

Relative Turán densities for ordered graphs: all and nothing

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Abstract

Reiher, Rödl, Sales, and Schacht initiated the study of relative Turán densities of ordered graphs and showed that it is more subtle and interesting than the unordered case. For an ordered graph F , its relative Turán density, $\rho_{<}(F)$, is the greatest α such that every ordered graph G has an F -free subgraph with at least $\alpha e(G)$ edges.

This paper contains two main results about relative Turán densities. First, we find a family of host graphs that is optimal for all F . Second, we characterise the ordered graphs with zero relative Turán density: precisely those with no monotone path of length two.

1 Introduction

Turán problems are some of the oldest and most fundamental in graph theory. The archetypal question asks, given a graph F and host graph G , for the greatest edge density of an F -free subgraph of G . That is, for the value of

$$\rho(F, G) := \max \left\{ \frac{e(G')}{e(G)} : G' \subseteq G, F \not\subseteq G' \right\}.$$

The classical Turán problem, solved by the theorems of Mantel [Man07], Turán [Tur41], and Erdős, Stone, and Simonovits [ES46, ES66], addresses the case where the host graph G is complete. Together they show that the *Turán density* of a graph F is

$$\pi(F) := \lim_{n \rightarrow \infty} \rho(F, K_n) = 1 - \frac{1}{\chi(F) - 1}. \quad (1)$$

Turán problems on non-complete host graphs have been widely studied (see for example [Chu92, SV03, KRS04, AKS07, FÖ11, CG16, Sch16]) although the problems are typically

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much more difficult. For example, Erdős's \$100 problem [Erd84, Erd90] from 1984 of determining $\lim_{d \rightarrow \infty} \rho(C_4, Q_d)$ remains open to this day [Blo].

Given the interest in this class of problems, it is natural to ask for the infimum of $\rho(F, G)$ over all host graphs G , which is called the *relative Turán density* of F and denoted $\rho(F)$. That is, to ask for the greatest $\alpha \in [0, 1]$ such that every graph G has an F -free subgraph on at least $\alpha e(G)$ edges. However, this infimum is always given by $\pi(F)$ and so complete graphs are universally hardest to make F -free¹.

Reiher, Rödl, Sales, and Schacht [RRSS25] proved that the situation is more interesting and nuanced for ordered graphs, which are graphs whose vertex set is equipped with a total ordering. For a survey on extremal problems in ordered graphs see Tardos's ICM talk [Tar18] or its extended version [Tar19]. Analogously to graphs, the Turán problem for ordered graphs asks, given an ordered graph F and ordered host graph G , for $\rho_{<}(F, G)$, the greatest density of an F -free subgraph of G . When the host graph is complete, Pach and Tardos [PT06] answered the problem to leading order by proving the ordered analogue of the Erdős-Stone-Simonovits theorem:

$$\pi_{<}(F) := \lim_{n \rightarrow \infty} \rho_{<}(F, K_n) = 1 - \frac{1}{\chi_{<}(F) - 1},$$

where $\chi_{<}(F)$ is the *ordered/interval chromatic number*. While it remains true that the relative Turán density, $\rho_{<}(F)$, of an ordered graph satisfies

$$\rho_{<}(F) := \inf_G \rho_{<}(F, G) \leq \inf_n \rho_{<}(F, K_n) = 1 - \frac{1}{\chi_{<}(F) - 1}, \quad (2)$$

Reiher, Rödl, Sales, and Schacht showed that equality does not hold in general. They did so by proving that for a certain family of graphs $R(m, d)$ (see § 2.3), first constructed in [ARS22], the monotone path of length k , $\vec{P}_{k+1}$, satisfies

$$\rho_{<}(\vec{P}_{k+1}) = \lim_{d \rightarrow \infty} \lim_{m \rightarrow \infty} \rho_{<}(\vec{P}_{k+1}, R(m, d)) = \frac{1}{2} \left(1 - \frac{1}{k}\right),$$

while $\pi_{<}(\vec{P}_{k+1}) = 1 - 1/k$ is twice as large. In particular, the family of complete ordered graphs is not a universally optimal host for the Turán problem in ordered graphs, and it seems that generally, $\rho_{<}(F)$ is much harder to determine than $\rho(F)$. Thus, it is perhaps surprising that for every ordered graph F , whatever value $\rho_{<}(F)$ may take, that value is achieved by the very family $R(m, d)$. This is our first result.

Theorem 1.1. *For every ordered graph F , $\rho_{<}(F) = \lim_{d \rightarrow \infty} \lim_{m \rightarrow \infty} \rho_{<}(F, R(m, d))$.*

This theorem reduces determining $\rho_{<}(F)$ to solving the Turán problem on the specific host graph $R(m, d)$. Our next concern is to develop tools to solve this problem. The ordered graphs $R(m, d)$ are quite sparse when d is large, which creates technical difficulties. We rectify this by showing that, to determine $\rho_{<}(F)$, it suffices to find F in graphs on $\{0, 1\}^d$ that have a powerful local density property we call *(α, C) -richness*. The definition of (α, C) -richness is deferred to § 2.3.

Theorem 1.2. *For every ordered graph F ,*

$$\rho_{<}(F) = \inf \{ \alpha \in [0, 1] : \exists C \text{ such that every } (\alpha, C)\text{-rich graph contains } F \}.$$

¹The upper bound $\inf_G \rho(F, G) \leq 1 - 1/(\chi(F) - 1)$ follows from (1). To see that $\inf_G \rho(F, G) \geq 1 - 1/(\chi(F) - 1)$, colour the vertices of G randomly with $\chi(F) - 1$ colours and keep only the edges between vertices of distinct colours.

In upcoming work we will use this theorem to investigate $\rho_{<}(F)$ for many ordered graphs F . In the present paper, we use it to determine which ordered graphs have vanishing relative Turán density. It follows from $\rho_{<}(\vec{P}_3) = 1/4$ that the relative Turán density of an ordered graph which contains a monotone path of length two is positive (and, in fact, at least $1/4$). Of course, (2) implies that any ordered graph with interval chromatic number two has zero relative Turán density. However, the class of ordered graphs with no monotone path of length two is much larger (it contains graphs with arbitrarily large interval chromatic number). King, Lidický, Ouyang, Pfender, Wang, and Xiang [KLO⁺25] showed that certain families of ordered matchings have zero relative Turán density. We prove that this holds for every ordered graph with no monotone path of length two, which provides a complete characterisation.

Theorem 1.3. *For every ordered graph F , $\rho_{<}(F) = 0$ if and only if F does not contain a monotone path of length two.*

The rest of the paper is organised as follows. We introduce some ideas and tools in § 2 including the definitions of *richness* and $R(m, d)$. In § 3, we prove Theorems 1.1 and 1.2. § 4 is dedicated to the proof of Theorem 1.3. We conclude by discussing some open problems in § 5.

2 Tools and terminology

As mentioned in the introduction, our host graph $R(m, d)$ originates from [ARS22] and was used already in [RRSS25] to determine $\rho_{<}$ for ascending paths. In both articles, $R(m, d)$ is constructed recursively in d by an amalgamation of two copies of $R(m, d - 1)$. While this allows for elegant inductive proofs, it can make it difficult to see the structure of the graph as a whole. In this section, we will set up the notation, tools, and terminology required to construct and study $R(m, d)$ in a more explicit manner without recursion.

2.1 Ordered graphs

Let G be an ordered graph. Given $u \in V(G)$, we say an edge uv is a *forward edge* for u if $u < v$, and v is a *forward neighbour* of u in this case. We define *backward edges* and *backward neighbours* analogously, if $v < u$.

2.2 Binary ordering

We will often consider ordered graphs whose vertices are labelled by binary strings of the form $x \in \{0, 1\}^d$, indexed as $x = x_1 x_2 \dots x_d$. For distinct binary strings $x, y \in \{0, 1\}^d$, we define

$$\delta(x, y) := \min\{i: x_i \neq y_i\},$$

and so, for example, $\delta(001110, 001101) = 5$. We will always view $\{0, 1\}^d$ as being ordered lexicographically.

Definition 2.1 (lexicographic ordering). For distinct $x, y \in \{0, 1\}^d$, define $x < y$ if $x_{\delta(x, y)} = 0$ and $y_{\delta(x, y)} = 1$.

If one interprets each string as a binary number (so x corresponds to $\sum_i x_i 2^{d-i}$), then this corresponds to the usual order on the non-negative integers. The set $\{0, 1\}^d$ with its lexicographic ordering splits naturally into intervals.

Definition 2.2 (fundamental intervals). For $\ell \in \{0, 1, \dots, d\}$, let $\mathcal{P}_\ell$ be the family of equivalence classes of the relation on $\{0, 1\}^d$ given by $\delta(u, v) > \ell$. Each $I \in \mathcal{P}_\ell$ is an interval in $\{0, 1\}^d$ consisting of strings that agree on the first ℓ bits. We call these intervals the *fundamental intervals at level ℓ* . Given $I \in \mathcal{P}_\ell$, we denote the half of I in $\mathcal{P}_{\ell+1}$ for which the $(\ell + 1)$ th digit is 0 by $\text{Left}(I)$, and the other half by $\text{Right}(I)$.

Note that $|\mathcal{P}_\ell| = 2^\ell$ and each interval $I \in \mathcal{P}_\ell$ has size $2^{d-\ell}$. Observe that $\mathcal{P}_b$ refines² $\mathcal{P}_a$ for $a \leq b$.

