CHARACTERIZING CONVEX TRACE RANGES IN FINITE ATOMIC VON NEUMANN ALGEBRAS

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Abstract. The paper is devoted to characterizing convex trace ranges in finite atomic von Neumann algebras. The main result provides us with the necessary and sufficient condition for the range of a faithful normal trace on a finite atomic von Neumann algebra to be convex. In order to prove this result we will prove the following result, which has independent interest. Let $\mathbf{a} = (a_1, \ldots, a_n, \ldots)$ be a non-increasing positive sequence such that $\sum\limits_{n=1}^{\infty} a_n = 1$. Then each real number $0 \leq r \leq 1$ can be represented in the form $r = \sum\limits_{n=1}^{\infty} \varepsilon_n a_n,\ \ \varepsilon_n \in \{0, 1\}, n \geq 1$, if and only if the sequence $\mathbf{a}$ satisfies $a_n \leq 1 - \sum\limits_{k=1}^{n} a_k$ for all $n \geq 1$. A set K of all sequences that satisfy the last property can be represented as a convex weak-compact subset of $\ell_1 = c_0^*$. We will describe the set of all extreme points of K.

1. Introduction

Recall that for a Banach space $\mathcal{X}$ and a measure space (Ω, Σ), where Σ is a σ-algebra of subsets of Ω, the classical Lyapunov convexity theorem [5, Theorem I] states that if $\mathcal{X}$ is finite-dimensional and if μ is a non-atomic σ-additive $\mathcal{X}$-valued measure defined on Σ then its range $\mu(\Sigma)$ is compact and convex. Along with the Brouwer–Kakutani–Fan–Glicksberg fixed point theorems, and several versions of the Hahn–Banach theorem, Lyapunov's theorem [5,6] on the range of a nonatomic finite-dimensional vector measure plays a crucial role in modern mathematical economics, especially in general equilibrium and game theory (see [1–4, 7]).

Let $\mathcal{M}$ be an atomic von Neumann algebra with a faithful normal tracial state τ. We can find a sequence of mutually orthogonal minimal projections $e_1, \ldots, e_n, \ldots$ in $\mathcal{M}$ such that $\sum\limits_{n=1}^{\infty} e_n = \mathbf{1}$.

Let

$$t_n := \tau(e_n),\ n \geq 1. \tag{1.1}$$

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If necessary, after enumerating, we may assume that $t_1 \geq t_2 \geq \dots t_n \geq \dots$.

Let $\mathbf{a} = (a_1, \dots, a_n, \dots)$ be a non-increasing sequence of positive real numbers such that

$$a_n \leq \sum_{k=n+1}^{\infty} a_k \tag{1.2}$$

for all $n \geq 1$. If the series $\sum\limits_{n=1}^{\infty} a_n$ converges, then we can consider the sequence $\mathbf{a}$ as an element of ℓ_1, and set

$$\|\mathbf{a}\|_1 = \sum_{n=1}^{\infty} a_n.$$

In particular, if $\|\mathbf{a}\|_1 = 1$, then relations (1.2) rewrites as follows

$$a_n \leq 1 - \sum_{k=1}^{n} a_k \tag{1.3}$$

for all $n \geq 1$.

The next result provides us with the necessary and sufficient condition for the range of a faithful normal trace on a finite atomic von Neumann algebra to be convex.

Theorem 1.1. *Let $\mathcal{M}$ be an atomic von Neumann algebra with a faithful normal tracial state τ and let $\{t_n\}_{n\geq 1}$ be a sequence defined as in (1.1). Then the following assertions are equivalent.*

1. *$\tau(P(\mathcal{M}))$ is convex.*
2. *the sequence $\{t_n\}_{n\geq 1}$ satisfies (1.3).*

2. Proof of the main result

Let $\mathcal{M}$ be an atomic von Neumann algebra with a faithful normal tracial state τ. Then it has the form

$$\mathcal{M} \equiv \bigoplus_{k\in F} \mathbb{M}_k,$$

where $\mathbb{M}_k$ is the factor of type I_k and F is an infinite subset of the integers $\mathbb{N}$.

