Maximal orders optimal embedding of central simple algebras over number fields

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Abstract

Given a number field F and R be the ring of integers of F, the problem of embedding a field extension K/F into a central simple algebra B is classical. This paper proves that when the central simple algebra has degree p, the R-order $S \subset K$ can be optimal embedded into all maximal R-orders $O \subset B$, unless satisfies the optimal selectivity condition.

1 Introduction

First, we present some historical background on this problem. The most famous and earliest result in this area is the Albert–Brauer–Hasse–Noether Theorem.

Theorem 1.1 (Albert–Brauer–Hasse–Noether). *Let B be a quaternion algebra over a number field F, and let K/F be a quadratic extension of F. Then there is an embedding of K/F into B if and only if no prime of F which ramifies in B splits in K.*

We can generalize the quaternion algebras in this theorem to central simple algebras of degree p (p is an odd prime)[10, Theorem 32.15]:

Theorem 1.2. *Let B be a central simple algebra of degree p over a number field F, and $[K : F] = p$. Then there is an embedding of K/F into B if and only if no prime of F which ramifies in B splits in K.*

Throughout this paper, we consistently assume the condition $K \subset B$. Concerning the question of whether any R-order S in K can be integrally embedded into some R-order O in a quaternion algebra B, Chinburg and Friedman [3] was the first to prove selectivity when O is a maximal order; Chan and Xu [2] and Guo and Qin [5] independently proved selectivity when O is an Eichler order. If we replace the quaternion algebra B with a more general central simple algebra B, then Linowitz and Shemanske [6] proved the selectivity of maximal orders when B has degree p. Further, Linowitz and Shemanske [7] also proved the selectivity of embeddings of $\mathcal{O}_K$ into maximal orders in central simple algebras of arbitrary degree.

In the previous section, we reviewed results concerning selectivity of embeddings. In this paper, our primary focus is the selectivity of optimal embeddings. The notion of optimal embeddings was first introduced by Eichler [4], who also provided criteria for determining the existence of optimal embeddings into square-free level Eichler orders in quaternion algebras. Maclachlan [8] later established the first optimal selectivity results for square-free level Eichler orders in quaternion algebras. Subsequently, Voight [11] extended the theory by proving optimal selectivity for arbitrary level Eichler orders.

The goal of this paper is to establish the optimal selectivity of maximal orders in central simple algebras of degree p.

Theorem 1.3 (Theorem 3.7)**.** *Let B be a central simple algebra of degree p over a numble field F. R be the ring of integers of F and K/F be a field extension of degree p inside B. Suppose $S \subset K$ is an R-order, S can be optimal embedding an maximal R-order $O \subset B$. Then the following statements hold.*

(1) $\operatorname{Gen} O$ *is optimally selective for S if and only if $K \subseteq H_{GN(O)}$.*

(2)If $\operatorname{Gen} O$ *is optimally selective for S, then S can embeddding $O' \in \operatorname{Gen} O$ for exactly $\frac{1}{p}$ of the types $[O'] \in \operatorname{Typ} O$.*

The main objective of this paper is to refine Voight's work by extending the quaternion version of the optimal selection theorem to central simple algebras.

Notation. Throughout this paper, p denotes an odd prime and $\mathfrak{p}$ denotes a finite prime of F. If M is a finite dimensional F-vector space or a finitely generated R-module, $M_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic completion of M. Let $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ be the profinite completion of $\mathbb{Z}$. If X is a finite dimensional $\mathbb{Q}$-vector space or a finitely generated $\mathbb{Z}$-module, define $\widehat{X} = X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

2 Optimal embedding and class field

Regarding the problem of embedding fields into central simple algebras, Linowitz and Shemanske resolved the embedding problem for maximal orders [6]. This paper aims to refine general embeddings into optimal embeddings and obtain analogous results. First, we give the definition of an optimal embedding.

Definition 2.1. Let B be a central simple algebra of degree p over a numble field F and K/F be a field extension of degree p inside B and R be the ring of integers of F. Suppose $S \subset K$ is an R-order, $O \subset B$ is an R-order. An R-algebra embedding $\varphi : S \to O$ is optimal if $\varphi(K) \cap O = \varphi(S)$.

Let

$$\operatorname{Emb}_R(S, O) := \{\text{Optimal embeddings } S \hookrightarrow O\}$$

When no confusion can result, we drop the subscript R and write simply $\operatorname{Emb}(S, O)$.

If we fix one embedding $K \subset B$, then by the Skolem–Noether Theorem, any other embedding is obtained by conjugation by an element of $B^\times$. Let

$$E := \{\beta \in B^\times \mid \beta^{-1} K \beta \cap O = \beta^{-1} S \beta\}$$

So we have $K^\times \backslash E \cong \operatorname{Emb}(S, O)$.

Let $O^\times \le \Gamma \le N_{B^\times}(O)$, we can consider Γ acts on the right on $\mathrm{Emb}(S, O)$ by conjugation. For $\gamma \in \Gamma$ and $\phi \in \mathrm{Emb}(S, O)$

$$(\phi\gamma)(\alpha) := \gamma^{-1}\phi(\alpha)\gamma$$

Similarly, let

$$\mathrm{Emb}(S, O; \Gamma) := \{\Gamma\text{-conjugacy classes of optimal embeddings } S \hookrightarrow O\}$$

and

$$m(S, O; \Gamma) := \#\mathrm{Emb}(S, O; \Gamma)$$

We have $K^\times \backslash E/\Gamma \cong \mathrm{Emb}(S, O; \Gamma)$.

