q).$$

□

Lemma 4.4. *Let R_S be defined by (4.5). For $N \leq n_0+3$ and $t \leq \mathfrak{t}$ we have*

$$\|R_S(t)\|_N \lesssim i_q^{1/2-\delta} \begin{cases} 1, & N=0 \\ \lambda_{q+1}^N \delta_{q+1}^{1/2}, & N>0 \end{cases},$$

$$\|D_{t,q+1} R_S(t)\|_{N-2} \lesssim i_{q+1}^{-1/2-\delta} \begin{cases} 1 \vee \delta_{q+1}^{1/2}, & N=2 \\ \lambda_{q+1}^{N-2} \delta_{q+1}^{1/2}, & N>2 \end{cases}$$
$$+ i_q^{1/2-\delta} \left[\lambda_{q+1} \rho (\delta_{q+1} \vee \delta_{q+1}^{1/2})\right] \lambda_{q+1}^{N-2}.$$

Proof. We recall that, by definition:

$$R_S = w \mathring{\odot} (z_{q+1} - z_l) + u_q \mathring{\odot} \overline{z} + \overline{z} \mathring{\otimes} \overline{z}, \quad \overline{z} := z_{q+1} - z_q.$$

Therefore, the desired bounds follow from Lemma 3.8, Proposition 3.3, and Corollary 3.5, as well as

$$\|D_{t,q+1} u_q\|_N \leq \|D_{t,q} u_q\|_N + \|(u_{q+1} - u_q) \cdot \nabla u_q\|_N \lesssim i_q^{-1/2-\delta} + \lambda_{q+1}^N \lambda_q \, \delta_q. \tag{4.14}$$

□

4.1.2. *Proof of Proposition 4.1: Conclusion.* We are ready to show Proposition 4.1. First, bounds (4.9) and (4.10) follow directly from previous Lemmas 4.2 to 4.4, after some some standard computations. Instead, the bounds (4.7) and (4.8) are due to Lemmas 4.2 to 4.4 and Proposition 3.3 combined with

$$\|Xw(t)\|_N \lesssim \sum_{j=0}^{N} \|X(t)\|_j \|w(t)\|_{N-j}$$

applied with either $X(t) := R_{q+1}(t) - \frac{2}{3} f(t) \mathrm{Id}$ or $X(t) := D_{t,q+1}\left(R_{q+1}(t) - \frac{2}{3} f(t) \mathrm{Id}\right)$.

4.2. **Estimates on the pressure.** Our next proposition concerns the new pressure p_{q+1}. We have the following:

Proposition 4.5. *There exists $b_0 := b_0(\alpha) > 1$ such that, for any $b \in (1, b_0)$, there is $a_0 > 1$ such that for every $a > a_0$, $t \leq \mathfrak{t}$ the following estimates hold:*

$$\|p_{q+1}(t)\|_N \leq \overline{M} \lambda_{q+1}^N \delta_{q+1}, \quad \|D_{t,q+1} p_{q+1}(t)\|_{N-2} \leq \overline{M} \lambda_{q+1}^{N-1} \delta_{q+1}^{3/2}, \quad \textit{for } N \in \{1,2,\ldots,n_0+3\}, \tag{4.15}$$
$$\|p_{q+1}(t) - p_q(t)\|_N \leq \overline{M} \lambda_{q+1}^N \delta_{q+1} \qquad \textit{for } N \in \{0,1\}.$$

Proof. Since $p_{q+1} - p_q = -\left(\frac{2}{3} w \cdot (z_{q+1} - z_l) + \frac{2}{3} u_q \cdot \overline{z} + \frac{|\overline{z}|^2}{3}\right)$ by construction, the estimates on $\|p_{q+1}\|_N$ and $\|p_{q+1} - p_q\|_N$ can be easily obtained by mimicking those on R_S. Analogously,

$$\begin{aligned} \|D_{t,q+1} p_{q+1}\|_{N-2} &\lesssim \lambda_{q+1}^{N-1} \delta_{q+2} \delta_{q+1}^{1/2} + \|D_{t,q} p_q\|_{N-2} + \|(u_{q+1} - u_q) \cdot \nabla p_q\|_{N-2} \\ &\lesssim \lambda_{q+1}^{N-1} \delta_{q+2} \delta_{q+1}^{1/2} + \lambda_q^{N-1} \delta_q^{3/2} + \lambda_{q+1}^{N-2} \lambda_q \delta_q \delta_{q+1}^{1/2} \leq \overline{M} \lambda_{q+1}^{N-1} \delta_{q+1}^{3/2}, \end{aligned}$$

where the last inequality is due to picking b_0 and a_0 properly. □

5. Construction of the new current

In the previous sections we have constructed the new velocity v_{q+1}, pressure p_{q+1}, and Reynolds stress R_{q+1}. Moreover, the noise z_{q+1} is given by (2.2). The new current ϕ_{q+1} is defined by imposing that $(v_{q+1}, p_{q+1}, R_{q+1}, \phi_{q+1}, z_{q+1})$ satisfies the relaxed local energy equality in (2.3), namely

$$\begin{aligned} \partial_t \frac{|v_{q+1}|^2}{2} + v_{q+1} \cdot \nabla p_{q+1} + v_{q+1} \cdot \operatorname{div}\big((v_{q+1} + z_{q+1}) \otimes (v_{q+1} + z_{q+1})\big) = -E' \\ + \frac{1}{2} D_{t,q+1}\big(\operatorname{Tr}(R_{q+1})\big) + \operatorname{div}\big(R_{q+1} v_{q+1}\big) + R_{q+1} : \nabla z_{q+1}^T + \operatorname{div}(\phi_{q+1}). \end{aligned} \tag{5.1}$$

Since $(v_q, p_q, R_q, \phi_q, z_q)$ solves itself the relaxed local energy inequality, we have

$$
\begin{aligned}
(5.2)\quad \partial_t \frac{|v_{q+1}|^2}{2} &+ v_{q+1}\cdot \nabla p_{q+1} + v_{q+1}\cdot \operatorname{div}\big(u_{q+1}\otimes u_{q+1}\big) = -E' \\
&+ \operatorname{div}(R_{q+1}v_{q+1}) + R_{q+1} : \nabla z_{q+1}^T \\
&+ D_{t,q}\left(\frac{\operatorname{Tr}(R_q)+|w|^2+2(v_q-v_l)\cdot w}{2}\right) + \operatorname{div}\left(\phi_q + \frac{|w|^2}{2}w - R_{q+1}w + (\overline{z}\otimes v_q)w\right) \\
&+ \operatorname{div}\left((R_q + w\otimes w - R_{q+1})(v_q - v_l) + [\overline{z}\otimes\overline{z} + u_q\odot\overline{z}]\, v_l\right) \\
&+ \operatorname{div}\left(\left(\frac{|v_q|^2+|w|^2}{2}\right)\overline{z} + \frac{|v_q-v_l|^2}{2}w\right) \\
&+ [(w_o\otimes w_o - c_0) - \mathcal{R}(\operatorname{div}(w_o\otimes w_o - c_0) + D_{t,l}w + w\cdot\nabla u_l)] : \nabla(v_l+z_q)^T - R_{q+1} : \nabla\overline{z}^T \\
&- \big(w\,\mathring{\odot}\,(z_{q+1}-z_l) + u_q\,\mathring{\odot}\,\overline{z} + \overline{z}\otimes\overline{z}\big) : \nabla z_q^T + v_q\cdot\operatorname{div}(u_q\otimes\overline{z} + \overline{z}\otimes z_{q+1}) \\
&- (z_{q+1}-z_l)\cdot\operatorname{div}(w\otimes v_l) + (v_q - v_l)\cdot\operatorname{div}(w\otimes\overline{z}) + w\cdot\operatorname{div}\Big\{(p_{q+1}-p_l)\mathrm{Id} + R_l \\
&+ R_{comm} - z\otimes v_l + (z_{q+1}-z_l)\otimes z_q + z_l\otimes(z_q - z_l) - v_l\otimes(z_l - z_q) + u_{q+1}\otimes\overline{z}\Big\},
\end{aligned}
$$

where $R_{comm} := u_l\otimes u_l - \mathbf{P}_{\le l^{-1}}(u_q\otimes u_q)$, and we have used multiple times the divergence-free property of all the vector fields at stake, as well as the following standard algebraic identities valid for generic matrices M and vector fields v, u:

$$
\begin{aligned}
\operatorname{div}(Mv) &= \operatorname{div}(M)\cdot v + M : \nabla v^T, \\
u\otimes v : \nabla w^T &= v\cdot\operatorname{div}(u\otimes w).
\end{aligned}
$$

Equation (4.6) implies that

$$
\begin{aligned}
D_{t,q}\Big(\frac{\operatorname{Tr}(R_q)+|w|^2+2(v_q-v_l)\cdot w}{2}\Big) &= D_{t,q}\left(\frac{\operatorname{Tr}(R_{q+1}) + \operatorname{Tr}(w_o\otimes w_o - c_0)}{2}\right) - \partial_t f \\
&= D_{t,q+1}\left(\frac{\operatorname{Tr}(R_{q+1})}{2}\right) - \operatorname{div}\left((w+\overline{z})\frac{\operatorname{Tr}(R_{q+1})}{2}\right) \\
&\quad + \operatorname{div}\left(\mathcal{R}\left(D_{t,l}\frac{\operatorname{Tr}(w_o\otimes w_o - c_0)}{2}\right)\right) \\
&\quad + \fint_{\mathbb{T}^3} D_{t,l}\frac{\operatorname{Tr}(w_o\otimes w_o - c_0)}{2}\,dx \\
&\quad + \operatorname{div}\left((v_q - v_l)\frac{\operatorname{Tr}(w_o\otimes w_o - c_0)}{2}\right) - \partial_t f.
\end{aligned}
$$