Let $\mathcal{A}$ be a maximal abelian von Neumann subalgebra (masa) in $\mathcal{M}$. By [8, Theorem 2.4.3 and Remark 2.4.4], for any masa $\mathcal{A}$ in $\mathcal{M}$ there is a unitary $w \in \mathcal{M}$ such that $\mathcal{A} = w\mathcal{B}w^*$. Since any projection e in $\mathcal{M}$ is contained in some masa, there exists a projection $q \in \mathcal{A}$ such that $e \sim q$, that is, $e = vqv^*$ for a unitary $v \in \mathcal{M}$. Thus,

$$\tau(e) = \tau(vqv^*) = \tau(q).$$

Hence,

$$\tau\left(P(\mathcal{A})\right) = \tau\left(P(\mathcal{M})\right). \tag{2.1}$$

Since $\mathcal{A}$ is an atomic masa, we can find a sequence of mutually orthogonal minimal projections $q_1, \ldots, q_n, \ldots$ in $\mathcal{A}$ such that $\sum_{n=1}^{\infty} q_n = \mathbf{1}$. Moreover, any projection $q \in \mathcal{A}$ can be represented in the form of

$$q = \sum_{n=1}^{\infty} \varepsilon_n q_n, \tag{2.2}$$

where $\varepsilon_n \in \{0, 1\}$ for all $n \geq 1$. Furthermore, instead the sequence $\{e_n\}_{n\geq 1}$ we can consider the sequence $\{q_n\}_{n\geq 1}$, that is, we can set

$$t_n := \tau(q_n),\ n \geq 1.$$

If necessary, after enumerating, we may assume that $t_1 \geq t_2 \geq \ldots t_n \geq \ldots$. Combing (2.1) and (2.2) we have that

$$\tau(P(\mathcal{M})) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n t_n : \varepsilon_n \{0, 1\}, n \geq 1 \right\}. \tag{2.3}$$

Taking into account equation (2.3), we see that Theorem 1.1 follows directly from the following result, which has independent interest.

Theorem 2.1. *Let $\mathbf{a} = (a_1, \ldots, a_n, \ldots)$ be a positive sequence such that $\|\mathbf{a}\|_1 = 1$. Then each real number $0 \leq r \leq 1$ can be represented in the form*

$$r = \sum_{n=1}^{\infty} \varepsilon_n a_n,\ \ \varepsilon_n \in \{0, 1\}, n \geq 1, \tag{2.4}$$

if and only if the sequence $\mathbf{a}$ satisfies (1.3).

Proof. 'if' part. Let $r \in [0, 1]$. Define sequences $\{\varepsilon_0, \ldots, \varepsilon_n, \ldots\} \subset \{0, 1\}$ and $\{r_0, \ldots, r_n, \ldots\} \subset [0, 1]$ by the following recurrence formula:

$$\begin{aligned} &\varepsilon_0 = 0,\ \ r_0 = 0, \\ &\varepsilon_n = \begin{cases} 0, & \text{if } r - r_{n-1} < a_n \\ 1, & \text{otherwise} \end{cases}, && n \geq 1, \\ &r_n = r_{n-1} + \varepsilon_n a_n, && n \geq 1. \end{aligned}$$

Let us show that

$$r = \sum_{n=1}^{\infty} \varepsilon_n a_n.$$

It suffices to show that

$$0 \leq r - r_n \leq 1 - \sum_{k=1}^{n} a_k,\ n \geq 1. \tag{2.5}$$

Let $n = 1$. If $a_1 > r - r_0 = r$, we have that $\varepsilon_1 = 0$. Hence, $r_1 = r_0 + \varepsilon_1 a_1 = 0$. Thus

$$0 \leq r - r_1 = r < a_1 = 1 - \sum_{k=2}^{\infty} a_k \overset{(1.3)}{\leq} 1 - a_1.$$

If $a_1 \leq r - r_0$, we have that $\varepsilon_1 = 1$ and $r_1 = r_0 + a_1 = a_1$. Further, we obtain that

$$0 \leq r - r_1 \quad = \quad r - a_1 \leq 1 - a_1,$$

because $0 \leq r \leq 1$.