In the following discussion, we always assume that there exists a canonical optimal embedding $S \hookrightarrow O$, that is, E is nonempty.

Let B be a central simple algebra of degree p over F and let O be a R-order. The *genus* $\mathrm{Gen}\, O$ of O is the set of R-orders in B locally isomorphic to O. The *type set* $\mathrm{Typ}\, O$ of O is the set of isomorphism classes of orders in the genus of O. If two orders O and O' are locally isomorphic, then there exists an element $\widehat{v} \in \widehat{B}^\times$ such that $\widehat{O'} = \widehat{v}^{-1}\widehat{O}\widehat{v}$, which defines the map

$$\begin{aligned} \mathrm{Gen}\, O &\longrightarrow \widehat{B}^\times \\ O' &\longrightarrow \widehat{v} \end{aligned}$$

This map defines a bijection $\mathrm{Gen}\, O \cong \widehat{B}^\times/N_{\widehat{B}^\times}(\widehat{O})$, and there is also another bijection $\mathrm{Typ}\, O \cong B^\times \backslash \widehat{B}^\times/N_{\widehat{B}^\times}(\widehat{O})$. Next, we prove that the reduced norm of this central simple algebra B is a bijection on $\mathrm{Typ}\, O$.

Theorem 2.1 ([6, Theorem 3.1]). *The reduced norm on B induces a bijection:*

$$Nr : \mathrm{Typ}\, O \cong B^\times \backslash \widehat{B}^\times/N_{\widehat{B}^\times}(\widehat{O}) \longrightarrow F^\times \backslash \widehat{F}^\times/Nr(N_{\widehat{B}^\times}(\widehat{O}))$$

Proof. First, we prove that this reduced norm is surjective. Let $(f_\mathfrak{p})_\mathfrak{p} \in \widehat{F}^\times$, if $B_\mathfrak{p} = M_p(F_\mathfrak{p})$, $O_\mathfrak{p} = \alpha_\mathfrak{p}^{-1} M_p(R_\mathfrak{p})\alpha_\mathfrak{p}$ is an maximal order, then $b_\mathfrak{p} = \alpha_\mathfrak{p}^{-1}\mathrm{diag}(f_\mathfrak{p}, 1, \cdots, 1)\alpha_\mathfrak{p}$, if $B_\mathfrak{p}$ is an division ring, find $b_\mathfrak{p} \in Nr^{-1}(f_\mathfrak{p})$. Only finite $\mathfrak{p}$ such that $b_\mathfrak{p} \notin O_\mathfrak{p}$, so $(b_\mathfrak{p})_\mathfrak{p} \in \widehat{B}^\times$ and $Nr((b_\mathfrak{p})_\mathfrak{p}) = (f_\mathfrak{p})_\mathfrak{p}$.

To verify Nr is injective we need prove $Nr^{-1}(F^\times Nr(N_{\widehat{B}^\times}(\widehat{O}))) = B^\times \widehat{B}^1 N_{\widehat{B}^\times}(\widehat{O})$, where $\widehat{B}^1$ is the kernel of the norm map $Nr : \widehat{B}^\times \to \widehat{F}^\times$. Obviously, $Nr(B^\times \widehat{B}^1 N_{\widehat{B}^\times}(\widehat{O})) \subseteq F^\times Nr(N_{\widehat{B}^\times}(\widehat{O}))$. Let $Nr(B^\times \widehat{\beta} N_{\widehat{B}^\times}(\widehat{O})) \in F^\times Nr(N_{\widehat{B}^\times}(\widehat{O}))$, then $Nr(\widehat{\beta}) \in F^\times Nr(N_{\widehat{B}^\times}(\widehat{O}))$, let $\widehat{\beta} = f \cdot nr(\widehat{\gamma}), f \in F^\times, \widehat{\gamma} \in N_{\widehat{B}^\times}(\widehat{O})$. Recall p is odd, so no infinite prime of K ramifies in B, use Hasse-Schilling-Maass Theorem, we can find $nr(b) = f, b \in B^\times$. Finally, $B^\times \widehat{\beta} N_{\widehat{B}^\times}(\widehat{O}) = B^\times b^{-1}\widehat{\beta}\widehat{\gamma}^{-1} N_{\widehat{B}^\times}(\widehat{O}) \in B^\times \widehat{B}^1 N_{\widehat{B}^\times}(\widehat{O})$.

