Friction in Stochastic Inflation

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Abstract. We solve time-reversed stochastic inflation in the semi-infinite flat potential with a constant drift term and derive an exact expression for the probability distribution of the curvature fluctuations. It exhibits exponential decaying tails which contrast to the Levy-like power law behaviour encountered without friction. Such a non-vanishing drift acts as a regulator for the conventional "forward" stochastic δN-formalism, which is otherwise ill-defined in the unbounded and flat potentials typical of plateau models of inflation. This setup therefore allows us to compare the curvature distribution derived from both approaches, reverse and forward in time. Up to similar exponential tails, we find quantitative differences. In particular, in the classical-like limit of very large drift, the tails become Gaussian but only in the time-reversed picture. As a toy model of eternal inflation, we finally discuss the case of negative drift in which inflation never ends for many field trajectories. The forward approach becomes pathological whereas the reverse formalism gives back a finite curvature distribution with always exponential tails. All these differences end up being related to the very definition of the background which is ambiguous when a classical trajectory does not exist.

Keywords: Time-reversed Stochastic Inflation, Semi-infinite potential, Constant friction

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1 Introduction

Within the inflationary paradigm, cosmological perturbations are of quantum mechanical origin, generated during the semi-classical regime of Cosmic Inflation [1–12]. In this regime, the space-time is classical and in a quasi-exponential accelerated expansion: the inevitable outcome of a self-gravitating scalar field having a non-vanishing potential energy [13–18]. On top of this homogeneous background, the field-metric system is experiencing small quantum fluctuations which are the seeds of the large scale structures of the universe [19–21]. As of today, this mechanism is one of simplest and most favoured explanation of the current cosmological observations and the preferred potential shapes are plateau-like [22–24].

Quantum diffusion refers to another possible regime of Cosmic Inflation, the one in which quantum fluctuations dominate over the classical dynamics. This may occur either when quantum fluctuations are large, i.e., when the Hubble parameter during inflation approaches the Planck scale, or, when the classical dynamics becomes irrelevant, as in nearly flat potentials. The expanding space-time can become strongly inhomogeneous and the concept of a classical background disappears [25–29]. Cosmological observations provide strong bounds on the inflationary dynamics, it has to be semi-classical on the observed length scales. However, it is still possible that quantum diffusion has something to do with the very large or the very small length scales [30–34]. Although not new, the topic has received renewed interest in the recent years as a mechanism to produce primordial black holes (PBHs) [35]. Indeed, a small flat domain within the field potential $V(\phi)$, located towards, or after, the end of the semi-classical regime is sufficient to create small scale curvature fluctuations which are large enough to collapse into PBHs [36–45].

When quantum fluctuations dominate, it is still possible to (approximately) describe the dynamics of the field and of the space-time by using the so-called stochastic inflation formalism [46–53]. It consists in considering the cumulated contribution of all sub-Hubble quantum fluctuations as a Gaussian white noise ξ acting on the super-Hubble dynamics of a coarse-grained inflaton field ϕ. For not too steep potentials, and out of the ultra slow-roll unstable regime, ϕ can be shown to follow a Langevin equation [54]

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{1}{3H^2}\frac{\mathrm{d}V}{\mathrm{d}\phi} + \frac{H}{2\pi}\xi(N), \tag{1.1}$$

where we have taken Planck units ($M_{\mathrm{Pl}} = 1$). Here $H(\phi)$ stands for the Hubble parameter during inflation, $N = \ln a$ is the number of e-folds, a being a scale factor for the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Up to the noise term, this equation is manifestly the same as the one of a homogeneous slow rolling scalar field but this is a non-trivial property coming from the fact that the time variable has been chosen, on purpose, to be the number of e-folds. Indeed, it is the only choice consistent with quantum field theory expectations [55, 56]. Stochastic realisations of the field trajectory drawn from Eq. (1.1) are now mapping the quantum histories of the field-metric system.

Within the so-called separate universe picture, stochastic realisations of Eq. (1.1) can also be associated with different regions of space-time which, due to the coarse-graining, are necessarily separated by super-Hubble distances. For super-Hubble scalar inhomogeneities, one can always recast the metric as $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}e^{2\zeta(\boldsymbol{x})}\,\mathrm{d}x^i\,\mathrm{d}x^j$ where $\zeta(\boldsymbol{x})$ is a conserved quantity [57–60]. As a result, defining the e-fold number $N(\boldsymbol{x},t) \equiv a(t)e^{\zeta(\boldsymbol{x})}$ in each of these comoving regions yields to seemingly-like homogeneous equations even though inhomogeneities are present. The δN-formalism is precisely based on this observation [57, 61–64]. Starting from a coordinate choice ensuring an initial flat slicing and evolving the field up to a uniform energy density hypersurface, curvature fluctuations on super-Hubble scales are given by $\zeta(\boldsymbol{x}) = N(\boldsymbol{x},t) - N_\flat$, where $N_\flat$ stands for a reference unperturbed number of e-folds. The stochastic δN-formalism generalises these results to stochastic inflation [65–69]. Because the field trajectories are stochastic, the number of e-folds $\mathcal{N}$ elapsed from an initial field value ϕ_0 to ϕ_{qw}, the field value on the "quantum wall" where quantum diffusion ends, is a stochastic variable and so it is for the "forward" curvature perturbation $\zeta_{\mathrm{fw}} \equiv \mathcal{N} - N_\flat$. Combined with Eq. (1.1), the stochastic δN-formalism allows for the determination of the statistical properties of ζ_{fw}. The formalism has been applied in the context of PBH formation, where it has been shown that a short period of stochastic inflation occurring within a flat (or quasi-flat) and bounded region of the potential, the so-called "quantum well", leads to an enhanced curvature probability distribution having exponential tails instead of the typical Gaussian statistics [70–73].

Let us remark that, in the aforementioned stochastic δN-formalism, the number of e-folds is counted forward, from an initial field state ϕ_0 to the end of quantum diffusion at ϕ_{qw}. In spite of its name, it therefore shares more resemblances with the so-called forward δn-formalism rather than with the genuine classical δN-formalism [74]. Indeed, curvature perturbations, as measured by observers attached to the end of inflation hypersurfaces, are given by the fluctuations of the number of e-folds counted backward, namely from the end of inflation to the initial state [62]. For instance, one can use the (backward) formalism to recover that the squeezed bispectrum measured by local observers exactly vanishes for single-field inflation [75]. Most of the time, these subtleties can be ignored as, in both schemes, the differences in e-fold numbers are tracing the differences in ζ. This is particularly true in the context of PBH where the formation criteria are associated with the compaction function which involves differential operators acting on ζ [76–78]. Another remark concerns the value of $N_\flat$. In practical applications, stochastic effects usually occur on top of a classical trajectory, and, in these situations, $N_\flat$ is taken as the classical number of e-folds. An exception to this case is the quantum well where the potential is exactly flat, but bounded [70]. In there, it seems natural to take $N_\flat = \langle\mathcal{N}\rangle$ which is a well defined quantity. Moreover, it leads to $\langle\zeta_{\mathrm{fw}}\rangle = 0$, which is satisfactory as the space-time over which quantum fluctuations are integrated out is considered to be globally flat (see Refs. [79–81] for non-vanishing curvature).

Current cosmological observations favour plateau-like models for the semi-classical regime of inflation. When extrapolated outside the observed cosmological window, plateau-models asymptote to a flat and unbounded potential shape in which stochastic inflation has to take place. As can be seen in Eq. (1.1), when $V(\phi)$ is exactly constant, the coarse-grained field is only driven by the noise term and one can show that $\langle\mathcal{N}\rangle = \infty$. The stochastic curvature fluctuation $\zeta_{\mathrm{fw}} = \mathcal{N} - \langle\mathcal{N}\rangle$ becomes ill-defined and no classical trajectory exists to set a privileged $N_\flat$. This reflects the fact that the system is being driven by pure quantum effects.

In Ref. [82], we have proposed a time-reversed approach to deal with these exactly flat and unbounded potentials. The main idea is to solve the stochastic inflation backward, from the end of the quantum diffusion regime to the initial state, i.e., imposing from the very beginning that observers have to be in the regions of the universe where inflation ended. As detailed in Ref. [82], time-reversing stochastic inflation happens to be equivalent to condition, and then to marginalise, the stochastic realisations of Eq. (1.1) by their lifetimes ΔN_0, the realisations of $\mathcal{N}$. Combined with the δN-formalism, this method has allowed us to derive an exact form for the probability distribution $P(\zeta|\phi_0)$, which ends up being normalisable with power-law tails $P(\zeta|\phi_0) \propto |\zeta|^{-3/2}$. Such a distribution does not have any finite moment, which is compatible with the fact that $\langle\mathcal{N}\rangle = \infty$ in the forward picture. Let us remark that conditioning the stochastic realisations by ΔN_0 avoids specifying the, here ill-defined, reference e-fold number $N_\flat$ to measure ζ. Instead, curvature fluctuations are calculated over all realisations producing the same lifetime, for all lifetimes. If we allow ourselves to link the time-reversal approach to the underlying quantum processes from which Eq. (1.1) is derived, we are somehow enforcing that curvature fluctuations are only measured once the quantum states have collapsed into an actual amount of expansion given by ΔN_0. In the absence of any classical background, this is quite satisfying as local observers have to be defined by some quantum observable and a measurement: the total amount of accelerated expansion. In other words, the observed background is now defined as one of its possible quantum realisations.

In this work, we solve time-reversed stochastic inflation for a flat semi-infinite potential in presence of a constant friction. This corresponds to a non-vanishing first term on the right-hand side of Eq. (1.1), which can be viewed as considering a small linear tilt of the potential. As shown below, the constant friction, or drift, term does not alter, at all, the time-reversed quantum diffusion but regularises the probability distribution of the lifetimes: $\langle\mathcal{N}\rangle < \infty$. Therefore, even if the system does not really have a classical background, a non-vanishing friction permits a comparison of the probability distribution for ζ obtained between the forward and time-reversed formalism.

The paper is organised as follow. In Section 2, we briefly recap the time-reversed stochastic inflation formalism, as developed in Ref. [82]. The time-reversed equations, in presence of a constant drift, are then derived and exactly solved. In Section 3, we calculate the probability distribution of the curvature fluctuations at given lifetime, and marginalised over all lifetimes. In Section 4, we quantitatively compare the probability distributions for ζ obtained in the time-reversed and forward approaches. Although they exhibit the same qualitative behaviour, exponential decaying tails multiplied by power-law, they do differ by order unity factors in the exponents. Other statistical properties, such as the mode and the asymmetry between positive and negative curvatures, are found to be always different. We finally discuss the extreme case of a negative drift for which eternal inflation occurs. The differences between the two approaches are exacerbated, the reverse picture predicting finite curvature distributions whereas the forward one ends up being pathological. We explain these results and draw our conclusion in Section 5.