Now, assume that (2.5) holds for $n - 1$, that is,

$$0 \leq r - r_{n-1} \leq 1 - \sum_{k=1}^{n-1} a_k. \tag{2.6}$$

If $a_n > r - r_{n-1}$, we have that $\varepsilon_n = 0$, and therefore $r_n = r_{n-1} + \varepsilon_n a_n = r_{n-1}$. Therefore

$$0 \leq r - r_n = r - r_{n-1} < a_n \overset{(1.3)}{\leq} 1 - \sum_{k=1}^{n} a_k.$$

If $a_n \leq r - r_{n-1}$, we have that $\varepsilon_n = 1$, and $r_n = r_{n-1} + a_n$. Hence

$$0 \quad \leq \quad r - r_n = r - (r_{n-1} + a_n) = r - r_{n-1} - a_n \overset{(2.6)}{\leq} 1 - \sum_{k=1}^{n-1} a_k - a_n = 1 - \sum_{k=1}^{n} a_k.$$

Since $\sum\limits_{k=1}^{\infty} a_k = 1$, we have $r_n \uparrow r$, and the proof of the 'if' is complete.

'only if' part. Assume by contrary, that is, there exists n such that

$$\sum_{k=n+1}^{\infty} a_k < a_n.$$

Let us take a number r such that

$$\sum_{k=n+1}^{\infty} a_k < r < a_n.$$

By the assumption we find expansion

$$r := \sum_{k=1}^{\infty} \varepsilon_k a_k.$$

If $\varepsilon_k = 1$ for some $k \in \{1, \ldots, n\}$, we have that

$$r \geq a_k \geq a_n > r.$$

If $\varepsilon_k = 0$ for all $k \in \{1, \ldots, n\}$, we have that

$$r = \sum_{k=n+1}^{\infty} \varepsilon_k a_k \leq \sum_{k=n+1}^{\infty} a_k < r.$$

So, in both cases we have contradiction. Hence, the sequence $\mathbf{a}$ satisfies (1.3). The proof is completed. $\square$

2.1. **Extreme boundary.**

Consider the set K of all sequences with the property (1.3) with the sum no larger than 1. Then K can be seen as a subset of $\ell_1 \equiv c_0^*$. It is clear that K is bounded, and closed with respect to weak topology on ℓ_1. So, it is a convex (c_0, ℓ_1)-compact set. By the Krein-Milman theorem K is a weak-closure of the convex hull of extreme boundary, that is, $K = \overline{\mathrm{co}(\mathrm{ext} K)}^w$.

In this subsection we describe the extreme boundary $\mathrm{ext}K$ of K.

Let $k_1, \dots, k_n, \dots$ be a sequence of integers such that $k_n \geq 2$ for all $n \geq 1$. Set

$$\mathbf{a} = \left(\underbrace{\frac{1}{k_1}, \dots, \frac{1}{k_1}}_{k_1 - 1}, \underbrace{\frac{1}{k_1 k_2}, \dots, \frac{1}{k_1 k_2}}_{k_2 - 1}, \dots, \underbrace{\frac{1}{k_1 \cdots k_n}, \dots, \frac{1}{k_1 \cdots k_n}}_{k_n - 1}, \dots \right). \tag{2.7}$$

Theorem 2.2. *Let* $\mathbf{a} \in K$. *Then* $\mathbf{a}$ *is an extreme point of* K *if and only if it is in the form* (2.7).

Let us first check that a sequence $\mathbf{a}$ defined as in (2.7) belong to K.

Proposition 2.3. *A sequence* $\mathbf{a}$ *defined as in* (2.7) *is in* K.

Proof. Let us first show that

$$\sum_{n=1}^{\infty} \frac{k_n - 1}{k_1 \cdots k_n} = 1.$$

We will check that

$$s_n = \sum_{i=1}^{n} \frac{k_i - 1}{k_1 \cdots k_i} = 1 - \frac{1}{k_1 \cdots k_n} \tag{2.8}$$

for all $n \geq 1$. Taking into account

$$\frac{k_i - 1}{k_1 \cdots k_i} = \frac{1}{k_1 \cdots k_{i-1}} - \frac{1}{k_1 \cdots k_i},\ i \geq 2,$$

we obtain that

$$s_n = \left(1 - \frac{1}{k_1}\right) + \left(\frac{1}{k_1} - \frac{1}{k_1 k_2}\right) + \ldots + \left(\frac{1}{k_1 \cdots k_{n-1}} - \frac{1}{k_1 \cdots k_n}\right) = 1 - \frac{1}{k_1 \cdots k_n},$$

which gives us (2.8). Equality (2.8) also shows that

$$\sum_{m \geq 2} \frac{k_m - 1}{k_1 \cdots k_m} = \frac{1}{k_1 \cdots k_{n-1}}$$

for all $n \geq 2$, which implies that $\mathbf{a}$ satisfies (1.3). The proof is completed. $\square$

