EXISTENCE AND NONEXISTENCE OF VISCOSITY SOLUTIONS FOR A CLASS OF DEGENERATE/SINGULAR EIGENVALUE TYPE EQUATIONS

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Abstract. This paper is devoted to a complete classification on the existence and nonexistence results of viscosity solutions to the general Dirichlet problem for a class of eigenvalue type equations. With the distance function included in the right-hand side, this type of equations can be degenerate and (or) singular near the boundary of uniformly convex domains. One highlight is that all cases related to the exponent of the distance function are investigated. Moreover, when viscosity solutions exist, we derive a series of global estimates based on the distance function. The key ingredients of this paper include adaptions of the Perron method and comparison principle as well as constructions of suitable classical sub-solutions and super-solutions.

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Contents

1. Introduction

In this paper, we study the existence and nonexistence of viscosity solutions to the Dirichlet problem:

$$F\left(\lambda_1(D^2u),\cdots,\lambda_n(D^2u)\right)=g(x)(d(x))^\alpha \quad \text{in } \Omega, \tag{1.1}$$

$$u=\varphi \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega\subseteq\mathbb{R}^n (n\geq 2)$ refers to a convex domain having non-empty boundary $\partial\Omega$; $u:\Omega\to\mathbb{R}$ is a convex function; $\lambda_1(D^2u),\cdots,\lambda_n(D^2u)$ represent all the eigenvalues of the Hessian D^2u; F denotes a given operator; g and φ are given functions; and $d(x)$ indicates the distance from the point x to the boundary $\partial\Omega$:

$$d(x)=\operatorname{dist}(x,\partial\Omega).$$

1.1. **Assumptions, eigenvalue analysis, and examples.** The following four sets of assumptions are made regarding the domain Ω, operator F, right-hand-side coefficient function g, and boundary function φ, respectively:

($\mathbf{A_1}$) Ω is a uniformly convex domain of class C^2, where the uniform convexity of a domain is a concept from differential geometry, which states that all principal curvatures of the boundary $\partial\Omega$, denoted by $\kappa_1,\cdots,\kappa_{n-1}$, are bounded away from zero.

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($\mathbf{A_2}$) $F(\lambda_1,\cdots,\lambda_n)\in C(\mathbb{R}^n)$; $F(\lambda_1,\cdots,\lambda_n)$ satisfies the semi-elliptic condition, that is,

$$F(\lambda_1(A),\cdots\lambda_n(A))\geq F(\lambda_1(B),\cdots\lambda_n(B)) \tag{1.3}$$

for any n-order positive semi-definite matrices A and B with $A\geq B$ (this symbol means that $A-B$ is positive semi-definite); and either one of the following two boundedness conditions holds:

(i) there exist some constants $\Lambda_0>0$, $a\geq 0$ and $b>0$ such that

$$F\left(\lambda_1(D^2u),\cdots,\lambda_n(D^2u)\right)\geq \Lambda_0(\lambda_{\min}(D^2u))^a(\lambda_{\max}(D^2u))^b,$$

(ii) there exist some constants $\Lambda_0>0$, $a\geq 0$ and $b>0$ such that

$$F\left(\lambda_1(D^2u),\cdots,\lambda_n(D^2u)\right)\leq \Lambda_0(\lambda_{\min}(D^2u))^a(\lambda_{\max}(D^2u))^b,$$

where $\lambda_{\min}(D^2u):=\min\{\lambda_1(D^2u),\cdots,\lambda_n(D^2u)\}$ and $\lambda_{\max}(D^2u):=\max\{\lambda_1(D^2u),\cdots,\lambda_n(D^2u)\}$.

($\mathbf{A_3}$) $g(x)\in C(\overline{\Omega})$; and there exist some constants $0<m_0\leq M_0$ such that

$$m_0\leq g(x)\leq M_0,\quad \forall x\in\overline{\Omega}.$$

($\mathbf{A_4}$) $\varphi(x)\in C(\partial\Omega)$ is a function defined on $\partial\Omega$; and φ has at least one continuous convex extension to $\overline{\Omega}$, implying that there exists a convex function $\varphi_*\in C(\overline{\Omega})$ such that $\varphi_*\big|_{\partial\Omega}=\varphi$.

Under these assumptions, (1.1) can describe a class of fully nonlinear elliptic equations, and the right-hand side of (1.1) can be degenerate or singular on the boundary $\partial\Omega$. It is evident from the assumption ($\mathbf{A_3}$) that if $\alpha>0$, then $g(x)(d(x))^\alpha\to 0$ as $x\to\partial\Omega$, indicating the degenerate case; if $\alpha<0$, then $g(x)(d(x))^\alpha\to+\infty$ as $x\to\partial\Omega$, reflecting the singular case.

For the equation (1.1) with a degenerate or singular right-hand side at the boundary, the tangential and normal eigenvalues of D^2u might exhibit degeneracy or singularity on the boundary at different speeds. In fact, we have the following eigenvalue analysis on the boundary, given that $\partial\Omega$ is uniformly convex. On one hand, when the right-hand side of (1.1) degenerates, it follows that the normal eigenvalues also degenerate, whereas the tangential eigenvalues typically do not. In this case, all tangential eigenvalues are larger than all normal eigenvalues as the boundary is approached. On the other hand, when the right-hand side of (1.1) becomes singular, it holds that the normal eigenvalues are definitely singular, while the tangential eigenvalues may be singular (if the right-hand side exhibits strong singularity) or non-singular (if the right-hand side has weak singularity). In this scenario, all normal eigenvalues are larger than all tangential eigenvalues near the boundary. Generally speaking, the eigenvalues of D^2u always can be split into two groups with distinct orders of speed: l small and $n-l$ large for some $l\in\{1,\cdots,n-1\}$, i.e.

$$\lambda_1(D^2u)\sim\cdots\sim\lambda_l(D^2u)\ll\lambda_{l+1}(D^2u)\sim\cdots\sim\lambda_n(D^2u), \tag{1.4}$$

and then we have

$$\{\lambda_1(D^2u),\cdots,\lambda_n(D^2u)\}\sim\{\underbrace{\lambda_{\min}(D^2u),\cdots,\lambda_{\min}(D^2u)}_{l},\underbrace{\lambda_{\max}(D^2u),\cdots,\lambda_{\max}(D^2u)}_{n-l}\}, \tag{1.5}$$

where the symbol "$\sim$" signifies identical orders of speed.

Since u is a convex function, there are several usual operators that meet the requirements of ($\mathbf{A_2}$). For instance, we observe the following: For any $t,t_1,\cdots,t_n>0$, both the operators $(\lambda_1(D^2u))^t+\cdots+(\lambda_n(D^2u))^t$ and $(\lambda_1(D^2u))^{t_1}\cdots(\lambda_n(D^2u))^{t_n}$ fulfill ($\mathbf{A_2}$) due to the following inequalities:

$$(\lambda_{\max}(D^2u))^t\leq(\lambda_1(D^2u))^t+\cdots+(\lambda_n(D^2u))^t\leq n(\lambda_{\max}(D^2u))^t,$$
$$(\lambda_{\min}(D^2u))^{\sum_{i=1}^n t_i-\min\{t_1,\cdots,t_n\}}(\lambda_{\max}(D^2u))^{\min\{t_1,\cdots,t_n\}}\leq(\lambda_1(D^2u))^{t_1}\cdots(\lambda_n(D^2u))^{t_n}\leq(\lambda_{\max}(D^2u))^{\sum_{i=1}^n t_i}.$$

In general, if

$$F(\lambda_1(D^2u),\cdots,\lambda_n(D^2u))=\sum_{(t_1,\cdots,t_n)\in S}C(t_1,\cdots,t_n)(\lambda_1(D^2u))^{t_1}\cdots(\lambda_n(D^2u))^{t_n}$$

is well-defined with $C(t_1,\cdots,t_n)>0$ for all $(t_1,\cdots,t_n)\in S$ (which represents an index set), then this F also satisfies ($\mathbf{A_2}$).

Another classical type of examples for F is the k-Hessian operator. For any $k \in \{1, \cdots, n\}$, the k-Hessian operator is given by the k-th elementary symmetric polynomial of eigenvalues of the Hessian:

$$\sigma_k(D^2u) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1}(D^2u) \cdots \lambda_{i_k}(D^2u).$$

Indeed, one can check

$$\binom{n-1}{k-1}(\lambda_{\min}(D^2u))^{k-1}\lambda_{\max}(D^2u) \le F(\lambda_1(D^2u), \cdots, \lambda_n(D^2u)) \le \binom{n}{k}(\lambda_{\max}(D^2u))^k.$$

Notably, $\sigma_1(D^2u) = \Delta u$ corresponds to the Laplace operator, while $\sigma_n(D^2u) = \det D^2u$ corresponds to the Monge-Ampère operator. The k-Hessian operator is fully nonlinear when $k \ge 2$, and is elliptic when restricted to k-admissible functions.

This leads to a variety of special cases and geometric applications of (1.1). For example, the k-Hessian type equations [15, 26] in the form of

$$\sigma_k(D^2u) = g(x)(d(x))^\alpha$$

can be related to the k-Yamabe problems in conformal geometry [24, 26] and special Lagrangian equations in calibrated geometry [27]. Notably, the following quadratic Hessian equation (sigma-2 equation) [22] is also included:

$$\sigma_2(D^2u) = 1.$$

When $k = n$, the Monge-Ampère type equations [7], taking the form

$$\det D^2u = g(x)(d(x))^\alpha,$$

figure in geometric optics problems [9]. More specially, a typical example of (1.1) arising in affine geometry is

$$\det D^2u = 1,$$

for which the Legendre transform of solution in the entire space, due to the Jörgens-Calabi-Pogorelov theorem [3, 17, 21], constructs a complete parabolic affine sphere. Various examples and widespread applications can also be found in [7, 8, 26].

1.2. **Review of previous results.** During the last half-century, existence and nonexistence theories of solutions to fully nonlinear elliptic equations have been investigated by many researchers. We refer the readers to [1, 2, 4, 10, 11, 14–16, 18, 23, 25] for example for earlier works on related topics. In Theorem 3 of [4], the existence of a unique continuous convex generalized solution to the problem

$$\begin{aligned} \det D^2u &= f(x) \quad \text{in } \Omega \\ u &= \varphi \qquad \text{on } \partial\Omega \end{aligned}$$

was established by Cheng and Yau, where Ω is a bounded C^2 strictly convex domain, $\varphi \in C^2(\partial\Omega)$, and f belongs to class C^k ($k \ge 3$) and behaves near the boundary as follows:

$$0 < f(x) \le A(d(x))^\alpha$$

for some constants $A > 0$ and $\alpha > -n-1$. In Theorem 2 of [2], given the smoothness and some appropriate conditions on the domain Ω, operator F, right-hand-side function f and boundary function φ respectively, Caffarelli, Nirenberg and Spruck obtained the existence of a unique smooth admissible solution to the problem

$$\begin{aligned} F\left(\lambda_1(D^2u), \cdots, \lambda_n(D^2u)\right) &= f(x) \quad \text{in } \Omega \\ u &= \varphi \qquad \text{on } \partial\Omega, \end{aligned}$$

where it is noteworthy that the right-hand-side function f is required to be smooth and positive, which also means that the equation cannot be degenerate or singular. In comparison, the equations considered in this paper may exhibit degeneracy or singularity, and we aim to comprehensively classify the existence and nonexistence of solutions based on the values of α. We note that the continuity of the right-hand-side coefficient function $g(x)$ in $(\mathbf{A_3})$ does not guarantee the existence of a classical solution to the problem (1.1)-(1.2). Instead, our focus will be on the existence and nonexistence of viscosity solutions to this problem rather than classical solutions.

The Perron method is known as a powerful technique used to establish the existence of viscosity solutions. This method might first appear in [12], introduced by Ishii for the study of Hamilton-Jacobi equations, and shortly afterwards, was applied to infer existence of viscosity solutions of the Dirichlet problem of fully nonlinear elliptic equations in [13]. The main idea of this method involves constructing a family of viscosity sub-solutions or viscosity super-solutions that serve as candidates for the viscosity solution. By considering the supremum of viscosity sub-solutions or the infimum of viscosity super-solutions, one can identify a function that satisfies the conditions of being a viscosity solution. The whole process also heavily relies on using comparison principle. The Perron method, comparison principle as well as constructions of suitable classical sub-solutions and super-solutions will play an important role in this paper.

This paper also builds upon and extends our earlier works [19, 20]. In [19], assuming the existence of solution, we derived the *a priori* Hölder regularity estimates for the Dirichlet problem of fully nonlinear elliptic equations:

$$\begin{aligned} F(\lambda_1(D^2u), \cdots, \lambda_n(D^2u)) &= f(x, u, \nabla u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

These estimates were obtained under certain structure conditions on the domain Ω, operator F and right-hand-side function f respectively. In particular, f was required to satisfy the following bound:

$$\begin{aligned} &\text{for some constants } A \geq 0,\ \alpha \in \mathbb{R},\ \gamma \in \mathbb{R},\ \beta \geq \max\{n+1, n+1+\gamma\}, \text{ there holds} \\ &0 \leq f(x, z, q) \leq A(d(x))^{\beta-n-1}|z|^{-\alpha}(1+|q|^2)^{\frac{\gamma}{2}}, \quad \forall (x, z, q) \in \Omega \times (-\infty, 0) \times \mathbb{R}^n, \end{aligned}$$

where $\beta - n - 1 \geq 0$ always holds, implying that the exponent of $d(x)$ must be nonnegative. In contrast to [19], the results in this paper will thoroughly analyze all cases involving the exponent of $d(x)$ for the problem (1.1)-(1.2). Furthermore, we will establish a nonexistence result in this paper when the exponent of $d(x)$ is sufficiently small, specifically when it is no greater than $-a - 2b$, i.e. $\alpha \leq -a - 2b$ based on the parameter notations in (1.1) and ($\mathbf{A_2}$).

Our recent work [20] relates to the study of viscosity solution to the Dirichlet problem of Monge-Ampère type equations:

$$\begin{aligned} \det D^2u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.6}$$

Under certain structure conditions on the domain Ω and right-hand-side function f, we established the existence, uniqueness and interior regularity of the viscosity solution to this problem. We recall that f therein satisfies that

$$\begin{aligned} &\text{for some constants } A > 0,\ \alpha \in \mathbb{R},\ \beta \geq n+1,\ \gamma < \min\{n+\alpha, \beta-n+1\}, \text{ there holds} \\ &0 < f(x, z, q) \leq A(d(x))^{\beta-n-1}|z|^{-\alpha}(1+|q|^2)^{\frac{\gamma}{2}}, \quad \forall (x, z, q) \in \Omega \times (-\infty, 0) \times \mathbb{R}^n, \end{aligned}$$

where $\beta - n - 1 \geq 0$ also means that the exponent of $d(x)$ needs to be nonnegative. Unlike [20], for the problem (1.1)-(1.2), given that $\partial\Omega \in C^2$ is uniformly convex, we will construct different auxiliary functions as the exponent of $d(x)$ varies, and moreover in the critical case $\alpha = -b$ based on the parameter notations in (1.1) and ($\mathbf{A_2}$), we will obtain a global log-type estimate for the viscosity solution.

1.3. **Main results and outline of this paper.** According to the assumption ($\mathbf{A_4}$), one can extend φ from $\partial\Omega$ to $\overline{\Omega}$. We are going to uniquely extend φ to $\overline{\Omega}$ through one Monge-Ampère equation. In fact, by the existence result for Monge-Ampère equations with continuous boundary data (see Theorem 2.14 in [7]), there exists a unique convex function $\psi \in C(\overline{\Omega})$ that solves the Dirichlet problem

$$\begin{aligned} \det D^2\psi &= 0 \quad \text{in } \Omega, \\ \psi &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{1.7}$$

Remark 1.1. *In some sense, the extension function $\psi \in C(\overline{\Omega})$ can be regarded as a weak solution to the following homogeneous equation with Dirichlet boundary condition:*

$$\begin{aligned} \Lambda_0 \left(\lambda_{\min}(D^2\psi)\right)^a \left(\lambda_{\max}(D^2\psi)\right)^b &= 0 \quad \textit{in } \Omega, \\ \psi &= \varphi \quad \textit{on } \partial\Omega, \end{aligned} \tag{1.8}$$

This observation will be easy to explain if we assume $\psi \in C^2(\Omega)$. We first note that

$$\lambda_1(D^2\psi)\cdots\lambda_n(D^2\psi) = \det D^2\psi = 0 \quad \text{in } \Omega.$$

Since ψ is convex, it holds in Ω that

$$\lambda_i(D^2\psi) \geq 0, \quad \forall i = 1, \cdots, n.$$

Now we can infer that

$$\lambda_{\min}(D^2\psi) = 0 \quad \text{in } \Omega.$$

Therefore, ψ actually satisfies (1.8) for any constants $\Lambda_0 > 0$, $a > 0$ and $b > 0$.

Now we are ready to state our main results. A series of existence results can be elaborated as the following theorem. Global estimates for viscosity solutions via distance function estimates are also obtained.

