ON MODAL COMPANIONS OF LOGICS WITH STRONG NEGATION

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ABSTRACT. $\mathsf{BS4}$ is a natural Belnapian conservative extension of Lewis' modal system $\mathsf{S4}$ via strong negation. In [23] it was proved that the translation $\mathrm{T}_{\mathbf{B}}$ that naturally generalises the Gödel–Tarski translation T embeds faithfully Nelson's logic $\mathsf{N4}^{\perp}$ into $\mathsf{BS4}$. So it is natural to define a modal companion of a logic extending $\mathsf{N4}^{\perp}$ as an extension of $\mathsf{BS4}$. In this paper we construct a representation of an $\mathsf{N4}^{\perp}$-lattice similar to the representation of a Heyting algebra as an open elements algebra for a suitable topoboolean algebra. Using this algebraic result we construct a wide class of $\mathsf{N4}^{\perp}$-extensions, elements of which have modal companions. In particular, all $\mathsf{N3}^{\perp}$-extensions have modal companions. Also we prove that there are a continuum of $\mathsf{N4}^{\perp}$-extensions that have no modal companions.

Keywords: algebraic logic, strong negation, Belnapian modal logic, modal companion, twist-structure, Kleene algebra.

1. Introduction

Since the discovery that intuitionistic logic Int is faithfully embedded into the normal modal logic $\mathsf{S4}$ via Gödel–Tarski translation (see [5, Theorem 3.83] for details), the notion of modal companion is linked with extensions of Int and normal extensions of $\mathsf{S4}$. Transferring different properties from a superintuitionistic logic to its modal companions and *vice versa* became an important branch of investigations. The picture of results obtained in this branch is well-presented in the survey [4].

Our paper will be devoted to modal companions for extensions of Nelson's constructive logic. Originally D. Nelson [17] defined a constructive logic with stong negation, which is denoted now by $\mathsf{N3}^{\perp}$, as an alternative formalization of Brouwer's intuitionistic logic. Later R. Thomason [27] developed a Kripke style semantics for $\mathsf{N3}^{\perp}$, N. Belnap suggested his "useful four-valued logic" [1] based on what is called now the Belnap–Dunn matrix $\mathsf{BD4}$, and M. Dunn [8] proved that $\mathsf{BD4}$

provides a semantics for First-Degree Entailment FDE. These facts allows to consider $\mathsf{N3}^{\perp}$ as well as its paraconsistent version $\mathsf{N4}^{\perp}$ as many-valued versions of intuitionistic logic. More exactly, Nelson's constructive logic $\mathsf{N4}^{\perp}$ is determined by the same class of Kripke frames as intuitionistic logic, but models of this logic are augmented with four-valued valuations. In each possible world a formula may have one of four truth values: T (True), F (False), N (Neither), B (Both) of the Belnap–Dunn matrix $\mathsf{BD4}$; and values of conjunction, disjunction and strong negation of formulas are computed via the respective operations of $\mathsf{BD4}$. In case of $\mathsf{N3}^{\perp}$ the truth value B is excluded, a formula can not be true and false simultaneously in each world.

The Belnapian modal logic BK [23] extends the least normal modal logic K via strong negation $\sim$ in exactly the same way as the logic $\mathsf{N4}^{\perp}$ extends Int. The Kripke style semantics for BK also can be obtained from that of K via replacement of two-valued valuations by four-valued ones. The semantics of necessity and possibility operators is defined according to Fitting's approach [10] to many-valued modalities. An algebraic semantics for BK and its extensions was developed in [20] and [26]. The structure of the lattice of logics extending BK was investigated in [22]. Further, first-order versions of BK were studied in [13], and the "bilattice" version of BK was considered in [25]. It should be noted that only two of four truth values of the Belnap-Dunn matrix can be expressed in the language of BK: T and F; but enriching the language with the constants N and B leads to unexpected results (see [21] for details).

It was proved in [23] that $\mathsf{N4}^{\perp}$ is faithfully embedded into the logic $\mathsf{BS4} = \mathsf{BK} + \{\Box p \to p, \Box p \to \Box\Box p\}$ via the translation $\mathrm{T}_{\mathbf{B}}$, which is a natural extension of the well-known Gödel–Tarski translation T [12] to the language with strong negation. Due to this result it would be natural to define a modal companion of an $\mathsf{N4}^{\perp}$-extension as a $\mathsf{BS4}$-extension and we are curious about transferring results about modal companions of superintuitionistic logics to extensions of $\mathsf{N4}^{\perp}$. The efforts to study modal companions of $\mathsf{N4}^{\perp}$-extensions were made in [14] and [28]. These efforts demonstrated that finding modal companions of constructive logics meets essential complications. As a result in [28] it only was proved that the so-called special extensions of $\mathsf{N4}^{\perp}$ have modal companions.

In this paper we prove that logics from a wide class of $\mathsf{N4}^{\perp}$-extensions, which includes, in particular, all $\mathsf{N3}^{\perp}$-extensions, have modal companions. Also we find a continuum of logics that have no modal companions. We get the results by constructing some regular presentation of models of $\mathsf{N4}^{\perp}$ as subsets of models of $\mathsf{BS4}$, which is similar to the presentation of Heyting algebras as open elements algebras of topoboolean algebras.

The paper is structured as follows. In Section 2 we define the logics of interest, their algebraic semantics and the notion of modal companion. In Section 3 we investigate algebraic relations between models of $\mathsf{BS4}$ and models of $\mathsf{N4}^{\perp}$. In Section 4 we distinguish a wide class of logics and prove that all logics from this class have modal companions by using algebraic results from Section 3; also we show that there are a continuum of logics without modal companions.

2. Preliminaries

In the present section we give definitions of logics of interest and of their algebraic semantics. We recall the classical results about modal companions of superintuitionistic logics and provide definitions of the translation $\mathrm{T}_{\mathbf{B}}$ used to define modal companions of $\mathsf{N4}^{\perp}$-extensions. We will assume throughout that the reader is familiar with basic notions of universal algebra. We refer to [3] for details.

2.1. **Logics.** We work with the following propositional languages:

$$\mathcal{L}_i = \{\wedge^2, \vee^2, \to^2, \perp^0\}, \qquad \mathcal{L}_\sim = \mathcal{L}_i \cup \{\sim^1\},$$
$$\mathcal{L}^\Box = \mathcal{L}_i \cup \{\Box^1\}, \qquad \mathcal{L}_\sim^\Box = \mathcal{L}_i \cup \{\sim^1, \Box^1, \Diamond^1\}.$$

We denote a countable set of propositional variables by Prop. Let $\mathrm{Fm}_{\mathcal{L}}$ be the absolutely free algebra (the formula-algebra) of a language $\mathcal{L}$ built up over Prop. *Formulas* of the language $\mathcal{L}$ are elements of $\mathrm{Fm}_{\mathcal{L}}$.

A set $L \subseteq \mathrm{Fm}_{\mathcal{L}}$ is called a *logic in the language* $\mathcal{L}$ if it satisfies the following property:

1. For $\mathcal{L} \in \{\mathcal{L}_i, \mathcal{L}_\sim\}$, L is closed under the *substitution* (SUB) and the *modus ponens* (MP) rules:
$$\text{(SUB)} \quad \frac{\varphi(p_1, \ldots, p_n)}{\varphi(\psi_1, \ldots, \psi_n)} \qquad \text{(MP)} \quad \frac{\varphi, \quad \varphi \to \psi}{\psi}$$
2. In case $\mathcal{L} = \mathcal{L}^\Box$, L is closed under (SUB), (MP) and the *monotonicity rule for* $\Box$ ($\mathrm{RM}_\Box$):
$$(\mathrm{RM}_\Box) \quad \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$
3. Finally, if $\mathcal{L} = \mathcal{L}_\sim^\Box$, then L is closed under (SUB), (MP), ($\mathrm{RM}_\Box$) and the *monotonicity rule for* $\Diamond$ ($\mathrm{RM}_\Diamond$):
$$(\mathrm{RM}_\Diamond) \quad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$$

For a logic L in a language $\mathcal{L}$ we will denote the set of all logics in $\mathcal{L}$ containing L by $\mathcal{E}L$. $\mathcal{E}L$ is a complete lattice w.r.t. inclusion, because the intersection of an arbitrary set of logics in $\mathcal{L}$ is also a logic in $\mathcal{L}$. So $\mathcal{E}L$ is called the *lattice of extensions of* L. Also for a set of formulas $X \subseteq \mathrm{Fm}_{\mathcal{L}}$ we will denote the least logic in $\mathcal{L}$ containing L and X by $L + X$.

We use the following standard abbreviations:

$$\neg\varphi := \varphi \to \bot \qquad \text{(intuitionistic negation);}$$
$$\Diamond\varphi := \neg\Box\neg\varphi \qquad \text{(only for formulas in } \mathrm{Fm}_{\mathcal{L}^\Box}\text{);}$$
$$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi) \qquad \text{(equivalence);}$$
$$\varphi \Leftrightarrow \psi := (\varphi \leftrightarrow \psi) \wedge ({\sim}\varphi \leftrightarrow {\sim}\psi) \qquad \text{(strong equivalence).}$$

Now we introduce logics of interest. The *intuitionistic logic* Int is the least logic in the language $\mathcal{L}_i$ containing the following axioms:

1. $p \to (q \to p)$
2. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
3. $(p \wedge q) \to p$
4. $(p \wedge q) \to q$
5. $p \to (q \to (p \wedge q))$
6. $p \to (p \vee q)$
7. $q \to (p \vee q)$
8. $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$
9. $\bot \to p$

The *classical logic* is defined as $\mathsf{CL} = \mathsf{Int} + \{p \vee \neg p\}$.

Constructive Nelson's logic $\mathsf{N4}^\bot$ is the least logic in the language $\mathcal{L}_\sim$ containing:

I) the axioms of Int
II) the axioms of strong negation:
   1. ${\sim}(p \vee q) \leftrightarrow ({\sim}p \wedge {\sim}q)$
   2. ${\sim}(p \wedge q) \leftrightarrow ({\sim}p \vee {\sim}q)$
   3. ${\sim}(p \to q) \leftrightarrow (p \wedge {\sim}q)$
   4. ${\sim}{\sim}p \leftrightarrow p$
   5. ${\sim}\bot$

The modal logic $\mathsf{S4}$ is the least logic in the language $\mathcal{L}^\Box$ containing:

I) the axioms of CL
II) the following modal axioms:
   1. $\Box(p \to p)$
   2. $(\Box p \wedge \Box q) \to \Box(p \wedge q)$
   3. $\Box p \to p$
   4. $\Box p \to \Box\Box p$

Grzegorczyk's logic is defined as $\mathsf{Grz} = \mathsf{S4} + \{\Box(\Box(p \to \Box p) \to p) \to p\}$.

The modal logic $\mathsf{BS4}$ is the least logic in the language $\mathcal{L}^{\Box}_{\sim}$ containing:

I) the axioms of CL
II) the axioms of strong negation
III) the modal axioms of $\mathsf{S4}$
IV) the interplay axioms:
   1. $\neg\Box p \leftrightarrow \Diamond\neg p$
   2. $\neg\Diamond p \leftrightarrow \Box\neg p$
   3. $\Box p \Leftrightarrow \sim\Diamond\sim p$
   4. $\Diamond p \Leftrightarrow \sim\Box\sim p$

It is known (see [19]) that each $L \in \mathcal{E}\mathsf{N4}^{\perp}$ is closed under the positive replacement rule

$$\text{(PR)} \quad \frac{\varphi_1 \leftrightarrow \psi_1, \dots, \varphi_n \leftrightarrow \psi_n}{\chi(\varphi_1, \dots, \varphi_n) \leftrightarrow \chi(\psi_1, \dots, \psi_n)},$$

where $\varphi_j, \psi_j \in \mathrm{Fm}_{\mathcal{L}_\sim}$ and $\chi \in \mathrm{Fm}_{\mathcal{L}_i}$. The logic $\mathsf{N4}^{\perp}$ is not closed under the usual replacement rule without restriction on χ, but each $L \in \mathcal{E}\mathsf{N4}^{\perp}$ is closed under the weak replacement rule

$$\text{(WR)} \quad \frac{\varphi_1 \Leftrightarrow \psi_1, \dots, \varphi_n \Leftrightarrow \psi_n}{\chi(\varphi_1, \dots, \varphi_n) \leftrightarrow \chi(\psi_1, \dots, \psi_n)},$$

where $\varphi_j, \psi_j, \chi \in \mathrm{Fm}_{\mathcal{L}_\sim}$.