2 Time-reversed stochastic inflation

2.1 General formalism

The coarse-grained field evolution of equation (1.1) is an Itô stochastic differential equation of the form

$$\mathrm{d}\phi = F[\phi(N), N]\,\mathrm{d}N + G[\phi(N), N]\,\mathrm{d}W\,, \tag{2.1}$$

where $\mathrm{d}W = \xi(N)\,\mathrm{d}N$ is a Wiener process, $F = -V_{,\phi}/(3H^2)$ is the so-called drift term and $G = H/(2\pi)$ the diffusion amplitude. The transition probability distribution associated with Eq. (2.1) is solution of the Fokker-Planck (or forward Kolmogorov) equation given by [83]

$$\frac{\partial P(\phi, N|\phi_0, N_0)}{\partial N} = -\frac{\partial}{\partial \phi}\left[F(\phi, N)P(\phi, N|\phi_0, N_0)\right] + \frac{1}{2}\frac{\partial^2}{\partial \phi^2}\left[G^2(\phi, N)P(\phi, N|\phi_0, N_0)\right], \tag{2.2}$$

where $N > N_0$ and $P(\phi, N_0|\phi_0, N_0) = \delta(\phi - \phi_0)$. Solving this equation with adequate boundary conditions gives the probability of finding field values ϕ at the time N starting from the state ϕ_0 at time N_0. In the semi-infinite potential, if we denote by ϕ_{qw} the field value at which quantum diffusion ends ($\phi_{\mathrm{qw}} \leq \phi_0$), it corresponds to an absorbing boundary condition $P(\phi_{\mathrm{qw}}, N|\phi_0, N_0) = 0$, referred to as the "quantum wall" in Ref. [82].

Time-reversed stochastic inflation describes the evolution of the same process but with ϕ emerging at $\phi = \phi_{\mathrm{qw}}$ and randomly evolving towards a sink located at $\phi = \phi_0$. The starting times have to be the ending times of the forward processes, i.e., the first passage times N_{qw} at which each realisation of Eq. (2.1) reaches the absorbing boundary at $\phi = \phi_{\mathrm{qw}}$. These random times are in one-to-one mapping with the lifetimes of the reverse processes. Denoting by $P_{\mathrm{LT}}(N_{\mathrm{qw}}|\phi_0, N_0)$ the probability distribution of the N_{qw}, the survival probability $S(N)$ at time N reads [67]

$$S(N|\phi_0, N_0) = \int_{\phi_{\mathrm{qw}}}^{+\infty} P(\phi, N|\phi_0, N_0)\,\mathrm{d}\phi = 1 - \int_{N_0}^{N} P_{\mathrm{LT}}(N_{\mathrm{qw}}|\phi_0, N_0)\,\mathrm{d}N_{\mathrm{qw}}\,. \tag{2.3}$$

Differentiating by N and using Eq. (2.2) gives

$$P_{\mathrm{LT}}(N|\phi_0, N_0) = -\frac{\partial S(N|\phi_0, N_0)}{\partial N} = \frac{1}{2}\left.\frac{\partial\left[G^2(\phi, N)P(\phi, N|\phi_0, N_0)\right]}{\partial \phi}\right|_{\phi_{\mathrm{qw}}}, \tag{2.4}$$

where regularity has been assumed at infinity. The time-reversed number of e-folds is defined by

$$\Delta N \equiv N_{\mathrm{qw}} - N, \tag{2.5}$$

ranging from $\Delta N = 0$, when the field is on the quantum wall, to $\Delta N_0 = N_{\mathrm{qw}} - N_0$ when it reaches the sink at ϕ_0. Let us stress that $\mathcal{N}$ of the forward formalism and $\Delta N_0 = N_{\mathrm{qw}} - N_0$ are the same quantities, the probability distribution of the lifetimes and first passage times are identical and given by Eq. (2.4).

As shown in Refs. [82, 84, 85], the time-reversed process follows an Itô stochastic equation which is completely characterised by the probability distribution $\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$, with $\Delta N \le \Delta N_0$, solution of the time-reversed Fokker-Planck equation

$$\begin{aligned}\frac{\partial \bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)}{\partial \Delta N} = &-\frac{\partial}{\partial \phi}\left[\bar{F}(\phi, \Delta N)\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)\right] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial \phi^2}\left[G^2(\phi, N)\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)\right],\end{aligned} \tag{2.6}$$

where the reversed drift term $\bar{F}(\phi, \Delta N)$ reads

$$\bar{F}(\phi, \Delta N) = -F(\phi, N) + \frac{1}{P(\phi, N|\phi_0, N_0)}\frac{\partial}{\partial \phi}\left[G^2(\phi, N)P(\phi, N|\phi_0, N_0)\right], \tag{2.7}$$

the diffusion coefficient G being unchanged. Notice the appearance of the forward transition probability $P(\phi, N|\phi_0, N_0)$ in this expression, requiring to first solve Eq. (2.2) before moving on determining $\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$ and thereby rendering the time-reversed approach technically more involved.

2.2 Quantum diffusion with friction

For a self-gravitating scalar field, the first Friedmann-Lemaître equation can be recast into [29, 86]

$$H^2 = \frac{V(\phi)}{3 - \epsilon_1}, \tag{2.8}$$

where the first Hubble flow function maps the field kinetic energy

$$\epsilon_1 = \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}N}\right)^2. \tag{2.9}$$

For an exactly flat semi-infinite potential at $\phi \ge \phi_{\mathrm{qw}}$, one has $V = V_0$ and the drift term F in Eq. (2.1) vanishes. The stochastic field ϕ is then driven by pure quantum diffusion with $G = H/(2\pi)$ and, provided $V_0 \ll 1$ (sub-Planckian values), one has $\epsilon_1 \ll 1$ and $H^2 \simeq H^2_{\mathrm{inf}} = V_0/3$ is constant. Time-reversed stochastic inflation for this case has been exactly solved in Ref. [82].

In order to tame the effects associated with pure quantum diffusion, we now consider a non-vanishing drift term F in Eq. (2.1). Such a term can appear if, instead of being exactly flat, the potential is slightly tilted at $\phi > \phi_{\mathrm{qw}}$. For instance, considering

$$V(\phi) = V_0\left(1 + \alpha\frac{\Delta\phi}{\Delta\phi_\infty}\right), \tag{2.10}$$

where α is a dimensionless number, $\Delta\phi \equiv \phi - \phi_{\mathrm{qw}}$ and $\Delta\phi_\infty = \phi_\infty - \phi_{\mathrm{qw}}$ is a (large) field excursion domain, one has

$$\frac{V_{,\phi}}{3H^2} \simeq \frac{V_{,\phi}}{V} \simeq \frac{\alpha}{\Delta\phi_\infty}, \tag{2.11}$$

provided $\Delta\phi \ll \Delta\phi_\infty/\alpha$. The previous approximation assumes that $2\epsilon_1 \simeq \alpha^2/\Delta\phi_\infty^2 \ll 1$, which also demands that $\Delta\phi_\infty/\alpha \gg 1$ (Planck units). When both conditions are satisfied, $V(\phi) \simeq V_0$ and $H^2 \simeq H_{\rm inf}^2 = V_0/3$ remains constant. To summarize, if α is of order unity, for a very large excursion domain $\Delta\phi_\infty$, and provided the stochastic field remains such that $\phi - \phi_{\rm qw} < \Delta\phi_\infty$, stochastic inflation is described by a Langevin equation with constant friction and diffusion terms

$$\mathrm{d}\phi = -\frac{\alpha}{\Delta\phi_\infty}\,\mathrm{d}N + \frac{H_{\rm inf}}{2\pi}\xi\,\mathrm{d}N\,. \tag{2.12}$$

This is the same equation as for the so-called "tilted quantum well" discussed in the context of PBH in Refs. [41, 71, 72]. However, because the quantum diffusion domain here is semi-infinite, as opposed to bounded, Eq. (2.12) breaks down if the stochastic field develops excursions larger than $\Delta\phi_\infty$. Indeed, neither $H(\phi)$ nor $V(\phi)$ can be considered constant in these cases. Nonetheless, and for reasons that will become clear later on, such a situation is under control in the time-reversed formalism as it ends up being dependent of the lifetimes. Moreover, it is always possible to take very large $\Delta\phi_\infty$ values, or, to transit to an exactly flat potential at $\Delta\phi > \Delta\phi_\infty/\alpha$.

From now on, we will be interested in solving Eqs. (2.2) and (2.6) with[1]

$$F = -\frac{\alpha}{\Delta\phi_\infty}\,, \qquad G = \frac{H_{\rm inf}}{2\pi}\,. \tag{2.13}$$

2.3 Forward transition probability

The forward transition probability $P(\phi, N|\phi_0, N_0)$ is the solution of Eq. (2.2) submitted to the absorbing boundary and initial conditions

$$P(\phi_{\rm qw}, N|\phi_0, N_0) = 0, \qquad P(\phi, N_0|\phi_0, N_0) = \delta(\phi - \phi_0)\,. \tag{2.14}$$

It can be obtained by a Fourier transform, using the superposition principle, or the method of images, to enforce the absorbing condition. One obtains, for $N \geq N_0$ and $\phi \geq \phi_{\rm qw}$,

$$P(\phi, N|\phi_0, N_0) = \frac{1}{\sqrt{2\pi}G\sqrt{N-N_0}}\left[e^{-\frac{\left(\phi-\phi_0+\alpha\frac{N-N_0}{\Delta\phi_\infty}\right)^2}{2G^2(N-N_0)}} - e^{-\frac{2\alpha(\phi_{\rm qw}-\phi_0)}{G^2\Delta\phi_\infty}} \times e^{-\frac{\left(\phi-2\phi_{\rm qw}+\phi_0+\alpha\frac{N-N_0}{\Delta\phi_\infty}\right)^2}{2G^2(N-N_0)}}\right]. \tag{2.15}$$

The probability distribution of the first-passage times $N_{\rm qw}$ at $\phi = \phi_{\rm qw}$ is given by Eq. (2.4). This is also the probability distribution of the lifetimes $\Delta N_0 = N_{\rm qw} - N_0$ and one gets

$$P_{\rm LT}(\Delta N_0|\phi_0) = P_{\rm LT}(N_{\rm qw}|\phi_0, N_0) = \frac{\Delta\phi_0}{\sqrt{2\pi}\,G\,\Delta N_0^{3/2}}\,e^{-\frac{\left(\Delta\phi_0-\alpha\frac{\Delta N_0}{\Delta\phi_\infty}\right)^2}{2G^2\Delta N_0}}, \tag{2.16}$$

[1]Notice that the friction term has the dimension of a mass, it reads $F = -\alpha M_{\rm Pl}^2/\Delta\phi_\infty$ with the Planck mass explicit.

where the initial field value is now in reference to the quantum wall

$$\Delta\phi_0 \equiv \phi_0 - \phi_{\rm qw}. \tag{2.17}$$

Taking $\alpha = 0$ in Eq. (2.16), one recovers the distribution of the lifetimes for the exactly flat and semi-infinite potential [82]. In particular, the large lifetime limit for $\alpha = 0$ gives $P_{\rm LT}(\Delta N_0|\phi_0) \propto \Delta N_0^{-3/2}$ and one recovers that $\langle \mathcal{N} \rangle = \langle \Delta N_0 \rangle = \infty$. On the contrary, for $\alpha > 0$, one gets an exponential behaviour at infinity ensuring the existence of all moments.