Theorem 1.2 (Existence Theorem)**.** *Under the assumptions* $(\mathbf{A_1})$, $(\mathbf{A_2})$-(i), $(\mathbf{A_3})$ *and* $(\mathbf{A_4})$, *we have a list of existence and estimation conclusions based on varying values of α:*

(i) If $\alpha \in (-b, +\infty)$, then the problem (1.1)-(1.2) admits a viscosity solution $u(x) \in C(\overline{\Omega})$, and there exists some constant $\overline{C_0} > 0$ such that

$$u(x) \geq \psi(x) - \overline{C_0}\, d(x), \quad \forall x \in \overline{\Omega}.$$

(ii) If $\alpha = -b$, then the problem (1.1)-(1.2) admits a viscosity solution $u(x) \in C(\overline{\Omega})$, and there exists some constants $\overline{C_0}$, $\overline{C_1} > 0$ such that

$$u(x) \geq \psi(x) - \overline{C_0}\, d(x)\left(-\log\left(\overline{C_1}\, d(x)\right)\right)^{\frac{b}{a+b}}, \quad \forall x \in \overline{\Omega}.$$

(iii) If $\alpha \in (-a - 2b, -b)$, then the problem (1.1)-(1.2) admits a viscosity solution $u(x) \in C(\overline{\Omega})$, and there exists some constant $\overline{C_0} > 0$ such that

$$u(x) \geq \psi(x) - \overline{C_0}\,(d(x))^{\frac{a+2b+\alpha}{a+b}}, \quad \forall x \in \overline{\Omega}.$$

Our second theorem addresses the issue of nonexistence:

Theorem 1.3 (Nonexistence Theorem)**.** *Under the assumptions* $(\mathbf{A_1})$, $(\mathbf{A_2})$-(ii), $(\mathbf{A_3})$ *and* $(\mathbf{A_4})$, *we have one nonexistence conclusion: if $\alpha \in (-\infty, -a-2b]$, then the problem (1.1)-(1.2) does not admit any viscosity solutions, and consequently, it also does not admit classical solutions.*

The following corollary follows directly from Theorems 1.2 and 1.3:

Corollary 1.4 (Special Case)**.** *Under the assumptions* $(\mathbf{A_1})$, $(\mathbf{A_3})$ *and* $(\mathbf{A_4})$, *all the conclusions in Theorems 1.2 and 1.3 remain valid for the following problem:*

$$\Lambda_0\left(\lambda_{\min}(D^2u)\right)^a\left(\lambda_{\max}(D^2u)\right)^b = g(x)(d(x))^\alpha \quad \text{in } \Omega, \tag{1.9}$$

$$u = \varphi \qquad \text{on } \partial\Omega, \tag{1.10}$$

for any constants $\Lambda_0 > 0$, $a \geq 0$ and $b > 0$.

Remark 1.5. *For the problem (1.1)-(1.2), by combining Theorems 1.2 and 1.3, we achieve a complete classification for $\alpha \in (-\infty, +\infty)$, i.e. the exponent of $d(x)$ in (1.1). For the problem (1.9)-(1.10), this configuration also provides a complete classification on the existence and nonexistence results for the viscosity solution. Intuitively, when α is large (tending more towards the degenerate case), viscosity solutions exist. In contrast, when α is small (tending more towards the singular case), viscosity solutions do not exist. The critical value that distinguishes these two scenarios is $-a-2b$. We will explain these intuitions in Section 6.*

The rest of this paper is organized as follows. In Section 2, we present basic concepts, preliminary results and necessary estimates for the problem (1.1)-(1.2). In Section 3, we establish a key lemma that provides a transition from classical sub-solutions to viscosity solutions. Based on this transition and constructions of suitable classical sub-solutions, Section 4 is dedicated to the proofs for existence results and global estimates of viscosity solutions in Theorem 1.2. We devote Section 5 to demonstrate the nonexistence result in Theorem 1.3 through a contradiction by constructing a suitable classical super-solution. Finally, some intuitions and discussions regarding our results are provided in Section 6.

2. Preliminaries

Throughout this paper, we adhere to the following convention: for any function Φ, we always use the notation $D^2\Phi > (<)\, 0$ to represent that the Hessian matrix $(D^2\Phi)$ is positive (negative) definite and use the notation $D^2\Phi \geq (\leq)\, 0$ to represent that the Hessian matrix $(D^2\Phi)$ is positive (negative) semi-definite.

2.1. **Classical solutions and viscosity solutions.** Let us begin this subsection by clarifying the concepts of classical solutions and viscosity solutions to the problem (1.1)-(1.2).

Definition 2.1. *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a convex function. We refer to u as a classical solution (sub-solution, super-solution) to (1.1) over Ω if*

$$F\left(\lambda_1(D^2u), \cdots, \lambda_n(D^2u)\right) = (\geq, \leq)\, g(x)(d(x))^\alpha \quad \text{in } \Omega.$$

u is called a classical solution to the problem (1.1)-(1.2) over Ω if it is a classical solution to (1.1) over Ω and satisfies (1.2) on $\partial\Omega$. Moreover, we say that u is a classically strict sub-solution (super-solution) to (1.1) over Ω if

$$F\left(\lambda_1(D^2u), \cdots, \lambda_n(D^2u)\right) > (<)\, g(x)(d(x))^\alpha \quad \text{in } \Omega.$$

Definition 2.2. *Let $u \in C(\overline{\Omega})$ be a convex function. We refer to u as a viscosity sub-solution (super-solution) to (1.1) over Ω if for any point $x_0 \in \Omega$, any open neighborhood $U(x_0) \subset \Omega$ and any convex function $\phi \in C^2(U(x_0))$ satisfying*

$$(\phi - u)(x) \geq\, (\leq)\, (\phi - u)(x_0), \quad \forall x \in U(x_0),$$

there holds

$$F\left(\lambda_1(D^2\phi(x_0)), \cdots, \lambda_n(D^2\phi(x_0))\right) \geq\, (\leq)\, g(x_0)(d(x_0))^\alpha.$$

u is said to be a viscosity solution to (1.1) over Ω if it is both a viscosity sub-solution and a viscosity super-solution to (1.1) over Ω. u is called a viscosity solution to the problem (1.1)-(1.2) over Ω if it is a viscosity solution to (1.1) over Ω and satisfies (1.2) on $\partial\Omega$.

See the lectures of Crandall collected in [5] for a general theory on viscosity solutions. Next, we will demonstrate the equivalent condition and comparison principle for viscosity solutions to (1.1) as the following two lemmas.

Lemma 2.3. *Under the semi-elliptic assumption in $(\mathbf{A_2})$, if $u \in C^2(\Omega)$ is a convex function, then the following two statements are equivalent:*

(i) u is a viscosity solution (sub-solution, super-solution) to (1.1) over Ω;
(ii) u is a classical solution (sub-solution, super-solution) to (1.1) over Ω.

Proof. The proofs for all three cases are not easily unified. However, we can derive the results for the sub-solution case and the super-solution case separately, and combining these two results directly leads to the conclusion for the solution case.

We first infer (ii) from (i). Let u be a viscosity sub-solution (super-solution) to (1.1) over Ω. For any $x_0 \in \Omega$ and any $x \in U(x_0)$, we take $\phi(x) = u(x)$. Since $u \in C^2(\Omega)$, there holds $\phi \in C^2(U(x_0))$. As a result of $\phi(x) = u(x)$, we have $\phi(x) - u(x) = 0 = \phi(x_0) - u(x_0)$ and therefore

$$(\phi - u)(x) \geq\, (\leq)\, (\phi - u)(x_0).$$

By Definition 2.2, it follows that

$$F\left(\lambda_1(D^2\phi(x_0)), \cdots, \lambda_n(D^2\phi(x_0))\right) \geq\, (\leq)\, g(x_0)(d(x_0))^\alpha,$$

which together with $\phi(x) = u(x)$ implies that

$$F\left(\lambda_1(D^2u(x_0)), \cdots, \lambda_n(D^2u(x_0))\right) \geq\, (\leq)\, g(x_0)(d(x_0))^\alpha.$$

Since $x_0 \in \Omega$ is arbitrary, we infer that u is a classical sub-solution (super-solution) to (1.1) over Ω.

From (ii) to (i), we let u to be a classical sub-solution (super-solution) to (1.1) over Ω. Then for any $x_0 \in \Omega$, we have

$$F\left(\lambda_1(D^2u(x_0)), \cdots, \lambda_n(D^2u(x_0))\right) \geq (\leq)\, g(x_0)(d(x_0))^\alpha.$$

For any convex function $\phi \in C^2(U(x_0))$ satisfying

$$(\phi - u)(x) \geq (\leq)\, (\phi - u)(x_0), \quad \forall x \in U(x_0),$$

it is clear that $\phi - u$ takes its minimum (maximum) at x_0, which yields

$$D(\phi-u)\big|_{x_0}=0, \quad D^2(\phi-u)\big|_{x_0}\geq(\leq)\,0.$$

Due to (1.3), $F(\lambda_1,\cdots,\lambda_n)$ is semi-elliptic, we further obtain

$$D\phi(x_0)=Du(x_0), \quad F\left(\lambda_1(D^2\phi(x_0)),\cdots,\lambda_n(D^2\phi(x_0))\right)\geq(\leq)\,F\left(\lambda_1(D^2u(x_0)),\cdots,\lambda_n(D^2u(x_0))\right).$$

It then follows that

$$F\left(\lambda_1(D^2\phi(x_0)),\cdots,\lambda_n(D^2\phi(x_0))\right)\geq(\leq)\,F\left(\lambda_1(D^2u(x_0)),\cdots,\lambda_n(D^2u(x_0))\right)\geq(\leq)\,g(x_0)(d(x_0))^\alpha,$$

which means that u is a viscosity sub-solution (super-solution) to (1.1) over Ω.

The proof of this lemma is now complete. □

Lemma 2.4 (Comparison Principle)**.** *If $u\in C(\overline{\Omega})$ is a viscosity solution to (1.1) over Ω and $W(x)\in C^2(\Omega)\cap C(\overline{\Omega})$ is convex function satisfying*

$$F\left(\lambda_1(D^2W(x)),\cdots,\lambda_n(D^2W(x))\right)>(<)\,g(x)(d(x))^\alpha, \qquad \forall x\in\Omega, \tag{2.1}$$

$$W(x)\leq(\geq)\,u(x), \qquad \forall x\in\partial\Omega, \tag{2.2}$$

then

$$W(x)\leq(\geq)\,u(x), \qquad \forall x\in\overline{\Omega}. \tag{2.3}$$

Here, based on Definition 2.1, the condition (2.1) is also equivalent to the condition that W is a classically strict sub-solution (super-solution) to (1.1) over Ω.

Proof. To prove (2.3), it suffices to prove that

$$\max_{x\in\overline{\Omega}}(W-u)(x)\leq 0 \quad (\text{ or } \min_{x\in\overline{\Omega}}(W-u)(x)\geq 0).$$

The proof is by contradiction. Suppose there exists a point $x_0\in\overline{\Omega}$ such that

$$\max_{x\in\overline{\Omega}}(W-u)(x)=(W-u)(x_0)>0 \quad (\text{ or } \min_{x\in\overline{\Omega}}(W-u)(x)=(W-u)(x_0)<0).$$

Due to (2.2), we have $(W-u)\big|_{\partial\Omega}\leq(\geq)\,0$, and hence $x_0\in\Omega$. Now x_0 is the maximum (minimum) point of $(W-u)(x)$ in the interior of Ω. Since u is a viscosity solution to (1.1) over Ω, then u is also a viscosity super-solution (sub-solution) to (1.1) over Ω. Given that $W\in C^2(\Omega)$, by Definition 2.2, we can further derive

$$F\left(\lambda_1(D^2W(x_0)),\cdots,\lambda_n(D^2W(x_0))\right)\leq(\geq)\,g(x_0)(d(x_0))^\alpha,$$

which contradicts our condition (2.1). We have thus completed the proof of the lemma. □

2.2. **Construction of approximation functions.** In this subsection, we construct a family of Monge-Ampère equations with continuous boundary data, generating a family of approximation functions via their solutions.

Lemma 2.5 (Approximation function ψ_k over $\overline{\Omega}$)**.** *Let Ω be a bounded and strictly convex domain in $\mathbb{R}^n$. Let φ satisfy the assumption ($\mathbf{A_4}$). Then there exists a family of convex functions $\{\psi_k\}_{k\in\mathbb{N}^+}$ such that*

(i) $\psi_k\in C^\infty(\Omega)\cap C(\overline{\Omega})$;
(ii) $\psi_k\big|_{\partial\Omega}=\varphi$;
(iii) ψ_k converges locally uniformly to ψ over Ω.

Proof. For any $k\in\mathbb{N}^+$, let ψ_k be the solution to the nonhomogeneous Dirichlet problem

$$\begin{aligned} \det D^2\psi_k &= \tfrac{1}{k} \quad \text{in } \Omega,\\ \psi_k &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

Since the constant $\frac{1}{k}$ can be considered a smooth function, then we obtain that $\psi_k\in C^\infty(\Omega)\cap C(\overline{\Omega})$ is a convex function. It is clear that $\frac{1}{k}\to 0$ as $k\to\infty$, and hence for Borel measures, we have $\frac{1}{k}dx\to^* 0dx$ as $k\to\infty$. By the stability result for Monge-Ampère equations with continuous boundary data (see Proposition 2.16 in [7]), it follows that ψ_k converges locally uniformly in Ω to the unique solution ψ of the Dirichlet problem (1.7). Therefore, we have proved the lemma. □

Remark 2.6. *For any $k \in \mathbb{N}^+$, since ψ_k constructed in Lemma 2.5 is a smooth convex function, then*

$$D^2\psi_k \text{ is a positive semi-definite matrix, i.e. } D^2\psi_k \geq 0.$$

Moreover, it follows that for any nonzero vector $\boldsymbol{\xi}$, we have $\boldsymbol{\xi}^T D^2\psi_k\, \boldsymbol{\xi} \geq 0$.

2.3. **Distance function estimates.** For $\mu > 0$, let us set

$$\Gamma_\mu = \{x \in \overline{\Omega}: \ d(x) < \mu\}.$$

As a consequence of Lemma 14.16 and Lemma 14.17 in [8], we have the following lemma that relates the C^2 smoothness of $d(x)$ in Γ_μ to that of the boundary $\partial\Omega$ and expresses the gradient and Hessian of $d(x)$ at points close to $\partial\Omega$.

Lemma 2.7 (Near boundary properties of distance function $d(x)$)**.** *Under the assumption ($\mathbf{A_1}$), we have the following conclusions:*

(i) There exists some constant $\mu_0 = \mu_0(\Omega) > 0$ such that

$$d(x) \in C^2(\Gamma_{\mu_0});$$

(ii) Let $x_0 \in \Gamma_{\mu_0}$ and $y_0 \in \partial\Omega$ satisfy $d(x_0) = |x_0 - y_0|$. Then by choosing some suitable coordinate system, the gradient and Hessian of $d(x)$ at the point x_0 can be expressed as

$$\begin{aligned} Dd(x_0) &= (\underbrace{0, \cdots, 0}_{n-1}, 1), \\ D^2 d(x_0) &= \operatorname{diag}\left(\frac{-\kappa_1(y_0)}{1-\kappa_1(y_0)d(x_0)}, \cdots, \frac{-\kappa_{n-1}(y_0)}{1-\kappa_{n-1}(y_0)d(x_0)}, 0\right), \end{aligned}$$

where $\kappa_1(y_0), \cdots, \kappa_{n-1}(y_0)$ represent all the $(n-1)$ principal curvatures of $\partial\Omega$ at y_0.

Proof. Since Ω is a uniformly convex domain having non-empty boundary $\partial\Omega$, we know that Ω is a bounded domain. In fact, $\partial\Omega$ is an $(n-1)$-dimensional hypersurface, and all the $(n-1)$ principal curvatures of $\partial\Omega$ are bounded from below by a positive constant, which we take to be c at present. Then for any unit tangent vector X, its Ricci curvature has the following positive lower bound:

$$\operatorname{Ric}(X, X) \geq (n-2)c^2.$$

Using the Bonnet–Myers theorem, we infer that $\partial\Omega$ is compact and its diameter satisfies

$$\operatorname{diam}(\partial\Omega) := \sup_{p,q\in\partial\Omega} \operatorname{dist}(p, q) \leq \frac{\pi}{c},$$

which together with the uniform convexity of Ω further implies that Ω is bounded.