For modal logics it is easy to check that each $L \in \mathcal{E}\mathsf{S4}$ or $L \in \mathcal{E}\mathsf{BS4}$ is closed under the normalisation rule

$$\text{(RN)} \quad \frac{\varphi}{\Box\varphi}.$$

The logic $\mathsf{BS4}$ is not closed under the usual replacement rule, similarly to $\mathsf{N4}^{\perp}$, but (see [23]) each logic $L \in \mathcal{E}\mathsf{BS4}$ is closed under the positive and the weak variants of this rule:

$$\text{(PR)} \quad \frac{\varphi_1 \leftrightarrow \psi_1, \dots, \varphi_n \leftrightarrow \psi_n}{\theta(\varphi_1, \dots, \varphi_n) \leftrightarrow \theta(\psi_1, \dots, \psi_n)},$$

and

$$\text{(WR)} \quad \frac{\varphi_1 \Leftrightarrow \psi_1, \dots, \varphi_n \Leftrightarrow \psi_n}{\chi(\varphi_1, \dots, \varphi_n) \leftrightarrow \chi(\psi_1, \dots, \psi_n)},$$

where $\theta \in \mathrm{Fm}_{\mathcal{L}^{\Box}}$ and $\varphi_j, \psi_j, \chi \in \mathrm{Fm}_{\mathcal{L}^{\Box}_{\sim}}$.

2.2. **Algebraic semantics.** Note that we use the same abbreviations for operations $\neg, \Diamond, \leftrightarrow, \Leftrightarrow$ in algebras $\mathbf{A} = \langle A; \mathcal{L}^{\mathbf{A}} \rangle$ as for formulas.

An algebra $\mathbf{A} = \langle A; \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \to_{\mathbf{A}}, \perp_{\mathbf{A}} \rangle$ is called a *Heyting algebra*, if its reduct $\langle A; \wedge_{\mathbf{A}}, \vee_{\mathbf{A}} \rangle$ is a lattice with order relation $\leqslant_{\mathbf{A}}$, $\perp_{\mathbf{A}}$ is the least element w.r.t. $\leqslant_{\mathbf{A}}$, and for arbitrary $a, b, c \in A$ the following holds:

$$a \wedge_{\mathbf{A}} b \leqslant_{\mathbf{A}} c \iff a \leqslant_{\mathbf{A}} b \rightarrow_{\mathbf{A}} c$$

Sometimes we will omit the subscript $\mathbf{A}$, when it is clear which algebra we work in.

Note that each Heyting algebra $\mathbf{A}$ is a distributive lattice, it has the greatest element $1_{\mathbf{A}}$, and for all $a \in A$ we have $a \rightarrow a = 1_{\mathbf{A}}$ (see, e.g., [24]). Also the following hold for all $a, b \in A$:

(1) $a \rightarrow b = 1_{\mathbf{A}} \iff a \leqslant_{\mathbf{A}} b$
(2) $a \wedge (a \rightarrow b) = a \wedge b$
(3) $a \leqslant_{\mathbf{A}} b \Longrightarrow \neg b \leqslant_{\mathbf{A}} \neg a$
(4) $a \leqslant_{\mathbf{A}} \neg\neg a$

A non-empty set $F \subseteq A$ is called a *filter of* $\mathbf{A}$ if it satisfies the following properties:

$$(1)\ b \in F,\ b \leqslant a \Rightarrow a \in F; \qquad (2)\ a, b \in F \Rightarrow a \wedge b \in F.$$

By analogy, a non-empty set $I \subseteq A$ is called an *ideal of* $\mathbf{A}$ if the dual properties hold:

$$(1)\ b \in I,\ a \leqslant b \Rightarrow a \in I; \qquad (2)\ a, b \in I \Rightarrow a \vee b \in I.$$

The set of all filters (ideals) of $\mathbf{A}$ is denoted by $\mathcal{F}(\mathbf{A})$ ($\mathcal{I}(\mathbf{A})$).

An element $a \in A$ is called *dense* if one of the following three equivalent properties holds:

$$(1)\ \neg a = \bot; \quad (2)\ \neg\neg a = 1; \quad (3)\ a = b \vee \neg b \text{ for some } b \in A.$$

It is well-known that the set of all dense elements of $\mathbf{A}$ is a filter, and we denote it by $F_d(\mathbf{A})$. We denote the set of all filters which contain $F_d(\mathbf{A})$ as a subset by $\mathcal{F}_d(\mathbf{A})$.

A Heyting algebra $\mathbf{A}$ is called a *Boolean algebra* if $a \vee \neg a = 1$ for all $a \in A$ or, equivalently, if $F_d(\mathbf{A}) = \{1\}$.

An algebra $\mathbf{B} = \langle B; \wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \rightarrow_{\mathbf{B}}, \bot_{\mathbf{B}}, \Box_{\mathbf{B}} \rangle$ is called a *topological Boolean algebra* or a TBA if its reduct $\langle B; \wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \rightarrow_{\mathbf{B}}, \bot_{\mathbf{B}} \rangle$ is a Boolean algebra and the following hold for all $a, b \in B$ (some subscripts are omitted):

$$\Box 1_{\mathbf{B}} = 1_{\mathbf{B}}, \qquad \Box(a \wedge b) = \Box a \wedge \Box b,$$
$$\Box a \leqslant_{\mathbf{B}} a, \qquad \Box a \leqslant_{\mathbf{B}} \Box\Box a.$$

Note that in an arbitrary TBA $\mathbf{B}$ the following hold for all $a, b \in B$ (see, e.g., [24]):

(1) $\Diamond \bot_{\mathbf{B}} = \bot_{\mathbf{B}},\ \Box \bot_{\mathbf{B}} = \bot_{\mathbf{B}},\ \Diamond 1_{\mathbf{B}} = 1_{\mathbf{B}}$
(2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
(3) $a \leqslant_{\mathbf{B}} \Diamond a$

(4) $\Diamond\Diamond a \leqslant_{\mathbf{B}} \Diamond a$
(5) $a \leqslant_{\mathbf{B}} b \Longrightarrow (\Box a \leqslant_{\mathbf{B}} \Box b$ and $\Diamond a \leqslant_{\mathbf{B}} \Diamond b)$
(6) $\Box\Box a = \Box a$ and $\Diamond\Diamond a = \Diamond a$
(7) $\Box a \vee \Box b \leqslant_{\mathbf{B}} \Box(a \vee b)$ and $\Diamond(a \wedge b) \leqslant_{\mathbf{B}} \Diamond a \wedge \Diamond b$

A *filter (ideal) of* TBA is defined as for Heyting algebras. We call a filter (ideal) of $\mathbf{B}$ *open (closed)* if it is closed under $\Box_{\mathbf{B}}$ ($\Diamond_{\mathbf{B}}$). The set of all open filters (closed ideals) of $\mathbf{B}$ is denoted by $\mathcal{F}_{\Box}(\mathbf{B})$ ($\mathcal{I}_{\Diamond}(\mathbf{B})$).

Let $\mathbf{B}$ be a TBA. The set $\mathsf{G}(\mathbf{B}) := \{a \in B \mid \Box a = a\}$ of all *open elements* contains $\bot_{\mathbf{B}}$ and is closed under $\vee, \wedge$. The algebra

$$\mathcal{G}(\mathbf{B}) = \langle \mathsf{G}(\mathbf{B}); \vee_{\mathcal{G}(\mathbf{B})}, \wedge_{\mathcal{G}(\mathbf{B})}, \rightarrow_{\mathcal{G}(\mathbf{B})}, \bot_{\mathcal{G}(\mathbf{B})} \rangle$$

where

$$a \vee_{\mathcal{G}(\mathbf{B})} b := a \vee b \qquad\qquad a \wedge_{\mathcal{G}(\mathbf{B})} b := a \wedge b$$
$$a \rightarrow_{\mathcal{G}(\mathbf{B})} b := \Box(a \rightarrow b) \qquad\qquad \bot_{\mathcal{G}(\mathbf{B})} := \bot_{\mathbf{B}}$$

is called the *algebra of open elements of* $\mathbf{B}$. It is known that $\mathcal{G}(\mathbf{B})$ is a Heyting algebra (see, e.g., [11, pp. 61–62]).

For a TBA $\mathbf{B}$ we denote its subalgebra generated by all open elements by $\mathbf{B}^s$. It is well-known that for each Heyting algebra $\mathbf{A}$ there is one up to isomorphism TBA $\mathbf{C}$ such that $\mathcal{G}(\mathbf{C}) = \mathbf{A}$ and $\mathbf{C}^s = \mathbf{C}$ (see, e.g., [11, Theorem 3.8]). We denote this TBA by $s(\mathbf{A})$.

Let $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, \bot \rangle$ be a Heyting algebra. The algebra $\mathbf{A}^{\bowtie} = \langle A \times A; \vee_{\mathbf{A}^{\bowtie}}, \wedge_{\mathbf{A}^{\bowtie}}, \rightarrow_{\mathbf{A}^{\bowtie}}, \bot_{\mathbf{A}^{\bowtie}}, \sim_{\mathbf{A}^{\bowtie}} \rangle$ is called the *full twist-structure over a Heyting algebra* $\mathbf{A}$, where the operations of $\mathbf{A}^{\bowtie}$ are defined as follows:

$$(a,b) \vee_{\mathbf{A}^{\bowtie}} (c,d) = (a \vee c, b \wedge d) \qquad\qquad \sim_{\mathbf{A}^{\bowtie}}(a,b) = (b,a)$$
$$(a,b) \wedge_{\mathbf{A}^{\bowtie}} (c,d) = (a \wedge c, b \vee d) \qquad\qquad \bot_{\mathbf{A}^{\bowtie}} = (\bot, 1)$$
$$(a,b) \rightarrow_{\mathbf{A}^{\bowtie}} (c,d) = (a \rightarrow c, a \wedge d)$$

If we have a TBA $\mathbf{B} = \langle B, \vee, \wedge, \rightarrow, \bot, \Box \rangle$, then the algebra $\mathbf{B}^{\bowtie} = \langle B \times B; \vee_{\mathbf{B}^{\bowtie}}, \wedge_{\mathbf{B}^{\bowtie}}, \rightarrow_{\mathbf{B}^{\bowtie}}, \bot_{\mathbf{B}^{\bowtie}}, \sim_{\mathbf{B}^{\bowtie}}, \Box_{\mathbf{B}^{\bowtie}}, \Diamond_{\mathbf{B}^{\bowtie}} \rangle$ is called the *full twist-structure over a TBA* $\mathbf{B}$, where the operations of $\mathcal{L}_{\sim}$ are defined as for the full twist-structure over a Heyting algebra, and the modal operations are defined as:

$$\Box_{\mathbf{B}^{\bowtie}}(a,b) = (\Box a, \Diamond b) \qquad\qquad \Diamond_{\mathbf{B}^{\bowtie}}(a,b) = (\Diamond a, \Box b)$$

Finally, if we have a Heyting algebra (a TBA) $\mathbf{C}$, then a subalgebra $\mathcal{B}$ of $\mathbf{C}^{\bowtie}$ is called a *twist-structure over* $\mathbf{C}$ if $\pi_1(\mathcal{B}) = C$, where π_1 denotes the first projection function. Also we denote the second projection function by π_2 in what follows.