2.4 Time-reversal

Applying the time-reversal formalism detailed in Section 2.1, one needs to solve Eq. (2.6) with a reverse drift term given by Eq. (2.7). From the constancy of Eq. (2.13), one gets

$$\bar{F} = \frac{\alpha}{\Delta\phi_\infty} + G^2 \frac{\partial}{\partial\phi}\left[\ln P(\phi, N|\phi_0, N_0)\right], \tag{2.18}$$

where the forward transition probability distribution is given in Eq. (2.15). This one can be more conveniently expressed in terms of the time-reversed e-fold numbers ΔN and the field values in reference to the quantum wall

$$\Delta\phi \equiv \phi - \phi_{\rm qw}, \tag{2.19}$$

as

$$P(\phi, N|\phi_0, N_0) = \frac{e^{-\frac{\alpha^2(\Delta N_0 - \Delta N)^2 + 2\alpha\Delta\phi_\infty(\Delta\phi - \Delta\phi_0)}{2G^2\Delta\phi_\infty^2}}}{\sqrt{2\pi}\, G\sqrt{\Delta N_0 - \Delta N}} \left[e^{-\frac{(\Delta\phi - \Delta\phi_0)^2}{2G^2(\Delta N_0 - \Delta N)}} - e^{-\frac{(\Delta\phi + \Delta\phi_0)^2}{2G^2(\Delta N_0 - \Delta N)}} \right]. \tag{2.20}$$

Plugging Eq. (2.20) into Eq. (2.18), one obtains, after some algebra,

$$\begin{aligned} \bar{F}(\phi, \Delta N) &= -\frac{\Delta\phi - \Delta\phi_0}{\Delta N_0 - \Delta N} + \frac{2\Delta\phi_0}{\Delta N_0 - \Delta N} \frac{e^{-\frac{2\Delta\phi_0\Delta\phi}{G^2(\Delta N_0 - \Delta N)}}}{1 - e^{-\frac{2\Delta\phi_0\Delta\phi}{G^2(\Delta N_0 - \Delta N)}}} \\ &= -\frac{\Delta\phi}{\Delta N_0 - \Delta N} + \frac{\Delta\phi_0}{\Delta N_0 - \Delta N} \coth\left[\frac{\Delta\phi_0\Delta\phi}{G^2(\Delta N_0 - \Delta N)}\right]. \end{aligned} \tag{2.21}$$

There is no longer any term depending on α in this expression showing that the reverse drift $\bar{F}$ is independent of the forward (constant) drift. As a matter of fact, Eq. (2.21) is the same as the one derived in Ref. [82] [see Eq. (3.12)] for the flat semi-infinite potential. As a consequence, the solution of the time-reversed Fokker-Planck equation (2.6), satisfying the initial and boundary conditions

$$\bar{P}(\phi, \Delta N = 0|\phi_0, \Delta N_0) = \delta(\Delta\phi), \quad \bar{P}(\phi, \Delta N = \Delta N_0|\phi_0, \Delta N_0) = \delta(\Delta\phi - \Delta\phi_0), \tag{2.22}$$

is *exactly* the same as the one derived for the flat potential. Introducing the rescaled variables

$$\hat{\chi} \equiv \frac{\Delta\phi}{G\sqrt{\Delta N_0}}, \qquad \hat{\chi}_0 \equiv \frac{\Delta\phi_0}{G\sqrt{\Delta N_0}}, \qquad \tau \equiv \frac{\Delta N}{\Delta N_0}, \tag{2.23}$$

it reads [82, 87]

$$\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0) = \frac{\sqrt{2/\pi}}{\tau^{3/2}\sqrt{1-\tau}} \frac{\hat{\chi}}{\hat{\chi}_0 G\sqrt{\Delta N_0}} \sinh\left(\frac{\hat{\chi}\hat{\chi}_0}{1-\tau}\right) e^{-\frac{\hat{\chi}^2 + \tau^2\hat{\chi}_0^2}{2\tau(1-\tau)}}. \tag{2.24}$$

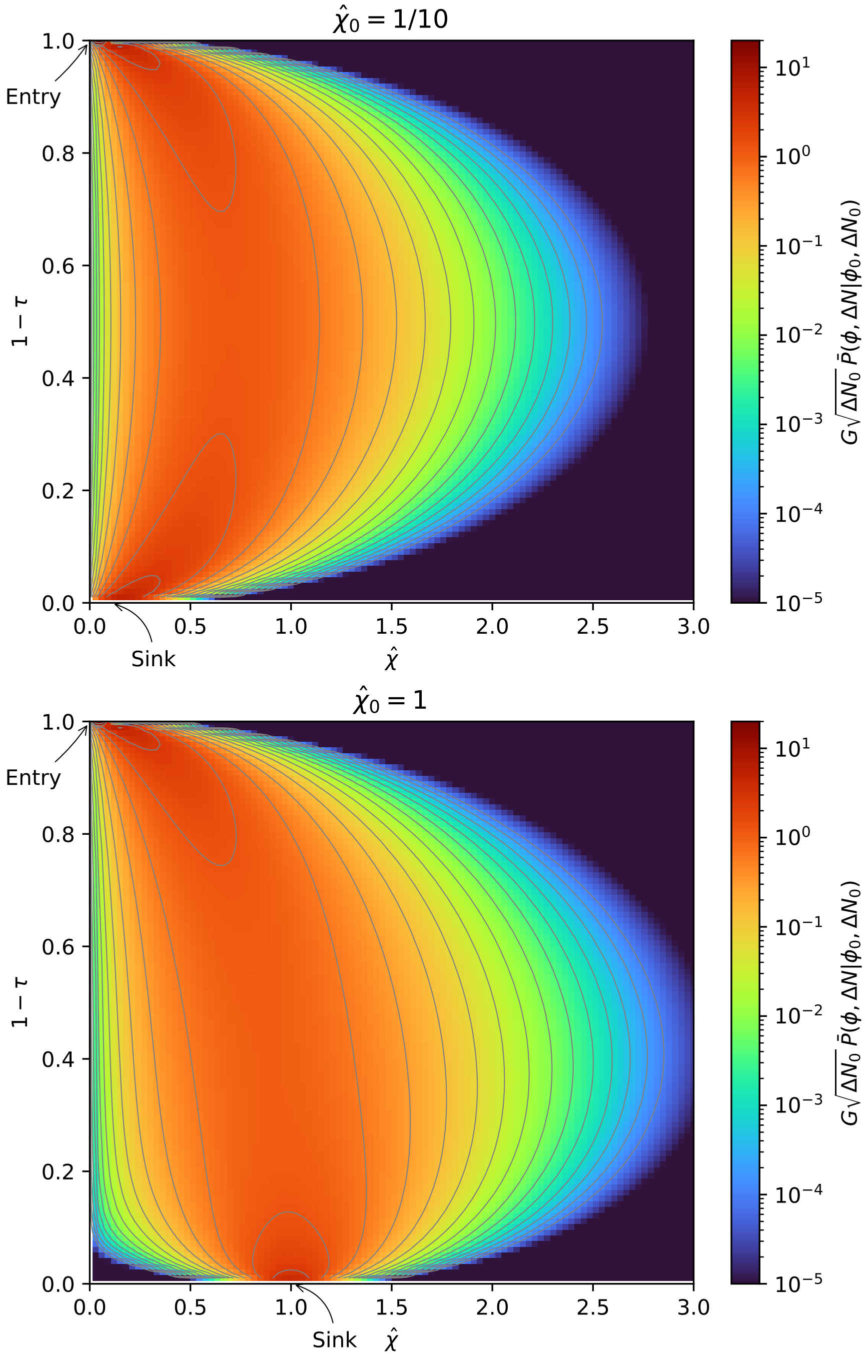

Figure 1. Contour plots of the rescaled reverse probability distribution $G\sqrt{\Delta N_0}\,\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$ as a function of $\hat{\chi} = \Delta\phi/(G\sqrt{\Delta N_0})$ and $\tau = \Delta N/\Delta N_0$ in the diffusion regime ($\hat{\chi}_0 \leq 1$). The field emerges at $\hat{\chi} = 0$ at time $\tau = 0$, diffuses in the domain before being absorbed at $\hat{\chi} = \hat{\chi}_0$ at $\tau = 1$.