The fact that Ω is bounded and of class C^2 yields that $\partial\Omega$ satisfies the following uniform interior sphere condition: there exists a constant $\mu_0 = \mu_0(\Omega) > 0$ such that for any $y_0 \in \partial\Omega$, there is a ball $B(y_0, r)$ with radius $r \geq \mu_0$ satisfying $\overline{B(y_0, r)} \cap (\mathbb{R}^n \setminus \Omega) = \{y_0\}$. Now it is clear that $B(y_0, r)$ is tangent to $\partial\Omega$ at y_0 and lies entirely inside Ω. Since all the principal curvatures of $\partial B(y_0, r)$ at y_0 are $\frac{1}{r}$, then all the principal curvatures of $\partial\Omega$ at y_0 satisfy

$$\kappa_i(y_0) \leq \frac{1}{r} \leq \frac{1}{\mu_0}, \quad \forall i = 1, \cdots, n-1. \tag{2.4}$$

Due to the C^2 regularity of $\partial\Omega$, for any $x \in \Gamma_{\mu_0}$, there exists a unique $y \in \partial\Omega$ such that $d(x) = |x-y|$. Let $N(y)$ and $T(y)$ denote respectively the unit inward normal to $\partial\Omega$ at y and the tangent hyperplane to $\partial\Omega$ at y. Let $\kappa_1(y), \cdots, \kappa_{n-1}(y)$ represent all the $(n-1)$ principal curvatures of $\partial\Omega$ at y. Now we choose a principal coordinate system at $y \in \partial\Omega$. By a rotation of coordinates, we can assume that the x_n coordinate axis lies in the direction $N(y)$, and the $x_1, \cdots, x_{n-1}$ coordinate axes lie along principal directions corresponding to $\kappa_1(y), \cdots, \kappa_{n-1}(y)$ at y. In fact, in some neighborhood $\mathcal{U}(y)$ of y, $\partial\Omega$ is given by $x_n = h(x')$, where $x' = (x_1, \cdots, x_{n-1})$, $h \in C^2(T(y) \cap \mathcal{U}(y))$, $Dh(y') = 0$ and

$$D^2 h(y') = \operatorname{diag}(\kappa_1(y), \cdots, \kappa_{n-1}(y)). \tag{2.5}$$

The unit inward normal vector $\overline{N}(y') = N(y)$ at $y = (y', h(y'))$ is then given by

$$N_i(y) = \frac{-D_i h(y')}{\sqrt{1+|Dh(y')|^2}}, \quad \forall i = 1, \cdots, n-1, \tag{2.6}$$

$$N_n(y) = \frac{1}{\sqrt{1+|Dh(y')|^2}}.$$

Hence we have

$$D_j\overline{N}_i(y') = -\kappa_i(y)\delta_{ij}, \quad \forall i,j = 1,\cdots,n-1.$$

We note that x is the closest point to y in Ω and must lie along the inward normal direction as follows:

$$x = y + tN(y), \tag{2.7}$$

where $t \in \mathbb{R}$. For a fixed point $x_0 \in \Gamma_{\mu_0}$, we let $y_0 = y(x_0)$. Define a mapping $H = (H_1,\cdots,H_n) : (T(y_0)\cap\mathcal{U}(y_0))\times\mathbb{R} \to \mathbb{R}^n$ by

$$H(y',t) = y + tN(y), \qquad y = (y', h(y')).$$

Clearly, $y_0 \in \partial\Omega$ and $H \in C^1((T(y_0)\cap\mathcal{U}(y_0))\times\mathbb{R})$. Due to (2.5) and (2.6), the gradient of H at $(y_0', d(x_0))$ is given by

$$DH = \mathrm{diag}(1-\kappa_1(y_0)d(x_0),\cdots,1-\kappa_{n-1}(y_0)d(x_0),1).$$

Since $x_0 \in \Gamma_{\mu_0}$, we also notice that

$$d(x_0) < \mu_0. \tag{2.8}$$

By virtue of (2.4) and (2.8), it follows that the Jacobian of H at $(y_0', d(x_0))$ is positive:

$$\det DH = (1-\kappa_1 d(x_0))\cdots(1-\kappa_{n-1}d(x_0)) > 0.$$

Thus using the inverse mapping theorem implies that $y' \in C^1(\mathcal{V}(x_0))$ for some neighborhood $\mathcal{V}(x_0)$, and by (2.7), there holds

$$Dd(x) = N(y(x)) = \overline{N}(y'(x)) \in C^1(\mathcal{V}(x_0)), \quad \forall x \in \mathcal{V}(x_0).$$

Hence, $d \in C^2(\mathcal{V}(x_0))$ and thus we can derive $d \in C^2(\Gamma_{\mu_0})$, which leads us to (i).

We note that the gradient of H^{-1} at x_0 is given by

$$DH^{-1} = \mathrm{diag}\Big(\frac{1}{1-\kappa_1(y_0)d(x_0)},\cdots,\frac{1}{1-\kappa_{n-1}(y_0)d(x_0)},1\Big).$$

Now it is clear that

$$\begin{aligned}
&Dd(x_0) = N(y_0) = (\underbrace{0,\cdots,0}_{n-1},1),\\
&D_{in}d(x_0) = 0, \quad \forall i = 1,\cdots,n,\\
&D_{ij}d(x_0) = D_jN_i\circ y(x_0) = D_l\overline{N}_i(y_0)D_jy_l(x_0) = \frac{-\kappa_i(y_0)\delta_{ij}}{1-\kappa_i(y_0)d(x_0)}, \quad \forall i,j = 1,\cdots,n-1.
\end{aligned}$$

This yields (ii) immediately. The proof of this lemma is now finished. $\square$

Based on Lemma 2.7, though $d(x)$ is not uniformly convex near $\partial\Omega$, we can show that

$$-4\,\mathrm{diam}(\Omega)\cdot d(x) + (d(x))^2$$

is uniformly convex near $\partial\Omega$. Our construction of this function is inspired by [6], and a detailed analysis of the distance function to a smooth hypersurface can be found in Section 5 therein.

Lemma 2.8 (Near boundary properties of function $-4\,\mathrm{diam}(\Omega)\cdot d(x) + (d(x))^2$)**.** *Under the assumption* $(\mathbf{A_1})$, *there exists some constant* $\mu_0 = \mu_0(\Omega) > 0$ *such that the function*

$$-4\,\mathrm{diam}(\Omega)\cdot d(x) + (d(x))^2$$

is C^2 *and uniformly convex over* Γ_{μ_0}.

Proof. Here, we adopt the notations used in the proof of Lemma 2.7.

By virtue of Lemma 2.7 (i), there exists $\mu_0 = \mu_0(\Omega) > 0$ such that $d(x) \in C^2(\Gamma_{\mu_0})$. Hence we have

$$-4\,\mathrm{diam}(\Omega)\cdot d(x) + (d(x))^2 \in C^2(\Gamma_{\mu_0}).$$

For any $x_0 \in \Gamma_{\mu_0}$, direct computations lead us to

$$\begin{aligned}
&D\left(-4\,\mathrm{diam}(\Omega)\cdot d(x) + (d(x))^2\right)\big|_{x=x_0} = -4\,\mathrm{diam}(\Omega)\cdot Dd(x_0) + 2d(x_0)\cdot Dd(x_0),\\
&D^2\left(-4\,\mathrm{diam}(\Omega)\cdot d(x) + (d(x))^2\right)\big|_{x=x_0} = (4\,\mathrm{diam}(\Omega) - 2d(x_0))\cdot\left(-D^2d(x_0)\right) + 2Dd(x_0)\otimes Dd(x_0).
\end{aligned}$$

Using Lemma 2.7 (ii), we note that

$$
\begin{aligned}
-D^2 d(x_0) &= \operatorname{diag}\left(\frac{\kappa_1(y_0)}{1-\kappa_1(y_0)d(x_0)}, \cdots, \frac{\kappa_{n-1}(y_0)}{1-\kappa_{n-1}(y_0)d(x_0)}, 0\right),\\
Dd(x_0)\otimes Dd(x_0) &= \operatorname{diag}(\underbrace{0,\cdots,0}_{n-1},1).
\end{aligned}
$$

In view of the fact that $\partial\Omega$ is uniformly convex and C^2, there exist constants $\overline{m_0}$ and $\overline{M_0}$ satisfying $0 < \overline{m_0} \le \overline{M_0}$ such that

$$\overline{m_0} \le \kappa_i(y_0) \le \overline{M_0}, \quad \forall i = 1, \cdots, n-1.$$

Now we take $\mu_0' = \min\{\mu_0, \frac{1}{2\overline{M_0}}\}$ and still denote it as μ_0. Then for any $x_0 \in \Gamma_{\mu_0}$, we have

$$1 - \kappa_i(y_0)\cdot d(x_0) \ge 1 - \overline{M_0}\cdot\mu_0 > \frac{1}{2} > 0,$$

and hence

$$\frac{\kappa_i(y_0)}{1-\kappa_i(y_0)\cdot d(x_0)} \ge \overline{m_0} > 0.$$

It follows that

$$-D^2 d(x_0) \ge \operatorname{diag}\left(\overline{m_0}, \cdots, \overline{m_0}, 0\right), \qquad \forall x_0 \in \Gamma_{\mu_0}.$$

Based on the observation

$$4\operatorname{diam}(\Omega) - 2d(x_0) \in [2\operatorname{diam}(\Omega), 4\operatorname{diam}(\Omega)],$$

we further derive

$$
\begin{aligned}
&(4\operatorname{diam}(\Omega) - 2d(x_0))\cdot\left(-D^2 d(x_0)\right) + 2Dd(x_0)\otimes Dd(x_0)\\
\ge\ &\operatorname{diag}\left(2\operatorname{diam}(\Omega)\cdot\overline{m_0}, \cdots, 2\operatorname{diam}(\Omega)\cdot\overline{m_0}, 0\right) + \operatorname{diag}\left(0,\cdots,0,2\right)\\
=\ &\operatorname{diag}\left(2\operatorname{diam}(\Omega)\cdot\overline{m_0}, \cdots, 2\operatorname{diam}(\Omega)\cdot\overline{m_0}, 2\right), \qquad \forall x_0 \in \Gamma_{\mu_0}.
\end{aligned}
$$

Up to now, we can deduce

$$D^2\left(-4\operatorname{diam}(\Omega)\cdot d(x) + (d(x))^2\right)\big|_{x=x_0} \ge \operatorname{diag}\left(2\operatorname{diam}(\Omega)\cdot\overline{m_0}, \cdots, 2\operatorname{diam}(\Omega)\cdot\overline{m_0}, 2\right), \quad \forall x_0 \in \Gamma_{\mu_0},$$

which means that $-4\operatorname{diam}(\Omega)\cdot d(x) + (d(x))^2$ is uniformly convex over Γ_{μ_0}. This concludes the proof of the lemma. $\square$

Lemma 2.9 (Extension function ϕ_0 over $\overline{\Omega}$)**.** *If Ω is a uniformly convex domain of class C^2, then there exists a function ϕ_0 such that*

(i) $\phi_0 \in C^2(\overline{\Omega})$;
(ii) ϕ_0 is uniformly convex over $\overline{\Omega}$;
(iii) $\phi_0\big|_{\partial\Omega} = 0$.

Proof. The function $-4\operatorname{diam}(\Omega)\cdot d(x) + (d(x))^2$ introduced in Lemma 2.8 can be extended from the near-boundary region Γ_{μ_0} to the interior region $\overline{\Omega}\setminus\Gamma_{\mu_0}$ in such a way that the resulting function, denoted as ϕ_0, is C^2 and uniformly convex throughout $\overline{\Omega}$. Now ϕ_0 serves as the desired function for this lemma. $\square$

By Lemma 2.9, we have

$$\phi_0(x) < 0, \quad \forall x \in \Omega. \tag{2.9}$$

Denote

$$\eta_0 := \max_{x\in\overline{\Omega}} |\phi_0(x)|. \tag{2.10}$$

Hence $\eta_0 > 0$, and there always exists some point $x_0 \in \Omega$ such that

$$|\phi_0(x_0)| = \eta_0. \tag{2.11}$$

Now we are ready to derive the distance function estimate in the following lemma. For the $n = 1$ case, this estimate can be easily depicted as Figure 1.

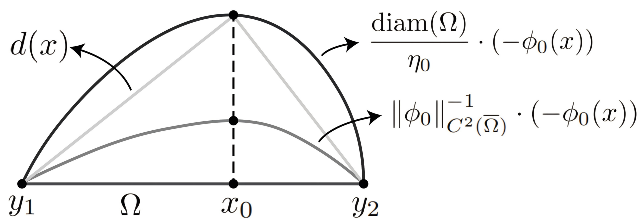

FIGURE 1. Estimate of distance function $d(x)$.

Lemma 2.10 (Estimate of distance function $d(x)$)**.** *There holds*

$$\|\phi_0\|^{-1}_{C^2(\overline{\Omega})} \cdot (-\phi_0(x)) \leq d(x) \leq \frac{\mathrm{diam}(\Omega)}{\eta_0} \cdot (-\phi_0(x)), \quad \forall x \in \Omega.$$

Proof. We note that for any $x \in \Omega$, there exists $z \in \partial\Omega$ such that $d(x) = |x - z|$. Since ϕ_0 is a convex function and $\phi_0(z) = 0$, we can derive

$$\frac{|\phi_0(x)|}{d(x)} = \frac{|\phi_0(x) - \phi_0(z)|}{|x - z|} \leq \|D\phi_0\|_{L^\infty(\Omega)} \leq \|\phi_0\|_{C^2(\overline{\Omega})}. \tag{2.12}$$

Fix any $x \in \Omega$ with $x \neq x_0$. We use L to denote the line connecting x_0 and x. Due to the convexity of Ω, there always exist two intersection points of L and $\partial\Omega$. Without loss of generality, we can assume that $L \cap \partial\Omega = \{y_1, y_2\}$ and x is between x_0 and y_1. Since ϕ_0 restricted to L is also a convex function, we have

$$\frac{|\phi_0(y_1) - \phi_0(x)|}{|y_1 - x|} \geq \frac{|\phi_0(y_1) - \phi_0(x_0)|}{|y_1 - x_0|}.$$

Using $\phi_0(y_1) = 0$ and (2.11), we further obtain

$$|\phi_0(x)| \geq \frac{|y_1 - x|}{|y_1 - x_0|} |\phi_0(x_0)| \geq \frac{d(x)}{\mathrm{diam}(\Omega)} \eta_0. \tag{2.13}$$

Combining (2.12) and (2.13) gives rise to

$$\|\phi_0\|^{-1}_{C^2(\overline{\Omega})} |\phi_0(x)| \leq d(x) \leq \frac{\mathrm{diam}(\Omega)}{\eta_0} |\phi_0(x)|, \quad \forall x \in \Omega.$$

Together with (2.9), this leads us to the desired result. □

2.4. **Eigenvalue estimates.** Different from [19], we will employ quadratic form to do eigenvalue estimates for the Hessian matrix of convex function in this paper.

For convenience, we provide general expressions in quadratic form for the minimum and maximum eigenvalues of a positive semi-definite matrix in the following lemma. The result is evident, so we will omit its proof.

Lemma 2.11 (Eigenvalue estimates in quadratic form)**.** *If A is a positive semi-definite matrix, then we have*

$$\lambda_{\min}(A) = \min_{\boldsymbol{\xi} \neq 0} \frac{\boldsymbol{\xi}^T A \boldsymbol{\xi}}{\boldsymbol{\xi}^T \boldsymbol{\xi}} = \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T A \boldsymbol{\xi},$$
$$\lambda_{\max}(A) = \max_{\boldsymbol{\xi} \neq 0} \frac{\boldsymbol{\xi}^T A \boldsymbol{\xi}}{\boldsymbol{\xi}^T \boldsymbol{\xi}} = \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T A \boldsymbol{\xi}.$$

Here, $\frac{\boldsymbol{\xi}^T A \boldsymbol{\xi}}{\boldsymbol{\xi}^T \boldsymbol{\xi}}$ is known as the Rayleigh quotient.

Based on Lemma 2.11, we have the following two remarks for ϕ_0 introduced in Lemma 2.9:

Remark 2.12. *Since $\phi_0 \in C^2(\overline{\Omega})$ is uniformly convex, then*

$$D^2\phi_0 \text{ is positive definite, i.e. } D^2\phi_0 > 0,$$

and we can denote

$$\underline{\rho_0} := \min_{x\in\overline{\Omega},\, |\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0(x)\,\boldsymbol{\xi},$$
$$\overline{\rho_0} := \max_{x\in\overline{\Omega},\, |\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0(x)\,\boldsymbol{\xi},$$

such that

$$\underline{\rho_0}\,\mathrm{I}_n \le D^2\phi_0 \le \overline{\rho_0}\,\mathrm{I}_n,$$

where I_n *is the* n*-order unit matrix, and the constants* $\underline{\rho_0}$ *and* $\overline{\rho_0}$ *satisfy* $0 < \underline{\rho_0} \le \overline{\rho_0}$.

Remark 2.13. *We note that all the* n *eigenvalues of matrix* $D\phi_0 \otimes D\phi_0$ *are* $0, \cdots, 0, |D\phi_0|^2$. *It follows that*

$$D\phi_0 \otimes D\phi_0 \text{ is positive semi-definite, i.e. } D\phi_0 \otimes D\phi_0 \ge 0.$$

Moreover, if we use $\boldsymbol{\xi_1}$ *and* $\boldsymbol{\xi_2}$ *to denote the unit eigenvectors of the matrix* $D\phi_0 \otimes D\phi_0$ *corresponding to the eigenvalues* 0 *and* $|D\phi_0|^2$ *respectively, then we have*

$$\min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D\phi_0 \otimes D\phi_0\,\boldsymbol{\xi} = \boldsymbol{\xi_1}^T D\phi_0 \otimes D\phi_0\,\boldsymbol{\xi_1} = 0,$$
$$\max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D\phi_0 \otimes D\phi_0\,\boldsymbol{\xi} = \boldsymbol{\xi_2}^T D\phi_0 \otimes D\phi_0\,\boldsymbol{\xi_2} = |D\phi_0|^2.$$

3. Passing from classical sub-solution to viscosity solution

This section is dedicated to proving a key lemma that facilitates the passage from classical sub-solutions to viscosity solutions for the problem (1.1)-(1.2).

Lemma 3.1 (Key Lemma)**.** *If the equation* (1.1) *admits a convex viscosity sub-solution* $v \in C(\overline{\Omega})$ *satisfying* $v\big|_{\partial\Omega} = \varphi$, *then it admits a convex viscosity solution* $u \in C(\overline{\Omega})$ *satisfying* $u\big|_{\partial\Omega} = \varphi$. *In particular, if the equation* (1.1) *admits a convex classical sub-solution* $v \in C(\overline{\Omega})$ *satisfying* $v\big|_{\partial\Omega} = \varphi$, *then it admits a convex viscosity solution* $u \in C(\overline{\Omega})$ *satisfying* $u\big|_{\partial\Omega} = \varphi$.

The proof of Lemma 3.1 is based on an adaption of the Perron method. We first define

$$S := \{v :\ v \in C(\overline{\Omega}),\ v\big|_{\partial\Omega} = \varphi,\ \text{ and } v \text{ is a convex viscosity sub-solution to (1.1) over } \Omega\}. \tag{3.1}$$

Since the condition of this lemma already gives $S \neq \emptyset$, our goal is reduced to showing that

$$u_0(x) := \sup_{v\in S} v(x), \quad \forall x \in \overline{\Omega} \tag{3.2}$$

is the desired convex viscosity solution to (1.1). The rest of the proof can be split into five steps.