Let E be a linear space and let Q be its convex subset. Recall that a convex subset F of Q is called a face of Q if for any $x \in F$ and $y, z \in Q$, $0 < \lambda < 1$ such that $x = \lambda y + (1-\lambda)z$, we have $y, z \in F$.

For a fixed integer $m \geq 2$ set

$$K_m = \left\{ \mathbf{a} = (a_n)_{n\geq 1} \in K : a_1 = \ldots = a_{m-1} = \frac{1}{m} \right\}.$$

Setting $k_n = m$ for all $n \geq 1$ in (2.7), we see that $\mathbf{a} \in K_m$. Furthermore, K_m is a non-empty convex compact subset in K.

Proposition 2.4. *K_m is a face of K.*

Proof. Let $\mathbf{b}, \mathbf{c} \in K$ and $0 < \lambda < 1$ be such that $\lambda \mathbf{b} + (1-\lambda)\mathbf{c} \in K_m$.

Let us first consider a case $m = 2$. By (1.3) we have that $b_1 \leq 1 - b_1$, that is, $b_1 \leq \frac{1}{2}$. Likewise, $c_1 \leq \frac{1}{2}$. Since $\lambda b_1 + (1-\lambda)c_1 = \frac{1}{2}$, it follows that $b_1 = c_1 = \frac{1}{2}$. This means that $\mathbf{b}, \mathbf{c} \in K$.

Now suppose that $m \geq 3$. We have that $\lambda b_i + (1-\lambda)c_i = \frac{1}{m}$ for all $i = 1, \ldots, m-1$. Note that $b_1 \geq \ldots \geq b_{m-1}$ and $c_1 \geq \ldots \geq c_{m-1}$. Assume that $b_i > b_{i+1}$ for some $i \in \{1, \ldots, m-1\}$. Then

$$\frac{1}{m} = \lambda b_i + (1-\lambda)c_i > \lambda b_{i+1} + (1-\lambda)c_{i+1} = \frac{1}{m},$$

which gives us a contradiction. Hence, $b_1 = \ldots = b_{m-1}$ and $c_1 = \ldots = c_{m-1}$. Using (1.3) we obtain that

$$b_{m-1} \leq 1 - \sum_{i=1}^{m-1} b_i = 1 - (m-1)b_{m-1},$$

that is, $b_1 = \ldots = b_{m-1} \leq \frac{1}{m}$. Likewise, $c_1 = \ldots = c_{m-1} \leq \frac{1}{m}$. Taking into account $\lambda b_1 + (1-\lambda)c_1 = \frac{1}{m}$, we obtain that

$$b_1 = c_1 = \ldots = b_{m-1} = c_{m-1} = \frac{1}{m}.$$

This means that $\mathbf{b}, \mathbf{c} \in K_m$, that is, K_m is a face. The proof is completed. $\square$

Let $m \geq 2$. Define a mapping $\Pi_m : K \to K_m$ as follows

$$\Pi_m(\mathbf{a}) = \left(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{m-1}, \frac{1}{m}a_1, \frac{1}{m}a_2, \ldots \right), \mathbf{a} \in K. \tag{2.9}$$

The next property directly follows from the definition. The mapping Π_m defined as in (2.9) is an affine isomorphism from K onto K_m. The inverse of the mapping defined as in (2.9) is in the form

$$\mathbf{a} = \left(\frac{1}{m}, \ldots, \frac{1}{m}, a_m, \ldots \right) \in K_m \mapsto \Pi_m^{-1}(\mathbf{a}) = (ma_m, ma_{m+1}, \ldots) \in K. \tag{2.10}$$

Proposition 2.5. *Let* $\mathbf{a} \in K$ *be such that* $a_1 = \ldots = a_{m-1} > a_m$*, where* $m \geq 2$*. Then the following assertions are equivalent.*

(a) $\mathbf{a} \in \mathrm{ext}K$;

(b) $a_1 = \ldots = a_{m-1} = \frac{1}{m}$ *and* $(ma_m, ma_{m+1}, \ldots) \in \mathrm{ext}K$.