Let $\widehat{\alpha}, \widehat{\beta} \in \widehat{B}^\times$ and $Nr(B^\times \widehat{\alpha} N_{\widehat{B}^\times}(\widehat{O})) = Nr(B^\times \widehat{\beta} N_{\widehat{B}^\times}(\widehat{O}))$, $\widehat{F}^\times$ is abelian, so $Nr(\widehat{\alpha}\widehat{\beta}^{-1}) \in F^\times Nr(N_{\widehat{B}^\times}(\widehat{O}))$. From the previous argument, it follows that $\widehat{\alpha}\widehat{\beta}^{-1} \in B^\times \widehat{B}^1 N_{\widehat{B}^\times}(\widehat{O})$. $B^\times \widehat{B}^1$ is the kernel $\widehat{B}^\times \to \widehat{F}^\times/F^\times$, therefore, $B^\times \widehat{B}^1$ is a normal subgroup of $\widehat{B}^\times$, and we have $\widehat{\alpha} \in \widehat{\beta} B^\times \widehat{B}^1 N_{\widehat{B}^\times}(\widehat{O}) = B^\times \widehat{B}^1 \widehat{\beta} N_{\widehat{B}^\times}(\widehat{O})$. By the strong approximation property for central simple algebras, we obtain $\widehat{B}^1 \subseteq B^\times\widehat{\gamma}N_{\widehat{B}^\times}(\widehat{O})\widehat{\gamma}^{-1}$ for all $\widehat{\gamma} \in \widehat{B}^\times$. Let $\widehat{\gamma} = \widehat{\beta}$, then $\widehat{\alpha} \in B^\times\widehat{B}^1\widehat{\beta}N_{\widehat{B}^\times}(\widehat{O}) \subseteq \widehat{\alpha} \in B^\times\widehat{\beta}N_{\widehat{B}^\times}(\widehat{O})$. By the symmetry between $\widehat{\alpha}$ and $\widehat{\beta}$, the proof of injectivity is complete. $\square$

It is well known that the type number of an order is finite, and $\widehat{F}^\times$ is an abelian group, we can further describe the structure of $\mathrm{Typ}\, O$.

Theorem 2.2. *Let $GN(O) = F^\times Nr(N_{\widehat{B}^\times}(\widehat{O}))$. Then $\mathrm{Typ}\, O \cong \widehat{F}^\times/GN(O)$ and $\mathrm{Typ}\, O$ is an elementary abelian group of exponent p.*

Proof. For all local order $O_\mathfrak{p}$, $F_\mathfrak{p}^\times O_\mathfrak{p}^\times \subseteq N_{B_\mathfrak{p}^\times}(O_\mathfrak{p})$, so $F_\mathfrak{p}^{\times p} \subseteq Nr(N_{B_\mathfrak{p}^\times}(O_\mathfrak{p}))$ and

$$\widehat{F}^\times/GN(O) \leq \widehat{F}^\times/(\widehat{F}^\times)^p$$

Moreover, since we know that the class set of an order is finite [10, Theorem 26.4], it follows that the type number is finite. This completes the proof. $\square$

Corollary 2.3. *The subgroup $GN(O) \leq \widehat{F}^\times$ is a finite-index open subgroup containing $F_{>0}^\times$, the group of totally positive elements of $F^\times$.*

We abbreviate conjugation $K^\beta = \beta^{-1}K\beta$ for conciseness. Let

$$\widehat{E} := \{\widehat{\beta} \in \widehat{B}^\times \mid \widehat{K}^{\widehat{\beta}} \cap \widehat{O} = \widehat{S}^{\widehat{\beta}}\}$$

The inclusion $B^\times \hookrightarrow \widehat{B}^\times$ gives an inclusion $E/\Gamma \hookrightarrow \widehat{E}/\widehat{\Gamma}$. We can precisely describe the image of this map by using Theorem 2.1.

Lemma 2.4 ([11, Lemma 31.3.5])**.** *We have*

$$E/\Gamma = \{\widehat{\beta}\widehat{\Gamma} \in \widehat{E}/\widehat{\Gamma} \mid Nr(\widehat{\beta}) \in G(\Gamma)\} \subseteq \widehat{E}/\widehat{\Gamma}$$

where $G(\Gamma) := F^\times Nr(\widehat{\Gamma})$. That is to say, if $\widehat{\beta} \in \widehat{E}$, then there exists $\beta \in E$ such that $\beta\widehat{\Gamma} = \widehat{\beta}\widehat{\Gamma}$ if and only if $Nr(\widehat{\beta}) \in G(\Gamma)$.

Proof. By Theorem 2.1, we know

$$B^\times\backslash\widehat{B}^\times/\widehat{\Gamma} \longrightarrow F^\times\backslash\widehat{F}^\times/Nr(\widehat{\Gamma})$$

is bijection. So forall $\widehat{\beta}\widehat{\Gamma}$ we can find $\beta \in B^\times$ (without the condition that $\beta \in E$) such that $\beta\widehat{\Gamma} = \widehat{\beta}\widehat{\Gamma}$. Let $\beta = \widehat{\beta}\widehat{\gamma} \in \widehat{\beta}\widehat{\Gamma}$, because $\widehat{\Gamma} \subseteq N_{\widehat{B}^\times}(\widehat{O})$, we have

$$\widehat{K}^\beta \cap \widehat{O} = \widehat{K}^{\widehat{\beta}\widehat{\gamma}} \cap \widehat{O}^{\widehat{\gamma}} = \widehat{S}^{\widehat{\beta}\widehat{\gamma}} = \widehat{S}^\beta$$

and intersecting with B we find $\beta \in E$. $\square$

To apply the Artin map in class field theory, we need to establish the following lemma.

