UNORTHODOX ALGEBRAS AND THEIR ASSOCIATED UNORTHODOX LOGICS

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ABSTRACT. This paper grew out of our investigation into a simple, but natural, question: Can "$F \to T$" be different from "F" and "T"? To this end, we introduce five "unorthodox" algebras that will play a major role, not only in providing a positive answer to the question, but also in their similarity to the 2-element Boolean algebra **2**. Yet, they are remarkably different from **2** in many respects. In this paper, we will examine these five algebras both algebraically and logically. We define, and initiate an investigation into, a subvariety, called $\mathbb{RUNO}1$, of the variety of De Morgan semi-Heyting algebras and show that $\mathbb{RUNO}1$ is, in fact, the variety generated by the five algebras. Then several applications of this theorem are given. It is shown that $\mathbb{RUNO}1$ is a discriminator variety and that all five algebras are primal. It is also shown that every subvariety of $\mathbb{RUNO}1$ satisfies the Strong Amalgamation Property and the property that epimorphisms are surjective (ES). It is shown that the lattice of subvarieties of $\mathbb{RUNO}1$ is a Boolean lattice of 32 elements. The bases for all the subvarieties of $\mathbb{RUNO}1$ are also given. We introduce a new logic called $\mathcal{RUNO}1$ and show that it is algebraizable with the variety $\mathbb{RUNO}1$ as its equivalent algebraic semantics. We then present axiomatizations for all 32 axiomatic extensions of $\mathcal{RUNO}1$ and deduce that all the axiomatic extensions are decidable. The paper ends with some open problems.

1. Introduction

Let "true" and "false" abbreviate to "T" and "F" (or, "1" and "0") respectively, and let $\to$ denote an implication. Let us start with a very simple, but natural, question:

Question A: What is "$F \to T''$?, or equivalently, what is "$0 \to 1''$?

Of course, a student who is taking the first course in logic would, in most likelihood, be told by her/his professor that $F \to T = T$, or in the other notation, $0 \to 1 = 1$, as the answer to the question. This answer, together with $1 \to 1 = 1$, $0 \to 0 = 1$, and $1 \to 0 = 0$, would lead, of course, to the well-known "orthodox" (i.e. Boolean) implication on the Boolean algebra **2** and, thereby, to the the variety $\mathbb{V}(\mathbf{2})$ of Boolean algebras and the associated "orthodox" (i.e. classical) logic. A curious student, however,

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might ask his/her professor the following follow-up question, in response to the professor's previous answer:

Question B: Can "$0 \to 1$" be different from 1?

If the student was taking the logic course before the middle of the nineteenth century, it is very likely that he/she would have been told by the professor that it couldn't be different from T (or 1); but if the student is taking the course in 2025, the professor would have certainly answered question B by saying, "YES, indeed, it is possible". This paper grew out of an attempt to give an elaborate and convincing justification to the answer "YES."

In order to answer Question B completely satisfactorily, however, we need to first refine the Question B into two questions as follows:

Question B1: Can "$0 \to 1 = 0$, so that the resulting logic is "interesting"?

Question B2: Can $0 \to 1$ be different from both 0 and 1, so that the resulting logic is also "interesting")?

Let us first consider, briefly, an answer to **Question B1**. For this purpose, consider the following 2-element algebra, with $0 < 1$, in the language $\langle \vee, \wedge, \to, 0, 1 \rangle$:

$$\overline{\mathbf{2}}: \quad \begin{array}{c|cc} \to & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

Observe that it is indeed the case that $0 \to 1 = 0$ in the algebra $\overline{\mathbf{2}}$. So, we could call this algebra "**anti-Boolean**." YET, $\overline{\mathbf{2}}$, and the variety $\mathbb{V}(\overline{\mathbf{2}})$ that it generates, as well as the corresponding logic are quite interesting in the sense that they share several important properties with the Boolean (i.e., orthodox) algebra $\mathbf{2}$, the variety $\mathbb{V}(\mathbf{2})$ and its corresponding logic namely classical logic. For instance,

- $\overline{\mathbf{2}}$ is a primal algebra, just as the Boolean algebra $\mathbf{2}$ is.
- $\mathbb{V}(\overline{\mathbf{2}})$ has an interesting logic. It is a **connexive logic** in the sense that Aristotle's Theses: $\vdash \neg(\neg\alpha \to \alpha)$, and $\vdash \neg(\alpha \to \neg\alpha)$ and Boethius Theses: $\vdash (\alpha \to \beta) \to \neg(\alpha \to \neg\beta)$ and $(\alpha \to \neg\beta) \to \neg(\alpha \to \beta)$ are theorems in that logic.
- More interestingly, the variety $\mathbb{V}(\overline{\mathbf{2}})$ turns out to be **term-equivalent** to $\mathbb{V}(\mathbf{2})$, the variety of Boolean algebras! Hence, we could say that the classical logic is "connexive" too.

Thus the answer to **Question B1** is a definite "YES". For all these claims, and much more, about $\overline{\mathbf{2}}$, $\mathbb{V}(\overline{\mathbf{2}})$, its associated connexive logic, and other related connexive logics, we refer the reader to the recent papers [22] and [24].

The main focus of this paper[1] is to address **Question B2**. Since the variety of Boolean algebras is known to have an axiomatization in the language $\mathbf{L} = \langle \vee, \wedge, \to, ', 0, 1\rangle$, we will address **Question B2** in the language **L**, together with some of its far reaching ramifications. The following FIVE algebras in the language **L** will play a major role in providing a positive answer to **Question B2**. These algebras can be viewed as logical matrices with 1 as the only designated truth value. If we think of the classical logic as "orthodox", then we could, informally speaking, think of the following algebras as "unorthodox" as their $\to$ operation is such that $0 \to 1$ is a fixed point under the (unary) De Morgan operation $'$ and hence is neither 1 nor 0. (The precise definition of 'unorthodox" will be given later in the paper.) The first four of these five algebras are chain-based and hence $\wedge$, $\vee$ are not shown.

1.1. "Unorthodox Algebras".

All five algebras are of type **L**$= \langle \vee, \wedge, \to, ', 0, 1\rangle$. The first four have the 3-element chain as their lattice-reduct and the fifth one has the 4-element Boolean lattice as its lattice-reduct.

- **ALGEBRA A1**: $0 < 2 < 1$,

$\to$	0	1	2
0	1	2	1
1	0	1	2
2	0	1	1

$'$	0	1	2
	1	0	2

- **ALGEBRA A2**: $0 < 2 < 1$,

$\to$:	0	1	2
0	1	2	1
1	0	1	2
2	0	2	1

$'$:	0	1	2
	1	0	2

[1] The contents of this paper were presented by the author in invited lectures at the conference on Non-Classical Logics, Lodz, Poland, Sept 5-8, 2024 and at the International Conference on Algebra, Logic, and their Applications, Yerevan, Armenia, 13-19 October 2024, as well as at Mathematical Congress of Americas, Miami, Florida, 21-25 July 2025.

- **ALGEBRA A3**: $0 < 2 < 1$,

$\to$:	0	1	2
0	1	2	2
1	0	1	2
2	0	1	1

$'$:	0	1	2
	1	0	2

- **ALGEBRA A4**: $0 < 2 < 1$,

$\to$	0	1	2
0	1	2	2
1	0	1	2
2	0	2	1

$'$	0	1	2
	1	0	2

- **ALGEBRA A5**: $2 \vee 3 = 1$, $2 \wedge 3 = 0$,

$\to$	0	1	2	3
0	1	2	1	2
1	0	1	2	3
2	3	2	1	0
3	2	1	2	1

$'$	0	1	2	3
	1	0	2	3

Remark 1.1.

(a) *All these algebras share the following properties with the the Boolean (i.e., orthodox) algebra* **2**:
   (1) $0 \to 1 = 2$ *(different from* 0 *and* 1*).*
   (2) $(0 \to 1)' = 0 \to 1$. *Hence, we can consider these algebras as "unorthodox."*
   (3) *They have no proper subalgebras. In particular, none of the five algebras have the Boolean algebra* **2** *as a subalgebra.*
   (4) *They have no nontrivial automorphism.*

(b) *The elements in red on the* $\to$ *table of the first four algebras indicate the places where they differ from the* $\to$ *table of the 3-element Heyting algebra, while those of the fifth algebra indicate where they differ from the* $\to$ *table of the 4-element Boolean algebra.*

In this paper, we will examine these five algebras both algebraically and logically. So, we first investigate the variety generated by the above five algebras; more specifically, we will examine the following problem:

PROBLEM C: Find an equational base for the variety generated by the above five algebras.

We then study the structure of the lattice of subvarieties of the variety $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}, \mathbf{A5})$ generated by the five algebras. We also introduce logics associated with the subvarieties of this variety and investigate the properties, such as amalgamation property, being a discriminator variety, decidability of the logics and so on.

More specifically, the paper is organized as follows: Section 2 recalls definitions, notation and preliminary results about De Morgan semi-Heyting algebras that are needed later in the paper. In Section 3, we define and initiate an investigation into a subvariety, called $\mathbb{RUNO}1$, of the variety of De Morgan semi-Heyting algebras. Section 4 presents a characterization of the subdirectly irreducible algebras in $\mathbb{RUNO}1$ and as a consequence, we show that $\mathbb{RUNO}1$ is, in fact, the variety generated by the five algebras, thus solving PROBLEM C. Then several applications of this theorem are given. It is shown that $\mathbb{RUNO}1$ is a discriminator variety and that all five algebras are primal. It is also shown that every subvariety of $\mathbb{RUNO}1$ satisfies the Strong Amalgamation Property and the property that epimorphisms are surjective (ES). Sections 5 and 6 give further applications. It is shown, in Section 5, that the lattice of subvarieties of $\mathbb{RUNO}1$ is a Boolean lattice of 32 elements. We also present bases for all 32 subvarieties of $\mathbb{RUNO}1$. In section 6 we give several more identities that hold in $\mathbb{RUNO}1$ and in its subvarieties.

The rest of the paper deals with the variety $\mathbb{RUNO}1$ from a logical point of view. Section 7 will recall, by way of a quick introduction to implicative logics in the sense of Rasiowa, and to abstract algebraic logic due to Blok and Pigozzi, the notions of algebraizability of a logic and its equivalent algebraic semantics as well as the most fundamental results connecting them. In Section 8, we recall from [16] a known algebraizable logic called $\mathcal{DMSH}$ corresponding to De Morgan semi-Heyting algebras, which plays a crucial role in the rest of the paper. We then introduce a new logic called $\mathcal{RUNO}1$ as an axiomatic extension of the logic $\mathcal{DMSH}$. It is then deduced that the logic $\mathcal{RUNO}1$ is implicative and hence algebraizable, with the variety $\mathbb{RUNO}1$ as its equivalent algebraic semantics. Section 9 presents bases for all axiomatic extensions of $\mathcal{RUNO}1$, from which it follows that all its axiomatic extensions are decidable. The paper ends with some open problems in Section 10.

