Enumeration With Nice Roman Domination Properties

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Abstract Although Extension Perfect Roman Domination is NP-complete, all minimal (with respect to the pointwise order) perfect Roman dominating functions can be enumerated with polynomial delay. This algorithm uses a bijection between minimal perfect Roman dominating functions and Roman dominating functions and the fact that all minimal Roman dominating functions can be enumerated with polynomial delay. This bijection considers the set of vertices with value 2 under the functions. In this paper, we will generalize this idea by defining so called *nice Roman Domination properties* for which we can employ this method. With this idea, we can show that all minimal *maximal Roman Dominating functions* can be enumerated with polynomial delay in $\mathcal{O}(1.9332^n)$ time. Furthermore, we prove that enumerating all minimal connected/total Roman dominating functions on cobipartite graphs can be achieved with polynomial delay. Additionally, we show the existence of a polynomial-delay algorithm for enumerating all minimal connected Roman dominating function on interval graphs. We show some downsides to this method as well.

1 Introduction

Enumeration problems have many different applications, as for example data mining, machine learning [14] or biology/chemistry [24]. One possible property of enumeration algorithms is polynomial delay. An enumeration algorithm has polynomial delay if the time between two new outputs is polynomially bounded. This is interesting for multi-agent systems, where one agent enumerates and the other processes the output. In this scenario, we know that the second agent does not need to wait for too long and use unnecessary resources. One example for polynomial delay enumeration is enumerating minimal Roman dominating functions [3]. This is interesting, since it is open for more than 4 decades if there exists an enumeration algorithm for minimal dominating sets with polynomial delay (or even output polynomial time).

Roman Domination is a well-studied variation of Dominating Set, see [4,7,9,12,13,20,21,22,26,28], being motivated by a defense strategy of the Roman Empire [30]. In this scenario, you want to put up to 2 legions per region such that each region has at least one legion on itself or there is a neighbored region with two legions on it. This can be modeled on graphs where each region is represented by its own vertex and vertices are connected if and only if the corresponding regions are neighbors. The distribution of the legions can be represented by a function $f\colon V \to \{0,1,2\}$ for the set of regions V. $f\colon V \to \{0,1,2\}$ is called a *Roman dominating function* (Rdf for short) on a graph $G = (V,E)$ if, for all $v \in V$ with $f(v) = 0$, there is a $u \in V$ with $f(u) = 2$ and $\{v,u\} \in E$. A function $f\colon V \to \{0,1,2\}$ can also be represented as a partition of V into $(V_0(f), V_1(f), V_2(f))$, where $V_i(f) \coloneqq \{v \in V \mid f(v) = i\}$ for all $i \in \{0,1,2\}$.

The polynomial delay algorithm in [3] uses Extension Roman Domination, which is polynomial time solvable. Using the extension version of a problem is a usual strategy to obtain a polynomial-delay algorithm. In this paper, we will only consider monotonically increasing problems (We enumerate minimal elements with respect to an ordering). Therefore, we will explain the idea of extension problems with respect to these. For more information, we refer to [6]. The idea of an extension problem is to find a minimal solution which is greater than or equal to a given presolution with respect to the given partial order. Then, Extension X Roman Domination (where X specifies the variant, which is empty in the classical version) is defined as follows:

Problem name: Extension X Roman Domination, or Ext X RD for short.
Given: A graph $G = (V,E)$ and a function $f \in \{0,1,2\}^V$.
Question: Is there a minimal X Roman dominating function g on G with $f \le g$?

Here, $f \le g$ is understood pointwisely: for all $v \in V$, $f(v) \le g(v)$. In [23], it is proven that enumerating minimal perfect Roman dominating functions (or pRdfs for short; a pRdf is a Roman dominating function f such that all vertices in $V_0(f)$ have exactly one neighbor in $V_2(f)$) can be done with polynomial delay even though the extension version is NP-complete. The enumeration algorithm is based on a bijection B between minimal Roman dominating functions and minimal perfect Roman dominating function on the same graph. For a minimal Roman dominating function f, $V_2(f) = V_2(B(f))$. In this paper, we want to generalize this idea for more Roman domination variations like total, connected or maximal. To this end, we will separate the choice of vertices with value 1 and value 2. Therefore, we define nice Roman domination properties in Section 3. We also prove how to make use of this by enumerating all minimal functions of a variation with a nice Roman domination property and a fixed set of vertices with value 2. From this situation, some enumeration properties can be inherited to enumerating all minimal functions of this variation. In Section 4, we use these results to prove that all minimal maximal Roman dominating functions can be enumerated with polynomial delay and in $\mathcal{O}(1.9332^n)$. We use this technique also for enumerating minimal total Roman dominating functions/connected Roman dominating functions on cobipartite in Section 5. This section also includes a polynomial delay algorithm for enumerating minimal connected Roman dominating functions in interval graphs. We also show that this approach sometimes fails unless $\mathsf{P} = \mathsf{NP}$ in Section 6.

2 Definitions and Notation

By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the non-negative integers. For all $n \in \mathbb{N}$, $[n] \coloneqq \{1, \ldots, n\}$. $\binom{A}{k}$ denotes all subsets of a set A of size exactly $k \in \mathbb{N}$. For two sets A, B, the set of all functions $f : A \to B$ is denoted by B^A. A special function is the *characteristic function* $\chi_A : B \to \mathbb{N}$ such that for all $v \in B$, $\chi_A(v) = 1$ if and only if $v \in A$ and $\chi_A(v) = 0$ otherwise.