Therefore, (5.1) holds for ϕ_{q+1} and $f(t)$ given by:

$$
\begin{aligned}
&\phi_{q+1} := \phi_A + \phi_{D_1} + \phi_{D_2} + \phi_{S_1} + \phi_{S_2}, \\
(5.3)\qquad &f_{q+1}(t) := f_A(t) + f_{D_2}(t) + f_{S_2}(t) := \int_0^t \fint_{\mathbb{T}^3} (\varphi_A(x,s) + \varphi_{D_2}(x,s) + \varphi_{S_2}(x,s))\,dx\,ds,
\end{aligned}
$$

where the terms at the right-hand sides above are defined as follows: the *antidivergence current*

$$
\begin{aligned}
(5.4)\qquad \phi_A &:= \overline{\mathcal{R}}\varphi_A, \\
\varphi_A &:= D_{t,l}\frac{\operatorname{Tr}(w_o\otimes w_o - c_0)}{2} + \operatorname{div}\left(\phi_l + \frac{|w_o|^2}{2}w_o\right) + (w_o\otimes w_o - c_0) : \nabla(v_l + z_q)^T \\
&\quad + w\,\mathring{\odot}\,(z_{q+1}-z_l) : \nabla z_q^T - (z_{q+1}-z_l)\cdot\operatorname{div}(w\otimes v_l) + (v_q - v_l)\cdot\operatorname{div}(w\otimes\overline{z}) \\
&\quad + w\cdot\operatorname{div}\left(\mathbf{P}_{\le l^{-1}}R_q + R_{comm} - \overline{z}\otimes v_l + (z_{q+1}-z_l)\otimes z_q\right) \\
&\quad + w\cdot\operatorname{div}(u_l\otimes(z_q - z_l) + u_{q+1}\otimes\overline{z}) + w\cdot\nabla(p_q - p_l),
\end{aligned}
$$

contains all the terms that will be treated by means of the properties of the *antidivergence* operator $\overline{\mathcal{R}} := -\nabla(-\Delta)^{-1}$, that to smooth scalar fields g associates a vector field satisfying $\operatorname{div}\overline{\mathcal{R}}g = g - \fint_{\mathbb{T}^3} g$;

the *stochastic currents*

$$\phi_{S_1} := \left(\frac{\mathrm{Tr}(R_{q+1} - \frac{3}{2}f\,\mathrm{Id})}{2} + \frac{|v_q|^2 + |w|^2}{2}\right)\overline{z} + (\overline{z}\otimes v_q)w + (\overline{z}\otimes\overline{z} + u_q\odot\overline{z})v_l + \overline{p}\,w\,, \tag{5.5}$$

$$\begin{aligned} \phi_{S_2} &:= \overline{\mathcal{R}}\varphi_{S_2}, \\ \varphi_{S_2} &:= R_A : \nabla(z_q - z_l)^T + \left(u_q \mathring{\odot}\,\overline{z} + \overline{z}\otimes\overline{z}\right) : \nabla z_q^T \\ &\quad + v_q\cdot\mathrm{div}(u_q\otimes\overline{z} + \overline{z}\otimes z_{q+1}) - \left(R_{q+1} - \frac{3}{2}f\,\mathrm{Id}\right) : \nabla\overline{z}^T, \end{aligned} \tag{5.6}$$

containing all the terms with explicit dependence of the noise z; and the *deterministic currents*

$$\begin{aligned} \phi_{D_1} &:= w\frac{\mathrm{Tr}(R_{q+1} - \frac{3}{2}f\,\mathrm{Id})}{2} + (v_q - v_l)\frac{\mathrm{Tr}(w_o\otimes w_o - c_0)}{2} + \phi_q - \phi_l \\ &\quad + \frac{|w|^2}{2}w - \frac{|w_o|^2}{2}w_o - \left(R_{q+1} - \frac{3}{2}f\,\mathrm{Id}\right)w + \frac{|v_q - v_l|^2}{2}w \\ &\quad + \left(R_q - w\otimes w - R_{q+1} - \rho\mathrm{Id} + \frac{3}{2}f\,\mathrm{Id}\right)(v_q - v_l)\,, \end{aligned} \tag{5.7}$$

$$\phi_{D_2} := \overline{\mathcal{R}}\varphi_{D_2}, \quad \varphi_{D_2} := R_A : \nabla u_l^T, \tag{5.8}$$

containing all the remaining terms that would be there even without external random forcing. Obviously, if $z \neq 0$ then the deterministic currents ϕ_{D_1} and ϕ_{D_2} are random objects nonetheless, being so all the other objects involved in the construction. Furthermore, the terms ϕ_{D_1} and ϕ_{S_1} can be written in the form $\overline{\mathcal{R}}\varphi_{D_1}$ and $\overline{\mathcal{R}}\varphi_{S_1}$ respectively, where $\varphi_{D_1} = \mathrm{div}\phi_{D_1}$ and $\varphi_{S_1} = \mathrm{div}\phi_{S_2}$ have zero space average. On the other hand, the currents φ_A, φ_{S_2}, and φ_{D_2} have non-zero space average.

5.1. Estimates on the current.

Our main proposition on the current is the following Proposition 5.1. The same caveat as before Proposition 3.3 applies also here: the proof of this results could be skipped at a first reading. Roughly speaking, taking either a space derivative or an advective derivative of the new current ϕ_{q+1} "costs" a factor λ_{q+1}. On the other hand, the scalar function f and its time derivative go to zero as $\lambda_{q+1}^{-\gamma}\delta_{q+2}$ and $\lambda_{q+1}^{-\gamma}\delta_{q+2}\delta_{q+1}^{1/2}$, respectively.

Proposition 5.1. *There exists $b_0 := b_0(\alpha) > 1$ such that, for any $b \in (1, b_0)$, there are parameters $\gamma := (b-1)^2$, $a_0 > 1$ and $n_0, h_0 \in \mathbb{N}$ sufficiently large with the following property. For every $a > a_0$, $t \leq \mathfrak{t}$ and $N \leq 2$ it holds*

$$\|\phi_{q+1}(t)\|_N \leq \lambda_{q+1}^{N-3\gamma/2}\delta_{q+2}^{3/2}, \tag{5.9}$$

$$\|D_{t,q+1}\phi_{q+1}(t)\|_{N-1} \leq \lambda_{q+1}^{N-3\gamma/2}\delta_{q+2}^{3/2}\delta_{q+1}^{1/2}, \tag{5.10}$$

$$|f(t)| \leq \frac{1}{100}\lambda_{q+1}^{-\gamma}\delta_{q+2},$$

$$|\partial_t f(t)| \leq \frac{1}{100}\lambda_{q+1}^{-\gamma}\delta_{q+2}\delta_{q+1}^{1/2}.$$

5.1.1. *Proof of Proposition 5.1: bounds on ϕ_A.*

Lemma 5.2. *Let ϕ_A, f_A be defined by (5.4) and (5.3). For $N \leq 2$ and $t \leq \mathfrak{t}$ we have*

$$\|\phi_A(t)\|_N \lesssim \lambda_{q+1}^N\left[\lambda_{q+1}^{-1} + (\lambda_{q+1}\mu)^{-n_0-1}\right]\frac{\delta_{q+1}}{\epsilon},$$

$$\|D_{t,q+1}\phi_A(t)\|_{N-1} \lesssim \lambda_{q+1}^{N-1}\left[(\epsilon\lambda_{q+1})^{-1} + \lambda_{q+1}^{-n_0}\mu^{-n_0-1} + (l\lambda_{q+1})^{-h_0}\lambda_{q+1}^2\lambda_q\delta_q^{1/2}\right]\frac{\delta_{q+1}}{\epsilon},$$

$$|f_A(t)| + |f_A'(t)| \lesssim \frac{\delta_{q+1}}{\epsilon}(\lambda_{q+1}\mu)^{-n_0-1}.$$

Proof. The result is due to Propositions A.3 and A.4 applied to $F = \phi_A$. Indeed, recall the decomposition of ϕ_A given by (5.4). Following the notation of Proposition A.3, let us define $a_{m,k} := a_{m,k}^1 + a_{m,k}^2 + a_{m,k}^3$

where

$$\begin{aligned}
a^1_{m,k} &:= D_{t,l}\left(\frac{\mathrm{Tr}(c_{m,k})}{2}\right) + \mathrm{div}(d_{m,k}) + c_{m,k} : \nabla(v_l + z_q)^T + \left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right) \mathring{\odot} (z_{q+1} - z_l) : \nabla z_q^T \\
&\quad + \left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right) \cdot \mathrm{div}\left((z_{q+1} - z_l) \otimes z_q\right) - (z_{q+1} - z_l) \cdot \left[\left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right) \cdot \nabla v_l\right] \\
&\quad + \left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right) \cdot \left[-(\overline{z} \cdot \nabla) v_l + u_l \cdot \nabla(z_q - z_l) + u_{q+1} \cdot \nabla \overline{z}\right] \\
&\quad + (v_q - v_l) \cdot \left[\left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right) \cdot \nabla \overline{z}\right], \\
a^2_{m,k} &:= \left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right) \mathrm{div}(\mathbf{P}_{\leq l^{-1}} R_q + R_{comm}), \\
a^3_{m,k} &:= \left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right) \cdot \nabla(p_q - p_l).
\end{aligned}$$

We intend to apply Proposition A.3 with $\lambda = \lambda_{q+1}$, a sequence

$$s_j = \frac{\delta_{q+1}}{\epsilon} \mu^{-j},$$

and $\mathring{a}_k, a_F$ defined accordingly. As for the terms of the form $a^1_{m,k}$, the assumptions of the proposition are readily checked by Lemma 3.8, Lemma 3.4, and the inductive estimates on v_{q+1}. The assumptions of the proposition for terms of the form $a^2_{m,k}$ can be checked by Lemma 3.8, inductive estimate (2.6), and [DLK22, Lemma A.3]. Finally, for $a^3_{m,k}$ we invoke Lemma 3.8 and Proposition 4.5. Observe that the constant a_F from Proposition A.3 depends only on finitely many derivatives of the Mikado flows ψ_I, in particular we absorb it into the $\lesssim$ in the bound on $\|\phi_A(t)\|_N$.

Let us move to the estimate on $\|D_{t,q+1}\phi_A(t)\|_{N-1}$. We apply Proposition A.4 with λ, s_j as above and $c = \epsilon$. In this case, we check the assumptions on $D_{t,l} a^1_{m,k}$ by the inductive estimates, Lemma 3.8, the bound (3.28), Lemma 3.4, and Corollary 3.5. In a similar fashion, in order to control the assumptions on $D_{t,l} a^2_{m,k}$ we use Bernstein inequality, (3.27) and [DLK22, Lemma A.4] to control, on one hand:

$$\begin{aligned}
\|D_{t,l}(\mathrm{div} R_{comm})\|_N &\lesssim l^{-N} \|D_{t,l}(\mathrm{div} R_{comm})\|_0 \\
&\lesssim \|D_{t,l}(u_q - u_l)\|_0 \|u_q\|_1 + l\|u_q\|_1 \left[\|D_{t,l} u_l\|_1 + \|u_q\|_1^2\right] + l^2 \|u_q\|_1^2 \|u_q\|_2 \\
&\lesssim l^{1-N} \left[\lambda_q^3 \delta_q^{3/2} + \lambda_q \delta_q^{1/2} i_q^{-1/2-\delta}\right],
\end{aligned}$$

where the second inequality follows by the same computations shown in [DLK22, Lemma 5.2], and on the other

$$\begin{aligned}
\|D_{t,l}(\mathrm{div} \mathbf{P}_{\leq l^{-1}} R_q)\|_N &\leq \|\mathbf{P}_{\leq l^{-1}} D_{t,l} R_q\|_{N+1} + \|[u_l \cdot \nabla, \mathbf{P}_{\leq l^{-1}}] R_q\|_{N+1} + \|(\nabla u_l) \nabla (\mathbf{P}_{\leq l^{-1}} R_q)\|_N \\
&\lesssim l^{-N} \|D_{t,q} R_q\|_1 + \|(u_q - u_l) \cdot \nabla R_q\|_1 + l^{-N} \|u_q\|_1 \|R_q\|_1 \\
&\lesssim l^{-N} \lambda_q^2 \delta_{q+1} \delta_q^{1/2}.
\end{aligned}$$