We define the following sets for a twist-structure $\mathcal{B}$ over a Heyting algebra (a TBA) $\mathbf{C}$:

$$\nabla(\mathcal{B}) := \{a \vee b \mid (a,b) \in \mathcal{B}\}, \quad \Delta(\mathcal{B}) := \{a \wedge b \mid (a,b) \in \mathcal{B}\}.$$

It is known (see [18, 20]) that $\nabla(\mathcal{B}) \in \mathcal{F}_d(\mathbf{C})$ ($\mathcal{F}_\Box(\mathbf{C})$) and $\Delta(\mathcal{B}) \in \mathcal{I}(\mathbf{C})$ ($\mathcal{I}_\Diamond(\mathbf{C})$). Conversely, for every $\nabla \in \mathcal{F}_d(\mathbf{C})$ ($\mathcal{F}_\Box(\mathbf{C})$) and $\Delta \in \mathcal{I}(\mathbf{C})$ ($\mathcal{I}_\Diamond(\mathbf{C})$) we can define the set

$$X = \{(a,b) \in C \times C \mid a \vee b \in \nabla, a \wedge b \in \Delta\},$$

and it is known that X is closed under the operations of $\mathbf{C}^{\bowtie}$ and $\pi_1(X) = C$. So we can define the twist-structure over $\mathbf{C}$ with X as the domain, and we denote it by $Tw(\mathbf{C}, \nabla, \Delta)$. Moreover, it is known that

$$\nabla(Tw(\mathbf{C}, \nabla, \Delta)) = \nabla, \quad \Delta(Tw(\mathbf{C}, \nabla, \Delta)) = \Delta$$

and

$$Tw(\mathbf{C}, \nabla(\mathcal{B}), \Delta(\mathcal{B})) = \mathcal{B},$$

so twist-structures are uniquely determined by these sets, which we call the *invariants of twist-structures*. The sets $\nabla(\mathcal{B})$ and $\Delta(\mathcal{B})$ are called the *filter of* $\mathcal{B}$ and the *ideal of* $\mathcal{B}$ respectively.

Now, let $\mathcal{L} \in \{\mathcal{L}_i, \mathcal{L}^\Box, \mathcal{L}_\sim, \mathcal{L}_\sim^\Box\}$, $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}$, and let $\mathcal{A}$ ($\mathcal{K}$) be:

1. a Heyting algebra (some class of Heyting algebras), if $\mathcal{L} = \mathcal{L}_i$;
2. a TBA (some class of TBAs), if $\mathcal{L} = \mathcal{L}^\Box$;
3. a twist-structure over a Heyting algebra $\mathbf{A}$ (some class of twist-structures over Heyting algebras), if $\mathcal{L} = \mathcal{L}_\sim$;
4. a twist-structure over a TBA $\mathbf{A}$ (some class of twist-structures over TBAs), if $\mathcal{L} = \mathcal{L}_\sim^\Box$.

Also let us denote the class of all Heyting algebras (the class of all twist-structures over Heyting algebras) by $\mathcal{HA}$ ($\mathcal{HA}^{\bowtie}$) and the class of all TBAs (the class of all twist-structures over TBAs) by $\mathcal{TBA}$ ($\mathcal{TBA}^{\bowtie}$). Recall that for an algebra $\mathcal{A} = \langle A; \mathcal{L}^\mathcal{A}\rangle$ an $\mathcal{A}$-*valuation* is a homomorphism $v\colon \mathrm{Fm}_\mathcal{L} \to \mathcal{A}$.

Let's define the validity of formulas (sets of formulas) in algebras (classes of algebras):

Definition 2.2.1.

1. If $\mathcal{L} \in \{\mathcal{L}_i, \mathcal{L}^\Box\}$, $\mathcal{A} \models \varphi$ iff $v(\varphi) = 1_\mathcal{A}$ for every $\mathcal{A}$-valuation v
2. If $\mathcal{L} \in \{\mathcal{L}_\sim, \mathcal{L}_\sim^\Box\}$, $\mathcal{A} \models \varphi$ iff $\pi_1 v(\varphi) = 1_\mathbf{A}$ for every $\mathcal{A}$-valuation v
3. $\mathcal{A} \models \Gamma$ iff $\mathcal{A} \models \varphi$ for every $\varphi \in \Gamma$
4. $\mathrm{L}\mathcal{A} := \{\varphi \in \mathrm{Fm}_\mathcal{L} \mid \mathcal{A} \models \varphi\}$
5. $\mathcal{K} \models \varphi$ iff $\mathcal{A} \models \varphi$ for every $\mathcal{A} \in \mathcal{K}$
6. $\mathcal{K} \models \Gamma$ iff $\mathcal{K} \models \varphi$ for every $\varphi \in \Gamma$
7. $\mathrm{L}\mathcal{K} := \bigcap\{\mathrm{L}\mathcal{A} \mid \mathcal{A} \in \mathcal{K}\}$

Remark 1. Note that if we have some twist-structure $\mathcal{A}$ over a Heyting algebra (a TBA) and some $\mathcal{A}$-valuation v, then $\pi_1(v(\psi(p_1,\ldots,p_n))) = \psi(\pi_1(v(p_1)),\ldots,\pi_1(v(p_n)))$ for every $\psi \in \mathrm{Fm}_{\mathcal{L}_i}$ ($\psi \in \mathrm{Fm}_{\mathcal{L}^\Box}$), because the operations of twist-structure are agreed with the operations of Heyting algebra (TBA) on the first component.

The following theorem links logics and algebras:

Theorem 2.2.1.

1. $\mathrm{L}\mathcal{HA} = \mathsf{Int}$ *and* $\mathrm{L}\mathcal{A}$ $(\mathrm{L}\mathcal{K}) \in \mathcal{E}\mathsf{Int}$*, if* $\mathcal{L} = \mathcal{L}_i$ *(see, e.g., [24]).*
2. $\mathrm{L}\mathcal{TBA} = \mathsf{S4}$ *and* $\mathrm{L}\mathcal{A}$ $(\mathrm{L}\mathcal{K}) \in \mathcal{E}\mathsf{S4}$*, if* $\mathcal{L} = \mathcal{L}^\Box$ *(see, e.g., [5]).*
3. $\mathrm{L}\mathcal{HA}^{\bowtie} = \mathsf{N4}^\perp$ *and* $\mathrm{L}\mathcal{A}$ $(\mathrm{L}\mathcal{K}) \in \mathcal{E}\mathsf{N4}^\perp$*, if* $\mathcal{L} = \mathcal{L}_\sim$ *(see [18]).*
4. $\mathrm{L}\mathcal{TBA}^{\bowtie} = \mathsf{BS4}$ *and* $\mathrm{L}\mathcal{A}$ $(\mathrm{L}\mathcal{K}) \in \mathcal{E}\mathsf{BS4}$*, if* $\mathcal{L} = \mathcal{L}^\Box_\sim$ *(see [23]).*

We call an algebra $\mathcal{A}$ a *model of logic* L *in the language* $\mathcal{L}$ if $\mathcal{A} \models L$. A model $\mathcal{A}$ of L is called *proper* if $\mathrm{L}\mathcal{A} = L$.

For each language $\mathcal{L} \in \{\mathcal{L}_\sim, \mathcal{L}^\Box_\sim\}$ and logic $L \in \mathcal{E}\mathsf{N4}^\perp$ or $L \in \mathcal{E}\mathsf{BS4}$ in the language $\mathcal{L}$ we define a special twist-structure which is called the *Lindenbaum twist-structure of* L. First, we define a binary relation $\equiv_L$ on $\mathrm{Fm}_{\mathcal{L}}$:

$$\varphi \equiv_L \psi \iff \varphi \leftrightarrow \psi \in L.$$

It is obvious that $\equiv_L$ is an equivalence relation. Denote the equivalence class of φ w.r.t. $\equiv_L$ by $[\varphi]_L$. We define the following operations $\vee_{\mathcal{T}(L)}$, $\wedge_{\mathcal{T}(L)}$, $\to_{\mathcal{T}(L)}$ and $\perp_{\mathcal{T}(L)}$ (also $\Box_{\mathcal{T}(L)}$ and $\Diamond_{\mathcal{T}(L)}$) on the equivalence classes:

$$\begin{array}{cc}
[\varphi]_L \vee_{\mathcal{T}(L)} [\psi]_L := [\varphi \vee \psi]_L & [\varphi]_L \wedge_{\mathcal{T}(L)} [\psi]_L := [\varphi \wedge \psi]_L \\
[\varphi]_L \to_{\mathcal{T}(L)} [\psi]_L := [\varphi \to \psi]_L & \perp_{\mathcal{T}(L)} := [\perp]_L \\
(\Box_{\mathcal{T}(L)}[\varphi]_L := [\Box\varphi]_L) & (\Diamond_{\mathcal{T}(L)}[\varphi]L := [\Diamond\varphi]_L)
\end{array}$$

These operations are well-defined because L is closed under (PR). Also the definition of $\Diamond_{\mathcal{T}(L)}$ is correct w.r.t. $\Diamond\varphi := \neg\Box\neg\varphi$ because of the interplay axioms 1–2.

$\mathsf{Int} \subseteq \mathsf{N4}^\perp$ ($\mathsf{S4} \subseteq \mathsf{BS4}$) implies that the algebra

$$\mathcal{T}(L) := \left\langle \{[\varphi]_L \mid \varphi \in \mathrm{Fm}_{\mathcal{L}}\}; \vee_{\mathcal{T}(L)}, \wedge_{\mathcal{T}(L)}, \to_{\mathcal{T}(L)}, \perp_{\mathcal{T}(L)}(, \Box_{\mathcal{T}(L)}) \right\rangle,$$

is a Heyting algebra (a TBA). And in that case

$$1_{\mathcal{T}(L)} = [\varphi]_L \to [\varphi]_L = [\varphi \to \varphi]_L = L\,.$$

It is known that the following subset of $\mathcal{T}(L)^2$ is closed under the operations of $\mathcal{T}(L)^{\bowtie}$ (see [18, 23]):

$$W_L := \{([\varphi]_L, [\sim\varphi]_L) \mid \varphi \in \mathrm{Fm}_{\mathcal{L}}\}.$$

It is obvious that $\pi_1(W_L) = \{[\varphi]_L \mid \varphi \in \mathrm{Fm}_{\mathcal{L}}\}$, so the *Lindenbaum twist-structure* $\mathcal{W}_L$ is the twist-structure over $\mathcal{T}(L)$ with W_L as the domain. We denote its invariants by

$$\nabla_L := \nabla(\mathcal{W}_L) = \{[\varphi \vee {\sim}\varphi]_L \mid \varphi \in \mathrm{Fm}_{\mathcal{L}}\},$$
$$\Delta_L := \Delta(\mathcal{W}_L) = \{[\varphi \wedge {\sim}\varphi]_L \mid \varphi \in \mathrm{Fm}_{\mathcal{L}}\}.$$

It is known that $\mathcal{W}_L$ is a proper model of L.

2.3. **The translations and modal companions.** Let's define the Gödel–Tarski translation $\mathrm{T}\colon \mathrm{Fm}_{\mathcal{L}_i} \to \mathrm{Fm}_{\mathcal{L}^\Box}$ by induction of the construction of a formula:

$$\begin{aligned} \mathrm{T}(p) &:= \Box p, & \mathrm{T}(\bot) &:= \bot, \\ \mathrm{T}(\varphi \wedge \psi) &:= \mathrm{T}(\varphi) \wedge \mathrm{T}(\psi), & \mathrm{T}(\varphi \vee \psi) &:= \mathrm{T}(\varphi) \vee \mathrm{T}(\psi), \\ \mathrm{T}(\varphi \to \psi) &:= \Box(\mathrm{T}(\varphi) \to \mathrm{T}(\psi)), \end{aligned}$$

where $p \in \mathrm{Prop}$. Notice that the definition of operations in the algebra of open elements reproduces the definition of the translation T for connectives $\vee$, $\wedge$, $\to$, $\bot$. Thus, we have for every $\varphi \in \mathrm{Fm}_{\mathcal{L}_i}$ (see, e.g., [11, Lemma 3.10])

$$\mathcal{G}(\mathbf{B}) \models \varphi \iff \mathbf{B} \models \mathrm{T}\varphi. \tag{Top}$$

Now, let $L \in \mathcal{E}\mathsf{Int}$. A logic $M \in \mathcal{E}\mathsf{S4}$ is called a *modal companion of* L, if T faithfully embeds L into M: for every $\varphi \in \mathrm{Fm}_{\mathcal{L}_i}$

$$\varphi \in L \iff \mathrm{T}\varphi \in M.$$

It is well-known that for each $L \in \mathcal{E}\mathsf{Int}$, the $\mathsf{S4}$-extensions $\tau L := \mathsf{S4} + \{\mathrm{T}\varphi \mid \varphi \in L\}$ and $\sigma L := \tau L + \mathsf{Grz}$ are the least (see [7]) and the greatest (see [2, 9]) modal companions of L respectively.

