The Quantum Agreement Theorem*

María García Díaz † Adam Brandenburger ‡ Giannicola Scarpa §

Version 11/26/25

Abstract

We formulate and prove an Agreement Theorem for quantum mechanics (QM), describing when two agents, represented by separate laboratories, can or cannot maintain differing probability estimates of a shared quantum property of interest. Building on the classical framework (Aumann, 1976), we define the modality of "common certainty" through a hierarchy of certainty operators acting on each agent's Hilbert space. In the commuting case – when all measurements and event projectors commute – common certainty leads to equality of the agents' conditional probabilities, recovering a QM analog of the classical theorem. By contrast, when non-commuting operators are allowed, the certainty recursion can stabilize with different probabilities. This yields common certainty of disagreement (CCD) as a distinctive QM phenomenon. Agreement is restored once measurement outcomes are recorded in a classical register. The classical Agreement Theorem can therefore be seen as emergent from the quantum world via recording. We establish an impossibility result stating that QM forbids a scenario where one agent is certain that a property of interest occurs, and is also certain that the other agent is certain that the property does not occur. In this sense, QM admits non-classical disagreement, but disagreement is still bounded in a disciplined way. We argue that our analysis offers a rigorous approach to the longstanding issue of how to understand intersubjectivity across agents in QM.

Soon after the formalism of quantum mechanics (QM) was in place, the question of the role of the observer arose. Bohr argued that the outcome of an experiment, and even which physical quantities can meaningfully be said to possess values, naturally depends on the experimental context, while Einstein famously objected that such "observer-created" facts conflict with scientific realism (Held, 1998). Wigner, (1961) sharpened the issue with his Friend Paradox. If Wigner treats his colleague and her laboratory as a quantum system in superposition, he must assign an entangled state to a situation in which the colleague herself has already witnessed a definite outcome. This thought experiment suggests that observer dependence might even yield intersubjective contradiction.

The debate has continued. Elaborating on Wigner's Friend, Frauchiger and Renner (2018) recently presented a scenario exhibiting an especially strong form of intersubjective disagreement, which causes the authors to question the consistency of QM. (See also Brukner, 2018; Nurgalieva and del Rio, 2018; Proietti et al., 2019). Under quantum Bayesianism (QBism), full intersubjectivity is allowed in any measurement setting (Fuchs and Schack, 2013; Fuchs, Mermin, and Schack, 2014). QBism is also interested in how agents might actively create conditions for agreement among them (Schack, 2024), such as when Alice first performs a measurement and can then predict with probability one the outcome that Bob will observe if he performs a measurement on her post-measurement state. In relational QM, every state of a system is only defined relative to another system – usually, an observer (Rovelli, 1996; Di Biagio and Rovelli, 2021). But in relational QM, too, the question of intersubjective agreement has been studied and a specific postulate, called "cross-perspective links" (Adlam and Rovelli, 2022), put forward that ensures agreement about past quantum events.

This paper aims to add to our understanding of intersubjectivity in QM by formally describing the epistemic states of agents, that is, what they know or believe about underlying properties of interest, know or believe about what other agents know or believe, and so on, potentially to arbitrarily many levels of reasoning. To do so, we draw on the well-established sub-field of game theory that formalizes the epistemic

*We thank Piefrancesco La Mura for very important inspiration for this project. M.G.D. and G.S. acknowledge financial support from the following grants: PID2020-113523GB-I00 and PID2023-146758NB-I00 funded by MICIU/AEI/10.13039/501100011033. A.B. acknowledges financial support from NYU Stern School of Business, NYU Shanghai, and J.P. Valles. ChatGPT 5, Grok 4, and MS Copilot were used to help in searching for examples and in editing text.

†Departamento de Matemática Aplicada a la Ingeniería Industrial, Universidad Politécnica de Madrid, 28006 Madrid, Spain, maria.garcia.diaz@upm.es

‡Stern School of Business, Tandon School of Engineering, NYU Shanghai, New York University, New York, NY 10012, U.S.A., adam.brandenburger@stern.nyu.edu

§Escuela Técnica Superior de Ingeniería de Sistemas Informáticos, Universidad Politécnica de Madrid, 28031 Madrid, Spain, g.scarpa@upm.es

states of agents (Aumann, 1976; Aumann and Brandenburger, 1995; Dekel and Siniscalchi, 2015). Thus, our results operate at this intersection of the logic of belief and the physics of measurement in a quantum setting.

In our first scenario, Alice and Bob each perform local measurements in their respective labs and, updating via the Born-Lüders rule, obtain conditional probabilities for some property of shared interest. The probability estimates by Alice and Bob will generally differ, which is natural. Next suppose that Alice can calculate with certainty what Bob's probability is. This number may be different from her own probability, which is again natural. Likewise with the roles of Alice and Bob interchanged. The issue of disagreement between Alice and Bob comes into sharp focus when we suppose that there is "common certainty" of their respective probabilities, that is, each agent is certain of the probability estimates, is certain the other agent is certain, and so on ad infinitum.

We first study a commuting regime where Alice's and Bob's measurements commute with each other, and each commutes with the property of interest. Our first result is: *In the commuting regime, if Alice and Bob have common certainty of their respective probability estimates of a shared property of interest E, then their estimates must agree.* We then turn to a non-commuting regime, where an agent's measurements do not necessarily commute with the property of interest. Our second result is: *In a non-commuting regime, a quantum system can exhibit common certainty of disagreement (CCD).* To show this, we produce a qutrit-qubit-qubit system in a particular entangled state where Alice's measurements do not commute with the property of interest E, and it is common certainty that Alice assigns probability $1/2$ to E and Bob assigns probability 1 to E.

Our Commutative Agreement Theorem encompasses the fundamental Agreement Theorem of classical epistemics (Aumann, 1976), which states that if two agents share a common prior on a classical probability space, and their respective probability estimates of an event of interest are common knowledge, then these estimates must agree. (In the Supplementary Material, we explain the shift from knowledge to certainty modality, needed to treat quantum systems.) The Classical Agreement Theorem has been used, for example, to establish epistemic conditions for Nash equilibrium (Aumann and Brandenburger, 1995; Pacuit and Roy, 2025) and investigate market efficiency (Milgrom and Stokey, 1982; Sebenius and Geanakoplos, 1983; Gizatulina and Hellman, 2019).

We show that, in the non-commuting case, recording the outcomes of Alice's and Bob's measurements in a classical register restores agreement. This offers a new interpretation of the Classical Agreement Theorem as true even in a quantum world, under the physical act of classical recording. This is a different viewpoint from classical epistemics, which has not taken into account the physical environment in which agents operate.

Our final set of results provides bounds on disagreement. We establish a 0 - 1 impossibility result: *There is no positive-probability state where Alice is certain of E and certain that Bob is certain of not-E.* By contrast, in the Supplementary Material, we produce a superquantum no-signaling box (Popescu and Rohrlich, 1994, that exhibits 0 - 1 disagreement. (These two results extend earlier work by Contreras-Tejada et al., 2021. By Abramsky and Brandenburger, 2011, the no-signaling example can also be cast in quasi-classical terms using signed probabilities, as we show.) Evidently, while QM permits "spooky" disagreement across agents in the form of CCD, it is not a case of anything goes: The extreme of 0 - 1 disagreement is ruled out. As for the possibility of 0 - 1 disagreement in no-signaling systems, we see this phenomenon as adding to the well-known list of implausible superquantum behaviors, such as trivialized communication complexity (van Dam, 2005), violations of information causality (Pawlowksi et al., 2009), and signaling in the classical limit (Rohrlich, 2014). Finally, we obtain two robustness results for the Commutative Agreement Theorem, showing smooth behavior under small departures from a shared initial state and exact common certainty (cf. Hellman, 2013; Monderer and Samet, 1989).

We believe that our results help shed light on the issue of intersubjectivity in QM surveyed above. First, a clear distinction emerges between the commuting and non-commuting regimes: CCD is impossible in the former case and possible in the latter case. On the one hand, our findings challenge the notion that intersubjective disagreement should never be tolerated in a physical theory. The explicit construction of CCD shows that QM permits two agents to sustain different probability estimates of the same event, even under common certainty. We see this phenomenon as in line with the kind of perspective-dependent facts that QBism (Fuchs and Schack, 2013) and relational QM (Rovelli, 1996) were formulated to accommodate, and which more traditional interpretations of QM struggle to accept. Far from being a pathology, CCD appears as a natural consequence of non-commuting measurements and provides direct support for interpretations that treat quantum probabilities as agent-relative. In employing formal epistemics, our work may also suggest new tools for the further development of these interpretations.

On the other hand, we obtain bounds on the extent of disagreement that is possible. Our 0 - 1 impossibility theorem limits disagreement, even in the non-commuting case. So, we show that quantum theory is not entirely undisciplined and rules out certain epistemic scenarios.

We next present our framework, the commuting scenario we analyze, and our first result on common certainty and agreement. We then present two examples – the first is an illustration of the commuting case, while the second is non-commuting and exhibits CCD. We then examine restoration of agreement via a classical register and bounds on disagreement. We conclude with a discussion of some interpretational aspects, related work, and possible future directions.