Finally, as for the terms $D_{t,l} a^3_{m,k}$, we have by Lemma 3.8

$$\begin{aligned}
\|D_{t,l} a^3_{m,k}\|_j &\lesssim \sum_{h=0}^j \left\|D_{t,l}\left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right)\right\|_{j-h} \|p_q - p_l\|_{h+1} \\
&\quad + \sum_{h=0}^j \left\|\left(b_{m,k} + \frac{e_{m,k}}{\lambda_{q+1}}\right)\right\|_{j-h} \left(\|D_{t,l}(p_q - p_l)\|_{h+1} + \|(\nabla u_l) \nabla (p_q - p_l)\|_h\right) \\
&\lesssim \lambda_q \delta_q \rho^{1/2} \epsilon^{-1} \mu^{-j},
\end{aligned}$$

where in the last line we have used the identity

$$D_{t,l}(p_q - p_l) = D_{t,q} p_q - (D_{t,q} p_q)_l + (u_l - u_q) \cdot \nabla p_q + (u_q \cdot \nabla p_q)_l - u_l \cdot \nabla u_l,$$

and the following bound, obtained by [DLK22, Lemma A.3] and Bernstein inequality:

$$\|D_{t,l}(p_q - p_l)\|_M \lesssim l^{1-M} \lambda_q^2 \delta_q^{3/2}.$$

At last, let us consider $|f_A(t)|+|f_A'(t)|$. Due to the spatial and temporal partitions of unity, arguing as in Proposition 3.3 it is sufficient to estimate $f_{m,k}(t) := \int_0^t \int_{\mathbb{T}^3} a_{m,k}(x,s) e^{i\lambda_{q+1}\xi_m(x,s)\cdot k}\,dx\,ds$. Then, [DLK22, Lemma A.2] implies that

$$|f_A(t)|+|f_A'(t)| \lesssim \lambda_{q+1}^{-n_0+1}(s_{n_0+1}+s_0\, l^{-n_0-1}) \lesssim \frac{\delta_{q+1}}{\epsilon}(\lambda_{q+1}\mu)^{-n_0-1}.$$

□

5.1.2. *Proof of Proposition 5.1: bounds on ϕ_D.*

Lemma 5.3. *Let ϕ_{D_1} be defined by (5.7). For $N \le 2$ and $t \le \mathfrak{t}$ we have*

$$\begin{aligned}
\|\phi_{D_1}(t)\|_N &\lesssim \lambda_{q+1}^N \left[l\,\lambda_q \delta_q^{1/2}\rho + (\lambda_{q+1}\mu)^{-1}\rho^{3/2}\right] \\
&\quad + l^{1-N}\lambda_q^{1-\gamma}\delta_{q+1}^{3/2}(l^{-1}\delta_q^{1/2} l_{\text{temp}}+1) + \lambda_{q+1}^{N-2\gamma}\delta_{q+2}^{3/2},\\
\|D_{t,q+1}\phi_{D_1}(t)\|_{N-1} &\lesssim \lambda_{q+1}^{N-2\gamma}\delta_{q+2}^{3/2}\delta_{q+1}^{1/2} + l^{1-N}\lambda_q^{-\gamma}\delta_{q+1}^{3/2} l_{\text{temp}}^{-1}\\
&\quad + \lambda_{q+1}^N[l\delta_{q+1}^{3/2}\lambda_q\delta_q^{1/2} + \lambda_{q+1}^{-1}\lambda_q^{1-\gamma}\delta_{q+1}^{3/2}\rho^{1/2} + (\lambda_{q+1}\mu)^{-1}\rho^2]
\end{aligned}$$

Proof. We bound each term in the decomposition (5.7) separately, starting from

$$\bar{\phi}_{D_1} := w\frac{\operatorname{Tr}(R_{q+1}-\frac{3}{2}f\,\mathrm{Id})}{2} - \left(R_{q+1}-\frac{3}{2}f\,\mathrm{Id}\right)w.$$

In this case, the bounds (4.7) and (4.8) imply that

$$\|\bar{\phi}_{D_1}\|_N \lesssim \lambda_{q+1}^{N-3\gamma/2}\delta_{q+2}^{3/2}, \qquad \|D_{t,q+1}\bar{\phi}_{D_1}\|_{N-1} \lesssim \lambda_{q+1}^{N-3\gamma/2}\delta_{q+2}^{3/2}\delta_{q+1}^{1/2}.$$

Moving to the other terms, we have by (2.4), Lemma 3.8, (3.43) and the bound $\mu^{-1} < \lambda_{q+1}$ that

$$\left\|(v_q-v_l)\frac{\operatorname{Tr}(w_o\otimes w_o - c_0)}{2}\right\|_N \lesssim l\lambda_q\delta_q^{1/2}\rho\lambda_{q+1}^N, \tag{5.11}$$

while the advective derivative is controlled by

$$\begin{aligned}
\left\|D_{t,q+1}\left((v_q-v_l)\frac{\operatorname{Tr}(w_o\otimes w_o - c_0)}{2}\right)\right\|_{N-1} &\lesssim \|D_{t,l}(v_q-v_l)\operatorname{Tr}(w_o\otimes w_o-c_0)\|_{N-1}\\
&\quad + \|(v_q-v_l)D_{t,l}\operatorname{Tr}(w_o\otimes w_o-c_0)\|_{N-1}\\
&\quad + \|((u_{q+1}-u_l)\cdot\nabla)\left((v_q-v_l)\operatorname{Tr}(w_o\otimes w_o-c_0)\right)\|_{N-1}\\
&\lesssim \lambda_{q+1}^N l\,\rho^{3/2}\lambda_q\delta_q^{1/2},
\end{aligned}$$

where in the last line we have used (5.11) above, Lemma 3.8, the bounds (3.27), (3.23), and Proposition 3.3. By similar computations, we can control also

$$\begin{aligned}
\left\|\frac{|v_q-v_l|^2}{2}w\right\|_N &\lesssim \lambda_{q+1}^N l^2\lambda_q^2\delta_q\,\rho^{1/2},\\
\left\|D_{t,q+1}\left(\frac{|v_q-v_l|^2}{2}w\right)\right\|_{N-1} &\lesssim \lambda_{q+1}^N l^2\lambda_q^2\delta_q\,\rho.
\end{aligned}$$

The next term in the decomposition (5.7) we deal with is $\phi_q-\phi_l$. In this case, we can repeat the computations reported in Lemma 4.2, obtaining

$$\begin{aligned}
\|\phi_q-\phi_l\|_N &\lesssim l^{1-N}\lambda_q^{1-\gamma}\delta_{q+1}^{3/2}(l^{-1}\delta_q^{1/2}l_{\text{temp}}+1),\\
\|D_{t,q+1}(\phi_q-\phi_l)\|_{N-1} &\lesssim \lambda_q^{N-\gamma}\delta_{q+1}^{3/2}\delta_q^{1/2} + \lambda_{q+1}^{N-1}\lambda_q^{1-\gamma}\delta_{q+1}^{3/2}\rho^{1/2} + l^{1-N}\lambda_q^{-\gamma}\delta_{q+1}^{3/2}l_{\text{temp}}^{-1}.
\end{aligned}$$

The cubic term in the velocity perturbation is controlled by means of Proposition 3.3 as follows:

$$\begin{aligned}
\left\|\frac{|w|^2w-|w_o|^2w_o}{2}\right\|_N &\lesssim \lambda_{q+1}^N(\lambda_{q+1}\mu)^{-1}\rho^{3/2},\\
\left\|D_{t,q+1}\left(\frac{|w|^2w-|w_o|^2w_o}{2}\right)\right\|_{N-1} &\lesssim \lambda_{q+1}^N(\lambda_{q+1}\mu)^{-1}\rho^2.
\end{aligned}$$

Finally, the remaining term $\tilde{\phi}_{D_1}$ can be rewritten as $\tilde{\phi}_{D_1} = \bar{R}(v_q - v_l)$, where

$$\begin{aligned}\bar{R} &:= R_q - w \otimes w - R_{q+1} - \rho \mathrm{Id} + \frac{3}{2} f \,\mathrm{Id} \\ &= -\sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} c_{m,k} e^{i\lambda_{q+1}k\cdot\xi_m} - R_A - R_S - w \odot (v_q - v_l),\end{aligned}$$

and then we can control it by Lemmas 3.8, 4.3 and 4.4 as follows

$$\|\tilde{\phi}_{D_1}\|_N \lesssim \lambda_{q+1}^N l \,\lambda_q \delta_q^{1/2} \rho,$$

whereas the advective derivative can be controlled by using the aforementioned lemmas, Proposition 3.3, and the estimates (3.23), (3.27)

$$\begin{aligned}\|D_{t,q+1}\tilde{\phi}_{D_1}\|_{N-1} &\leq \|(v_q - v_l) D_{t,q+1}(R_A + R_S + w \odot (v_q - v_l))\|_{N-1} \\ &\quad + \left\| (v_q - v_l) \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \left(D_{t,l}(c_{m,k}) e^{i\lambda_{q+1}k\cdot\xi_m} + (u_{q+1} - u_l)\cdot\nabla(c_{m,k} e^{i\lambda_{q+1}k\cdot\xi_m}) \right) \right\|_{N-1} \\ &\quad + \left\| (D_{t,l}(v_q - v_l) + (u_{q+1} - u_l)\cdot\nabla(v_q - v_l))\, \bar{R} \right\|_{N-1} \\ &\lesssim \lambda_{q+1}^{N-1} \lambda_q \delta_q^{1/2} \left(\lambda_{q+1} l \,\delta_{q+1}^{3/2} + \rho^{3/2} \right).\end{aligned}$$

□

Lemma 5.4. *Let ϕ_{D_2}, f_{D_2} be defined by (5.8) and (5.3). For $N \leq 2$, it holds for any $t \leq \mathfrak{t}$ that*

$$\|\phi_{D_2}(t)\|_N \lesssim \frac{\rho}{\mu} \lambda_{q+1}^N \lambda_q \delta_q^{1/2} (\lambda_{q+1}^{-2} + \mu^{-n_0-1}\lambda_{q+1}^{-n_0-1}), \tag{5.12}$$

$$\begin{aligned}\|D_{t,q+1}\phi_{D_2}(t)\|_{N-1} &\lesssim \frac{\rho}{\mu} \lambda_{q+1}^{N-1} \Bigg[\rho^{1/2} \lambda_q \delta_q^{1/2} (\lambda_{q+1}^{-1} + \lambda_{q+1}(\lambda_{q+1}\mu)^{-n_0-1}) \\ &\quad + \left(\frac{\lambda_q \delta_q^{1/2} \epsilon^{-1}}{\lambda_{q+1}^2} + \frac{\lambda_q^2 \delta_q \, l}{\lambda_{q+1}} \right) + (\lambda_{q+1}\mu)^{-n_0-1} \left(\lambda_{q+1} \lambda_q \delta_q^{1/2} \right) \Bigg], \\ |f_{D_2}(t)| + |f'_{D_2}(t)| &\lesssim \frac{\rho}{\mu} \lambda_q \delta_q^{1/2} (\lambda_{q+1}\mu)^{-n_0-1}.\end{aligned} \tag{5.13}$$