In this paper, we only consider simple undirected graphs $G = (V, E)$, where V denotes the vertex set and $E \subseteq \{\{v, u\} \mid v, u \in V\}$ is the set of edges. The by $A \subseteq V$ induced subgraph of G is defined as $G[A] = (A, E \cap \binom{A}{2})$. For all $v \in V$, $N(v) \coloneqq \{u \mid \{v, u\} \in E\}$ is called the (open) neighborhood of v with respect to G. The closed neighborhood of $v \in V$ is denoted by $N[v] \coloneqq \{v\} \cup N(v)$. For a set $A \subseteq V$, we define the neighborhood by the union $N(A) = \bigcup_{v \in A} N(v)$ or $N[A] = \bigcup_{v \in A} N[v]$. $P_{G,A}(v) = N[v] \setminus N[A \setminus \{v\}]$ is the set of private neighbors of $v \in A \subseteq V$ with respect to A on G. We also consider hypergraphs $H = (X, S)$, where X is the so-called universe and $S \subseteq 2^X$ is the set of hyperedges.

We call a set $D \subseteq V$ a dominating set of the graph $G = (V, E)$ if, for all $v \in V$, $N[v] \cap D \neq \emptyset$. This can be viewed as a special case of hitting sets. A hitting set $D \subseteq X$ of the hypergraph $H = (X, S)$ if for all $s \in S$, $s \cap D \neq \emptyset$. There are many different variations of dominating sets on a graph $G = (V, E)$, as connected dominating set, cds for short, (D has to be connected), or total dominating set (D is a hitting set of $(V, \{N(v) \mid v \in V\})$). Similar to dominating set, Roman domination has also many variations

- Italian Domination or Roman $\{2\}$-domination ($f \colon V \to \{0, 1, 2\}$ is an Italian dominating function if for each $v \in V_0(f)$, the values under f of the neighbors sum up to at least 2) [8,15];
- Perfect Roman Domination ($f \colon V \to \{0, 1, 2\}$ is a perfect Roman dominating function if for each $v \in V_0(f)$, there exists exactly one neighbor of value 2) [16];
- Unique Response Roman Domination ($f \colon V \to \{0, 1, 2\}$ is a unique response Roman dominating function if f is a perfect Roman dominating function and for each $v \in V_1(f) \cup V_2(f)$, v has no neighbor in $V_2(f)$) [27];
- Total Roman Domination ($f \colon V \to \{0, 1, 2\}$ is a total Roman dominating function if f is a Roman dominating function and $G[V_1(f) \cup V_2(f)]$ has no isolates) [1]
- Connected Roman Domination ($f \colon V \to \{0, 1, 2\}$ is a connected Roman dominating function if f is a Roman dominating function and $G[V_1(f) \cup V_2(f)]$ is connected) [19].
- Maximal Roman Domination (the Roman dominating function $f \colon V \to \{0, 1, 2\}$ is a maximal Roman dominating function if $V_0(f)$ is no dominating set) [29].

Further, we consider some notation of enumeration. As for decision problems, we need an alphabet Σ. Let $A \subseteq \Sigma^* \times \Sigma^*$ be a binary relation. An enumeration problem Enum A is defined as enumerating all elements in $A(x) \coloneqq \{y \in \Sigma^* \mid A(x, y)\}$ for a given instance $x \in \Sigma^*$ without repetitions.

The analysis of the running time for enumeration is divided into input-sensitive (running time with respect to the input size $|x|$) and output-sensitive (running time with respect to the input and output size $|x| + |A(x)|$). We mentioned already polynomial delay which is a output-sensitive property. For a polynomial delay algorithm there exists a polynomial $p\colon \mathbb{N} \to \mathbb{N}$ such that the running time to the first output, between two consecutive outputs and after the last is bounded in $p(|x|)$ for the input x. Another possible property of an enumeration algorithm is output-polynomial time where there exists a polynomial $p\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the running time is bounded in $p(|x|, |A(x)|)$ for the input x. It should be mentioned that a polynomial delay algorithm is also output polynomial. Enumerating minimal hitting sets on a hypergraph, also known as minimal transversal enumeration problem (denoted by Enum Tr), is a very famous enumeration problem, for which it is open for more than four decades whether there is an output-polynomial time algorithm.

As for classical complexity, there are also reductions for enumeration problems. Now we consider two of these. For this purpose let Enum A and Enum B be two enumeration problems. A pair of polynomial-time computable functions f, g, is called a *parsimonious reduction* if, for each instance x of Enum A, $f(x)$ is an instance of Enum B such that $g(x)$ is a bijection between $A(x)$ and $B(f(x))$. See [31] for more details.