Agreement Theorem

Consider a setting where two agents, Alice and Bob, have laboratories labeled A and B, respectively. There is a quantum system made up of one subsystem in each of their labs and a third subsystem in a third lab, labeled C, which is not accessible to the agents. Alice and Bob each perform a measurement in their lab, and they are interested in estimating a property of the subsystem in the third lab. Figure 1 depicts this setting.

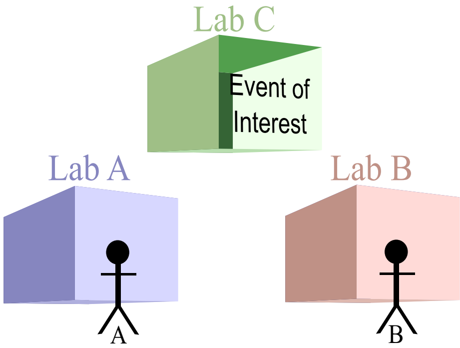

Figure 1: Three-laboratory scenario

Formally, we have a tripartite finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where the first part is accessible to Alice and the second part to Bob, and the third part is inaccessible. The quantum state is described by a density matrix $\rho \in \mathcal{B}(\mathcal{H})$, and this state is prior knowledge shared by Alice and Bob.

Alice and Bob perform projective measurements on their respective systems, in order to estimate the property of interest in the third system $\mathcal{H}_C$. That is, for each agent X (Alice or Bob), there is a projective measurement $\Pi_X = \{P_X^k\}_{k=1}^{\ell_X}$ with projectors $P_X^k \in \mathcal{B}(\mathcal{H})$. Each projector for Alice takes the form $P_A^i = \hat{P}_A^i \otimes \mathbb{I}_B \otimes \mathbb{I}_C$, for $i = 1, \ldots, \ell_A$, and each projector for Bob takes the form $P_B^j = \mathbb{I}_A \otimes \hat{P}_B^j \otimes \mathbb{I}_C$, for $j = 1, \ldots, \ell_B$, where each $\hat{P}_X^k \in \mathcal{B}(\mathcal{H}_X)$, and $\mathbb{I}_X$ and $\mathbb{I}_C$ are the identity operators. The property of interest E in the inaccessible lab is modeled as a projector $P_E = \mathbb{I}_A \otimes \mathbb{I}_B \otimes \hat{P}_E \in \mathcal{B}(\mathcal{H})$ with $\hat{P}_E \in \mathcal{B}(\mathcal{H}_C)$. In this setting, each pair of projectors for Alice and Bob commute with each other:

$$[P_A^i, P_B^j] = 0 \text{ for all } i, j, \tag{1}$$

and each projector P_X^k commutes with P_E:

$$[P_X^k, P_E] = 0 \text{ for all } k. \tag{2}$$

Next, fix a projector $P_F \in \mathcal{B}(\mathcal{H})$. The (conditional) probability that agent X assigns to property F, given the outcome P_X^k of the measurement, is given by the Born rule with a Lüders update:

$$\mathrm{Prob}[P_F; P_X^k] = \mathrm{Tr}\Big(P_F \frac{P_X^k\, \rho\, P_X^k}{\mathrm{Tr}(P_X^k\, \rho)}\Big), \tag{3}$$

whenever the denominator is nonzero. If this probability is 1, we say that agent X is *certain* of the property F (Monderer and Samet, 1989). Now let q be a given probability and consider the set of indices k for which agent X assigns probability q to the property F, after observing the outcome P_X^k:

$$K_X(F; q) = \big\{k : \mathrm{Tr}(P_X^k\, \rho) > 0 \text{ and } \mathrm{Prob}[P_F; P_X^k] = q\big\}. \tag{4}$$

(We suppress the dependence on ρ.) Next define the operator that agent X assigns probability q:

$$Q_X(F; q) = \sum_{k \in K_X(F;\, q)} P_X^k, \tag{5}$$

which, as a sum of mutually orthogonal projectors, is again a projector. The probability that agent Y

assigns to $Q_X(F)$, given the outcome P_Y^k, is $\text{Prob}[Q_X(F); P_Y^k]$. If this probability is 1, we say that agent Y is certain that agent X assigns probability q to property F. A particular instance is $q = 1$, in which case we say agent Y is certain that agent X is certain of property F. We also write $C_X(F)$ for $Q_X(F; 1)$.

We are now ready to define common certainty. Fix probabilities q_A and q_B for Alice and Bob, respectively. Let:

$$A_0 = Q_A(E; q_A), \tag{6}$$
$$B_0 = Q_B(E; q_B), \tag{7}$$

and recursively define higher-order certainty operators, for all $n \geq 0$:

$$A_{n+1} = A_n\, C_A(B_n), \tag{8}$$
$$B_{n+1} = B_n\, C_B(A_n). \tag{9}$$

(See the Supplementary Material for details.) By finiteness of the projective measurements Π_X, for $X = A, B$, we know there is a finite integer N such that the recursion stabilizes: $A_n = A_N$ and $B_n = B_N$ for all $n \geq N$. Writing $A_* = A_N$ and $B_* = B_N$, we call $C_* = A_*\, B_*$ the *common certainty* projector. It is the fixed point of the certainty recursion, capturing the infinite iteration "Alice is certain that Bob is certain that Alice is certain ... that the property E has probability q_A (resp. q_B)." Since A_* and B_* commute, we can also write $C_* = B_*\, A_*$, a fact we will use later. If $q_A = q_B = 1$, we simply say that there is common certainty of E. We now have the ingredients to define the epistemic condition of interest.

Definition: *Fix a property of interest E and probability estimates q_A and q_B for Alice and Bob, respectively. Then q_A and q_B are common certainty at state ρ if* $\text{Tr}(C_*\, \rho) > 0$.

This definition ensures the common-certainty recursion is non-vacuous by requiring that the operator C_* has positive weight at the given state ρ.

Commutative Agreement Theorem: *If it is common certainty at state ρ that Alice assigns q_A to E and Bob assigns q_B to E, then $q_A = q_B$.*

The proof has four main steps. In the Supplementary Material, we first show that common certainty implies that Bob's certainty operator does not disturb Alice's post-measurement state. That is, after projecting onto the subspace where Alice is certain, the further application of Bob's certainty operator leaves the state unchanged. Formally:

$$\frac{B_* A_*\, \rho\, A_* B_*}{\text{Tr}(B_* A_*\, \rho)} = \frac{A_*\, \rho\, A_*}{\text{Tr}(A_*\, \rho)}. \tag{10}$$

Second, this invariance ensures that the probability of the property E is the same whether evaluated before or after Bob's certainty operator is applied:

$$\text{Tr}\Big(P_E \frac{A_*\, \rho\, A_*}{\text{Tr}(A_*\, \rho)}\Big) = \text{Tr}\Big(P_E \frac{B_* A_*\, \rho\, A_* B_*}{\text{Tr}(B_* A_*\, \rho)}\Big). \tag{11}$$

Third, by symmetry, the same reasoning holds with Alice and Bob interchanged:

$$\text{Tr}\Big(P_E \frac{B_*\, \rho\, B_*}{\text{Tr}(B_*\, \rho)}\Big) = \text{Tr}\Big(P_E \frac{A_* B_*\, \rho\, B_* A_*}{\text{Tr}(A_* B_*\, \rho)}\Big). \tag{12}$$

Since A_* and B_* commute, the right-hand sides of Equations 11 and 12 are equal. It follows that the left-hand sides are also equal.

The final step is to observe that because P_E commutes with Alice's measurement projectors, her probability estimate q_A on her actual outcome P_A^i agrees with the coarse-grained conditional on her common-certainty projector:

$$\text{Pr}[P_E; P_A^i] = \text{Tr}\Big(P_E \frac{A_*\, \rho\, A_*}{\text{Tr}(A_*\, \rho)}\Big). \tag{13}$$

Repeating the argument for Bob and using the equality of the left-hand sides of Equations 11 and 12, we conclude that $q_A = q_B$.

An important point is that even though the recursion of certainty operators stabilizes at a finite stage, owing to the finiteness of the measurement sets, the full force of common certainty is essential. It is already known in the classical setting that, for any finite order n, there are scenarios where Alice and Bob achieve nth-order certainty of their respective probability estimates, while still disagreeing on their values (Geanakoplos and Polemarchakis, 1982; Aumann and Brandenburger, 1995). A general statement of our theorem therefore requires the complete, infinite hierarchy of certainty – captured by the common-certainty operator C_* – to ensure agreement. We also note that common-certainty probabilities are not, in general, the same as pooled probabilities that are obtained if Alice and Bob combine their respective measurement outcomes. (See the Supplementary Material.)

Our Commutative Agreement Theorem holds in the tripartite setting of Figure 1, understood as satisfying the conditions of Equations 1–2. In the Supplementary Material, we describe a two-lab setting and a single-lab setting where the theorem again applies, subject to some modifications.

Commuting and Non-Commuting Examples

To clarify the scope and implications of the our Commutative Agreement Theorem, we present two examples. The purpose of the first example, which satisfies the commutativity conditions of Equations 1–2, is to demonstrate that the certainty recursion does not necessarily stabilize immediately. This establishes that our first theorem has nontrivial epistemic content in that agents may need to refine what they are certain about. In the second example, Alice's measurement does not commute with the property of interest, and we obtain *common certainty of disagreement* (*CCD*), in contrast with our first theorem.