Proof. Based on the decompositions (5.8) and (4.3), we have that

$$\phi_{D_2} = \overline{\mathcal{R}}\left[\mathcal{R}\left(G_1 + G_2 + G_3\right) : \nabla u_l^T\right],$$

where we have denoted $G_1 := \mathrm{div}(w_o \otimes w_o - c_0)$, $G_2 := D_{t,l} w$, and $G_3 := w \cdot \nabla u_l$. Notice that, for every $i = 1,2,3$, $\mathcal{R}\mathbf{P}_{\geq 3\lambda_{q+1}/8} G_i : \nabla u_l^T$ has frequencies of size at least $3\lambda_{q+1}/8 - l^{-1} \gtrsim \lambda_{q+1}$ and $\mathcal{R}\mathbf{P}_{<3\lambda_{q+1}/8} G_i : \nabla u_l^T$ has frequencies of size at most $3\lambda_{q+1}/8 + l^{-1} \lesssim \lambda_{q+1}$. Therefore, by the bound on the antidivergence $\|\overline{\mathcal{R}}\mathbf{P}_{\gtrsim\lambda_{q+1}} f\|_N \lesssim \lambda_{q+1}^{-1} \|f\|_N$ we have

$$\begin{aligned}\left\|\overline{\mathcal{R}}\left[\mathcal{R}\left(G_i\right) : \nabla u_l^T\right]\right\|_N &\leq \left\|\overline{\mathcal{R}}\left[\mathcal{R}\mathbf{P}_{\geq 3\lambda_{q+1}/8} G_i : \nabla u_l^T\right]\right\|_N + \left\|\overline{\mathcal{R}}\left[\mathcal{R}\mathbf{P}_{<3\lambda_{q+1}/8} G_i : \nabla u_l^T\right]\right\|_N \\ &\lesssim \frac{1}{\lambda_{q+1}} \left\|\mathcal{R}\mathbf{P}_{\geq 3\lambda_{q+1}/8} G_i : \nabla u_l^T\right\|_N + \lambda_{q+1}^N \left\|\mathcal{R}\mathbf{P}_{<3\lambda_{q+1}/8} G_i : \nabla u_l^T\right\|_0 \\ &\lesssim \frac{1}{\lambda_{q+1}^2} \sum_{j=0}^{N} \|G_i\|_j \, l^{N-j} \|u_q\|_1 + \lambda_{q+1}^N \|\mathbf{P}_{<3\lambda_{q+1}/8} G_i\|_0 \|u_q\|_1.\end{aligned} \tag{5.14}$$

Now, recalling the definition on the antidivergence stress R_A given by (4.3), we observe that each G_i has already been treated in the proof of Lemma 4.3. Therefore, the estimates therein and (3.43) imply that

$$\|G_i\|_j \lesssim \frac{\rho}{\mu} \lambda_{q+1}^j. \tag{5.15}$$

Instead, (A.8) implies that $\mathbf{P}_{<3\lambda_{q+1}/8} G_i = -\sum_{m,k} \epsilon_{n_0}^{\lambda}(k\cdot\xi_m, a_{m,k}^i) e^{i\lambda_{q+1}k\cdot\xi_m}$, so that by (A.15)

$$\|\mathbf{P}_{<3\lambda_{q+1}/8} G_i\|_0 \lesssim \frac{\rho}{\mu} \mu^{-n_0-1} \lambda_{q+1}^{-n_0-1}. \tag{5.16}$$

Therefore, (5.14), (5.15), and (5.16) produce the following bound

$$\|\phi_{D_2}\|_N \lesssim \frac{\rho}{\mu}\lambda_{q+1}^N\lambda_q\delta_q^{1/2}(\lambda_{q+1}^{-2}+\mu^{-n_0-1}\lambda_{q+1}^{-n_0-1}).$$

Now, let us turn to Eq. (5.13). Denote $H_i := \mathcal{R}\mathbf{P}_{\geq 3\lambda/8}G_i : \nabla u_l^T$, then

$$\begin{aligned}\|D_{t,l}\mathcal{R}\mathbf{P}_{\geq\lambda_{q+1}}H_i\|_{N-1} &\leq \|\mathcal{R}\left(\mathbf{P}_{\geq\lambda_{q+1}}D_{t,l}H_i+[u_l\cdot\nabla,\mathbf{P}_{\geq\lambda_{q+1}}]H_i\right)\|_{N-1}\\ &\quad+\|[u_l\cdot\nabla,\mathcal{R}]\mathbf{P}_{\geq\lambda_{q+1}}H_i\|_{N-1}\\ &\lesssim \frac{1}{\lambda_{q+1}}\|D_{t,l}H_i+[u_l\cdot\nabla,\mathbf{P}_{\geq\lambda_{q+1}}]H_i\|_{N-1}+\|[u_l\cdot\nabla,\mathcal{R}]\mathbf{P}_{\geq\lambda_{q+1}}H_i\|_{N-1}\\ &\lesssim \frac{1}{\lambda_{q+1}}\|D_{t,l}H_i\|_{N-1}+\lambda_{q+1}^{N-3}\|u_q\|_1\|H_i\|_1+\sum_{j_1+j_2\leq N-1} l\,\|\nabla u_l\|_{j_1}\|H_i\|_{j_2},\end{aligned}\tag{5.17}$$

where the first inequality is due to $\mathcal{R}$, the second to a trivial adaptation of [DLK22, Lemma A.4], Remark A.5 and $(l\lambda_{q+1})^{-h_0}\lambda_{q+1}^2 \leq l$. By a similar reasoning as above, we can estimate $\|D_{t,l}(\mathcal{R}\mathbf{P}_{\geq\lambda_{q+1}}G_i)\|_M$. Indeed, it holds:

$$\|D_{t,l}(\mathcal{R}\mathbf{P}_{\geq\lambda_{q+1}}G_i)\|_M \lesssim \frac{1}{\lambda_{q+1}}\|D_{t,l}G_i\|_M+\lambda_{q+1}^{M-2}\|u_l\|_1\|G_i\|_1+\sum_{M_1+M_2\leq M} l\|\nabla u_l\|_{M_1}\|G_i\|_{M_2}.$$

Thanks to the previous inequality, Eqs. (3.28) and (4.13), we have that

$$\begin{aligned}\|D_{t,l}H_i\|_{N-1} &\leq \|D_{t,l}(\mathcal{R}\mathbf{P}_{\geq\lambda_{q+1}}G_i):\nabla u_l^T\|_{N-1}+\|(\mathcal{R}\mathbf{P}_{\geq\lambda_{q+1}}G_i):D_{t,l}\nabla u_l^T\|_{N-1}\\ &\lesssim \lambda_{q+1}^{N-1}\frac{\rho}{\mu}\left(\frac{\lambda_q\delta_q^{1/2}\epsilon^{-1}}{\lambda_{q+1}^2}+\frac{\lambda_q^2\delta_q\, l}{\lambda_{q+1}}\right)\end{aligned}\tag{5.18}$$

Instead by standard computations, it holds true

$$\|H_i\|_M \lesssim \lambda_{q+1}^{M-1}\frac{\rho}{\mu}\lambda_q\delta_q^{1/2}.\tag{5.19}$$

Therefore, Eqs. (5.17) to (5.19) imply that

$$\|D_{t,l}\mathcal{R}\mathbf{P}_{\geq\lambda_{q+1}}H_i\|_{N-1} \lesssim \lambda_{q+1}^{N-1}\frac{\rho}{\mu}\left(\frac{\lambda_q\delta_q^{1/2}\epsilon^{-1}}{\lambda_{q+1}^2}+\frac{\lambda_q^2\delta_q\, l}{\lambda_{q+1}}\right).$$

Instead, Bernstein inequality, Eqs. (5.16) and (A.24) imply that

$$\begin{aligned}\|D_{t,l}\mathcal{R}(\mathcal{R}\mathbf{P}_{\lesssim\lambda_{q+1}}G_i:\nabla u_l^T)\|_{N-1} &\lesssim \lambda_{q+1}^{N-1}[\|D_{t,l}(\mathcal{R}\mathbf{P}_{\lesssim\lambda_{q+1}}G_i:\nabla u_l^T)\|_0\\ &\quad+\|[u_l\cdot\nabla,\mathcal{R}](\mathcal{R}\mathbf{P}_{\lesssim\lambda_{q+1}}G_i:\nabla u_l^T)\|_0]\\ &\lesssim \lambda_{q+1}^{N-1}\Big[\left(\|D_{t,l}\mathbf{P}_{\lesssim\lambda_{q+1}}G_i\|_0+\|[u_l\cdot\nabla,\mathcal{R}](\mathcal{R}\mathbf{P}_{\lesssim\lambda_{q+1}}G_i)\|_0\right)\|u_l\|_1\\ &\quad+\|\mathbf{P}_{\lesssim\lambda_{q+1}}G_i\|_0\|D_{t,l}\nabla u_l\|_0+\lambda_{q+1}\|u_l\|_0\|u_l\|_1\|\mathbf{P}_{\lesssim\lambda_{q+1}}G_i\|_0\Big]\\ &\lesssim \lambda_{q+1}^{N-1}\frac{\rho}{\mu}(\lambda_{q+1}\mu)^{-n_0-1}\left[\lambda_{q+1}\lambda_q\delta_q^{1/2}\right]\end{aligned}$$

At last, Eqs. (3.23) and (5.12)

$$\|(u_{q+1}-u_l)\cdot\nabla\phi_{D_2}\|_{N-1} \lesssim \lambda_{q+1}^N\frac{\rho^{3/2}}{\mu}\lambda_q\delta_q^{1/2}(\lambda_{q+1}^{-2}+(\lambda_{q+1}\mu)^{-n_0-1}).$$

This concludes the estimates for ϕ_{D_2}. Regarding the estimates on f_{D_2}, notice that

$$\int_{\mathbb{T}^3}\left(\mathcal{R}(G_i):\nabla u_l^T\right)(x,s)\,dx=\int_{\mathbb{T}^3}\left(\mathcal{R}(\mathbf{P}_{<3\lambda_{q+1}/8}G_i):\nabla u_l^T\right)(x,s)\,dx.$$

Therefore, (5.16) implies that

$$|f_{D_2}(t)|+|f_{D_2}'(t)| \lesssim \frac{\rho}{\mu}\lambda_q\delta_q^{1/2}(\lambda_{q+1}\mu)^{-n_0-1}.$$

□

5.1.3. *Proof of Proposition 5.1: bounds on* ϕ_S.