Lemma 2.5. *Let K/F be a Galois extension of number fields, and let $\underline{F}^\times$ and $\underline{K}^\times$ denote the idele groups of F and K, respectively, then*

$$\widehat{F}^\times / F_K^\times Nm(\widehat{K}^\times) \longrightarrow \underline{F}^\times / F^\times Nm(\underline{K}^\times)$$

is a bijection, where $F_K^\times := \{f \in F^\times \mid f_v > 0$ for all real place v ramified in $K\}$.

Proof. First, we claim

$$\widehat{F}^\times / F_{>0}^\times \longrightarrow \underline{F}^\times / F^\times F_{\infty,>0}^\times$$

is a bijection, where

$$F_{\infty,>0}^\times := \{(f_v)_v \in \underline{F}^\times \mid f_v > 0 \text{ for all real place } v, f_v = 1 \text{ for all finite place } v\}$$

and

$$\begin{array}{c} \widehat{F}^\times \longrightarrow \underline{F}^\times \\ \prod_{\mathfrak{p}}{}' F_{\mathfrak{p}} \longrightarrow \prod_{\mathfrak{p}}{}' F_{\mathfrak{p}} \times \prod_{v \text{ infinite}} 1 \end{array}$$

We can see that although $\widehat{F}^\times \to \underline{F}^\times$ is not surjective, but the composed map $\widehat{F}^\times \to \underline{F}^\times / F^\times F_{\infty,>0}^\times$ is surjective. For each $(f_v)_v \in \underline{F}^\times$, we first determine the signs at the real places. Then, by the weak approximation theorem, we choose $g \in F^\times$ so that $f_v g_v > 0$ satisfies the required sign conditions at the real places. Next, we use $(h_v)_v \in F_{\infty,>0}^\times$ such that $h_v = f_v/g_v$ to adjust the corresponding infinite places. Finally, we find $(\alpha_{\mathfrak{p}})_{\mathfrak{p}} \in \widehat{F}^\times$ such that $\alpha_{\mathfrak{p}} = f_{\mathfrak{p}}/g_{\mathfrak{p}}$. This proves that the map is surjective. It is clear that the kernel of this map is $F_{>0}^\times$, which completes the proof of our claim.

We have $F_{\infty,>0}^\times \leq Nm(\underline{K}^\times)$, so the map

$$\widehat{F}^\times \to \underline{F}^\times / F^\times Nm(\underline{K}^\times)$$

is also surjective, and by observing that the kernel of this map is $F_K^\times Nm(\widehat{K}^\times)$, the lemma is proved. □

With this lemma established, we can extend the conclusions of global class field theory:

Corollary 2.6. *There is a bijection:*

$$\begin{array}{c} \{K/F \text{ finite abelian}\} \leftrightarrow \{H \leq \widehat{F}^\times \text{ is an open finite-index subgroup containing } F_{>0}^\times\} \\ K \to F_K^\times Nm(\widehat{K}^\times) \end{array}$$

This shows that $GN(O)$ is an open subgroup of finite index in $\widehat{F}^\times$, corresponds to a class field $H_{GN(O)}$. Finally, we present the definition of being optimally selective as the conclusion of this chapter.

Definition 2.2. If exist $O' \in \operatorname{Gen} O$ such that

$$\operatorname{Emb}(S, O') = \emptyset$$

then we say that $\operatorname{Gen} O$ is optimally selective for S.

3 Selectivity sandwich

We can refine Voight's *selectivity sandwich*[11]

$$F_K^\times Nr(\widehat{K}^\times) \leq F^\times Nr(\widehat{K}^\times)Nr(N_{\widehat{B}^\times}(\widehat{O})) \leq F^\times Nr(\widehat{E}) \leq \widehat{F}^\times \tag{*}$$

First, if K/F is galois, according to class field theory, we can deduce that

$$[\widehat{F}^\times : F_K^\times Nr(\widehat{K}^\times)] = |\mathrm{Gal}(K/F)| = p$$

This tells us that among the three inequalities in $(*)$, only one can be strict, while the remaining two must be equalities. Next, we consider the meanings represented by each of these three inequalities. The following two lemmas are merely straightforward higher-degree generalizations of those in [11, 31.4].

Lemma 3.1. *If K/F is galois, $F_K^\times Nr(\widehat{K}^\times) = F^\times Nr(\widehat{K}^\times)Nr(N_{\widehat{B}^\times}(\widehat{O}))$ if ane only if $K \subseteq H_{GN(O)}$.*

Proof. We know that the class field corresponding to $F_K^\times Nr(\widehat{K}^\times)$ is K, and the class field corresponding to $F^\times Nr(\widehat{K}^\times)Nr(N_{\widehat{B}^\times}(\widehat{O}))$ is $K \cap H_{GN(O)}$. This completes the proof. □

Lemma 3.2. *Let $O' \in \mathrm{Gen}\, O$, $[O']$ denote the type of O', $[O']$ be represented by the class $B^\times \widehat{v} N_{\widehat{B}^\times}(\widehat{O})$. Then $\mathrm{Emb}(S, O') \neq \emptyset$ if and only if $Nr(\widehat{v}) \in F^\times Nr(\widehat{E})$.*

Proof. Let

$$E' = \{\beta' \in B^\times | K^{\beta'} \cap O' = S^{\beta'}\}, \widehat{E}' = \{\widehat{\beta'} \in \widehat{B}^\times | \widehat{K}^{\widehat{\beta'}} \cap \widehat{O}' = \widehat{S}^{\widehat{\beta'}}\}$$

then $\mathrm{Emb}(S, O') = K^\times \backslash E', \mathrm{Emb}(\widehat{S}, \widehat{O'}) = \widehat{K}^\times \backslash \widehat{E}'$.