Example 1: The global state is pure and entangled within each of Alice's and Bob's labs, with its entanglement extending to the third, inaccessible lab containing a single qubit. Alice and Bob each perform local rank-2 projective measurements on their respective subsystems. Specifically, Alice's Hilbert space decomposes into two orthogonal two-dimensional subspaces and Bob's Hilbert space into three orthogonal two-dimensional subspaces:

$$\mathcal{H}_A = \mathcal{A}_0 \oplus \mathcal{A}_1, \quad \mathcal{H}_B = \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \mathcal{B}_2. \tag{14}$$

Thus, $\mathcal{H}_A \cong \mathbb{C}^4$ and $\mathcal{H}_B \cong \mathbb{C}^6$. For each i, j, let $|\phi_{\mathcal{A}_i,\mathcal{B}_j}\rangle$ denote a maximally entangled Bell state on the two-dimensional subspace $\mathcal{A}_i \otimes \mathcal{B}_j$. The global state involves the inaccessible qubit in $\mathcal{H}_C$, which is entangled with the sectors of $\mathcal{H}_A$ and $\mathcal{H}_B$. Specifically, set $\rho = |\psi\rangle\langle\psi|$ where:

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(|\phi_{\mathcal{A}_0,\mathcal{B}_0}\rangle \, |0\rangle_{\mathcal{H}_C} + |\phi_{\mathcal{A}_1,\mathcal{B}_1}\rangle \, |0\rangle_{\mathcal{H}_C} + |\phi_{\mathcal{A}_1,\mathcal{B}_2}\rangle \, |1\rangle_{\mathcal{H}_C} \right). \tag{15}$$

Alice measures $\Pi_A = \{P_A^0, P_A^1\}$ with $P_A^i = \Pi_{\mathcal{A}_i} \otimes \mathbb{I}_B \otimes \mathbb{I}_C$, while Bob measures $\Pi_B = \{P_B^0, P_B^1, P_B^2\}$ with $P_B^j = \mathbb{I}_A \otimes \Pi_{\mathcal{B}_j} \otimes \mathbb{I}_C$, where $\Pi_{\mathcal{A}_i}$ and $\Pi_{\mathcal{B}_j}$ are the orthogonal projectors onto the respective two-dimensional subspaces A_i and B_j. The property of interest concerns the qubit in lab C:

$$P_E(\theta) = \mathbb{I}_A \otimes \mathbb{I}_B \otimes |\phi_\theta\rangle\langle\phi_\theta| \;\text{ where }\; |\phi_\theta\rangle = \cos\Big(\frac{\theta}{2}\Big) \, |0\rangle + \sin\Big(\frac{\theta}{2}\Big) \, |1\rangle \, . \tag{16}$$

Calculation yields:

$$\Pr[P_E; P_A^0] = \Pr[P_E; P_B^0] = \Pr[P_E; P_B^1] = \cos^2\Big(\frac{\theta}{2}\Big), \quad \Pr[P_E; P_A^1] = \frac{1}{2}, \quad \Pr[P_E; P_B^2] = \sin^2\Big(\frac{\theta}{2}\Big). \tag{17}$$

Therefore, choosing $q_A = q_B = \cos^2(\theta/2)$ where $\theta \neq \pi/2 + \pi z$, for an integer z, we have $A_0 = P_A^0$ and $B_0 = P_B^0 + P_B^1$. A further calculation gives $C_A(B_0) = P_A^0$ and $C_B(A_0) = P_B^0$, from which, using Equations 8–9, we find $A_1 = P_A^0$ and $B_1 = P_B^0$. The recursion stabilizes at this step, so that $C_* = A_* B_* = P_A^0 P_B^0$. Moreover, $\mathrm{Tr}(C_* \rho) = 1/3 \; > 0$, so the probability assessment $\cos^2(\theta/2)$ is common certainty at ρ.

Notice that the recursion does not stabilize immediately. (Variants of the example could be designed that require additional steps until stabilization.) Also, we obtain a range of examples between $q = 1$ ($\theta = 0$) and $q = 0$ ($\theta = \pi$), showing that our Commutative Agreement Theorem is not restricted to edge cases.

Example 2: We remain in the three-lab setting. Alice holds a qutrit, Bob's system decomposes into two orthogonal subspaces, and the third lab contains an inaccessible qubit, as follows:

$$\mathcal{H}_A = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2, \quad \mathcal{H}_B = \mathcal{B}_0 \oplus \mathcal{B}_1, \quad \mathcal{H}_C = \mathcal{C}_0 \oplus \mathcal{C}_1, \tag{18}$$

where each $\mathcal{A}_i = \mathrm{span}\{|i\rangle_A\}$, for $i = 0, 1, 2$, each $\mathcal{B}_j = \mathrm{span}\{|\beta_j\rangle_B\}$, for $j = 0, 1$, where $\{|\beta_0\rangle_B, |\beta_1\rangle_B\}$ is an orthonormal basis of $\mathcal{H}_B$, and each $\mathcal{C}_k = \mathrm{span}\{|k\rangle_C\}$, for $k = 0, 1$. The global state is $\rho = |\psi\rangle\langle\psi|$ where:

$$|\psi\rangle = \frac{1}{\sqrt{3}} \big(|0\rangle_A \, |\beta_0\rangle_B \, |0\rangle_C + |1\rangle_A \, |\beta_0\rangle_B \, |0\rangle_C + |2\rangle_A \, |\beta_1\rangle_B \, |1\rangle_C \big). \tag{19}$$

Alice measures $\Pi_A = \{P_A^0, P_A^1, P_A^2\}$ with $P_A^i = \Pi_{\mathcal{A}_i} \otimes \mathbb{I}_B \otimes \mathbb{I}_C$, while Bob measures $\Pi_B = \{P_B^0, P_B^1\}$ with $P_{B_j} = \mathbb{I}_A \otimes \Pi_{\mathcal{B}_j} \otimes \mathbb{I}_C$. The property of interest acts on $\mathcal{H}_A \otimes \mathcal{H}_C$ and is chosen not to commute with Alice's measurement:

$$P_E = |\phi\rangle\langle\phi|_{AC} \otimes \mathbb{I}_B \text{ where } |\phi\rangle_{AC} = \frac{|0\rangle_A |0\rangle_C + |1\rangle_A |0\rangle_C}{\sqrt{2}}. \tag{20}$$

All projectors for Alice and Bob commute, Bob's projectors commute with P_E, but Alice's projectors do not all commute with P_E: $[P_A^0, P_E] \neq 0$ and $[P_A^1, P_E] \neq 0$. Calculation yields:

$$\Pr[P_E; P_A^0] = \Pr[P_E; P_A^1] = \frac{1}{2}, \quad \Pr[P_E; P_A^2] = 0, \quad \Pr[P_E; P_B^0] = 1, \quad \Pr[P_E; P_B^1] = 0. \tag{21}$$

Therefore, choosing $q_A = 1/2$ and $q_B = 1$, we have $A_0 = P_A^0 + P_A^1$ and $B_0 = P_B^0$. Moreover, $C_A(B_0) = P_A^0 + P_A^1$ and $C_B(A_0) = P_B^0$, so that the recursion stabilizes immediately. Since $\mathrm{Tr}(A_* B_* \rho) = \mathrm{Tr}(B_* A_* \rho) = \mathrm{Tr}(P_B^0 \rho) = 2/3 > 0$, we conclude that it is common certainty that Alice holds the probability estimate $q_A = 1/2$ and Bob holds the estimate $q_B = 1$, in contrast with our Commutative Agreement Theorem. That is, we have exhibited a scenario of CCD. Later, we will establish a bound on the extent of disagreement possible in such scenarios.

It is instructive to see that Example 2 breaks only the final step (Equation 13) in the proof of the Commutative Agreement Theorem. It can be checked that Alice's probability of E conditioned on A_* is 1. But along her individual outcome branches, non-commutativity with P_E reduces the conditional probability to 1/2 (and to 0 on the third branch).

We chose a 3-2-2-level set-up in Example 2 because it naturally produces both CCD, as we have just seen, and also a branch exhibiting common certainty of agreement. For the latter, set $q_A = q_B = 0$ and check that the corresponding certainty recursion stabilizes immediately with $\mathrm{Tr}(C_* \rho) = 1/3 > 0$. It will be useful to have both branches when we revisit the example in the next section.

Common Certainty with a Classical Register

In a regime that violates commutativity of the measurements with the property of interest, the certainty recursion can stabilize with $q_A \neq q_B$, resulting in CCD, as Example 2 illustrated. We now show that recording the agents' measurement outcomes in a classical register restores commutativity and agreement. Our Classical Register Theorem therefore sheds new light on the Classical Agreement Theorem: In a quantum world, the classical result emerges from the physical act of classical recording.