Lemma 5.5. *Let* ϕ_{S_1} *be defined by* (5.5)*. For* $N \le 2$*, it holds for any* $t \le \mathfrak{t}$ *that*

$$\|\phi_{S_1}(t)\|_N \lesssim l^{-N} i_q^{1/2-\delta} + \lambda_{q+1}^{N-2\gamma}\delta_{q+2}^{3/2} + i_q^{1/2-\delta}\begin{cases}1, & N=0\\ \lambda_{q+1}^N\delta_{q+1}^{1/2}, & N>0\end{cases},$$

$$\begin{aligned}\|D_{t,q+1}\phi_{S_1}(t)\|_{N-1} &\lesssim i_q^{1/2-\delta}\lambda_{q+1}^N\delta_{q+1} + \lambda_{q+1}^{N-2\gamma}\delta_{q+2}^{3/2}\delta_{q+1}^{1/2} + \lambda_{q+1}^{N-1}\delta_{q+2}^{3/2}\rho^{1/2}\epsilon^{-1}\\ &\quad + i_{q+1}^{-1/2-\delta}\begin{cases}1, & N=1\\ \lambda_{q+1}^{N-1}\delta_{q+1}^{1/2}\vee l^{1-N}, & N>1\end{cases}\end{aligned}$$

Proof. We divide the right-hand side of (5.5) into several terms, each to be controlled separately. First, the trace term can be controlled using the bounds (4.9), (4.10), and (3.26), more specifically we have

$$\left\|\mathrm{Tr}\left(R_{q+1} - \frac{3}{2} f\,\mathrm{Id}\right)\overline{z}\right\|_N \lesssim \lambda_{q+1}^N\delta_{q+2}\, i_q^{1/2-\delta}, \tag{5.20}$$

$$\left\|D_{t,q+1}\left[\mathrm{Tr}\left(R_{q+1} - \frac{3}{2} f\,\mathrm{Id}\right)\overline{z}\right]\right\|_{N-1} \lesssim \lambda_{q+1}^{N-1}\delta_{q+2}(i_{q+1}^{-1/2-\delta} + i_q^{1/2-\delta}\lambda_{q+1}). \tag{5.21}$$

Next, notice that

$$\frac{|v_q|^2 + |w|^2}{2}\overline{z} + (\overline{z}\otimes v_q)w = \frac{|v_{q+1}|^2}{2}\overline{z},$$

for which we have the estimates

$$\left\|\frac{|v_{q+1}|^2}{2}\overline{z}\right\|_N \lesssim i_q^{1/2-\delta}\begin{cases}1, & N=0\\ \lambda_{q+1}^N\delta_{q+1}^{1/2}, & N>0\end{cases},$$

$$\left\|D_{t,q+1}\left(\frac{|v_{q+1}|^2}{2}\overline{z}\right)\right\|_{N-1} \lesssim \lambda_{q+1}^N\delta_{q+1} i_q^{1/2-\delta} + i_{q+1}^{-1/2-\delta}\begin{cases}1, & N=0\\ \lambda_{q+1}^{N-1}\delta_{q+1}^{1/2}, & N>0\end{cases},$$

where the second inequality we have used Lemma 3.4 and Eqs. (3.26), (4.9) and (4.15) and

$$D_{t,q+1}v_{q+1} = \mathrm{div}\left(R_{q+1} - \frac{3}{2} f\,\mathrm{Id}\right) - \nabla p_{q+1} + u_{q+1}\cdot\nabla z_{q+1}.$$

Finally, upon recalling the definition (4.2) of the pressure increment $\overline{p}$, the remaining term $\tilde{\phi}_{S_1} := (\overline{z}\otimes\overline{z} + u_q\odot\overline{z})v_l + \overline{p}w$ is controlled by means of Proposition 3.3, Lemma 3.4, the bounds (2.4), (3.26), (3.23), (4.14), (3.27), and the following inequality:

$$\|D_{t,q+1}v_l\|_{N-1} \le \|D_{t,l}v_l\|_{N-1} + \|(u_{q+1}-u_l)\cdot\nabla v_l\|_{N-1} \lesssim \lambda_{q+1}^{N-1}\lambda_q\,\delta_q. \tag{5.22}$$

Namely, we have

$$\|\tilde{\phi}_{S_1}\|_N \lesssim l^{-N} i_q^{1/2-\delta} + \lambda_{q+1}^{N-2\gamma}\delta_{q+2}^{3/2}, \tag{5.23}$$

$$\begin{aligned}\|D_{t,q+1}\tilde{\phi}_{S_1}\|_{N-1} &\lesssim i_{q+1}^{-1/2-\delta} l^{1-N} + i_q^{1/2-\delta}\lambda_{q+1}^{N-1}\lambda_q\delta_q\\ &\quad + \lambda_{q+1}^{N-2\gamma}\delta_{q+2}^{3/2}\delta_{q+1}^{1/2} + \lambda_{q+1}^{N-1}\delta_{q+2}^{3/2}\rho^{1/2}\epsilon^{-1}.\end{aligned} \tag{5.24}$$

□

Lemma 5.6. *Let* ϕ_{S_2}, f_{S_2} *be defined by* (5.6) *and* (5.3)*. For* $N \le 2$*, it holds for any* $t \le \mathfrak{t}$ *that*

$$\|\phi_{S_2}(t)\|_N \lesssim \lambda_{q+1}^N\delta_{q+2} i_q^{1/2-\delta} + i_q^{1/2-\delta}\begin{cases}1, & N=0\\ \lambda_q^N\delta_q^{1/2}, & N>0\end{cases},$$

$$\|D_{t,q+1}\phi_{S_2}(t)\|_{N-1} \lesssim \lambda_{q+1}^{N-1}\lambda_q\delta_q^{1/2} i_q^{1/2-\delta} + \lambda_{q+1}^{N-1}\delta_{q+2} i_{q+1}^{-1/2-\delta} + l\lambda_{q+1}^N\delta_{q+2} + i_{q+1}^{-1/2-\delta}\begin{cases}1, & N=1\\ \lambda_q^{N-1}\delta_q^{1/2}, & N>1\end{cases},$$

$$|f_{S_2}(t)| + |f_{S_2}'(t)| \lesssim i_q^{1/2-\delta}.$$

Proof. Since we will only use that $\|\overline{\mathcal{R}}\|_{L^\infty\to L^\infty} \lesssim 1$, the estimates on f_{S_2} follow from the estimate on $\|\phi_{S_2}\|_0$.

Let us start with an estimate on $\bar{\phi}_{S_2} := \overline{\mathcal{R}}[R_A : \nabla(z_q - z_l)^T]$. In this case, (4.9) implies that

$$\|\bar{\phi}_{S_2}\|_N \lesssim l\lambda_{q+1}^{N-\gamma}\delta_{q+2}.$$

As for the advective derivative, we have that

$$\begin{aligned}\|D_{t,q+1}\bar{\phi}_{S_2}\|_{N-1} &\leq \left\|\overline{\mathcal{R}}[D_{t,q+1}\left(R_A : \nabla(z_q - z_l)^T\right)]\right\|_{N-1} + \left\|[u_{q+1}\cdot\nabla, \overline{\mathcal{R}}](R_A : \nabla(z_q - z_l)^T)\right\|_{N-1}\\ &\leq \left\|D_{t,q+1}\left(R_A : \nabla(z_q - z_l)^T\right)\right\|_{N-1} + \sum_{j=0}^{N-1}\|u_{q+1}\|_j\|R_A : \nabla(z_q - z_l)^T\|_{N-j}\\ &\leq l\lambda_{q+1}^{N-1}\delta_{q+2}\left(i_q^{-1/2-\delta} + \lambda_{q+1}\right),\end{aligned}$$

where the last inequality is due to Lemma 3.4 and the estimates (4.9) and (4.10).

The next term we study is $\tilde{\phi}_{S_2} := \overline{\mathcal{R}}[(u_q \mathbin{\mathring{\odot}} \overline{z} + \overline{z}\otimes\overline{z}) : \nabla z_q^T]$, which is controlled similarly to bounds (5.23) and (5.24) previously obtained for $\tilde{\phi}_{S_1}$, with the difference that (5.22) is now replaced by the bound $\|D_{t,q+1}\nabla z_q\|_N \lesssim i_q^{-1/2-\delta} + \begin{cases}1, & N=0\\ \lambda_{q+1}^N\delta_{q+1}^{1/2}, & N>0\end{cases}$, consequence of Lemma 3.4, giving

$$\begin{aligned}\|\tilde{\phi}_{S_2}\|_N &\lesssim i_q^{1/2-\delta}\begin{cases}1, & N=0\\ \lambda_q^N\delta_q^{1/2}, & N>0\end{cases},\\ \|D_{t,q+1}\tilde{\phi}_{S_2}\|_{N-1} &\lesssim \lambda_{q+1}^{N-1}\lambda_q\delta_q^{1/2}\, i_q^{1/2-\delta} + i_{q+1}^{-1/2-\delta}\begin{cases}1, & N=1\\ \lambda_q^{N-1}\delta_q^{1/2}, & N>1\end{cases}.\end{aligned}$$

Furthermore, similar arguments leading to (5.20) and (5.21) also yield

$$\begin{aligned}\left\|\overline{\mathcal{R}}\left[\left(R_{q+1} - \frac{3}{2}f\,\mathrm{Id}\right) : \nabla\overline{z}^T\right]\right\|_N &\lesssim \lambda_{q+1}^N\delta_{q+2}\, i_q^{1/2-\delta},\\ \left\|D_{t,q+1}\overline{\mathcal{R}}\left[\left(R_{q+1} - \frac{3}{2}f\,\mathrm{Id}\right) : \nabla\overline{z}^T\right]\right\|_{N-1} &\lesssim \lambda_{q+1}^{N-1}\delta_{q+2}(i_{q+1}^{-1/2-\delta} + i_q^{1/2-\delta}\lambda_{q+1}).\end{aligned}$$