2. Preliminaries

In this section, we recall some useful results regarding De Morgan semi-Heyting algebras. Observe that all the five algebras mentioned above have semi-Heyting reducts.

Semi-Heyting algebras were introduced by the author in 1984-85 as an abstraction of Heyting algebras. The first results on these algebras, however, were not published until 2007 (see [37]).

An algebra $L = \langle L, \wedge, \vee, \to, 0, 1\rangle$ is a semi-Heyting algebra if the following conditions hold:

(SH1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with 0 and 1,
(SH2) $x \wedge (x \to y) \approx x \wedge y$,
(SH3) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,
(SH4) $x \to x \approx 1$.

A semi-Heyting algebra is a Heyting algebra if it satisfies:

$$(x \wedge y) \to x \approx 1.$$

We will denote the variety of semi-Heyting algebras by $\mathbb{SH}$ and that of Heyting algebras by $\mathbb{H}$.

Semi-Heyting algebras share some important properties with Heyting algebras; for instance, semi-Heyting algebras are distributive and pseudocomplemented; the congruences on them are determined by filters and the variety of semi-Heyting algebras is arithmetical (for more such properties, see [1, 2, 3, 37]).

A key feature of semi-Heyting algebras is the following [3]: Every semi-Heyting algebra $\langle A, \vee, \wedge, \to, 0, 1\rangle$ gives rise naturally to a Heyting algebra $\langle A, \vee, \wedge, \to_H, 0, 1\rangle$ by defining the implication $x \to_H y$ as $x \to (x \wedge y)$.

The following lemmas will be useful in the sequel.

Lemma 2.1. *Let $\mathbf{L} = \langle L, \wedge, \vee, \to, 0, 1\rangle$ be a semi-Heyting algebra. For $a, b, c \in L, a \wedge b \leq c$ if and only if $a \leq b \to (b \wedge c)$.*

Lemma 2.2. [12, Corollary 3.9] *Let $\mathbf{L} = \langle L, \wedge, \vee, \to, 0, 1\rangle$ be a semi-Heyting algebra. For $a, b \in L$, $a \to_H b = 1$ if and only if $a \leq b$.*

3. The variety $\mathbb{RUNO}1$

In this section we define, and investigate, an important subvariety, called "$\mathbb{RUNO}1$" of the variety of dually hemimorphic semi-Heyting algebras.

Definition 3.1. *An algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, ', 0, 1\rangle$ is a semi-Heyting algebra with a dual hemimorphism (or dually hemimorphic semi-Heyting algebra) if $\mathbf{A}$ satisfies the following conditions:*

$(E1)$ *$\langle A, \wedge, \vee, \to, 0, 1\rangle$ is a semi-Heyting algebra,*
$(E2)$ *$0' \approx 1$,*
$(E3)$ *$1' \approx 0$,*
$(E4)$ *$(x \wedge y)' \approx x' \vee y'$ ($\wedge$-De Morgan law).*

We will denote the variety of dually hemimorphic semi-Heyting algebras by $\mathbb{DHMSH}$. An algebra $\mathbf{A} \in \mathbb{DHMSH}$ is a dually quasi-De Morgan semi-Heyting algebra if $\mathbf{A}$ satisfies the following:

(DQD1) *$(x \vee y)'' \approx x'' \vee y''$;*
(DQD2) *$x'' \leq x$.*

The variety of dually quasi-De Morgan semi-Heyting algebras is denoted by $\mathbb{DQDSH}$.

An algebra $\mathbf{A} \in \mathbb{DQDSH}$ is said to be De Morgan if $\mathbf{A} \models x'' \approx x$. An algebra $\mathbf{A} \in \mathbb{DHMSH}$ is said to be **unorthodox** *if $\mathbf{A}$ satisfies the axiom: $(0 \to 1)' = 0 \to 1$.*

The variety of unorthodox De Morgan semi-Heyting algebras is denoted by $\mathbb{UNO}$*. An algebra* $\mathbf{A} \in \mathbb{UNO}$ *is called regular if* $\mathbf{A}$ *satisfies the identity:*
$x \wedge x'^{*\prime} \leq y \vee y^*$ *(R).*
The variety of regular unos is denoted by $\mathbb{RUNO}$*. An algebra* $\mathbf{A} \in \mathbb{DHMSH}$ *is of level 1 if* $\mathbf{A}$ *satisfies the identity:* $x \wedge x'^* \approx (x \wedge x'^*)'^*$*.*
If $\mathbb{V}$ *is a subvariety of* $\mathbb{DHMSH}$*, then* $\mathbb{V}1$ *denotes the subvariety of* $\mathbb{V}$ *consisting of algebras from* $\mathbb{V}$ *of level 1. Thus* $\mathbb{RUNO}1$ *denotes the subvariety of* $\mathbb{RUNO}$ *consisting of algebras from* $\mathbb{RUNO}$ *of level 1.*

The significance of the variety $\mathbb{RUNO}1$ is revealed by the following theorem:

Theorem 3.2. *The algebras* $\mathbf{A}i \in \mathbb{RUNO}1$*,* $i = 1, 2, \cdots, 5$*.*

Proof It is routine to verify that the axioms of $\mathbb{RUNO}1$ are true in each $\mathbf{A}i$, $i = 1, 2, \cdots, 5$. □

We wish to show that the converse of the above theorem holds as well. To achieve this goal, we need to characterize the subdirectly irreducible algebras in $\mathbb{RUNO}1$, Toward this end, it will be helpful to know the maximum height of subdirectly irreducible algebras in $\mathbb{RUNO}1$.

3.1. **On the height of algebras of** $\mathbb{RUNO}1$ **satisfying (SC).**

The purpose of this subsection is to show that the height of members of $\mathbb{RUNO}1$ that satisfy the condition (SC) (see below) is at most two–a result which will be useful in describing the subdirectly irreducible algebras in $\mathbb{RUNO}1$. Let $x^+ := x'^{*\prime}$.

Unless stated otherwise, the lemmas below will have the following hypothesis:
"Let $\mathbf{A} \in \mathbb{RUNO}1$ satisfy:
(SC) $x \neq 1 \Rightarrow x \wedge x'^* = 0$.
Let $a := 0 \to 1$, and $b \in A$ such that $a < b < 1$."

Lemma 3.3. $b' \leq b$*.*

Proof From $a \leq b$ we have $b' \leq a'$, implying $b' \leq a$ as $a' = a$ by hypothesis, which yields $b' \leq b$, as $a \leq b$. □

Lemma 3.4. $b' \vee b'^* = b \vee b'^*$*.*

Proof As $b \neq 1$, $b \wedge b'^* = 0$ by (SC). Hence, $b' \vee b^+ = 0'$, whence we have

$$b' \vee b^+ = 1. \tag{3.1}$$

Now,

$$\begin{aligned} b' \vee b'^* &= (b' \vee b'^*) \vee (b \wedge b^+) && \text{by regularity} \\ &= (b' \vee b'^* \vee b) \wedge (b' \vee b'^* \vee b'^{*\prime}) \\ &= (b'^* \vee b) \wedge (b' \vee b'^* \vee b^+) && \text{by Lemma 3.3} \\ &= b \vee b'^* && \text{by (3.1),} \end{aligned}$$

proving the lemma. □

Lemma 3.5. $a \wedge (b' \vee b'^*) = b'$.

Proof Since $b \neq 1$, we have $b \wedge b'^* = 0$ by (SC), which implies $a \wedge b'^* = 0$, as $a \leq b$, implying

$$a \wedge b'^* = 0. \tag{3.2}$$

Hence we have

$$\begin{aligned} a \wedge (b' \vee b'^*) &= (a \wedge b') \vee (a \wedge b'^*) \\ &= (a \wedge b') \vee 0 \\ &= a' \wedge b' \\ &= b' && \text{by (3.2),} \end{aligned}$$

proving the lemma. □

We are now ready to prove the following theorem which will be useful in the next section.

Theorem 3.6. *Let* $\mathbf{A} \in \mathbb{RUNO}1$ *satisfy:*
(SC) $\quad x \neq 1 \Rightarrow x \wedge x'^* = 0$.
Then, the height of $\mathbf{A}$*,* $h(\mathbf{A}) \leq 2$.

Proof Since $a' = a$, it is clear that $a \neq 0$. Now, suppose $h(\mathbf{A}) > 2$. Then there exists an element $b \in A$ such that $0 < a < b < 1$. We wish to arrive at a contradiction. Since $b' \vee b'^* = b \vee b'^*$ from Lemma 3.4, we get from Lemma 3.5 that $a \wedge (b \vee b'^*) = b'$, implying $(a \wedge b) \vee (a \wedge b'^*) = b'$. Therefore, in view of (3.2), we obtain a = b'. Hence $a = a' = b'' = b$, which is a contradiction, completing the proof. □

4. A Characterization of Subdirectly irreducible Algebras in $\mathbb{RUNO}1$

Our main goal in this section is to characterize the subdirectly irreducible algebras in $\mathbb{RUNO}1$.

The following theorem, which is a special case of a result proved in [39], is needed here.

Theorem 4.1. [39, Corollary 4.1] *Let* $\mathbf{A} \in \mathbb{DMSH}1$ *with* $|A| \geq 2$*. Then the following are equivalent:*

(1) $\mathbf{A}$ *is simple;*
(2) $\mathbf{A}$ *is subdirectly irreducible;*
(3) *For every* $x \in A$*, if* $x \neq 1$*, then* $x \wedge x'^* = 0$ *(SC).*

We are now ready to characterize the subdirectly irreducible algebras in $\mathbb{RUNO}1$.

Theorem 4.2. *Let* $\mathbf{A} \in \mathbb{RUNO}1$ *with* $|A| > 2$*. Then the following are equivalent*

(1) $\mathbf{A}$ *is simple*
(2) $\mathbf{A}$ *is subdirectly irreducible*
(3) *For every $x \in A$, if $x \neq 1$, then $x \wedge x'^{*} = 0$ (SC).*
(4) $\mathbf{A} \in \{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3, \mathbf{A}4, \mathbf{A}5\}$.

Proof We know that (1), (2) and (3) are equivalent by Theorem 4.1. Now, suppose (3) holds in $\mathbf{A}$. Then, by Theorem 3.6 we know the height is ≤ 2. Since $|A| > 2$, we conclude that the height of A =2. Then it follows that $|A| = 3$ or $|A| = 4$. If $|A| = 3$ then it is easy to see that $\mathbf{A} \in \{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3, \mathbf{A}4\}$. If $|A| = 4$, then $\mathbf{A} = \mathbf{A}5$, up to isomorphism. Thus, (4) holds in $\mathbf{A}$, whence (3) implies (4), while the statement that (4) implies (3) is routine to verify. Thus the proof is complete. $\square$

Corollary 4.3. $\mathbb{RUNO}1 = \mathbb{V}(\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3, \mathbf{A}4, \mathbf{A}5\})$.