Suppose $\mathrm{Emb}(S, O') \neq \emptyset$, there exist $\beta' \in E$ and

$$\widehat{K}^{\beta'} \cap \widehat{O}' = \widehat{K}^{\beta'} \cap \widehat{O}^{\widehat{v}^{-1}} = \widehat{S}^{\beta'}$$

so $\widehat{\beta} = \beta' \widehat{v} \in \widehat{E}$, and $Nr(\widehat{v}) = Nr(\beta'^{-1})Nr(\widehat{\beta}) \in F^\times Nr(\widehat{E})$.

Conversely, suppose $Nr(\widehat{v}) \in F^\times Nr(\widehat{E})$, then exist $a \in F^\times$ and $\widehat{\beta} \in \widehat{E}$ such that $Nr(\widehat{v}) = aNr(\widehat{\beta})$. Let $\widehat{\beta'} = \widehat{\beta}\widehat{v}^{-1}$, we have

$$\widehat{K}^{\widehat{\beta'}} \cap \widehat{O}' = \widehat{K}^{\widehat{\beta}\widehat{v}^{-1}} \cap \widehat{O}^{\widehat{v}^{-1}} = \widehat{S}^{\widehat{\beta}\widehat{v}^{-1}} = \widehat{S}^{\widehat{\beta'}}$$

so $\widehat{\beta'} \in \widehat{E}'$ and $Nr(\widehat{\beta'}) = Nr(\widehat{\beta}\widehat{v}^{-1}) = a^{-1} \in F^\times$. By Lemma 2.4, it follows immediately that E' is nonempty.

□

Corollary 3.3. *If $K \not\subseteq H_{GN(O)}$, S can embedding all order $O' \in \mathrm{Gen}\, O$.*

Proof. If K/F is not galois, then

$$[\widehat{F}^{\times} : F_K^{\times} Nr(\widehat{K}^{\times})] = |\mathrm{Gal}((K \cap F^{ab})/F)| = 1$$

We can directly obtain $F^{\times} Nr(\widehat{E}) = \widehat{F}^{\times}$.

If K/F is galois and $K \not\subseteq H_{GN(O)}$. According to Lemma 3.1, when $K \not\subseteq H_{GN(O)}$, we obtain

$$F_K^{\times} Nr(\widehat{K}^{\times}) < F^{\times} Nr(\widehat{K}^{\times}) Nr(N_{\widehat{B}^{\times}}(\widehat{O}))$$

which in turn implies $F^{\times} Nr(\widehat{E}) = \widehat{F}^{\times}$. Combining this with Lemma 3.2, we conclude the result. $\square$

We are more interested in whether the converse holds; that is, if $K \subseteq H_{GN(O)}$, whether we can deduce that $\mathrm{Gen}\, O$ is optimally selective for S. To achieve this, we must show that equality holds in the middle inequality in $(*)$. For this purpose, we compute the local optimal embeddings.

$F_{\mathfrak{p}}$ is a local field and $R_{\mathfrak{p}}$ is a discrete valuation ring so the $R_{\mathfrak{p}}$-order $S_{\mathfrak{p}}$ is free. We can find an $R_{\mathfrak{p}}$-basis $\{e_1, e_2, \cdots, e_p\}$, without loss of generality let $e_1 = 1$.

Lemma 3.4. *Let $\varphi : S_{\mathfrak{p}} \hookrightarrow \mathrm{M}_n(R_{\mathfrak{p}})$, $\varphi(e_i) = A_i \in \mathrm{M}_n(R_{\mathfrak{p}})$. Use $\widetilde{A_i} \in \mathrm{M}_n(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$ to denote the representative of each element of A_i modulo the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. The following conditions are equivalent.*

(1) φ is optimal embedding.

(2) Forall $i \leq n$, $\sum_{i \neq j} R_{\mathfrak{p}} A_j + A_i \notin \mathrm{M}_n(\pi R_{\mathfrak{p}})$.

(2') Forall $i \leq n$, $\sum_{i \neq j} (R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \widetilde{A_j} + \widetilde{A_i} \neq 0$.

(3) Exist $s_i, t_i \in \{1, 2, \cdots, n\}$, and

$$X := X_{s_i,t_i} = \begin{pmatrix} (A_1)_{s_1t_1} & (A_2)_{s_1t_1} & \cdots & (A_n)_{s_1t_1} \\ (A_1)_{s_2t_2} & (A_2)_{s_2t_2} & \cdots & (A_n)_{s_2t_2} \\ \vdots & \vdots & \ddots & \vdots \\ (A_1)_{s_nt_n} & (A_2)_{s_nt_n} & \cdots & (A_n)_{s_nt_n} \end{pmatrix}$$

where $(A_i)_{s_jt_j}$ denotes the element in the s_j-th row and t_j-th column of the matrix A_i, such that $\det X \in R_{\mathfrak{p}}^{\times}$.