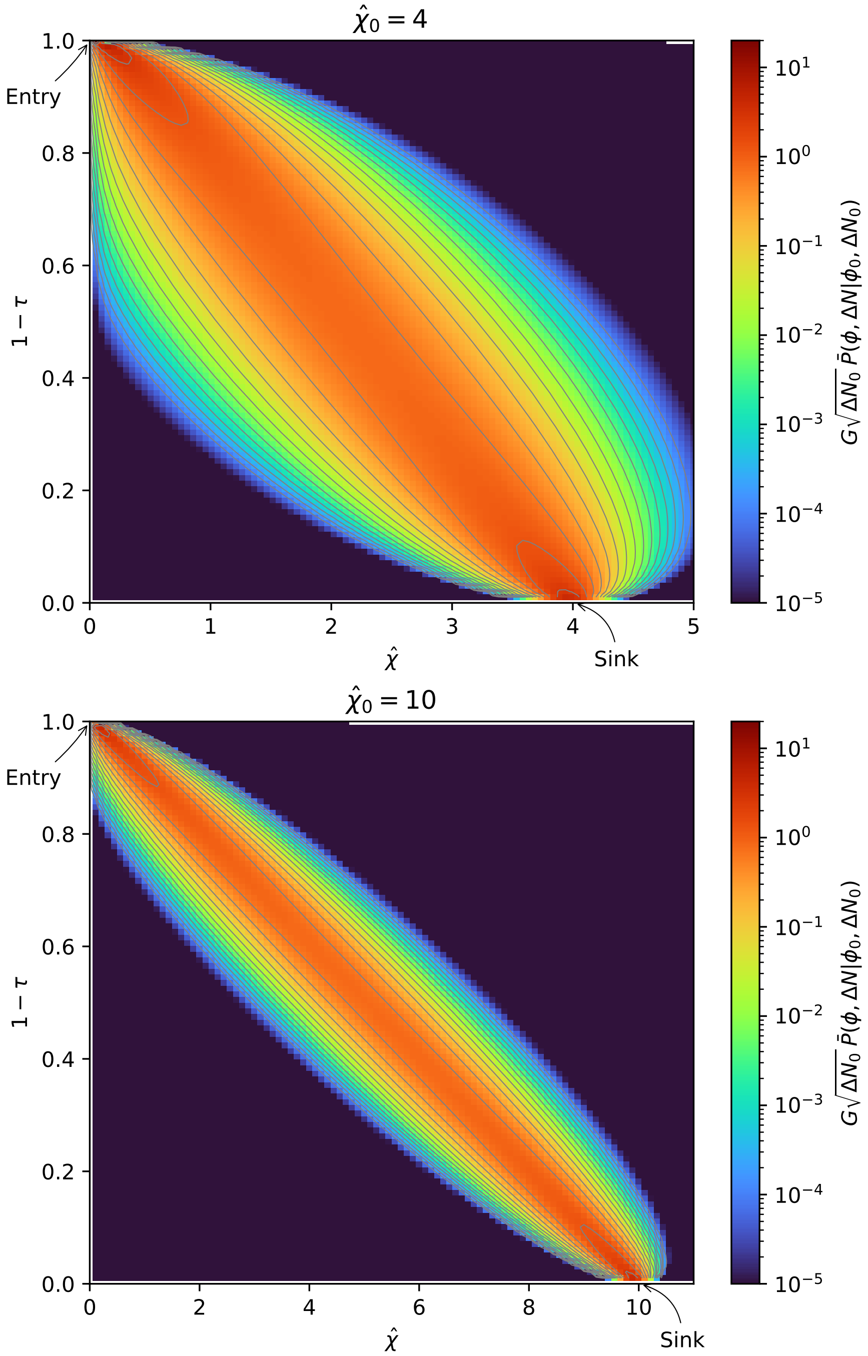

Figure 2. Contour plots of the rescaled reverse probability distribution $G\sqrt{\Delta N_0}\,\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$ as a function of $\hat{\chi} = \Delta\phi/(G\sqrt{\Delta N_0})$ and $\tau = \Delta N/\Delta N_0$ in the fluxing regime ($\hat{\chi}_0 > 1$). The field emerges at $\hat{\chi} = 0$ at time $\tau = 0$ and straightly jumps to $\hat{\chi} = \hat{\chi}_0$ at $\tau = 1$.

The new time variable $\tau \in [0,1]$ is the reverse e-fold number in unit of the lifetime ΔN_0. Similarly, $\hat{\chi}$ are field values, in reference to the quantum wall, in unit of the typical diffusion excursion domain, $G\sqrt{\Delta N_0}$, where the diffusion coefficient G is given by the Hubble parameter during inflation. As described at length in Ref. [82], Eq. (2.24) completely characterises time-reversed stochastic inflation: the field emerges at the quantum wall $\phi_{\rm qw}$, at time $\Delta N = 0$, and randomly evolves to reach the sink ϕ_0 at a reverse e-fold time ΔN matching the lifetime ΔN_0 of the forward process. As explicit in Eq. (2.24), the conditioning is with respect to ϕ_0 *and* to the lifetime ΔN_0. In other words, we have now sorted, and partitioned, all forward realisations in which stochastic inflation ends by ΔN_0. Let us remark that it should not come as a surprise that the constant drift we are considering here does not impact the reverse process. Indeed, conditioning a Brownian motion, or a drifted Brownian motion, on its final state is known to lead to the same process: the so-called Brownian bridge, which is thus independent of the original drift [87].

The rescaled reverse probability distribution $G\sqrt{\Delta N_0}\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$ has been represented in Fig. 1 in the diffusion dominated regime ($\hat{\chi}_0 \leq 1$) and in Fig. 2 for the fluxing regime ($\hat{\chi}_0 > 1$). In the diffusion dominated case, $\hat{\chi}_0 < 1$ implies that the sink ϕ_0 is located well within the typical excursion domain $G\sqrt{\Delta N_0}$ associated with quantum diffusion lasting ΔN_0 e-folds. Therefore, the reverse process has plenty of time to transit from the quantum wall to the sink and goes on visiting the surrounding with the extra time. On the contrary, for $\hat{\chi}_0 \gg 1$, one has $\Delta\phi_0 \gg G\sqrt{\Delta N_0}$ and solely the straightest stochastic realisations are selected: quantum diffusion is much reduced, this is the fluxing regime.

2.5 To drift or not to drift

As discussed in Section 2.2, the Langevin equation (2.12) with constant drift can be considered as a good approximation of the one stemming from the tilted potential of Eq. (2.10) but only within the domain $\Delta\phi \ll \Delta\phi_\infty/\alpha$ (provided that $\Delta\phi_\infty/\alpha \gg 1$). As can be seen in Figs. 1 and 2, this is completely under control and fixed by the value of $\hat{\chi}_0$, i.e., by both ϕ_0 and ΔN_0. Moreover, because the time-reversed process is the same as the one with $\alpha = 0$, one might assume that patching the tilted potential to an exactly flat semi-infinite plateau at $\Delta\phi = \Delta\phi_\infty/\alpha$ would still give the same reverse probability distribution $\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$ as in Eq. (2.24). This is not entirely correct as sudden transitions in the potential are known to trigger sub-Hubble quantum fluctuations different than the ones required for having a Gaussian white noise $\xi(N)$ [88, 89]. Nonetheless, for a smooth transition between the tilted potential and the exactly flat one at $\Delta\phi = \Delta\phi_\infty/\alpha$, it is reasonable to assume that $\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$ will be only slightly distorted in that region. In other words, the constant friction we are considering, and thus the linear tilt of the potential, has essentially only one effect which is to change the probability distribution of the lifetimes without affecting the strong quantum randomness of the system. This intuitively implies that considering a small tilt or no tilt at all does not drastically change the quantum dynamics in the semi-infinite potentials.

In the next section, we derive the probability distribution of the curvature fluctuations using the reverse-time stochastic δN-formalism and show that, up to the tails, it is indeed similar to one stemming from the flat semi-infinite potential.

3 Quantum-generated curvature distribution

In order to extract the curvature distribution, we apply the adapted version of the stochastic δN-formalism to the time-reversed process, as exposed in Ref. [82]. The reference $N_\flat$ is taken as the stochastic average within the reverse processes, and $\zeta = N - N_0 - \langle N - N_0 \rangle$ is promoted to a stochastic variable conditioned by the lifetime ΔN_0, i.e.,

$$\zeta = \langle \Delta N \rangle - \Delta N. \tag{3.1}$$

As represented in Figs. 1 and 2, at given lifetime ΔN_0, quantum diffusion allows for many different realisations and, for a given field value ϕ, ΔN is a stochastic quantity. As explicit in Eq. (3.1), ζ is measured by the fluctuations of the reverse number of e-folds around the mean, in a background defined by the lifetime ΔN_0 of the processes.

3.1 Curvature fluctuations at given lifetime

The reverse probability distribution $\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$ being identical to the one derived for the flat semi-infinite potential, the calculations required to determine the curvature fluctuations, at given ϕ_0 and ΔN_0, are the same as in Ref. [82]. As such, we only summarise below the main steps on how to obtain the probability distribution $P(\zeta|\phi_0, \Delta N_0)$.

As visible in Figs. 1 and 2, a given field value can be reached by many trajectories thereby promoting ΔN to a random variable. Therefore, one has first to determine $P(\Delta N|\phi, \phi_0, \Delta N_0)$, which is given by $\bar{P}(\phi, \Delta N|\phi_0, \Delta N_0)$ up to a normalisation factor which has to be calculated. This probability distribution allows us to calculate $\langle \Delta N \rangle = \langle \tau \rangle \Delta N_0$ and one gets [82]

$$\langle \tau \rangle = \Delta N_0 \int_0^1 \tau \, P(\tau \Delta N_0|\phi, \phi_0, \Delta N_0) \, \mathrm{d}\tau = \sqrt{\frac{\pi}{2}} \, \hat{\chi} \, e^{\frac{\hat{\chi}_0^2}{2}} \frac{\mathrm{erf}\left(\frac{2\hat{\chi} + \hat{\chi}_0}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{\hat{\chi} + |\hat{\chi} - \hat{\chi}_0|}{\sqrt{2}}\right)}{e^{-\hat{\chi}(|\hat{\chi} - \hat{\chi}_0| + \hat{\chi} - \hat{\chi}_0)} - e^{-2\hat{\chi}(\hat{\chi} + \hat{\chi}_0)}} . \tag{3.2}$$

From Eq. (3.1), the joint distribution $P(\phi, \zeta|\phi_0, \Delta N_0)$ is obtained by plugging the relation $\Delta N = \Delta N_0 \langle \tau \rangle - \zeta$ in Eq. (2.24). Defining the curvature fluctuation in unit of the lifetime

$$\hat{\zeta} \equiv \frac{\zeta}{\Delta N_0} , \tag{3.3}$$

one obtains

$$\begin{aligned} P\left(\phi, \hat{\zeta}|\phi_0, \Delta N_0\right) = & \frac{1}{G\sqrt{\Delta N_0}} \frac{1}{\sqrt{2\pi}} \frac{\hat{\chi}}{\hat{\chi}_0} \frac{\Theta\left(\langle \tau \rangle - \hat{\zeta}\right) - \Theta\left(\langle \tau \rangle - \hat{\zeta} - 1\right)}{\left(\langle \tau \rangle - \hat{\zeta}\right)^{3/2} \sqrt{1 - \langle \tau \rangle + \hat{\zeta}}} \\ & \times \exp\left\{ -\frac{\left[\hat{\chi} - \left(\langle \tau \rangle - \hat{\zeta}\right) \hat{\chi}_0\right]^2}{2\left(\langle \tau \rangle - \hat{\zeta}\right)\left(1 - \langle \tau \rangle + \hat{\zeta}\right)} \right\} \left[1 - \exp\left(-\frac{2\hat{\chi}\hat{\chi}_0}{1 - \langle \tau \rangle + \hat{\zeta}}\right)\right] . \end{aligned} \tag{3.4}$$

The Heaviside functions appearing in this expression come from the reverse process enforcing that $0 < \Delta N < \Delta N_0$, which gets translated into $0 < \langle \tau \rangle - \hat{\zeta} < 1$ in the δN-formalism. Because $\langle \tau \rangle$ is a (complicated) function of $\hat{\chi}$, see Eq. (3.2), these functions act as complicated windows on field values selecting only the domains compatible with the given curvature.