We define the translation $\mathrm{T}_{\mathbf{B}}\colon \mathrm{Fm}_{\mathcal{L}_\sim} \to \mathrm{Fm}_{\mathcal{L}^\Box_\sim}$:

$$\begin{aligned} \mathrm{T}_{\mathbf{B}}(p) &:= \Box p, & \mathrm{T}_{\mathbf{B}}({\sim}p) &:= \Box{\sim}p, \\ \mathrm{T}_{\mathbf{B}}(\bot) &:= \bot, & \mathrm{T}_{\mathbf{B}}({\sim}\bot) &:= {\sim}\bot, \\ \mathrm{T}_{\mathbf{B}}(\varphi \wedge \psi) &:= \mathrm{T}_{\mathbf{B}}(\varphi) \wedge \mathrm{T}_{\mathbf{B}}(\psi), & \mathrm{T}_{\mathbf{B}}({\sim}(\varphi \wedge \psi)) &:= \mathrm{T}_{\mathbf{B}}({\sim}\varphi) \vee \mathrm{T}_{\mathbf{B}}({\sim}\psi), \\ \mathrm{T}_{\mathbf{B}}(\varphi \vee \psi) &:= \mathrm{T}_{\mathbf{B}}(\varphi) \vee \mathrm{T}_{\mathbf{B}}(\psi), & \mathrm{T}_{\mathbf{B}}({\sim}(\varphi \vee \psi)) &:= \mathrm{T}_{\mathbf{B}}({\sim}\varphi) \wedge \mathrm{T}_{\mathbf{B}}({\sim}\psi), \\ \mathrm{T}_{\mathbf{B}}(\varphi \to \psi) &:= \Box(\mathrm{T}_{\mathbf{B}}(\varphi) \to \mathrm{T}_{\mathbf{B}}(\psi)), & \mathrm{T}_{\mathbf{B}}({\sim}(\varphi \to \psi)) &:= \mathrm{T}_{\mathbf{B}}(\varphi) \wedge \mathrm{T}_{\mathbf{B}}({\sim}\psi). \end{aligned}$$

It is easy to see that the restriction of $\mathrm{T}_{\mathbf{B}}$ to $\mathrm{Fm}_{\mathcal{L}_i}$ coincides with T. Moreover, $\mathrm{T}_{\mathbf{B}}$ moves strong negation through positive connectives and removes even iterations of strong negation, so $\sim$ occurs in $\mathrm{T}_{\mathbf{B}}\varphi$ only in front of propositional variables and $\bot$.

In [23] it was proved that $\mathrm{T}_{\mathbf{B}}$ faithfully embeds $\mathsf{N4}^\bot$ into $\mathsf{BS4}$, so it is natural to define a *modal companion of* $L \in \mathcal{E}\mathsf{N4}^\bot$ as a $\mathsf{BS4}$-extension, in which L is faithfully embedded by $\mathrm{T}_{\mathbf{B}}$.

By analogy with Int-extensions, for $L \in \mathcal{E}\mathsf{N4}^{\perp}$ we can define the candidate $\tau_{\mathbf{B}} L := \mathsf{BS4} + \{\mathrm{T}_{\mathbf{B}}\varphi \mid \varphi \in L\}$ to be the least modal companion of L. And it is not hard to see that if L has some modal companion then $\tau_{\mathbf{B}} L$ is the least modal companion of L.

3. Preliminary algebraic observations

The main goal of the present work, which gives motivation to the following algebraic results, is to explore which $\mathsf{N4}^{\perp}$-extensions have modal companions.

Recall the idea how to get the results on modal companions of superintuitionistic logics:

1. For a Heyting algebra $\mathbf{A}$ take a TBA $\mathbf{B}$ such that $\mathcal{G}(\mathbf{B}) = \mathbf{A}$.
2. In view of (Top) we have that $\mathrm{L}\mathbf{B}$ is a modal companion of $\mathrm{L}\mathbf{A}$.
3. Now just take $\mathbf{A}$ to be some proper model of $L \in \mathcal{E}\mathsf{Int}$.

In the present section we provide a representaion of twist-structures over Heyting algebras as subsets of twist-structures over TBAs defined in some regular way, for which we have the property of preserving the truth similar to (Top) but relative to the translation $\mathrm{T}_{\mathbf{B}}$.

3.1. From $\mathsf{BS4}$-twist-structures to $\mathsf{N4}^{\perp}$-twist-structures.

In this subsection we construct a twist-structure over a Heyting algebra starting from a twist-structure over a TBA. Let's fix an arbitrary TBA $\mathbf{B} = \langle B; \vee, \wedge, \to, \bot, \Box\rangle$ with the greatest element 1, $\nabla \in \mathcal{F}_{\Box}(\mathbf{B})$ and $\Delta \in \mathcal{I}_{\Diamond}(\mathbf{B})$. We will consider the twist-structure $\mathbf{T} := Tw(\mathbf{B}, \nabla, \Delta)$ throughout this subsection.

We will search for a twist-structure over the algebra of open elements $\mathcal{G}(\mathbf{B})$ or its subalgebra. This motivates us to consider the following set:

$$\mathsf{G}_2(\mathbf{T}) := \{(a, b) \in \mathbf{T} \mid \Box a = a, \Box b = b\}.$$

We define then $\Gamma(\mathbf{T}) := \pi_1(\mathsf{G}_2(\mathbf{T}))$. Also we consider a set

$$\Lambda(\mathbf{B}, \nabla) := \{a \in \mathsf{G}(\mathbf{B}) \mid a \vee \Box\neg a \in \nabla\},$$

the sense of which will be clear later.

Lemma 3.1.1. *The following hold:*

1. $a \wedge \Box\neg a = \bot$ *for all* $a \in B$*, so* $a \wedge \Box\neg a \in \Delta$*;*
2. $\Gamma(\mathbf{T}) \subseteq \mathsf{G}(\mathbf{B})$*;*
3. $\mathsf{G}_2(\mathbf{T})$ *is closed under* $\vee_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \wedge_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \sim_{\mathcal{G}(\mathbf{B})^{\bowtie}}$*, and* $\bot_{\mathcal{G}(\mathbf{B})^{\bowtie}} \in \mathsf{G}_2(\mathbf{T})$*;*
4. $\pi_2(\mathsf{G}_2(\mathbf{T})) = \Gamma(\mathbf{T})$*;*
5. $\Gamma(\mathbf{T})$ *is closed under* $\vee_{\mathcal{G}(\mathbf{B})}, \wedge_{\mathcal{G}(\mathbf{B})}$ *and* $\bot_{\mathcal{G}(\mathbf{B})} \in \Gamma(\mathbf{T})$*;*
6. $\Lambda(\mathbf{B}, \nabla) \subseteq \Gamma(\mathbf{T})$*;*
7. $\Lambda(\mathbf{B}, \nabla)$ *is a subalgebra of* $\mathcal{G}(\mathbf{B})$*.*

Proof. Items 1, 2, 3 are obvious. Item 5 is a consequence of item 3. Item 4 holds because $\mathsf{G}_2(\mathbf{T})$ is closed under $\sim_{\mathcal{G}(\mathbf{B})^{\bowtie}}$.

6. If $x \in \Lambda(\mathbf{B}, \nabla)$, then $x \vee \Box\neg x \in \nabla$ and $\Box x = x$. In view of (1) $x \wedge \Box\neg x \in \Delta$. So $(x, \Box\neg x) \in \mathbf{T}$ by definition of $Tw(\mathbf{B}, \nabla, \Delta)$. And $x \in \Gamma(\mathbf{T})$.

7. Suppose $a, b \in \Lambda(\mathbf{B}, \nabla)$. So, obviously,

$$a \vee_{\mathcal{G}(\mathbf{B})} b,\ a \wedge_{\mathcal{G}(\mathbf{B})} b,\ a \to_{\mathcal{G}(\mathbf{B})} b,\ \bot_{\mathcal{G}(\mathbf{B})} \in \mathsf{G}(\mathbf{B}).$$

For $\vee_{\mathcal{G}(\mathbf{B})}$ we have

$$\begin{aligned}(a \vee_{\mathcal{G}(\mathbf{B})} b) \vee \Box\neg(a \vee_{\mathcal{G}(\mathbf{B})} b) &= (a \vee b) \vee \Box\neg(a \vee b) = \\ &= (a \vee b) \vee \Box(\neg a \wedge \neg b) = (a \vee b) \vee (\Box(\neg a) \wedge \Box(\neg b)) = \\ &= (a \vee \Box(\neg a) \vee b) \wedge (b \vee \Box(\neg b) \vee a) \geqslant (a \vee \Box\neg a) \wedge (b \vee \Box\neg b) \in \nabla.\end{aligned}$$

The case of $\wedge_{\mathcal{G}(\mathbf{B})}$ is considered as follows:

$$\begin{aligned}(a \wedge_{\mathcal{G}(\mathbf{B})} b) \vee \Box\neg(a \wedge_{\mathcal{G}(\mathbf{B})} b) &= (a \wedge b) \vee \Box\neg(a \wedge b) = \\ &= (a \wedge b) \vee \Box(\neg a \vee \neg b) \geqslant (a \wedge b) \vee (\Box(\neg a) \vee \Box(\neg b)) = \\ &= (a\vee\Box(\neg a)\vee\Box(\neg b))\wedge(b\vee\Box(\neg b)\vee\Box(\neg a)) \geqslant (a\vee\Box\neg a)\wedge(b\vee\Box\neg b) \in \nabla.\end{aligned}$$

Finally, consider the implication:

$$\begin{aligned}(a \to_{\mathcal{G}(\mathbf{B})} b) \vee \Box\neg(a \to_{\mathcal{G}(\mathbf{B})} b) &= \Box(a \to b) \vee \Box\neg\Box(a \to b) = \\ &= \Box(\neg a \vee b) \vee \Box\Diamond\neg(\neg a \vee b) = \Box(\neg a \vee b) \vee \Box\Diamond(a \wedge \neg b) \geqslant \\ &\geqslant (\Box(\neg a) \vee \Box b) \vee \Box(a \wedge \neg b) = \Box(\neg a) \vee b \vee (a \wedge \Box\neg b) = \\ &= (a \vee \Box(\neg a) \vee b) \wedge (b \vee \Box(\neg b) \vee \Box(\neg a)) \geqslant (a \vee \Box\neg a) \wedge (b \vee \Box\neg b) \in \nabla.\end{aligned}$$

Obviously,

$$\bot_{\mathcal{G}(\mathbf{B})} \vee \Box\neg\bot_{\mathcal{G}(\mathbf{B})} = \bot \vee \Box\neg\bot = 1 \in \nabla. \qquad \Box$$

We will explore when $\mathsf{G}_2(\mathbf{T})$ is the domain of some twist-structure over a Heyting algebra. So it is important to consider its invariants. Define the sets:

$$\begin{aligned}\nabla_{\mathsf{G}}(\mathbf{T}) &:= \{a \vee b \mid (a, b) \in \mathsf{G}_2(\mathbf{T})\},\\ \Delta_{\mathsf{G}}(\mathbf{T}) &:= \{a \wedge b \mid (a, b) \in \mathsf{G}_2(\mathbf{T})\}.\end{aligned}$$

The following lemma gives other descriptions of these sets.

Lemma 3.1.2.

1. $\nabla_{\mathsf{G}}(\mathbf{T}) = \nabla \cap \mathsf{G}(\mathbf{B}) = \nabla \cap \Gamma(\mathbf{T}) = \nabla \cap \Lambda(\mathbf{B}, \nabla)$.
2. $\Delta_{\mathsf{G}}(\mathbf{T}) = \Delta \cap \mathsf{G}(\mathbf{B}) = \Delta \cap \Gamma(\mathbf{T})$.