Proof. (a)⇒(b). Let $\mathbf{a} \in \mathrm{ext}K$ and $a = a_1 = \ldots = a_{m-1} > a_m$. By (1.3) we have that

$$a = a_{m-1} \leq 1 - \sum_{i=1}^{m-1} a_i = 1 - (m-1)a,$$

that is, $a \leq \frac{1}{m}$. Assume that $a < \frac{1}{m}$. We will find $0 < \delta, \varepsilon < 1$ such that

$$\mathbf{b} = ((1+\delta)a_1, \ldots, (1+\delta)a_{m-1}, (1-\varepsilon)a_m, (1-\varepsilon)a_{m+1}, \ldots),$$

$$\mathbf{c} = ((1-\delta)a_1, \ldots, (1-\delta)a_{m-1}, (1+\varepsilon)a_m, (1+\varepsilon)a_{m+1}, \ldots),$$

both should be in K. We must take ε and δ such that

$$(m-1)\delta a = \varepsilon(1-(m-1)a),$$

in order to guarantee $\|\mathbf{b}\|_1 = \|\mathbf{c}\|_1 = 1$. Taking into account $a < 1-(m-1)a$ and $a = a_{m-1} > a_m$ we can choose these numbers such that

$$(1+\delta)a_{m-1} \leq (1-\varepsilon)(1-(m-1)a), \ (1-\delta)a_{m-1} \geq (1+\varepsilon)a_m.$$

The last two conditions ensure that both $\mathbf{b}$ and $\mathbf{c}$ are non-increasing sequence and satisfy (1.3).

Since $\mathbf{a} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$ and $\mathbf{b} \neq \mathbf{c}$, we obtain a contradiction with that $\mathbf{a} \in \mathrm{ext}K$. So, $a_1 = \ldots = a_{m-1} = \frac{1}{m}$.

Since affine isomorphism preserves extreme points of convex sets, by Proposition 2.4, we obtain that $(ma_m, ma_{m+1}, \ldots) = \Pi_m^{-1}(\mathbf{a}) \in \mathrm{ext}K$.

(b)⇒(a). Let $\mathbf{a} \in K$ be such that $a_1 = \ldots = a_{m-1} = \frac{1}{m}$ and $\mathbf{x} = (ma_m, ma_{m+1}, \ldots) \in \mathrm{ext}K$. Let $\mathbf{b}, \mathbf{c} \in K$ and $0 < \lambda < 1$ be such that $\mathbf{a} = \lambda\mathbf{b} + (1-\lambda)\mathbf{c}$. By a similar way as in the proof of Proposition 2.4 we obtain that

$$b_1 = c_1 = \ldots = b_{m-1} = c_{m-1} = \frac{1}{m}.$$

This means that $\mathbf{b}, \mathbf{c} \in K_m$. Taking into account that

$$\mathbf{x} = \Pi_m^{-1}(\mathbf{a}) = \lambda\Pi_m^{-1}(\mathbf{b}) + (1-\lambda)\Pi_m^{-1}(\mathbf{c}),$$

$\mathbf{x} = (ma_m, ma_{m+1}, \ldots) \in \mathrm{ext}K$ and Π_m is affine isomorphism, we obtain that $\Pi_m^{-1}(\mathbf{b}) = \Pi_m^{-1}(\mathbf{c})$. Hence, $\mathbf{a} = \mathbf{b} = \mathbf{c}$, that is, $\mathbf{a} \in \mathrm{ext}K$. The proof is completed. □

Now we are in position to present the proof of Theorem 2.2.

Proof of Theorem 2.2. 'if' part. Let $\mathbf{a} \in K$ be in the form (2.7) and let $\mathbf{a} = \lambda\mathbf{b} + (1-\lambda)\mathbf{c}$, where $\mathbf{b}, \mathbf{c} \in K$, $0 < \lambda < 1$. Since $\mathbf{a} \in K_{k_1}$ and K_{k_1} is a face, it follows that $\mathbf{b}, \mathbf{c} \in K_{k_1}$. Thus $b_1 = c_1 = \ldots = b_{k_1-1} = c_{k_1-1} = \frac{1}{k_1}$. Then $\Pi_{k_1}^{-1}(\mathbf{a})$ is also in the form (2.7), that is, $\Pi_{k_1}^{-1}(\mathbf{a}) \in K_{k_2}$, where $k_2 \geq 2$.