2.3 Graphs on $\{0, 1\}^d$ and $\{0, 1\}^d \times [m]$

We now define notation related to a graph G on vertex set $\{0, 1\}^d$, ordered lexicographically. We say that $uv \in E(G)$ is a *level ℓ -edge* if $\delta(u, v) = \ell$. We denote the set of level ℓ -edges of G by $E_\ell(G)$ and their number by $e_\ell(G)$. When A is a subset of $\{0, 1\}^d$ we write $E_{\ell, G}(A)$ and $e_{\ell, G}(A)$ to mean $E_\ell(G[A])$ and $e_\ell(G[A])$, respectively, where $G[A]$ is the induced subgraph of G with vertex-set A . We may omit G from the subscripts when the context is clear. To help us normalise the number $e_{\ell, G}(A)$ correctly, we write

$$\tau_\ell(A) := |\{(u, v) \in A \times A : u < v, \delta(u, v) = \ell\}|$$

for the total number of pairs $\{u, v\} \subset A$ for which $\delta(u, v) = \ell$ (note that this is independent of G). We write

$$\tau_{\ell, d} := \tau_\ell(\{0, 1\}^d) = 2^{2d-\ell-1}.$$

We are particularly interested in levels on which G has a large proportion of all possible edges.

Definition 2.3 (rich levels and graphs). For a graph G on $\{0, 1\}^d$ and $\alpha \in [0, 1]$, we say that a level $\ell \in [d]$ is *α -rich with respect to G* if $e_\ell(G) \geq \alpha \tau_{\ell, d}$. We say that G is *(α, C) -rich* for some $C > 0$ if it contains at least C levels that are α -rich.

We will also need to discuss ordered graphs on $\{0, 1\}^d \times [m]$. Their vertex set is ordered lexicographically: $(x, i) < (y, j)$ if $x < y$ or $x = y$ and $i < j$. For each $x \in \{0, 1\}^d$, we define $B_x = \{x\} \times [m]$ and refer to the B_x as *blocks*. Just as for graphs on $\{0, 1\}^d$, we say an edge uv is a *level ℓ -edge* if $u \in B_x$ and $v \in B_y$ where $\delta(x, y) = \ell$ and we write $e_\ell(G)$ for the number of level ℓ -edges in a graph G on $\{0, 1\}^d \times [m]$.

Proposition 2.4. For all $d, m \in \mathbb{N}$, there exists a graph $R(m, d)$ on $\{0, 1\}^d \times [m]$ such that the following holds. For every $d \in \mathbb{N}$, $\varepsilon > 0$, and for all sufficiently large m :

- (i) for every $\ell \in [d]$, $e_\ell(R(m, d)) = (1 \pm \varepsilon)2^{d-1}m^2$ and so $e(R(m, d)) = (1 \pm \varepsilon)d2^{d-1}m^2$;
- (ii) for every $x, y \in \{0, 1\}^d$ and every $P \subseteq B_x$ and $Q \subseteq B_y$, each of size at least $m^{2/3}$, the number of edges in $R(m, d)$ between P and Q is at most $(1 + \varepsilon)2^{-d+\delta(x, y)}|P||Q|$.

Proof. For all $d, m \in \mathbb{N}$, consider the random graph $\mathcal{R}(m, d)$ on $\{0, 1\}^d \times [m]$ where an $(x, i)(y, j)$ is present with probability $2^{-d+\delta(x, y)}$ independently of all other edges. Two

²A partition $\mathcal{P}$ *refines* a partition $\mathcal{Q}$ if, for all $P \in \mathcal{P}$, there is some $Q \in \mathcal{Q}$ such that $P \subseteq Q$.

simple applications of Chernoff's inequality show that for each $d \in \mathbb{N}$ and sufficiently large m , $\mathcal{R}(m, d)$ has the properties above with high probability. We take $R(m, d)$ to be an arbitrary sample of $\mathcal{R}(m, d)$, such that for m sufficiently large, both properties are satisfied. $\square$

Here is one useful corollary of the first property that relates to richness. Note that the denominator $2^{d-1}m^2$ is roughly $e_\ell(R(m, d))$.

Proposition 2.5. *Let $d \in \mathbb{N}$ and $\varepsilon > 0$. Then, for all sufficiently large m and every $\alpha \in [0, 1]$, every subgraph G of $R(m, d)$ with at least $(\alpha + 2\varepsilon) \cdot e(R(m, d))$ edges satisfies*

$$\frac{1}{d} \sum_{\ell=1}^d \frac{e_\ell(G)}{2^{d-1}m^2} \geq \alpha + \varepsilon.$$

Proof. For each level $\ell \in [d]$, we define $p_\ell = e_\ell(G) / (2^{d-1}m^2)$. Now, [Proposition 2.4\(i\)](#) with input $\varepsilon/2$ implies

$$\frac{1}{d} \sum_{\ell} p_\ell = \frac{1}{d \cdot 2^{d-1}m^2} \cdot \sum_{\ell} e_\ell(G) \geq \frac{e(G)}{(1 + \varepsilon/2) \cdot e(R(m, d))} \geq \frac{\alpha + 2\varepsilon}{1 + \varepsilon/2} \geq \alpha + \varepsilon. \quad \square$$

2.4 The regularity lemma

We will use Szemerédi's regularity lemma [[Sze78](#)] together with some associated machinery in order to reduce the problem of proving upper bounds for $\rho_{<}(F)$ to finding a copy of F in a rich graph (see [Lemma 3.6](#)). For a pair of disjoint vertex sets $X, Y \subseteq V(G)$, we use $d(X, Y) = e(X, Y) / |X|^{-1}|Y|^{-1}$ to denote the *density* between X and Y .

For $\eta > 0$, we say a pair (A, B) of disjoint vertex sets is *η -regular* if for all $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| \geq \eta|A|$ and $|Y| \geq \eta|B|$, we have

$$|d(X, Y) - d(A, B)| \leq \eta.$$

A partition $\mathcal{Q} = \{V_1, \dots, V_t\}$ of $V(G)$ is an *equipartition* if $||V_i| - |V_j|| \leq 1$ for all i and j . This implies that $\lfloor |V(G)|/t \rfloor \leq |V_i| \leq \lceil |V(G)|/t \rceil$ for all i . Finally, $\mathcal{Q}$ is *η -regular* if (V_i, V_j) is η -regular for all but at most $\eta \binom{t}{2}$ pairs $i < j$.

We need a version of the regularity lemma where the resulting η -regular partition refines some initial equipartition. The proof of this follows from the proof of the usual regularity lemma (see, for example, [[KS96](#), Lemma 1.15]).

Lemma 2.6 (refined regularity lemma). *For every $\eta > 0$ and $r \in \mathbb{N}$, there exists $L \in \mathbb{N}$ such that the following holds. Let G be a graph and $\mathcal{Q}$ be an equipartition of $V(G)$ into r parts. Then there exists an η -regular equipartition $\mathcal{P}$ of G that refines $\mathcal{Q}$, and such that each part of $\mathcal{Q}$ is refined into at most L parts.*

We will also need the following building lemma (see, for example, [[KS96](#), Theorem 2.1]).

Lemma 2.7 (graph building lemma). *Let H be a graph (on vertex set $\{1, 2, \dots, |V(H)|\}$) and $\lambda \in (0, 1)$. For all sufficiently small $\eta > 0$ the following holds. Suppose $V_1, \dots, V_{|V(H)|}$ are sufficiently large pairwise disjoint vertex sets such that (V_i, V_j) is η -regular of density at least λ for each $ij \in E(H)$. Then there is a copy of H with each vertex in the corresponding V_i .*

3 All: universally optimal host graphs

In this section we prove both our universality result, [Theorem 1.1](#), and as the reduction to rich graphs, [Theorem 1.2](#). Since it will be useful for determining $\rho_{<}(F)$ in future work, we prove the following mild strengthening.

Theorem 3.1. *For every ordered graph F ,*

$$\begin{aligned}\rho_{<}(F) &= \inf \left\{ \alpha \in [0, 1] : \forall \text{ large } d, \text{ every } G \text{ on } \{0, 1\}^d \text{ with } \frac{1}{d} \sum_{\ell} \frac{e_{\ell}(G)}{\tau_{\ell, d}} \geq \alpha \text{ contains } F \right\} \\ &= \inf \{ \alpha \in [0, 1] : \exists C \text{ such that every } (\alpha, C)\text{-rich graph contains } F \}.\end{aligned}$$

In [§ 3.1](#) we prove [Lemma 3.2](#) which gives lower bounds for $\rho_{<}(F)$ while in [§ 3.2](#) we prove [Lemma 3.6](#) which gives upper bounds for $\rho_{<}(F)$. Finally, in [§ 3.3](#) we put them together to prove [Theorems 1.1, 1.2, and 3.1](#).

3.1 Random embeddings

In this subsection we prove [Lemma 3.2](#), about random embeddings of graphs into $R(m, d)$, which we will use to prove the lower bound on $\rho_{<}(F)$ in [Theorem 1.1](#). Before delving into the details of the somewhat technical proof, we first describe our general strategy and motivate certain choices that might otherwise seem arbitrary.

Consider the following proof that for every *unordered* graph F , $\rho(F) = \inf_n \rho(F, K_n)$. It suffices to show that for every graph H , $\rho(F, K_n) \leq \rho(F, H)$ for sufficiently large n . For this purpose, let Φ be a uniformly random embedding of H into K_n . Then, for each edge $e \in E(H)$, $\Phi(e)$ is a uniformly random edge of K_n . Therefore, for any subgraph G of K_n with more than $\rho(F, H)e(K_n)$ edges, the probability that $\Phi(e) \in E(G)$ is greater than $\rho(F, H)$. Hence, by Markov's inequality, there is an embedding ϕ of H into K_n such that $G' = G \cap \phi(H)$ contains more than $\rho(F, H)e(H)$ edges of H , which implies that G' , and hence G , contains F .