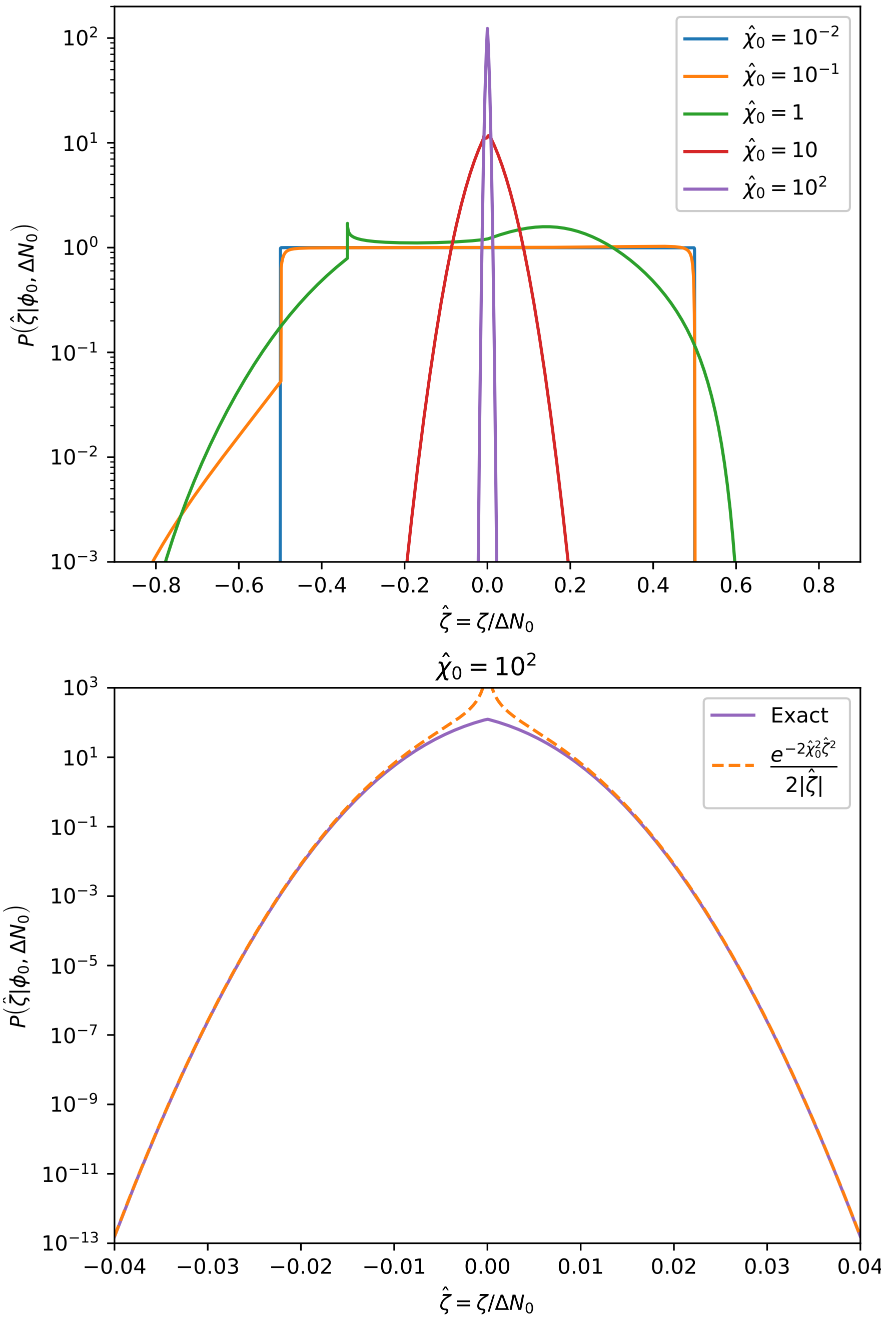

Figure 3. Probability distribution of the rescaled curvature fluctuations $\hat{\zeta} = \zeta/\Delta N_0$, at given lifetime ΔN_0, as a function of its sole parameter $\hat{\chi}_0 = \Delta\phi_0/(G\sqrt{\Delta N_0})$ (top panel). It asymptotes to the rectangle function of Eq. (3.7) in the large diffusion limit (small $\hat{\chi}_0$), and to a Dirac-like distribution in the fluxing regime (large $\hat{\chi}_0$). In between, it develops a complicated shape, some visible kicks coming from the field-domain selection functions appearing in Eq. (3.4). The bottom panel is a zoom on the distribution in the fluxing regime, for $\hat{\chi}_0 = 10^2$. The tails of the distribution are becoming Gaussian and decay as $e^{-2\chi_0^2\hat{\zeta}^2}/(2|\hat{\zeta}|)$.

Finally, a marginalisation over all field values gives the probability distribution of $\hat{\zeta}$ given ϕ_0 and ΔN_0

$$P\left(\hat{\zeta}|\phi_0, \Delta N_0\right) = \int_0^{+\infty} P\left(\phi, \hat{\zeta}|\phi_0, \Delta N_0\right) \mathrm{d}\Delta\phi = G\sqrt{\Delta N_0} \int_0^{+\infty} P\left(\phi, \hat{\zeta}|\phi_0, \Delta N_0\right) \mathrm{d}\hat{\chi} \,. \quad (3.5)$$

Let us stress that $\hat{\zeta}$ in these expressions is the stochastic variable. From Eq. (3.3), one has

$$P(\zeta|\phi_0, \Delta N_0) = \frac{1}{\Delta N_0} P\left(\hat{\zeta}|\phi_0, \Delta N_0\right). \quad (3.6)$$

Moreover, the functional forms of Eqs. (3.2) and (3.4) imply that $P\left(\hat{\zeta}|\phi_0, \Delta N_0\right)$ is a function of $\hat{\chi}_0$ only. The integral over $\hat{\chi}$ in Eq. (3.5) has to be performed numerically and we have plotted the results in Fig. 3. However, in the diffusion regime displayed in Fig. 1, one has $\hat{\chi}_0 \ll 1$ and Eq. (3.2) implies that $\langle\tau\rangle \to 1/2$. Plugging $\langle\tau\rangle = 1/2$ in Eq. (3.4), the integral of Eq. (3.5) is explicit and reads [82]

$$P\left(\hat{\zeta}|\phi_0, \Delta N_0\right) \underset{\langle\tau\rangle\to 1/2}{=} \Theta\left(\frac{1}{2} - \hat{\zeta}\right) - \Theta\left(-\frac{1}{2} - \hat{\zeta}\right). \quad (3.7)$$

In the pure diffusion limit, all values of $-1/2 < \hat{\zeta} < 1/2$ are equiprobable, and so it is for $-\Delta N_0/2 < \zeta < \Delta N_0/2$. This is confirmed by the numerical results displayed in Fig. 3, but could equally well be spotted by eyes in the top panel of Fig. 1.

As evident in Fig. 3, in the fluxing regime, for $\hat{\chi}_0 \gg 1$, the curvature fluctuations are much reduced and the probability distribution for $\hat{\zeta}$ approaches a Dirac-like distribution. In fact, it is possible to show that, in the limit of large $\hat{\chi}_0$, one has $\langle\tau\rangle \simeq \hat{\chi}/\hat{\chi}_0$. Plugging this approximation into Eq. (3.4), the various integrals appearing in Eq. (3.5) can be analytically performed[2] to show that

$$\lim_{\hat{\chi}_0|\hat{\zeta}|\to\infty} P\left(\hat{\zeta}|\chi_0, \Delta N_0\right) \simeq \frac{e^{-2\hat{\chi}_0^2\hat{\zeta}^2}}{2|\hat{\zeta}|}. \quad (3.8)$$

The tails of the curvature distribution at given lifetime are becoming Gaussian. The approximation of Eq. (3.8) has been represented in the bottom panel of Fig. 3, for $\hat{\chi}_0 = 100$, together with the distribution obtained from an exact numerical integration.

3.2 Marginalisation over lifetimes

The results summarised in the previous section are identical to the one associated with the exactly flat semi-infinite potential. The friction term modifying only the probability distribution of the lifetimes, it enters into the marginalisation of $P(\zeta|\phi_0, \Delta N_0)$ over ΔN_0. Explicitly, using Eq. (3.6), one has

$$P(\zeta|\phi_0) = \int_0^{+\infty} P\left(\hat{\zeta}|\phi_0, \Delta N_0\right) \frac{P_{\mathrm{LT}}(\Delta N_0|\phi_0)}{\Delta N_0} \mathrm{d}\Delta N_0 \,, \quad (3.9)$$

where the distribution of the lifetimes is given in Eq. (2.16). We should remember that $P\left(\hat{\zeta}|\phi_0, \Delta N_0\right) = P\left(\hat{\zeta}|\hat{\chi}_0\right)$ ends up being a function of $\hat{\chi}_0$ only, which suggests to change

[2] See Section 3.2 and appendix B of Ref. [82] on how to deal with these integrals.

the integration variable from ΔN_0 to $\hat{\chi}_0 = \Delta\phi_0/(G\sqrt{\Delta N_0})$ (at fixed $\Delta\phi_0$). One has to pay attention to the argument of this function, which is

$$\hat{\zeta} = \frac{\zeta}{\Delta N_0} = \frac{\zeta}{\chi_0^2} \times \hat{\chi}_0^2, \tag{3.10}$$

where we have defined

$$\chi_0 \equiv \frac{\Delta\phi_0}{G}, \tag{3.11}$$

the initial field value, in reference to the quantum wall, in unit of the quantum diffusion G. After some algebra, Eq. (3.9) reads[3]

$$P(\zeta|\phi_0) = \frac{e^{\hat{\alpha}}}{\chi_0^2}\sqrt{\frac{2}{\pi}} \int_0^{\frac{\chi_0}{\sqrt{|\zeta|}}} P\left(\frac{\zeta}{\chi_0^2}\hat{\chi}_0^2 \middle| \hat{\chi}_0\right) \hat{\chi}_0^2 e^{-\frac{\hat{\chi}_0^2}{2} - \frac{\hat{\alpha}^2}{2\hat{\chi}_0^2}} \,\mathrm{d}\hat{\chi}_0\,, \tag{3.12}$$

in which we have introduced the rescaled drift coefficient[4]

$$\hat{\alpha} \equiv \frac{\alpha}{G^2}\frac{\Delta\phi_0}{\Delta\phi_\infty} = \frac{\alpha}{G}\frac{\chi_0}{\Delta\phi_\infty}\,. \tag{3.13}$$

Without performing any calculations, it is clear that, up to the $e^{\hat{\alpha}}$ normalisation factor, the friction term $\hat{\alpha}$ acts as a regulator for $\hat{\chi}_0$ small (diffusion regime) while having minimal impact for $\hat{\chi}_0 \gg 1$ (fluxing regime). Moreover, Eq. (3.12) shows that $P(\zeta|\phi_0)$ is a function of $\hat{\alpha}$ and ζ/χ_0^2 only.