To proceed, let R denote the finite set of classical transcripts from protocol runs. The classical register corresponds to the Hilbert space:

$$\mathcal{H}_R = \mathrm{span}\{|r\rangle_R : r \in R\}, \tag{22}$$

with associated commutative algebra:

$$\mathcal{A}_R = \big\{\mathbb{I}_{\mathcal{H}} \otimes \sum_{r \in S} |r\rangle\langle r|_R : S \subseteq R\big\}, \tag{23}$$

where $\mathbb{I}_{\mathcal{H}}$ is the identity on the three-lab Hilbert space. The recording process is modeled as a completely positive trace-preserving (CPTP) map $J : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_R)$, defined by:

$$\rho' = J(\rho) = \sum_{r \in R} M_r \, \rho \, M_r^\dagger \otimes |r\rangle\langle r|_R\,, \tag{24}$$

where M_r is the Kraus operator linked to transcript r. The resulting state ρ' is block-diagonal in the pointer basis of $\mathcal{H}_R$, thereby representing a measurement with classically recorded outcomes.

We next demonstrate that, once the outcomes are embedded in this commutative register algebra, the certainty recursion satisfies the commutativity conditions necessary for our Quantum Agreement Theorem. To make this explicit, write:

$$f_A : R \to \{1, \ldots, \ell_A\}, \; f_B : R \to \{1, \ldots, \ell_B\}, \tag{25}$$

for the functions extracting from each transcript r the outcomes registered by Alice and Bob. Define their respective projectors on the register:

$$\Pi_R^{(i,A)} = \sum_{\{r \,:\, f_A(r) = i\}} |r\rangle\langle r|_R\,, \quad \Pi_R^{(j,B)} = \sum_{\{r \,:\, f_B(r) = j\}} |r\rangle\langle r|_R\,. \tag{26}$$

In the full system, these operators lie in the commutative algebra $\mathcal{A}_R$ of Equation 23, and therefore:

$$\big[\mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(i,A)}, \mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(j,B)}\big] = 0,\ \big[\mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(\cdot)}, P_E \otimes \mathbb{I}_R\big] = 0, \tag{27}$$

which says that the commutativity assumptions of Equations 1-2 are now satisfied on $\mathbb{I}_{\mathcal{H}} \otimes \mathcal{H}_R$.

For any projector $P \in \mathbb{I}_{\mathcal{H}} \otimes \mathcal{A}_R$ and any property F acting on the system, the conditional probability of F given P is the analogous formula to Equation 3:

$$\Pr[P_F \otimes \mathbb{I}_R; P] = \frac{\mathrm{Tr}\big[(P_F \otimes \mathbb{I}_R)\, P\, \rho'\, P\big]}{\mathrm{Tr}(P\, \rho')}, \tag{28}$$

where we use the recorded state ρ'. With this definition, we can proceed to define analogs of the earlier certainty recursion of Equations 6-9. Letting C'_* be the stabilized common-certainty operator, our Commutative Agreement Theorem carries over directly.

Definition: *Fix a property of interest E and probability estimates q'_A and q'_B for Alice and Bob, respectively. Then q'_A and q'_B are recorded common certainty at (recorded) state ρ' if $\mathrm{Tr}(C'_*\, \rho') > 0$.*

Classical Register Theorem: *If it is recorded common certainty at state ρ' that Alice assigns q'_A to E and Bob assigns q'_B to E, then $q'_A = q'_B$.*

We illustrate this result by running the classical register on Example 2 from earlier, where we previously found CCD.

Example 2 Contd.: In the recorded setting, we use the same local measurements as before, but now work on the enlarged space $\mathcal{H} \otimes \mathcal{H}_R$. We write the transcripts in the form $r = (i, j)$ where $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$. The recorded state is:

$$\rho' = \sum_{i,j} M_{(i,j)}\, \rho\, M_{(i,j)}^{\dagger} \otimes |(i,j)\rangle\langle(i,j)|_R\,, \tag{29}$$

with nonzero blocks $(0, 0)$, $(1, 0)$, and $(2, 1)$, each with weight $1/3$. The conditional probabilities of E, calculated from ρ', are:

$$\Pr[P_E \otimes \mathbb{I}_R\,; \mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(i=0,A)}] = \Pr[P_E \otimes \mathbb{I}_R\,; \mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(i=1,A)}] = \frac{1}{2},\ \Pr[P_E \otimes \mathbb{I}_R\,; \mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(i=2,A)}] = 0, \tag{30}$$

$$\Pr[P_E \otimes \mathbb{I}_R\,; \mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(j=0,B)}] = \frac{1}{2},\ \Pr[P_E \otimes \mathbb{I}_R\,; \mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(j=1,B)}] = 0. \tag{31}$$

Choosing $q'_A = q'_B = \frac{1}{2}$, we obtain the immediately-stabilized recursion:

$$A'_* = A'_0 = \mathbb{I}_{\mathcal{H}} \otimes (\Pi_R^{(i=0,A)} + \Pi_R^{(i=1,A)}),\ B'_* = B'_0 = \mathbb{I}_{\mathcal{H}} \otimes \Pi_R^{(j=0,B)}, \tag{32}$$

where $\mathrm{Tr}(C'_*\, \rho') = \frac{2}{3} > 0$. Thus, there is recorded common certainty at ρ' that $q'_A = q'_B = \frac{1}{2}$. It can be checked that the other common-certainty branch in Example 2, with agreement $q_A = q_B = 0$, is preserved under recording as a branch with $q'_A = q'_B = 0$ and probability $\mathrm{Tr}(C'_*\, \rho') = \frac{1}{3} > 0$.

Scenario	**Probability of E**
Prior (pre-measurement)	$\mathrm{Tr}(P_E\, \rho) = \frac{2}{3}$
Common certainty of disagreement	Alice: $q_A = \frac{1}{2}$, Bob: $q_B = 1$
Agreement branch I (post-register)	Alice: $q_A = \frac{1}{2}$, Bob: $q_B = \frac{1}{2}$
Agreement branch II (post-register)	Alice: $q_A = 0$, Bob: $q_B = 0$

Table 1: Non-Commuting and Classical-Register Probabilities

Table 1 summarizes the calculations for Example 2. We see how the classical register restores agreement. The CCD branch has been classicalized to $q_A = q_B = 1/2$. (Note that this agreed-on probability is different from the prior probability of E.) The common-certainty branch with agreement, at $q_A = q_B = 0$, has been carried over unchanged, as we expect.

Bounds on Disagreement

We now examine the question of the extent of disagreement across agents allowed by QM. We first state a result that applies to the general case, commuting or non-commuting, that rules out "extreme" disagreement between agents. This is true even for second-level certainty, let alone common certainty. (The theorem generalizes an earlier result in Contreras-Tejada et al., 2021, where the 0 - 1 phenomenon is called singular disagreement.)

0 - 1 Impossibility Theorem: *There is no positive-probability state where Alice is certain of E and certain that Bob is certain of not-E. Formally:*

$$\mathrm{Tr}\big(Q_A(E;\,1)\,C_A(Q_B(E;\,0))\,\rho\big) = 0. \tag{33}$$

This result does not use any commutativity assumptions and therefore imposes an epistemic constraint independent of any compatibility conditions.

Our impossibility result has bite. The Supplementary Material contains an example of a no-signaling box (Popescu and Rohrlich, 1994) where 0-1 disagreement does arise: Alice is certain of E and certain that Bob is certain of not-E. We also present an equivalent formulation of the example on phase space where the sole non-classicality is the use of signed probabilities (Abramsky and Brandenburger, 2011).

Next, we show that our Commutative Agreement Theorem is robust to small changes to the set-up. We first examine the case where Alice and Bob start from different initial states ρ_A and ρ_B, in place of the same initial state ρ as previously. Such a difference might arise from, for example, noise, channel disturbance, or the presence of an eavesdropper. Building on a classical analysis by Hellman (2013), it can be shown that in our commuting regime:

$$|q_A - q_B| \leq \frac{2\,\|\rho_A - \rho_B\|_1}{\max\{\mathrm{Tr}(B_* A_*\,\rho_A),\mathrm{Tr}(A_* B_*\,\rho_B)\}}, \tag{34}$$

where $\|\cdot\|_1$ is the trace norm of the difference between two quantum states. This formula allows us to incorporate perturbations of the quantum state. For example, in the eavesdropping scenario, after Alice has performed her measurement, a third agent, Eve, disturbs the state from ρ_A to ρ_B before Bob makes his measurement. Inequality 34 shows that the resulting departure from agreement is controlled.

The second robustness test relaxes the certainty modality. Given a measurement outcome P_X^k for agent X, we follow the classical treatment in Monderer and Samet (1989) and say that X is $(1-\epsilon)$-*certain* of the property F if, in place of probability 1 in Equation 3, we have:

$$\mathrm{Prob}[P_F; P_X^k] \geq 1 - \epsilon, \tag{35}$$

for a small positive number ϵ. Using Equation 35 in place of Equation 3 in the certainty recursion of Equations 6–9, we arrive at the condition of *common* $(1-\epsilon)$-*certainty* of F. From this, one can show in our commuting regime: If it is common $(1-\epsilon)$-certainty at state ρ that Alice assigns q_A to E and Bob assigns q_B to E, then:

$$|q_A - q_B| \leq 2\epsilon. \tag{36}$$

This second bound is relevant because, in practice, agents must confront finite statistics, detector inefficiencies, and coarse-gained outcomes, making $(1-\epsilon)$-certainty more realistic. Inequality 36 shows that the departure from agreement is again smooth. Proofs of both robustness results are in the Supplementary Material.