Step 1: $u_0\big|_{\partial\Omega} = \varphi$.

This is clear because it holds for any $x \in \partial\Omega$ that

$$u_0(x) = \sup_{v\in S} v(x) = \sup_{v\in S} \varphi(x) = \varphi(x).$$

Step 2: u_0 *is a convex function over* $\overline{\Omega}$.

For any $x_0, y_0 \in \overline{\Omega}$, $t_0 \in (0,1)$, and $z_0 = t_0x_0 + (1-t_0)y_0 \in \Omega$, we can infer from the definition (3.2) that there exists a sequence $\{v_k\} \subset S$ such that

$$\lim_{k\to\infty} v_k(z_0) = u_0(z_0).$$

Since v_k is convex, there holds

$$v_k(z_0) \le t_0 v_k(x_0) + (1-t_0)v_k(y_0).$$

It follows that

$$\begin{aligned} u_0(z_0) &= \lim_{k\to\infty} v_k(z_0) = \overline{\lim_{k\to\infty}}\, v_k(z_0) \\ &\le \overline{\lim_{k\to\infty}}\, (t_0v_k(x_0) + (1-t_0)v_k(y_0)) \\ &\le t_0u_0(x_0) + (1-t_0)u_0(y_0). \end{aligned}$$

Thus u_0 is a convex function over $\overline{\Omega}$.

Step 3: $u_0 \in C(\overline{\Omega})$.

From Step 1, we have $u_0\big|_{\partial\Omega} = \varphi$. Using Step 2 and the openness of Ω, we note that $u_0 \in C(\Omega)$. It remains to prove that u_0 is also continuous on $\partial\Omega$.

For any $z_0 \in \Omega$, let $\mathcal{C}_{z_0}(x)$ be the cone function generated by the point $(z_0, u_0(z_0))$ and the set of all boundary value points $\{(x, \varphi(x)) : x \in \partial\Omega\}$. By virtue of Step 2, we have $\mathcal{C}_{z_0}(x) \geq u_0(x)$ for any $x \in \overline{\Omega}$. Clearly, we also have $\mathcal{C}_{z_0}(x) \in C(\overline{\Omega})$ and $\mathcal{C}_{z_0}(x)\big|_{\partial\Omega} = \varphi$. Thus we can derive

$$\lim_{x \to y \in \partial\Omega} \mathcal{C}_{z_0}(x) = \varphi(y). \tag{3.3}$$

Now we arbitrarily take $v_0 \in S$. Using the definition (3.1) of S then gives rise to $v_0(x) \in C(\overline{\Omega})$ and $v_0(x)\big|_{\partial\Omega} = \varphi$. Hence we obtain

$$\lim_{x \to y \in \partial\Omega} v_0(x) = \varphi(y). \tag{3.4}$$

By the definition (3.2) of u_0, we note that $u_0(x) \geq v_0(x)$ for any $x \in \overline{\Omega}$. It follows that

$$\mathcal{C}_{z_0}(x) \geq u_0(x) \geq v_0(x), \quad \forall x \in \overline{\Omega}, \tag{3.5}$$

For the $n = 1$ case, this relation can be easily illustrated by Figure 2, where $y_1, y_2 \in \partial\Omega$, both $\mathcal{C}_{z_0}(x)$ and

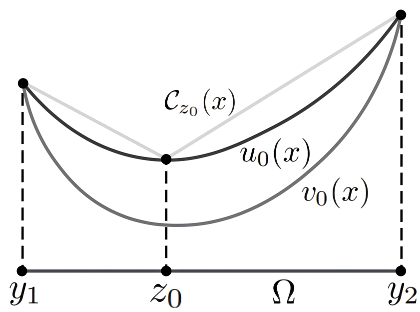

FIGURE 2. Relations among the functions $\mathcal{C}_{z_0}(x)$, $u_0(x)$, and $v_0(x)$.

$u_0(x)$ take the minimum at z_0. Combining (3.3), (3.4) and (3.5) together, we can apply the squeeze theorem to get

$$\lim_{x \to y \in \partial\Omega} u_0(x) = \varphi(y),$$

which implies that u_0 is continuous on $\partial\Omega$. Up to now, we have proved that u_0 is continuous over $\overline{\Omega}$.

Step 4: u_0 *is a viscosity sub-solution to* (1.1) *over* Ω.

The proof of this step is by contradiction. We suppose u_0 does not satisfy the condition of viscosity sub-solution at some point $\overline{x_0} \in \Omega$. Then there exists an open neighborhood $U_0(\overline{x_0}) \subset \Omega$ and a convex function $\overline{\phi_0} \in C^2(U_0(\overline{x_0}))$ satisfying

$$(\overline{\phi_0} - u_0)(x) \geq (\overline{\phi_0} - u_0)(\overline{x_0}), \quad \forall x \in U_0(\overline{x_0}) \tag{3.6}$$

such that

$$F\left(\lambda_1(D^2\overline{\phi_0}(\overline{x_0})), \cdots, \lambda_n(D^2\overline{\phi_0}(\overline{x_0}))\right) < g(\overline{x_0})(d(\overline{x_0}))^\alpha.$$

Denote

$$\overline{\zeta_0} := g(\overline{x_0})(d(\overline{x_0}))^\alpha - F\left(\lambda_1(D^2\overline{\phi_0}(\overline{x_0})), \cdots, \lambda_n(D^2\overline{\phi_0}(\overline{x_0}))\right).$$

It follows that $\overline{\zeta_0} > 0$ and

$$F\left(\lambda_1(D^2\overline{\phi_0}(\overline{x_0})), \cdots, \lambda_n(D^2\overline{\phi_0}(\overline{x_0}))\right) < g(\overline{x_0})(d(\overline{x_0}))^\alpha - \frac{\overline{\zeta_0}}{2}.$$

Since $F \in C(\mathbb{R}^n)$, $g(x) \in C(\overline{\Omega})$, $d(x) \in C(\overline{\Omega})$ and $\overline{\phi_0} \in C^2(U_0(\overline{x_0}))$, then there exists $\overline{\delta_0} > 0$ such that for any $x \in B(\overline{x_0}, \overline{\delta_0}) \subset U_0(\overline{x_0})$, we have

$$F\left(\lambda_1(D^2\overline{\phi_0}(x)), \cdots, \lambda_n(D^2\overline{\phi_0}(x))\right) < g(x)(d(x))^\alpha - \frac{\overline{\zeta_0}}{4}, \quad \forall x \in B(\overline{x_0}, \overline{\delta_0}).$$

Let

$$\overline{\phi_1}(x) := \overline{\phi_0}(x) + \frac{\overline{\varepsilon}|x-\overline{x_0}|^2}{2} - \left(\overline{\phi_0} - u_0\right)(\overline{x_0}) - \frac{\overline{\varepsilon}\,\overline{\delta_0}^2}{8}, \quad \forall x \in B(\overline{x_0}, \overline{\delta_0}), \tag{3.7}$$

where $\overline{\varepsilon} \in (0,1]$ is a constant to be determined later. We notice that

$$D^2\overline{\phi_1}(x) = D^2\overline{\phi_0}(x) + \overline{\varepsilon}\,\mathrm{I}_n, \qquad \lambda_k\left(D^2\overline{\phi_1}(x)\right) = \lambda_k\left(D^2\overline{\phi_0}(x)\right) + \overline{\varepsilon}, \quad \forall k = 1, \cdots, n.$$

Denote

$$R_0 := \left\|\overline{\phi_0}(x)\right\|_{C^2(\overline{B(\overline{x_0},\overline{\delta_0})})}.$$

Then we have

$$\begin{aligned}
&\left(\lambda_1(D^2\overline{\phi_1}(x)), \cdots, \lambda_n(D^2\overline{\phi_1}(x))\right) \in [0, R_0+1]^n, \quad \forall x \in B(\overline{x_0}, \overline{\delta_0}),\\
&\left(\lambda_1(D^2\overline{\phi_0}(x)), \cdots, \lambda_n(D^2\overline{\phi_0}(x))\right) \in [0, R_0+1]^n, \quad \forall x \in B(\overline{x_0}, \overline{\delta_0}).
\end{aligned}$$

By virtue of $F \in C(\mathbb{R}^n)$, it is clear that F is uniformly continuous on $[0, R_0+1]^n$, which means there exists a constant $\overline{\varepsilon_0} > 0$ such that for any $0 < \overline{\varepsilon} \le \overline{\varepsilon_0}$, we have

$$\left|F\left(\lambda_1(D^2\overline{\phi_1}(x)), \cdots, \lambda_n(D^2\overline{\phi_1}(x))\right) - F\left(\lambda_1(D^2\overline{\phi_0}(x)), \cdots, \lambda_n(D^2\overline{\phi_0}(x))\right)\right| < \frac{\overline{\zeta_0}}{8}, \quad \forall x \in B(\overline{x_0}, \overline{\delta_0}).$$

Up to now, we get

$$F\left(\lambda_1(D^2\overline{\phi_1}(x)), \cdots, \lambda_n(D^2\overline{\phi_1}(x))\right) < g(x)(d(x))^\alpha - \frac{\overline{\zeta_0}}{8}, \quad \forall x \in B(\overline{x_0}, \overline{\delta_0}). \tag{3.8}$$

Without loss of generality, we now take $\overline{\varepsilon} = \overline{\varepsilon_0}$. Consider

$$U_1 := \{x \in B(\overline{x_0}, \overline{\delta_0}) : \ \overline{\phi_1}(x) < u_0(x)\}. \tag{3.9}$$

Hence, for any $x \in U_1$, using definition (3.7) now leads us to

$$\overline{\phi_0}(x) + \frac{\overline{\varepsilon_0}|x-\overline{x_0}|^2}{2} - \left(\overline{\phi_0} - u_0\right)(\overline{x_0}) - \frac{\overline{\varepsilon_0}\,\overline{\delta_0}^2}{8} < u_0(x),$$

that is,

$$\frac{\overline{\varepsilon_0}|x-\overline{x_0}|^2}{2} < \left(\overline{\phi_0} - u_0\right)(\overline{x_0}) - \left(\overline{\phi_0} - u_0\right)(x) + \frac{\overline{\varepsilon_0}\,\overline{\delta_0}^2}{8}.$$

which along with (3.6) further gives

$$|x - \overline{x_0}| < \frac{\overline{\delta_0}}{2}, \qquad \text{i.e. } x \in B(\overline{x_0}, \frac{\overline{\delta_0}}{2}).$$

Now we obtain

$$U_1 \subset B(\overline{x_0}, \frac{\overline{\delta_0}}{2}) \subset\subset B(\overline{x_0}, \overline{\delta_0}). \tag{3.10}$$

In view of (3.2) and (3.7), we note that

$$\begin{aligned}
u_0(\overline{x_0}) &= \sup_{v \in S} v(\overline{x_0}),\\
\overline{\phi_1}(\overline{x_0}) &= u_0(\overline{x_0}) - \frac{\overline{\varepsilon_0}\,\overline{\delta_0}^2}{8}.
\end{aligned}$$

Then by definition of supremum, there exists $v_1 \in S$ such that

$$v_1(\overline{x_0}) > u_0(\overline{x_0}) - \frac{\overline{\varepsilon_0}\,\overline{\delta_0}^2}{16} > u_0(\overline{x_0}) - \frac{\overline{\varepsilon_0}\,\overline{\delta_0}^2}{8} = \overline{\phi_1}(\overline{x_0}).$$

Let

$$U_2 := \{x \in B(\overline{x_0}, \overline{\delta_0}) : \ \overline{\phi_1}(x) < v_1(x)\}.$$

Since $v_1(x) \le u_0(x)$ holds for any $x \in \Omega$, then by (3.9) and (3.10), we have

$$U_2 \subset U_1 \subset\subset B(\overline{x_0}, \overline{\delta_0}) \subset \Omega.$$

We turn to consider $\overline{\phi_1} - v_1$ on U_2. By virtue of $\overline{\phi_1} - v_1 \in C(B(\overline{x_0}, \overline{\delta_0}))$, there always exists some point $z_0 \in \overline{U_2}$ such that

$$\left(\overline{\phi_1} - v_1\right)(z_0) = \min_{x \in \overline{U_2}} \left(\overline{\phi_1} - v_1\right)(x).$$

Together with $\left(\overline{\phi_1} - v_1\right)\big|_{\partial U_2} = 0$, we can derive $z_0 \in U_2$, which implies that $\overline{\phi_1} - v_1$ indeed takes its minimum at z_0 in the interior of U_2. Due to $\overline{\phi_0} \in C^2(U_0(\overline{x_0}))$ and (3.7), we have $\overline{\phi_1} \in C^2(U_2)$. We also note that $v_1 \in S$ is a viscosity sub-solution to (1.1) over Ω. Then Definition 2.2 leads us to

$$F\left(\lambda_1(D^2\overline{\phi_1}(z_0)), \cdots, \lambda_n(D^2\overline{\phi_1}(z_0))\right) \geq g(z_0)(d(z_0))^\alpha,$$

which contradicts (3.8). Thus we are ready to conclude that u_0 is a viscosity sub-solution to (1.1) over Ω.

Step 5: *u_0 is a viscosity super-solution to* (1.1) *over* Ω.

By the above four steps, we already obtain $u_0 \in S$. The proof of this step is also by contradiction. We suppose u_0 does not satisfy the condition of viscosity super-solution at some point $\underline{x_0} \in \Omega$. Then there exists an open neighborhood $V_0(\underline{x_0}) \subset \Omega$ and a convex function $\underline{\phi_0} \in C^2(V_0(\underline{x_0}))$ satisfying

$$(\underline{\phi_0} - u_0)(x) \leq (\underline{\phi_0} - u_0)(\underline{x_0}), \quad \forall x \in V_0(\underline{x_0})$$

such that

$$F\left(\lambda_1(D^2\underline{\phi_0}(\underline{x_0})), \cdots, \lambda_n(D^2\underline{\phi_0}(\underline{x_0}))\right) > g(\underline{x_0})(d(\underline{x_0}))^\alpha.$$

Denote

$$\underline{\zeta_0} := F\left(\lambda_1(D^2\underline{\phi_0}(\underline{x_0})), \cdots, \lambda_n(D^2\underline{\phi_0}(\underline{x_0}))\right) - g(\underline{x_0})(d(\underline{x_0}))^\alpha.$$

Then we have $\underline{\zeta_0} > 0$ and

$$F\left(\lambda_1(D^2\underline{\phi_0}(\underline{x_0})), \cdots, \lambda_n(D^2\underline{\phi_0}(\underline{x_0}))\right) > g(\underline{x_0})(d(\underline{x_0}))^\alpha + \frac{\underline{\zeta_0}}{2}.$$

Since $F \in C(\mathbb{R}^n)$, $g(x) \in C(\overline{\Omega})$, $d(x) \in C(\overline{\Omega})$ and $\underline{\phi_0} \in C^2(V_0(\underline{x_0}))$, then there exists $\underline{\delta_0} > 0$ such that for any $x \in B(\underline{x_0}, \underline{\delta_0}) \subset V_0(\underline{x_0})$, we have

$$F\left(\lambda_1(D^2\underline{\phi_0}(x)), \cdots, \lambda_n(D^2\underline{\phi_0}(x))\right) > g(x)(d(x))^\alpha + \frac{\underline{\zeta_0}}{4}, \quad \forall x \in B(\underline{x_0}, \underline{\delta_0}).$$

Let

$$\underline{\phi_1}(x) := \underline{\phi_0}(x) - \frac{\underline{\varepsilon}|x - \underline{x_0}|^2}{2} - \left(\underline{\phi_0} - u_0\right)(\underline{x_0}) + \frac{\underline{\varepsilon}\underline{\delta_0}^2}{8}, \quad \forall x \in B(\underline{x_0}, \underline{\delta_0}), \tag{3.11}$$

where $\underline{\varepsilon} > 0$ is a constant to be determined later. It is clear that

$$D^2\underline{\phi_1}(x) = D^2\underline{\phi_0}(x) - \underline{\varepsilon}\,\mathrm{I}_n, \qquad \lambda_k\left(D^2\underline{\phi_1}(x)\right) = \lambda_k\left(D^2\underline{\phi_0}(x)\right) - \underline{\varepsilon}, \quad \forall k = 1, \cdots, n.$$

Similar to the arguments used in Step 4, we can derive that there exists a constant $\underline{\varepsilon_0} > 0$ such that for any $0 < \underline{\varepsilon} \leq \underline{\varepsilon_0}$, we have

$$\left|F\left(\lambda_1(D^2\underline{\phi_1}(x)), \cdots, \lambda_n(D^2\underline{\phi_1}(x))\right) - F\left(\lambda_1(D^2\underline{\phi_0}(x)), \cdots, \lambda_n(D^2\underline{\phi_0}(x))\right)\right| < \frac{\underline{\zeta_0}}{8}, \quad \forall x \in B(\underline{x_0}, \underline{\delta_0}).$$

Hence we obtain

$$F\left(\lambda_1(D^2\underline{\phi_1}(x)), \cdots, \lambda_n(D^2\underline{\phi_1}(x))\right) > g(x)(d(x))^\alpha + \frac{\underline{\zeta_0}}{8}, \quad \forall x \in B(\underline{x_0}, \underline{\delta_0}). \tag{3.12}$$

Without loss of generality, we now take $\underline{\varepsilon} = \underline{\varepsilon_0}$. Consider

$$V_1 := \{x \in B(\underline{x_0}, \underline{\delta_0}) : \ \underline{\phi_1}(x) > u_0(x)\}. \tag{3.13}$$