At last, Lemma 3.4, (4.14), and the following inequalities

$$\begin{aligned}\|D_{t,q+1}v_q\|_N &\lesssim \lambda_{q+1}^N\lambda_q\delta_q,\\ \|D_{t,q+1}\nabla\overline{z}\|_N &\lesssim i_{q+1}^{-1/2-\delta} + i_q^{1/2-\delta}\begin{cases}1, & N=0\\ \lambda_{q+1}^N\delta_{q+1}^{1/2}, & N>0\end{cases},\\ \|D_{t,q+1}\nabla z_{q+1}\|_N &\lesssim i_{q+1}^{-1/2-\delta} + \begin{cases}1, & N=0\\ \lambda_{q+1}^N\delta_{q+1}^{1/2}, & N>0\end{cases},\end{aligned}$$

imply that

$$\begin{aligned}\left\|\overline{\mathcal{R}}\Big[v_q\cdot\operatorname{div}(u_q\otimes\overline{z} + \overline{z}\otimes z_{q+1})\Big]\right\|_N &\lesssim i_q^{1/2-\delta}\begin{cases}1, & N=0\\ \lambda_q^N\delta_q^{1/2}, & N>0\end{cases},\\ \left\|D_{t,q+1}\overline{\mathcal{R}}\Big[v_q\cdot\operatorname{div}(u_q\otimes\overline{z} + \overline{z}\otimes z_{q+1})\Big]\right\|_{N-1} &\lesssim i_q^{1/2-\delta}\lambda_{q+1}^{N-1}\lambda_q\,\delta_q^{1/2} + i_{q+1}^{-1/2-\delta}\begin{cases}1, & N=1\\ \lambda_q^{N-1}\delta_q^{1/2}, & N>1\end{cases}.\end{aligned}$$

□

5.1.4. *Proof of Proposition 5.1: Conclusion.* Bounds (5.9) and (5.10) follow from Lemma 5.2, Lemma 5.3, Lemma 5.4, Lemma 5.5, Lemma 5.6 above and our choice of parameters, up to taking the constant $a_0 > 1$ large enough so to absorb all the implicit constants coming from the aforementioned lemmas. As for the estimates on f, we have $f := f_A + f_{D_2} + f_{S_2}$ and thus by Lemma 5.2, Lemma 5.4, and Lemma 5.6:

$$|\partial_t f(t)| \leq |\partial_t f_A(t) + \partial_t f_{D_2}(t) + \partial_t f_{S_2}(t)| \lesssim \lambda_{q+1}^{-3\gamma/2}\delta_{q+2}\delta_{q+1}^{1/2}.$$

Finally, the claim on f follows from $|f(t)| \leq T\sup_{[0,t]}|\partial_t f(t)|$ and from picking $a_0 > 1$ large enough.

Appendix A. Appendix

A.1. **Estimates for transport equations.** We report some well-known estimates regarding smooth solutions $f : \mathbb{R}^3 \to \mathbb{R}$ of the transport equation

$$\begin{cases} \partial_t f + v \cdot \nabla f = g, \\ f(\cdot, t_0) = f_0, \end{cases}$$

where $v : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field and $g : \mathbb{R}^3 \to \mathbb{R}$ is a smooth scalar field. The following proposition is a slight modification of [BDLSV19, Proposition B.1].

Proposition A.1. *In the same setting as above, for any $\alpha \in [0,1]$ it holds*

$$\|f_t\|_\alpha \lesssim e^{\alpha(t-t_0)\|v_{[t_0,t]}\|_1} \left(\|f_0\|_\alpha + \int_{t_0}^t \|g_s\|_\alpha ds \right), \qquad \|v_{[t_0,t]}\|_1 := \sup_{s\in[t_0,t]} \|v_s\|_{C^1_x}.$$

In addition, if $|t-t_0|\|v_{[t_0,t]}\|_1 \le 1$, then for any $N \ge 1$ and $\alpha \in [0,1)$

$$[f_t]_{N+\alpha} \lesssim [f_0]_{N+\alpha} + |t-t_0|[v_{[t_0,t]}]_{N+\alpha}[f_0]_1 + \int_{t_0}^t \left([g_s]_{N+\alpha} + (t-s)[v_{[t_0,t]}]_{N+\alpha}[g_s]_1 \right) ds, \tag{A.1}$$

where $[v_{[t_0,t]}]_{N+\alpha} := \sup_{s\in[t_0,t]}[v_s]_{N+\alpha}$.

Moreover, the spatial inverse ϕ of the Lagrangian flux:

$$\phi_t(x) = (X_t)^{-1}(x), \quad \dot{X}_t = v(t, X_t), \quad X(x, t_0) = x, \quad \forall x \in \mathbb{R}^3,$$

satisfies the bounds

$$\begin{aligned} \|\nabla\phi(t) - \mathrm{Id}\|_0 &\lesssim |t-t_0|\|v_{[t_0,t]}\|_1, \\ [\phi(t)]_N &\lesssim |t-t_0|[v_{[t_0,t]}]_N, \quad \forall N \ge 2. \end{aligned} \tag{A.2}$$

A.2. **Antidivergence estimates.** In this subsection, we report a technical generalization of [DLK22, Corollary 8.2]. Let us start by recalling the following lemma from [DLK22].

Lemma A.2. *([DLK22, Lemma 8.1]). Let $\mathfrak{m} \in \mathcal{S}(\mathbb{R}^3)$ and define the Fourier multiplier operator $T_\mathfrak{m}$, acting an smooth functions $f \in C^\infty(\mathbb{T}^3, \mathbb{C})$ (smoothness meant in real sense) as follows:*

$$\mathcal{F}[T_\mathfrak{m}(f)](k) = \mathcal{F}[f](k)\mathfrak{m}(k), \quad k \in \mathbb{Z}^3.$$

Then, for any $n_0 \in \mathbb{N}$, $\lambda > 0$ and any scalar functions $a, \xi \in C^\infty(\mathbb{T}^3)$, the function $T_\mathfrak{m}(ae^{i\lambda\xi})$ can be decomposed as

$$T_\mathfrak{m}(ae^{i\lambda\xi}) = \left[a\,\mathfrak{m}(\lambda\nabla\xi) + \sum_{k=1}^{2n_0} C_k^\lambda(\xi, a) : (\nabla^k\mathfrak{m})(\lambda\nabla\xi) + \epsilon_{n_0}(\xi, a) \right] e^{i\lambda\xi},$$

for some tensor-valued coefficents $C_k^\lambda(\xi, a)$ and smooth function ϵ_{n_0} given by

$$\begin{aligned} \epsilon_{n_0}(\xi, a)(x) := &\sum_{n_1+n_2=n_0+1} \frac{(-1)^{n_1} c_{n_1,n_2}}{n_0!} \\ &\cdot \int_0^1 \int_{\mathbb{R}^3} \overset{\vee}{\mathfrak{m}}(y) e^{-i\lambda\nabla\xi(x)\cdot y} ((y\cdot\nabla)^{n_1} a)(x-ry) e^{i\lambda Z[\xi]_{x,y}(r)} \beta_{n_2}[\xi](r)(1-r)^{n_0} dy\, dr, \end{aligned} \tag{A.3}$$

where c_{n_1,n_2} are coefficients depending only on n_1, n_2, and $\beta_n[\xi]$ is defined by the following expressions:

$$\beta_n[\xi](r) := B_n(i\lambda Z'(r), \cdots, i\lambda Z^{(n)}(r)),$$

$$Z(r) := Z[\xi]_{x,y}(r) := r\int_0^1 (1-s)(y\cdot\nabla)^2 \xi(x-rsy)\, ds, \quad x, y \in \mathbb{R}^3, \tag{A.4}$$

$$B_n(x_1, \dots, x_n) := \sum_{k=1}^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \tag{A.5}$$

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) := \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{n-k+1!}\right)^{j_{n-k+1}}, \tag{A.6}$$

and the last summation is over $\{j_k\} \subset \mathbb{N}$ such that

$$j_1 + \cdots + j_{n-k+1} = k, \quad j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.$$

Proposition A.3. *Let $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ and l, ϵ, λ be positive constants. Let $v_l := \mathbf{P}_{\le l^{-1}} v$ and consider, for any $m \in \mathbb{Z}$, the solution ξ_m to the vector-valued transport equation*

$$\begin{cases} \partial_t \xi_m + v_l \cdot \nabla \xi_m = 0, \\ \xi_m(x, \epsilon(m - 1/8)) = x, \quad \forall x \in \mathbb{R}^3. \end{cases}$$

Let $F := \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{m\in\mathbb{Z}} a_{m,k} e^{i\lambda \xi_m \cdot k}$, where the functions $a_{m,k}$ satisfy the following conditions:

1. $\mathrm{Supp}_t(a_{m,k}) \subset (\epsilon(m - 1/8), \epsilon(m + 9/8))$, *and $\epsilon > 0$ is such that $2\epsilon \|v\|_{C_t C^1_x} \le 1/4$;*
2. *there exist a positive real-valued sequence $(\mathring{a}_k)_{k\in\mathbb{N}}$, and constants $a_F > 0$, $n_0 \in \mathbb{N}$ with the following property: for every $j \le 2 \vee (n_0 + 1)$ there exists a finite constant s_j such that*
$$\sum_k |k|^{(n_0+2)\vee N} \mathring{a}_k \le a_F, \quad \|a_{m,k}(t)\|_j \le s_j \mathring{a}_k, \quad \forall (m,k) \in \mathbb{Z} \times (\mathbb{Z}^3 \setminus \{0\});$$
3. *the sequence $(s_n)_{n\in\mathbb{N}}$ is non-decreasing and the following inequalities hold for every $n \le m$:*
$$l^{-m} s_n \le s_{m+n}, \quad s_n \lambda^m \le s_0 \lambda^{n+m}, \quad s_m \lambda^{-m} \le s_n \lambda^{-n}.$$

Then, for every $N \in \{0, 1, 2\}$ and every $t \in \mathbb{R}$ we have

$$\|\mathcal{R}F(t)\|_N \lesssim \left(s_0 \lambda^{N-1} + \lambda^{-n_0-1} s_{n_0+1} \lambda^N\right) a_F. \tag{A.7}$$

Proof. First of all, let us quickly show (more details can be found at [DLK22]) that

$$F = \mathbf{P}_{\gtrsim 3\lambda/8} \left(\sum_{m,k} a_{m,k} e^{i\lambda k \cdot \xi_m} \right) - \sum_{m,k} \epsilon^\lambda_{n_0}(k \cdot \xi_m, a_{m,k}) e^{i\lambda k\cdot\xi_m}, \tag{A.8}$$

where

$$\mathbf{P}_{\gtrsim 3\lambda/8} = \sum_{2^j \ge 3\lambda/8} \mathbf{P}_{2^j},$$

$$\epsilon^\lambda_{n_0}(k \cdot \xi_m, a_{m,k}) = \sum_{2^j \ge 3\lambda/8} \epsilon_{n_0,j}(k \cdot \xi_m, a_{m,k}), \tag{A.9}$$

with $\epsilon_{n_0,j}$ being obtained by Lemma A.2 applied to the Fourier multiplier operator $\mathbf{P}_{2^j}$. Notice that, by construction, ξ_m is smooth and space-periodic, since v_l is so.