(3') Exist $s_i, t_i \in \{1, 2, \cdots, n\}$, and

$$\widetilde{X} := \widetilde{X_{s_i,t_i}} = \begin{pmatrix} (\widetilde{A_1})_{s_1t_1} & (\widetilde{A_2})_{s_1t_1} & \cdots & (\widetilde{A_n})_{s_1t_1} \\ (\widetilde{A_1})_{s_2t_2} & (\widetilde{A_2})_{s_2t_2} & \cdots & (\widetilde{A_n})_{s_2t_2} \\ \vdots & \vdots & \ddots & \vdots \\ (\widetilde{A_1})_{s_nt_n} & (\widetilde{A_2})_{s_nt_n} & \cdots & (\widetilde{A_n})_{s_nt_n} \end{pmatrix}$$

where $(\widetilde{A_i})_{s_jt_j}$ denotes the element in the s_j-th row and t_j-th column of the matrix $\widetilde{A_i}$, such that $\det X \neq 0$.

Proof. (2)↔(2'), (3)↔(3') is trivial.

(1)→(2), if exist $k_j \in R_{\mathfrak{p}}$ such that $\sum k_j A_j + A_i \in \mathrm{M}_n(\pi R_{\mathfrak{p}})$. Then $\sum \pi^{-1} k_j e_j + \pi^{-1} e_i \in K_{\mathfrak{p}} - S_{\mathfrak{p}}$ and $\varphi(\sum \pi^{-1} k_j e_j + \pi^{-1} e_i) \in \mathrm{M}_n(R_{\mathfrak{p}})$, contradiction.

(2)→(1), if $\varphi : S_{\mathfrak{p}} \hookrightarrow \mathrm{M}_n(R_{\mathfrak{p}})$ is not optimal embedding, then exist k_i such that $\varphi(\sum k_i e_i) \in \mathrm{M}_n(R_{\mathfrak{p}})$ and $\nu(k_j) < 0$. Assume $\forall i, \nu(k_j) \leq \nu(k_i)$, then $\varphi(\sum k_j^{-1} k_i e_i) \in \mathrm{M}_n(\pi R_{\mathfrak{p}})$, contradiction.

(2')↔(3'), condition (2') means that $\widetilde{A_i}$ is linearly independent over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, while condition (3') means that one can always select n components among these vectors such that the resulting matrix is of full rank. It is clear that these two conditions are equivalent. □

Example We can directly enumerate the equivalent conditions in the case of $n = 2$. Let $\varphi(e_1) = E$ and

$$\varphi(e_2) = A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It can be seen that swapping the positions of s_i, t_i and s_j, t_j corresponds to swapping two rows in X. This only changes the determinant by a sign and does not affect whether it lies in the unit group. Therefore, in subsequent arrangements, we always require that s_i, t_i be ordered according to lexicographic order. We can compute

$$\det X_{11,12} = \begin{vmatrix} 1 & a \\ 0 & b \end{vmatrix} = b, \det X_{11,21} = \begin{vmatrix} 1 & a \\ 0 & c \end{vmatrix} = c, \det X_{11,22} = \begin{vmatrix} 1 & a \\ 1 & d \end{vmatrix} = d - a$$

$$\det X_{12,21} = \begin{vmatrix} 0 & b \\ 0 & c \end{vmatrix} = 0, \det X_{12,22} = \begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix} = -b, \det X_{21,22} = \begin{vmatrix} 0 & c \\ 1 & d \end{vmatrix} = -c$$

Thus, we can conclude that φ is optimal if and only if at least one of b, c, or $d - a$ lies in $R_{\mathfrak{p}}^{\times}$.

Theorem 3.5. *The following statements hold.*

(1) If $K \subseteq H_{GN(O)}$, then $m(S_{\mathfrak{p}}, \mathrm{M}_p(R_{\mathfrak{p}}); \mathrm{GL}_p(R_{\mathfrak{p}})) = 1$;

(2) Let $B_{\mathfrak{p}}$ be a division algebra and $O_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ its valuation ring. If $K_{\mathfrak{p}}$ is a field and $S_{\mathfrak{p}}$ is integrally closed, then $m(S_{\mathfrak{p}}, O_{\mathfrak{p}}; N_{B_{\mathfrak{p}}^{\times}}(O_{\mathfrak{p}})) = 1$. If $K_{\mathfrak{p}}$ is not a field or $S_{\mathfrak{p}}$ is not integrally closed ,then $\mathrm{Emb}(S_{\mathfrak{p}}, O_{\mathfrak{p}}) = \emptyset$.