Then for any $x \in V_1$, by using the similar arguments in the Step 4, we can infer that

$$V_1 \subset B(\underline{x_0}, \frac{\underline{\delta_0}}{2}) \subset\subset B(\underline{x_0}, \underline{\delta_0}) \subset \Omega. \tag{3.14}$$

Define

$$\underline{u_0}(x) := \begin{cases} \underline{\phi_1}(x) & \text{if } x \in V_1, \\ u_0(x), & \text{if } x \in \overline{\Omega} \setminus V_1. \end{cases} \tag{3.15}$$

In fact, by (3.13) and (3.14), we have

$$\underline{u_0}(x) = \begin{cases} \max\{u_0(x), \underline{\phi_1}(x)\} & \text{if } x \in B(\underline{x_0}, \underline{\delta_0}), \\ u_0(x), & \text{if } x \in \overline{\Omega} \setminus B(\underline{x_0}, \underline{\delta_0}). \end{cases}$$

We also note that

$$\max\{u_0(x), \underline{\phi_1}(x)\} = u_0(x), \quad \forall x \in \partial B(\underline{x_0}, \underline{\delta_0}).$$

Then it follows that $\underline{u_0} \in C(\overline{\Omega})$. We have already proved in Step 4 that u_0 is a viscosity sub-solution to (1.1) over Ω. In view of $\underline{\phi_0} \in C^2(U_0(\underline{x_0}))$ and (3.11), we have $\underline{\phi_1} \in C^2(B(\underline{x_0}, \underline{\delta_0}))$. By virtue of (3.12) and Definition 2.1, $\underline{\phi_1}$ is a classical sub-solution to (1.1) over $B(\underline{x_0}, \underline{\delta_0})$, and then by Lemma 2.3, $\underline{\phi_1}$ is also a viscosity sub-solution to (1.1) over $B(\underline{x_0}, \underline{\delta_0})$. Therefore, $\max\{u_0, \underline{\phi_1}\}$ is a viscosity sub-solution to (1.1) over $B(\underline{x_0}, \underline{\delta_0})$, and thus $\underline{u_0}$ is a viscosity sub-solution to (1.1) over $B(\underline{x_0}, \underline{\delta_0})$. Using (3.15) and (1.2), we notice that

$$\underline{u_0}\big|_{\partial\Omega} = u_0\big|_{\partial\Omega} = \varphi.$$

Thus, by (3.1), we can obtain $\underline{u_0} \in S$. Consequently, by (3.2), we have

$$u_0(x) = \sup_{v \in S} v(x) \geq \underline{u_0}(x), \quad \forall x \in \Omega. \tag{3.16}$$

On the other hand, by (3.13) and (3.15), there holds

$$\underline{u_0}(x) = \underline{\phi_1}(x) > u_0(x), \quad \forall x \in V_1,$$

which is a contradiction to (3.16) and (3.14). Hence we can summarize that u_0 is a viscosity super-solution to (1.1) over Ω.

Up to now, u_0 is the viscosity solution u as desired, and thus we have completed the proof of Lemma 3.1.

4. Proof of Existence Theorem

The proof of Theorem 1.2 is divided into four cases according to the value of α. For each case, we first construct a classically strict sub-solution W_k by using the eigenvalue estimates in quadratic form; Lemma 3.1 is then employed to establish the existence of a convex viscosity solution u, while the Comparison Principle (Lemma 2.4) ensures that $u \geq W_k$; finally, we apply Lemma 2.10 to obtain the lower bound estimates for u via the distance function $d(x)$.

4.1. **Case 1:** $\alpha \in [0, +\infty)$**.** Denote

$$W_k = C\phi_0 + \psi_k,$$

where $C > 0$ is a constant to be determined. Then we have

$$D^2 W_k = C D^2\phi_0 + D^2\psi_k.$$

By Remark 2.6, Lemma 2.11 and Remark 2.12, we can derive

$$\lambda_{\min}(D^2W_k) = \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2 W_k\, \boldsymbol{\xi} = \min_{|\boldsymbol{\xi}|=1} \left(C\boldsymbol{\xi}^T D^2\phi_0\, \boldsymbol{\xi} + \boldsymbol{\xi}^T D^2\psi_k\, \boldsymbol{\xi}\right) \geq C \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0\, \boldsymbol{\xi} \geq C\underline{\rho_0},$$

$$\lambda_{\max}(D^2W_k) = \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2 W_k\, \boldsymbol{\xi} = \max_{|\boldsymbol{\xi}|=1} \left(C\boldsymbol{\xi}^T D^2\phi_0\, \boldsymbol{\xi} + \boldsymbol{\xi}^T D^2\psi_k\, \boldsymbol{\xi}\right) \geq C \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0\, \boldsymbol{\xi} \geq C\underline{\rho_0},$$

For any $x \in \Omega$, using the assumptions $(\mathbf{A_2})$-(i) and $(\mathbf{A_3})$ together with the fact $d(x) \leq \operatorname{diam}(\Omega)$, we can infer

$$\begin{aligned} & F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} \\ \geq{} & \Lambda_0 \left(\lambda_{\min}(D^2W_k)\right)^a \left(\lambda_{\max}(D^2W_k)\right)^b \cdot M_0^{-1}(d(x))^{-\alpha} \\ \geq{} & \Lambda_0 \left(C\underline{\rho_0}\right)^a \left(C\underline{\rho_0}\right)^b \cdot M_0^{-1}(\operatorname{diam}(\Omega))^{-\alpha} \\ ={} & C^{a+b}\Lambda_0 M_0^{-1} \underline{\rho_0}^{a+b}(\operatorname{diam}(\Omega))^{-\alpha}, \quad \forall x \in \Omega. \end{aligned}$$

Hence there exists some constant $C_0 > 0$ such that for any $C \geq C_0$ (without loss of generality, we can take $C = C_0$ here), we have

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} > 1, \quad \forall x \in \Omega,$$

i.e.

$$F\left(\lambda_1(D^2W_k),\cdots,\lambda_n(D^2W_k)\right) > g(x)(d(x))^{\alpha}, \quad \forall x\in\Omega. \tag{4.1}$$

Up to now, it is evident that

$$W_k = C_0\phi_0 + \psi_k \tag{4.2}$$

is a classically strict sub-solution to (1.1) over Ω.

By Lemma 2.9, there holds $\phi_0\big|_{\partial\Omega} = 0$. In view of Lemma 2.5, we have $\psi_k\big|_{\partial\Omega} = \varphi$. Then it is clear from (4.2) that $W_k \in C(\overline{\Omega})$ and

$$W_k\big|_{\partial\Omega} = \psi_k\big|_{\partial\Omega} = \varphi.$$

Now, by virtue of (4.1) and Lemma 3.1, the equation (1.1) admits a convex viscosity solution $u\in C(\overline{\Omega})$ satisfying $u\big|_{\partial\Omega} = \varphi$, i.e. the problem (1.1)-(1.2) admits a viscosity solution $u\in C(\overline{\Omega})$. Using (4.1) again, we can obtain from the Comparison Principle (Lemma 2.4) that

$$W_k(x) \le u(x), \quad \forall x\in\overline{\Omega},$$

i.e.

$$u(x) \ge C_0\phi_0(x) + \psi_k(x), \quad \forall x\in\overline{\Omega}.$$

Due to Lemma 2.10, we derive

$$d(x) \ge \|\phi_0\|^{-1}_{C^2(\overline{\Omega})}\cdot(-\phi_0(x)), \quad \forall x\in\overline{\Omega},$$

that is,

$$\phi_0(x) \ge \|\phi_0\|_{C^2(\overline{\Omega})}\cdot(-d(x)), \quad \forall x\in\overline{\Omega}.$$

Hence, we can deduce that

$$u(x) \ge \psi_k(x) - C_0\,\|\phi_0\|_{C^2(\overline{\Omega})}\,d(x), \quad \forall x\in\overline{\Omega}.$$

Denote $\overline{C_0} = C_0\,\|\phi_0\|_{C^2(\overline{\Omega})}$. Thus, by letting $k\to\infty$, we can conclude that

$$u(x) \ge \psi(x) - \overline{C_0}\,d(x), \quad \forall x\in\overline{\Omega}.$$

4.2. **Case 2:** $\alpha\in(-b,0)$**.** In this case, we have $-\alpha\in(0,b)$, and then by Lemma 2.10, we obtain

$$(d(x))^{-\alpha} \ge \|\phi_0\|^{\alpha}_{C^2(\overline{\Omega})}\cdot(-\phi_0(x))^{-\alpha}, \quad \forall x\in\Omega. \tag{4.3}$$

Now we consider

$$W_k = C_1\phi_0 + C_2\left(-\phi_0\right)^{1+\theta} + \psi_k,$$

where $C_1>0$, $C_2>0$ and $\theta\in(0,1)$ are constants to be determined. Our first step is to show that there exists $\theta_0\in(0,1)$ such that when $\theta=\theta_0$, if C_1 and C_2 are sufficiently large, W_k is a classically strict sub-solution to (1.1) over Ω.

It is clear that

$$\begin{aligned} DW_k &= C_1D\phi_0 - C_2(1+\theta)\left(-\phi_0\right)^{\theta}D\phi_0 + D\psi_k,\\ D^2W_k &= \left[C_1 - C_2(1+\theta)\left(-\phi_0\right)^{\theta}\right]D^2\phi_0 + C_2(1+\theta)\theta\left(-\phi_0\right)^{\theta-1}D\phi_0\otimes D\phi_0 + D^2\psi_k. \end{aligned}$$

Without loss of generality, we let

$$C_2 = \frac{C_1}{2(1+\theta){\eta_0}^{\theta}},$$

where C_1 and θ are still to be determined. By (2.11), we have $0<-\phi_0\le\eta_0$. Now we can derive

$$\begin{aligned} C_1 - C_2(1+\theta)\left(-\phi_0\right)^{\theta} &\ge \frac{C_1}{2},\\ C_2(1+\theta)\theta &= \frac{C_1\theta}{2{\eta_0}^{\theta}}. \end{aligned}$$

Hence

$$D^2W_k \ge \frac{C_1}{2}D^2\phi_0 + \frac{C_1\theta}{2{\eta_0}^{\theta}}\left(-\phi_0\right)^{\theta-1}D\phi_0\otimes D\phi_0 + D^2\psi_k.$$

Similar to the last case, using Lemma 2.11, Remark 2.6, Remark 2.12 and Remark 2.13, we can deduce that

$$\lambda_{\min}(D^2W_k) = \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2W_k\, \boldsymbol{\xi} \geq \frac{C_1}{2} \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0\, \boldsymbol{\xi} \geq \frac{C_1}{2} \underline{\rho_0}, \tag{4.4}$$

$$\lambda_{\max}(D^2W_k) = \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2W_k\, \boldsymbol{\xi} \geq \frac{C_1\theta}{2{\eta_0}^\theta} (-\phi_0)^{\theta-1} \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D\phi_0 \otimes D\phi_0\, \boldsymbol{\xi} = \frac{C_1\theta}{2{\eta_0}^\theta} (-\phi_0)^{\theta-1} |D\phi_0|^2. \tag{4.5}$$

Now we need to bound $|D\phi_0|$ in the estimate of $\lambda_{\max}(D^2W_k)$ carefully. For this purpose, we split Ω as two disjoint domains:

$$\Omega := \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 := \left\{x \in \Omega : -\frac{\eta_0}{2} \leq \phi_0(x) < 0\right\}$$

represents the domain near the boundary, and

$$\Omega_2 := \left\{x \in \Omega : -\eta_0 \leq \phi_0(x) < -\frac{\eta_0}{2}\right\}$$

represents the domain away from the boundary.

Case 2-1: In Ω_1, $D\phi_0$ is nondegenerate. We claim that

$$|D\phi_0(x)| \geq \frac{\eta_0}{2l_0}, \qquad \forall x \in \Omega_1. \tag{4.6}$$

Now we prove this claim. By virtue of (2.9) and (2.11), we have $\phi_0(x_0) = -\eta_0$. For any $x \in \Omega_1$, we let $\boldsymbol{e} = \frac{x-x_0}{|x-x_0|}$ and $L_{\boldsymbol{e}} = \{y = x + t\boldsymbol{e} : t \in \mathbb{R}\}$. We notice that ϕ_0 restricted to the segment $L_{\boldsymbol{e}} \cap \partial\Omega$ is also a convex function. Therefore we obtain

$$|D\phi_0(x)| \geq \boldsymbol{e} \cdot D\phi_0(x) \geq \frac{\phi_0(x) - \phi_0(x_0)}{|x - x_0|} \geq \frac{-\frac{\eta_0}{2} - (-\eta_0)}{l_0} = \frac{\eta_0}{2l_0}, \qquad \forall x \in \Omega_1,$$

which proves the claim.

For any $x \in \Omega_1$, using the assumptions $(\mathbf{A_2})$-(i) and $(\mathbf{A_3})$ together with (4.3), (4.4), (4.5) and (4.6), we infer that

$$\begin{aligned}
&F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} \\
&\geq \Lambda_0 \left(\lambda_{\min}(D^2W_k)\right)^a \left(\lambda_{\max}(D^2W_k)\right)^b \cdot M_0^{-1} (d(x))^{-\alpha} \\
&\geq \Lambda_0 \left(\frac{C_1}{2}\underline{\rho_0}\right)^a \left(\frac{C_1\theta}{2{\eta_0}^\theta} (-\phi_0)^{\theta-1} \left(\frac{\eta_0}{2l_0}\right)^2\right)^b M_0^{-1} \|\phi_0\|^\alpha_{C^2(\overline{\Omega})} \cdot (-\phi_0(x))^{-\alpha} \\
&= \frac{1}{2^{a+3b}} C_1^{a+b} \Lambda_0 M_0^{-1} \underline{\rho_0}^a \theta^b {\eta_0}^{(2-\theta)b} {l_0}^{-2b} \|\phi_0\|^\alpha_{C^2(\overline{\Omega})} \cdot (-\phi_0)^{b(\theta-1)-\alpha}, \quad \forall x \in \Omega_1.
\end{aligned}$$

Let us take θ_0 such that $b(\theta_0 - 1) - \alpha = 0$, i.e.

$$\theta_0 = \frac{b+\alpha}{b} \in (0,1).$$

When $\theta = \theta_0$, it is now clear that

$$\begin{aligned}
&F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} \\
&\geq \frac{1}{2^{a+3b}} C_1^{a+b} \Lambda_0 M_0^{-1} \underline{\rho_0}^a {\theta_0}^b {\eta_0}^{(2-\theta_0)b} {l_0}^{-2b} \|\phi_0\|^\alpha_{C^2(\overline{\Omega})}, \quad \forall x \in \Omega_1.
\end{aligned}$$

Hence there exists some constant $C_{2,1} > 0$ such that for any $C_1 \geq C_{2,1}$, we have

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} > 1, \quad \forall x \in \Omega_1,$$

i.e.

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) > g(x)(d(x))^\alpha, \quad \forall x \in \Omega_1. \tag{4.7}$$

Case 2-2: In Ω_2, $D\phi_0$ can be degenerate. By virtue of (2.9) and (2.11), ϕ_0 indeed takes its minimum at $x_0 \in \Omega_2$. Then we have $D\phi_0(x_0) = 0$, and hence we need to reconsider $\lambda_{\max}(D^2W_k)$ in Ω_2:

$$\lambda_{\max}(D^2W_k) = \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2W_k\,\boldsymbol{\xi} \geq \frac{C_1}{2} \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0\,\boldsymbol{\xi} \geq \frac{C_1}{2}\underline{\rho_0}. \tag{4.8}$$

For any $x \in \Omega_2$, we note that

$$-\phi_0(x) > \frac{\eta_0}{2}.$$

Then using the assumptions $(\mathbf{A_2})$-(i) and $(\mathbf{A_3})$ together with (4.3), (4.4) and (4.8), we obtain

$$\begin{aligned}
&F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} \\
\geq\ & \Lambda_0 \left(\lambda_{\min}(D^2W_k)\right)^a \left(\lambda_{\max}(D^2W_k)\right)^b \cdot M_0^{-1} (d(x))^{-\alpha} \\
\geq\ & \Lambda_0 \left(\frac{C_1}{2}\underline{\rho_0}\right)^a \left(\frac{C_1}{2}\underline{\rho_0}\right)^b M_0^{-1} \|\phi_0\|^\alpha_{C^2(\overline{\Omega})} \cdot (-\phi_0(x))^{-\alpha} \\
\geq\ & \Lambda_0 \left(\frac{C_1}{2}\underline{\rho_0}\right)^a \left(\frac{C_1}{2}\underline{\rho_0}\right)^b M_0^{-1} \|\phi_0\|^\alpha_{C^2(\overline{\Omega})} \cdot \left(\frac{\eta_0}{2}\right)^{-\alpha} \\
=\ & \frac{1}{2^{a+b-\alpha}} C_1^{a+b} \Lambda_0 M_0^{-1} \underline{\rho_0}^{a+b} \|\phi_0\|^\alpha_{C^2(\overline{\Omega})} \cdot {\eta_0}^{-\alpha}, \quad \forall x \in \Omega_2.
\end{aligned}$$

Hence there exists some constant $C_{2,2} > 0$ such that for any $C_1 \geq C_{2,2}$, we have

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} > 1, \quad \forall x \in \Omega_2,$$

i.e.

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) > g(x)(d(x))^\alpha, \quad \forall x \in \Omega_2. \tag{4.9}$$

Up to now, combining (4.7) and (4.9) together, we can conclude that for any $C_1 \geq C_0 := \max\{C_{2,1}, C_{2,2}\}$ (without loss of generality, we can take $C_1 = C_0$ here), there holds

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) > g(x)(d(x))^\alpha, \quad \forall x \in \Omega_1 \cup \Omega_2 = \Omega. \tag{4.10}$$

Therefore we obtain that

$$W_k = C_0\phi_0 + \frac{C_0}{2(1+\theta_0){\eta_0}^{\theta_0}} (-\phi_0)^{1+\theta_0} + \psi_k \tag{4.11}$$

is a classically strict sub-solution to (1.1) over Ω.