Note that this argument does not work for ordered graphs because for many a graph H , a uniformly random embedding of H into K_n will exhibit strong biases towards certain edges and against others. For example, if H is a long ascending path, then random embeddings will be biased in favour of edges xy for which x and y are fairly close in the ordering. What is more, because there exist graphs F for which $\rho_{<}(F, R(m, d))$ is strictly less than $\rho_{<}(F, K_n)$, we have $R(m, d)$ as a specific example of an ordered graph for which it is impossible to find a distribution of the embeddings of it into K_n without strongly favouring certain edges of K_n .

Thus, we endeavour instead to construct “edge unbiased” distributions on the embeddings of arbitrary ordered graphs into $R(m, d)$ although we will hide some of the complexity of this task by using the language of edge levels introduced in the previous section. The existence of such a distribution is the key lemma of this subsection.

Lemma 3.2 (tiling). *For every ordered graph H and all $\varepsilon > 0$, there exists $C \in \mathbb{N}$ such that, for all $d \in \mathbb{N}$ and subsets $\mathcal{L} \subset [d]$ of size at least C , there exists a random variable Φ on the embeddings of H into the complete ordered graph on $\{0, 1\}^d$ such that the following holds. For all but an ε -proportion of $\ell \in \mathcal{L}$, all but an ε -proportion of $x, y \in \{0, 1\}^d$ with $\delta(x, y) = \ell$*

satisfy

$$\mathbb{P}(xy \in \Phi(H)) \geq (1 - \varepsilon) \frac{e(H)}{|\mathcal{L}| \tau_{\ell,d}}.$$

To give an idea of the proof of [Lemma 3.2](#), let H be an arbitrary ordered graph, K the complete graph on $\{0, 1\}^d$, and $\mathcal{L} \subset [d]$ large. We want to construct a random variable Φ on the embeddings $\phi: H \hookrightarrow K$ such that for each $uv \in E(H)$, the level $\ell := \delta(\phi(u), \phi(v))$ is approximately uniformly distributed on $\mathcal{L}$, and conditioned on ℓ , the edge $\phi(uv)$ is approximately³ uniformly distributed on $\tau_{\ell,d}$.

The first condition shows that our distribution cannot be the uniform distribution over all embeddings of H into K as this would favour the lower levels with more edges. Instead, labelling the vertices of H as $u_1 < \dots < u_h$, we first choose a subset of levels $\mathcal{L}' := \{\ell_1 < \dots < \ell_{h-1}\} \in \mathcal{L}$ and subsequently choose $\phi(u_1) < \dots < \phi(u_h)$ uniformly at random under the constraint $\delta(\phi(u_i), \phi(u_j)) = \ell_i$ for $i < j$.

Choosing the set of levels $\mathcal{L}'$ still requires care. For example, if $\mathcal{L}'$ is a uniformly random subset of $\mathcal{L}$ of size $h - 1$, then ℓ_1 and the distribution of edges involving u_1 exhibit strong biases towards lower levels. More subtly, it is also not valid to choose ℓ_1 uniformly at random and the remaining ℓ_i to be the $h - 2$ next higher levels of $\mathcal{L}$. Indeed, if h is large and $u_1 u_h$ is the only edge of H , then, for every supported embedding ϕ , the image $\phi(E(H))$ can contain only those edges xy at level $\ell_1 \in \mathcal{L}$ for which $y_{\ell_2} = \dots = y_{\ell_{h-1}} = 1$ for the $h - 2$ indices $\ell_2, \dots, \ell_{h-1}$ immediately following ℓ_1 in $\mathcal{L}$.

Our resolution to these difficulties is to sample $\mathcal{L}'$ by choosing first an “interval” of consecutive levels of $\mathcal{L}$ uniformly at random, and then choosing $\mathcal{L}'$ uniformly at random from the $(h - 1)$ -sets contained in the interval. The interval will be short compared to $\mathcal{L}$ so that each ℓ_i is roughly uniformly distributed in $\mathcal{L}$, but long compared to h so that we do not induce strong biases on the indices of the vertices incident to embedded edges.

For a rigorous proof of [Lemma 3.2](#), we need the following three quantitative lemmas whose proofs are fairly straightforward and can be found in [§ A](#).

Lemma 3.3. *For all $\alpha \in (0, 1]$, $\varepsilon > 0$, and $k \in \mathbb{N}$, there exist $\eta > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$\binom{\lfloor (\alpha - \eta)n \rfloor}{k} \geq (\alpha^k - \varepsilon) \binom{n}{k}.$$

Lemma 3.4. *For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$ and for all but an ε -proportion of $x \in \{0, 1\}^n$, every interval $J \subset [n]$ of length at least $\ln^2 n$ satisfies*

$$\left| \sum_{i \in J} x_i - \frac{|J|}{2} \right| < \varepsilon |J|.$$

Lemma 3.5. *For all $\alpha, \varepsilon > 0$, and $k \in \mathbb{N}$, there exist $\eta > 0$ and $N \in \mathbb{N}$ such that for all $n > N$ the following holds. Suppose that $f: [n] \rightarrow [0, 1]$ has the property that for all subintervals $J \subset [n]$ of length at least $\lfloor \eta n \rfloor$,*

$$|\mathbb{E}_{t \in J} f(t) - \alpha| \leq \eta. \tag{3}$$

³Note that it is not possible to achieve an exactly uniform distribution for either the level or the edge within the level. For example, only the lexicographically smallest edge of H can be mapped to the lexicographically smallest edge of K .

Then, for all $x, y \in [k]$,

$$\sum_{t=\lceil \eta n \rceil}^{\lfloor (1-\eta)n \rfloor} f(t) \binom{t-1}{x} \binom{n-t}{y} \geq (1-\varepsilon) \alpha \binom{n}{x+y+1}.$$

Proof of Lemma 3.2. Let $\varepsilon > 0$ and let H be an ordered graph. Defining $h := |V(H)|$, we may assume without loss of generality that $V(H) = [h]$ with the usual ordering. Let us now determine C . First, we apply Lemma 3.3 with $\alpha_{3.3} = 1/2$, $\varepsilon_{3.3} = \varepsilon/4$, and $k_{3.3} = h$ to obtain $N_{3.3}$ and $\eta_{3.3}$. Independently, we apply Lemma 3.5 with $\alpha_{3.5} = 1/2$, $\varepsilon_{3.5} = \varepsilon/4$, and $k_{3.5} = h$, to obtain $N_{3.5}$ and $\eta_{3.5}$. Then, we let $\eta = \min(\varepsilon/2, \eta_{3.5}, \eta_{3.3})$ and apply Lemma 3.4 with $\varepsilon_{3.4} = \eta$ to obtain $N_{3.4}$.

Finally, let C be an integer that is larger than $\eta^{-6} \max(N_{3.5}, N_{3.4}, N_{3.3})$ and sufficiently large to satisfy $\ln^2(C) < \eta^6 C - 1$.

Given $d \in \mathbb{N}$ and $\mathcal{L} \subset [d]$ of size $L > C$, we let $w = \lceil \eta^2 L \rceil$, $M = \lceil \eta^4 L \rceil$, and $m = \lceil \eta^6 L \rceil$ so that $w < \eta L$, $M < \eta w$, and $m < \eta M$, while $m - 1 > \ln^2 L$ and $m > \max(N_{3.5}, N_{3.4}, N_{3.3})$. Furthermore, we define $\iota: [L] \rightarrow \mathcal{L}$ as the enumeration of $\mathcal{L}$ in order.

We sample Φ as follows. First, choose an integer a in $[0, L - w)$ uniformly at random, and let $b = a + w$. Next, choose $\mathcal{L}' \subset \iota((a, b])$ of size h uniformly at random, and denote the elements of $\mathcal{L}'$ in ascending order by $\ell_1, \dots, \ell_h$. Finally, the vertices $v_1 < \dots < v_h$ are chosen uniformly at random under the restriction that $\delta(v_i, v_{i+1}) = \ell_i$ for all $i \in [h - 1]$ and that $(v_h)_{\ell_h}$, i.e., the coordinate of v_h with index ℓ_h , is 0. Such a choice of $v_1, \dots, v_h$ has the form displayed in Figure 1 where the A_i and B_i are binary strings of the correct length: $|A_1| = \ell_1 - 1$; $|A_i| = \ell_i - \ell_{i-1} - 1$ for each $i \in [2, h]$; $|B_i| = d - \ell_i$ for each $i \in [h]$.

		ℓ_1		ℓ_2		ℓ_3	$\cdots$	ℓ_{h-1}		ℓ_h	
v_1 :	A_1	0	B_1								
v_2 :	A_1	1	A_2	0	B_2						
v_3 :	A_1	1	A_2	1	A_3	0	B_3				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$					$\vdots$	
v_h :	A_1	1	A_2	1	A_3	1	$\cdots$	1	A_h	0	B_h

Figure 1. The random embedding Φ

Given the ℓ_i there is a bijection between the set of tuples of vertices, $(v_1, \dots, v_h)$, satisfying the restriction, and the set of tuples of binary strings (of the correct lengths), $(A_1, \dots, A_h, B_1, \dots, B_h)$. Hence, given $\ell_1, \dots, \ell_h$, the random choice of the v_i is equivalent to sampling each of $A_1, \dots, A_h$ and $B_1, \dots, B_h$ independently and uniformly at random (with the correct lengths).