The idea of our technique will be based on the notion of an *e-reduction* [10]: Let Σ be an alphabet, Enum A, Enum B $\in$ EnumP be enumeration problems over the alphabet Σ. We say Enum A *e-reduces* to Enum B (Enum A $\leq_e$ Enum B for short) if there exists a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ as well as a relation $\tau \subseteq \Sigma^* \times \Sigma^* \times \Sigma^*$, such that the following conditions hold:

1. $A(x) = \bigcup_{y \in B(f(x))} \tau(x, y, \cdot)$, where $\tau(x, y, \cdot) := \{z \in \Sigma^* \mid \tau(x, y, \cdot)\}$ for all $x, y \in \Sigma^*$,
2. For all $y \in B(f(x))$, either $\tau(x, y, \cdot) = \emptyset$ or $\emptyset \subsetneq \tau(x, y, \cdot) \subseteq A(x)$ and $\tau(x, y, \cdot)$ can be enumerated with polynomial delay in $|x|$; moreover $|\{y \in B(f(x)) \mid \tau(x, y, \cdot) = \emptyset\}|$ is polynomial bounded in $|x|$.
3. For all $z \in A(x)$, $\{y \in \Sigma^* \mid \tau(x, y, z)\} \subseteq B(f(x))$ and $|\tau(x, y, \cdot)|$ is bounded polynomially in $|x|$.

3 Nice Roman Domination Property

In this section, we will set the theoretical basis of this paper. The main goal is to prove Theorem 1. For this paper, we assume that a property $\mathcal{P}$ is a set of functions mapping the vertex set of a graph to $\{0, 1, 2\}$. Instead of $f \in \mathcal{P}$, we say f has property $\mathcal{P}$.[1]. To this end, we start by defining nice Roman domination properties. We call a property $\mathcal{P}$ a *nice Roman domination property* (*nrdp* for short) if for every $G = (V, E)$ and every $f \in \{0, 1, 2\}^V$ with property $\mathcal{P}$ fulfills the following constraints:

[1] Actually, we have to say f has property $\mathcal{P}$ with respect to a graph G, but the graph is known due to the context

1. f is a Roman dominating function.
2. For all $v \in V_0(f)$, $f + \chi_{\{v\}}$ has property $\mathcal{P}$.
3. For all $v \in V_2(f)$, $f - \chi_{\{v\}}$ has property $\mathcal{P}$ if and only if $P_{G[V \setminus V_1(f)], V_2(f)}(v) \subseteq \{v\}$, i.e. $f - \chi_{\{v\}}$ does not have property $\mathcal{P}$ if and only if v has a private neighbor, but itself, on the graph $G[V_0(f) \cup V_2(f)]$.

For a nice Roman domination property $\mathcal{P}$, a graph G and an $A \subseteq V$, define $C_{\mathcal{P},G}[A]$ as the set of minimal $f \in \{0,1,2\}^V$ with respect to $\leq$ and the property $\mathcal{P}$ such that $V_2(f) = A$. Now, we are ready to formulate our main theorem.

Theorem 1. *Let $\mathcal{P}$ be a nice Roman domination property and $\mathcal{G}$ be a class of graphs. If there exists an algorithm $\mathcal{A}$ that enumerates all elements in $C_{\mathcal{P},G}[A]$ in output-polynomial time (resp. with polynomial delay) for all $G = (V,E) \in \mathcal{G}$ and $A \subseteq V$, then there exists an algorithm to enumerate all minimal elements with property $\mathcal{P}$ on any $G \in \mathcal{G}$ in output-polynomial time (resp. with polynomial delay).*

Theorem 1 is a modification of $\leq_e$-reductions for nice Roman domination property to get stronger results and even output-polynomial time. In order to prove this theorem, we first show a couple of lemmas. The next lemma follows from [3, Lemma 7].

Lemma 1. *Let $G = (V,E)$ be a graph. $F(A) := 2 \cdot \chi_A + \chi_{V \setminus N[A]}$ is a bijection between all sets $A \subseteq V$ such that for all $v \in V$, $P_{G[N[A]],A}(v) \nsubseteq \{v\}$ and all minimal Roman dominating functions of G.*

Lemma 2. *Let $\mathcal{P}$ be a nice Roman domination property, $G = (V,E)$ be graph and $A \subseteq B \subseteq V$. If $C_{\mathcal{P},G}[B] \neq \emptyset$ then $C_{\mathcal{P},G}[A] \neq \emptyset$.*

Proof. Let $g \in C_{\mathcal{P},G}[B]$ and for each $v \in B$ let $Y_v = P_{G,B}(v) \cap V_0(g)$. Define $Y_A := \bigcup_{v \in A} Y_v$ and $f := \chi_{V \setminus Y_A} + \chi_A$. We want to show that f has property $\mathcal{P}$. Observe $V_0(f) = Y_A \subseteq N(A)$. Hence, f is a Roman dominating function. By Constraint 2 of nice Roman domination property and adding $\chi_{\{u\}}$ for $u \in V_0(g) \setminus Y_A$ to g, $\widetilde{g} := g + \chi_{V_0(g) \setminus Y_A}$ has property $\mathcal{P}$. Furthermore, $\widetilde{g} = f + \chi_{B \setminus A} \geq f$, as $V_2(\widetilde{g}) = B = V_2(f) \cup B \setminus A$, $V_0(\widetilde{g}) = Y_A = V_0(f)$ and $B \setminus A \subseteq V_1(f)$ as they are no private neighbors of elements in A with respect to B. By definition of f and Y_A, $N(u) \cap V_0(f) = \emptyset$ for all $u \in B \setminus A$. By an inductive argument, Constraint 3 of nice Roman domination property implies $\widetilde{g} - \chi_{B \setminus A} = f$ has property $\mathcal{P}$.