By (4.11), Lemma 2.5 and Lemma 2.9, we have $W_k \in C(\overline{\Omega})$ and

$$W_k\big|_{\partial\Omega} = \psi_k\big|_{\partial\Omega} = \varphi.$$

Now, by virtue of (4.10) and Lemma 3.1, the equation (1.1) admits a convex viscosity solution $u \in C(\overline{\Omega})$ satisfying $u\big|_{\partial\Omega} = \varphi$, i.e. the problem (1.1)-(1.2) admits a viscosity solution $u \in C(\overline{\Omega})$. Using (4.10) again, we can obtain from the Comparison Principle (Lemma 2.4) that

$$W_k(x) \leq u(x), \qquad \forall x \in \overline{\Omega}.$$

Then by the definition (4.11) and Lemma 2.10, we can derive

$$\begin{aligned}
u(x) &\geq C_0\phi_0(x) + \frac{C_0}{2(1+\theta_0){\eta_0}^{\theta_0}} (-\phi_0(x))^{1+\theta_0} + \psi_k(x) \\
&\geq C_0\phi_0(x) + \psi_k(x) \\
&\geq C_0 \|\phi_0\|_{C^2(\overline{\Omega})} \cdot (-d(x)) + \psi_k(x), \qquad \forall x \in \overline{\Omega}.
\end{aligned}$$

Denote $\overline{C_0} = C_0 \|\phi_0\|_{C^2(\overline{\Omega})}$. Thus, by letting $k \to \infty$, we arrive at the result that

$$u(x) \geq \psi(x) - \overline{C_0}\, d(x), \quad \forall x \in \overline{\Omega}.$$

To conclude, the above Case 1 and Case 2 lead us to Theorem 1.2 (i).

4.3. **Case 3:** $\alpha = -b$**.** We are going to construct a classically strict sub-solution in the following form:

$$W_k = -C(-\overline{\phi_0})\left[-\log(-\overline{\phi_0})\right]^{\theta} + \psi_k, \tag{4.12}$$

where $C > 0$ and $\theta \in (0,1]$ are constants to be determined, and

$$\overline{\phi_0} = \frac{\phi_0}{3\eta_0}.$$

By virtue of (2.9) and (2.11), we note that $-\overline{\phi_0}(x) \in (0, \frac{1}{3})$ for any $x \in \Omega$. Along with Lemma 2.10, we also have

$$\begin{aligned}(d(x))^b &\geq \|\phi_0\|_{C^2(\overline{\Omega})}^{-b} \cdot (-\phi_0(x))^b, \\ &\geq \|\phi_0\|_{C^2(\overline{\Omega})}^{-b} \cdot \left(-3\eta_0\overline{\phi_0}(x)\right)^b, \qquad \forall x \in \Omega.\end{aligned} \tag{4.13}$$

By (4.12), direct computations give rise to

$$\begin{aligned}DW_k &= \left\{C\left[-\log(-\overline{\phi_0})\right]^{\theta} - C\theta\left[-\log(-\overline{\phi_0})\right]^{\theta-1}\right\} \cdot \nabla\overline{\phi_0} + D\psi_k, \\ D^2W_k &= \left\{C\left[-\log(-\overline{\phi_0})\right]^{\theta} - C\theta\left[-\log(-\overline{\phi_0})\right]^{\theta-1}\right\} \cdot D^2\overline{\phi_0} \\ &\quad + \left\{C\theta(-\overline{\phi_0})^{-1}\left[-\log(-\overline{\phi_0})\right]^{\theta-1} - C\theta(\theta-1)(-\overline{\phi_0})^{-1}\left[-\log(-\overline{\phi_0})\right]^{\theta-2}\right\} \cdot D\overline{\phi_0} \otimes D\overline{\phi_0} + \nabla^2\psi_k.\end{aligned} \tag{4.14}$$

In order to bound $-\log(-\overline{\phi_0})$, we also split Ω as two disjoint domains.

$$\Omega := \Omega_3 \cup \Omega_4.$$

Here, we use

$$\Omega_3 := \{x \in \Omega : -\overline{\eta_0} \leq \phi_0(x) < 0\}$$

to represent the domain near the boundary, and use

$$\Omega_4 := \{x \in \Omega : -\eta_0 \leq \phi_0(x) < -\overline{\eta_0}\}$$

to represent the domain away from the boundary, where

$$\overline{\eta_0} = \min\{2e^{-2\theta}\eta_0, \frac{\eta_0}{2}\}.$$

Case 3-1: In Ω_3, $D\phi_0$ is nondegenerate. We also claim that

$$|D\phi_0(x)| \geq \frac{\eta_0}{2l_0}, \qquad \forall x \in \Omega_3. \tag{4.15}$$

In fact, this claim can be proved in a similar manner as (4.6) with only one modification due to the definition of Ω_3:

$$|D\phi_0(x)| \geq \frac{\phi_0(x) - \phi_0(x_0)}{|x - x_0|} \geq \frac{-\overline{\eta_0} - (-\eta_0)}{l_0} \geq \frac{-\frac{\eta_0}{2} - (-\eta_0)}{l_0} = \frac{\eta_0}{2l_0}, \qquad \forall x \in \Omega_3.$$

For $x \in \Omega_3$, it is obvious that $-2e^{-2\theta}\eta_0 \leq \phi_0 < 0$, then $0 < -\overline{\phi_0} \leq e^{-2\theta}$, and hence

$$-\log(-\overline{\phi_0}) \geq 2\theta. \tag{4.16}$$

It follows that

$$C\left[-\log(-\overline{\phi_0})\right]^{\theta} = C\left[-\log(-\overline{\phi_0})\right] \cdot \left[-\log(-\overline{\phi_0})\right]^{\theta-1} \geq 2C\theta\left[-\log(-\overline{\phi_0})\right]^{\theta-1},$$

which implies

$$C\theta\left[-\log(-\overline{\phi_0})\right]^{\theta-1} \leq \frac{1}{2}C\left[-\log(-\overline{\phi_0})\right]^{\theta}.$$

Together with (4.16), this yields

$$\frac{1}{2}C\left[-\log(-\overline{\phi_0})\right]^{\theta} \leq C\left[-\log(-\overline{\phi_0})\right]^{\theta} - C\theta\left[-\log(-\overline{\phi_0})\right]^{\theta-1} \leq C\left[-\log(-\overline{\phi_0})\right]^{\theta}. \tag{4.17}$$

Since $\theta \in (0,1]$, we also have

$$-C\theta(\theta-1)(-\overline{\phi_0})^{-1}\left[-\log(-\overline{\phi_0})\right]^{\theta-2} \geq 0. \tag{4.18}$$

Based on Remark 2.12 and Remark 2.13, we note that

$$D^2\overline{\phi_0} = \frac{1}{3\eta_0} D^2\phi_0 > 0,$$
$$D\overline{\phi_0} \otimes D\overline{\phi_0} = \frac{1}{9{\eta_0}^2} D\phi_0 \otimes D\phi_0 \geq 0.$$

Hence, using (4.17) and (4.18) leads us to

$$D^2 W_k \geq \frac{1}{2} C \left[-\log(-\overline{\phi_0})\right]^{\theta} \cdot D^2\overline{\phi_0} + C\theta(-\overline{\phi_0})^{-1} \left[-\log(-\overline{\phi_0})\right]^{\theta-1} \cdot D\overline{\phi_0} \otimes D\overline{\phi_0} + \nabla^2 \psi_k.$$

Now we are ready to deduce that

$$\begin{aligned}
\lambda_{\min}(D^2 W_k) &= \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2 W_k \, \boldsymbol{\xi} \\
&\geq \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T \left(\frac{1}{2} C \left[-\log(-\overline{\phi_0})\right]^{\theta} \cdot D^2\overline{\phi_0} \right) \boldsymbol{\xi} \\
&= \frac{1}{2} C \left[-\log(-\overline{\phi_0})\right]^{\theta} \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\overline{\phi_0} \, \boldsymbol{\xi} \\
&\geq \frac{1}{6\eta_0} C \left[-\log(-\overline{\phi_0})\right]^{\theta} \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0 \, \boldsymbol{\xi} \\
&\geq \frac{1}{6\eta_0} C \left[-\log(-\overline{\phi_0})\right]^{\theta} \underline{\rho_0}.
\end{aligned} \tag{4.19}$$

By virtue of Remark 2.13 and (4.15), we also obtain

$$\begin{aligned}
\lambda_{\max}(D^2 W_k) &= \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2 W_k \, \boldsymbol{\xi} \\
&\geq \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T \left(C\theta(-\overline{\phi_0})^{-1} \left[-\log(-\overline{\phi_0})\right]^{\theta-1} \cdot D\overline{\phi_0} \otimes D\overline{\phi_0} \right) \boldsymbol{\xi} \\
&= C\theta(-\overline{\phi_0})^{-1} \left[-\log(-\overline{\phi_0})\right]^{\theta-1} \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D\overline{\phi_0} \otimes D\overline{\phi_0} \, \boldsymbol{\xi} \\
&\geq \frac{1}{9{\eta_0}^2} C\theta(-\overline{\phi_0})^{-1} \left[-\log(-\overline{\phi_0})\right]^{\theta-1} \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D\phi_0 \otimes D\phi_0 \, \boldsymbol{\xi} \\
&= \frac{1}{9{\eta_0}^2} C\theta(-\overline{\phi_0})^{-1} \left[-\log(-\overline{\phi_0})\right]^{\theta-1} |D\phi_0|^2 \\
&\geq \frac{1}{36{l_0}^2} C\theta(-\overline{\phi_0})^{-1} \left[-\log(-\overline{\phi_0})\right]^{\theta-1}.
\end{aligned} \tag{4.20}$$

Using the assumptions $(\mathbf{A_2})$-(i) and $(\mathbf{A_3})$ together with (4.13), (4.15), (4.19) and (4.20), we derive

$$\begin{aligned}
& F\left(\lambda_1(D^2 W_k), \cdots, \lambda_n(D^2 W_k)\right) \cdot \left[g(x)(d(x))^{\alpha}\right]^{-1} \\
\geq\ & \Lambda_0 \left(\lambda_{\min}(D^2 W_k)\right)^a \left(\lambda_{\max}(D^2 W_k)\right)^b \cdot M_0^{-1} \left(d(x)\right)^b \\
\geq\ & \Lambda_0 \left\{ \frac{1}{6\eta_0} C \left[-\log(-\overline{\phi_0})\right]^{\theta} \underline{\rho_0} \right\}^a \left\{ \frac{1}{36{l_0}^2} C\theta(-\overline{\phi_0})^{-1} \left[-\log(-\overline{\phi_0})\right]^{\theta-1} \right\}^b M_0^{-1} \|\phi_0\|_{C^2(\overline{\Omega})}^{-b} \cdot \left(-3\eta_0\overline{\phi_0}(x)\right)^b \\
=\ & \frac{1}{6^a \times 12^b} C^{a+b} \Lambda_0 M_0^{-1} \underline{\rho_0}^a \theta^b {\eta_0}^{b-a} {l_0}^{-2b} \|\phi_0\|_{C^2(\overline{\Omega})}^{-b} \left[-\log(-\overline{\phi_0})\right]^{(a+b)\theta-b}, \quad \forall x \in \Omega_3.
\end{aligned}$$

Let us take θ_0 such that $(a+b)\theta_0 - b = 0$, i.e.

$$\theta_0 = \frac{b}{a+b} \in (0,1].$$

When $\theta = \theta_0$, it is now clear that

$$\begin{aligned}
& F\left(\lambda_1(D^2 W_k), \cdots, \lambda_n(D^2 W_k)\right) \cdot \left[g(x)(d(x))^{\alpha}\right]^{-1} \\
\geq\ & \frac{1}{6^a \times 12^b} C^{a+b} \Lambda_0 M_0^{-1} \underline{\rho_0}^a {\theta_0}^b {\eta_0}^{b-a} {l_0}^{-2b} \|\phi_0\|_{C^2(\overline{\Omega})}^{-b}, \quad \forall x \in \Omega_3.
\end{aligned}$$

Hence there exists some constant $C_{3,1} > 0$ such that for any $C \geq C_{3,1}$, we have

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} > 1, \quad \forall x \in \Omega_3,$$

i.e.

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) > g(x)(d(x))^\alpha, \quad \forall x \in \Omega_3. \tag{4.21}$$

Case 3-2: In Ω_4, $D\phi_0$ can be degenerate. By (2.9) and (2.11), ϕ_0 indeed takes its minimum at $x_0 \in \Omega_4$. It follows that $D\phi_0(x_0) = 0$ and therefore we need to reconsider $\lambda_{\max}(D^2W_k)$.

Now we still take $\theta = \theta_0 = \frac{b}{a+b} \in (0,1]$. For $x \in \Omega_4$, it is clear that $-\eta_0 \leq \phi_0 < -\overline{\eta_0} = -\min\{2e^{-2\theta_0}\eta_0, \frac{\eta_0}{2}\}$, then

$$\min\left\{e^{-2\theta_0}, \frac{1}{6}\right\} < -\overline{\phi_0} \leq \frac{1}{3}, \tag{4.22}$$

and hence

$$-\log(-\overline{\phi_0}) \geq -\log\frac{1}{3} = \log 3. \tag{4.23}$$

We can infer that

$$\begin{aligned} C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} &= C\left[-\log(-\overline{\phi_0})\right] \cdot \left[-\log(-\overline{\phi_0})\right]^{\theta_0 - 1} \\ &\geq C\log 3 \cdot \left[-\log(-\overline{\phi_0})\right]^{\theta_0 - 1} \\ &= \frac{\log 3}{\theta_0} \cdot C\theta_0\left[-\log(-\overline{\phi_0})\right]^{\theta_0 - 1}, \end{aligned}$$

which yields

$$C\theta_0\left[-\log(-\overline{\phi_0})\right]^{\theta_0 - 1} \leq \frac{\theta_0}{\log 3} \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0}.$$

Using (4.23) again, we obtain

$$C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} - C\theta\left[-\log(-\overline{\phi_0})\right]^{\theta_0 - 1} \geq \left(1 - \frac{\theta_0}{\log 3}\right) \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0}. \tag{4.24}$$

Here, $1 - \frac{\theta_0}{\log 3} > 0$ due to $\theta_0 \in (0,1]$ and $\log 3 > 1$. Hence, using (4.14), (4.23) and (4.24) leads us to

$$D^2W_k \geq \left(1 - \frac{\theta_0}{\log 3}\right) \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} \cdot D^2\overline{\phi_0} + C\theta_0(-\overline{\phi_0})^{-1}\left[-\log(-\overline{\phi_0})\right]^{\theta_0 - 1} \cdot D\overline{\phi_0} \otimes D\overline{\phi_0} + \nabla^2\psi_k.$$

Now we are ready to estimate $\lambda_{\min}(D^2W_k)$ and $\lambda_{\max}(D^2W_k)$ in Ω_4 in the same manner:

$$\begin{aligned} \lambda_{\max}(D^2W_k) \geq \lambda_{\min}(D^2W_k) &= \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2W_k\, \boldsymbol{\xi} \\ &\geq \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T \left(\left(1 - \frac{\theta_0}{\log 3}\right) \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} \cdot D^2\overline{\phi_0}\right) \boldsymbol{\xi} \\ &= \left(1 - \frac{\theta_0}{\log 3}\right) \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\overline{\phi_0}\, \boldsymbol{\xi} \\ &= \frac{1}{3\eta_0}\left(1 - \frac{\theta_0}{\log 3}\right) \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0\, \boldsymbol{\xi} \\ &\geq \frac{1}{3\eta_0}\left(1 - \frac{\theta_0}{\log 3}\right) \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} \underline{\rho_0}. \end{aligned} \tag{4.25}$$

Using the assumptions $(\mathbf{A_2})$-(i) and $(\mathbf{A_3})$ together with (4.13), (4.25), (4.22) and (4.23), we derive

$$\begin{aligned} & F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} \\ \geq\ & \Lambda_0\left(\lambda_{\min}(D^2W_k)\right)^a \left(\lambda_{\max}(D^2W_k)\right)^b \cdot M_0^{-1}(d(x))^b \\ \geq\ & \Lambda_0\left\{\frac{1}{3\eta_0}\left(1 - \frac{\theta_0}{\log 3}\right) \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} \underline{\rho_0}\right\}^a \left\{\frac{1}{3\eta_0}\left(1 - \frac{\theta_0}{\log 3}\right) \cdot C\left[-\log(-\overline{\phi_0})\right]^{\theta_0} \underline{\rho_0}\right\}^b \\ & \cdot M_0^{-1}\|\phi_0\|_{C^2(\overline{\Omega})}^{-b} \cdot \left(-3\eta_0\overline{\phi_0}(x)\right)^b \end{aligned}$$

$$= \frac{1}{3^a}\left(1-\frac{\theta_0}{\log 3}\right)^{a+b} \cdot C^{a+b}\Lambda_0 M_0^{-1}\underline{\rho_0}^{a+b}\eta_0{}^{-a}\,\|\phi_0\|_{C^2(\overline{\Omega})}^{-b}\left[-\log(-\overline{\phi_0})\right]^{(a+b)\theta_0}\cdot\left(-\overline{\phi_0}\right)^b$$
$$\geq \frac{1}{3^a}\left(1-\frac{\theta_0}{\log 3}\right)^{a+b} \cdot C^{a+b}\Lambda_0 M_0^{-1}\underline{\rho_0}^{a+b}\eta_0{}^{-a}\,\|\phi_0\|_{C^2(\overline{\Omega})}^{-b}(\log 3)^{(a+b)\theta_0}\cdot\left(\min\left\{e^{-2\theta_0},\frac{1}{6}\right\}\right)^b,\quad \forall x\in\Omega_4.$$

Hence there exists some constant $C_{3,2}>0$ such that for any $C\geq C_{3,2}$, we have

$$F\left(\lambda_1(D^2W_k),\cdots,\lambda_n(D^2W_k)\right)\cdot\left[g(x)(d(x))^\alpha\right]^{-1}>1,\quad \forall x\in\Omega_4,$$

i.e.

$$F\left(\lambda_1(D^2W_k),\cdots,\lambda_n(D^2W_k)\right)>g(x)(d(x))^\alpha,\quad \forall x\in\Omega_4. \tag{4.26}$$

Putting (4.21) and (4.26) together, we can summarize that for any $C\geq C_0:=\max\{C_{3,1},C_{3,2}\}$ (without loss of generality, we can take $C=C_0$ here), there holds

$$F\left(\lambda_1(D^2W_k),\cdots,\lambda_n(D^2W_k)\right)>g(x)(d(x))^\alpha,\quad \forall x\in\Omega_3\cup\Omega_4=\Omega. \tag{4.27}$$

Now we obtain that

$$W_k=-C_0(-\overline{\phi_0})\left[-\log(-\overline{\phi_0})\right]^{\theta_0}+\psi_k \tag{4.28}$$

is a classically strict sub-solution to (1.1) over Ω.