Remark 4.4. *Thus, we have achieved an axiomatization of the variety $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3, \mathbf{A}4, \mathbf{A}5\})$, thus solving* **PROBLEM C** *posed earlier.*

In the rest of this section we present some remarkable properties of the variety $\mathbb{RUNO}1$ as consequences of Corollary 4.3.

4.1. **Discriminator.**

A ternary term $t(x, y, z)$ is a discriminator term for an algebra A if, for $a, b, c \in A$,

$$t(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{if } a \neq b. \end{cases}$$

A variety $\mathbb{V}$ of algebras is a discriminator variety if there is a class $\mathbb{K}$ of algebras which generates $\mathbb{V}$ such that there is a ternary term $t(x, y, z)$ which is a discriminator term for every member of $\mathbb{K}$.

Theorem 4.5. $\mathbb{RUNO}1$ *is a discriminator variety.*

Proof Let $d_1(x) = x \wedge x'^{*}$. Let $T(x, y, z) := [z \wedge d_1((x \vee y) \to (x \wedge y))] \vee [x \wedge \{d_1((x \vee y) \to (x \wedge y))\}^{*}]$. Then it is routine to verify that the term $T(x, y, z)$ is a ternary discriminator term on each of the algebras $\mathbf{Ai}$, $\mathbf{i}$=1,2, $\cdots$, 5. Then the theorem follows from Corollary 4.3. $\square$

The following theorem, due to Bulman-Fleming, Keimel, and Werner. is well-known (See [9, Page 165, Theorem 9.4] for a proof).

Theorem 4.6. *Every member of a discriminator variety is isomorphic to a Boolean product of simple algebras from the variety.*

The following corollary is immediate from Theorem 4.6 and Theorem 4.5.

Corollary 4.7. *Every member of* $\mathbb{RUNO}1$ *is isomorphic to a Boolean product of* $\mathbf{Ai}, \mathbf{i} = 1, 2, \cdots, 5$.

Corollary 4.8. *The algebras* $\mathbf{Ai}, \mathbf{i} = 1, 2, \cdots, 5$ *are primal (just like the two element Boolean algebra).*

Corollary 4.9. *Every algebra in the variety* $\mathbb{V}(\mathbf{Ai})$, $\mathbf{i} = 1, 2, \cdots, 5$ *is a Boolean Power of* $\mathbf{A}i$.

Theorem 4.10. (Burris [8]) Every discriminator variety has unary unification type.

Corollary 4.11. *All subvarieties of* $\mathbb{RUNO}1$ *have unitary unification type.*

It was shown in [10] that the first-order theory of finitely generated discriminator variety is decidable. Hence the following corollary is immediate.

Corollary 4.12. *The first-order theory of each* $\mathbb{V}(\mathbf{Ai}), \mathbf{i} = 1, 2, \cdots, 5$, *is decidable.*

4.2. **Properties AP, SAP, ES.**

Amalgamation Property (AP, for short) is an important property for varieties to have. We refer the reader to [23] for its history and its connection to algebraic logic (see also [21]).

In this subsection we wish to investigate whether $\mathbb{V}(\mathbf{A}i)$, $i = 1, 2, \cdots, 5$ have AP, SAP and ES.

Definition 4.13. *By a diagram in a class* $\mathbb{K}$ *of algebras we mean a quintuple* $\langle A, f, B, g, C\rangle$ *with* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathbb{K}$ *and* $f : \mathbf{A} \mapsto \mathbf{B}$ *and* $g : \mathbf{A} \mapsto \mathbf{C}$ *embeddings. By an amalgam of this diagram in* $\mathbb{K}$ *we mean a triple* $\langle f_1, g_1, \mathbf{D}\rangle$ *with* $\mathbf{D} \in \mathbb{K}$ *and with* $f_1 : \mathbf{B} \mapsto \mathbf{D}$ *and* $g_1 : C \mapsto D$ *embeddings such that* $f_1 \circ f = g_1 \circ g$. *If such an amalgam exists, then we say that the diagram is amalgamable in* $\mathbb{K}$. *If such an amalgam exists and satisfies the condition:* $f_1(B) \cap g_1(C) = f_1 circ f_(A)$, *then we say that the diagram is strongly amalgamable in* $\mathbb{K}$. *We say that* $\mathbb{K}$ *has the Amalgamation Property (AP, for short) if every diagram in* $\mathbb{K}$ *is amalgamable.* $\mathbb{K}$ *has the Strong Amalgamation Property (SAP, for short) if every diagram in* $\mathbb{K}$ *is strongly amalgamable.*

Remark 4.14. *Since* f *and* g *in a diagram are embeddings, we can, without loss of generality, simply think of a diagram* $\langle A, f, B, g, C\rangle$ *as a triple* $\langle A, B, C\rangle$ *such that* $\mathbf{A} \subseteq \mathbf{B} \cap \mathbf{C}$. *Accordingly, the definition of (AP) can be restated.*

For the rest of this subsection, we let $\mathbb{S} := \{\mathbf{Ai}, \mathbf{i} = 1, 2, \cdots, 5\}$. Note that $\mathbb{S}$ is prcisely the the class of simple, non-singleton members of $\mathbb{RUNO}1$.

Lemma 4.15. *Every non-empty subclass of* $\mathbb{S}$ *has AP.*

Proof. Let $\mathbb{S}' \subset \mathbb{S}$ be nonempty. Recall that every member of $\mathbb{S}'$ has no proper subalgebras. Hence the only possible diagrams are those in which all three algebras are equal to each other; and such diagrams can easily be amalgamated, which implies that $\mathbb{S}'$ has AP. $\square$

Theorem 4.16. (Werner [45, page 27]) *A discriminator variety has the amalgamation property iff the class of its simple, non-singleton members has the amalgamation property.*

The following corollary is immediate from Theorem 4.5, Lemma 4.15 and Theorem 4.16.

Corollary 4.17. *Every subvariety of* $\mathbb{RUNO}1$ *has the amalgamation property.*

Definition 4.18 (Epimorphism)**.** *Let* $\mathbb{K}$ *be a class of algebras, and* $\mathbf{A}, \mathbf{B} \in \mathbb{K}$*. A homomorphism* $h : A \to B$ *is called a* $\mathbb{K}$*-epimorphism, or K-epi, if it satisfies the following condition: For any* $\mathbf{C} \in \mathbb{K}$ *and any pair of homomorphisms* $k, k' : \mathbf{B} \to \mathbf{C}$*, if* $k \circ h = k' \circ h$*, then* $k = k'$*. We say that* $\mathbb{K}$ *has the property: Epimorphisms are surjective (ES, for short) if every epimorphism on every algebra in* $\mathbb{K}$ *is surjective (i.e., onto).*

Since all members of $\mathbb{S}$ are primal, the following lemma is immediate.

Lemma 4.19. *Every non-empty subclass of* $\mathbb{S}$ *has ES.*

Theorem 4.20. *Comer* [11] *Let V be a discriminator variety such that the class of its simple members has AP and ES. Then V satisfies ES.*

Corollary 4.21. *. Every subvariety of* $\mathbb{RUNO}1$ *satisfies ES.*

Proof. Let $\mathbb{V}$ be a subvariety of $\mathbb{RUNO}1$. Recall that the variety $\mathbb{V}$ is a discriminator variety. Also, $\mathbb{S}$ has AP by Lemma 4.15 and has ES by Lemma 4.19. Hence, we conclude, in view of Theorem 4.20 that $\mathbb{V}$ has ES. ☐

Theorem 4.22. *Isbell* [28] *Let V be a variety. If V has ES and AP, then V has SAP.*

Corollary 4.23. *Every subvariety of* $\mathbb{RUNO}1$ *satisfies SAP.*

Proof. Let $\mathbb{V}$ be a subvariety of $\mathbb{RUNO}1$. Then $\mathbb{V}$ has AP by Corollary 4.17 and has ES by Corollary 4.21. Hence by Theorem 4.22 we conclude that $\mathbb{V}$ has SAP. ☐

5. The lattice of subvarieties of $\mathbb{RUNO}1$

In this section we present a concrete description of the structure of the lattice of subvarieties of $\mathbb{RUNO}1$ and also give an (equational) base for each of the thirty nontrivial, proper subvarieties of $\mathbb{RUNO}1$.

Corollary 5.1. *The lattice* $\mathbf{L}$ *of subvarieties of the variety* $\mathbb{RUNO}1$ *is a Boolean lattice of size* 2^5 (=32) elements.

Proof Since, by Corollary 4.3, $\mathbb{RUNO}1$ is generated by $\{\mathbf{Ai} : i = 1, \cdots, 5\}$, and all the generators are primal by Corollary 4.8, it follows that the five generators are atoms of the distributive lattice $\mathbf{L}$. Then, using Jónsson's theorem [29], we can conclude that

- of height 0: The trivial variety;
- of height 1: $\mathbb{V}(\mathbf{A}i)$, $i = 1, 2, \cdots, 5$ as atoms;
- of height 2: subvarieties generated by pairs;
- of height 3: subvarieties generated by (unordered) triples;

- of height 4: subvarieties generated by (unordered) quadruples of these algebras;
- of height 5: the variety $\mathbb{RUNO}1$.

It is clear that the lattice $\mathbf{L}$ is complemented and hence is Boolean. Thus we conclude that $\mathbf{L}$ is isomorphic to the power set of $\{1, 2, 3, 4, 5\}$; hence $\mathbf{L}$ is a Boolean lattice of size 2^5 $(= 32)$ elements. $\square$

5.1. Bases for subvarieties of $\mathbb{RUNO}1$.

In this section, we will give bases for all the subvarieties of $\mathbb{RUNO}1$. Here "a base for a class $\mathbb{K}$ of algebras" means "a base for the variety $\mathbb{V}(\mathbb{K})$ of algebras." In particular, "a base for a single algebra $\mathbf{A}$" means a base for $\mathbb{V}(\mathbf{A})$. Also, a "base" in the following theorem means "a base relative to $\mathbb{RUNO}1$. Note also that $a := 0 \to 1$, $b := 0 \to a$.

Theorem 5.2. *Bases for all non-trivial and proper subvarieties of* $\mathbb{RUNO}1$ *are as follows:*

(1) **BASES FOR SINGLE ALGEBRAS :**

(i) **A BASE FOR $\{\mathbf{A}1\}$**
 (a) $(0 \to 1) \to 1 \approx 1$;
 (b) $0 \to (0 \to 1) \approx 1$.

(ii) **A BASE FOR $\{\mathbf{A}2\}$:**
 (a) $0 \to (0 \to 1) \approx 1$;
 (b) $(0 \to 1) \to 1 \approx 0 \to 1$;
 (c) $(0 \to 1)^* \approx 0$.

(iii) **A BASE FOR $\{\mathbf{A}3\}$:**
 (a) $x \to (x \to (x \to y)) \approx x \to (x \to y)$.