Proof. First, prove part(1). We begin by establishing the existence through the construction of an optimal embedding. Let $\phi : S_{\mathfrak{p}} \hookrightarrow \mathrm{End}_{R_{\mathfrak{p}}}(S_{\mathfrak{p}}) = \mathrm{M}_p(R_{\mathfrak{p}})$ given by the regular representation of $S_{\mathfrak{p}}$ on itself, in the basis$\{e_1, e_2, \cdots, e_p\}$ we have:

$$e_1 \longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, e_i \longrightarrow \begin{pmatrix} 0 & * & \cdots & * \\ \vdots & * & \ddots & * \\ 1 & * & \cdots & * \\ \vdots & * & \ddots & * \\ 0 & * & \cdots & * \end{pmatrix}$$

In the corresponding matrix of e_i wthe i-th entry of the first column is 1, while all other entries in the first column are 0. This embedding is optimal, because if $\alpha = \sum k_i e_i, k_i \in F_{\mathfrak{p}}$, then

$$\phi(\alpha) = \begin{pmatrix} k_1 & * & \cdots & * \\ \vdots & * & \ddots & * \\ k_i & * & \cdots & * \\ \vdots & * & \ddots & * \\ k_n & * & \cdots & * \end{pmatrix}$$

$\phi(\alpha) \in \mathrm{M}_p(R_\mathfrak{p})$ if and only if $k_i \in R_\mathfrak{p}$. In other words, $\phi(K_\mathfrak{p}) \cap \mathrm{M}_p(R_\mathfrak{p}) = \phi(S_\mathfrak{p})$.

We now proceed to prove that the optimal embedding is unique up to conjugation by $\mathrm{GL}_p(R_\mathfrak{p})$. Assume $\varphi : S_\mathfrak{p} \hookrightarrow \mathrm{M}_p(R_\mathfrak{p}), \varphi(e_i) = A_i \in \mathrm{M}_p(R_\mathfrak{p})$ is optimal embedding. We can find $\widetilde{U} \in \mathrm{GL}_p(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ such that $\widetilde{U}^{-1}\widetilde{A_p}\widetilde{U}$ is in standard form over $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Choose a lift U of $\widetilde{U}$. Clearly, U belongs to $\mathrm{GL}_p(R_\mathfrak{p})$. By conjugating A_i with U, we obtain a new A_i'. In the subsequent proofs, we will use A_i' in place of A_i to ensure that $\widetilde{A_p}$ is in standard form.

Because $K \subseteq H_{GN(O)}$ and $H_{GN(O)}/F$ is unramified field extension, $K_\mathfrak{p} \cong F_\mathfrak{p}^p$ or $K_\mathfrak{p}/F_\mathfrak{p}$ is inert local field extension.

If $K_\mathfrak{p} \cong F_\mathfrak{p}^p$, then the minimal polynomial of each e_i is a product of distinct linear factors, which means that all $\widetilde{A_i}$ can be simultaneously diagonalized. Without loss of generality, we assume that all $\widetilde{A_i}$ are diagonal matrices. By the Lemma 3.4, we know that if φ is optimal, then there must exist some $\widetilde{X_{s_i,t_i}}$ such that $\det \widetilde{X_{s_i,t_i}} \neq 0$. However, since all $\widetilde{A_i}$ are diagonal, this implies that there is only one possible $\widetilde{X_{s_i,t_i}}$ that satisfies this condition, namely $\widetilde{X_{11,22,\cdots,pp}}$.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p)^T, \alpha_i \in R_\mathfrak{p}, V = (A_1\alpha, A_2\alpha, \cdots, A_p\alpha)$. We can compute

$$\det V = \sum_{s_i=i,t_i} (\det X_{s_i,t_i} \prod_i \alpha_{t_i})$$

Let $\alpha_i = 1, \det V = \sum_{s_i=i,t_i} \det X_{s_i,t_i}$, only one $\det X_{11,22,\cdots,pp} \in R_\mathfrak{p}^\times$, so $\det V \in R_\mathfrak{p}^\times$ and $V \in \mathrm{GL}_p(R_\mathfrak{p})$. In this case, $V^{-1}A_iV$ is the regular representation of e_i with respect to the basis $\{e_1, e_2, \cdots, e_p\}$.

If $K_\mathfrak{p}/F_\mathfrak{p}$ is inert local field extension, then the minimal polynomial of e_p is an irreducible polynomial. By the Lemma 3.4, we also know $R_\mathfrak{p} + e_p \notin \mathfrak{p}_K$, which implies that the characteristic polynomial of $\widetilde{A_p}$ is still an irreducible polynomial. It is well known that if the characteristic polynomial of a matrix $\widetilde{A_p}$ is irreducible, then any matrix that commutes with $\widetilde{A_p}$ must be a polynomial in $\widetilde{A_p}$. This implies that $\widetilde{A_i}$ is a polynomial in $\widetilde{A_p}$, so assume $\widetilde{A_i} = \sum k_{i,j}\widetilde{A_p}^j$.

Because $\widetilde{A_p}$ is in standard form. Then

$$\widetilde{A_i} = \begin{pmatrix} k_{i,0} & * & \cdots & * \\ \vdots & * & \ddots & * \\ k_{i,j-1} & * & \cdots & * \\ \vdots & * & \ddots & * \\ k_{i,p-1} & * & \cdots & * \end{pmatrix}$$

We claim $\det \widetilde{X_{11,21,\cdots,p1}} \neq 0$, otherwise $\widetilde{A_i}$ would be linearly dependent, in which case any $\det \widetilde{X_{s_i,t_i}}$ would be zero, contradicting the assumption in our Lemma 3.4(3').

Similarly, let $\alpha_1 = 1, \alpha_j = 0$ then $\det V = \det X_{11,21,\cdots,p1} \in R_{\mathfrak{p}}^{\times}$, in this case, $V^{-1}A_iV$ is the regular representation of e_i with respect to the basis $\{e_1, e_2, \cdots, e_p\}$.