For the remainder of the proof, we fix $i < j$ in $[h]$. For $y \in \{0, 1\}^d$ and $\beta \in \{0, 1\}$, let

$$S_{y,\beta} = \{\ell \in [d] : y_\ell = \beta\}.$$

We say that y is *good*, if for every interval $J \subset [L]$ of length at least $m - 1$, and $\beta \in \{0, 1\}$, we have

$$|\iota(J) \cap S_{y,\beta}| \geq \frac{1-\eta}{2}|J|.$$

Recall that $m > \ln^2 L$ and $L \geq N_{3.4}$. Thus, if we project elements of $\{0, 1\}^d$ onto their coordinates indexed by $\mathcal{L}$, then Lemma 3.4 implies that all but $\eta 2^d$ vertices in $\{0, 1\}^d$ are good. Thus, to complete the proof it is enough to show that for all good y and all $x < y$ such that $\delta(x, y)$ is in $\iota([w, L - w])$, the probability of v_i being mapped to x and v_j being mapped to y is at least $\frac{1-\eta}{L^{\tau_{\lambda,d}}}$. Fix x and y with these properties.

Let $\mathfrak{L}$ be the set of $\{\lambda_1 < \dots < \lambda_h\} \subset \mathcal{L}$ such that $y_\lambda = 1$ for all $\lambda \in \{\lambda_1, \dots, \lambda_{j-1}\}$, $\lambda_i = \delta(x, y)$, and $y_{\lambda_j} = 0$. Note that the first two conditions then also imply that $x_\lambda = 1$ for all $\lambda \in \{\lambda_1, \dots, \lambda_{i-1}\}$ and $x_{\lambda_i} = 0$. We have

$$\mathbb{P}(v_i = x, v_j = y) = \sum_{\{\lambda_1 < \dots < \lambda_h\} \in \mathfrak{L}} \mathbb{P}(\ell_1 = \lambda_1, \dots, \ell_h = \lambda_h, v_i = x, v_j = y). \quad (4)$$

Happily, conditioning on the event that $\ell_1 = \lambda_1, \dots, \ell_h = \lambda_h$ for fixed $\{\lambda_1 < \dots < \lambda_h\} \in \mathfrak{L}$, it is easy to compute the probability that $v_i = x$ and $v_j = y$. Namely, sampling $v_1, \dots, v_h$ by their bitstrings $A_1, \dots, A_h$ and $B_1, \dots, B_h$ as illustrated in Figure 1, this happens if and only if

- A_r coincides with the substring of y on coordinates $[\lambda_{r-1} + 1, \lambda_r - 1]$ for all $r \in [j]$,
- B_i coincides with the substring of x on coordinates $[\lambda_i + 1, d]$, and
- B_j coincides with the substring of y on coordinates $[\lambda_j + 1, d]$.

Now, since the A_t and B_t are independently and uniformly random bit strings, the first event happens with probability $1/2^{\lambda_j - j}$, the second with probability $1/2^{d - \lambda_i - 1} = 1/2^{d - \delta(x, y)}$, and the third with probability $1/2^{d - \lambda_j}$. Thus, (4) becomes

$$\begin{aligned} \mathbb{P}(v_i = x, v_j = y) &= \sum_{\{\lambda_1 < \dots < \lambda_h\} \in \mathfrak{L}} \mathbb{P}(\ell_1 = \lambda_1, \dots, \ell_h = \lambda_h) \frac{1}{2^{2d - \delta(x, y) - j}} \\ &= \mathbb{P}(\{\ell_1 < \dots < \ell_h\} \in \mathfrak{L}) \cdot \frac{2^{j-1}}{2^{2d - \delta(x, y) - 1}}. \end{aligned}$$

Since $\tau_{\delta(x, y), d} = 2^{2d - \delta(x, y) - 1}$, it remains to show that

$$\mathbb{P}(\{\ell_1 < \dots < \ell_h\} \in \mathfrak{L}) \geq \frac{1 - \varepsilon}{2^{j-1}L}.$$

To simplify our notation in this task, let $\mathcal{E}$ be the event that $\{\ell_1 < \dots < \ell_h\} \in \mathfrak{L}$, let $\kappa = \iota^{-1}(\lambda)$, $S_0 = \iota^{-1}(S_{y,0})$, and $S_1 = \iota^{-1}(S_{y,1})$. For each $a \in [0, L - w]$, there are $\binom{w}{h}$ possible choices for $\ell_1, \dots, \ell_h$. To count those that are in $\mathfrak{L}$, we count the number of sets $\{\lambda_1, \dots, \lambda_h\} \subset \iota((a, a + w])$ with $\lambda_j = \iota(r)$ for each $r \in (\kappa, a + w] \cap S_0$ separately. We must have $\lambda_i = \delta(x, y)$, while $\lambda_1, \dots, \lambda_{i-1}$ can be chosen from $\iota(S_1 \cap (a, \kappa - 1])$

arbitrarily, $\lambda_{i+1}, \dots, \lambda_{j-1}$ can be chosen from $\iota(S_1 \cap (\kappa, r-1])$, and $\lambda_{j+1}, \dots, \lambda_h$ can be chosen from $\iota((r, a+w])$. Thus,

$$\mathbb{P}(\mathcal{E}) = \frac{1}{(L-w)\binom{w}{h}} \sum_{a=0}^{L-w} \binom{|S_1 \cap (a, \kappa-1]|}{i-1} \sum_{r=\kappa+1}^{a+w} 1_{S_0}(r) \binom{|S_1 \cap (\kappa, r-1]|}{j-i-1} \binom{a+w-r}{h-j}. \quad (5)$$

Since y is good, whenever $\kappa \geq a+m$ and $r \geq \kappa+m$, we have both

$$\begin{aligned} |S_1 \cap (a, \kappa-1]| &\geq \frac{1-\eta}{2}(\kappa-1-a), \\ |S_1 \cap (\kappa, r-1]| &\geq \frac{1-\eta}{2}(r-1-\kappa). \end{aligned}$$

Inserting this into (5) for valid values of a and r , we obtain

$$\mathbb{P}(\mathcal{E}) \geq \frac{1}{(L-w)\binom{w}{h}} \sum_{a=\kappa-w+M}^{\kappa-M} \binom{\lceil \frac{1-\eta}{2}(\kappa-1-a) \rceil}{i-1} \sum_{r=\kappa+m}^{a+w} 1_{S_0}(r) \binom{\lceil \frac{1-\eta}{2}(r-1-\kappa) \rceil}{j-i-1} \binom{a+w-r}{h-j}.$$

Since i and j are at most h , and $m > N_{3.3}$, we may apply Lemma 3.3 to the above which yields

$$\mathbb{P}(\mathcal{E}) \geq \frac{1-\varepsilon/2}{2^{j-2}(L-w)\binom{w}{h}} \sum_{a=\kappa-w+M}^{\kappa-M} \binom{\kappa-1-a}{i-1} \sum_{r=\kappa+m}^{a+w} 1_{S_0}(r) \binom{r-1-\kappa}{j-i-1} \binom{a+w-r}{h-j}. \quad (6)$$

Shifting the index of the inner sum by κ , the inner sum becomes

$$\sum_{t=m}^{a+w-\kappa} 1_{S_0}(t+\kappa) \binom{t-1}{j-i-1} \binom{a+w-\kappa-t}{h-j}.$$

However, since y is good, $a+w-\kappa \geq M > N_{3.5}$, and $m < \eta M$, we know from Lemma 3.5 that the above is at least

$$\frac{1-\varepsilon/4}{2} \binom{a+w-\kappa}{h-i}.$$

Inserting this into (6), we obtain

$$\mathbb{P}(\mathcal{E}) \geq \frac{1-3\varepsilon/4}{2^{j-1}(L-w)\binom{w}{h}} \sum_{a=\kappa-w+M}^{\kappa-M} \binom{\kappa-1-a}{i-1} \binom{a+w-\kappa}{h-i}. \quad (7)$$

It only remains to show that the inner sum is approximately equal to $\binom{w}{h}$. For this purpose, we make an index transformation $a = \kappa - t$, which transforms the inner sum into

$$\sum_{t=M}^{w-M} \binom{t-1}{i-1} \binom{w-t}{h-i}.$$

However, since $M < \eta w$, applying [Lemma 3.5](#) with $f \equiv 1$ shows that this is at least $(1 - \varepsilon/4)^{\binom{w}{h}}$, so that (7) yields

$$\mathbb{P}(\mathcal{E}) \geq \frac{1 - \varepsilon}{2^{j-1}(L - w)} \geq \frac{1 - \varepsilon}{2^{j-1}L},$$

as desired. $\square$

3.2 Reduction to graphs on $\{0, 1\}^d$

We now prove [Lemma 3.6](#) which reduces obtaining upper bounds on $\rho_{<}(F)$ to finding F in all graphs with sufficiently large ‘average richness’. This will make use of the regularity lemma machinery from [§ 2.4](#).

Lemma 3.6 (reduction to $\{0, 1\}^d$). *Let $\alpha \in [0, 1]$ and let F be an ordered graph. Suppose that, for all $\varepsilon > 0$ and all sufficiently large d (in terms of ε and F), every graph G on $\{0, 1\}^d$ that satisfies*

$$\frac{1}{d} \sum_{\ell=1}^d \frac{e_{\ell}(G)}{\tau_{\ell,d}} \geq \alpha + \varepsilon$$

contains an ordered copy of F . Then $\lim_{d \rightarrow \infty} \lim_{m \rightarrow \infty} \rho_{<}(F, R(m, d)) \leq \alpha$.

Proof. Let $\alpha \in [0, 1]$ and F be an ordered graph. Fix a small $\varepsilon > 0$ and let d be sufficiently large as in the hypothesis of the lemma.