(iv) **A BASE FOR $\{\mathbf{A}4\}$:**
 (a) $((0 \to 1) \to 1)' \approx 0 \to (0 \to 1)$.

(v) **A BASE FOR $\{\mathbf{A}5\}$:**
 (a) $(0 \to 1)^* \to 1 \approx 1$.

(2) **BASES FOR PAIRS OF ALGEBRAS :**

(i) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}2\}$:**
 (a) $(0 \to 1)^* \approx 0$;
 (b) $(0 \to (0 \to 1) \approx 1$.

(ii) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}3\}$:**
 (a) $(0 \to 1)^* \approx 0$;
 (b) $(0 \to 1) \to 1 \approx 1$.

(iii) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}4\}$:**

(a) $(0 \to 1)^* \approx 0$;
(b) $(0 \to (0 \to 1)) \approx ((0 \to 1) \to 1)$.

(iv) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}5\}$:**
(a) $(0 \to (0 \to 1)) \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$;
(b) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$;
(c) $(0 \to 1)^* \to ((0 \to 1) \to 1) \approx (0 \to 1)$.

(v) **A BASE FOR $\{\mathbf{A}2, \mathbf{A}3\}$:**
(a) $(0 \to 1)^* \approx 0$;
(b) $(0 \to (0 \to 1)) \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$;
(c) $(a \to b) \to a \approx b$.

(vi) **A BASE FOR $\{\mathbf{A}2, \mathbf{A}4\}$:**
(a) $(0 \to 1)^* \approx 0$;
(b) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$;
(c) $(a \to b) \to a \approx b$.

(vii) **A BASE FOR $\{\mathbf{A}2, \mathbf{A}5\}$:**
(a) $(0 \to 1) \to (0 \to (0 \to 1)) \approx 0 \to 1$;
(b) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$;
(c) $(a \to b) \to a \approx b$.

(viii) **A BASE FOR $\{\mathbf{A}3, \mathbf{A}4\}$:**
(a) $(0 \to 1)^* \approx 0$;
(b) $(0 \to (0 \to 1) \approx 0 \to 1$;
(c) $(a \to b) \to a \approx b$.

(ix) **A BASE FOR $\{\mathbf{A}3, \mathbf{A}5\}$**
(a) $(0 \to (0 \to 1)) \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$;
(b) $(0 \to 1)^* \to ((0 \to 1) \to 1) \approx (0 \to 1)$;
(c) $(a \to b) \to a \approx b$.

(x) **A BASE FOR $\{\mathbf{A}4, \mathbf{A}5\}$**
(a) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$;
(b) $(0 \to 1)^* \to ((0 \to 1) \to 1) \approx (0 \to 1)$;
(c) $(a \to b) \to a \approx b$.

(3) **BASES FOR TRIPLES OF ALGEBRAS :**
(i) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3\}$:**
(a) $(0 \to 1)^* \approx 0$;
(b) $(0 \to (0 \to 1)) \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$.

(ii) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}4\}$:**
(a) $(0 \to 1)^* \approx 0$;

(b) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$.

(iii) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}5\}$:**
(a) $(0 \to (0 \to 1)) \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$;
(b) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$.

(iv) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}3, \mathbf{A}4\}$:**
(a) $(0 \to 1)^* \approx 0$;
(b) $0 \to (0 \to (0 \to 1) \approx 0 \to 1$;
(c) $(0 \to 1)^* \to ((0 \to 1) \to 1) \approx (0 \to 1)$.

(v) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}3, \mathbf{A}5\}$:**
(a) $(0 \to 1)^* \to ((0 \to 1) \to 1) \approx (0 \to 1)$;
(b) $(0 \to (0 \to 1)) \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$.

(vi) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}4, \mathbf{A}5\}$:**
(a) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$;
(b) $(0 \to 1)^* \to ((0 \to 1) \to 1) \approx (0 \to 1)$.

(vii) **A BASE FOR $\{\mathbf{A}2, \mathbf{A}3, \mathbf{A}4\}$:**
(a) $(0 \to 1)^* \approx 0$;
(b) $(a \to b) \to a \approx b$.

(viii) **A BASE FOR $\{\mathbf{A}2, \mathbf{A}3, \mathbf{A}5\}$:**
(a) $(0 \to (0 \to 1)) \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$;
(b) $(a \to b) \to a \approx b$.

(ix) **A BASE FOR $\{\mathbf{A}2, \mathbf{A}4, \mathbf{A}5\}$:**
(a) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$;
(b) $(a \to b) \to a \approx b$.

(x) **A BASE FOR $\{\mathbf{A}3, \mathbf{A}4, \mathbf{A}5\}$:**
(a) $(0 \to 1)^* \to ((0 \to 1) \to 1) \approx (0 \to 1)$;
(b) $(a \to b) \to a \approx b$.

(4) **BASES FOR QUADRUPLES OF ALGEBRAS :**

(i) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3, \mathbf{A}4\}$:**
(a) $(0 \to 1)^* \approx 0$.

(ii) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3, \mathbf{A}5\}$:**
(a) $(0 \to (0 \to 1)) \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$.

(iii) **A BASE FOR $\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}4, \mathbf{A}5\}$:**

(a) $(0 \to (0 \to 1)) \geq ((0 \to 1) \to 1)$.

(iv) **A BASE FOR** $\{\mathbf{A}1, \mathbf{A}3, \mathbf{A}4, \mathbf{A}5\}$**:**
(a) $(0 \to 1)^* \to ((0 \to 1) \to 1) \approx (0 \to 1)$.

(v) **A BASE FOR** $\{\mathbf{A}2, \mathbf{A}3, \mathbf{A}4, \mathbf{A}5\}$**:**
(a) $(a \to b) \to a \approx b$.

Proof We illustrate the proofs by proving (1)(i): Routine verification shows that the algebra $\mathbf{A}1$ satisfies (a) and (b), while both (a) and (b) fail in the remaining four algebras , proving that (a) and (b) form an (equational) base for $\mathbb{V}(\mathbf{A})$. Similar arguments apply for the remaining parts of the theorem. □

Corollary 5.3. *Let* $\mathbb{A} \in \mathbb{RUNO}1$ *such that* $\mathbb{A} \models (0 \to 1) \to 1 \approx 1$. *Then* $\mathbb{A} \models (0 \to 1)^* \approx 0$.

Proof Let $\mathbf{A}$ be as in the hypothesis. Then it follows from the preceding theorem that $\mathbf{A} \in \mathbb{V}(\mathbf{A1}, \mathbf{A3})$. Observe that the algebras $\mathbf{A}1$ and $\mathbf{A}3$ satisfy the conclusion. So, the conclusion holds in $\mathbf{A}$. □

6. More Identities

We will give several more consequences to Theorem 4.1 in the rest of this section. The following corollaries to Theorem 4.1 (or to Corollary 4.3) are immediate. Recall that $a := 0 \to 1$, $b := 0 \to a$. Also, let $c := 0 \to b$.

Corollary 6.1.
(1) $\mathbb{RUNO}1 \models x^* \vee x^{**} \approx 1$.
(2) $\mathbb{RUNO}1 \models x'^{*'*'*} \approx x'^*$.
(3) $\mathbb{RUNO}1 \models x^{*'*'*'} \approx x^{*'}$.
(4) $\mathbb{RUNO}1 \models [((x \to y) \to y) \to y] \to y \approx (x \to y) \to y$.
(5) $\mathbb{RUNO}1 \models x \to [x \to (x \to (x \to y))] \approx x \to (x \to y)$.
(6) $\mathbb{RUNO}1 \models 0 \to (0 \to (0 \to 1)) \approx (0 \to 1)$.
(7) $\mathbb{RUNO}1 \models (0 \to (0 \to 1))^* \approx 0$.
(8) $\mathbb{RUNO}1 \models (0 \to 1)^{*'} \to (0 \to 1) \approx 0 \to 1$.
(9) $\mathbb{RUNO}1 \models x^* \vee (y \to z) \approx (x^* \vee y) \to (x^* \vee z)$ *(JID*)*. .
(10) $\mathbb{RUNO}1 \models ((0 \to 1) \to 1) \to 1 \approx (0 \to 1) \to 1$.
(11) $\mathbb{RUNO}1 \models c \approx a$.
(12) $\mathbb{RUNO}1 \models 0 \to 1 \ \leq \ 0 \to (0 \to 1)$.
(13) $\mathbb{RUNO}1 \models x^* \to (y^* \to z^*) \approx y^* \to (x^* \to z^*)$ *(Weak Exchange)*.
(14) $\mathbb{RUNO}1 \models (0 \to 1) \to (0 \to 1)^* \approx 0$.

Corollary 6.2.
(a) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models (0 \to 1)^* \approx 0$.
(b) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models x^{*'} \approx x^{**}$.
(c) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models [(0 \to 1) \to 1]^{**} \approx 1$.

(d) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models (0 \to 1) \to [0 \to (0 \to 1)]^* \approx [0 \to (0 \to 1)]^*$.
(e) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models [0 \to (0 \to 1)]^* \to (0 \to 1)^{**} \approx 0 \to 1$.
(f) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models x'^{*\prime *} \approx x'^{*}$.
(g) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models (0 \to 1) \to (0 \to 1)^* \approx (0 \to 1)^*$.
(h) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models (0 \to 1)^{*\prime *} \approx (0 \to 1)^*$.
(i) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models [0 \to (0 \to (0 \to 1))]'^{*} \to (0 \to (0 \to 1)) \approx 0 \to 1$.
(j) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}) \models 0 \to (0 \to 1)^* \approx 1$.
(k) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A5}) \models [0 \to (0 \to 1)] \to (0 \to 1)^{**} \approx (0 \to 1)^{**}$.
(l) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A4}, \mathbf{A5}) \models [0 \to (0 \to 1)] \geq [(0 \to 1) \to 1]$.
(m) $\mathbb{V}(\mathbf{A1}, \mathbf{A3}, \mathbf{A4}, \mathbf{A5}) \models (0 \to 1)^* \to [(0 \to 1) \to 1] \approx 0 \to 1$.
(n) $\mathbb{V}(\mathbf{A2}, \mathbf{A3}, \mathbf{A4}, \mathbf{A5}) \models (a \to b) \to b \approx a$.
(o) $\mathbb{V}(\mathbf{A2}, \mathbf{A3}, \mathbf{A4}, \mathbf{A5}) \models (a \to b) \approx b \to a$.
(p) $\mathbb{V}(\mathbf{A2}, \mathbf{A3}, \mathbf{A4}, \mathbf{A5}) \models (a \to b) \to a \approx b$.

Corollary 6.3.