For any $x\in\Omega$, by Lemma 2.9, we note that

$$\lim_{x\to\partial\Omega}\left(-\overline{\phi_0}\right)=\lim_{x\to\partial\Omega}\left(-\frac{\phi_0}{3\eta_0}\right)=0,$$

which further yields

$$\lim_{x\to\partial\Omega}(-\overline{\phi_0})\left[-\log(-\overline{\phi_0})\right]^{\theta_0}=\lim_{t_1\to0}t_1\left(-\log t_1\right)^{\theta_0}=\lim_{t_1\to0}\frac{\left(\log\frac{1}{t_1}\right)^{\theta_0}}{\frac{1}{t_1}}=\lim_{t_2\to+\infty}\frac{(\log t_2)^{\theta_0}}{t_2}=0.$$

Then by the definition (4.28), Lemma 2.5 and Lemma 2.9, we have $W_k\in C(\overline{\Omega})$ and

$$W_k\big|_{\partial\Omega}=\psi_k\big|_{\partial\Omega}=\varphi.$$

Thus, using (4.27) and Lemma 3.1, the equation (1.1) admits a convex viscosity solution $u\in C(\overline{\Omega})$ satisfying $u\big|_{\partial\Omega}=\varphi$. Using (4.27) again, we can obtain from the Comparison Principle (Lemma 2.4) that

$$W_k(x)\leq u(x),\qquad \forall x\in\overline{\Omega}.$$

Then by definition (4.28), we can derive

$$u(x)\geq -C_0(-\overline{\phi_0}(x))\left[-\log(-\overline{\phi_0}(x))\right]^{\theta_0}+\psi_k(x),\qquad \forall x\in\overline{\Omega}.$$

By virtue of Lemma 2.10, we note that

$$-\overline{\phi_0}(x)=\frac{1}{3\eta_0}(-\phi_0(x))\leq\frac{1}{3\eta_0}\|\phi_0\|_{C^2(\overline{\Omega})}\cdot d(x),\qquad \forall x\in\overline{\Omega},$$

and

$$-\overline{\phi_0}(x)=\frac{1}{3\eta_0}(-\phi_0(x))\geq\frac{1}{3\eta_0}\frac{\eta_0}{\operatorname{diam}(\Omega)}\cdot d(x)=\frac{1}{3\operatorname{diam}(\Omega)}\cdot d(x),\qquad \forall x\in\overline{\Omega}.$$

The latter also gives

$$\left[-\log(-\overline{\phi_0}(x))\right]^{\theta_0}\leq\left[-\log\left(\frac{1}{3\operatorname{diam}(\Omega)}\cdot d(x)\right)\right]^{\theta_0},\qquad \forall x\in\overline{\Omega}.$$

Thus it follows that

$$u(x)\geq-\frac{C_0}{3\eta_0}\|\phi_0\|_{C^2(\overline{\Omega})}\cdot d(x)\cdot\left[-\log\left(\frac{1}{3\operatorname{diam}(\Omega)}\cdot d(x)\right)\right]^{\theta_0}+\psi_k(x),\qquad \forall x\in\overline{\Omega}.$$

Denote $\overline{C_0}=\frac{C_0}{3\eta_0}\|\phi_0\|_{C^2(\overline{\Omega})}$ and $\overline{C_1}=\frac{1}{3\operatorname{diam}(\Omega)}$. Therefore, by letting $k\to\infty$, we can conclude that

$$u(x)\geq\psi(x)-\overline{C_0}d(x)\left[-\log\left(\overline{C_1}d(x)\right)\right]^{\theta_0},\qquad \forall x\in\overline{\Omega},$$

where $\theta_0=\frac{b}{a+b}$. This gives rise to Theorem 1.2 (ii).

4.4. **Case 4:** $\alpha \in (-a-2b, -b)$**.** Now we consider

$$W_k = -C\left(-\phi_0\right)^{\theta} + \psi_k,$$

where $C \geq 0$ and $\theta \in (0,1)$ are constants to be determined.

It follows that

$$\begin{aligned} DW_k &= C\theta\left(-\phi_0\right)^{\theta-1} D\phi_0 + D\psi_k, \\ D^2W_k &= C\theta\left(-\phi_0\right)^{\theta-1} D^2\phi_0 + C\theta(1-\theta)\left(-\phi_0\right)^{\theta-2} D\phi_0 \otimes D\phi_0 + D^2\psi_k. \end{aligned}$$

Analogously, using Remark 2.6, Lemma 2.11, Remark 2.12 and Remark 2.13, we can derive that

$$\lambda_{\min}(D^2W_k) = \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2W_k\,\boldsymbol{\xi} \geq C\theta\left(-\phi_0\right)^{\theta-1} \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0\,\boldsymbol{\xi} \geq C\theta\left(-\phi_0\right)^{\theta-1} \underline{\rho_0}, \tag{4.29}$$

$$\lambda_{\max}(D^2W_k) = \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2W_k\,\boldsymbol{\xi} \geq C\theta(1-\theta)\left(-\phi_0\right)^{\theta-2} \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D\phi_0 \otimes D\phi_0\,\boldsymbol{\xi} = C\theta(1-\theta)\left(-\phi_0\right)^{\theta-2} |D\phi_0|^2. \tag{4.30}$$

Similar to (4.3), we also have

$$\left(d(x)\right)^{-\alpha} \geq \|\phi_0\|^{\alpha}_{C^2(\overline{\Omega})} \cdot \left(-\phi_0(x)\right)^{-\alpha}, \quad \forall x \in \Omega. \tag{4.31}$$

In the same manner as Case 2, we also split Ω as

$$\Omega = \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 = \left\{x \in \Omega : -\frac{\eta_0}{2} \leq \phi_0(x) < 0\right\}, \quad \Omega_2 = \left\{x \in \Omega : -\eta_0 \leq \phi_0(x) < -\frac{\eta_0}{2}\right\}.$$

Case 4-1: In Ω_1, $D\phi_0$ is nondegenerate. By (4.6), we also have

$$|D\phi_0(x)| \geq \frac{\eta_0}{2l_0}, \quad \forall x \in \Omega_1. \tag{4.32}$$

For any $x \in \Omega_1$, using the assumptions $(\mathbf{A_2})$-(i) and $(\mathbf{A_3})$ along with (4.29), (4.30), (4.31) and (4.32), we obtain

$$\begin{aligned} &F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot \left[g(x)(d(x))^{\alpha}\right]^{-1} \\ \geq\ & \Lambda_0 \left(\lambda_{\min}(D^2W_k)\right)^a \left(\lambda_{\max}(D^2W_k)\right)^b \cdot M_0^{-1} \left(d(x)\right)^{-\alpha} \\ \geq\ & \Lambda_0 \left(C\theta\left(-\phi_0\right)^{\theta-1} \underline{\rho_0}\right)^a \left(C\theta(1-\theta)\left(-\phi_0\right)^{\theta-2} \left(\frac{\eta_0}{2l_0}\right)^2\right)^b M_0^{-1} \|\phi_0\|^{\alpha}_{C^2(\overline{\Omega})} \cdot \left(-\phi_0(x)\right)^{-\alpha} \\ =\ & \frac{1}{2^{2b}} C^{a+b} \Lambda_0 M_0^{-1} \underline{\rho_0}^a \theta^{a+b} (1-\theta)^b {\eta_0}^{2b} {l_0}^{-2b} \|\phi_0\|^{\alpha}_{C^2(\overline{\Omega})} \cdot \left(-\phi_0\right)^{(a+b)\theta-(a+2b+\alpha)}, \quad \forall x \in \Omega_1. \end{aligned}$$

Let us take θ_0 such that $(a+b)\theta_0 - (a+2b+\alpha) = 0$, i.e.

$$\theta_0 = \frac{a+2b+\alpha}{a+b} \in (0,1).$$

When $\theta = \theta_0$, it is now clear that

$$\begin{aligned} &F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot \left[g(x)(d(x))^{\alpha}\right]^{-1} \\ \geq\ & \frac{1}{2^{2b}} C^{a+b} \Lambda_0 M_0^{-1} \underline{\rho_0}^a \theta^{a+b} (1-\theta)^b {\eta_0}^{2b} {l_0}^{-2b} \|\phi_0\|^{\alpha}_{C^2(\overline{\Omega})}, \quad \forall x \in \Omega_1. \end{aligned}$$

Hence there exists some constant $C_{4,1} > 0$ such that for any $C \geq C_{4,1}$, we have

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot \left[g(x)(d(x))^{\alpha}\right]^{-1} > 1, \quad \forall x \in \Omega_1,$$

i.e.

$$F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) > g(x)(d(x))^{\alpha}, \quad \forall x \in \Omega_1. \tag{4.33}$$

Case 4-2: In Ω_2, $D\phi_0$ can be degenerate. Now we still take $\theta_0 = \frac{a+2b+\alpha}{a+b} \in (0,1)$.

Since $D\phi_0(x_0) = 0$ at $x_0 \in \Omega_2$, we also need to re-estimate $\lambda_{\max}(D^2 W_k)$ in Ω_2:

$$\begin{aligned}
\lambda_{\max}(D^2 W_k) &= \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2 W_k \, \boldsymbol{\xi} \\
&\geq \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T \left(C\theta_0 \, (-\phi_0)^{\theta_0 - 1} \, D^2 \phi_0 \right) \boldsymbol{\xi} \\
&= C\theta_0 \, (-\phi_0)^{\theta_0 - 1} \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2 \phi_0 \, \boldsymbol{\xi} \\
&\geq C\theta_0 \, (-\phi_0)^{\theta_0 - 1} \, \underline{\rho_0}.
\end{aligned} \tag{4.34}$$

Then using the assumptions $(\mathbf{A_2})$-(i) and $(\mathbf{A_3})$ together with (4.31), (4.29) and (4.34), we infer that

$$\begin{aligned}
&F\left(\lambda_1(D^2 W_k), \cdots, \lambda_n(D^2 W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} \\
&\geq \Lambda_0 \left(\lambda_{\min}(D^2 W_k)\right)^a \left(\lambda_{\max}(D^2 W_k)\right)^b \cdot M_0^{-1} \, (d(x))^{-\alpha} \\
&\geq \Lambda_0 \left(C\theta_0 \, (-\phi_0)^{\theta_0 - 1} \, \underline{\rho_0} \right)^a \left(C\theta_0 \, (-\phi_0)^{\theta_0 - 1} \, \underline{\rho_0} \right)^b M_0^{-1} \, \|\phi_0\|^\alpha_{C^2(\overline{\Omega})} \cdot (-\phi_0(x))^{-\alpha} \\
&= C^{a+b} \Lambda_0 M_0^{-1} \, \underline{\rho_0}^{a+b} {\theta_0}^{a+b} \, \|\phi_0\|^\alpha_{C^2(\overline{\Omega})} \cdot (-\phi_0)^{(a+b)\theta_0 - (a+b+\alpha)}, \quad \forall x \in \Omega_2.
\end{aligned}$$

Since $\theta_0 = \frac{a+2b+\alpha}{a+b}$, there holds

$$(a+b)\theta_0 - (a+b+\alpha) = b.$$

For any $x \in \Omega_2$, we note that

$$-\phi_0(x) > \frac{\eta_0}{2}.$$

Thus we have

$$(-\phi_0)^{(a+b)\theta_0 - (a+b+\alpha)} = (-\phi_0)^b > \left(\frac{\eta_0}{2}\right)^b.$$

Now we can derive

$$F\left(\lambda_1(D^2 W_k), \cdots, \lambda_n(D^2 W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} \geq \frac{1}{2^b} C^{a+b} \Lambda_0 M_0^{-1} \underline{\rho_0}^a \overline{\rho_0}^b {\theta_0}^{a+b} \, \|\phi_0\|^\alpha_{C^2(\overline{\Omega})} \cdot {\eta_0}^b, \quad \forall x \in \Omega_2.$$

Hence there exists some constant $C_{4,2} > 0$ such that for any $C \geq C_{4,2}$, we have

$$F\left(\lambda_1(D^2 W_k), \cdots, \lambda_n(D^2 W_k)\right) \cdot [g(x)(d(x))^\alpha]^{-1} > 1, \quad \forall x \in \Omega_2,$$

i.e.

$$F\left(\lambda_1(D^2 W_k), \cdots, \lambda_n(D^2 W_k)\right) > g(x)(d(x))^\alpha, \quad \forall x \in \Omega_2. \tag{4.35}$$

Up to now, combining (4.33) and (4.35) together, we can conclude that for any $C \geq C_0 := \max\{C_{4,1}, C_{4,2}\}$ (without loss of generality, we can take $C = C_0$ here), there holds

$$F\left(\lambda_1(D^2 W_k), \cdots, \lambda_n(D^2 W_k)\right) > g(x)(d(x))^\alpha, \quad \forall x \in \Omega_1 \cup \Omega_2 = \Omega. \tag{4.36}$$

Then it is clear that

$$W_k = -C_0 \, (-\phi_0)^{\theta_0} + \psi_k \tag{4.37}$$

is a classically strict sub-solution to (1.1) over Ω.

Using the definition (4.37), Lemma 2.5 and Lemma 2.9, we have $W_k \in C(\overline{\Omega})$ and

$$W_k\big|_{\partial\Omega} = \psi_k\big|_{\partial\Omega} = \varphi.$$

Now, by (4.36) and Lemma 3.1, the equation (1.1) admits a convex viscosity solution $u \in C(\overline{\Omega})$ satisfying $u\big|_{\partial\Omega} = \varphi$, i.e. the problem (1.1)-(1.2) admits a viscosity solution $u \in C(\overline{\Omega})$. Using (4.36) again, we can deduce from the Comparison Principle (Lemma 2.4) that

$$W_k(x) \leq u(x), \qquad \forall x \in \overline{\Omega}.$$

Hence we can infer from (4.37) and Lemma 2.10 that

$$\begin{aligned}
u(x) &\geq -C_0 \, (-\phi_0(x))^{\theta_0} + \psi_k(x) \\
&\geq -C_0 \, \|\phi_0\|^{\theta_0}_{C^2(\overline{\Omega})} \cdot (d(x))^{\theta_0} + \psi_k(x), \qquad \forall x \in \overline{\Omega}.
\end{aligned}$$

Denote $\overline{C_0} = C_0 \|\phi_0\|^{\theta_0}_{C^2(\overline{\Omega})}$. Thus, by letting $k \to \infty$, we get

$$u(x) \geq \psi(x) - \overline{C_0}\,(d(x))^{\theta_0}, \quad \forall x \in \overline{\Omega},$$

where $\theta_0 = \frac{a+2b+\alpha}{a+b}$. This yields Theorem 1.2 (iii).

Thus we have completed the proof of Theorem 1.2.

5. Proof of Nonexistence Theorem

Let us prove Theorem 1.3 by contradiction. We suppose that the problem (1.1)-(1.2) admits a viscosity solution u. By Definition 2.2, we have $u \in C(\overline{\Omega})$.

Consider

$$W = -C\,(-\phi_0)^{\theta} + \max_{\partial\Omega} \varphi, \tag{5.1}$$

where $C > 0$ and $\theta > 0$ are constants to be determined. We point out that the term $\max_{\partial\Omega} \varphi$ is indeed a constant. Let us turn to prove that for any $C > 0$, there exists $\theta = \theta(C)$ sufficiently small such that W is a classically strict super-solution to (1.1).