Let m be sufficiently large in terms of d , ε , and F . Let $R := R(m, d)$ as defined in [Proposition 2.4](#). Let $R' \subseteq R$ with $e(R') \geq (\alpha + 14\varepsilon) \cdot e(R)$. We may and will assume that $V(R') = V(R)$. It suffices to prove that $F \subseteq R'$ as then

$$\rho_{<}(F, R) \leq \alpha + 14\varepsilon,$$

which implies $\lim_{d \rightarrow \infty} \lim_{m \rightarrow \infty} \rho_{<}(F, R(m, d)) \leq \alpha$, since ε was arbitrary. For each level $\ell \in [d]$, we define $p_{\ell} = e_{\ell}(R') / (2^{d-1}m^2)$. Now [Proposition 2.5](#) guarantees that, provided m is sufficiently large in terms of d and ε ,

$$\frac{1}{d} \sum_{\ell} p_{\ell} \geq \alpha + 7\varepsilon.$$

Let η be the parameter obtained from the graph building lemma, [Lemma 2.7](#), applied to F with density $\varepsilon 2^{-d}$. We may and will assume that $\eta \leq \varepsilon 2^{-d}$. Note that the blocks $\mathcal{B} := \{B_x : x \in \{0, 1\}^d\}$ form an equipartition of R' into 2^d parts. Let $L \in \mathbb{N}$ obtained from the refined regularity lemma, [Lemma 2.6](#), applied to η and 2^d and let $\mathcal{P}$ be the resulting η -regular equipartition of R' that refines $\mathcal{B}$. Note that L is a function of ε , d , and F only. Provided m is sufficiently large compared to L , the equipartition $\mathcal{P}$ has split each block into an equal number of parts. We call this number m' (and so $|\mathcal{P}| = 2^d m'$ and each part has size $\lfloor m/m' \rfloor$ or $\lceil m/m' \rceil$) and note that $m' \leq L$.

We clean R' as follows:

- we delete all edges between pairs of parts whose density is at most $\varepsilon 2^{-d}$,
- we delete all edges between pairs of parts that are not η -regular.

The total number of deleted edges is at most

$$\begin{aligned} \varepsilon 2^{-d} |V(R')|^2 + \eta \binom{2^d m'}{2} \left\lceil \frac{|V(R')|}{2^d m'} \right\rceil^2 &\leq (\eta + \varepsilon 2^{-d}) |V(R')|^2 \\ &\leq 2\varepsilon \cdot 2^{-d} (m 2^d)^2 = 4\varepsilon \cdot 2^{d-1} m^2. \end{aligned}$$

Let R'' be the resulting graph. For each level ℓ , we have

$$\frac{e_\ell(R'')}{2^{d-1} m^2} \geq \frac{e_\ell(R')}{2^{d-1} m^2} - 4\varepsilon = p_\ell - 4\varepsilon.$$

Consider the graph H whose vertex set is $\mathcal{P}$ and where two parts are adjacent if they have an edge between them in R'' (in fact, due to the cleaning process, the pair of parts will be η -regular with density at least $\varepsilon 2^{-d}$). We will order H as follows: parts in different blocks are ordered in the same way as their blocks (so if $P \subset B_x$ and $Q \subset B_y$ where $x < y$, then $P < Q$) and parts within blocks are ordered arbitrarily. Since each block is split into m' parts by $\mathcal{P}$, we may view H as having vertex set $\{0, 1\}^d \times [m']$ and talk about its level ℓ -edges for every $\ell \in [d]$.

Now, provided m is sufficiently large compared to L , property (ii) of [Proposition 2.4](#) implies that, between two parts $P, Q \in \mathcal{P}$, the number of level ℓ -edges in $R(m, d)$ is at most

$$(1 + \varepsilon) 2^{-d+\ell} |P||Q| \leq (1 + \varepsilon) 2^{-d+\ell} \lceil m/m' \rceil^2 \leq (1 + 2\varepsilon) 2^{-d+\ell} (m/m')^2.$$

Therefore, the number of level ℓ -edges of R'' between any two parts is also at most $(1 + 2\varepsilon) 2^{-d+\ell} (m/m')^2$. This implies that, for every $\ell \in [d]$

$$\begin{aligned} e_\ell(H) &\geq \frac{e_\ell(R'')}{(1 + 2\varepsilon) 2^{-d+\ell} (m/m')^2} \\ &= \frac{1}{1 + 2\varepsilon} \cdot \frac{e_\ell(R'')}{2^{d-1} m^2} \cdot \tau_{\ell, d} \cdot (m')^2 \\ &\geq \frac{1}{1 + 2\varepsilon} \cdot (p_\ell - 4\varepsilon) \cdot \tau_{\ell, d} \cdot (m')^2 \\ &\geq (p_\ell - 6\varepsilon) \cdot \tau_{\ell, d} \cdot (m')^2. \end{aligned}$$

For each $x \in \{0, 1\}^d$, we independently pick a random vertex v_x from $\{x\} \times [m']$. Let $G = H[\{v_x : x \in \{0, 1\}^d\}]$ and note that G can be viewed as an ordered graph on $\{0, 1\}^d$. Furthermore, each edge of H is in G with probability $1/(m')^2$ and so

$$\mathbb{E} \left[\frac{1}{d} \sum_{\ell} \frac{e_\ell(G)}{\tau_{\ell, d}} \right] = \frac{1}{d} \sum_{\ell} \frac{e_\ell(H)}{\tau_{\ell, d} \cdot (m')^2} \geq \frac{1}{d} \sum_{\ell} (p_\ell - 6\varepsilon) \geq \alpha + \varepsilon.$$

Thus there is a choice of the v_x for which G satisfies $\frac{1}{d} \sum_{\ell} e_\ell(G) / \tau_{\ell, d} \geq \alpha + \varepsilon$. This G , by hypothesis, contains an ordered copy of F . This means that H contains F as an ordered subgraph with vertices in distinct blocks. Recall that the edges of H correspond to pairs of parts in $\mathcal{P}$ that are η -regular and have density at least $\varepsilon 2^{-d}$ in R' . By our choice of η , [Lemma 2.7](#) guarantees that R' contains a copy of F . Due to the way we ordered H and the fact that the vertices of F are all in distinct blocks, this copy of F in R' is ordered in the correct way. Thus R' contains F as an ordered subgraph, as required. $\square$

3.3 Proof of Theorems 1.1, 1.2, and 3.1

With the previous two sections in hand, we are ready to prove Theorems 1.1, 1.2, and 3.1. First note that Lemma 3.6 implies

$$\begin{aligned} & \lim_{d \rightarrow \infty} \lim_{m \rightarrow \infty} \rho_{<}(F, R(m, d)) \\ & \leq \inf \left\{ \alpha \in [0, 1] : \forall \text{ large } d \text{ every } G \text{ on } \{0, 1\}^d \text{ with } \frac{1}{d} \sum_{\ell} \frac{e_{\ell}(G)}{\tau_{\ell, d}} \geq \alpha \text{ contains } F \right\}. \end{aligned}$$

Now let $\alpha_0 = \inf \{ \alpha \in [0, 1] : \exists C \text{ such that every } (\alpha, C)\text{-rich graph contains } F \}$ and fix $\varepsilon > 0$. There is a constant C_0 such that every $(\alpha_0 + \varepsilon/2, C_0)$ -rich graph contains F . Suppose that G is an ordered graph on $\{0, 1\}^d$ which satisfies $\frac{1}{d} \sum_{\ell} e_{\ell}(G)/\tau_{\ell, d} \geq \alpha_0 + \varepsilon$. Then, since $e_{\ell}(G)/\tau_{\ell, d} \in [0, 1]$, at least an $\varepsilon/2$ -proportion of $\ell \in [d]$ satisfy $e_{\ell}(G)/\tau_{\ell, d} \geq \alpha_0 + \varepsilon/2$ and so G is $(\alpha_0 + \varepsilon/2, \varepsilon d/2)$ -rich. Provided d is sufficiently large, $\varepsilon d/2 \geq C_0$ and so G is $(\alpha_0 + \varepsilon/2, C_0)$ -rich and hence contains F . Thus,

$$\begin{aligned} & \inf \left\{ \alpha \in [0, 1] : \forall \text{ large } d \text{ every } G \text{ on } \{0, 1\}^d \text{ with } \frac{1}{d} \sum_{\ell} \frac{e_{\ell}(G)}{\tau_{\ell, d}} \geq \alpha \text{ contains } F \right\} \\ & \leq \inf \{ \alpha \in [0, 1] : \exists C \text{ such that every } (\alpha, C)\text{-rich graph contains } F \}. \end{aligned}$$

It remains to prove that

$$\inf \{ \alpha \in [0, 1] : \exists C \text{ such that every } (\alpha, C)\text{-rich graph contains } F \} \leq \rho_{<}(F), \quad (8)$$

since this would give the following chain of inequalities,

$$\begin{aligned} \rho_{<}(F) & \leq \lim_{d \rightarrow \infty} \lim_{m \rightarrow \infty} \rho_{<}(F, R(m, d)) \\ & \leq \inf \left\{ \alpha \in [0, 1] : \forall \text{ large } d \text{ every } G \text{ on } \{0, 1\}^d \text{ with } \frac{1}{d} \sum_{\ell} \frac{e_{\ell}(G)}{\tau_{\ell, d}} \geq \alpha \text{ contains } F \right\} \\ & \leq \inf \{ \alpha \in [0, 1] : \exists C \text{ such that every } (\alpha, C)\text{-rich graph contains } F \} \\ & \leq \rho_{<}(F), \end{aligned}$$

where the first inequality follows from $\rho_{<}(F) = \inf_G \rho_{<}(F, G) \leq \rho_{<}(F, R(m, d))$ for all m and d . Since $\rho_{<}(F)$ appears at both ends of the chain, there is equality throughout which gives the statements of Theorems 1.1, 1.2, and 3.1.