(a) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A5}) \models 0 \to (0 \to 1) \approx 1$.
(b) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A5}) \models x \to [x \to (x \to y)] \approx x \to y$.
(c) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A5}) \models 0 \to (0 \to 1) \approx 1$.
(d) $\mathbb{V}(\mathbf{A1}, \mathbf{A3}, \mathbf{A4}) \models (a \to b) \to a \approx a$.
(e) $\mathbb{V}(\mathbf{A1}, \mathbf{A3}, \mathbf{A4}) \models [0 \to (0 \to 1)] \leq [(0 \to 1) \to 1]$.
(f) $\mathbb{V}(\mathbf{A1}, \mathbf{A3}, \mathbf{A4}) \models (0 \to 1) \to [0 \to (0 \to 1)] \approx 1$.
(g) $\mathbb{V}(\mathbf{A2}, \mathbf{A4}, \mathbf{A5}) \models (0 \to 1) \to 1 \approx 0 \to 1$.
(h) $\mathbb{V}(\mathbf{A2}, \mathbf{A4}, \mathbf{A5}) \models [(x \to y) \to y] \to y \approx x \to y$.
(i) $\mathbb{V}(\mathbf{A2}, \mathbf{A4}, \mathbf{A5}) \models 0 \to [0 \to (0 \to 1)] \approx (0 \to 1) \to 1$.
(j) $\mathbb{V}(\mathbf{A2}, \mathbf{A4}, \mathbf{A5}) \models 0 \to [0 \to (0 \to 1)] \approx [(0 \to 1) \to 1] \to 1$.
(k) $\mathbb{V}(\mathbf{A3}, \mathbf{A4}, \mathbf{A5}) \models (0 \to 1)^* \to (0 \to 1) \approx 0 \to 1$.
(l) $\mathbb{V}(\mathbf{A3}, \mathbf{A4}, \mathbf{A5}) \models (0 \to 1)^* \to [0 \to (0 \to 1)] \approx 0 \to (0 \to 1)$.

Corollary 6.4.

(1) $\mathbb{V}(\mathbf{A1}, \mathbf{A3}) \models (0 \to 1) \to 1 \approx 1$.
(2) $\mathbb{V}(\mathbf{A1}, \mathbf{A4}) \models [0 \to (0 \to 1)] \approx (0 \to 1) \to 1$.
(3) $\mathbb{V}(\mathbf{A2}, \mathbf{A5}) \models (0 \to 1) \to [0 \to (0 \to 1)] \approx 0 \to 1$.
(4) $\mathbb{V}(\mathbf{A3}, \mathbf{A4}) \models 0 \to (0 \to 1) \approx 0 \to 1$.

Corollary 6.5.

(a) $\mathbb{V}(\mathbf{A1}) \models (0 \to 1) \to [0 \to (0 \to 1)] \approx 0 \to (0 \to 1)$.
(b) $\mathbb{V}(\mathbf{A3}) \models (0 \to 1) \to 1 \approx 1$.
(c) $\mathbb{V}(\mathbf{A3}) \models 0 \to (0 \to 1) \approx 0 \to 1$.
(d) $\mathbb{V}(\mathbf{A5}) \models (0 \to 1)^{*\prime} \approx (0 \to 1)^*$.
(e) $\mathbb{V}(\mathbf{A5}) \models (0 \to 1)^{**} \approx (0 \to 1)$.
(f) $\mathbb{V}(\mathbf{A5}) \models 0 \to [0 \to (0 \to 1)] \approx 0 \to (0 \to 1)^*$.
(g) $\mathbb{V}(\mathbf{A5}) \models (0 \to 1)^* \to 1 \approx 1$.
(h) $\mathbb{V}(\mathbf{A5}) \models x \to (y \to z) \approx y \to (x \to z)$ *(Exchange)*.
(i) $\mathbb{V}(\mathbf{A5}) \models x' \vee (y \to z) \approx (x' \vee y) \to (x' \vee z)$ *(JID)*.
(j) $\mathbb{V}(\mathbf{A5}) \models [(0 \to 1) \to 1]^{**} \approx 0 \to 1$.

(k) $\mathbb{V}(\mathbf{A5}) \models (0 \to 1) \vee (0 \to 1)^* \approx 1$.

Remark 6.6. *Each of the equations in Corollary 6.5 gives a 1-base for the variety generated by the corresponding algebra.*

7. Toward a logic for the variety $\mathbb{RUNO}1$

In this section we introduce and investigate a propositional logic, in Hilbert-style, called $\mathcal{RUNO}1$. It will be shown that $\mathcal{RUNO}1$ is implicative in the sense of Rasiowa, and hence is algebraizable (in the sense of Blok and Pigozzi) with the variety $\mathbb{RUNO}1$ of regular unorthodox algebras of level 1 as its equivalent algebraic semantics.

7.1. **Abstract Algebraic Logic: Basic definitions and results.**

In this subsection, we recall the basic definitions and results of Abstract Algebraic Logic that will play a crucial role.

- Let us fix the language $\mathbf{L} = \langle \vee, \wedge, \to, \sim, \bot, \top \rangle$.
- The set of **formulas** in $\mathbf{L}$ will be denoted by $Fm_{\mathbf{L}}$.
- The set of formulas $Fm_{\mathbf{L}}$ can be turned into an algebra of formulas, denoted by $\mathbf{Fm_L}$, in the usual way.
- Γ denotes a set of formulas and lower case Greek letters denote single formulas.
- The homomorphisms from the formula algebra $\mathbf{Fm_L}$ into an $\mathbf{L}$-algebra (i.e, an algebra of type $\mathbf{L}$) $\mathbf{A}$ are called **interpretations** (or **valuations**) in A. The set of all such interpretations is denoted by $Hom(\mathbf{Fm_L}, \mathbf{A})$.
- If $h \in Hom(\mathbf{Fm_L}, \mathbf{A})$ then the *interpretation of a formula* α under h is its image $h(\alpha) \in A$.
- $h[\Gamma]$ denotes the set $\{h\phi \mid \phi \in \Gamma\}$.

Consequence Relations and Logics:

- A **consequence relation** on $Fm_{\mathbf{L}}$ is a binary relation $\vdash$ between sets of formulas and formulas that satisfies the following conditions:
  for all Γ, $\Delta \subseteq Fm_{\mathbf{L}}$ and $\phi \in Fm_{\mathbf{L}}$:
  (i) $\phi \in \Gamma$ implies $\Gamma \vdash \phi$,
  (ii) $\Gamma \vdash \phi$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash \phi$,
  (iii) $\Gamma \vdash \phi$ and $\Delta \vdash \beta$ for every $\beta \in \Gamma$ imply $\Delta \vdash \phi$.

- A consequence relation $\vdash$ is **finitary** if $\Gamma \vdash \phi$ implies $\Gamma' \vdash \phi$ for some finite $\Gamma' \subseteq \Gamma$.

- A consequence relation $\vdash$ is **structural** if $\Gamma \vdash \phi$ implies $\sigma[\Gamma] \vdash \sigma(\phi)$ for every substitution $\sigma \in Hom(\mathbf{Fm_L}, \mathbf{Fm_L})$. ($\sigma[\Gamma] := \{\sigma(\alpha) : \alpha \in \Gamma\}$).

- A **logic** (or **deductive system**) is a pair $\mathcal{L} := \langle \mathbf{L}, \vdash_{\mathcal{L}} \rangle$, where $\mathbf{L}$ is a propositional language and $\vdash_{\mathcal{L}}$ is a finitary, structural consequence relation on $\mathrm{Fm}_{\mathbf{L}}$.
- **Logics in a Hilbert-style**:
  One way to present a logic $\mathcal{L}$ is by displaying it (syntactically) in **Hilbert-style**; that is, giving its axioms and rules of inference which induce a consequence relation $\vdash_{\mathcal{L}}$ as follows:
  $\Gamma \vdash_{\mathcal{L}} \phi$ if there is a **(formal) proof** (or, a **derivation**) of ϕ from Γ, where a proof is defined as a sequence of formulas $\phi_1, \dots, \phi_n$, $n \in \mathbb{N}$, such that $\phi_n = \phi$, and for every $i \leq n$, one of the following conditions holds:
  (i) $\phi_i \in \Gamma$,
  (ii) there is an axiom ψ and a substitution σ such that $\phi_i = \sigma\psi$,
  (iii) there is a rule $\langle \Delta, \psi \rangle$ and a substitution σ such that $\phi_i = \sigma\psi$ and $\sigma[\Delta] \subseteq \{\phi_j : j < i\}$.
- Identities in $\mathbf{L}$ are ordered pairs of $\mathbf{L}$-formulas that will be written in the form $\alpha \approx \beta$.
- If $\bar{x}$ is a sequence of variables and h is an interpretation in $\mathbf{A}$, then we write $\bar{a}$ for $h(\bar{x})$.
- $\mathbf{A} \models \alpha \approx \beta$ if and only if $\alpha^{\mathbf{A}}(\bar{a}) = \beta^{\mathbf{A}}(\bar{a})$, for all $\bar{a} \in \mathbf{A}$.
- For a class $\mathbb{K}$ of algebras of type $\mathbf{L}$, $\mathbb{K} \models \alpha \approx \beta$ if and only if $\mathbf{A} \models \alpha \approx \beta$, for all $\mathbf{A} \in \mathbb{K}$.
- We define the relation $\models_{\mathbb{K}}$ between a set Γ of identities and an identity $\alpha \approx \beta$ as follows:
  $\Gamma \models_{\mathbb{K}} \alpha \approx \beta$ if and only if for every $\mathbf{A} \in \mathbb{K}$ and for every interpretation $\bar{a}$ of the variables of $\Gamma \cup \{\alpha \approx \beta\}$ in $\mathbf{A}$,
  if $\phi^{\mathbf{A}}(\bar{a}) = \psi^{\mathbf{A}}(\bar{a})$, for every $\phi \approx \psi \in \Gamma$, then
  $\alpha^{\mathbf{A}}(\bar{a}) = \beta^{\mathbf{A}}(\bar{a})$.

- The relation $\models_{\mathbb{K}}$ is called the **(semantic) equational consequence relation determined by** $\mathbb{K}$.
- In this case, we say that $\alpha \approx \beta$ is a $\mathbb{K}$-consequence of Γ.

7.2. Algebraizability of a logic.

Algebraic Semantics for a Logic:

Let $\langle L, \vdash_L \rangle$ be a logic and K a class of L-algebras. K is called an **algebraic semantics** for $\langle L, \vdash_L \rangle$ if $\vdash_L$ can be interpreted in $\vdash_K$ in the following sense:

There exists a finite set $\delta_i(p) \approx \epsilon_i(p)$, for $i < n$, of identities with a single variable p such that, for all $\Gamma \cup \phi \subseteq Fm$ and each $j < n$,

$$\Gamma \vdash_L \phi \Leftrightarrow \{\delta_i[\psi/p] \approx \epsilon_i[\psi/p], i < n, \psi \in \Gamma\} \models_K \delta_j[\phi/p] \approx \epsilon_j[\phi/p],$$

where $\delta[\psi/p]$ denotes the formula obtained by the substitution of ψ at every occurrence of p in δ.