We now show that from f, we can construct some $h \in C_{\mathcal{P},G}[A]$. If f is minimal, set $h = f$. Otherwise, assume there is a $h \in \{0,1,2\}^V$ with property $\mathcal{P}$ on G, $h \leq f$ and a $u \in V$ such that $h(u) < f(u)$. If $f(u) = 2$ then the elements in Y_u are not dominated by h, as they private neighbors of u. This is not possible and, therefore, $V_2(h) = V_2(f)$. This implies $h = f - \chi_M$ for an $M \subseteq V_1(f)$, $M \neq \emptyset$, and $h \in C_{\mathcal{P},G}[A]$. Thus, $C_{\mathcal{P},G}[A] \neq \emptyset$. □

Now we will show a connection between $C_{\mathcal{P},G}$ and extension problems.

Lemma 3. *Let $G = (V,E)$ be a graph and $A \subseteq V$. $C_{\mathcal{P},G}[A] \neq \emptyset$ if and only if there is a minimal $g \in \{0,1,2\}^V$ with property $\mathcal{P}$ on G such that $2 \cdot \chi_A + \chi_{V \setminus N[A]} \leq g$. Further, for all $h \in C_{\mathcal{P},G}[A]$, $2 \cdot \chi_A + \chi_{V \setminus N[A]} \leq h$.*

Proof. Let $C_{\mathcal{P},G}[A] \neq \emptyset$. By Lemma 1, $f \coloneqq 2 \cdot \chi_A + \chi_{V \setminus N[A]}$ is the only minimal Roman dominating function with $V_2(f) = A$. Thus, every Roman dominating function g on G with $V_2(g) = A$ fulfills $f \leq g$. Conversely, let $g \in \{0,1,2\}^V$ be minimal with the property $\mathcal{P}$ on G with $f \leq g$. This implies $V_0(g) \subseteq V_0(f) = N[A] \setminus A$. If there exists an $x \in V_2(g) \setminus A$, x has no private neighbor in $V_0(g)$ and g is not minimal (see Constraint 3 of nice Roman domination properties). Hence, $C_{\mathcal{P},G}[A] \neq \emptyset$ and for all $g \in C_{\mathcal{P},G}[A]$, $f \leq g$. □

Let us shortly follow a corollary before proving Theorem 1. We use this result to discuss the why our approach is helpful.

Corollary 1. *Let $\mathcal{P}$ be a nice Roman domination property, $G = (V, E)$ a graph and $f \in \{0,1,2\}^V$ with $N(V_1(f)) \cap V_2(f) = \emptyset$. Then, there is an $h \in \{0,1,2\}$ minimal with the property $\mathcal{P}$ and $f \leq h$ if and only if $C_{\mathcal{P},G}[V_2(f)] \neq \emptyset$.*

Proof. The if part follows by Lemma 3 and $f \leq 2 \cdot \chi_A + \chi_{V \setminus N[A]}$. So, let us assume there is minimal $h \in \{0,1,2\}$ with property $\mathcal{P}$ with $f \leq h$ but $C_{\mathcal{P},G}[V_2(f)] = \emptyset$. Then $V_2(f) \subseteq V_2(h)$ and $C_{\mathcal{P},G}[V_2(h)] \neq \emptyset$, which contradicts Lemma 2. □

Now we can turn our attention back to Theorem 1.

Proof (of Theorem 1). We will only consider the output-polynomial time case. The proof for the polynomial-delay case is analogous. Hence, let $\mathcal{A}$ be the algorithm for enumerating $C_{\mathcal{P},G}[A]$ for given $G = (V, E) \in \mathcal{G}$ and $A \subseteq V$ and p the polynomial such that $p(|V| + |E|, |C_{\mathcal{P},G}[A]|)$ is the running time of the algorithm. Without loss of generality, all the multiplicative constants of the monomials are positive. Otherwise, we could use the polynomial with the absolute values of the multiplicative constants instead.

The idea of the algorithm for this proof is based on a branching algorithm. We will branch on each vertex if it is in A. Every time we add a vertex to A, we compute $C_{\mathcal{P},G}[A]$. If $C_{\mathcal{P},G}[A] = \emptyset$, we can skip this branch. This case can be checked in polynomial time, as we can enumerate all elements in output-polynomial time. If $C_{\mathcal{P},G}[A] \neq \emptyset$, we enumerate the elements and go on with the next vertex.

Now we want to prove that this algorithm works. So, let $G = (V, E) \in \mathcal{G}$. Clearly, $\bigcup_{A \subseteq V} C_{\mathcal{P},G}[A]$ is exactly the set of all minimal elements with property $\mathcal{P}$ on G. By Lemma 2, we can skip the branches with $C_{\mathcal{P},G}[A] = \emptyset$. Therefore, we only need to show that the algorithm runs in output-polynomial time. To do this, we will prove the following claim:

Claim. The algorithm considers at most $|V|^2$ many consecutive sets $A \subseteq V$ with $C_{\mathcal{P},G}[A] = \emptyset$ before the algorithm either stops or finds a $B \subseteq V$ with $C_{\mathcal{P},G}[B] \neq \emptyset$.

Proof. If $C_{\mathcal{P},G}[\emptyset] = \emptyset$ then the algorithm stops directly by Lemma 2. So, assume there is a $B' \subseteq V$ such that $C_{\mathcal{P},G}[B']$ is not empty. In this case, the algorithm would now go through at most $|V|$ vertices which it did not consider so far to obtain B'. For each of these vertices $v \in V \setminus B'$, the algorithm either has found $C_{\mathcal{P},G}[B' \cup \{v\}] \neq \emptyset$ or would cut this branch and go on with the next vertex.