Next, prove part (2). We know if $B_{\mathfrak{p}}$ is a division $F_{\mathfrak{p}}$-algebra and $O_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ its valuation ring, then $O_{\mathfrak{p}}$ is the unique maximal $R_{\mathfrak{p}}$-order in $B_{\mathfrak{p}}$ and the integral closure of $R_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$. If $K_{\mathfrak{p}}$ is not a field, then there is no embedding $K_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$. If $S_{\mathfrak{p}}$ is not integrally closed, then there is no optimal embedding. So $N_{B_{\mathfrak{p}}^{\times}}(O_{\mathfrak{p}}) = B_{\mathfrak{p}}^{\times}$, the proof of part (2) is complete. □

Corollary 3.6. *Let B be a central simple algebra of degree p over a numble field F and K/F be a field extension of degree p inside B. Suppose $K \subseteq H_{GN(O)}$, $S \subset K$ is an R-order, $O \subset B$ is an maximal R-order, S can be embedding $O' \in \operatorname{Gen} O$. Then $m(\widehat{S}, \widehat{O}; N_{\widehat{B}^{\times}}(\widehat{O})) = 1$, moreover $Nr(\widehat{K}^{\times})Nr(N_{\widehat{B}^{\times}}(\widehat{O})) = Nr(\widehat{E})$.*

Proof.

$$m(\widehat{S}, \widehat{O}; N_{\widehat{B}^{\times}}(\widehat{O})) = \prod_{\mathfrak{p}} m(S_{\mathfrak{p}}, O_{\mathfrak{p}}; N_{B_{\mathfrak{p}}^{\times}}(O_{\mathfrak{p}}))$$

According to the structure theorem for central simple algebras, we conclude that either $B_{\mathfrak{p}} \cong \mathrm{M}_p(F_{\mathfrak{p}})$ or $B_{\mathfrak{p}}$ is a division algebra. Then, applying the previous lemma, we compute that $m(S_{\mathfrak{p}}, O_{\mathfrak{p}}; N_{B_{\mathfrak{p}}^{\times}}(O_{\mathfrak{p}})) = 1$. If $B_{\mathfrak{p}}$ is a division algebra, then

$$E_{\mathfrak{p}} = B_{\mathfrak{p}}^{\times} = K_{\mathfrak{p}}^{\times} N_{B_{\mathfrak{p}}^{\times}}(O_{\mathfrak{p}})$$

If $B_{\mathfrak{p}} \cong \mathrm{M}_p(F_{\mathfrak{p}})$, since all maximal orders in $\mathrm{M}_p(F_{\mathfrak{p}})$ are conjugate to each other, the reduced norm $Nr(E_{\mathfrak{p}})$ remains identical regardless of which maximal order's corresponding $E_{\mathfrak{p}}$ is chosen. Therefore, without loss of generality, we may assume that $O_{\mathfrak{p}} = \mathrm{M}_p(R_{\mathfrak{p}})$, so

$$Nr(E_{\mathfrak{p}}) = Nr(K_{\mathfrak{p}}^{\times})Nr(\mathrm{GL}_p(R_{\mathfrak{p}})) \leq Nr(K_{\mathfrak{p}}^{\times})Nr(N_{B_{\mathfrak{p}}^{\times}}(O_{\mathfrak{p}}))$$

And $E_{\mathfrak{p}} \geq K_{\mathfrak{p}}^{\times} N_{B_{\mathfrak{p}}^{\times}}(O_{\mathfrak{p}})$, we conclude $Nr(E_{\mathfrak{p}}) = Nr(K_{\mathfrak{p}}^{\times})Nr(N_{B_{\mathfrak{p}}^{\times}}(O_{\mathfrak{p}}))$, therefore

$$Nr(\widehat{K}^{\times})Nr(N_{\widehat{B}^{\times}}(\widehat{O})) = Nr(\widehat{E})$$

□

This finally shows that if O is a maximal order and $K \subseteq H_{GN(O)}$ holds, then the middle inequality in $(*)$ is always an equality.

Theorem 3.7. *Let B be a central simple algebra of degree p over a numble field F. R be the ring of integers of F and K/F be a field extension of degree p inside B. Suppose $S \subset K$ is an R-order, S can be optimal embedding an maximal R-order $O \subset B$. Then the following statements hold.*

(1) $\operatorname{Gen} O$ is optimally selective for S if and only if $K \subseteq H_{GN(O)}$.

(2) If $\operatorname{Gen} O$ is optimally selective for S, then S can embeddding $O' \in \operatorname{Gen} O$ for exactly $\frac{1}{p}$ of the types $[O'] \in \operatorname{Typ} O$.

Proof. Corollary 3.3 tells us that when $K \nsubseteq H_{GN(O)}$ holds, S can embedding all order $O' \in \operatorname{Gen} O$. While when $K \subseteq H_{GN(O)}$ holds, Corollary 3.6 shows that the middle inequality in $(*)$ is always an equality. This forces the final inequality to be strict. Consequently, in this case, S can only select those order embeddings lying in the subgroup $F^\times Nr(\widehat{E}) \leq \widehat{F}^\times$, which is precisely a subgroup of index p. This establishes both the selectivity and the number of possible selections in this situation.

□

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