Proof of (8). Let $\varepsilon > 0$. It suffices to prove that there exists C such that every $(\rho_{<}(F) + \varepsilon, C)$ -rich graph contains F . By the definition of $\rho_{<}(F)$, there exists a graph H such that all subgraphs H' of H with at least $(\rho_{<}(F) + \varepsilon/4) \cdot e(H)$ edges contain an ordered copy of F . Let C be the constant produced by Lemma 3.2 for H and $\varepsilon/4$. Consider now an $(\rho_{<}(F) + \varepsilon, C)$ -rich graph G on $\{0, 1\}^d$ for some positive integer d and denote the set of rich levels of G by $\mathcal{L}$. It suffices to show that G contains an ordered copy of F .

Denote the complete graph on $V(G) = \{0, 1\}^d$ by K and let Φ be a random embedding of H into K as given by Lemma 3.2 applied to H , $\varepsilon/4$, and d . Let Z_{ℓ} be the set of pairs $xy \in \{0, 1\}^d$ with $\delta(x, y) = \ell$ that satisfy $\mathbb{P}(xy \in \Phi(H)) \geq (1 - \varepsilon/4)e(H)/(|\mathcal{L}|\tau_{\ell, d})$. Recall that $\tau_{\ell, d}$ is the number of pairs $xy \in \{0, 1\}^d$ with $\delta(x, y) = \ell$. The conclusion of

Lemma 3.2 implies that the set $\mathcal{L}'$ of $\ell \in \mathcal{L}$ for which $|Z_\ell| \geq (1 - \varepsilon/4)\tau_{\ell,d}$ has size at least $(1 - \varepsilon/4)|\mathcal{L}|$. Now

$$\begin{aligned} \mathbb{E}[e(G \cap \Phi(H))] &= \sum_{\ell=1}^d \sum_{xy \in E_\ell(G)} \mathbb{P}(xy \in \Phi(H)) \\ &\geq \sum_{\ell \in \mathcal{L}'} \sum_{xy \in E_\ell(G) \cap Z_\ell} \frac{(1 - \varepsilon/4) \cdot e(H)}{|\mathcal{L}| \tau_{\ell,d}} \\ &\geq \sum_{\ell \in \mathcal{L}'} (e_\ell(G) - \varepsilon \tau_{\ell,d}/4) \frac{(1 - \varepsilon/4) \cdot e(H)}{|\mathcal{L}| \tau_{\ell,d}}. \end{aligned}$$

But every level $\ell \in \mathcal{L}$ satisfies $e_\ell(G) \geq (\rho_{<}(F) + \varepsilon)\tau_{\ell,d}$ and so the above is at least

$$\sum_{\ell \in \mathcal{L}'} \frac{(\rho_{<}(F) + 3\varepsilon/4)(1 - \varepsilon/4) \cdot e(H)}{|\mathcal{L}|} \geq \sum_{\ell \in \mathcal{L}'} \frac{(\rho_{<}(F) + \varepsilon/2) \cdot e(H)}{|\mathcal{L}|},$$

which, as $|\mathcal{L}'| \geq (1 - \varepsilon/4)|\mathcal{L}|$, is at least $(\rho_{<}(F) + \varepsilon/4) \cdot e(H)$. Thus, there is an embedding of H into K such that $e(G \cap \Phi(H)) \geq (\rho_{<}(F) + \varepsilon/4) \cdot e(H)$. Now $G \cap \Phi(H)$ can be viewed as a subgraph of H and so, by the definition of H , it contains an ordered copy of F . Hence, G contains an ordered copy of F , as desired. $\square$

4 Nothing: relative ordered Turán at density zero

In this section we will prove [Theorem 1.3](#), which we recall here for the reader's convenience.

Theorem 1.3. *For every ordered graph F , $\rho_{<}(F) = 0$ if and only if F does not contain a monotone path of length two.*

It is useful to think of graphs F without monotone paths of length two as analogous to bipartite graphs for classical Turán densities. Indeed, since no vertex of F can have both neighbours smaller and larger than itself, we can partition $V(F)$ into two sets A and B which have no smaller and no larger neighbours, respectively. The family of such ordered graphs include all those with interval chromatic number two but is much larger (indeed it contains ordered graph of arbitrarily large interval chromatic number).

This structure implies that every ordered graph with no monotone path of length two embeds into some ordered graph H_k ($k \in \mathbb{N}$), which we define in [§ 4.1](#). Hence, to prove [Theorem 1.3](#) it suffices to show that $\rho_{<}(H_k) = 0$ for all k . Gishboliner showed this when $k = 2$, see [\[KLO⁺25, Conclusion\]](#). We prove it for all k in [§ 4.2](#).

4.1 The graph H_k

For a positive integer k , let H_k be the ordered graph with vertex set $[k] \times \{0, 1\}$ (ordered lexicographically) and edge set $\{(x, 0)(y, 1) : x \leq y\}$. See [Figure 2](#).

It is clear that H_k does not contain a monotone path of length two and that it has interval chromatic number $k + 1$. In particular, $\pi_{<}(H_k) = 1 - 1/k$ and so, subject to proving [Theorem 1.3](#), the sequence $(H_k)_{k \in \mathbb{N}}$ consists of ordered graphs with vanishing relative Turán density but Turán densities tending to one.

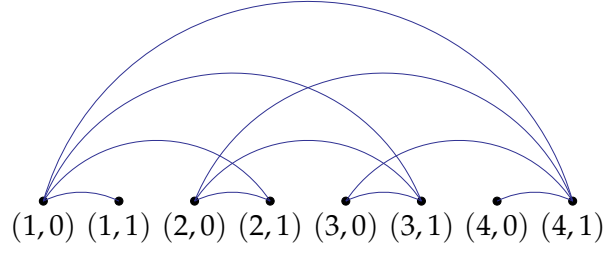


Figure 2. The ordered graph H_4

Lemma 4.1. *Every ordered graph F on k vertices that does not contain a monotone path of length two is an ordered subgraph of H_k .*

Proof. For each vertex v of F , let $\ell_{<}(v)$ be the length of the longest increasing ordered path in F that ends at v . Since F does not contain a monotone path of length two, each vertex v of F satisfies $\ell_{<}(v) \in \{0, 1\}$.

Let the vertices of F be $v_1 < \dots < v_k$. Define the function $f: V(F) \rightarrow V(H_k)$ by

$$f(v_i) = (i, \ell_{<}(v_i)),$$

for each i . We claim this is an ordered embedding of F into H_k . Due to the lexicographic order on $V(H_k)$, the function f is order-preserving. Next, let $v_i v_j$ ($i < j$) be an edge of F . We have $1 \geq \ell_{<}(v_j) \geq \ell_{<}(v_i) + 1 \geq 1$ and so $\ell_{<}(v_i) = 0$ and $\ell_{<}(v_j) = 1$. Therefore $f(v_i) = (i, 0)$ and $f(v_j) = (j, 1)$. By the definition of H_k , the vertices $f(v_i)$ and $f(v_j)$ are adjacent, as required. $\square$

This immediately yields the following corollary.

Corollary 4.2. *Every ordered graph F on k vertices that does not contain a monotone path of length two satisfies $\rho_{<}(F) \leq \rho_{<}(H_k)$.*

Corollary 4.2 reduces Theorem 1.3 to showing that $\rho_{<}(H_k) = 0$ for every k .

4.2 Proof of Theorem 1.3

The second part of the proof of Theorem 1.3 is to show that $\rho_{<}(H_k) = 0$ for all k . Given a graph G with many rich levels, we first set aside for each vertex all forward edges of the highest level in which the vertex has any forward edges.

Our next step, encapsulated in Lemma 4.3, is to find a vertex x and a level ℓ such that, for the forward level ℓ -neighbourhood Y of x , there are many backward edges from Y at levels lower than ℓ . Note that Y is contained in a fundamental interval I at level $\ell + 1$. What is more, the subgraph G' of $G[I]$ in which we remove all edges of G that are not backward edges from Y has again many rich levels. Therefore, we will be able to find an embedding ϕ of H_{k-1} into G' by induction.

Then, we use the forward edges of the highest level from x in G , which we set aside earlier, to find a neighbour y of x that is smaller than every vertex in I . Finally, we extend ϕ to an embedding of H_k by mapping $(1, 0)$ to x and mapping $(1, 1)$ to y .