The identities $\delta_i \approx \epsilon_i$, for $i < n$, are called **defining identities** for $\langle L, \vdash_L \rangle$ and $\mathbb{K}$.

Equivalent Algebraic Semantics and Algebraizable Logic:

Definition 7.1. *Let $\mathcal{S}$ be a logic over a language* **L** *and $\mathbb{K}$ an algebraic semantics of S with defining equations $\delta_i(p) \approx \epsilon_i(p)$, $i \leq n$. Then,* **K** *is an* **equivalent algebraic semantics** *of* **S** *if there exists a* **finite** *set $\{\Delta_j(p,q) : j \leq m\}$ of formulas in two variables satisfying the condition:*

For every $\phi \approx \psi \in Eq_{\mathbf{L}}$,
$\phi \approx \psi \models_K \{\delta_i(\Delta_j(\phi,\psi)) \approx \epsilon_i(\Delta_j(\phi,\psi)) : i \leq n, j \leq m\}$ *and*
$\{\delta_i(\Delta_j(\phi,\psi)) \approx \epsilon_i(\Delta_j(\phi,\psi)) : i \leq n, j \leq m\} \models_K \phi \approx \psi$.

The set $\{\Delta_j(p,q) : j \leq m\}$ is called an **equivalence system**.

A logic is **algebraizable (in the sense of Blok and Pigozzi)** *if it has an equivalent algebraic semantics.*

7.3. Algebraizabilty of Implicative Logics. :

The following definition is well known (see [32, page 179]).

Definition 7.2. [32] *Let $\mathcal{L}$ be a logic in a language that includes a binary connective $\to$, either primitive or defined by a term in exactly two variables. Then $\mathcal{L}$ is called an implicative logic with respect to the binary connective $\to$ if the following conditions are satisfied:*

(IL1) $\vdash_{\mathcal{L}} \alpha \to \alpha$.
(IL2) $\alpha \to \beta, \beta \to \gamma \vdash_{\mathcal{L}} \alpha \to \gamma$.
(IL3) *For each connective f in the language of arity $n > 0$,*
$$\left\{ \begin{array}{c} \alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n, \\ \beta_1 \to \alpha_1, \ldots, \beta_n \to \alpha_n \end{array} \right\} \vdash_{\mathcal{L}} f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n).$$
(IL4) $\alpha, \alpha \to \beta \vdash_{\mathcal{L}} \beta$.
(IL5) $\alpha \vdash_{\mathcal{L}} \beta \to \alpha$.

Definition 7.3. *(Rasiowa) Let $\mathcal{L}$ be an implicative logic in* $\mathbf{L}$ *with* $\to$*.*
An $\mathcal{L}$**-algebra** *is an algebra* $\mathbf{A}$ *in the language* $\mathbf{L}$ *that has an element* 1 *with the following properties:*

(LALG1) *For all* $\Gamma \cup \{\phi\} \subseteq Fm$ *and all* $h \in Hom(\mathbf{Fm_L}, \mathbf{A})$*,*
if $\Gamma \vdash_{\mathcal{L}} \phi$ *and* $h\Gamma \subseteq \{1\}$ *then* $h\phi = 1$*,*

(LALG2) *For all* $a, b \in A$*, if* $a \to b = 1$ *and* $b \to a = 1$ *then* $a = b$*.*

The class of $\mathcal{L}$*-algebras is denoted by* $Alg^*(\mathcal{L})$*.*

Theorem 7.4. *(The algebraizability theorem of Blok and Pigozzi)*
Let $\mathcal{L}$ *be an implicative logic and* $\mathbb{K} = Alg^*(L)$*. For all* $\Gamma \cup \{\phi\} \subseteq Fm$*, and for all* $\Sigma \cup \{\epsilon \approx \delta\} \subseteq \mathbf{Id}$*, the following properties hold:*

(ALG1) $\Gamma \vdash_{\mathcal{L}} \phi$ *if and only if* $\{\alpha \approx 1 : \alpha \in \Gamma\} \models_{Alg^*(\mathcal{L})} \phi \approx 1$*.*

(ALG2) $\Sigma \models_K \epsilon \approx \delta$ *if and only if*
$$\{\alpha \to \beta, \beta \to \alpha : \alpha \approx \beta \in \Sigma\} \vdash_{\mathcal{L}} \{\epsilon \to \delta, \delta \to \epsilon\}.$$

(ALG3) $\phi \dashv\vdash_{\mathcal{L}} \{\phi \to 1, 1 \to \phi\}$*.*

(ALG4) $\epsilon \approx \delta \models_{\mathbb{K}} \{\epsilon \to \delta \approx 1, \delta \to \epsilon \approx 1\}$ *and*
$$\{\epsilon \to \delta \approx 1, \delta \to \epsilon \approx 1\} \models_{\mathbb{K}} \epsilon \approx \delta.$$

Remark 7.5. *(ALG2) and (ALG3) are derivable from (ALG1) and (ALG4) and conversely.*

The following theorem, which is a restatement of the preceding theorem, provides a "translation algorithm" to go back and forth between an implicative logic $\mathcal{L}$ and $Alg^*(\mathcal{L})$.

Theorem 7.6. *(BP89), (Fo16) Every implicative logic* $\mathcal{L}$ *is algebraizable (in the sense of Blok and Pigozzi) with respect to the class* $Alg^*(\mathcal{L})$ *and the algebraizability is witnessed by the set of defining identities* $E(x) = \{x \approx x \to x\}$ *and the set of equivalence formulas* $\Delta(x, y) = \{x \to y, y \to x\}$*.*

Axiomatic Extensions of Algebraizable logics:

A logic $\mathcal{L}'$ is an **axiomatic extension** of $\mathcal{L}$ if $\mathcal{L}$ is obtained by adding new axioms but keeping the rules of inference the same as in $\mathcal{L}$.
Let $Ext(\mathcal{L})$ denote the lattice of axiomatic extensions of a logic $\mathcal{L}$ and $\mathbf{L_V}(\mathbb{V})$ denote the lattice of subvarieties of a variety $\mathbb{V}$ of algebras.

The following theorems due to Blok and Pigozzi [7] are useful here.

Theorem 7.7. [7] , [25, Theorem 3.33] *Let* $\mathcal{L}$ *be an algebraizable logic whose equivalent algebraic semantics* $\mathbb{V}$ *is a variety. Then* $Ext(\mathcal{L})$ *is dually isomorphic to* $\mathbf{L_V}(\mathbb{V})$*.*

Theorem 7.8. *(Blok and Pigozzi 1989) Let* $\mathcal{L}$ *be an algebraizable logic and* $\mathcal{L}'$ *be an axiomatic extension of L. Then* $\mathcal{L}'$ *is also algebraizable with the same transformers* τ *and* ρ *of* $\mathcal{L}$*.*

8. A logic for the variety $\mathbb{RUNO}1$

In 2022, [16] introduced, among others, the logic $\mathcal{DMSH}$ which arose in an attempt to "logicize" the variety of De Morgan semi-Heyting algebras.

This logic will play a crucial role in the rest of this paper. We will now recall this logic from [16].

8.1. **De Morgan Semi-Heyting Logic.**

The logic $\mathcal{DMSH}$:

LANGUAGE: $\langle \vee, \wedge, \to, \sim, \bot, \top \rangle$

Let $\alpha \to_H \beta := \alpha \to (\alpha \wedge \beta)$ and $\alpha \leftrightarrow_H \beta := (\alpha \to_H \beta) \wedge (\beta \to_H \alpha)$.

AXIOMS:
(a) SEMI-HEYTING LOGIC:

1 $\alpha \to_H (\alpha \vee \beta)$,
2 $\beta \to_H (\alpha \vee \beta)$,
3 $(\alpha \to_H \gamma) \to_H [(\beta \to_H \gamma) \to_H (\alpha \vee \beta) \to_H \gamma)]$,
4 $(\alpha \wedge \beta) \to_H \alpha$,
5 $(\gamma \to_H \alpha) \to_H [(\gamma \to_H \beta) \to_H (\gamma \to_H (\alpha \wedge \beta))]$,
6 $\top$,
7 $\bot \to_H \alpha$,
8 $[(\alpha \wedge \beta) \to_H \gamma] \to_H [\alpha \to_H (\beta \to_H \gamma)]$,
9 $[\alpha \to_H (\beta \to_H \gamma)] \to_H [(\alpha \wedge \beta) \to_H \gamma]$,
10 $(\alpha \to_H \beta) \to_H [(\beta \to_H \alpha) \to_H ((\alpha \to \gamma) \to_H (\beta \to \gamma))]$,
11 $(\alpha \to_H \beta) \to_H [(\beta \to_H \alpha) \to_H ((\gamma \to \beta) \to_H (\gamma \to \alpha))]$.

(b) De Morgan Axioms

12 $\top \leftrightarrow_H \sim \bot$,
13 $\sim (\alpha \wedge \beta) \leftrightarrow_H (\sim \alpha \vee \sim \beta)$.
14 $\sim (\alpha \vee \beta) \leftrightarrow_H (\sim \alpha \wedge \sim \beta)$,
15 $(\sim\sim \alpha) \leftrightarrow_H \alpha$.

RULES OF INFERENCE:

(SMP) From ϕ and $\phi \to_H \gamma$, deduce γ (semi-Modus Ponens),

(SCP) From $\phi \to_H \gamma$, deduce $\sim \gamma \to_H \sim \phi$ (semi-Contraposition rule).

The following theorem was proved in [16].

Theorem 8.1. *The logic $\mathcal{DMSH}$ is an implicative logic with respect to the connective $\to_H$.*

From Theorem 7.6 and Theorem 8.1, we have the following corollary which was already mentined in [16].

Corollary 8.2. *The logic $\mathcal{DMSH}$ is algebraizable and the variety $\mathbb{DMSH}$ is the equivalent algebraic semantics for $\mathcal{DMSH}$ with the the two transformers τ and ρ given by $\tau(x) = E(x) = \{x \approx x \to_H x\} = \{x \approx 1\}$ and $\rho(x, y) = \Delta(x, y) = \{x \to_H y, y \to_H x\} = \{x \leftrightarrow_H y\}$.*

8.2. **The logic $\mathcal{RUNO}1$.**

The logic $\mathcal{RUNO}1$ will be defined as an axiomatic extension of the logic $\mathcal{DMSH}$, obtained by adding the following additional axioms:
Define $\neg$ by: $\neg\alpha := \alpha \to 0$.

(Additional) AXIOMS:

16 $\sim (0 \to 1) \leftrightarrow_H (0 \to 1)$ (Unorthodoxy),
17 $(\alpha \wedge \sim \neg \sim \alpha) \vee (\beta \vee \neg\beta) \leftrightarrow_H (\beta \vee \neg\beta)$ (Regularity),
18 $\neg \sim (\alpha \wedge \neg \sim \alpha) \leftrightarrow_H \alpha \wedge \neg \sim \alpha$ (Level 1).