Before turning to the numerical integration of Eq. (3.12), let us derive an analytical approximation for the diffusion regime. This is the limit $\hat{\chi}_0 \ll 1$, which will be enforced as soon as $|\zeta|/\chi_0^2 \gg 1$, i.e., in the tails of the probability distribution for ζ. Plugging Eq. (3.7) into Eq. (3.12), defining the new integration variable $z = \hat{\chi}_0^2/\chi_0^2 = 1/\Delta N_0$, one gets

$$P(\zeta|\phi_0) \underset{|\zeta|\gg\chi_0^2}{=} \frac{\chi_0 e^{\hat{\alpha}}}{\sqrt{2\pi}} \mathcal{I}_1\left(x = \frac{1}{2|\zeta|}, \beta = \frac{\hat{\alpha}^2}{2\chi_0^2}, \gamma = \frac{\chi_0^2}{2}\right), \tag{3.14}$$

where

$$\mathcal{I}_n(x,\beta,\gamma) \equiv \int_0^x z^{n-\frac{1}{2}}\, e^{-\frac{\beta}{z} - \gamma z} \,\mathrm{d}z\,. \tag{3.15}$$

The family of definite integrals $\mathcal{I}_n(x,\beta,\gamma)$ can be analytically calculated from another family of known integrals (see appendix B of Ref. [82])

$$I_n(\beta,\gamma) \equiv \int_0^1 z^{n-\frac{1}{2}}\, e^{-\frac{\beta}{z} - \gamma z} \,\mathrm{d}z\,. \tag{3.16}$$

A change of variable gives

$$\mathcal{I}_n(x,\beta,\gamma) = x^{n+\frac{1}{2}} I_n\left(\frac{\beta}{x}, \gamma x\right), \tag{3.17}$$

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[3] The upper bound comes from the window functions of Eq. (3.4) enforcing $0 < \langle\tau\rangle - \hat{\zeta} < 1$ with $\langle\tau\rangle \in [0,1]$.

[4] Notice that $\hat{\alpha}$ is dimensionless, restoring the Planck mass, it indeed reads $\hat{\alpha} = \frac{\alpha M_{\mathrm{Pl}}^2}{G^2}\frac{\Delta\phi_0}{\Delta\phi_\infty}$.

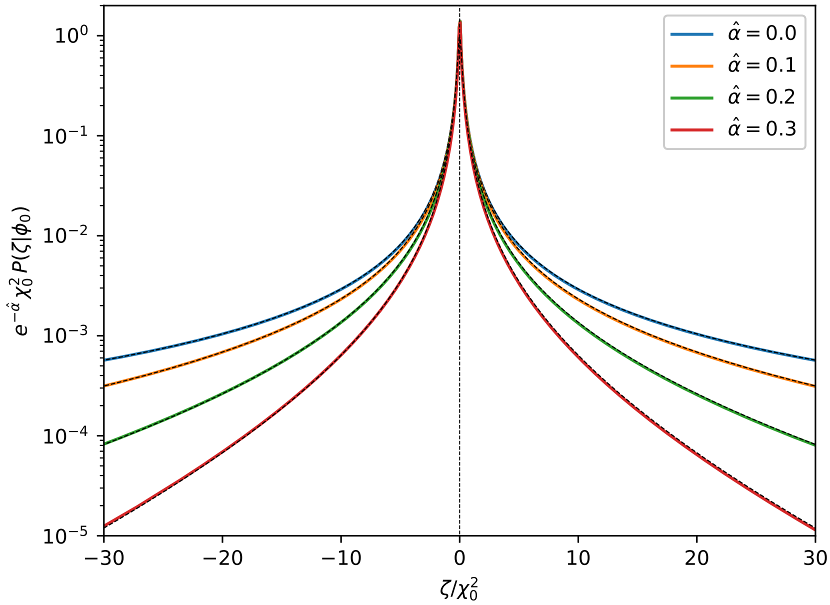

Figure 4. Probability distribution of the curvature fluctuations ζ (solid lines) as a function of ζ/χ_0^2 for various values of the rescaled drift coefficient $\hat{\alpha}$, defined in Eq. (3.13). The dashed lines are the analytical approximation of Eq. (3.19), obtained in the large diffusion limit. The tails are asymptotically exponentials. See Fig. 5 for a zoom within the domain of small $|\zeta|/\chi_0^2$.

from which one gets [90]

$$\mathcal{I}_1(x,\beta,\gamma) = -\frac{\sqrt{x}e^{-\frac{\beta}{x}-\gamma x}}{\gamma} + \frac{\sqrt{\pi}\left(1+2\sqrt{\beta\gamma}\right)}{4\gamma\sqrt{\gamma}}e^{-2\sqrt{\beta\gamma}}\,\mathrm{erfc}\left(\sqrt{\frac{\beta}{x}}-\sqrt{\gamma x}\right) - \frac{\sqrt{\pi}\left(1-2\sqrt{\beta\gamma}\right)}{4\gamma\sqrt{\gamma}}e^{2\sqrt{\beta\gamma}}\,\mathrm{erfc}\left(\sqrt{\frac{\beta}{x}}+\sqrt{\gamma x}\right), \tag{3.18}$$

where erfc( ) is the complementary error function. Plugging this expression into Eq. (3.14), one finally obtains

$$\chi_0^2 e^{-\hat{\alpha}} P(\zeta|\phi_0) \underset{|\zeta|\gg\chi_0^2}{=} \frac{\hat{\alpha}+1}{2}e^{-\hat{\alpha}}\,\mathrm{erfc}\left(\hat{\alpha}\frac{\sqrt{|\zeta|}}{\chi_0}-\frac{\chi_0}{2\sqrt{|\zeta|}}\right) + \frac{\hat{\alpha}-1}{2}e^{\hat{\alpha}}\,\mathrm{erfc}\left(\hat{\alpha}\frac{\sqrt{|\zeta|}}{\chi_0}+\frac{\chi_0}{2\sqrt{|\zeta|}}\right) - \frac{1}{\sqrt{\pi}}\frac{\chi_0}{\sqrt{|\zeta|}}e^{-\frac{\chi_0^2}{4|\zeta|}-\hat{\alpha}^2\frac{|\zeta|}{\chi_0^2}}. \tag{3.19}$$

A consistency check can be made by considering the limit of vanishing drift, $\hat{\alpha} = 0$, for which we recover, as it should, the large lifetime limit for the flat semi-infinite potential [82]

$$\chi_0^2 P(\zeta|\phi_0) \underset{|\zeta|\gg\chi_0^2}{=} \mathrm{erf}\left(\frac{\chi_0}{2\sqrt{|\zeta|}}\right) - \frac{\chi_0}{\sqrt{|\zeta|}}\frac{e^{-\frac{\chi_0^2}{4|\zeta|}}}{\sqrt{\pi}}. \tag{3.20}$$

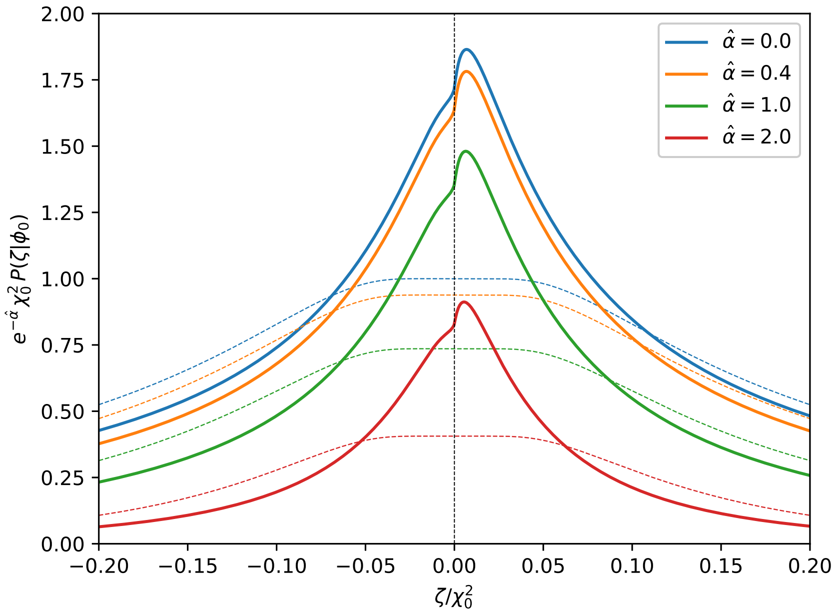

Figure 5. Probability distribution of the curvature fluctuations ζ (solid lines) as a function of ζ/χ_0^2 for various values of the rescaled drift $\hat{\alpha}$. Up to a change of amplitude, the shape of the distribution at small curvature is similar to the the one of the flat semi-infinite potential ($\hat{\alpha} = 0$). It is skewed and the mode is at a positive small value which depends on $\hat{\alpha}$ (see Fig. 6). Despite the asymmetry, one has $\langle\zeta\rangle = 0$. The dashed lines are the analytical approximation of Eq. (3.19) providing a good fit of the tails (see Fig. 4).

In Fig. 4, we have plotted $P(\zeta|\phi_0)$ as a function of ζ/χ_0^2, for various values of $\hat{\alpha}$, obtained from a numerical integration of Eq. (3.12) (solid lines). The large diffusion approximation, Eq. (3.19), is represented as dashed lines and both distributions are undistinguishable in the tails. The probability distribution for ζ in the region of small ζ/χ_0^2 is represented in Fig. 5, for various values of $\hat{\alpha}$ (solid lines). As expected, up to a change in amplitude, the shape is mostly the same as for the flat semi-infinite potential ($\hat{\alpha} = 0$). In particular, the distribution is skewed and the mode $\zeta_{\mathrm{mode}}/\chi_0^2$ is at a positive small value which depends on $\hat{\alpha}$. The dependence of $\zeta_{\mathrm{mode}}/\chi_0^2$ with respect to $\hat{\alpha}$ has been computed and is represented in Fig. 6. It goes to zero in the large drift limit as $\hat{\alpha}^{-2}$. Notice that, even though the large diffusion approximation does not capture these features, the width of its plateau matches the typical spread of the exact distribution (see dashed curves in Fig. 5).

In order to better understand the behaviour of $P(\zeta|\phi_0)$ in the tails, we can further expand Eq. (3.19) in the limit of $\hat{\alpha}\sqrt{|\zeta|}/\chi_0 \gg 1$. After some tedious algebra, one obtains[5]

$$\chi_0^2 e^{-\hat{\alpha}} P(\zeta|\phi_0) \simeq \frac{3}{16\sqrt{\pi}\,\hat{\alpha}^2}\left(\frac{\chi_0}{\sqrt{|\zeta|}}\right)^5 e^{-\hat{\alpha}^2\frac{|\zeta|}{\chi_0^2}}. \tag{3.21}$$

[5]Expanding the erfc(z) functions of Eq. (3.19) at large z, while paying attention to keep the dependence in $\chi_0/\sqrt{|\zeta|}$ in the arguments, all terms up to order three cancel. The expansion has to be pushed up to the fifth order to get a non-vanishing leading order term (a well-known computer algebra software fails in doing this expansion correctly).