Lemma 4.3. *For all $\varepsilon > 0$, there exists $\eta > 0$ such that the following holds for all $C \geq 12/\varepsilon$. Every (ε, C) -rich graph G contains a fundamental interval I , a vertex $x \in \text{Left}(I)$, and a subgraph G' of $G[\text{Right}(I)]$ such that G' is $(\eta, \varepsilon C/24)$ -rich and the larger vertex of every edge of G' is in $N^+(x)$.*

Proof. Let G be an (ε, C) -rich graph and let $\mathcal{L} \subseteq [d]$ be the set of ε -rich levels, so $|\mathcal{L}| \geq C$. For a vertex $y \in V(G)$, we write $d_{\ell, G}^-(y)$ for the backward level ℓ -degree of y , i.e. the number of $x \in V(G)$ such that $xy \in E(G)$, $x < y$, and $\delta(x, y) = \ell$. For $\ell \in \mathcal{L}$ and $y \in V(G)$, define $f(\ell, y) := d_{\ell, G}^-(y)/2^{d-\ell}$ and note that this is in $[0, 1]$. Now, since each $\ell \in \mathcal{L}$ is ε -rich, for every $\ell \in \mathcal{L}$,

$$\frac{1}{|G|} \sum_{y \in V(G)} f(\ell, y) = \frac{1}{|G|} \sum_{y \in V(G)} \frac{d_{\ell, G}^-(y)}{2^{d-\ell}} = \frac{e_\ell(G)}{2^{2d-\ell}} = \frac{1}{2} \frac{e_\ell(G)}{\tau_{\ell, d}} \geq \varepsilon/2.$$

Thus,

$$\frac{1}{|G|} \sum_{y \in V(G)} \left(\frac{1}{|\mathcal{L}|} \sum_{\ell \in \mathcal{L}} f(\ell, y) \right) \geq \varepsilon/2,$$

and so, since the parenthesised expression is always in $[0, 1]$, at least an $\varepsilon/6$ -proportion of $y \in V(G)$ have $|\mathcal{L}|^{-1} \sum_{\ell \in \mathcal{L}} f(\ell, y) \geq \varepsilon/3$. Denote the set of such y by Y_1 . Since $f(\ell, y) \in [0, 1]$, for every $y \in Y_1$, at least an $\varepsilon/6$ -proportion of $\ell \in \mathcal{L}$ satisfy $f(\ell, y) \geq \varepsilon/6$. For each $y \in Y_1$, let $\mathcal{L}_y$ be the set of such ℓ . Partition each $\mathcal{L}_y$ into two halves $\mathcal{L}_y^{\text{low}}$ and $\mathcal{L}_y^{\text{high}}$ where $\mathcal{L}_y^{\text{high}}$ consists of the largest $\lceil |\mathcal{L}_y|/2 \rceil$ elements of $\mathcal{L}_y$. Note that $|\mathcal{L}_y| \geq \varepsilon/6 \cdot |\mathcal{L}| \geq 2$ and so $|\mathcal{L}_y^{\text{low}}| \geq |\mathcal{L}_y|/3 \geq \varepsilon/18 \cdot |\mathcal{L}|$.

By averaging, there is some $\ell^* \in \mathcal{L}$ that is in at least an $\varepsilon/18$ -proportion of the $\mathcal{L}_y^{\text{low}}$ ($y \in Y_1$). Let Y_2 be the set of $y \in Y_1$ such that $\ell^* \in \mathcal{L}_y^{\text{low}}$. Note that $|Y_2| \geq \varepsilon/18 \cdot |Y_1| \geq \varepsilon^2/108 \cdot |G|$. Since the fundamental intervals at level ℓ^* partition $V(G)$, there is some $J \in \mathcal{P}_{\ell^*}$ for which $|Y_2 \cap J| \geq \varepsilon^2/108 \cdot |J|$. Let $I \in \mathcal{P}_{\ell^*-1}$ be the unique fundamental interval at level $\ell^* - 1$ which contains J . Every $y \in Y_2$ satisfies $d_{\ell^*, G}^-(y) \geq \varepsilon/6 \cdot 2^{d-\ell^*} > 0$ and so $J = \text{Right}(I)$. Furthermore, every $y \in Y_2 \cap J$ is adjacent (using level ℓ^* -edges) to at least an $\varepsilon/6$ -proportion of $\text{Left}(I)$. By averaging, there is a vertex $x \in \text{Left}(I)$ that is adjacent to at least an $\varepsilon/6$ -proportion of $Y_2 \cap J$.

Let Y_3 be the set of neighbours of x in $Y_2 \cap J$ and so $|Y_3| \geq \varepsilon/6 \cdot |Y_2 \cap J| \geq \varepsilon^3/648 \cdot |J|$. Every $\mathcal{L}_y^{\text{high}}$ ($y \in Y_3$) has size at least $|\mathcal{L}_y|/2 \geq \varepsilon/12 \cdot |\mathcal{L}|$. Thus, at least an $\varepsilon/24$ -proportion of $\ell \in \mathcal{L}$ are in at least an $\varepsilon/24$ -proportion of $\mathcal{L}_y^{\text{high}}$ ($y \in Y_3$). Let $\mathcal{L}' \subseteq \mathcal{L}$ be the set of such ℓ .

Now, for every $\ell \in \mathcal{L}'$, there is some $y \in Y_3$ with $\ell \in \mathcal{L}_y^{\text{high}}$. Since $y \in Y_3 \subseteq Y_2$, we have $\ell^* \in \mathcal{L}_y^{\text{low}}$ and so $\ell^* < \ell$. Thus $\mathcal{L}' \subseteq \mathcal{L} \cap (\ell^*, d]$.

Let G' be the subgraph of $G[\text{Right}(I)]$ induced by all edges of $G[\text{Right}(I)]$ whose larger vertex is in Y_3 (and so in $N^+(x)$). Note that G' is a graph on $\{0, 1\}^{d-\ell^*}$ and the number of a level in G' is ℓ^* less than the number of a level in G . We claim that, for every $\ell \in \mathcal{L}'$, the level $\ell - \ell^*$ is η -rich in G' , where $\eta = \varepsilon^5/46656$, and so G' is $(\eta, \varepsilon/24 \cdot C)$ -rich, as

desired. Indeed, for $\ell \in \mathcal{L}'$,

$$\begin{aligned}
e_{\ell-\ell^*}(G') &= \sum_{y \in Y_3} d_{\ell-\ell^*, G'}^-(y) = \sum_{y \in Y_3} d_{\ell, G}^-(y) \geq \sum_{\substack{y \in Y_3: \\ \ell \in \mathcal{L}_y^{\text{high}}}} d_{\ell, G}^-(y) \\
&\geq \sum_{\substack{y \in Y_3: \\ \ell \in \mathcal{L}_y^{\text{high}}}} \varepsilon/6 \cdot 2^{d-\ell} \geq \varepsilon/24 \cdot |Y_3| \cdot \varepsilon/6 \cdot 2^{d-\ell} = \varepsilon^2/144 \cdot |Y_3| \cdot 2^{d-\ell} \\
&\geq \varepsilon^2/144 \cdot \varepsilon^3/648 \cdot |J| \cdot 2^{d-\ell} = \eta/2 \cdot 2^{d-\ell^*} \cdot 2^{d-\ell}.
\end{aligned}$$

Now, $1/2 \cdot 2^{d-\ell^*} \cdot 2^{d-\ell} = 2^{2d-\ell-\ell^*-1} = 2^{2(d-\ell^*)-(\ell-\ell^*)-1} = \tau_{\ell-\ell^*, d-\ell^*}$, and so level $\ell - \ell^*$ is η -rich in G' , as desired. $\square$

With this in hand, we can prove [Theorem 1.3](#).

Proof of Theorem 1.3. By [Theorem 1.2](#) and [Corollary 4.2](#), it is enough to show that for any k and $\varepsilon > 0$, there exists C such that every (ε, C) -rich graph contains H_k . We prove this by induction on k , noting that the base case $k = 1$ is trivial. Therefore, assume that $k > 1$ and that the hypothesis is true for $k - 1$.

Let $\varepsilon > 0$, and apply [Lemma 4.3](#) with $\varepsilon_{4.3} = \varepsilon/2$ to obtain η . Now, let C_0 be sufficiently large such that every (η, C_0) -rich graph contains H_{k-1} , let $C := 24C_0/\varepsilon + 2/\varepsilon$, and let G be a (ε, C) -rich graph. Denote the ε -rich levels of G by $\mathcal{L}$.

For each $x \in V(G)$, let ℓ_x be the maximal $\ell \in [d]$ for which x is the smaller vertex of a level ℓ -edge (if x is isolated we let $\ell_x = 0$). We construct a subgraph $G_1 \subset G$ by removing for each vertex $x \in V(G)$, every (forward) edge at level ℓ_x to which x is incident. Writing p_ℓ for the proportion of vertices $x \in V(G)$ with $\ell_x = \ell$, we have $\sum_{\ell \in \mathcal{L}} p_\ell \leq 1$. Furthermore, the total number of edges that we remove from G at level ℓ is at most $p_\ell \tau_{\ell, d}$. Thus, the number of levels ℓ from which we remove more than $\varepsilon \tau_{\ell, d}/2$ edges is at most $2/\varepsilon$ and so G_1 is still $(\varepsilon/2, 24C_0/\varepsilon)$ -rich.

By our choice of η and [Lemma 4.3](#), there exists a fundamental interval I , a vertex $x \in \text{Left}(I)$ and an (η, C_0) -rich subgraph $G' \subset G_1[\text{Right}(I)]$ such that each vertex of G' with a backward neighbour in G' is in $N^+(x)$.

By the induction hypothesis, there is an embedding ϕ of H_{k-1} into G' , and moreover, the vertices $(1, 1), \dots, (k-1, 1)$ of H_{k-1} are all mapped into $N^+(x)$ by ϕ . Furthermore, since x is not isolated in G_1 , it was not isolated in G either. Thus, if I is a fundamental interval at level $a \in [d]$, then the edges between x and $\text{Right}(I)$ are at level $a + 1$, and $\ell_x > a + 1$. Hence, x has a neighbour $y \in \text{Left}(I)$, and in particular $y < \phi(v)$ for all $v \in H_{k-1}$.