It is straightforward to see that

$$\begin{aligned}
DW &= C\theta\,(-\phi_0)^{\theta-1} D\phi_0,\\
D^2W &= C\theta\,(-\phi_0)^{\theta-1} D^2\phi_0 + C\theta(1-\theta)\,(-\phi_0)^{\theta-2} D\phi_0 \otimes D\phi_0.
\end{aligned}$$

Now we can apply Lemma 2.11, Remark 2.12, Remark 2.13 and (2.10) to obtain

$$\begin{aligned}
\lambda_{\min}(D^2W) &= \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2 W\, \boldsymbol{\xi} = \min_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T \left(C\theta\,(-\phi_0)^{\theta-1} D^2\phi_0 + C\theta(1-\theta)\,(-\phi_0)^{\theta-2} D\phi_0 \otimes D\phi_0 \right) \boldsymbol{\xi}\\
&\leq \boldsymbol{\xi}_{\mathbf{1}}^T \left(C\theta\,(-\phi_0)^{\theta-1} D^2\phi_0 + C\theta(1-\theta)\,(-\phi_0)^{\theta-2} D\phi_0 \otimes D\phi_0 \right) \boldsymbol{\xi}_{\mathbf{1}}\\
&= C\theta\,(-\phi_0)^{\theta-1} \boldsymbol{\xi}_{\mathbf{1}}^T D^2\phi_0\, \boldsymbol{\xi}_{\mathbf{1}} + C\theta(1-\theta)\,(-\phi_0)^{\theta-2} \boldsymbol{\xi}_{\mathbf{1}}^T D\phi_0 \otimes D\phi_0\, \boldsymbol{\xi}_{\mathbf{1}}\\
&= C\theta\,(-\phi_0)^{\theta-1} \boldsymbol{\xi}_{\mathbf{1}}^T D^2\phi_0\, \boldsymbol{\xi}_{\mathbf{1}}\\
&\leq C\theta\,(-\phi_0)^{\theta-1} \overline{\rho_0},
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\lambda_{\max}(D^2W) &= \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2 W\, \boldsymbol{\xi} = \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T \left(C\theta\,(-\phi_0)^{\theta-1} D^2\phi_0 + C\theta(1-\theta)\,(-\phi_0)^{\theta-2} D\phi_0 \otimes D\phi_0 \right) \boldsymbol{\xi}\\
&\leq C\theta\,(-\phi_0)^{\theta-1} \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D^2\phi_0\, \boldsymbol{\xi} + C\theta(1-\theta)\,(-\phi_0)^{\theta-2} \max_{|\boldsymbol{\xi}|=1} \boldsymbol{\xi}^T D\phi_0 \otimes D\phi_0\, \boldsymbol{\xi}\\
&\leq C\theta\,(-\phi_0)^{\theta-1} \overline{\rho_0} + C\theta(1-\theta)\,(-\phi_0)^{\theta-2} |D\phi_0|^2\\
&= C\theta \left[(-\phi_0)\,\overline{\rho_0} + (1-\theta)|D\phi_0|^2 \right] (-\phi_0)^{\theta-2}\\
&\leq C\theta \left[\eta_0\overline{\rho_0} + (1-\theta)\|\phi_0\|^2_{C^2(\overline{\Omega})} \right] \cdot (-\phi_0)^{\theta-2}.
\end{aligned} \tag{5.3}$$

In view of Lemma 2.10, we also notice that

$$d(x) \leq \frac{\operatorname{diam}(\Omega)}{\eta_0} \cdot (-\phi_0(x)), \quad \forall x \in \Omega. \tag{5.4}$$

Then using the assumptions $(\mathbf{A_2})$-(ii) and $(\mathbf{A_3})$ together with (5.2), (5.3) and (5.4), we obtain

$$\begin{aligned}
&F\left(\lambda_1(D^2W_k), \cdots, \lambda_n(D^2W_k)\right) \cdot [g(x)(d(x))^{\alpha}]^{-1}\\
&\leq \Lambda_0 \left(\lambda_{\min}(D^2W_k)\right)^a \left(\lambda_{\max}(D^2W_k)\right)^b \cdot m_0^{-1}\,(d(x))^{-\alpha}\\
&\leq \Lambda_0 \left(C\theta\,(-\phi_0)^{\theta-1}\overline{\rho_0}\right)^a \left\{ C\theta \left[\eta_0\overline{\rho_0} + (1-\theta)\|\phi_0\|^2_{C^2(\overline{\Omega})}\right] \cdot (-\phi_0)^{\theta-2} \right\}^b m_0^{-1} \left(\frac{\operatorname{diam}(\Omega)}{\eta_0}\right)^{-\alpha} \cdot (-\phi_0(x))^{-\alpha}\\
&= C^{a+b}\theta^{a+b}\Lambda_0 m_0^{-1} (\operatorname{diam}(\Omega))^{-\alpha} {\eta_0}^{\alpha} \overline{\rho_0}^{a} \left[\eta_0\overline{\rho_0} + (1-\theta)\|\phi_0\|^2_{C^2(\overline{\Omega})}\right]^b \cdot (-\phi_0)^{(a+b)\theta-(a+2b+\alpha)}, \quad \forall x \in \Omega.
\end{aligned}$$

We note that

$$\lim_{\theta \to 0^+} \left[\eta_0\overline{\rho_0} + (1-\theta)\|\phi_0\|^2_{C^2(\overline{\Omega})}\right]^b = \left[\eta_0\overline{\rho_0} + \|\phi_0\|^2_{C^2(\overline{\Omega})}\right]^b.$$

Since $\alpha \le -a-2b$, we have $-(a+2b+\alpha) \ge 0$. Then it follows from (2.10) that

$$\lim_{\theta\to 0^+} (-\phi_0)^{(a+b)\theta-(a+2b+\alpha)} = (-\phi_0)^{-(a+2b+\alpha)} \le {\eta_0}^{-(a+2b+\alpha)}.$$

By virtue of the fact that

$$\lim_{\theta\to 0^+} \theta^{a+b} = 0,$$

we are now ready to derive that for any $C > 0$, there exists $\theta_0 = \theta_0(C) > 0$ sufficiently small (without loss of generality, we can require $\theta_0 \in (0,1]$) such that for any $\theta \in (0,\theta_0]$, there holds

$$F\left(\lambda_1(D^2W),\cdots,\lambda_n(D^2W)\right)\cdot\left[g(x)(d(x))^\alpha\right]^{-1} < 1, \quad \forall x\in\Omega,$$

that is,

$$F\left(\lambda_1(D^2W),\cdots,\lambda_n(D^2W)\right) < g(x)(d(x))^\alpha, \quad \forall x\in\Omega, \tag{5.5}$$

Now we take $\theta = \theta_0$. Then we obtain that

$$W = -C\left(-\phi_0\right)^{\theta_0} + \max_{\partial\Omega}\varphi \tag{5.6}$$

is a classically strict super-solution to (1.1) over Ω.

According to the definition (5.6) and Lemma 2.9, we have $W \in C(\overline{\Omega})$ and

$$W\big|_{\partial\Omega} = \max_{\partial\Omega}\varphi \ge \varphi.$$

Therefore, by (5.5) and the Comparison Principle (Lemma 2.4), we can infer that

$$W(x) \ge u(x), \qquad \forall x\in\overline{\Omega}.$$

In particular, it follows from (5.6), (2.10) and (2.11) that

$$u(x_0) \le W(x_0) = -C\left(-\phi_0(x_0)\right)^{\theta_0} + \max_{\partial\Omega}\varphi = -C{\eta_0}^{\theta_0} + \max_{\partial\Omega}\varphi.$$

Since $\theta_0 \in (0,1]$, we notice that

$${\eta_0}^{\theta_0} \ge \min\{{\eta_0}^1, {\eta_0}^0\} = \min\{\eta_0, 1\} > 0.$$

Thus for any $C > 0$, we can derive that

$$u(x_0) \le -C\min\{\eta_0,1\} + \max_{\partial\Omega}\varphi.$$

By letting $C \to +\infty$, we further obtain

$$u(x_0) \le -\left(\lim_{C\to+\infty} C\right)\min\{\eta_0,1\} + \max_{\partial\Omega}\varphi = -\infty,$$

which contradicts the fact that $u \in C(\overline{\Omega})$. The proof of Theorem 1.3 is now complete.

6. Further discussions

In the last section, we mainly discuss the intuitions behind why $-a-2b$ is a critical value of α that distinguishes between the existence and nonexistence of viscosity solutions to the problem (1.9)-(1.10). In fact, when $\alpha \in (-\infty, -a-2b]$, one can observe that the problem (1.9)-(1.10) never admits a viscosity solution u even if we assume the smoothness of Ω and φ, i.e. assume that Ω is of C^∞ and $\varphi \in C^\infty(\partial\Omega)$, which together with Lemma 2.5 further yields $\psi_k \in C^\infty(\overline{\Omega})$.

Now we let $\alpha \in (-\infty, -a-2b]$. In what follows, we use the singularity order to describe the speed that approaches the singularity (i.e. tends to infinity). We are going to analyze the specific singularity order of $\lambda_{\min}(D^2u)$ and $\lambda_{\max}(D^2u)$ by using the method of undetermined coefficients. Intuitively, from (1.9), we assume the following relation for the norm of the gradient Du near the boundary:

$$|Du| \sim \left(d(x)\right)^{-\widetilde{\theta}}, \quad \text{as } x\to\partial\Omega. \tag{6.1}$$

By choosing some suitable coordinate system, we can assume that

$$Du \sim \Big(\underbrace{0,\cdots,0}_{n-1}, -\left(d(x)\right)^{-\widetilde{\theta}}\Big), \tag{6.2}$$

which also implies

$$\partial_{x_n} u \sim -\big(d(x)\big)^{-\widetilde{\theta}}, \quad \text{as } x \to \partial\Omega. \tag{6.3}$$

By the analysis in (1.4)-(1.5), we note that near the boundary, the second tangential derivative has the same singularity order as $\lambda_{\min}(D^2u)$, and the second normal derivative has the same singularity order as $\lambda_{\max}(D^2u)$. Now we have the following singularity order relations when the boundary is approached:

$$D^2_{x'x'}u \sim \lambda_{\min}(D^2u), \quad \text{as } x \to \partial\Omega, \tag{6.4}$$

$$\partial_{x_nx_n} u \sim \lambda_{\max}(D^2u), \quad \text{as } x \to \partial\Omega. \tag{6.5}$$

Fix a point $z_0 \in \partial\Omega$. Up to suitable translations and rotations, we can assume that z_0 is the origin $\mathbf{0}$ and

$$\Omega \cap B_\rho(\mathbf{0}) = \left\{ (x', x_n) = (x_1, \cdots, x_n) : \, x_n \geq \widetilde{\varphi}(x') = \sum_{i=1}^{n-1} \frac{\kappa_i}{2} x_i^2 + o(|x'|^2) \right\},$$

where $\rho > 0$ is a constant, and $\widetilde{\varphi} : \mathbb{R}^{n-1} \to \mathbb{R}$ is a uniformly convex C^2 function such that

$$\widetilde{\varphi}(0) = 0, \qquad \nabla_{x'}\widetilde{\varphi}(0) = 0, \qquad \text{and} \qquad D^2_{x'x'}\widetilde{\varphi}(0) = \operatorname{diag}\big(\kappa_1, \cdots, \kappa_{n-1}\big).$$

Denote

$$w = u - \psi_k.$$

By (1.2) and Lemma 2.5, we have

$$w\big|_{\partial\Omega} = 0.$$

It follows that

$$D^2_{x'x'} w(\mathbf{0}) = -\partial_{x_n} w(\mathbf{0}) \operatorname{diag}\big(\kappa_1, \cdots, \kappa_{n-1}\big),$$

i.e.

$$D^2_{x'x'}u(\mathbf{0}) - D^2_{x'x'}\psi_k(\mathbf{0}) = -\partial_{x_n}u(\mathbf{0}) \operatorname{diag}\big(\kappa_1, \cdots, \kappa_{n-1}\big) + \partial_{x_n}\psi_k(\mathbf{0}) \operatorname{diag}\big(\kappa_1, \cdots, \kappa_{n-1}\big),$$

that is,

$$D^2_{x'x'}u(\mathbf{0}) = -\partial_{x_n}u(\mathbf{0}) \operatorname{diag}\big(\kappa_1, \cdots, \kappa_{n-1}\big) + D^2_{x'x'}\psi_k(\mathbf{0}) + \partial_{x_n}\psi_k(\mathbf{0}) \operatorname{diag}\big(\kappa_1, \cdots, \kappa_{n-1}\big).$$

Due to the smoothness of ψ_k over $\overline{\Omega}$, we note that both $D^2_{x'x'}\psi_k$ and $\partial_{x_n}\psi_k$ are bounded on the boundary. Hence the singularity order of $D^2_{x'x'}u$ on the boundary is primarily influenced by the singularity order of $\partial_{x_n}u$. Consequently, the second tangential derivative and the first normal derivative have the same singularity order near the boundary:

$$D^2_{x'x'}u \sim -\partial_{x_n}u, \quad \text{as } x \to \partial\Omega. \tag{6.6}$$

We recall that the gradient of the distance function $d(x)$ is precisely N, the unit inward normal to $\partial\Omega$. This implies that when x is near the boundary point $\mathbf{0} \in \partial\Omega$, we have

$$\partial_{x_n} d(x) \sim |Dd(x)| = 1, \quad \text{as } x \to \mathbf{0} \in \partial\Omega.$$

Using the chain rule leads us to

$$\partial_{x_nx_n}u(x) = \partial_{d(x)}\partial_{x_n}u(x) \cdot \partial_{x_n}d(x) \sim \partial_{d(x)}\partial_{x_n}u(x), \quad \text{as } x \to \mathbf{0} \in \partial\Omega.$$

Thus we have

$$\partial_{x_nx_n}u(x) \sim \partial_{d(x)}\partial_{x_n}u(x), \quad \text{as } x \to \mathbf{0} \in \partial\Omega.$$

Now by (6.3), we obtain that the second normal derivative has a singularity order that is one order greater than that of the first normal derivative near the boundary, that is,

$$\partial_{x_nx_n}u \sim \big(d(x)\big)^{-\widetilde{\theta}-1}, \quad \text{as } x \to \partial\Omega. \tag{6.7}$$

It follows from (6.3), (6.4), (6.5), (6.6) and (6.7) that

$$\lambda_{\min}(D^2u) \sim \big(d(x)\big)^{-\widetilde{\theta}}, \quad \text{as } x \to \partial\Omega,$$

$$\lambda_{\max}(D^2u) \sim \big(d(x)\big)^{-\widetilde{\theta}-1}, \quad \text{as } x \to \partial\Omega.$$

By virtue of (1.9), i.e.

$$\Lambda_0 \left(\lambda_{\min}(D^2u)\right)^a \left(\lambda_{\max}(D^2u)\right)^b = g(x)(d(x))^\alpha,$$

we can derive from the exponent of $d(x)$ on both sides to get

$$a \cdot (-\widetilde{\theta}) + b \cdot (-\widetilde{\theta} - 1) = \alpha.$$

Considering $\alpha \in (-\infty, -a - 2b]$ now, we infer that

$$\widetilde{\theta} \geq 1. \tag{6.8}$$

Up to now, in the case $\alpha \in (-\infty, -a - 2b]$, for any boundary point, we can choose proper coordinate system and summarize what we have obtained in (6.1)-(6.8) as the following singularity order relations:

$$\begin{aligned}
&\lambda_{\min}(D^2u) \sim D^2_{x'x'}u \sim -\partial_{x_n}u \sim |Du| \sim \big(d(x)\big)^{-\widetilde{\theta}}, \quad \text{as } x \to \partial\Omega, \\
&\lambda_{\max}(D^2u) \sim \partial_{x_nx_n}u \sim \big(d(x)\big)^{-\widetilde{\theta}-1}, \quad \text{as } x \to \partial\Omega,
\end{aligned}$$

where $\widetilde{\theta} \geq 1$. In particular, when $\alpha = -a - 2b$, we have the following relations of singularity order:

$$\begin{aligned}
&\lambda_{\min}(D^2u) \sim D^2_{x'x'}u \sim -\partial_{x_n}u \sim |Du| \sim \big(d(x)\big)^{-1}, \quad \text{as } x \to \partial\Omega, \\
&\lambda_{\max}(D^2u) \sim \partial_{x_nx_n}u \sim \big(d(x)\big)^{-2}, \quad \text{as } x \to \partial\Omega.
\end{aligned}$$

Suppose there exists a viscosity solution u. By Definition 2.2, we have $u \in C(\overline{\Omega})$. Let $z(t)$ for $0 \leq t \leq \varepsilon_0$ be a curve segment in $\overline{\Omega}$ starting from the point $z_0 \in \partial\Omega$ that follows the direction of the fastest decrease of $u(x)$. Hence $z(t)$ satisfies the following conditions:

$$\begin{aligned}
&z(0) = z_0, \\
&z(t) \in \overline{\Omega}, \quad \forall 0 \leq t \leq \varepsilon_0, \\
&\frac{d}{dt}u(z(t)) = -|Du(z(t))|, \quad \forall 0 \leq t \leq \varepsilon_0.
\end{aligned}$$

Without loss of generality, we can assume that t represents the arc length parameter of the curve $z(t)$. It is clear that the distance from the point $z(t)$ to the boundary is no greater than the arc length from the point $z(t)$ to the point $z(0)$ along the curve $z(t)$, which means

$$d(z(t)) \leq t.$$

Then due to (6.8), we can derive

$$\begin{aligned}
u(z(\varepsilon_0)) - u(z_0) = u(z(\varepsilon_0)) - u(z(0)) &= \int_0^{\varepsilon_0} \frac{d}{dt}u(z(t))dt = -\int_0^{\varepsilon_0} |Du(z(t))|dt \\
&\sim -\int_0^{\varepsilon_0} \big(d(z(t))\big)^{-\widetilde{\theta}}dt \leq -\int_0^{\varepsilon_0} t^{-\widetilde{\theta}}dt = -\infty,
\end{aligned}$$

which implies that

$$u(z(\varepsilon_0)) \leq u(z_0) - \infty = \varphi(z_0) - \infty = -\infty.$$

This contradicts the fact that $u \in C(\overline{\Omega})$. Therefore the problem (1.9)-(1.10) does not admit any viscosity solutions when $\alpha \in (-\infty, -a - 2b]$.

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