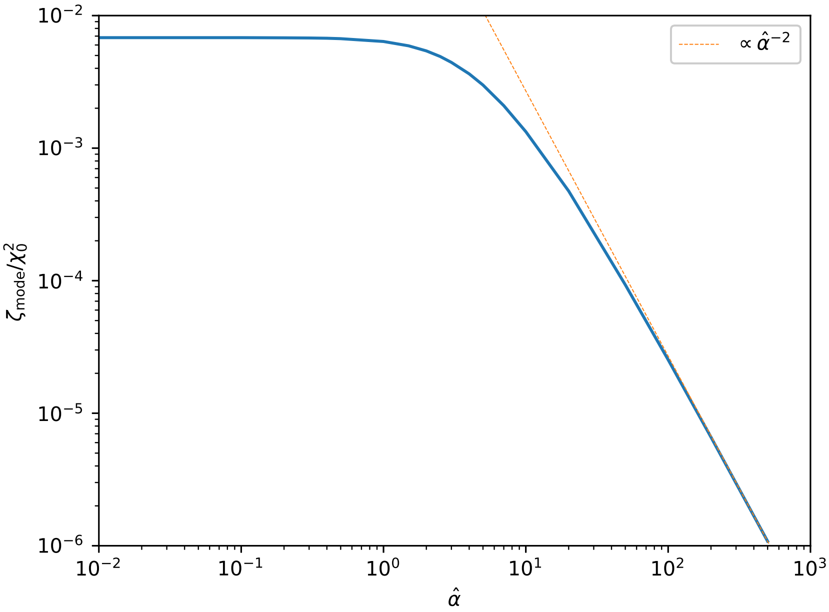

Figure 6. Numerical evaluation of the curvature at maximum probability, plotted as a function of the rescaled drift coefficient $\hat{\alpha}$. At small $\hat{\alpha}$, $\zeta_{\rm mode}/\chi_0^2 \simeq 6.9 \times 10^{-3}$, which is the value obtained for the flat semi-infinite potential without friction. At large values, $\zeta_{\rm mode} \to 0$ with a power law behaviour in $\hat{\alpha}^{-2}$.

As such, the asymptotic behaviour of the curvature distribution is a decaying exponential multiplied by the power-law $|\zeta|^{-5/2}$. Contrary to the flat semi-infinite case, the presence of the exponential factor ensures that all moments of $P(\zeta|\phi_0)$ are finite.

4 Discussion

4.1 Comparison with the forward formalism

In the forward stochastic δN-formalism, the curvature fluctuations are defined in reference to the mean lifetime, i.e., $\zeta_{\rm fw} = \mathcal{N} - \langle\mathcal{N}\rangle$ where $\mathcal{N}$ denotes the stochastic variable having for realisations the lifetimes ΔN_0. As highlighted in the Introduction, without the friction term $\langle\mathcal{N}\rangle = \infty$ and the forward formalism cannot be applied. However, in the setup considered in the present paper, the constant drift term regularises the divergence and enables a comparison of the probability distribution for ζ obtained between the forward and the time-reversed formalism.

Recapping that $P_{\rm LT}(\Delta N_0|\phi_0)$ can also be interpreted as the first passage time distribution for the forward process, the average of $\mathcal{N}$ is

$$\langle\mathcal{N}\rangle = \int_0^{+\infty} \Delta N_0 P_{\rm LT}(\Delta N_0|\phi_0)\,{\rm d}\Delta N_0 = \frac{\chi_0}{\sqrt{2\pi}} \int_0^{+\infty} \frac{1}{\sqrt{\Delta N_0}} e^{-\frac{1}{2}\left(\frac{\chi_0}{\sqrt{\Delta N_0}} - \frac{\hat{\alpha}\sqrt{\Delta N_0}}{\chi_0}\right)^2} {\rm d}\Delta N_0\,. \tag{4.1}$$

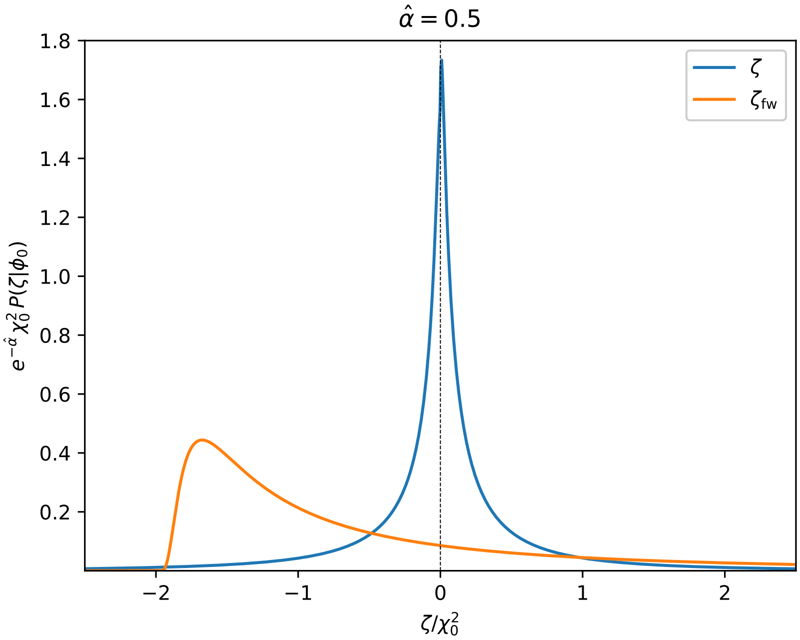

Figure 7. Comparison between the curvature probability distribution obtained from the time-reversed formalism (blue curve) and the one derived from the forward formalism (orange curve), for $\hat{\alpha} = 0.5$. The positive tails are power-law times exponential decay but with different exponents (see text). Notice that $P(\zeta_{\rm fw}|\phi_0)$ is exactly vanishing at $\zeta_{\rm fw}/\chi_0^2 = -1/\hat{\alpha}$ and undefined below.

The same change of variable as in Section 3.2, $z = 1/\Delta N_0$, allows us to recast this integral as

$$\langle \mathcal{N} \rangle = \frac{\chi_0 e^{\hat{\alpha}}}{\sqrt{2\pi}} I_0\left(\frac{\chi_0^2}{2}, \frac{\hat{\alpha}^2}{2\chi_0^2}\right) = \frac{\chi_0^2}{\hat{\alpha}}\,, \tag{4.2}$$

where $I_0(\beta,\gamma)$ is defined in Eq. (3.16). From $P(\zeta_{\rm fw}|\phi_0) = P_{\rm LT}(\zeta_{\rm fw} + \langle \mathcal{N} \rangle\,|\phi_0)$, the forward stochastic δN-formalism gives

$$\chi_0^2 e^{-\hat{\alpha}} P(\zeta_{\rm fw}|\phi_0) = \frac{\hat{\alpha}^{3/2}}{\sqrt{2\pi}\left(1 + \hat{\alpha}\dfrac{\zeta_{\rm fw}}{\chi_0^2}\right)^{3/2}}\, e^{-\frac{\hat{\alpha}}{2}\left(1 + \frac{1}{1+\hat{\alpha}\frac{\zeta_{\rm fw}}{\chi_0^2}}\right)} e^{-\hat{\alpha}^2 \frac{\zeta_{\rm fw}}{2\chi_0^2}}. \tag{4.3}$$

In the positive limit $\zeta_{\rm fw}/\chi_0^2 \gg 1$, the probability distribution behaves as

$$\chi_0^2 e^{-\hat{\alpha}} P(\zeta_{\rm fw}|\phi_0) \simeq \frac{1}{\sqrt{2\pi}} \left(\frac{\zeta_{\rm fw}}{\chi_0^2}\right)^{-3/2} e^{-\hat{\alpha}^2 \frac{\zeta_{\rm fw}}{2\chi_0^2}}\,, \tag{4.4}$$

a decaying exponential multiplied by a power-law. The exponential term is actually identical to the one of the tilted quantum-well in the so-called "wide" limit [41, 70, 71] whereas the power-law factor $\zeta_{\rm fw}^{-3/2}$ comes from the fact that the potential is, here, unbounded[6].

[6]For unbounded potentials, the poles of the characteristic function are not forming a discrete set as they do in the bounded case. As such, one cannot find pure exponential decaying behaviour for the associated probability density distribution.

However, the power-law index is $-3/2$ instead of $-5/2$ found in the time reverse approach. Also, the exponential decay rate has a factor of two difference compared to the one of Eq. (3.21). Another important difference is the strong asymmetry of Eq. (4.3) with respect to the sign of ζ_{fw}. Indeed, for $\zeta_{\mathrm{fw}} < 0$, $P(\zeta_{\mathrm{fw}}|\phi_0)$ behaves like a bump function, it is exactly vanishing at $\zeta_{\mathrm{fw}}/\chi_0^2 = -1/\hat{\alpha}$ and undefined below. Indeed, from the very definition of ζ_{fw}, $\mathcal{N}$ being a positive quantity, one always has $\zeta_{\mathrm{fw}} \geq -\langle\mathcal{N}\rangle$. The mode of $P(\zeta_{\mathrm{fw}}|\phi_0)$ is at a negative value, which can be quite large for $\hat{\alpha}$ small:

$$\left.\frac{\zeta_{\mathrm{fw}}}{\chi_0^2}\right|_{\mathrm{mode}} = -\frac{6}{\hat{\alpha}\left(3 + 2\hat{\alpha} + \sqrt{9 + 4\hat{\alpha}^2}\right)}. \tag{4.5}$$

Both $P(\zeta|\phi_0)$ and $P(\zeta_{\mathrm{fw}}|\phi_0)$ are represented in Fig. 7. Therefore, for the case of a flat unbounded potential with constant friction, the time-reversed and forward approaches do not predict the same curvature distribution, even in the tails.

This is not really surprising. The time-reversed formalism enforces a conditioning by the lifetime first, which defines our background as being one of its possible quantum realisations. As discussed in the introduction, this conditioning actually sets what are the cosmological observers in a quantum system. They are the ones measuring a total amount of accelerated expansion, i.e. a realisation of $\mathcal{N}$ having for value ΔN_0. We use the δN-formalism only to derive the curvature fluctuations at given lifetime, namely $P\left(\hat{\zeta}|\phi_0, \Delta N_0\right)$. Then, in a last step, the curvature fluctuations are weighted by the probability to be one of these observers, namely, by the probability of the lifetimes given in Eq. (2.16).