After this, the algorithm goes through all of the following cases: it deletes an element from B' and checks as before. It will go on with this process until it either finds a set B with $C_{\mathcal{P},G}[B] \neq \emptyset$ or it stops. As the algorithm can only delete at most $|B'| \leq |V|$ vertices from B' and after each deletion step it has to consider at most $|V|$ before deleting the next vertex, the algorithm considers at most $|V|^2$ consecutive sets $A \subseteq V$ with $C_{\mathcal{P},G}[A] = \emptyset$ before stopping. $\Diamond$

Let $\mathcal{B} := \{B \subseteq V \mid C_{\mathcal{P},G}[B] \neq \emptyset\}$. The claim implies that we consider at most $|V|^2 \cdot |\mathcal{B}|$ sets $A \subseteq V$ with $C_{\mathcal{P},G}[A] = \emptyset$. Hence, the running time is bounded by

$$\begin{aligned}&\left(\sum_{A \in \mathcal{B}} p(|V| + |E|, |C_{\mathcal{P},G}[A]|)\right) + |V|^2 \cdot |\mathcal{B}| \cdot p(|V| + |E|, 0)\\ \leq &\left(\sum_{B \in \mathcal{B}} |C_{\mathcal{P},G}[B]|\right) \cdot \left(p(|V| + |E|, \sum_{B \in \mathcal{B}} |C_{\mathcal{P},G}[B]|) + |V|^2\right) \cdot p(|V| + |E|, 0).\end{aligned}$$

This is a polynomial in $|V| + |E|$ and the output size. $\square$

It is important for the polynomial delay case that in the claim of Theorem 1 the sets are consecutive. Otherwise, the time between two outputs could be too long. This theorem only gives us an output-sensitive results. Now, we prove that if we can bound $C_{\mathcal{P},G}$ polynomially, we can also bound the input-sensitive running time.

Lemma 4. *Let $\mathcal{P}$ be a nice Roman domination property and $\mathcal{G}$ a class of graphs such that there is an algorithm enumerating all minimal Roman dominating functions on $G = (V, E) \in \mathcal{G}$ in $\mathcal{O}^*(r^{|V|})$ for $r \in \{x \in \mathbb{R} \mid x \geq 1\}$. If there is an algorithm $\mathcal{A}$ that enumerates all elements in $C_{\mathcal{P},G}[A]$ with polynomial delay and if there is polynomial p such that $|C_{\mathcal{P},G}[A]| \leq p(|V|)$ for each $G = (V, E) \in \mathcal{G}$ and $A \subseteq V$, then there is an algorithm to enumerate all minimal $g \in \{0, 1, 2\}^V$ with property $\mathcal{P}$ on $G = (V, E) \in \mathcal{G}$ in $\mathcal{O}^*(r^{|V|})$.*

Proof. Let $G = (V, E) \in \mathcal{G}$ be a graph. By Theorem 1 there exists a polynomial-delay enumeration algorithm to enumerate all minimal $g \in \{0, 1, 2\}^V$ with property $\mathcal{P}$ on $G = (V, E) \in \mathcal{G}$. By [3, Proposition 8], it is known that a minimal Roman dominating function is uniquely defined by the vertices of value 2. Together with Lemma 1 there is a bijection between all minimal Roman dominating functions on G and subsets $A \subseteq V$ such that for all $v \in A$, $P_{G,A}(v) \nsubseteq \{v\}$. Therefore, the number of elements with property $\mathcal{P}$ on G is bounded by $r^{|V|} \cdot p(|V|)$ for the given polynomial p. As the delay is bounded by a polynomial q, the running time of the algorithm is $q(|V|)p(|V|)r^{|V|}$ or $\mathcal{O}^*(r^n)$. $\square$

4 Maximal Roman Dominating Functions

In this section, we want to make use of the results in Section 3 by looking at maximal Roman dominating functions. For a graph $G = (V, E)$, a function

$f \in \{0,1,2\}^V$ is a *maximal Roman dominating function* (or mRdf for short) on G if and only if $N(v) \cap V_2(f) \neq \emptyset$ for all $v \in V_0(f)$ and $N[V_0(f)] \neq V$; see [29]. First, we will show a characterization result for minimal maximal Roman dominating functions.

Theorem 2. *Let $G = (V, E)$ be a graph and $f \in \{0,1,2\}^V$. f is a minimal maximal Roman dominating function if and only if the following constraints hold.*

1. *f is a maximal Roman dominating function.*
2. *For all $v \in V_1(f)$, either $N(v) \cap V_2(f) = \emptyset$ or $V \setminus N[V_0(f)] \subseteq N[v]$.*
3. *For all $v \in V_2(f)$, $P_{G[V_0(f) \cup V_2(f)], V_2(f)}(v) \nsubseteq \{v\}$.*

Proof. First we assume f is a minimal maximal Roman dominating function. Constraint 1 holds trivially. Assume there exists a $v \in V_2(f)$ such that v has no private neighbor but itself in $V_0(f) \cup V_2(f)$. Define $g := f - \chi_{\{v\}}$. Since $V_0(f) = V_0(g)$, $V_0(g)$ is no dominating set. So let $u \in V_0(f)$. If $u \notin N(v)$ then u is dominated by g as by f. For $u \in N(v)$, there exists a $w \in N(u) \cap (V_2(f) \setminus \{v\}) = N(u) \cap V_2(g)$. Hence, g is a maximal Roman dominating function. Thus, f would not be minimal. This proves Constraint 3.