Therefore, we can extend ϕ to an embedding of H_k into G by mapping $(1, 0) \in H_k$ to x , $(1, 1) \in H_k$ to y and $(i, j) \in H_k$ for $i \in \{2, \dots, k\}$ and $j \in \{0, 1\}$ to $\phi((i-1, j))$. This completes the proof. $\square$

5 In between: open problems

Every ordered graph with a monotone path of length two has relative Turán density at least $1/4$. Together with [Theorem 1.3](#) this implies that no ordered graph has a relative

Turán density in $(0, 1/4)$. Understanding the possible relative Turán densities would be very interesting. The corresponding problem for Turán densities in hypergraphs is a very well-known and difficult problem [Erd64, FR84, BT11, Pik14, CS25a, CS25b].

Problem 5.1. Understand the set $\{\rho_{<}(F) : F \text{ is an ordered graph}\}$.

Currently, the known members of this set are $1/2 \cdot (1 - 1/\ell)$ and $1 - 1/\ell$ ($\ell \in \mathbb{N}$) corresponding to F being a monotone path [RRSS25] and a clique, respectively.

Theorem 1.1 shows that the family $(R(m, d))_{m, d \in \mathbb{N}}$ is a universally optimal family of host graphs for the relative ordered Turán problem. Since the construction of $R(m, d)$ involves randomness, it is clear that the optimal family is not unique in a strict sense.

It is however possible, that any universally optimal family must satisfy its own version of **Lemma 3.2**. To make this precise, say that a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ is *universal* if all ordered graphs F satisfy $\rho_{<}(F) = \lim_{n \rightarrow \infty} \rho_{<}(F, G_n)$. For example, it readily follows from **Theorem 1.1** that if $(m_d)_{d \in \mathbb{N}}$ grows sufficiently fast, then $(R(m_d, d))_{d \in \mathbb{N}}$ is universal. Recall that one key observation⁴ behind the proof of **Theorem 1.1** was that every ordered graph can be embedded into $R(m, d)$ probabilistically in such a way that the edges of $R(m, d)$ are approximately equally likely to be hit. It is plausible that other universal sequences must also have this property.

Problem 5.2. Let $(G_n)_{n \in \mathbb{N}}$ be universal. Is it true that for all ordered graphs H , there exists a sequence of random embeddings $\Phi_n : H \rightarrow G_n$ such that

$$\lim_{n \rightarrow \infty} \left(\max_{e \in E(G_n)} \mathbb{P}(e \in \Phi_n(H)) - \frac{e(H)}{e(G_n)} \right) = 0?$$

Theorem 1.2 reduces the problem of proving $\rho_{<}(F, R(m, d)) \leq \alpha$ further to the more transparent problem of finding F in $(\alpha + \varepsilon, C)$ -rich graphs. However, our proof of this result uses the regularity lemma in such a way that makes it necessary to choose the parameter m in our host graph $R(m, d)$ as a tower of exponential height in $1/\varepsilon$.

We believe that it should be possible to bound the required m by a tower of the form $\exp^{o_{k_F}}(1/\varepsilon)$ where k_F depends only on F . Beyond that, other host graphs may be more efficient for some or all F and ε .

Problem 5.3. Given an ordered graph F and $\varepsilon > 0$, what is the minimum n such that there exists a graph G on n vertices for which $\rho(F, G) < \rho(F) + \varepsilon$.

In an upcoming paper, we will build on our work here, introducing new machinery for proving upper bounds on the relative Turán density of ordered graphs and finding explicit bounds for more families of graphs.

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⁴In **Lemma 3.2**, we avoided probabilistic technicalities by using the more concise language of rich levels, but a statement about $R(m, d)$ as expressed here can be shown using an essentially similar proof.

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A Auxiliary lemmas

Proof of Lemma 3.3. Let $\eta = \varepsilon/2k$, and note that $m = m(n) = \lfloor (\alpha - \eta)n \rfloor$ is much larger than k if n is large. Then, as n grows, we have

$$\begin{aligned}
\frac{\binom{m}{k}}{\binom{n}{k}} &= \frac{(1 - \mathcal{O}_k(1/m))m^k/k!}{(1 - \mathcal{O}_k(1/n))n^k/k!} \\
&= (1 - \mathcal{O}_k(1/n)) \left(\frac{m}{n}\right)^k \\
&\geq (1 - \mathcal{O}_k(1/n)) \left(\alpha - \eta - \frac{1}{n}\right)^k \\
&\geq \alpha^k - k\eta - \mathcal{O}_k(1/n),
\end{aligned}$$

which is at least $\alpha^k - \varepsilon$ when n is sufficiently large. □

Proof of Lemma 3.4. Let $\varepsilon > 0$ be fixed and let n be sufficiently large in terms of ε . Generate a uniformly random element $x = x_1 \dots x_n \in \{0, 1\}^n$ by independently sampling $x_i \in \text{Ber}(1/2)$ for each $i \in [n]$. For each interval $J \subset [n]$, consider the random variable $Z_J = \sum_{i \in J} x_i$ and note that $\mathbb{E}[Z_J] = \sum_{i \in J} \mathbb{P}(x_i = 1) = |J|/2$. Let $\mathcal{J}$ denote the set of integer intervals in n of size at least $\ln^2 n$. If $J \in \mathcal{J}$, then by a standard application of the Chernoff bound, we have

$$\mathbb{P}\left(\left|Z_J - \frac{|J|}{2}\right| \geq \varepsilon|J|\right) = \mathbb{P}(|Z_J - \mathbb{E}[Z_J]| \geq 2\varepsilon\mathbb{E}[Z_J]) \leq 2\exp\left(-\frac{2\varepsilon^2 \ln^2 n}{3}\right).$$

Note that for each $i \in [n]$ there are exactly $n - i + 1$ integer intervals in $[n]$ that start at i , and so in particular $|\mathcal{J}| \leq n(n+1)/2 \leq n^2$. Applying a union bound, we have $|Z_J - |J|/2| < \varepsilon|J|$ for all $J \in \mathcal{J}$ with probability at least $1 - n^2 \exp(-2\varepsilon^2 \ln^2 n/3) \geq 1 - \varepsilon$ for sufficiently large n . Therefore there exists at least a $(1 - \varepsilon)$ -proportion of elements of $\{0, 1\}^n$ for which $|\sum_{i \in J} x_i - |J|/2| < \varepsilon|J|$ holds for all $J \in \mathcal{J}$. This proves the lemma. □

Proof of Lemma 3.5. Throughout the proof η will be sufficiently small as a function of α , ε , and k , and n will be sufficiently large as a function of all other parameters. Let f , x and y be as in the claim, let $m = \lfloor \eta n \rfloor$, and let $t_0 = \lceil \sqrt{\eta n} \rceil$. We may assume

without loss of generality that $n - 2t_0 = Qm$ for some integer Q so that $(t_0, n - t_0] = (t_0, t_0 + m] \cup \dots \cup (t_0 + (h - 1)m, t_0 + Qm]$. Letting

$$g(t) = \binom{t-1}{x} \binom{n-t}{y},$$

we have that for all $t \in (t_0, n - t_0]$

$$\frac{g(t+1)}{g(t)} = \frac{t}{t-x} \frac{n-t-y}{n-t} \in \left(1 - \frac{y}{t_0}, 1 + \frac{x}{t_0 - x}\right).$$

Thus, we have for all $t_0 < s < t \leq n - t_0$ with $t - s \leq m$ that

$$\frac{g(t)}{g(s)} \geq \left(1 - \frac{y}{t_0}\right)^m \geq 1 - \frac{ym}{t_0} \geq 1 - \sqrt{\eta}y,$$

which is more than $1 - \varepsilon/2$ when η is sufficiently small. Furthermore, for sufficiently large n , we have that $x < t_0/2$ so that for all $t_0 < s < t \leq n - t_0$ with $t - s \leq m$

$$\frac{g(s)}{g(t)} \geq \left(1 - \frac{2x}{t_0}\right)^m \geq 1 - 2mx/t_0 \geq 1 - 2x\sqrt{\eta},$$

which is more than $1 - \varepsilon/2$ when η is sufficiently small. Therefore, we have

$$\begin{aligned} \sum_{t=t_0+1}^{n-t_0} f(t) \binom{t-1}{x} \binom{n-t}{y} &\geq \sum_{q=0}^{Q-1} \left(\min_{s \in (t_0+qm, t_0+(q+1)m]} \binom{s-1}{x} \binom{n-s}{y} \right) \sum_{t=t_0+qm+1}^{(q+1)m} f(t) \\ &\stackrel{(3)}{\geq} \sum_{q=0}^{Q-1} (\alpha - \eta)m \left(\min_{s \in (t_0+qm, t_0+(q+1)m]} \binom{s-1}{x} \binom{n-s}{y} \right) \\ &\geq \sum_{q=0}^{Q-1} (1 - \varepsilon/2)(\alpha - \eta) \sum_{t=t_0+qm+1}^{t_0+(q+1)m} \binom{t-1}{x} \binom{n-t}{y}, \end{aligned}$$

which is at least

$$(1 - 3\varepsilon/4)\alpha \sum_{t=t_0+1}^{n-t_0} \binom{t-1}{x} \binom{n-t}{y}. \quad (9)$$

The sum in (9) counts the number of sets $\{a_1 < \dots < a_{x+y+1}\} \subseteq [n]$ of size $x + y + 1$ such that a_{x+1} is in the interval $(t_0, n - t_0]$. Certainly, the number of such sets is at least the number of sets of size $x + y + 1$ that lie entirely in $(t_0, n - t_0]$, which is $\binom{n-2t_0}{x+y+1}$. But when η is sufficiently small and n sufficiently large, this is at least $(1 - \varepsilon/4)\binom{n}{x+y+1}$ by [Lemma 3.3](#), so that (9) is at least

$$(1 - \varepsilon)\alpha \binom{n}{x+y+1},$$

completing the proof. □