Proof. 1. Let's prove $\nabla \cap \mathsf{G}(\mathbf{B}) \subseteq \nabla_{\mathsf{G}}(\mathbf{T})$. If $x \in \nabla \cap \mathsf{G}(\mathbf{B})$, then $x \vee \bot = x \in \nabla$ and $x \wedge \bot = \bot \in \Delta$, so $(x, \bot) \in \mathbf{T}$. Moreover, $(x, \bot) \in \mathsf{G}_2(\mathbf{T})$, since $\Box x = x$. Thus, $x \vee \bot = x \in \nabla_{\mathsf{G}}(\mathbf{T})$.

Let's show that $\nabla_{\mathsf{G}}(\mathbf{T}) \subseteq \nabla \cap \Lambda(\mathbf{B}, \nabla)$. If $x \in \nabla_{\mathsf{G}}(\mathbf{T})$, then $x = a \vee b$ for $(a, b) \in \mathsf{G}_2(\mathbf{T})$. Obviously, $\Box x = x$ and $x = a \vee b \in \nabla$, so $x \vee \Box\neg x \in \nabla$ and $x \in \Lambda(\mathbf{B}, \nabla)$.

Items 2 and 6 of Lemma 3.1.1 imply:

$$\nabla_{\mathsf{G}}(\mathbf{T}) \subseteq \nabla \cap \Lambda(\mathbf{B}, \nabla) \subseteq \nabla \cap \Gamma(\mathbf{T}) \subseteq \nabla \cap \mathsf{G}(\mathbf{B}) \subseteq \nabla_{\mathsf{G}}(\mathbf{T}).$$

2. Let's prove $\Delta_{\mathsf{G}}(\mathbf{T}) \subseteq \Delta \cap \Gamma(\mathbf{T})$. If $x \in \Delta_{\mathsf{G}}(\mathbf{T})$, then $x = a \wedge b$ for $(a, b) \in \mathsf{G}_2(\mathbf{T})$. Firstly, $x = a \wedge b \in \Delta$. Secondly, item 3 of Lemma 3.1.1 implies $(a, b) \wedge_{\mathbf{B}^{\bowtie}} \sim_{\mathbf{B}^{\bowtie}} (a, b) = (a \wedge b, a \vee b) \in \mathsf{G}_2(\mathbf{T})$, so $x = a \wedge b \in \Gamma(\mathbf{T})$.

Let's show that $\Delta \cap \mathsf{G}(\mathbf{B}) \subseteq \Delta_{\mathsf{G}}(\mathbf{T})$. If $x \in \Delta \cap \mathsf{G}(\mathbf{B})$, then $x \vee 1 = 1 \in \nabla$, $x \wedge 1 = x \in \Delta$, so $(x, 1) \in \mathbf{T}$. Moreover, $(x, 1) \in \mathsf{G}_2(\mathbf{T})$, because $\Box x = x$, so $x \wedge 1 = x \in \Delta_{\mathsf{G}}(\mathbf{T})$.

From item 2 of Lemma 3.1.1 we have

$$\Delta_{\mathsf{G}}(\mathbf{T}) \subseteq \Delta \cap \Gamma(\mathbf{T}) \subseteq \Delta \cap \mathsf{G}(\mathbf{B}) \subseteq \Delta_{\mathsf{G}}(\mathbf{T}). \qquad \Box$$

The following lemma will be useful in Subsection 3.3.

Lemma 3.1.3. *$\neg_{\mathcal{G}(\mathbf{B})}\neg_{\mathcal{G}(\mathbf{B})} a \in \Delta_{\mathsf{G}}(\mathbf{T})$ for all $a \in \Delta_{\mathsf{G}}(\mathbf{T})$.*

Proof. In view of Lemma 3.1.2, $\Delta_{\mathsf{G}}(\mathbf{T}) = \Delta \cap \mathsf{G}(\mathbf{B})$. If $a \in \Delta_{\mathsf{G}}(\mathbf{T})$, then, obviously, $\neg_{\mathcal{G}(\mathbf{B})}\neg_{\mathcal{G}(\mathbf{B})} a \in \mathsf{G}(\mathbf{B})$. We have $a \in \Delta$, and then $\Diamond a \in \Delta$, since $\Delta \in \mathcal{I}_{\Diamond}(\mathbf{B})$. Since $\Box\Diamond a \leqslant \Diamond a$, we obtain

$$\Box\Diamond a = \Box\neg\Box\neg a = \neg_{\mathcal{G}(\mathbf{B})}\neg_{\mathcal{G}(\mathbf{B})} a \in \Delta. \qquad \Box$$

The following lemma gives us conditions for $\mathsf{G}_2(\mathbf{T})$ to be the domain of some twist-structure over a subalgebra of $\mathcal{G}(\mathbf{B})$.

Lemma 3.1.4.

$$\Gamma(\mathbf{T}) \subseteq \Lambda(\mathbf{B}, \nabla) \iff \mathsf{G}_2(\mathbf{T}) \textit{ is closed under } \to_{\mathcal{G}(\mathbf{B})^{\bowtie}}.$$

Proof. $\Longrightarrow$. Let $(a, b), (c, d) \in \mathsf{G}_2(\mathbf{T})$. We know that $c \wedge d \in \Delta$, so $\Box(a \to c) \wedge (a \wedge d) \in \Delta$ can be shown with ease:

$$\Box(a \to c) \wedge (a \wedge d) \leqslant (a \to c) \wedge (a \wedge d) =$$
$$= (a \wedge (a \to c)) \wedge d = a \wedge c \wedge d \leqslant c \wedge d \in \Delta.$$

Notice that

$$\Box(a \to c) \vee (a \wedge d) = (\Box(a \to c) \vee a) \wedge (\Box(a \to c) \vee d),$$

and we can easily show $\Box(a \to c) \vee d \in \nabla$:

$$\Box(a \to c) \vee d = \Box(\neg a \vee c) \vee d \geqslant \Box c \vee d = c \vee d \in \nabla.$$

In view of $a \in \Gamma(\mathbf{T})$ and $\Gamma(\mathbf{T}) \subseteq \Lambda(\mathbf{B}, \nabla)$ we have $\Box(a \to c) \vee a \in \nabla$:

$$\Box(a \to c) \vee a = \Box(\neg a \vee c) \vee a \geqslant \Box(\neg a) \vee a \in \nabla.$$

So $\Box(a \to c) \vee (a \wedge d) \in \nabla$, and from $a \wedge d \in \mathsf{G}(\mathbf{B})$ we have thus proved

$$(a, b) \to_{\mathcal{G}(\mathbf{B})^{\bowtie}} (c, d) = (\Box(a \to c), a \wedge d) \in \mathsf{G}_2(\mathbf{T}).$$

$\Longleftarrow$. If $a \in \Gamma(\mathbf{T})$, then $(a, b) \in \mathsf{G}_2(\mathbf{T})$ for some $b \in \mathsf{G}(\mathbf{B})$. We know that $\perp_{\mathcal{G}(\mathbf{B})^{\bowtie}} = (\perp, 1) \in \mathsf{G}_2(\mathbf{T})$, so, since $\mathsf{G}_2(\mathbf{T})$ is closed under $\to_{\mathcal{G}(\mathbf{B})^{\bowtie}}$,

$$(a, b) \to_{\mathcal{G}(\mathbf{B})^{\bowtie}} (\perp, 1) = (\Box(a \to \perp), a \wedge 1) = (\Box\neg a, a) \in \mathsf{G}_2(\mathbf{T}),$$

which implies $a \vee \Box\neg a \in \nabla$. Since $\Box a = a$, we get $a \in \Lambda(\mathbf{B}, \nabla)$. $\Box$

And now we explain how to extract a twist-structure over a Heyting algebra from a twist-structure over a TBA.

Proposition 3.1.5. *Let* $\Gamma(\mathbf{T}) = \Lambda(\mathbf{B}, \nabla)$. *Then:*

1. $\mathbf{\Gamma}(\mathbf{T}) = \langle \Gamma(\mathbf{T}); \vee_{\mathcal{G}(\mathbf{B})}, \wedge_{\mathcal{G}(\mathbf{B})}, \to_{\mathcal{G}(\mathbf{B})}, \perp_{\mathcal{G}(\mathbf{B})} \rangle$ *is a subalgebra of* $\mathcal{G}(\mathbf{B})$.
2. $\mathcal{G}_2(\mathbf{T}) = \langle \mathsf{G}_2(\mathbf{T}); \vee_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \wedge_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \to_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \perp_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \sim_{\mathcal{G}(\mathbf{B})^{\bowtie}} \rangle$ *is a twist-structure over the Heyting algebra* $\mathbf{\Gamma}(\mathbf{T})$. *More precisely,*

$$\mathcal{G}_2(\mathbf{T}) = Tw(\mathbf{\Gamma}(\mathbf{T}), \nabla_{\mathsf{G}}(\mathbf{T}), \Delta_{\mathsf{G}}(\mathbf{T})).$$

Proof. 1. This is just a consequence of item 7 of Lemma 3.1.1.

2. In view of item 4 of Lemma 3.1.1 $\mathsf{G}_2(\mathbf{T}) \subseteq \pi_1(\mathsf{G}_2(\mathbf{T})) \times \pi_2(\mathsf{G}_2(\mathbf{T})) = \Gamma(\mathbf{T}) \times \Gamma(\mathbf{T})$, and $\pi_1(\mathsf{G}_2(\mathbf{T})) = \Gamma(\mathbf{T})$ by definition. Obviously, the operations of $\mathcal{G}(\mathbf{B})^{\bowtie}$ coincide with operations of $\mathbf{\Gamma}(\mathbf{T})^{\bowtie}$. The condition $\Gamma(\mathbf{T}) = \Lambda(\mathbf{B}, \nabla)$ implies by Lemma 3.1.4 that $\mathsf{G}_2(\mathbf{T})$ is closed under $\to_{\mathcal{G}(\mathbf{B})^{\bowtie}}$. In view of item 3 of Lemma 3.1.1 $\mathsf{G}_2(\mathbf{T})$ is closed under the operations $\vee_{\mathcal{G}(\mathbf{B})^{\bowtie}}$, $\wedge_{\mathcal{G}(\mathbf{B})^{\bowtie}}$, $\sim_{\mathcal{G}(\mathbf{B})^{\bowtie}}$ and $\perp_{\mathcal{G}(\mathbf{B})^{\bowtie}} \in \mathsf{G}_2(\mathbf{T})$. So we have that $\mathcal{G}_2(\mathbf{T}) = \langle \mathsf{G}_2(\mathbf{T}); \vee_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \wedge_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \to_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \perp_{\mathcal{G}(\mathbf{B})^{\bowtie}}, \sim_{\mathcal{G}(\mathbf{B})^{\bowtie}} \rangle$ is a twist-structure over $\mathbf{\Gamma}(\mathbf{T})$. From $a \vee_{\mathcal{G}(\mathbf{B})} b = a \vee b$ and $a \wedge_{\mathcal{G}(\mathbf{B})} b = a \wedge b$ we get:

$$\mathcal{G}_2(\mathbf{T}) = Tw(\mathbf{\Gamma}(\mathbf{T}), \nabla_{\mathsf{G}}(\mathbf{T}), \Delta_{\mathsf{G}}(\mathbf{T})).$$ $\Box$

Definition 3.1.1. In case $\Gamma(\mathbf{T}) = \Lambda(\mathbf{B}, \nabla)$ we call the twist-structure $\mathcal{G}_2(\mathbf{T})$ the *algebra of open pairs of the twist-structure* $\mathbf{T}$.

3.2. **Preserving the truth.** We work with the same TBA $\mathbf{B}$, $\nabla \in \mathcal{F}_{\Box}(\mathbf{B})$, $\Delta \in \mathcal{I}_{\Diamond}(\mathbf{B})$ and $\mathbf{T} := Tw(\mathbf{B}, \nabla, \Delta)$.