Discussion

Our results operate at the interface of the logic of belief and the physics of measurement in a quantum setting. We establish that the quantum world supports an analog of the Agreement Theorem of classical epistemics (Aumann, 1976) in the commuting case, while we find the distinctive epistemic phenomenon of CCD in the non-commuting regime. We therefore identify a clear boundary between classical intersubjectivity (ruled our under common certainty) and quantum intersubjectivity (allowed under common certainty). We show that CCD reduces to agreement under classical recording, offering a way to understand the classical Agreement Theorem in a quantum world.

Importantly, QM bounds the disagreement that is possible across agents, via our 0 - 1 impossibility theorem. This result validates that our epistemic analysis has physical bite – there are epistemic states ruled out by QM. The impossibility result also distinguishes our approach from Frauchiger and Renner (2018). This latter work shows that chaining together statements of Alice's certainty across mutually incompatible measurement branches yields a scenario that looks exactly like 0 - 1 disagreement, which is then – very reasonably – argued to suggest that conceptual inconsistencies may lurk in QM. By insisting that Alice's certainties refer to the same underlying quantum state, we show, via our impossibility result, that the Frauchiger-Renner scenario cannot arise.

We still allow disagreement in QM, albeit of a milder nature than 0 - 1 disagreement. So, we see our findings as challenging the notion that intersubjective disagreement should not be allowed in physics and, in this way, push in a similar direction to the QBist and relational accounts of QM (Fuchs and Schack, 2013; Rovelli, 1996). By introducing the tools of formal epistemics, our approach may suggest new treatments of these approaches.

In previous work, Contreras-Tejada et al. (2021) formulate the agreement question in terms of a no-signaling box. Within this environment, a quantum agreement theorem follows and failed only for superquantum boxes. Liu, Chung, and Ramanathan (2024) generalize this result, while remaining in the commuting regime, to establish an agreement theorem for all quantum and almost-quantum regions. The present paper extends Contreras-Tejada et al. (2021) so that Alice's and Bob's operators can inhabit distinct, possibly non-commuting, operator algebras, and, as we have seen, this admits CCD even in a quantum setting. Leifer and Duarte (2022) treat the knowledge modality of the original classical Agreement Theorem, rather than the certainty modality, and establish the impossibility of common knowledge of disagreement in the setting of generalized probability theory (Barrett, 2007). (We comment on the difference between these modalities in the Supplementary Material.) Khrennikov and Basieva (2014) and Khrennikov (2015) study quantum-like agents interacting with quantum systems and are able to obtain CCD in this manner.

We note that CCD, as a distinctive kind of quantum strangeness, also becomes, under the modern view, a new quantum resource. We leave to future work possible applications to quantum computing and information (Nielsen and Chuang, 2011), to the design and performance of quantum-enabled economic games (Khan et al., 2018; Auletta et al., 2021), and to the detection of other manifestations of non-classicality (Smirne et al., 2018; Strasberg and García Díaz, 2019). We end with what we think is an intriguing possibility: The value of CCD as a quantum resource may stem precisely from the fact that agents have common certainty of it.

References

[1] Abramsky S., and A. Brandenburger, "The Sheaf-Theoretic Structure of Non-locality and Contextuality," *New Journal of Physics*, 13, 2011, 113036.

[2] Adlam, E., "What Does '(Non-)Absoluteness of Observed Events' Mean," *Foundations of Physics*, 54, 2024, 13.

[3] Adlam, E., and C. Rovelli, "Information is Physical: Cross-Perspective Links in Relational Quantum Mechanics," *Philosophy of Physics*, 2023, 1-19.

[4] Auletta, V., D. Ferraioli, A. Rai, G. Scarpa, and A. Winter, "Belief-Invariant and Quantum Equilibria in Games of Incomplete Information," *Theoretical Computer Science*, 895, 2021, 151-177.

[5] Aumann, R., "Agreeing to Disagree," *Annals of Statistics*, 4, 1976, 1236-1239.

[6] Aumann R., and A, Brandenburger, "Epistemic Conditions for Nash Equilibrium," *Econometrica*, 63, 1995, 1161-1180.

[7] Barrett, J., "Information Processing in Generalized Probabilistic Theories," *Physical Review A*, 75, 032304.

[8] Di Biagio, A., and C. Rovelli, "Stable Facts, Relative Facts," *Foundations of Physics*, 51, 2021, 30.

[9] Brandenburger, A., and E. Dekel, "Common Knowledge with Probability 1," *Journal of Mathematical Economics*, 3, 1987, 237-245.

[10] Brandenburger, A., P. Contreras-Tejada, P. La Mura, G. Scarpa, and K. Steverson, "Agreement and Disagreement in a Non-Classical World," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 382, 2024.

[11] Brukner, Č., "A No-Go Theorem for Observer-Independent Facts," *Entropy*, 20, 2018, 350.

[12] Contreras-Tejada, P., G. Scarpa, A. Kubicki, A. Brandenburger, P. La Mura, "Observers of Quantum Systems Cannot Agree to Disagree," *Nature Communications*, 12, 2021, 7021.

[13] Dekel, E., and M. Siniscalchi, “Epistemic Game Theory,” in Young, H.P., and S. Zamir (eds.), *Handbook of Game Theory with Economic Applications*, Vol. 4, 2015, 619-702.

[14] Frauchiger, D., and R. Renner, “Quantum Theory Cannot Consistently Describe the Use of Itself,” *Nature Communications*, 9, 2018, 3711.

[15] Fuchs, C., and R. Schack, “Quantum-Bayesian Coherence,” *Reviews of Modern Physics*, 85, 2013, 1693.

[16] Fuchs, C., N.D. Mermin, and R. Schack, “An Introduction to QBism With an Application to the Locality of Quantum Mechanics,” *American Journal of Physics*, 82, 2014, 749-754.

[17] Geanakoplos, J., and H. Polemarchakis, “We Cannot Disagree Forever,” *Journal of Economic Theory*, 28, 1982, 192-200.

[18] Gelfand, I., and M. Naimark, “On the Imbedding of Normed Rings into the Ring of Operators on a Hilbert space,” *Matematicheskii Sbornik*, 12, 1943, 197-217.

[19] Gizatulina, A., and Z. Hellman, “No Trade and Yes Trade Theorems for heterogeneous Priors,” *Journal of Economic Theory*, 182, 2019, 161-184.

[20] Held, C., *Die Bohr-Einstein-Debatte: Quantenmechanik und physikalische Wirklichkeit*, Brill, 1998.

[21] Hellman, Z., “Almost Common Priors,” *International Journal of Game Theory*, 42, 2013, 399.

[22] Khan, F., N. Solmeyer, B. Radhakrishnan, and T. Humble, “Quantum Games: a Review of the History, Current State, and Interpretation,” *Quantum Information Processing*, 17, 2018, 309.

[23] Khrennikov, A., and I. Basieva, “Possibility to Agree on Disagree from Quantum Information and Decision Making,” *Journal of Mathematical Psychology*, 62, 2014, 1-15.

[24] Khrennikov, A., “Quantum Version of Aumann’s Approach to Common Knowledge: Sufficient Conditions of Impossibility to Agree on Disagree,” *Journal of Mathemtical Economics*, 60, 2015, 89-104.

[25] Leifer M., and C. Duarte, “Generalizing Aumann’s Agreement Theorem,” 2022, at http://arxiv.org/abs/2202.02156.

[26] Liu, Y., H.-Y. Chung, and R. Ramanathan, “(Almost-)Quantum Bell Inequalities and Device-Independent Applications,” *Quantum*, 8, 2024, 1489.

[27] Milgrom P., and N. Stokey, “Information, Trade, and Common Knowledge,” *Journal of Economic Theory*, 26, 1982, 17-27.

[28] Monderer, D., and D. Samet, “Approximating Common Knowledge with Common Beliefs,” *Games and Economics Behavior*, 1, 1989, 170.

[29] Nielsen, M., and I. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary edition, Cambridge University Press, 2011.

[30] Nurgalieva, N., and L. del Rio, “Inadequacy of Modal Logic in Quantum Settings,” in Proceedings of QPL 2018, *EPTCS*, 287, 2019, 267-297.

[31] Pacuit, E., and O. Roy, “Epistemic Foundations of Game Theory”, in Zalta, E., and U. Nodelman (eds.), *The Stanford Encyclopedia of Philosophy*, Fall 2025 edition, at https://plato.stanford.edu/archives/fall2025/entries/epistemic-game/.

[32] Pawlowski, M., T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Zukowski, “Information Causality as a Physical Principle,” *Nature*, 461, 2009, 1101-1104.

[33] Popescu, S,, and D. Rohrlich, “Quantum Nonlocality as an Axiom,” *Foundations of Physics* 24, 1994, 379.

[34] Proietti, M., A. Pickston, F. Graffitti, P. Barrow, D. Kundys, C. Branciad, M. Ringbauer, and A. Fedrizzi, “Experimental Test of Local Observer Independence,” *Science Advances*, 5, 2019, 9832.

[35] Rohrlich, D., “Stronger-Than-Quantum Bipartite Correlations Violate Relativistic Causality in the Classical Limit,” 2014, at https://arxiv.org/pdf/1408.3125.