In order to show (A.8), let us consider $(m,k) \in \mathbb{Z} \times (\mathbb{Z}^3)^*$ and $j_k \in \mathbb{N}$ such that $2^{j_k} \ge \frac{5}{4}\lambda|k|$. Then

$$\mathbf{P}_{\gtrsim 3\lambda/8} = \sum_{3\lambda/8 \le 2^i \le 2^{j_k}} \mathbf{P}_{2^i} + \mathbf{P}_{> 2^{j_k}} =: T_{\mathfrak{m}_1} + \sum_{j > j_k} T_{\mathfrak{m}_j},$$

where the functions $\mathfrak{m}_1 \in \mathcal{S}(\mathbb{R}^3)$ and $T_{\mathfrak{m}_j} = \mathbf{P}_{2^j}$ satisfy

$$\mathrm{Supp}\ \mathfrak{m}_1 \subset B(0, 2^{j_k+1}), \quad \mathfrak{m}_1 \equiv 1 \text{ on } \overline{B(0, 2^{j_k})} \setminus B(0, \frac{3}{4}\lambda|k|), \quad \mathfrak{m}_j \equiv 0 \text{ on } \overline{B}(0, \frac{5}{4}\lambda|k|).$$

Moreover for every $t \in \mathrm{Supp}_t\, a_{m,k}$, Assumption 1 of this proposition and Eq. (A.2) imply that

$$\||(\nabla\xi_m(t))^T k| - |k|\|_0 \le 2\epsilon |k| \|v\|_{C_t C^1_x} \le \frac{|k|}{4} \implies \|\nabla(k \cdot \xi_m(t))\|_0 \in \left[\frac{3}{4}|k|, \frac{5}{4}|k|\right]$$

Therefore, Lemma A.2 applied to T_{m_1}, and noticing that $\mathfrak{m}_1(\lambda\nabla(k\cdot\xi_m)) \equiv 1$, $\nabla^h \mathfrak{m}_1(\lambda\nabla(k\cdot\xi_m)) \equiv 0$ for every $h \ge 1$, giving:

$$\begin{aligned} T_{\mathfrak{m}_1}(a_{m,k} e^{i\lambda k\cdot\xi_m}) &- \sum_{3\lambda/8 \le 2^i \le 2^{j_k}} \epsilon_{n_0,i}(k\cdot\xi_m, a_{m,k}) e^{i\lambda k\cdot\xi_m} \\ &= a_{m,k} \underbrace{\mathfrak{m}_1(\lambda\nabla(k\cdot\xi_m))}_{\equiv 1} e^{i\lambda k\cdot\xi_m} + \sum_{h=1}^{2n_0} C^\lambda_h(k\cdot\xi_m, a_{m,k}) : (\underbrace{\nabla^h \mathfrak{m}_1}_{\equiv 0})(\lambda\nabla(k\cdot\xi_m)) e^{i\lambda k\cdot\xi_m} \\ &= a_{m,k} e^{i\lambda k\cdot\xi_m}. \end{aligned}$$

At last, Eq. (A.8) follows from applying Lemma A.2 to $T_{\mathfrak{m}_j}$ for all $j > j_k$. Indeed,

$$T_{\mathfrak{m}_j}(a_{m,k} e^{i\lambda k\cdot\xi_m}) = \epsilon_{n_0,j}(k\cdot\xi_m, a_{m.k}) e^{i\lambda k\cdot\xi_m}$$

thanks to $(\nabla^h \mathfrak{m}_j)(\lambda\nabla(k\cdot\xi_m)) = 0$ for all $h \in \mathbb{N}$.

Now, for fixed (m,k), we want to estimate $\|\epsilon^\lambda_{n_0}(k\cdot\xi_m, a_{m,k})\|_0$. Based on Eq. (A.9), we have to consider $\epsilon_{n_0,j}(k\cdot\xi_m, a_{m.k})$, which is given by Eq. (A.3).

Let Z be defined by Eq. (A.4) and $Z_0(r) := r^{-1}Z(r)$, then

$$Z^{(n)} = nZ_0^{(n-1)} + Z_0^{(n)}, \quad Z^{(n)}(r) := \int_0^1 (1-s)(-s)^n (y\cdot\nabla)^{n+2}(k\cdot\xi_m)(x-rsy)\,ds. \tag{A.10}$$

Therefore, Assumption 1 and Proposition A.1 imply that

$$\|Z^{(n)}\|_0 \lesssim |y|^{n+1}\|k\cdot\xi_m\|_{n+1} + |y|^{n+2}\|k\cdot\xi_m\|_{n+2} \lesssim |y|^{n+1}l^{-n}|k|[1+|y|l^{-1}]. \tag{A.11}$$

Then, by means of Eqs. (A.5) and (A.6), we have that

$$\begin{aligned}\|\beta_{n_2}[k\cdot\xi_m](r)\|_0 &\lesssim \sum_{m=1}^{n_2} \lambda^m |y|^{m+n_2}|k|^m[1+|y|l^{-1}]^m l^{-n_2} \\ &\lesssim (|y|l^{-1})^{n_2}|k|^{n_0+1}\sum_{m=1}^{n_2}(\lambda|y|)^m\sum_{i=0}^{m}(|y|l^{-1})^i.\end{aligned} \tag{A.12}$$

Instead, Assumption 2 implies that

$$\|(y\cdot\nabla)^{n_1}a_{m,k}(x-ry)\|_0 \lesssim |y|^{n_1}|\mathring{a}_k|s_{n_1}. \tag{A.13}$$

Then, Eqs. (A.12) and (A.13) imply that

$$\begin{aligned}\|\epsilon_{n_0,j}\|_0 &\lesssim \sum_{n_1,n_2} s_{n_1}|k|^{n_0+1}|\mathring{a}_k|\sum_{m=1}^{n_2}\lambda^m\sum_{i=0}^m l^{-n_2-i}\int|\check{\mathfrak{m}}_j(y)||y|^{n_0+1+m+i}dy \\ &\lesssim \sum_{n_1,n_2} s_{n_1}|k|^{n_0+1}|\mathring{a}_k|\sum_{m=1}^{n_2}\lambda^m\sum_{i=0}^m l^{-n_2-i}2^{-j(n_0+1+m+i)},\end{aligned}$$

where the last inequality is due to $\|\check{\mathfrak{m}}_j(y)|y|^n\|_{L^1}\lesssim 2^{-jn}$. Therefore, we have that

$$\begin{aligned}\|\epsilon^\lambda_{n_0}\|_0 &\lesssim \sum_{n_1,n_2} s_{n_1}|k|^{n_0+1}|\mathring{a}_k|\sum_{m=1}^{n_2}\lambda^m\sum_{i=0}^m l^{-n_2-i}\sum_{2^j\gtrsim\lambda}2^{-j(n_0+1+m+i)} \\ &\lesssim |k|^{n_0+1}|\mathring{a}_k|s_{n_0+1}\lambda^{-n_0-1},\end{aligned} \tag{A.14}$$

where the last inequality makes use of Assumption 3. The assumption on the time supports, i.e. condition 1, implies that

$$\|\sum_{m,k}\epsilon^\lambda_{n_0}(k\cdot\xi_m, a_{m,k})\|_0 \lesssim \sum_k |k|^{n_0+1}|\mathring{a}_k|s_{n_0+1}\lambda^{-n_0-1} \le a_F s_{n_0+1}\lambda^{-n_0-1}. \tag{A.15}$$

So, Eq. (A.8) allows to conclude as follows:

$$\begin{aligned}\|\mathcal{R}F\|_N &\le \|\mathcal{R}\mathbf{P}_{\gtrsim 3\lambda/8}F\|_N + \|\mathcal{R}\sum_{m,k}\epsilon^\lambda_{n_0}(k\cdot\xi_m, a_{m,k})\|_N \\ &\lesssim \frac{\|F\|_N}{\lambda} + \lambda^N\|\sum_{m,k}\epsilon^\lambda_{n_0}(k\cdot\xi_m,a_{m,k})\|_0 \\ &\lesssim a_F[s_0\,\lambda^{N-1} + s_{n_0+1}\lambda^{-n_0-1}],\end{aligned}$$

where the second inequality is due to the Bernstein inequality, while the third one is due to

$$\|F\|_N \lesssim \sum_k\sum_{N_1+N_2\le N}|\mathring{a}_k|s_{N_1}|k|^{N_2}\lambda^{N_2} \le a_F\,s_0\,\lambda^N, \tag{A.16}$$

and the last inequality holds because of Assumption 3. □

Proposition A.4. *Under the same assumptions of Proposition A.3, let $l > \frac{100}{\lambda}$ and $w \in [C^\infty(\mathbb{T}^3)]^3$ such that there exists a non-decreasing sequence $(b_n)_{n\in\mathbb{N}}$ satisfying for every $n,m\in\mathbb{N}$*

$$b_n\lambda^m \le b_0\lambda^{m+n}, \quad \|w\|_n \le b_n.$$

Let us moreover suppose there exists $c>0$ such that

$$c\|D_{t,l}a_{m,k}(t)\|_j \le s_j\mathring{a}_k,$$

where $D_{t,l} := \partial_t + v_l \cdot \nabla$ *and* $j \le (n_0+1) \vee 2$, *and that for any* $n \in \mathbb{N}$

$$\|v\|_{n+1} \le l^{-n}\|v\|_1. \tag{A.17}$$

Then for every $N \le 1$ *and* $h_0 \in \mathbb{N} \setminus \{0,1\}$ *it holds*

$$\begin{aligned}\|D_t \mathcal{R}F(t)\|_N \lesssim \Big[&s_0(c\lambda)^{-1} + c^{-1}s_{n_0+1}\lambda^{-n_0-1} + s_0\lambda^{-1}\|v\|_1 \\ &+ (1+\lambda^{-n_0}s_{n_0+1})(b_0 + l\|v\|_1 + \|v\|_0) + s_0(l\lambda)^{-h_0}\lambda^2\|v\|_1\Big]\lambda^N a_F,\end{aligned} \tag{A.18}$$

where $D_t := \partial_t + (w+v)\cdot\nabla$.

Proof. By standard computations, we have that

$$D_t\mathcal{R}F = \mathcal{R}(D_{t,l}F) + [v_l\cdot\nabla, \mathcal{R}]F + [(w+(v-v_l))\cdot\nabla]\mathcal{R}F.$$

Since $D_{t,l}F = \sum_{k,m} D_{t,l}a_{m,k}e^{i\lambda k\cdot\xi_m}$, then the previous Proposition A.3 implies that

$$\|\mathcal{R}(D_{t,l}F)\|_N \lesssim a_F\lambda^N[s_0(c\lambda)^{-1} + s_{n_0+1}\lambda^{-n_0-1}c^{-1}]. \tag{A.19}$$

Instead,

$$\begin{aligned}\|[(w+(v-v_l))\cdot\nabla]\mathcal{R}F\|_N &\lesssim \sum_{N_1+N_2\le N}\|w+(v-v_l)\|_{N_1}\|RF\|_{N_2+1} \\ &\lesssim \sum_{N_1+N_2\le N}(b_{N_1} + \|v\|_1 l^{1-N_1})\lambda^{N_2}(1+s_{n_0+1}\lambda^{-n_0}) \\ &\lesssim (b_0 + l\|v\|_1)(1+s_{n_0+1}\lambda^{-n_0})\lambda^N,\end{aligned} \tag{A.20}$$

where the second inequality is due to the previous result, Bernstein inequality, (A.17) and the assumptions on w, while the third is due to $b_n\lambda^m \le b_0\lambda^{m+n}$.