The main goal of the present subsection is to find conditions guaranteeing that for every $\varphi \in \mathrm{Fm}_{\mathcal{L}_{\sim}}$ the following holds:

$$\text{(TwTop)} \qquad \mathcal{G}_2(\mathbf{T}) \models \varphi \iff \mathbf{T} \models \mathrm{T}_{\mathbf{B}}\varphi.$$

Assume that $\Gamma(\mathbf{T}) = \Lambda(\mathbf{B}, \nabla)$ and $v\colon \mathrm{Prop} \to \mathsf{G}_2(\mathbf{T})$. Then functions $h_1(v)\colon \mathrm{Fm}_{\mathcal{L}^{\Box}_{\sim}} \to \mathbf{T}$ and $h_2(v)\colon \mathrm{Fm}_{\mathcal{L}_{\sim}} \to \mathcal{G}_2(\mathbf{T})$ denote unique homomorphisms such that $h_1(v)\restriction_{\mathrm{Prop}}= v$ and $h_2(v)\restriction_{\mathrm{Prop}}= v$. Also, for each map $w\colon \mathrm{Prop} \to \mathbf{T}$, we denote by $h_1(w)$ the unique $\mathbf{T}$-valuation such that $h_1(w)\restriction_{\mathrm{Prop}}= w$.

Lemma 3.2.1. *Suppose that $\Gamma(\mathbf{T}) = \Lambda(\mathbf{B}, \nabla)$ and $v\colon \mathrm{Prop} \to \mathsf{G}_2(\mathbf{T})$. Then for every $\varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}$ it is true that $\pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}\varphi)) = \pi_1(h_2(v)(\varphi))$.*

Proof. It is a long but routine work to check this by induction of the construction of a formula. For instance, if $\pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}\varphi)) = \pi_1(h_2(v)(\varphi))$ and $\pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}{\sim}\psi)) = \pi_1(h_2(v)({\sim}\psi))$, then

$$\begin{aligned}\pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}({\sim}(\varphi \to \psi)))) &= \pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}(\varphi) \wedge \mathrm{T}_{\mathbf{B}}({\sim}\psi))) = \\ &= \pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}\varphi)) \wedge_{\mathbf{B}} \pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}{\sim}\psi)) = \\ &= \pi_1(h_2(v)(\varphi)) \wedge_{\mathcal{G}(\mathbf{B})} \pi_1(h_2(v)({\sim}\psi)) = \\ &= \pi_1(h_2(v)(\varphi)) \wedge_{\mathcal{G}(\mathbf{B})} \pi_1({\sim}_{\mathbf{\Gamma(T)}^{\bowtie}} h_2(v)(\psi)) = \\ &= \pi_1(h_2(v)(\varphi)) \wedge_{\mathcal{G}(\mathbf{B})} \pi_2(h_2(v)(\psi)) = \\ &= \pi_2(h_2(v)(\varphi) \to_{\mathbf{\Gamma(T)}^{\bowtie}} h_2(v)(\psi)) = \\ &= \pi_1({\sim}_{\mathbf{\Gamma(T)}^{\bowtie}}(h_2(v)(\varphi) \to_{\mathbf{\Gamma(T)}^{\bowtie}} h_2(v)(\psi))) = \pi_1(h_2(v)({\sim}(\varphi \to \psi))).\end{aligned}$$
$\square$

Proposition 3.2.2. *Suppose that $(\Box a, \Box b) \in \mathbf{T}$ for all $(a, b) \in \mathbf{T}$. Then $\mathsf{G}(\mathbf{B}) = \Gamma(\mathbf{T}) = \Lambda(\mathbf{B}, \nabla)$. Furthermore, for every $\varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}$,*

$$\text{(TwTop)} \qquad \mathcal{G}_2(\mathbf{T}) \models \varphi \iff \mathbf{T} \models \mathrm{T}_{\mathbf{B}}\varphi.$$

Proof. Let us show that $\mathsf{G}(\mathbf{B}) \subseteq \Lambda(\mathbf{B}, \nabla)$. If $x \in \mathsf{G}(\mathbf{B})$, then $(x, \neg x) \in \mathbf{T}$, because $x \wedge \neg x = \bot \in \Delta$, $x \vee \neg x = 1 \in \nabla$. So $(\Box x, \Box\neg x) = (x, \Box\neg x) \in \mathbf{T}$, which implies $x \vee \Box\neg x \in \nabla$. Thus, $x \in \Lambda(\mathbf{B}, \nabla)$.

Now, we prove the equivalence (TwTop).

$\Longleftarrow$. Let $\varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}$ and $\mathbf{T} \models \mathrm{T}_{\mathbf{B}}\varphi$. Consider an arbitrary map $v\colon \mathrm{Prop} \to \mathsf{G}_2(\mathbf{T})$. By Lemma 3.2.1, we have

$$\pi_1(h_2(v)(\varphi)) = \pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}\varphi)) = 1_{\mathbf{B}} = 1_{\mathcal{G}(\mathbf{B})}.$$

$\Longrightarrow$. Let $\varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}$ and $\mathcal{G}_2(\mathbf{T}) \models \varphi$. Consider an arbitrary map $v\colon \mathrm{Prop} \to \mathbf{T}$. Define $w\colon \mathrm{Prop} \to \mathsf{G}_2(\mathbf{T})$ by

$$w(p) := (\Box\pi_1(v(p)), \Box\pi_2(v(p))).$$

The map w is well-defined, since $(\Box a, \Box b) \in \mathbf{T}$ for all $(a, b) \in \mathbf{T}$.

We know that $\sim$ occurs in $\mathrm{T}_{\mathbf{B}}\varphi$ only in front of propositional variables and $\bot$. From $\mathrm{T}_{\mathbf{B}}(p) = \Box p$ and $\mathrm{T}_{\mathbf{B}}({\sim}p) = \Box{\sim}p$ we have that there is a $\Box$ before each occurrence of p or ${\sim}p$. Thus, if $\varphi = \varphi(p_1, \ldots, p_n)$, then there is a formula $\psi(p_1, \ldots, p_n, q_1, \ldots, q_n, s) \in \mathrm{Fm}_{\mathcal{L}^{\Box}}$ such that

$\mathrm{T}_{\mathbf{B}}\varphi = \psi(\Box p_1, \ldots, \Box p_n, \Box{\sim} p_1, \ldots, \Box{\sim} p_n, {\sim}\bot)$. So we have

$$\begin{aligned}
&\pi_1(h_1(v)(\mathrm{T}_{\mathbf{B}}\varphi)) = \\
&\quad = \pi_1(h_1(v)(\psi(\Box p_1, \ldots, \Box p_n, \Box{\sim} p_1, \ldots, \Box{\sim} p_n, {\sim}\bot))) = \\
&\quad = \psi(\Box\pi_1(v(p_1)), \ldots, \Box\pi_1(v(p_n)), \Box\pi_2(v(p_1)), \ldots, \Box\pi_2(v(p_n)), 1) = \\
&\quad = \psi(\Box\Box\pi_1(v(p_1)), \ldots, \Box\Box\pi_1(v(p_n)), \Box\Box\pi_2(v(p_1)), \ldots, \Box\Box\pi_2(v(p_n)), 1) = \\
&\quad = \psi(\Box\pi_1(w(p_1)), \ldots, \Box\pi_1(w(p_n)), \Box\pi_2(w(p_1)), \ldots, \Box\pi_2(w(p_n)), 1) = \\
&\quad = \pi_1(h_1(w)(\psi(\Box p_1, \ldots, \Box p_n, \Box{\sim} p_1, \ldots, \Box{\sim} p_n, {\sim}\bot))) = \\
&\quad = \pi_1(h_1(w)(\mathrm{T}_{\mathbf{B}}\varphi)) = \pi_1(h_2(w)(\varphi)) = 1_{\mathcal{G}(\mathbf{B})} = 1_{\mathbf{B}}. \quad \Box
\end{aligned}$$

Now we recall the Kripke semantics for Grz.

A *frame* is a pair $\mathcal{W} = \langle W, \leqslant \rangle$, where W is a non-empty set of *worlds*, and $\leqslant$ is a partial ordering on W.

A *model* over $\mathcal{W} = \langle W, \leqslant \rangle$ is a pair $\mathcal{M} = \langle \mathcal{W}, v \rangle$, where $v\colon \mathrm{Prop} \to 2^W$ is a *valuation* of propositional variables.

Given a model $\mathcal{M} = \langle \mathcal{W}, v \rangle$ over $\mathcal{W} = \langle W, \leqslant \rangle$ we define the truth relation between formulas of $\mathrm{Fm}_{\mathcal{L}^\Box}$ and worlds of W:

- $\mathcal{M}, x \models p \iff x \in v(p)$;
- $\mathcal{M}, x \not\models \bot$;
- $\mathcal{M}, x \models \varphi \wedge \psi \iff (\mathcal{M}, x \models \varphi$ and $\mathcal{M}, x \models \psi)$;
- $\mathcal{M}, x \models \varphi \vee \psi \iff (\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi)$;
- $\mathcal{M}, x \models \varphi \to \psi \iff (\mathcal{M}, x \models \varphi \Rightarrow \mathcal{M}, x \models \psi)$;
- $\mathcal{M}, x \models \Box\varphi \iff \forall y \in W\ (x \leqslant y \Rightarrow \mathcal{M}, y \models \varphi)$.

It is not hard to see that

$$\mathcal{M}, x \models \Diamond\varphi \iff \exists y \in W\ (x \leqslant y \text{ and } \mathcal{M}, y \models \varphi)\,.$$

A formula φ is *true* in a model $\mathcal{M}$, $\mathcal{M} \models \varphi$, if $\mathcal{M}, x \models \varphi$ for all $x \in W$; φ is *valid* in a frame $\mathcal{W}$, $\mathcal{W} \models \varphi$, if $\mathcal{M} \models \varphi$ for each model $\mathcal{M}$ over $\mathcal{W}$.

It is well-known (see, for example, [5, Theorem 5.51]) that for every $\varphi \in \mathrm{Fm}_{\mathcal{L}^\Box}$:

$$\varphi \in \mathsf{Grz} \iff \mathcal{W} \models \varphi \text{ for all finite frames } \mathcal{W}.$$

Lemma 3.2.3.

$$\big[\Box(p \vee q) \wedge (\Box p \vee \Box\Diamond\neg p) \wedge (\Box q \vee \Box\Diamond\neg q)\big] \to (\Box p \vee \Box q) \in \mathsf{Grz}.$$

Proof. Let $\mathcal{W}$ be an arbitrary finite frame, $\mathcal{M}$ be a model over $\mathcal{W}$ and $x \in W$.

We need to show that if

(1) $\mathcal{M}, x \models \Box(p \vee q)$,
(2) $\mathcal{M}, x \models \Box p \vee \Box\Diamond\neg p$ and

(3) $\mathcal{M}, x \models \Box q \vee \Box\Diamond\neg q$,

then $\mathcal{M}, x \models \Box p \vee \Box q$.

Suppose that $\mathcal{M}, x \not\models \Box p$ and $\mathcal{M}, x \not\models \Box q$. In view of (2) and (3) it means that $\mathcal{M}, x \models \Box\Diamond\neg p$ and $\mathcal{M}, x \models \Box\Diamond\neg q$.

There is a maximal $y \in W$ w.r.t. $\leqslant$ such that $x \leqslant y$, because W is finite. From $\mathcal{M}, x \models \Box\Diamond\neg p$ and $\mathcal{M}, x \models \Box\Diamond\neg q$ we get that $\mathcal{M}, y \models \Diamond\neg p$ and $\mathcal{M}, y \models \Diamond\neg q$.

From the maximality of y we have $\mathcal{M}, y \models \neg p$ and $\mathcal{M}, y \models \neg q$. But we also have $\mathcal{M}, y \models p \vee q$ because of (1): a contradiction. Thus, we have proved $\mathcal{M}, x \models \Box p \vee \Box q$. $\Box$

Lemma 3.2.4. *Suppose that* $\mathbf{B} \models \mathsf{Grz}$ *and* $\mathsf{G}(\mathbf{B}) = \Lambda(\mathbf{B}, \nabla)$. *Then* $(\Box a, \Box b) \in \mathbf{T}$ *for all* $(a, b) \in \mathbf{T}$.