[36] Rovelli, C., “Relational Quantum Mechanics,” *International Journal of Theoretical Physics*, 35, 1996, 1637.

[37] Schack, R., "When Will Two Agents Agree on a Quantum Measurement Outcome? Intersubjective Agreement in QBism," *International Journal of Theoretical Physics*, 63, 2024, 254.

[38] Sebenius J., and J. Geanakoplos, "Don't Bet on It: Contingent Agreements with Asymmetric Information," *Journal of the American Statistical Association*, 78, 1983, 424-426.

[39] Smirne, A., D. Egloff, M.G. Díaz, M.B. Plenio, and S.F. Huelga, "Coherence and Non-classicality of Quantum Markov Processes," *Quantum Science and Technology*, 4, 2019, 01LT01.

[40] Specker, E., "Die Logik Nicht Gleichzeitig Entscheidbarer Aussagen," *Dialectica*, 14, 1960, 239-246.

[41] Strasberg, P., and M. García Díaz, "Classical Quantum Stochastic Processes," *Physical Review A*, 100, 2019, 022120.

[42] van Dam, W. "Implausible Consequences of Superstrong Nonlocality," 2005, at https://arxiv.org/abs/quant-ph/0501159.

[43] Wigner, E., "Remarks on the Mind-Body Problem," in Good, I. (ed.), *The Scientist Speculates*, Heinemann, 1961.

Supplementary Material

S.1 Classical Agreement Theorem with the Certainty Modality

We begin with a review of the classical Agreement Theorem. The original statement due to Aumann (1976) is couched in terms of the knowledge modality and common knowledge. Here, we establish a different classical baseline by using the certainty modality: Alice is certain of an event E if she assigns probability 1 to E given her information (Brandenburger and Dekel, 1987; Monderer and Samet, 1989). This shift is important in aligning our formal arguments with real-world laboratory procedures, since a probability-1 assignment to an event is a statement about local frequencies in a lab.

In addition, it can be shown that common knowledge of probability assessments yields agreement in any phase-space model with signed probabilities (Brandenburger et al., 2024, Theorem 4.1). This result also aligns with Leifer and Duarte (2022). By Abramsky and Brandenburger (2011, Theorem 5.9), the same is then true for any no-signaling model. This gives another reason why we prefer the certainty modality. Using knowledge erases the boundary between quantum and superquantum systems (cf. the discussion of this boundary in the Introduction in the main text).

To proceed, fix a finite probability space (Ω, p) and information partitions $\mathcal{P}_A$ and $\mathcal{P}_B$ for Alice and Bob, respectively. We fix versions of conditional probability $p\big(\cdot \,|\, \mathcal{P}_A(\omega)\big)$ and $p\big(\cdot \,|\, \mathcal{P}_B(\omega)\big)$ for Alice and Bob, respectively, where $\mathcal{P}_A(\omega)$ and $\mathcal{P}_B(\omega)$ are the cells of Alice's and Bob's partitions that contain ω. (As an aside, this formulation extends without change – needing only the dominated convergence theorem – to the general measurable case where σ-fields replace partitions.)

Now, given an event E in Ω and probabilities q_A and q_B, let:

$$A_0 = \big\{\omega \in \Omega : p\big(E \,|\, \mathcal{P}_A(\omega)\big) = q_A\big\}, \tag{S.1}$$

$$B_0 = \big\{\omega \in \Omega : p\big(E \,|\, \mathcal{P}_B(\omega)\big) = q_B\big\}. \tag{S.2}$$

The event A_0 is the event that Alice assigns probability q_A to E; likewise for the event B_0. Next define recursively, for $n \geq 0$:

$$A_{n+1} = A_n \cap \big\{\omega \in \Omega : p(B_n \,|\, \mathcal{P}_A(\omega)) = 1\big\}, \tag{S.3}$$

$$B_{n+1} = B_n \cap \big\{\omega \in \Omega : p(A_n \,|\, \mathcal{P}_B(\omega)\big) = 1\big\}, \tag{S.4}$$

and:

$$C_\infty = \bigcap_{n=0}^{\infty} A_n \cap \bigcap_{n=0}^{\infty} B_n. \tag{S.5}$$

In words, it is common certainty at a state $\omega \in \Omega$ that Alice assigns probability q_A to E and Bob assigns probability q_B to E if $\omega \in C_\infty$. Note that a special case is that $q_A = q_B = 1$. Then we can simply say that the event E is common certainty at ω.

Theorem S.1. *If it is common certainty at state ρ that Alice assigns q_A to E and Bob assigns q_B to E, and $p(C_\infty) > 0$, then $q_A = q_B$.*

Proof. Since Ω is finite, and the sets A_n and B_n are non-increasing, there is an integer N such that $A_n = A_N$ and $B_n = B_N$ for all $n \geq N$. It follows from Equations S.3–S.4 that:

$$A_N \subseteq \{\omega \in \Omega : p(B_N \,|\, \mathcal{P}_A(\omega)) = 1\}, \tag{S.6}$$
$$B_N \subseteq \{\omega \in \Omega : p(A_N \,|\, \mathcal{P}_B(\omega)) = 1\}. \tag{S.7}$$

Observe that A_N is a union of cells of $\mathcal{P}_A$, which we write as $\cup\pi$. Equation S.6 then implies $p(B_N \,|\, \pi) = 1$ for each such π. From this:

$$p(A_N \cap B_N) = \sum_\pi p(\pi)\, p(B_N \,|\, \pi) = p(A_N). \tag{S.8}$$

Next, we use $A_N \subseteq A_0$ to write $p(E \,|\, \pi) = q_A$ for the same partition cells π. By the same argument as just given, we then get:

$$p(E \cap A_N) = q_A \times p(A_N). \tag{S.9}$$

Equation S.8 implies that for any event E:

$$p(E \cap A_N) = p(E \cap A_N \cap B_N), \tag{S.10}$$

which, together with Equation S.9 yields:

$$q_A \times p(A_N) = p(E \cap A_N \cap B_N), \tag{S.11}$$

or, using Equation S.8 again:

$$q_A \times p(A_N \cap B_N) = p(E \cap A_N \cap B_N). \tag{S.12}$$

Running exactly the same argument with Bob in place of Alice, we obtain:

$$q_B \times p(A_N \cap B_N) = p(E \cap A_N \cap B_N). \tag{S.13}$$

But $p(C_\infty) > 0$ implies $p(A_N \cap B_N) > 0$, so we conclude that $q_A = q_B$, as required. □

Figure 2 shows that common certainty of probability assessments need not coincide with the shared assessment Alice and Bob would obtain after pooling their information. Here, Alice's (resp. Bob's) partition is the rows (resp. columns). The event of interest E is $\{\omega_1, \omega_4\}$. At any state, it is common certainty that Alice and Bob both assign probability 1/2 to E. Yes, if they pooled their information, their shared assessment would be 1 or 0, depending on the true state.

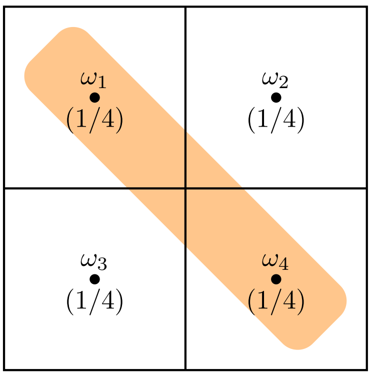

Figure 2: Information pooling

S.2 Proof of of the Commutative Agreement Theorem

Lemma S.1. *For all n, the operators A_n and B_n are projectors.*

Proof. Each of A_0 and B_0 is a sum of mutually orthogonal projectors, hence a projector. The certainty operators $C_A(B_n)$ and $C_B(A_n)$ are also sums of orthogonal projectors. Since a product of commuting projectors is again a projector, it follows that A_{n+1} and B_{n+1} are both projectors, as required. □

Lemma S.2. *For all n, m, we have $[A_n, B_m] = 0$. In particular, we have $[A_*, B_*] = 0$.*

Proof. All A_n lie in the algebra generated by $\{P_A^i\}$ and all B_m lie in the algebra generated by $\{P_B^j\}$. Since each P_A^i commutes with each P_B^j, the two algebras commute element-wise. Therefore, all their products commute. □

Lemma S.3. *For all n, we have:*

$$\mathrm{Tr}(A_n\,\rho) \geq \mathrm{Tr}(A_*\,\rho) \geq \mathrm{Tr}(C_*\,\rho) > 0, \tag{S.14}$$

and similarly for B_n.

Proof. Since $C_* = A_* B_*$ is the projector onto $\mathrm{ran}(A_*) \cap \mathrm{ran}(B_*)$, we have $0 \leq C_* \leq A_* \leq A_n$ in the Loewner (positive semidefinite) order, and similarly $0 \leq C_* \leq B_* \leq B_n$. Monotonicity of the trace with respect to the operator order and $\rho \geq 0$ give the inequalities and strict positivity. □

Lemma S.3 ensures that all conditional probabilities in the certainty recursion are well defined (no vanishing denominators).