Since we know the logic $\mathcal{RUNO}1$ is an axiomatic extension of $\mathcal{DMSH}$, we have the following consequence of Corollary 8.2.

Corollary 8.3. *The logic $\mathcal{RUNO}1$ is algebraizable, and the variety $\mathbb{RUNO}1$ is the equivalent algebraic semantics for $\mathcal{RUNO}1$, with the $E(x)$ and $\Delta(x, y)$ given by $E(x) := \{x \approx x \to_H x\} = \{x \approx 1\}$ and $\Delta(x, y) := \{x \to_H y, y \to_H x\} = \{x \leftrightarrow_H y\}$.*

We have the following important consequence of Corollary 8.3, Theorem 7.7 and Theorem 7.8.

Theorem 8.4.

(a) *The lattice $Ext(\mathcal{RUNO}1)$ of axiomatic extensions of the logic $\mathcal{RUNO}1$ is dually isomorphic to the lattice $\mathbf{L_V}(\mathbb{RUNO}\mathbf{1})$ of subvarieties of the variety $\mathbb{RUNO}1$.*
(b) *Every axiomatic extension of $\mathcal{RUNO}1$ is also algebraizable with its corresponding variety as its equivalent algebraic semantics.*

The above theorem justifies the use of the phrase "the logic corresponding to a subvariety $\mathbb{V}$ of $\mathbb{SH}$.

Let $\mathbf{Mod}(\mathcal{E}) := \{\mathbf{A} \in \mathbb{SH} : \mathbf{A} \models \delta \approx 1, \text{ for every } \delta \in \mathcal{E}\}$.

The following theorem is an immediate consequence of Corollary 8.3 and plays an important role in the next section.

Theorem 8.5. *Let $\mathcal{E}$ be an axiomatic extension of the logic $\mathcal{SH}$. Then*
(a) *$\mathcal{E}$ is also algebraizable with the same equivalence formulas and defining equations as those of the logic $\mathcal{SH}$.*
(b) *$\mathbf{Mod}(\mathcal{E})$ is an equivalent algebraic semantics of $\mathcal{E}$.*

9. Unorthodox Logics: Axiomatic Extensions of the logic $\mathcal{RUNO}1$

Let us call the axiomatic extensions of the logic $\mathcal{RUNO}1$ as "unorthodox logics". In this section, we provide axiomatizations (i.e., bases) for these unorthodox logics. The presentation of the bases is done according to the levels mentioned in Corollary 5.1 of the lattice of subvarieties of $\mathbb{RUNO}1$.

In the following theorem, we let $\alpha := \bot \to \top$, $\beta := \bot \to \alpha$ and $\gamma := \bot \to \beta$

Theorem 9.1.

(1) **BASES FOR LOGICS ASSOCIATED WITH SINGLE ALGEBRAS**
  (i) **A BASE FOR the logic $\mathcal{A}1$ of $\mathbb{V}(\mathbf{A1})$:**
   (a) $\alpha \to \top$,
   (b) β.
  (ii) **A BASE FOR the logic $\mathcal{A}2$ of $\mathbb{V}(\mathbf{A2})$:**
   (a) β
   (b) $\alpha \to \top \leftrightarrow_H \alpha$.
   (c) $\neg\alpha \leftrightarrow_H \bot$
  (iii) **A BASE FOR the logic $\mathcal{A}3$ of $\mathbb{V}(\mathbf{A3})$:**
   (a) $\phi \to (\phi \to (\phi \to \psi)) \leftrightarrow_H \phi \to (\phi \to \psi)$.
  (iv) **A BASE FOR the logic $\mathcal{A}4$ of $\mathbb{V}(\mathbf{A4})$:**
   $\sim (\alpha \to \top) \ \leftrightarrow_H \ \beta$.
  (v) **A BASE FOR the logic $\mathcal{A}5$ of $\mathbb{V}(\mathbf{A5})$:**
   $\neg\alpha \to \top$.

(2) **BASES FOR LOGICS OF PAIRS OF ALGEBRAS**
  (i) **A BASE FOR the logic of $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}2\})$:**
   (a) $\neg\alpha \leftrightarrow_H \bot$,
   (b) β.
  (ii) **A BASE FOR the logic of $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}3\})$:**
   (a) $\neg\alpha \leftrightarrow_H \bot$,
  (iii) **A BASE FOR the logic of $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}4\})$:**
   (a) $\neg\alpha \leftrightarrow_H \bot$,
   (b) $\beta \leftrightarrow_H \alpha \to \top$,
  (iv) **A BASE FOR the logic of $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}5\})$:**
   (a) $\beta \to \neg\neg\alpha \ \leftrightarrow_H \ \neg\neg\alpha$,
   (b) $\beta \ \geq \ \alpha \to \top$,
   (c) $\neg\alpha \to (\alpha \to \top) \leftrightarrow_H \alpha$.

(v) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}2, \mathbf{A}3\})$**:**

(a) $\neg\alpha \leftrightarrow_H \bot$,

(b) $\beta \to \neg\neg\alpha \ \leftrightarrow_H \ \neg\neg\alpha$,

(c) $(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(vi) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}2, \mathbf{A}4\})$ **:**

(a) $\neg\alpha \leftrightarrow_H \bot$,

(b) $\beta \ \geq \ \alpha \to \top$,

(c) $(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(vii) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}2, \mathbf{A}5\}$ **:**

(a) $\alpha \to \beta \leftrightarrow_H \alpha$,

(b) $\beta \ \geq \ \alpha \to \top$,

(c) $(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(viii) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}3, \mathbf{A}4\}$**:**

(a) $\neg\alpha \leftrightarrow_H \bot$,

(b) $\beta \leftrightarrow_H \alpha$,

(c) $(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(ix) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}3, \mathbf{A}5\}$**:**

(a) $\beta \to \neg\neg\alpha \leftrightarrow_H \neg\neg\alpha$,

(b) $\neg\alpha \to (\alpha \to \top) \leftrightarrow_H \alpha$,

(c) $(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(x) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}4, \mathbf{A}5\}$**:**

(a) $\beta \ \geq \ \alpha \to \top$,

(b) $\neg\alpha \to (\alpha \to \top) \leftrightarrow_H \alpha$,

(c) $(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(3) **BASES FOR LOGICS OF TRIPLES OF ALGEBRAS**

(i) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3\}$**:**

(a) $\neg\alpha \leftrightarrow_H \bot$,

(b) $\beta \to \neg\neg\alpha \leftrightarrow_H \neg\neg\alpha$.

(ii) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}4\}$**:**

(a) $\neg\alpha \leftrightarrow_H \bot$,

(b) $\beta \ \geq \ \alpha \to \top$.

(iii) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}5\}$**:**

(a) $\beta \to \neg\neg\alpha \leftrightarrow_H \neg\neg\alpha$,

(b) $\beta \ \geq \ \alpha \to \top$.

(iv) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}3, \mathbf{A}4\}$**:**

(a) $\neg\alpha \leftrightarrow_H \bot$;

(b) $\gamma \leftrightarrow_H \alpha$;

(c) $\neg\alpha \to (\alpha \to \top) \leftrightarrow_H \alpha$.

(v) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}3, \mathbf{A}5\}$**:**
(a) $\neg\alpha \to (\alpha \to \top) \leftrightarrow_H \alpha$,
(b) $\beta \to \neg\neg\alpha \leftrightarrow_H \neg\neg\alpha$.

(vi) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}4, \mathbf{A}5\}$**:**
(a) $\beta \ \geq \ \alpha \to \top$,
(b) $\neg\alpha \to (\alpha \to \top) \leftrightarrow_H \alpha$.

(vii) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}2, \mathbf{A}3, \mathbf{A}4\}$**:**
$\neg\alpha \leftrightarrow_H \bot$,
$(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(viii) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}2, \mathbf{A}3, \mathbf{A}5\}$**:**
(a) $\beta \to \neg\neg\alpha \leftrightarrow_H \neg\neg\alpha$,
(b) $(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(ix) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}2, \mathbf{A}4, \mathbf{A}5\}$**:**
$\beta \ \geq \ \alpha \to \top$,
$(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(x) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}3, \mathbf{A}4, \mathbf{A}5\}$**:**
(a) $\neg\alpha \to (\alpha \to \top) \leftrightarrow_H \alpha$,
(b) $(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

(4) **BASES FOR LOGICS OF QUADRUPLES OF ALGEBRAS***:*
(i) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3, \mathbf{A}4\}$**:**
$\neg\alpha \leftrightarrow_H \bot$,

(ii) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}3, \mathbf{A}5\}$**:**
$\beta \to \neg\neg\alpha \leftrightarrow_H \neg\neg\alpha$,

(iii) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}2, \mathbf{A}4, \mathbf{A}5\}$**:**
$\beta \ \geq \ \alpha \to \top$.

(iv) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}1, \mathbf{A}3, \mathbf{A}4, \mathbf{A}5\}$**:**
$\neg\alpha \to (\alpha \to \top) \leftrightarrow_H \alpha$.

(v) **A BASE FOR the logic of** $\mathbb{V}(\{\mathbf{A}2, \mathbf{A}3, \mathbf{A}4, \mathbf{A}5\}$**:**
$(\alpha \to \beta) \to \alpha \leftrightarrow_H \beta$.

Observe that all the logics mentioned in the preceding theorem are implicative since they are axiomatic extensions of $\mathcal{RUNO}1$. Hence the following corollary is immediate.

Corollary 9.2. *All the logics described in the above theorem are algebraizable.*

Corollary 9.3. *All the logics corresponing to the subvarieties of* $\mathbb{RUNO}1$ *are decidable.*

Using a slight generalization (see [19]) of Maximova's algebraic characterization of Disjunction Proprty, we can prove the following theorem.

Theorem 9.4. *Every consistent axiomatic extension of the logic* $\mathbb{RUNO}1$ *does not have the Disjunction Property.*

10. CONCLUDING REMARKS

In the preceding sections we saw:

(a) $\mathbb{V}(\mathbf{A1}, \mathbf{A2}, \mathbf{A3}, \mathbf{A4}, \mathbf{A5}) = \mathbb{RUNO}1$.
(b) $\mathbb{RUNO}1$ is a discriminator variety and its first-order theory is decidable.
(c) The algebras $\mathbf{Ai}$, $\mathbf{i} = 1, 2, \cdots, 5$ are primal.
(d) Each of the varieties $\mathbb{V}(\mathbf{Ai})$, $\mathbf{i} = 1, 2, \cdots, 5$ has the SAP and ES.
(e) The lattice of subvarieties of $\mathbb{RUNO}1$ is a Boolean lattice consisting of 32 subvarieties of $\mathbb{RUNO}1$.
(f) Equational axioms for all subvarieties of $\mathbb{RUNO}1$.
(g) The logic $\mathcal{RUNO}1$ for the variety $\mathbb{RUNO}1$.
(h) Axiomatizations for all axiomatic extensions of $\mathcal{RUNO}1$).