Since $\Pi_{k_1}^{-1}(\mathbf{a}) = \lambda\Pi_{k_1}^{-1}(\mathbf{b}) + (1-\lambda)\Pi_{k_1}^{-1}(\mathbf{c}) \in K_{k_2}$ and K_{k_2} is a face, we obtain that $\Pi_{k_1}^{-1}(\mathbf{a}) = \Pi_{k_1}^{-1}(\mathbf{b}) = \Pi_{k_1}^{-1}(\mathbf{c})$, in particular,

$$b_{k_1} = c_{k_1} = \ldots = b_{k_2-1} = c_{k_2-1} = \frac{1}{k_1 k_2}.$$

Continuing by this way we can find a sequence of integers $\{k_n\}_{n\geq 1}$ such that $\Pi_{k_n}^{-1}\cdots\Pi_{k_1}^{-1}(\mathbf{a}) = \Pi_{k_n}^{-1}\cdots\Pi_{k_1}^{-1}(\mathbf{b}) = \Pi_{k_n}^{-1}\cdots\Pi_{k_1}^{-1}(\mathbf{c})$ for all $n \geq 1$. Thus

$$b_{k_n} = c_{k_n} = \ldots = b_{k_{n+1}-1} = c_{k_{n+1}-1} = \frac{1}{k_1\cdots k_{n+1}}$$

for all $n \geq 1$, that is, $\mathbf{b} = \mathbf{c} = \mathbf{a}$. This means that $\mathbf{a} \in \mathrm{ext}K$.

'only if' part. Let $\mathbf{a} \in \mathrm{ext}K$. Let $k_1 \geq 2$ be an integer such that $a_1 = \ldots = a_{k_1-1} > a_{k_1}$. By Proposition 2.5 we obtain that $a_1 = \ldots = a_{m-1} = \frac{1}{k_1}$ and $(k_1 a_{k_1}, k_1 a_{k_1+1}, \ldots) \in \mathrm{ext}K$. Now we can find an integer $k_2 \geq 2$ such that $k_1 a_{k_1} = \ldots = k_1 a_{k_2-1} > k_1 a_{k_2}$. Again using Proposition 2.5 we obtain that $k_1 a_{k_1} = \ldots = k_1 a_{k_1-1} = \frac{1}{k_2}$ and $(k_2 k_1 a_{k_2}, k_2 k_1 a_{k_2+1}, \ldots) \in \mathrm{ext}K$. So,

$$a_1 = \ldots = a_{k_1-1} = \frac{1}{k_1}, \quad a_{k_1} = \ldots = a_{k_2-1} = \frac{1}{k_1 k_2}.$$

Continuing by this way we obtain that $\mathbf{a}$ is in the form (2.7). The proof is completed. □

Remark 2.6. Setting $k_n = 2$, $n \geq 1$ in (2.7) we obtain that

$$\mathbf{a} = \left(\frac{1}{2}, \ldots, \frac{1}{2^n} \ldots\right) \in \mathrm{ext}K.$$

Then (2.4) rewrites as follows

$$r = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}, \quad \varepsilon_n \in \{0,1\}, n \geq 1,$$

which gives the dyadic expansion of real number $r \in [0,1]$.

Further, if we take $k_n = m$, $n \geq 1$, where $m \geq 2$, using (2.7), from (2.4) we obtain that

$$r = \sum_{n=1}^{\infty} \frac{\epsilon_n}{m^n}, \quad \epsilon_n \in \{0,1,\ldots,m-1\}, n \geq 1,$$

which gives the m-ary expansion of real number $r \in [0,1]$.

In general, if we take an arbitrary sequence $k_1, \ldots, k_n, \ldots$ with $k_n \geq 2$, $n \geq 1$, (2.4) implies that

$$r = \sum_{n=1}^{\infty} \frac{\epsilon_n}{k_1\cdots k_n}, \quad \epsilon_n \in \{0,1,\ldots,k_n-1\}, n \geq 1.$$

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