Let $v \in V_1(f)$. Assume Constraint 2 does not hold for v. Define $g := f - \chi_{\{v\}}$. By assumption, $N(v) \cap V_2(g) = N(v) \cap V_2(f) \neq \emptyset$. As $N(u) \cap V_2(g) = N(u) \cap V_2(f) \neq \emptyset$ for $u \in V_0(f) = V_0(g) \setminus \{v\}$, g is a Roman dominating function. Furthermore, $V \setminus N[V_0(f)] \nsubseteq N[v]$ implies $V \setminus N[V_0(g)] = (V \setminus N[V_0(f)]) \setminus N[v] \neq \emptyset$. Hence, g is an maximal Roman dominating function and f was not minimal in the first place.

Conversely, assume f fulfills all conditions. By Constraint 1, f is a maximal Roman dominating function. So assume there is a maximal Roman dominating function $g \in \{0,1,2\}^V$ on G with $g \leq f$ and $g \neq f$. Thus, $V_0(f) \subseteq V_0(g)$, $V_2(g) \subseteq V_2(f)$ and $V_1(g) \cup V_2(g) \subseteq V_1(f) \cup V_1(g)$. Let $v \in V$ be a vertex with $g(v) < f(v)$. First we assume $f(v) = 2$. By Constraint 3, there is a $u \in N(v) \cap V_0(f) \subseteq N(v) \cap V_0(g)$ such that $N(u) \cap V_2(g) \subseteq N(u) \cap (V_2(f) \setminus \{v\}) = \emptyset$. This would imply that g is no (maximal) Roman dominating function. Hence, $f(v) = 1$. If $\emptyset = N(v) \cap V_2(f) (\subseteq N(v) \cap V_2(g))$, g is no Roman dominating function as $g(v) = 0$. Therefore, $V \setminus N[V_0(f)] \subseteq N[v]$. Then, $V \setminus N[V_0(g)] = \emptyset$ and $V_0(f)$ is a dominating set on G. In this case g would not be a maximal Roman dominating function. □

Corollary 2. *For a graph $G = (V, E)$ and a minimal maximal Roman dominating function $f \in \{0,1,2\}^V$ on G, $\emptyset \neq V \setminus N[V_0(f)] \subseteq V_1(f)$.*

Proof. As f is a maximal Roman dominating function, there exists a $v \in V \setminus N[V_0(f)]$. If $f(v) = 2$, then v would not have a private neighbor but itself, contradicting Constraint 3. □

To show that our approach is really helpful, we prove hardness for the extension problem. First, let us consider a non-trivial no-instance. Let $G = (\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{3,4\}\})$ be a path of 4 vertices with the function $f :=$

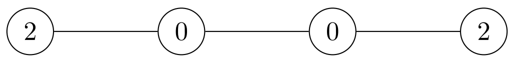

Figure 1: Non-trivial no-instance of Extension Maximum Roman Domination.

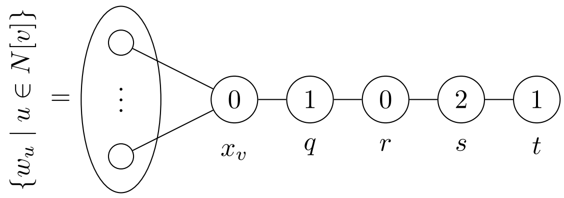

Figure 2: Construction for Theorem 3 with $v \in V$.

$2 \cdot \chi_{\{1,4\}}$. This instance is visualized by Figure 1. f is a Roman dominating function, but $V_0(f)$ is a dominating set. Furthermore, each vertex in $V_0(f)$ is a unique private neighbor of a vertex in $V_2(f)$. Theorem 2 implies, that for any $g \in \{0,1,2\}^{\{1,2,3,4\}}$ with $f \leq g$, g is not a minimal maximal Roman dominating function.

Theorem 3. *Extension Maximum Roman Domination is NP-complete even when $|V_1(f)| = 2$ ($|V_1(f)|$-Extension Maximum Roman Domination is para-NP-complete). $|V_2(f)|$-Extension Maximum Roman Domination is W[3]-hard (even on bipartite graphs).*

Proof. Extension Maximum Roman Domination is in NP, as the Turing machine can guess the values of g and then use Theorem 2 for polynomial-time verification. We will prove the hardness results by a reduction from Extension Dominating Set. Let $G = (V, E)$ be a graph and $U \subseteq V$ a vertex set. Define $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ with

$$\begin{aligned}\widetilde{V} &\coloneqq \{q, r, s, t\} \cup \{w_v, x_v, \mid v \in V\}\\ \widetilde{E} &\coloneqq \{\{w_v, x_u\} \mid v, u \in V, u \in N[v]\} \cup\\ &\quad\ \{\{x_v, q\} \mid v \in V\} \cup \{\{q, r\}, \{r, s\}, \{s, t\}\}.\end{aligned}$$

$\widetilde{G}$ is bipartite with the partition $\{q, s\} \cup \{w_v, \mid v \in V\}, \{r, t\} \cup \{x_v \mid v \in V\}$. Furthermore, define $f \in \{0,1,2\}^{\widetilde{V}}$ with

$$f = 2\chi_{\{s\} \cup \{w_v \mid v \in U\}} + \chi_{\{q,t\}}.$$

Claim. There exists a minimal dominating set D on G with $U \subseteq D$ if and only if there is minimal maximal Roman dominating function g on $\widetilde{G}$ with $f \leq g$.