The items (b), (c) and (d) above imply that the algebras $\mathbf{Ai}$, $\mathbf{i} = 1, 2, \cdots, 5$ share some strong properties with the Boolean algebra $\mathbf{2}$, from which we can conclude that each of the algebras $\mathbf{Ai}$, $\mathbf{i} = 1, 2, \cdots, 5$ (and their logic) could be a formidable competitor to the Boolean algebra $\mathbf{2}$ in computing. Computer chips based on three-valued logic of any (or all in parallel) of $\mathbf{Ai}$, $\mathbf{i} = 1, 2, \cdots, 4$, or based on four-valued logic of $\mathbf{A5}$, could be more useful in future, since they would have more expressive power than the binary chips.

OPEN PROBLEMS: Here are some open problems for further research.

PROBLEM 1: Find a duality (Priestley-type or otherwise) for each of the subvarieties of $\mathbb{RUNO}1$.

PROBLEM 2: Let $\mathbb{RUNO}2$ denote the variety of regular unorthodox algebras of level 2 (that is, satisfying the identity: $(x \wedge x'^{*})'^{*'*} \approx (x \wedge x'^{*})'^{*}$). Characterize the simple algebras in $\mathbb{RUNO}2$.

PROBLEM 3: Characterize the simple algebras in the variety $\mathbb{UNO}1$.

PROBLEM 4: Call an algebra $\mathbf{A} \in \mathbb{DMSH}1$ **contra-Boolean** if $\mathbf{A} \models 0 \to 1 \approx 0$. Let $\mathbb{RCB}1$ denote the subvariety of $\mathbb{DMSH}1$ consisting of regular contra-Boolean algebras of level 1. Characterize the simple algebras in the variety $\mathbb{RCB}1$.

A Historical Observation:

The fact that the first appearance of non classical logics only occurred around the end of the 19th century and in the beginning of the 20th century–more than 2000 years after Aristotle–seems to suggest that, perhaps, Question B was not considered until recently. In view of this observation, the following problem (or Question B, for that matter) might be of philosophical and/or historical interest:

PROBLEM: Why did the discovery of nonclassical logics not happen for more than 2000 years after Aristotle?

References

[1] M. Abad, J. M. Cornejo, and J. P. Diaz Varela. *The variety generated by semi-Heyting chains* Soft Comput (2011) 15:721-728

[2] M. Abad, J. M. Cornejo and J. P. Díaz Varela, *The variety of semi-Heyting algebras satisfying the equation* $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$, Rep. Math. Logic 46 (2011), 75-90.

[3] M. Abad, J. M. Cornejo, and J. P. Diaz Varela. *Semi-Heyting algebras term-equivalent to Gödel algebras.* Order, (2):625–642, 2013.

[4] M. E. Adams, H. P. Sankappanavar, and J. Vaz de Carvalho, Varieties of regular pseudocomplemented De Morgan Algebras, Order 37 (2020), 529-557 https://doi.org/10.1007/s11083-019-09518-y

[5] M. E. Adams, H. P. Sankappanavar, and J. Vaz de Carvalho, Regular double p-algebras, Mathematica Slovaca, vol. 69(1) (2019), 15-34. https://doi.org/10.1515/ms-2017-0200

[6] R. Balbes and PH. Dwinger, **Distributive lattices**, Univ. of Missouri Press, Columbia, 1974.

[7] W. J. Blok, D. Pigozzi, Algebraizable logics, Memoirs of the AMS, number 396, American Mathematical Society, Providence, RI, 1989.

[8] S. Burris, Discriminator Varieties and Symbolic Computation, J. Symbolic Computation 13 (1992), 175-207.

[9] S. Burris and H. P. Sankappanavar, **A course in universal algebra**, Springer–Verlag, New York, 1981. The free, corrected Millennium version (2012) is available online as a PDF file at math.uwaterloo.ca/ snburris.

[10] S. Burris and H. Werner, Sheaf constructions, and their elementary properties, Trans. Amer.Math Soc. 248 (1979), 269-309.

[11] S. Comer, Classes without the amalgamation property. Pacific J. Math., 28 (1969), 309-318.

[12] J. M. Cornejo. *Semi-intuitionistic logic.* Studia Logica, 98(1-2):9–25, 2011.

[13] J. M. Cornejo, L. F. Monteiro, H. P. Sankappanavar, and I. D. Viglizzo, A note on chain-based semi-Heyting algebras, Mathematical logic quarterly MLQ 66 (2020), no. 4, 409-417. DOI: 10.1002/malq.201900070

[14] JM Cornejo, M Kinyon, H. P. Sankappanavar, Regular Double p-Algebras: A converse to a Katriňák's Theorem, and Applications, arXiv preprint arXiv:2210.10387 (2022)

[15] J. M. Cornejo and I. D. Viglizzo, *On some semi-intuitionistic logics*, Studia Logica 103(2) (2015), 303-344.

[16] J. M. Cornejo and H. P. Sankappanavar, A logic for dually hemimorphic Semi-Heyting algebras and its axiomatic extensions, Bulletin of the Section of Logic Volume 51/4 (2022), 555-645. https://doi.org/10.18778/0138- 0680.2022.23

[17] J. M. Cornejo and H. P. Sankappanavar, Semi-Heyting algebras and Identities of Associative Type, Bulletin of the Section of Logic 48 (2019), no. 2, 117-135.

[18] J. M. Cornejo, H. P. Sankappanavar, Gautama and Almost Gautama Algebras and their associated logics, Transactions on Fuzzy Sets and Systems (TFSS) 2(2), (2023), 77-112.

[19] J. M. Cornejo, H. P. Sankappanavar, Disjunction Property in varieties of semi-Heyting algebras. Preprint 2023, 27 pages.

[20] J. M. Cornejo, M. Kinyon, H. P. Sankappanavar, Regular Double p-Algebras: A Converse to a Katriňák's Theorem and Applications, Mathematica Slovaca 73(6) (2023), 1373-1388

[21] J. M. Cornejo and H. P. Sankappanavar, Amalgamation Property in the subvarieties of Gautama and Almost Gautama algebras, arXiv:2409.13124v1 [math.LO]. https://doi.org/10.48550/arXiv.2409.13124

[22] J. M. Cornejo and H. P. Sankappanavar, Connexive logics and connexive semi-Heyting algebras. To appear in 2025 as a chapter in the volume dedicated to A.V. Kuznetsov, edited by A. Citkin and A. Muravitsky.

[23] J. Czelakowski and D. Pigozzi, Amalgamation and interpolation in abstract algebraic logic. In: Models, Algebras, and Proofs selected papers of the X Latin American symposium on mathematical logic held in Bogotá, edited by Xavier Caicedo and Carlos H. Montenegro, Lecture Notes in Pure and Applied Mathematics, Vol. 203, Marcel Dekker, Inc., New York, 1999.

[24] D. Fazio, A. Ledda, and F. Paoli, Intutionistic logic is a connexive logic. Special Issue: Frontiers of Connexive Logic. Edited by: Hitoshi Omori and Heinrich Wansing. Studia Logica 112 ((2024), 95-139. https://doi.org/10.1007/s11225-023-10044-7

[25] J.M. Font. *Abstract Algebraic Logic*. An Introductory Textbook. College Publications, 2016.

[26] G. Grätzer, **Lattice Theory**, W.H.Freeman and Co., San Francisco, 1971.

[27] G. Grätzer and H. Lakser, The structure of pseudocomplemented distributive lattices II: Congruence extension and amalgamation, Trans. Amer. Math. Soc. 156 (1971), 343–358.

[28] J. R. Isbell, Epimorphisms and dominions, Proc. Conf. Categorical Algebra, (La Jolla, Calif. 1965), Springer, New York, 1966, pp. 232-246.

[29] B. Jónsson, Algebras whose congruence lattices are distributive, Math. Scand. 21 (1967), 110–121.

[30] W. McCune, Prover9 and Mace 4, http://www.cs.unm.edu/mccune/prover9/

[31] A. A. Monteiro. *Sur les algèbres de Heyting symétriques*. Portugal. Math., 39(1-4):1–237, 1980. Special Issue in honor of António Monteiro.

[32] H. Rasiowa. *An algebraic approach to non-classical logics*. North-Holland Publishing Co., Amsterdam, 1974. Studies in Logic and the Foundations of Mathematics, Vol. 78.

[33] H. P. Sankappanavar. *Heyting algebras with dual pseudocomplementation* Pacific J. math (117): 405–415, 1985.

[34] H.P. Sankappanavar, *Pseudocomplemented Okham and De Morgan algebras*, Zeitschr. f. math. Logik und Grundlagen d. Math. 32 (1986), 385–394.

[35] H.P. Sankappanavar, *Semi-De Morgan algebras*, J. Symbolic. Logic 52 (1987), 712–724.

[36] H.P. Sankappanavar *Heyting algebras with a dual lattice endomorphism*. Zeitschr. f. math. Logik und Grundlagen d. Math. (33): 565–573, 1987.

[37] H. P. Sankappanavar. *Semi-Heyting algebras: an abstraction from Heyting algebras*. In Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress (Spanish), Actas

Congr. "Dr. Antonio A. R. Monteiro", pages 33–66, Bahía Blanca, 2007. Univ. Nac. del Sur.

[38] H. P. Sankappanavar. *Expansions of semi-Heyting algebras I: Discriminator varieties.* Studia Logica, 98(1-2):27–81, 2011.

[39] H. P. Sankappanavar. *Dually quasi-De Morgan Stone semi-Heyting algebras I. Regularity.* Categories, General Algebraic Structures and Applications, 2(1): 47–64, 2014.

[40] H. P. Sankappanavar. *Dually quasi-De Morgan Stone semi-Heyting algebras II. Regularity.* Categories, General Algebraic Structures and Applications, 2(1): 65–82, 2014.

[41] H. P. Sankappanavar. *A note on regular De Morgan semi-Heyting algebras.* Demonstracio mathematica, 2016.

[42] H. P. Sankappanavar. *JI-distributive dually quasi-De Morgan semi-Heyting and Heyting algebras*, Scientiae Mathematicae Japonicae 82, No.3 (2019), (245-271).

[43] H. P. Sankappanavar, (Chapter) A Few Historical Glimpses into the Interplay between Algebra and Logic, and investigations into Gautama Algebras. In: Handbook of Logical Thought in India, S. Sarukkai, M. K. Chakraborty (eds.), Springer Nature India Limited, 2022, pp. 979-1052.

[44] H. P. Sankappanavar, De Morgan semi-Heyting and Heyting algebras, New Trends in Algebra and Com-binatorics, Proceeding of the 3rd International Congress in Algebra and Combinatorics (ICAC2017),25-28, August 2017, Hong Kong, China, (2020), https://doi.org/10.1142/11694.

[45] H. Werner, Discriminator algebras, Studien zur Algebra und ihre Anwendungen, Band 6, Academie–Verlag, Berlin, 1978.

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