Proof. Suppose that $(a, b) \in \mathbf{T}$. Then $a \vee b \in \nabla$ and $a \wedge b \in \Delta$. We can easily show that $\Box a \wedge \Box b \in \Delta$:

$$\Box a \wedge \Box b \leqslant a \wedge b \in \Delta.$$

Since ∇ is an open filter, $\Box(a \vee b) \in \nabla$ holds. We have $\Box a \vee \Box\neg\Box a = \Box a \vee \Box\Diamond\neg a$ and $\Box b \vee \Box\neg\Box b = \Box b \vee \Box\Diamond\neg b$, so in view of $\mathsf{G}(\mathbf{B}) = \Lambda(\mathbf{B}, \nabla)$

$$\Box(a \vee b) \wedge (\Box a \vee \Box\Diamond\neg a) \wedge (\Box b \vee \Box\Diamond\neg b) \in \nabla.$$

From $\mathbf{B} \models \mathsf{Grz}$ and Lemma 3.2.3 we obtain that $\Box a \vee \Box b \in \nabla$. Since $\Box a \wedge \Box b \in \Delta$, we get $(\Box a, \Box b) \in \mathbf{T}$. $\Box$

Theorem 3.2.5. *Suppose that* $\mathbf{B} \models \mathsf{Grz}$ *and* $\mathsf{G}(\mathbf{B}) = \Lambda(\mathbf{B}, \nabla)$. *Then for every* $\varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}$

$$\mathcal{G}_2(\mathbf{T}) \models \varphi \iff \mathbf{T} \models \mathrm{T}_{\mathbf{B}}\varphi.$$

Proof. This is an immediate consequence of Lemma 3.2.4 and Proposition 3.2.2. $\Box$

3.3. **From $\mathsf{N4}^{\perp}$-twist-structures to $\mathsf{BS4}$-twist-structures.** In this subsection given a twist-structure $\mathcal{A}$ over a Heyting algebra we construct a twist-structure over a TBA such that $\mathcal{A}$ is a subalgebra of its algebra of open pairs.

We fix an arbitrary Heyting algebra $\mathbf{A} = \langle A; \vee_{\mathbf{A}}, \wedge_{\mathbf{A}}, \to_{\mathbf{A}}, \perp_{\mathbf{A}} \rangle$ with the greatest element $1_{\mathbf{A}}$, and we will search for a twist-structure over a TBA $\mathbf{B} := s(\mathbf{A})$. We write the operations of $\mathbf{A}$ and $\mathbf{B}$ without subscripts if it does not lead to a confusion. In what follows we always write $\neg$ for the negation operation of $\mathbf{A}$ and $\neg_{\mathbf{B}}$ for the negation operation of $\mathbf{B}$.

It is well-known (see, for example, [11, pp. 61–62]) that there is a bijective correspondence between open filters of $\mathbf{B}$ and filters of $\mathcal{G}(\mathbf{B})$ given by the following mutually inverse mappings:

$$\delta\colon \nabla \in \mathcal{F}_{\Box}(\mathbf{B}) \longmapsto \delta(\nabla) = \nabla \cap \mathsf{G}(\mathbf{B}) \in \mathcal{F}(\mathcal{G}(\mathbf{B}))$$
$$\rho\colon \nabla' \in \mathcal{F}(\mathcal{G}(\mathbf{B})) \longmapsto \rho(\nabla') = \{x \in B \mid \Box x \in \nabla'\} \in \mathcal{F}_{\Box}(\mathbf{B})$$

For each $\Delta \in \mathcal{I}(\mathbf{A})$ there is the least closed ideal of the TBA $\mathbf{B}$ containing Δ, which can be defined as

$$\sigma(\Delta) := \{x \in B \mid x \leqslant_{\mathbf{B}} \Diamond y \text{ for some } y \in \Delta\}.$$

It is easy to check that $\sigma(\Delta) \in \mathcal{I}_{\Diamond}(\mathbf{B})$, and it is obvious that $\sigma(\Delta)$ is contained in each closed ideal of $\mathbf{B}$ containing Δ.

We call an ideal Δ of a Heyting algebra $\mathbf{A}$ *closed*, if $\neg\neg a \in \Delta$ for all $a \in \Delta$. Not each ideal of $\mathbf{A}$ is closed. For instance, an ideal $\{x \in A \mid x \leqslant_{\mathbf{A}} a\}$ with $a \neq \neg\neg a$ is not closed. We can see from Lemma 3.1.3 that the ideal $\Delta_{\mathsf{G}}(\mathbf{T})$ of the algebra $\mathcal{G}_2(\mathbf{T})$ of open pairs is closed.

It is not hard to check that for $\Delta \in \mathcal{I}(\mathbf{A})$ the set

$$N(\Delta) := \{a \in A \mid a \leqslant_{\mathbf{A}} \neg\neg b \ \text{ for some } b \in \Delta\}.$$

is a closed ideal of $\mathbf{A}$, and it is a subset of each closed ideal of $\mathbf{A}$ containing Δ, so it is the least closed ideal of $\mathbf{A}$ containing Δ. We call this set the *closure of* Δ.

Lemma 3.3.1. $\sigma(\Delta) \cap A = N(\Delta)$.

Proof. $\subseteq$. If $x \in \sigma(\Delta) \cap A$, then $\Box x = x$ and $x \leqslant_{\mathbf{B}} \Diamond y$ for some $y \in \Delta$. So $x = \Box x \leqslant_{\mathbf{B}} \Box\Diamond y = \neg\neg y$, which means $x \in N(\Delta)$.

$\supseteq$. Let $x \in N(\Delta)$. Then $x \in A$ and $x \leqslant_{\mathbf{A}} \neg\neg y$ for some $y \in \Delta$. So $x \leqslant_{\mathbf{B}} \Box\Diamond y \leqslant_{\mathbf{B}} \Diamond y$, which means $x \in \sigma(\Delta)$. $\Box$

Now we fix $\nabla \in \mathcal{F}_d(\mathbf{A})$ and $\Delta \in \mathcal{I}(\mathbf{A})$.

Theorem 3.3.2. *Let* $\mathbf{T} = Tw(\mathbf{B}, \rho(\nabla), \sigma(\Delta))$. *Then for every* $\varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}$

$$\mathbf{T} \models \mathrm{T}_{\mathbf{B}}\varphi \iff Tw(\mathbf{A}, \nabla, N(\Delta)) \models \varphi.$$

Proof. We know that $\nabla \in \mathcal{F}_d(\mathbf{A})$, so for every $a \in A$ we have $a \vee \neg a = a \vee \Box\neg_{\mathbf{B}} a \in \nabla \subseteq \rho(\nabla)$. In view of $A = \mathsf{G}(\mathbf{B})$ we obtain $\mathsf{G}(\mathbf{B}) = \Lambda(\mathbf{B}, \rho(\nabla))$.

It is well-known that if $\mathbf{C}$ is a TBA such that $\mathbf{C}^s = \mathbf{C}$, then $\mathbf{C} \models \mathsf{Grz}$ (see, for example, [16, Corollary 3.5.14]). So $\mathbf{B} = s(\mathbf{A}) \models \mathsf{Grz}$.

Now, let us show that $\mathcal{G}_2(\mathbf{T}) = Tw(\mathbf{A}, \nabla, N(\Delta))$. Firstly, $A = \mathsf{G}(\mathbf{B}) = \Gamma(\mathbf{T}) = \Lambda(\mathbf{B}, \rho(\nabla))$, because $\mathsf{G}(\mathbf{B}) = \Lambda(\mathbf{B}, \rho(\nabla))$. So $\mathcal{G}_2(\mathbf{T})$ is a twist-structure over $\mathbf{A}$. By Lemma 3.1.2 we know that the invariants of this twist-structure are $\rho(\nabla) \cap A = \delta(\rho(\nabla)) = \nabla$ and $\sigma(\Delta) \cap A = N(\Delta)$ (by Lemma 3.3.1). Thus, $\mathcal{G}_2(\mathbf{T}) = Tw(\mathbf{A}, \nabla, N(\Delta))$.

And now, the conclusion of the theorem follows from $\mathbf{B} \models \mathsf{Grz}$, $\mathsf{G}(\mathbf{B}) = \Lambda(\mathbf{B}, \rho(\nabla))$ and Theorem 3.2.5. $\square$

From Lemma 3.1.3 and the proof of Theorem 3.3.2 we infer

Corollary 3.3.3. *A twist-structure over a Heyting algebra can be represented as the algebra of open pairs of some twist-structure over a* TBA *if and only if its ideal is closed.*

4. Modal companions

4.1. **Logics with modal companions.** An easy corollary of Theorem 3.3.2 is the following

Proposition 4.1.1. *If $L \in \mathcal{E}\mathsf{N4}^{\perp}$ has a proper model with closed ideal, then $\tau_{\mathbf{B}}L$ is a modal companion of L.*

Proof. Let $Tw(\mathbf{A}, \nabla, \Delta)$ be a proper model of L, and assume that $N(\Delta) = \Delta$. Let $\mathbf{T}$ denote $Tw(s(\mathbf{A}), \rho(\nabla), \sigma(\Delta))$. By Theorem 3.3.2, for every $\varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}$ we have

$$\mathbf{T} \models \mathrm{T}_{\mathbf{B}}\varphi \iff Tw(\mathbf{A}, \nabla, \Delta) \models \varphi.$$

So $\mathbf{T} \models \tau_{\mathbf{B}}L$, and if $\varphi \notin L$, then $\mathbf{T} \not\models \mathrm{T}_{\mathbf{B}}\varphi$ implies $\mathrm{T}_{\mathbf{B}}\varphi \notin \tau_{\mathbf{B}}L$. $\square$

We recall that the Lindenbaum twist-structure $\mathcal{W}_L$ over the Heyting algebra $\mathcal{T}(L)$ is a proper model of $L \in \mathcal{E}\mathsf{N4}^{\perp}$, and its invariants are

$$\nabla_L = \{[\varphi \vee {\sim}\varphi]_L \mid \varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}\}$$

and

$$\Delta_L = \{[\varphi \wedge {\sim}\varphi]_L \mid \varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}\}.$$

In view of the previous proposition we are curious for which $L \in \mathcal{E}\mathsf{N4}^{\perp}$ the ideal Δ_L is closed.

Proposition 4.1.2. *Let $L \in \mathcal{E}\mathsf{N4}^{\perp}$. Then*

$$\Delta_L \textit{ is closed } \iff \neg\neg(p \wedge {\sim}p) \leftrightarrow (p \wedge {\sim}p) \in L.$$

Proof. $\Longleftarrow$. This is obvious.

$\Longrightarrow$. If Δ_L is closed, then $\neg\neg(p_1 \wedge {\sim}p_1) \leftrightarrow (\psi \wedge {\sim}\psi) \in L$ for some $\psi \in \mathrm{Fm}_{\mathcal{L}_\sim}$. By the substitution rule (SUB) we can consider $\psi = \psi(p_1)$ such that it depends only on p_1.

Define the following ideal in $\mathcal{T}(L)$:

$$\Delta_1 := \{a \in \mathcal{T}(L) \mid a \leqslant_{\mathcal{T}(L)} [p_1 \wedge {\sim}p_1]_{\equiv_L}\}.$$

Let $\mathcal{C}$ denote $Tw(\mathcal{T}(L), \nabla_L, \Delta_1)$. The algebra $\mathcal{C}$ is a subalgebra of $Tw(\mathcal{T}(L), \nabla_L, \Delta_L)$, so if

$$Tw(\mathcal{T}(L), \nabla_L, \Delta_L) \models \neg\neg(p_1 \wedge \sim p_1) \leftrightarrow (\psi \wedge \sim\psi),$$

then

$$\mathcal{C} \models \neg\neg(p_1 \wedge \sim p_1) \leftrightarrow (\psi \wedge \sim\psi).$$

Consider the $\mathcal{C}$-valuation $v(p_1) := ([p_1]_{\equiv_L}, [\sim p_1]_{\equiv_L})$ and assume that $v(\psi(p_1)) = (a, b)$ for some $a, b \in \mathcal{T}(L)$. We have

$$\pi_1(v(\neg\neg(p_1 \wedge \sim p_1) \leftrightarrow (\psi \wedge \sim\psi))) = \\ = \neg\neg[p_1 \wedge \sim p_1]_{\equiv_L} \leftrightarrow (a \wedge b) = 1_{\mathcal{T}(L)},$$

so $\neg\neg[p_1 \wedge \sim p_1]_{\equiv_L} = (a \wedge b)$. From $a \wedge b \in \Delta_1$ we get

$$\neg\neg[p_1 \wedge \sim p_1]_{\equiv_L} \in \Delta_1,$$

or

$$\neg\neg[p_1 \wedge \sim p_1]_{\equiv_L} \leqslant_{\mathcal{T}(L)} [p_1 \wedge \sim p_1]_{\equiv_L},$$

which implies $\neg\neg(p_1 \wedge \sim p_1) \rightarrow (p_1 \wedge \sim p_1) \in L$. From $q \rightarrow \neg\neg q \in \mathsf{Int}$ we obtain $\neg\neg(p_1 \wedge \sim p_1) \leftrightarrow (p_1 \wedge \sim p_1) \in L$. □

Theorem 4.1.3. *Let L be a logic that extends*

$$\mathsf{N4}^{\perp} + \{\neg\neg(p \wedge \sim p) \leftrightarrow (p \wedge \sim p)\}.$$

Then $\tau_{\mathbf{B}} L$ is a modal companion of L.