Taking average and marginalising over lifetimes do not commute, and that is why the time-reversed approach leads to a finite result for $\hat{\alpha} = 0$ whereas the forward formalism becomes ill-defined. The forward approach assumes that all of the curvature fluctuations are generated from the randomness of the lifetimes and the distribution of $P(\zeta_{\mathrm{fw}}|\phi_0)$ is actually only mapping $P_{\mathrm{LT}}(\Delta N_0|\phi_0)$, up to a constant shift. From Eqs. (3.5) and (3.9), one can actually reinterpret the forward result within the time-reversed formalism as postulating a precise relation for $P\left(\hat{\zeta}|\phi_0, \Delta N_0\right)$ which would be

$$P\left(\hat{\zeta}_{\mathrm{fw}}|\phi_0, \Delta N_0\right) \equiv \delta\left(\hat{\zeta}_{\mathrm{fw}} - 1 + \frac{\langle\mathcal{N}\rangle}{\Delta N_0}\right). \tag{4.6}$$

Such a distribution would imply that no quantum fluctuations has to be present at given lifetime, which is not the case for the flat potential under scrutiny here, as evident in Figs. 1 and 2. We should nevertheless stress that, even if the non-vanishing drift term allows us to regularise the forward formalism, the system remains highly non-classical because the potential is still exactly flat. As discussed in the introduction, and in a way similar to the bispectrum measured by local observers, it is very well possible that the differences between the time-reversed and the forward formalism would become negligible in a classical background. However, at the time of writing, we do not have solutions of time-reversed stochastic inflation in non-flat, or even bounded, potentials.

4.2 Infinite drift limit

Although a proper classical limit cannot be defined within a flat potential, quantum fluctuations at given lifetime happen to be much reduced in the fluxing regime (see Section 2.4).

It is possible to force a very large fluxing by taking the limit of infinite drift, i.e., $\hat{\alpha} \to +\infty$. Indeed, Eq. (3.12) also reads

$$P(\zeta|\phi_0) = \frac{1}{\chi_0^2}\sqrt{\frac{2}{\pi}}\int_0^{\frac{\chi_0}{\sqrt{|\zeta|}}} P\left(\frac{\zeta}{\chi_0^2}\hat{\chi}_0^2\,\middle|\,\hat{\chi}_0\right)\hat{\chi}_0^2 e^{-\frac{\left(\hat{\chi}_0^2-\hat{\alpha}\right)^2}{2\hat{\chi}_0^2}}, \tag{4.7}$$

showing that, for $\hat{\alpha} \gg 1$, the exponential term is acting as a window function which is non-vanishing only for $\hat{\chi}_0^2 = \hat{\alpha}$. One can use either a saddle point, or a Dirac distribution, approximation for the exponential to obtain

$$P(\zeta|\phi_0) \simeq \frac{\hat{\alpha}}{\chi_0^2} P\left(\hat{\zeta} = \hat{\alpha}\frac{\zeta}{\chi_0^2}\,\middle|\,\hat{\chi}_0 = \sqrt{\hat{\alpha}}\right), \tag{4.8}$$

provided $|\zeta| < \hat{\alpha}^2\chi_0^2$ (and zero otherwise). Because $\hat{\chi}_0 = \sqrt{\hat{\alpha}}$, the infinite drift limit implies that we must also consider the infinite fluxing limit for the curvature distribution at given lifetime, i.e., Eq. (3.8). We therefore get a distribution with Gaussian tails

$$\lim_{\hat{\alpha}\to+\infty} P(\zeta|\phi_0) = \frac{1}{2|\zeta|} e^{-2\frac{\hat{\alpha}^3}{\chi_0^4}\zeta^2}. \tag{4.9}$$

In contrast, keeping only the dominant terms, the distribution of ζ_{fw} in the same limit asymptotes to a one-sided exponential distribution. From Eq. (4.3), one obtains

$$\lim_{\hat{\alpha}\to+\infty} P(\zeta_{\mathrm{fw}}|\phi_0) = \frac{\chi_0^3\,\Theta(\zeta_{\mathrm{fw}})}{\sqrt{2\pi}\zeta_{\mathrm{fw}}^{3/2}} e^{-\frac{\hat{\alpha}^2}{2\chi_0^2}\zeta_{\mathrm{fw}}}, \tag{4.10}$$

which still has the same exponential tails as the large $\zeta_{\mathrm{fw}}/\chi_0^2$ limit, and remains quite different from Eq. (4.9). The Heaviside distribution arises from the vanishing of $\langle\mathcal{N}\rangle$ when $\hat{\alpha} \to +\infty$, as explicit in Eq. (4.2). The infinite drift therefore acts as a classical-like limit but only for the reverse process. Once more, this is not surprising as the forward formalism is not sensitive to the fluctuations at given lifetime and, therefore, ζ_{fw} cannot be affected by the quantum fluctuations occurring, or not, at a given realisation of $\mathcal{N}$.

4.3 Negative drift

In order to further illustrate the differences between the forward and time-reversed approaches, we can consider another quite extreme case where the drift $\hat{\alpha} < 0$. In the regime where the friction term is approximately modelling a slightly tilted potential, as in Eq. (2.10), a negative drift would represent a potential tilted down at larger field values, allowing for field trajectories to never reach the end of quantum diffusion. Indeed, from Eqs. (2.3) and (2.16), one can derive the survival probability for non-vanishing friction. Its asymptotic limit reads

$$\lim_{N\to\infty} S(N) = \Theta(-\hat{\alpha})\left(1 - e^{-2|\hat{\alpha}|}\right). \tag{4.11}$$

It is non-vanishing for $\hat{\alpha} < 0$, and, for large negative values, only an exponentially suppressed fraction of all the realisations succeeds in crossing $\phi = \phi_{\mathrm{qw}}$.

For the forward formalism, such a situation is more pathological than the flat semi-infinite potential: the drift is pushing the field ϕ away from ϕ_{qw}. For instance, Eq. (4.2) implies that the mean number of (forward) e-folds is negative whereas Eq. (4.3) is only defined for $\zeta_{\rm fw} > -\chi_0^2/\hat{\alpha}$, which can be very large for small negative $\hat{\alpha}$. This would suggest that only very large positive curvature fluctuations can be generated for an infinitely small and negative drift.

For time-reversed stochastic inflation, Eq. (3.12) shows that only the term in $e^{\hat{\alpha}}$ behaves differently than in the case where $\hat{\alpha} > 0$, all other terms depending on $\hat{\alpha}^2$. As such, $P(\zeta|\phi_0)$ remains well-defined, has the same exponential tails as in Eq. (3.21), and, only the overall amplitude is reduced by the $e^{\hat{\alpha}}$ factor. However, Eq. (4.7) shows that, for negative drift, the Gaussian limit no longer exists. As discussed at length in Ref. [82], all of this is expected. The conditioning by the lifetimes ensures that we only count realisations in which inflation has ended. For negative drift, many realisations are actually inflating forever and never reach the quantum wall. But the time-reversed formalism only picks up the trajectories reaching the wall, which explains the finiteness of the curvature distribution, as well as the presence of the overall exponential damping factor $e^{\hat{\alpha}}$ in Eq. (3.12). Moreover, the realisations escaping the quantum region must be driven by quantum kicks, jumping over the stream set by $\hat{\alpha} < 0$. It justifies why the Gaussian limit exhibited in Eq. (4.9) cannot exist anymore, since it is precisely the diffusion that has to dominate in order to produce the kicks required for inflation to end.

5 Conclusion

In this work, we have solved time-reversed stochastic inflation for the flat semi-infinite potential in presence of a constant drift term. The stochastic δN-formalism, applied to the reverse process, has allowed us to derive the probability distribution $P(\zeta|\phi_0)$ of the curvature fluctuations in Eq. (3.12), which is plotted in Figs. 4 and 5. The drift term, encoded in the parameter $\hat{\alpha}$, appears to regularise the tails of the distribution $P(\zeta|\phi_0)$. It exhibits an exponential decay which contrasts to the Levy-like power-law tails studied in Ref. [82] for the driftless case $\hat{\alpha} = 0$. Considering the limit of infinite drift, we have found in Eq. (4.9) that $P(\zeta|\phi_0)$ develops Gaussian tails, which are typical of semi-classical processes. The large drift limit indeed corresponds to the large fluxing limit of the reverse process for which quantum fluctuations at given lifetime are much reduced.

Another interest of considering a constant drift term, in a flat semi-infinite potential, is that it regularises the forward formalism which is otherwise ill-defined. The curvature distribution is there identified with the fluctuations of the lifetime and $P(\zeta_{\rm fw}|\phi_0)$ has been derived in Eq. (4.3). Up to some similarity with the time-reversed result, as having exponential tails, it is quantitatively different, as apparent in Fig. 7. It is strongly asymmetric, behaving like a bump function for negative curvatures, and does not reach a Gaussian-like distribution in the infinite drift limit. As discussed in the introduction, both approaches differ by counting the stochastic number of e-folds either forward, from the initial state ϕ_0 to the absorbing wall at $\phi_{\rm qw}$, or reverse, from the final state $\phi_{\rm qw}$ towards the initial state ϕ_0. In the semi-classical limit, and slow-roll, it is known that both methods can lead to results differing by first order slow-roll corrections, which are therefore always small. Here slow-roll does not exist, the potential being exactly flat, quantum kicks are the driving dynamics of the system and they are only tampered by the added friction term. It is therefore not surprising that we find quite different results between the forward and reverse picture. It is possible that these differences become sub-dominant in more classical setups. Obtaining solutions for time-reversed stochastic inflation in more classical systems than the one presented in this work would be a way to confirm this hint, albeit technically challenging.

Let us also stress that the exponential positive tail of $P(\zeta|\phi_0)$ and $P(\zeta_{\rm fw}|\phi_0)$ happens to differ by a factor of two in the exponent. Such a factor may be quite important would we want to predict abundances of PBH in another context. In the time-reversed picture, one can trace this factor as coming from the fact that curvature fluctuations, at given lifetime ΔN_0, are in reference to $\langle \Delta N \rangle$. As such, they have a tendency to be centered on $\Delta N_0/2$, as can be seen in Fig. 1. On the contrary, in the forward picture, fluctuations at given lifetime are not accounted for, the lifetime alone is the main stochastic quantity and the curvature fluctuations are in reference to $\langle \mathcal{N} \rangle = \langle \Delta N_0 \rangle$. This difference in the reference point is at the origin of this factor of two difference in the exponential decay rates. It it also at the origin of the strong asymmetry of $P(\zeta_{\rm fw}|\phi_0)$ with respect to the sign of $\zeta_{\rm fw}$. As a result, the choice of $N_\flat$ in the definition of the stochastic δN-formalism is not innocuous when quantum diffusion dominates.

Finally, we have shown how a negative drift leads to finite results in the time-reversed approach while being pathological in the forward formalism. Such a negative drift is actually a simple case of eternal inflation in which many field trajectories never reach the end of quantum diffusion. Our findings illustrate how the conditioning by the lifetimes, hardcoded in the time-reversed stochastic inflation formalism, enforces the regularity of the extracted probability distributions. In more intuitive terms, why should we be concerned with trajectories in which inflation never ends? Assessing the statistics of cosmological observables should be performed in the subset of all possibilities leading to our universe, i.e., the Friedmann-Lemaître decelerating models. It seems to us that, in the absence of a well-defined classical background, considering all of its possible quantum realisations is the sensible alternative.

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