At last, we need to estimate

$$[v_l\cdot\nabla, \mathcal{R}]F = [v_l\cdot\nabla,\mathcal{R}]\mathbf{P}_{\gtrsim\lambda}F - [v_l\cdot\nabla,\mathcal{R}]\sum_{m,k}\epsilon^\lambda_{n_0}(k\cdot\xi_m, a_{m,k})e^{i\lambda k\cdot\xi_m} =: I_1 + I_2.$$

Since I_2 has localized active frequencies, the Bernstein inequality implies that

$$\begin{aligned}\|I_2\|_N &\lesssim \lambda^N\|I_2\|_0 \lesssim \lambda^N\|v_l\|_0\|\nabla\mathbf{P}_{\lesssim\lambda}F\|_0 \\ &\lesssim \lambda^{N+1}\|v\|_0\Big\|\sum_{m,k}\epsilon^\lambda_{n_0}(k\cdot\xi_m, a_{m,k})\Big\|_0 \\ &\lesssim \lambda^N\|v\|_0 s_{n_0+1}\lambda^{-n_0}a_F,\end{aligned} \tag{A.21}$$

where the last inequality is due to Eq. (A.15). Instead, by considering $\mathcal{R}$ as a Fourier multiplier with coefficients $\mathcal{F}(\mathcal{R})(k)$, see [RT25] for an explicit formula, we have that

$$\begin{aligned}-I_1 &= \sum_{2^j\gtrsim\lambda}\sum_{k,n} i\eta e^{ik\cdot x}\mathcal{F}(v_l)(k-\eta)\mathcal{F}[F_j](\eta)[\mathcal{F}(\mathcal{R})(k) - \mathcal{F}(\mathcal{R})(\eta)] \\ &= \sum_{2^j\gtrsim\lambda}\sum_{k,n} i\eta\mathcal{F}(v_l)(k-\eta)\mathcal{F}(F_j)(\eta)e^{ik\cdot x}\Big[\sum_{h=1}^{h_0}\frac{1}{h!}[(k-\eta)\cdot\nabla_\eta]^h\mathcal{F}(\mathcal{R})(\eta) \\ &+ \frac{1}{h_0!}\int_0^1[(k-\eta)\cdot\nabla_\eta]^{h_0+1}\mathcal{F}(\mathcal{R})(\eta+\sigma(k-\eta))(1-\sigma)^{h_0}d\sigma\Big] \\ &=: J_1 + J_2,\end{aligned}$$

where $F_j = \mathbf{P}_{2^j}F$. Now,

$$\begin{aligned}\|J_1\|_N &\lesssim \sum_{h=1}^{h_0}\|\nabla^h v_l\nabla\mathcal{R}_h\mathbf{P}_{\gtrsim\lambda}F\|_N \lesssim \sum_{h=1}^{h_0}\sum_{N_1+N_2\le N} l^{1-h-N_1}\|v\|_1\|F\|_{N_2+1}\lambda^{-h-1} \\ &\lesssim s_0\|v\|_1\lambda^{-2}\sum_{N_1+N_2\le N} l^{-N_1}\lambda^{N_2+1}a_F \lesssim a_F s_0\|v\|_1\lambda^{N-1},\end{aligned} \tag{A.22}$$

where $\mathcal{R}_h g$ has Fourier coefficient at frequency $k \in \mathbb{Z}^3\setminus\{0\}$ given by $\left[\nabla^h\mathcal{F}(\mathcal{R})\right](k)\mathcal{F}(g)(k)$, and the second-to-last inequality is due to Eq. (A.16).

At last,

$$
\begin{aligned}
\|J_2\|_N &\lesssim \sum_{2^j \gtrsim \lambda} \sum_{k,\eta} |k|^N |\eta| |\mathcal{F}(v_l)(k-\eta)| |\mathcal{F}(F_j)(\eta)| |k-\eta|^{h_0+1} |\eta - \sigma(\eta-k)|^{-h_0-2} \\
&\lesssim \sum_{2^j \gtrsim \lambda} \sum_{k,\eta} |\eta|^{N-h_0-1} |\mathcal{F}(F_j)(\eta)| |k-\eta|^{h_0} |\mathcal{F}(\nabla v_l)(k-\eta)| \\
&\lesssim \sum_{2^j \gtrsim \lambda} \frac{l^{-h_0}}{2^{j(h_0+1)}} \sum_{|k| \le 2^j} \|\nabla^N F_j\|_{L^2} \|\nabla v\|_{L^2} \lesssim (l\lambda)^{-h_0} \lambda^2 \|v\|_1 s_0\, a_F \lambda^N.
\end{aligned}
\tag{A.23}
$$

where the second inequality is due to $l^{-1} < \frac{\lambda}{100}$ and $|k-\eta| \le 2l^{-1}$, which imply that $|\eta - \sigma(\eta-k)| \ge \frac{|\eta|}{2}$ and $|k| \lesssim |\eta|$. All in all, Eqs. (A.19) to (A.23) prove the claim. □

Remark A.5. Notice that

$$
\sum_{h=1}^{h_0} \|\nabla^h v_l \nabla \mathcal{R}_h \mathbf{P}_{\gtrsim\lambda} F\|_N \lesssim \sum_{h=1}^{h_0} \sum_{N_1+N_2 \le N} l^{1-h} \|\nabla v_l\|_{N_1} \lambda^{-h} \|F\|_{N_2}.
$$

This inequality and Eq. (A.23) imply that

$$
\|[v_l \cdot \nabla, \mathcal{R}] \mathbf{P}_{\gtrsim\lambda} F\|_N \lesssim (l + (l\lambda)^{-h_0}\lambda^2) \sum_{N_1+N_2 \le N} \|\nabla v_l\|_{N_1} \|F\|_{N_2}.
$$

Proposition A.6. *Under the same assumption of Propositions A.3 and A.4, it holds that*

$$
\|D_{t,l} \sum_{m,k} \epsilon^{\lambda}_{n_0}(k\cdot\xi_m, a_{m,k}) e^{i\lambda k\cdot\xi_m}\|_0 \lesssim a_F (c^{-1} \vee \|v_l\|_1) s_{n_0+1} \lambda^{-n_0-1},
\tag{A.24}
$$

Proof. For fixed (m,k), we want to estimate $\|D_{t,l}\epsilon^{\lambda}_{n_0}\|_0$, given by Eq. (A.9). we have to consider $\epsilon_{n_0,j}(k\cdot\xi_m, a_{m.k})$, which is given by Eq. (A.3). Based on Eq. (A.10) and $Z_0(r) := r^{-1}Z(r)$, we have that

$$
\begin{aligned}
\|D_{t,l} Z_0^{(n)}\|_0 &\lesssim \| \int_0^1 (1-s)(-s)^n [v_l(x) - v_l(x-yrs)] \cdot \nabla_x [(y\cdot\nabla)^{n+2}(k\cdot\xi_m)(x-rsy)] ds\|_0 \\
&+ \| \int_0^1 (1-s)(-s)^n [D_{t,l}(y\cdot\nabla)^{n+2}] k\cdot\xi_m(x-rsy) ds\|_0 \\
&\lesssim \|v_l\|_1 |y|^{n+2} l^{-n-1} |k| [1 + l^{-1}|y|],
\end{aligned}
$$

where the last inequality is due to the following one, which is proved in [DLK22],

$$
\|D_{t,l}(y\cdot\nabla)^n g\|_0 \lesssim |y|^n \left[\|D_{t,l} g\|_n + \sum_{\substack{m_1+m_2=n \\ m_1 \ge 1}} \|v_l\|_{m_1} \|g\|_{m_2+1} \right].
\tag{A.25}
$$

Then, similarly to Eqs. (A.11) and (A.12), we have that

$$
\|D_{t,l} Z^{(n)}\|_0 \lesssim |y|^{n+1} l^{-n} |k| \|v_l\|_1 [1 + |y| l^{-1} + |y|^2 l^{-2}],
$$

$$
\|D_{t,l}\beta_{n_2}[k\cdot\xi_m]\|_0 \lesssim \|v_l\|_1 [1 + |y|^2 l^{-2}] (|y| l^{-1})^{n_2} |k|^{n_0+1} \sum_{m=1}^{n_2} (\lambda|y|)^m \sum_{i=0}^{m} (|y| l^{-1})^i.
\tag{A.26}
$$

Now, Eq. (3.31) and Proposition A.1 imply that

$$
\|D_{t,l}(e^{-i\lambda y\cdot\nabla(k\cdot\xi_m)})\|_0 \lesssim \lambda |y| \|v_l\|_1 |k|,
\tag{A.27}
$$

$$
\begin{aligned}
\|D_{t,l} e^{iZ(k\cdot\xi_m)}\|_0 &\lesssim \lambda \| \int_0^1 (1-s)(y\otimes y) : [(D_{t,l}\nabla^2(k\cdot\xi_m)(x-ry) \\
&\quad + ((v_l(x) - v_l(x-ry))\cdot\nabla)\nabla^2(k\cdot\xi_m)(x-ry) ds] e^{i\lambda Z(r)}\|_0 \\
&\lesssim \lambda |y|^2 |k| l^{-1} \|v_l\|_1 [1 + |y| l^{-1}].
\end{aligned}
\tag{A.28}
$$

Instead, Assumption 2 and Eq. (A.25) imply that

$$
\begin{aligned}
\|D_{t,l}[(y\cdot\nabla)^{n_1}a_{m,k}(x-ry)]\|_0 &\lesssim \|(v_l(x)-v_l(x-yr))\cdot\nabla[(y\cdot\nabla)^{n_1}a_{m,k}(x-ry)]\|_0\\
&\quad+\|(D_{t,l}(y\cdot\nabla)^{n_1}a_{m,k}(x-ry)\|_0\\
&\lesssim |y|^{n_1+1}\|v_l\|_1 s_{n_1+1}|\mathring{a}_k|+|y|^{n_1}|\mathring{a}_k|s_{n_1}(c^{-1}\vee\|v_l\|_1).
\end{aligned}
\tag{A.29}
$$

By following the computations used to get Eq. (A.14), Eqs. (A.26) to (A.29) imply that

$$\|D_{t,l}\epsilon^\lambda_{n_0}\|_0 \lesssim |k|^{n_0+2}\mathring{a}_k(c^{-1}\vee\|v_l\|_1)s_{n_0+1}\lambda^{-n_0-1}.$$

Then, the claim follows by just summing up in m and k. □

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