Proof. According to the previous proposition, $\mathcal{W}_L$ is a proper model of L with closed ideal. Now, the conclusion follows from Proposition 4.1.1. □

Theorem 4.1.3 immediately implies

Corollary 4.1.4. *Each extension of the explosive Nelson's logic $\mathsf{N3}^{\perp} = \mathsf{N4}^{\perp} + \{\neg(p \wedge \sim p)\}$ has a modal companion.*

Now we strengthen the results of [28] by placing restrictions on the axioms of a logic that will guarantee that the logic has a modal companion.

Lemma 4.1.5. *Let $\mathbf{A}$ be a Heyting algebra, $\nabla \in \mathcal{F}_d(\mathbf{A})$, $\Delta_1, \Delta_2 \in \mathcal{I}(\mathbf{A})$. And let $\varphi \in \mathrm{Fm}_{\mathcal{L}_\sim}$ be a formula such that*

$$(\#) \qquad \varphi = \psi(p_1, \ldots, p_n, q_1 \vee \sim q_1, \ldots, q_m \vee \sim q_m),$$

where $\psi(p_1, \ldots, p_n, q_1, \ldots, q_m) \in \mathrm{Fm}_{\mathcal{L}_i}$ and $\{p_1, \ldots, p_n\} \cap \{q_1, \ldots, q_m\} = \varnothing$. Then

$$Tw(\mathbf{A}, \nabla, \Delta_1) \models \varphi \iff Tw(\mathbf{A}, \nabla, \Delta_2) \models \varphi$$

Note that in case $m = 0$ formulas of the form (#) are just formulas of $\mathrm{Fm}_{\mathcal{L}_i}$.

Proof. It is enough to prove that

$$Tw(\mathbf{A}, \nabla, \Delta_1) \models \varphi \Longrightarrow Tw(\mathbf{A}, \nabla, \Delta_2) \models \varphi.$$

Given a valuation $v \colon \mathrm{Prop} \to Tw(\mathbf{A}, \nabla, \Delta_1)$, assume that $v(p_i) = (x_i, y_i)$ and $v(q_j) = (a_j, b_j)$. Then we have

$$\begin{aligned} &\pi_1(v(\varphi)) = \\ &= \psi(\pi_1(v(p_1)), \ldots, \pi_1(v(p_n)), \pi_1(v(q_1 \vee {\sim} q_1)), \ldots, \pi_1(v(q_m \vee {\sim} q_m))) = \\ &\qquad = \psi(x_1, \ldots, x_n, a_1 \vee b_1, \ldots, a_m \vee b_m). \end{aligned}$$

Let's show that $\psi(d_1, \ldots, d_n, c_1, \ldots, c_m) = 1_{\mathbf{A}}$ for all $d_1, \ldots, d_n \in \mathbf{A}$ and $c_1, \ldots, c_m \in \nabla$. Indeed, in view of $\pi_1(Tw(\mathbf{A}, \nabla, \Delta_1)) = \mathbf{A}$, we can define the $Tw(\mathbf{A}, \nabla, \Delta_1)$-valuation $v_0(p_i) := (d_i, d_i')$ for some $d_1', \ldots, d_n' \in \mathbf{A}$. The condition $c \in \nabla$ implies $c \vee \bot = c \in \nabla$ and $c \wedge \bot = \bot \in \Delta_1$, so we can define $v_0(q_j) := (c_j, \bot)$. Thus, we have by Remark 1

$$\pi_1(v_0(\varphi)) = \psi(d_1, \ldots, d_n, c_1, \ldots, c_m) = 1_{\mathbf{A}}.$$

Now for an arbitrary valuation $v \colon \mathrm{Prop} \to Tw(\mathbf{A}, \nabla, \Delta_2)$ such that $v(p_i) = (e_i, f_i)$ and $v(q_j) = (z_j, g_j)$ we have

$$\pi_1(v(\varphi)) = \psi(e_1, \ldots, e_n, z_1 \vee g_1, \ldots, z_m \vee g_m) = 1_{\mathbf{A}},$$

since $e_1, \ldots, e_n \in \mathbf{A}$ and $z_1 \vee g_1, \ldots, z_m \vee g_m \in \nabla$. □

Theorem 4.1.6. *Suppose that $L \in \mathcal{E}\mathsf{N4}^{\bot}$ can be axiomatized as*

$$L = \mathsf{N4}^{\bot} + X,$$

where X is some set of formulas of the form (#). Then $\tau_{\mathbf{B}} L$ is a modal companion of L.

Proof. The algebra $Tw(\mathcal{T}(L), \nabla_L, \Delta_L)$ is a subalgebra of the twist-structure $Tw(\mathcal{T}(L), \nabla_L, N(\Delta_L))$, so

$$\mathrm{L}Tw(\mathcal{T}(L), \nabla_L, \Delta_L) \supseteq \mathrm{L}Tw(\mathcal{T}(L), \nabla_L, N(\Delta_L)).$$

On the other hand, Lemma 4.1.5 guarantees that for every $\varphi \in X$,

$$Tw(\mathcal{T}(L), \nabla_L, \Delta_L) \models \varphi \Longrightarrow Tw(\mathcal{T}(L), \nabla_L, N(\Delta_L)) \models \varphi.$$

Thus,

$$L = \mathrm{L}Tw(\mathcal{T}(L), \nabla_L, \Delta_L) = \mathrm{L}Tw(\mathcal{T}(L), \nabla_L, N(\Delta_L)),$$

so $Tw(\mathcal{T}(L), \nabla_L, N(\Delta_L))$ is a proper model of L with closed ideal. Now, we apply Proposition 4.1.1. □

Remark 2. Theorem 4.1.6 implies that the so-called *special extensions of* $\mathsf{N4}^\perp$ having the form $\mathsf{N4}^\perp + L$ and the *normal special extensions of* $\mathsf{N4}^\perp$ of the form $\mathsf{N4}^\perp + L + \{\neg\neg(p \vee {\sim}p)\}$, where $L \in \mathcal{E}\mathsf{Int}$, have modal companions.

4.2. **Logics with no modal companions.** Consider the *Kleene axiom*[1] and the *modified Kleene axiom*:

$$\chi = (p \wedge {\sim}p) \to (q \vee {\sim}q),$$
$$\chi' = \neg\neg(p \wedge {\sim}p) \to (q \vee {\sim}q).$$

The *Kleene logic* is defined as $\mathsf{NK}^\perp := \mathsf{N4}^\perp + \{\chi\}$.

It is not hard to see that

$$Tw(\mathbf{A}, \nabla, \Delta) \models \chi \quad \iff \quad a \leqslant b \text{ for all } a \in \Delta \text{ and } b \in \nabla.$$

Thus, the twist-structure $Tw(\mathbf{3}, \nabla, \Delta)$ is a model of $\mathsf{NK}^\perp$, where $\mathbf{3} = \langle\{\perp, \heartsuit, 1\}; \vee, \wedge, \to, \perp\rangle$ with $\perp \leqslant \heartsuit \leqslant 1$, $\nabla = \{\heartsuit, 1\}$ and $\Delta = \{\perp, \heartsuit\}$.

Notice that $Tw(\mathbf{3}, \nabla, \Delta) \not\models \chi'$. Indeed, for the valuation v such that $v(p) = (\heartsuit, 1)$, $v(q) = (\heartsuit, \perp)$ the following holds:

$$\pi_1(v(\chi')) = \neg\neg\heartsuit \to \heartsuit = 1 \to \heartsuit = \heartsuit \neq 1.$$

So $\chi' \notin \mathsf{NK}^\perp$.

Lemma 4.2.1. *If $M \in \mathcal{E}\mathsf{BS4}$ and $\mathrm{T}_\mathbf{B}\chi \in M$, then $\mathrm{T}_\mathbf{B}\chi' \in M$.*

Proof. We have $\mathcal{W}_M \models \mathrm{T}_\mathbf{B}\chi$, where $\mathcal{W}_M$ is the Lindenbaum twist-structure of M. Since $\mathrm{T}_\mathbf{B}\chi = \Box((\Box p \wedge \Box{\sim}p) \to (\Box q \vee \Box{\sim}q))$, we also have $\mathcal{W}_M \models (\Box p \wedge \Box{\sim}p) \to (\Box q \vee \Box{\sim}q)$, which means that $\Box x \leqslant (\Box a \vee \Box b)$ for all $x \in \Delta_M$ and $(a, b) \in \mathcal{W}_M$.

Since $\Delta_M \in \mathcal{I}_\Diamond(\mathcal{T}(M))$, if $x \in \Delta_M$, then $\Diamond\Box x \in \Delta_M$, and we have that $\Box\Diamond\Box x \leqslant (\Box a \vee \Box b)$ for all $x \in \Delta_M$ and $(a, b) \in \mathcal{W}_M$. Thus,

$$\mathcal{W}_M \models \Box(\Box\neg\Box\neg(\Box p \wedge \Box{\sim}p) \to (\Box q \vee \Box{\sim}q)),$$

i.e. $\mathrm{T}_\mathbf{B}\chi' \in M$. $\Box$

We have that $\chi' \notin \mathsf{NK}^\perp$, but $\mathrm{T}_\mathbf{B}\chi \in \tau_\mathbf{B}\mathsf{NK}^\perp$, which implies $\mathrm{T}_\mathbf{B}\chi' \in \tau_\mathbf{B}\mathsf{NK}^\perp$ by Lemma 4.2.1. We obtain that $\tau_\mathbf{B}\mathsf{NK}^\perp$ is not a modal companion of $\mathsf{NK}^\perp$, so $\mathsf{NK}^\perp$ has no modal companions.

Moreover, $\mathbf{3}$ is a model of each L from the class $\{L \in \mathcal{E}\mathsf{Int} \mid \mathsf{Int} \subseteq L \subseteq \mathsf{HT} = \mathsf{Int} + \{p \vee (p \to q) \vee \neg q\}\} = \mathcal{E}\mathsf{Int} \setminus \{\mathsf{CL}, \mathrm{Fm}_{\mathcal{L}_i}\}$ whose cardinality is a continuum. So $\chi' \notin \mathsf{NK}^\perp + L$, but $\mathrm{T}_\mathbf{B}\chi' \in \tau_\mathbf{B}(\mathsf{NK}^\perp + L)$. And we have a continuum of logics that do not have modal companions.

[1]We call χ the Kleene axiom due to its similarity to the identity $x \wedge {\sim}x \leqslant y \vee {\sim}y$ that distinguishes Kleene algebras [6] in the variety of De Morgan algebras. Note that the term "Kleene algebra" also has a different meaning. See, for example, [15].

In conclusion we want to raise some natural questions to explore them later:

1. If $L \in \mathcal{E}\mathsf{N4}^{\perp}$ has a modal companion, does it have the greatest modal companion? If yes, how to axiomatise the greatest modal companion modulo $\tau_{\mathbf{B}}L$?
2. Is there a criteria when L has modal companions or not?
3. Whether the set of all $\mathsf{N4}^{\perp}$-extensions, which have modal companions, forms a sublattice in $\mathcal{E}\mathsf{N4}^{\perp}$?

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