Proof. Let D be a minimal dominating set (dominating set) on $G = (V, E)$ with $U \subseteq D$. Define $g_D = f + \chi_{\{w_v | v \in D \setminus U\}}$. Clearly, $f \leq g_D$. Vertex s is a neighbor of r. As D is a dominating set, for each $v \in V$, x_v has a neighbor $w_v \in \{w_u \mid u \in V\} \cap V_2(f)$. Thus, g_D is a Roman dominating function. Since $N[t] = \{s, t\} \subseteq V_1(g_D) \cup V_2(g_D)$, g_D is a maximal Roman dominating function. As each $v \in D$ has a private neighbor, for every $w_v \in V_2(f) \cap \{w_x \mid x \in V\} = \{w_v \mid v \in D\}$, there exists a private neighbor $x_u \in \{x_p \mid p \in V\}$. Furthermore, r is a private neighbor of s. Since $V_1(g_D) \cap N(V_2(g_D)) = \{t\}$ and $N(t) = \{s\}$, Theorem 2 implies that g_D is a minimal maximal Roman dominating function.

Now let g be a minimal maximal Roman dominating function with $f \leq g$. Since r is the only neighbor of s in $V_0(f)$, $g(r) = 0$ and $V_2(g) \cap N(r) = \{s\}$. Thus, $g(q) = 1$. If there is a $v \in V$ with $g(x_v) = 2$, q would contradict Constraint 2 of Theorem 2. Hence, for each $v \in V$, $N(w_v) \cap V_2(g) = \emptyset$ and $g(w_v) \neq 0$. For all $v \in V$, $N(x_v) \subseteq V_1(g) \cap V_2(g)$ which implies $\{x_v \mid v \in V\} \subseteq V_0(g)$. Otherwise, t would contradict Constraint 2 of Theorem 2. Therefore, for each $v \in V$ there exists a $u \in V$ such that $w_u \in V_2(g) \cap N(x_v)$. Thus, $D_g \coloneqq \{v \in V \mid w_v \in V_2(g)\}$ is a dominating set. Let $v \in D_g$. Since g is minimal, w_v has a private neighbor x_u. Hence, v has a private neighbor and D_g is minimal. ♢

The graph $\widetilde{G}$ has $2|V| + 4$ vertices. Therefore, the reduction can be implemented to run in polynomial time. Since $|V_2(f)| = |U| + 1$, this is also a FPT-reduction. □

Let $\mathcal{MRDF}$ denote the property of maximal Roman dominating functions. To apply earlier results, we now show that $\mathcal{MRDF}$ is nice.

Lemma 5. *$\mathcal{MRDF}$ is a nice Roman domination property.*

Proof. Clearly, every maximal Roman dominating function is also a Roman dominating function. Let $G = (V, E)$ be a graph and let f be a maximal Roman dominating function on G. Since domination and Roman domination are monotonically increasing (with respect to $\subseteq$ and $\leq$), $f + \chi_{\{v\}}$ is still a maximal Roman dominating function for all $v \in V_0(f)$. Therefore, we only need to consider the Constraint 3.

Let $v \in V_2(f)$ and $f_v \coloneqq f - \chi_{\{v\}}$. Clearly, $V_0(f_v) = V_0(f)$. Therefore, if f_v is no maximal Roman dominating function, there is a $u \in V_0(f_v)$ which is not dominated by $V_2(f_v) = V_2(f) \setminus \{v\}$. Hence, $u \in N(v)$ is a private neighbor of v. If $P_{G[V \setminus V_1(f)], V_2(f)}(v) \subseteq \{v\}$, then for each $w \in V_0(f) = V_0(f_v)$, there exists a $u \in (V_2(f) \setminus \{v\}) \cap N(w) = V_2(f_v) \cap N(w)$. Therefore, $V_0(f) = V_0(f_v)$ is dominated by $V_2(f) \setminus \{v\} = V_2(f_v)$. □

Lemma 6. *Let $G = (V, E)$ be a graph and let $A \subseteq V$. Then, $|C_{\mathcal{MRDF},G}[A]| \leq n$. Furthermore, $C_{\mathcal{MRDF},G}[A]$ can be computed in polynomial time.*

Proof. We can assume that each $v \in A$ has a private neighbor but itself. Otherwise, $C_{\mathcal{MRDF},G}[A] = \emptyset$ by Constraint 3 of Theorem 2. This property can be checked in polynomial time.

Define $f \in \{0, 1, 2\}^V$ with $f \coloneqq 2 \cdot \chi_A + \chi_{V \setminus N[A]}$. As $V_0(f) \subseteq N(A) = N(V_2(f))$, f is a (minimal) Roman dominating function. Since each $v \in V \setminus N[A]$

has no neighbor in V_2, $f \leq g$ for all $g \in C_{\mathcal{MRDF},G}[A]$. Therefore, if f is a maximal Roman dominating function, then $C_{\mathcal{MRDF},G}[A] = \{f\}$. So, assume there is no $v \in V_1(f)$ such that $N(v) \cap V_0(f)$ is not empty. Then define for all $v \in V \setminus A$, $g_v \in \{0,1,2\}^V$ with $g_v = f + \chi_{V_0(f) \cap N[v]}$. Clearly, $f \leq g_v$ and g_v is a Roman dominating function. Since $N[v] \cap V_0(g_v) = N[v] \cap (V_0(f) \setminus N[v])$, g_v is a maximal Roman dominating function.