Lemma S.4. *Let Q be a projector and σ a density operator. If $\mathrm{Tr}(Q\,\sigma) = 1$, then:*

$$Q\,\sigma\,Q = \sigma. \tag{S.15}$$

Proof. The condition $\mathrm{Tr}(Q\,\sigma) = 1$ implies that σ has support contained in $\mathrm{ran}(Q)$, from which it follows that $Q\,\sigma\,Q = \sigma$. □

Lemma S.5. *We have:*

$$\frac{B_* A_*\,\rho\,A_* B_*}{\mathrm{Tr}(B_* A_*\,\rho)} = \frac{A_*\,\rho\,A_*}{\mathrm{Tr}(A_*\,\rho)}. \tag{S.16}$$

Proof. Write:

$$A_* = \sum_{i \in L} P_A^i, \tag{S.17}$$

where L indexes those outcomes satisfying $\Pr[B_*; P_A^i] = 1$. For any such i, the conditional state $\rho_i = P_A^i\,\rho\,P_A^i / \mathrm{Tr}(P_A^i\,\rho)$ satisfies $\mathrm{Tr}(B_*\,\rho_i) = 1$. By Lemma S.4:

$$B_*\,\rho_i\,B_* = \rho_i, \tag{S.18}$$

or, equivalently:

$$B_*(P_A^i\,\rho\,P_A^i)B_* = P_A^i\,\rho\,P_A^i. \tag{S.19}$$

Summing over $i \in L$ and using commutativity of all projectors gives:

$$B_*(A_*\,\rho\,A_*)B_* = A_*\,\rho\,A_*. \tag{S.20}$$

Normalizing both sides yields the required equality. □

Lemma S.6. *We have:*

$$\mathrm{Tr}\Big(P_E \frac{A_*\,\rho\,A_*}{\mathrm{Tr}(A_*\,\rho)}\Big) = \mathrm{Tr}\Big(P_E \frac{B_* A_*\,\rho\,A_* B_*}{\mathrm{Tr}(B_* A_*\,\rho)}\Big). \tag{S.21}$$

Proof. Apply the linear functional $T \mapsto \mathrm{Tr}(P_E T)$ to both sides of Lemma S.5. □

Lemma S.7. *For any $i \in L$ with $\mathrm{Tr}(P_A^i\,\rho) > 0$:*

$$\Pr[E; P_A^i] = \mathrm{Tr}\Big(P_E \frac{A_*\,\rho\,A_*}{\mathrm{Tr}(A_*\,\rho)}\Big). \tag{S.22}$$

Proof. For each $i \in L$, we have $\Pr[E; P_A^i] = q_A$ by construction. Because P_E commutes with all P_A^i and these projectors are mutually orthogonal:

$$\mathrm{Tr}(P_E A_*\,\rho\,A_*) = \sum_{i \in L} \mathrm{Tr}(P_E P_A^i\,\rho\,P_A^i) = \sum_{i \in L} q_A \cdot \mathrm{Tr}(P_A^i\,\rho) = q_A \cdot \mathrm{Tr}(A_*\,\rho). \tag{S.23}$$

Dividing both sides by $\mathrm{Tr}(A_*\,\rho)$ yields the desired equality. □

The proof of the Commutative Agreement Theorem can now be finished just as in the main text.

We note that, in the setting of Equations 1–2 of the main text, a proof of the Commutative Agreement Theorem could be obtained via the Gelfand representation theorem (Gelfand and Naimark, 1943). This would involve some bookkeeping, and our operator proof works transparently within the Hilbert-space formalism. Our direct proof also sets the stage for our subsequent commuting and non-commuting examples, and for the further results in the paper.

S.3 Additional Scenarios

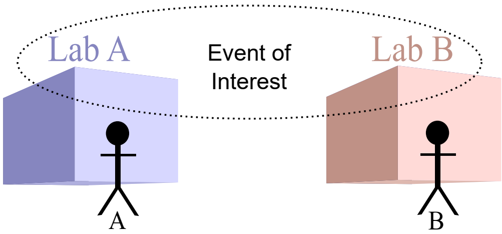

Figure 3: Two-laboratory scenario

For our first variant on the three-lab scenario in the main text, we suppose that Alice and Bob have their own labs (there is no third lab) and perform local measurements. The property of interest is modeled by a global projector $P_E \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. See Figure 3. In this scenario, the commutativity condition of Equation (1) continues to hold, but Equation (2) need not. The Agreement Theorem therefore holds in this setting whenever it makes sense to assume Equation (2) outright.

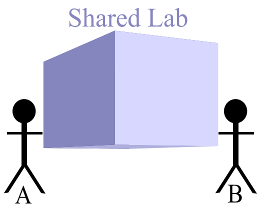

Figure 4: Single-laboratory scenario

For the second variant, we suppose that Alice and Bob share a lab in which there is a quantum system. Each performs a measurement, and they each estimate the probability of a property of interest of the system. See Figure 4. In this scenario, there is no reason to favor either Equation (1) or Equation (2). An Agreement Theorem follows if both conditions are assumed outright.

The Agreement Theorem also holds in any scenario where the following conditions obtain:

$$[P_A^i, \rho] = [P_B^j, \rho] = 0 \text{ for all } i, j \text{ and } [A_*, B_*] = 0. \tag{S.24}$$

S.4 Proof of the 0-1 Impossibility Theorem

We state the equivalent Born-Lüders update version of the statement of the theorem in the main text. Specifically, there is no state ρ and outcome projector P_A^i for Alice such that:

$$\mathrm{Tr}(P_A^i \rho) > 0, \quad \Pr[P_E; P_A^i] = 1, \quad \Pr[Q_B(E; 0); P_A^i] = 1 \tag{S.25}$$

Assume for contradiction that such a ρ and P_A^i exist. We first show that Alice's post-measurement state lies in the E-subspace. Let $S = \mathrm{supp}(\rho)$ and let P_S denote the projector onto S. Let Alice's updated state σ on outcome i be given by:

$$\sigma = \frac{P_A^i \rho P_A^i}{\mathrm{Tr}(P_A^i \rho)}. \tag{S.26}$$

Let $K = \mathrm{supp}(\sigma)$. Since ρ vanishes on $S^\perp$, we have $K \subseteq S$. The assumption $\Pr[P_E; P_A^i] = 1$ is equivalent to:

$$\mathrm{Tr}\big((\mathbb{I} - P_E)\sigma\big) = 0, \tag{S.27}$$

from which $v = P_E v$ for all $v \in K$. Therefore:

$$K \subseteq S \cap \mathrm{ran}(P_E). \tag{S.28}$$

Next, let Bob's updated state ρ_B^j on outcome j be given by:

$$\rho_B^j = \frac{P_B^j\, \rho\, P_B^j}{\mathrm{Tr}(P_B^j\, \rho)} \text{ where } \mathrm{Tr}(P_B^j\, \rho) > 0. \tag{S.29}$$

Let:

$$J_0 = \{j : \Pr[E; P_B^j] = 0 \text{ and } \mathrm{Tr}(P_B^j\, \rho) > 0\}, \tag{S.30}$$

and define Bob's "certain not-E" projector (paralleling the main text):

$$Q_B(E; 0) = \sum_{j \in J_0} P_B^j. \tag{S.31}$$

We now show that vectors in K are killed by each P_B^j with $j \in J_0$. Fix $j \in J_0$. Since ρ_B^j assigns probability 0 to E, we have:

$$P_E\, \rho_B^j\, P_E = 0, \tag{S.32}$$

which implies:

$$T_j = P_E P_B^j\, \rho\, P_B^j P_E = 0. \tag{S.33}$$

Factor $\rho = \rho^{1/2}\rho^{1/2}$ to write:

$$T_j = (\rho^{1/2} P_B^j P_E)^\dagger (\rho^{1/2} P_B^j P_E), \tag{S.34}$$

from which $T_j = 0$ implies:

$$\rho^{1/2} P_B^j P_E = 0. \tag{S.35}$$

Because $\rho^{1/2}$ is invertible on $S = \mathrm{supp}(\rho)$, we obtain:

$$P_S P_B^j P_E P_S = 0, \tag{S.36}$$

and therefore, for all $v \in S$:

$$P_B^j P_E v \in S^\perp. \tag{S.37}$$

Now let $v \in K$. Then $v \in S$ and, by Equation S.28, we get $v = P_E v$. Applying Equation S.37, we obtain:

$$P_B^j v \in S^\perp. \tag{S.38}$$

Since $K \subseteq S$, this means the component of $P_B^j v$ in K vanishes:

$$P_K P_B^j P_K = 0. \tag{S.39}$$

Summing over $j \in J_0$:

$$P_K Q_B(E; 0) P_K = 0. \tag{S.40}$$

To arrive at the contradiction, note that the assumption $\Pr[Q_B(E; 0); P_A^i] = 1$ is equivalent to:

$$\mathrm{Tr}(Q_B(E; 0)\, \sigma) = 1, \tag{S.41}$$

and, since $Q_B(E; 0)$ is a projector, Lemma S.4 implies:

$$Q_B(E; 0)\, \sigma\, Q_B(E; 0) = \sigma. \tag{S.42}$$

Therefore, $Q_B(E; 0)$ acts as the identity on $K = \mathrm{supp}(\sigma)$:

$$Q_B(E; 0)\big|_K = \mathbb{I}_K. \tag{S.43}$$

But by Equation S.40:

$$Q_B(E; 0)\big|_K = \sum_{j \in J_0} P_K P_B^j P_K = 0, \tag{S.44}$$

which contradicts Equation S.43, since $K \neq \{0\}$.

We next present a no-signaling box (Popescu and Rohlich, 1994) that exhibits 0-1 disagreement in the natural way that the epistemics are defined in this setting. (The example builds on Contreras-Tejada et al., 2021.) In Table 2, there are three possible measurements a, b, e, each with outcome 0 or 1. The compatible measurements are the pairs (a, b), (a, e), and (b, e). The probabilities of each of the sets of four joint outcomes are given in each row. No signaling is satisfied. For example, if the measurement pair (a, b) is undertaken, the marginal probability that b yields 1 is $1/4 + 1/4 = 1/2$. If the measurement pair (b, e) is undertaken, this marginal is $1/2 + 0 = 1/2$. The other no-signaling equalities can be checked similarly.

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
(a,e)	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
(b,e)	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 2: No-signaling box exhibiting 0-1 disagreement

Now, let the event of interest be $e = 1$ and suppose Alice obtains the outcome $a = 1$. Then, from the second row:

$$\Pr[e = 1|a = 1] = \frac{1/4}{0 + 1/4} = 1. \tag{S.45}$$

From the third row:

$$\Pr[e = 1|b = 0] = \frac{1/2}{0 + 1/2} = 1, \quad \Pr[e = 0|b = 1] = \frac{1/2}{1/2 + 0} = 1. \tag{S.46}$$

Finally, from the first row:

$$\Pr[b = 1|a = 1] = \frac{1/4}{0 + 1/4} = 1. \tag{S.47}$$

We conclude that if Alice measures a and gets the outcome 1, she is certain that $e = 1$. She is also certain that $b = 1$. From this, she is certain that Bob is certain that $e = 0$. We obtain 0-1 disagreement.

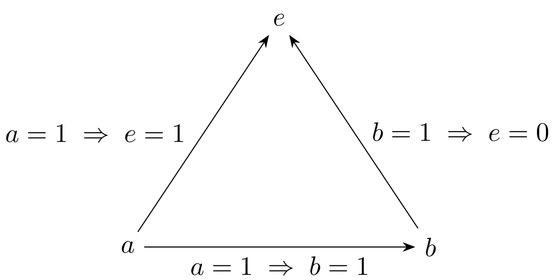

Figure 5: Implication structure

Interestingly, as Figure 5 indicates, the structure of this no-signaling example is exactly that of the Specker triangle (Specker, 1960). This connection between contextuality and 0-1 disagreement would seem to deserve further study.

We next recast this example in quasi-classical terms using signed probabilities on phase space (following Abramsky and Brandenburger, 2011, Theorem 5.4, and Brandenburger et al., 2024). Though an equivalent setting, the epistemic analysis may be more transparent this way. Let $X = \{a, b, e\}$ be the set of elementary measurements and $O = \{0, 1\}$ the set of outcomes. The phase space is the set of global assignments $\Omega = O^X$ and the signed probability measure λ on Ω is depicted in Table 3 below.

It can be checked that this table realizes the no-signaling box of Table 2. Also, we give Alice the partition $\{\{\omega : a(\omega) = 0\}, \{\omega : a(\omega) = 1\}\}$ and Bob the partition $\{\{\omega : b(\omega) = 0\}, \{\omega : b(\omega) = 1\}\}$. The event of interest is $E = \{\omega : e(\omega) = 1\}$. At the state ω_7, Alice assigns conditional probability 1 to E. The event that Bob assigns conditional probability 0 to E is $\{\omega : b(\omega) = 1\}$. At ω_7, Alice assigns conditional probability 1 to $\{\omega : b(\omega) = 1\}$. This establishes 0-1 disagreement in the phase-space setting.

S.5 Proofs of Bounds on Disagreement

To prove the quantum Inequality 34 in the main text, we build on the classical argument in Hellman (2013). To begin, assume $c_A = \mathrm{Tr}(C_* \rho_A) > 0$ and $c_B = \mathrm{Tr}(C_* \rho_B) > 0$. Also, write $e_A = \mathrm{Tr}(P_E C_* \rho_A)$ and

ω_i	a	b	e	λ
ω_0	0	0	0	$+\frac{3}{16}$
ω_1	0	0	1	$+\frac{5}{16}$
ω_2	0	1	0	$+\frac{5}{16}$
ω_3	0	1	1	$-\frac{1}{16}$
ω_4	1	0	0	$-\frac{3}{16}$
ω_5	1	0	1	$+\frac{3}{16}$
ω_6	1	1	0	$+\frac{3}{16}$
ω_7	1	1	1	$+\frac{1}{16}$

Table 3: Signed phase-space probability measure

$e_B = \mathrm{Tr}(P_E C_* \rho_B)$, so that $q_A = e_A/c_A$, $q_B = e_B/c_B$, and:

$$|q_A - q_B| = \frac{|c_B e_A - c_A e_B|}{c_A c_B}. \tag{S.48}$$

Next, set $x = c_B e_A$, $y = c_A e_B$, and $z = c_B e_B$, and use the triangle inequality:

$$|x - y| \leq |x - z| + |z - y|, \tag{S.49}$$

to obtain:

$$|q_A - q_B| \leq \frac{|c_B e_A - c_B e_B|}{c_A c_B} + \frac{|c_B e_B - c_A e_B|}{c_A c_B}. \tag{S.50}$$

Rewrite the first numerator in Equation S.50 as:

$$|c_B e_A - c_B e_B| = c_B\, |e_A - e_B| = c_B \left|\mathrm{Tr}(P_E C^* (\rho_A - \rho_B))\right|. \tag{S.51}$$

Now use the operator analog to the classical Hölder's inequality:

$$\left|\mathrm{Tr}(X^\dagger Y)\right| \leq \|X\|_1\, \|Y\|_\infty, \tag{S.52}$$

and the fact that $\|P_E C_*\|_\infty = 1$ to write:

$$|e_A - e_B| \leq \|\rho_A - \rho_B\|_1, \tag{S.53}$$

from which:

$$|c_B e_A - c_B e_B| \leq c_B\, \|\rho_A - \rho_B\|_1. \tag{S.54}$$

Turning to the second numerator in Equation S.50, write:

$$|c_B e_B - c_A e_B| = e_B\, |c_B - c_A| = e_B \left|\mathrm{Tr}(C^* (\rho_A - \rho_B))\right|. \tag{S.55}$$

A second application of Hölder's inequality together with $\|C_*\|_\infty = 1$ yields:

$$|c_B - c_A| \leq \|\rho_A - \rho_B\|_1. \tag{S.56}$$

Also, from $P_E \leq \mathbb{I}$ we obtain:

$$e_B = \mathrm{Tr}(P_E C_* \rho_B) \leq \mathrm{Tr}(C_* \rho_B) = c_B, \tag{S.57}$$

so that:

$$|c_B e_B - c_A e_B| \leq c_B\, \|\rho_A - \rho_B\|_1. \tag{S.58}$$

Substituting Inequalities S.54 and S.58 into Inequality S.50, we find:

$$|q_A - q_B| \leq \frac{c_B\, \|\rho_A - \rho_B\|_1 + c_B\, \|\rho_A - \rho_B\|_1}{c_A c_B} = \frac{2\, \|\rho_A - \rho_B\|_1}{c_A}. \tag{S.59}$$

Rerunning the argument with A and B interchanged gives:

$$|q_A - q_B| \leq \frac{2\,\|\rho_A - \rho_B\|_1}{c_B}. \tag{S.60}$$

Putting Inequalities S.59–S.60 together yields Inequality 34.

To prove the quantum Inequality 36 in the main text, we adapt the classical argument in Monderer and Samet (1989). Using Inequality 35, we can retrace the steps in Section S.2 to obtain:

$$\mathrm{Tr}(B_* A_*\, \rho) \geq (1-\epsilon)\,\mathrm{Tr}(A_*\, \rho). \tag{S.61}$$

Now, decompose A_* as $A_* = C_* + A_*(\mathbb{I} - B_*)$, from which:

$$\mathrm{Tr}(P_E A_*\, \rho) = \mathrm{Tr}(P_E C_*\, \rho) + \mathrm{Tr}\big(P_E A_*(\mathbb{I} - B_*)\, \rho\big). \tag{S.62}$$

Letting:

$$x = \mathrm{Pr}[P_E; C_*], \quad y = \mathrm{Pr}[P_E; A_*(\mathbb{I} - B_*)], \quad \alpha = \mathrm{Pr}[C_*; A_*], \tag{S.63}$$

we can use Equation S.62 to write:

$$q_A = x\alpha + y(1-\alpha). \tag{S.64}$$

Inequality S.61 tells us that $\alpha \geq 1 - \epsilon$, from which Equation S.64 yields the estimate:

$$q_A \geq x(1-\epsilon) + 0, \tag{S.65}$$

while, at the same time, we can also estimate:

$$q_A \leq x \cdot 1 + 1 \cdot \epsilon. \tag{S.66}$$

Since x is symmetric in A and B, we can repeat these bounds for q_B. It follows that:

$$|q_A - q_B| \leq x + \epsilon - x(1-\epsilon), \tag{S.67}$$

from which, since $x \leq 1$, Inequality 36 follows.