Assume there is a minimal maximal Roman dominating function h on G such that $V_2(h) = A$, but there is no $v \in V \setminus A$ with $h = g_v$. As h is a Roman dominating function, $f \leq h$. By Corollary 2, there is a $v \in V_1(h) \subseteq V \setminus A$, such that $v \notin N[V_0(h)]$. Hence, for each $v \in V \setminus A$, $g_v = f + \chi_{N[v] \cap V_0(f)} \leq h$. Thus, h is not minimal in the first place and $C_{\mathcal{MRDF},G}[A] \subseteq \{g_v \mid v \in V \setminus A\}$. Therefore, $|C_{\mathcal{MRDF},G}[A]| \leq |V \setminus A| \leq n$.

It can be checked in polynomial time if f and g_v (for all $v \in V \setminus V_2$) are minimal maximal Roman dominating functions. This implies that $C_{\mathcal{MRDF},G}[A]$ can be constructed in polynomial time. □

Theorem 1 and Lemma 4 imply with Theorem 11 of [3] the following:

Corollary 3. *There is a polynomial-space algorithm that enumerates all minimal maximal Roman dominating functions of a given graph of order n with polynomial delay, in time $\mathcal{O}(1.9332^n)$.*

The results in [2] imply the following:

Corollary 4. *There is a polynomial-space algorithm that enumerates all minimal maximal Roman dominating functions of a given graph of order n with polynomial delay and in time*

- $\mathcal{O}(1.8940^n)$ *on chordal graphs;*
- $\mathcal{O}(\sqrt{3}^n)$ *on interval graphs or on trees;*
- $\mathcal{O}(\sqrt[3]{3}^n)$ *on split or on cobipartite graphs.*

The next corollary follows by Corollary 1 and Lemma 6.

Corollary 5. *Extension Maximum Roman Domination is polynomial time solvable for the instance (G, f) if $N(V_1(f)) \cap V_2(f) = \emptyset$.*

Remark 1. In the proof of Theorem 3 we construct a graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$. For the hardness we makes use of a $t \in \widetilde{V}$ of value 1 for which a $s \in \widetilde{V}$ of value 2 is the only neighbor. Because of Constraint 2 of Theorem 2, there is no $w \in \widetilde{V} \setminus \{t\}$ of value 1 without a neighbor of value 0.

This is interesting since the instance would be polynomial time solvable, if the value of t would be 0 (see Corollary 5). Therefore, just one wrongly placed 1 is enough to make the extension problem hard. This is the motivation for considering the vertices of value 1 after choosing the vertices with the value 2.

5 Extension Total/Connected Roman Domination

For this chapter we want to consider total and connected Roman Domination. The so-called total Roman Domination parameter was introduced in [22]. A Roman dominating function $f \in \{0,1,2\}^V$ is a *total Roman dominating function* (or *tRdf* for short) with respect to G if, for each $v \in V_1(f) \cup V_2(f)$, $N(v) \cap (V_1(f) \cup V_2(f))$ is not empty. A *connected Roman dominating function* (or *cRdf* for short) with respect to G is a Roman dominating function such that $V_1(f) \cup V_2(f)$ is connected in G.[2] References [19,22] provide proofs for the NP-completeness for the natural decision problem of Total or Connected Roman Domination (given a graph $G = (V, E)$ and $k \in \mathbb{N}$, is there a total Roman dominating function or connected Roman dominating function f with $\sum_{v \in V} f(v) \leq k$). We start by proving a characterization result of a minimal total Roman dominating function/connected Roman dominating function:

Lemma 7. *Let $G = (V, E)$ be a graph and $f \in \{0,1,2\}^V$. f is a minimal total Roman dominating function if and only if the following conditions hold:*

1. *f is a total Roman dominating function,*
2. *for all $v \in V_1(f)$, $N(v) \cap V_2(f) = \emptyset$ or $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ has an isolated vertex and*
3. *for all $v \in V_2(f)$, $P_{G',V_2(f)}(v) \nsubseteq \{v\}$.*

Proof. First let f be a minimal total Roman dominating function. Hence, f is a total Roman dominating function. Assume there is a $v \in V_1(f)$ where $N(v) \cap V_2(f) \neq \emptyset$ and $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ has no isolate. Define $g \coloneqq f - \chi_{\{v\}}$. Clearly, each $x \in V_0(f) = V_0(g) \setminus \{v\}$ is dominated by g as it is by f. Since $N(v) \cap V_2(f) \neq \emptyset$, v is also dominated and g a Roman dominating function. As $G[V_1(g) \cup V_2(g)] = G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ has no isolates, g is a total Roman dominating function.

Now assume there is a $v \in V_2(f)$ such that $P_{G',V_2(f)}(v) \subseteq \{v\}$. Define $g \coloneqq f - \chi_{\{v\}}$. If u is a neighbor of v, there exists a vertex $w \in N_G[u] \cap (V_2(f) \setminus \{v\}) = N_G[u] \cap V_2(g)$. For $u \in V_0(g)$ with $u \notin N(v)$, there is a vertex $w \in V_2(f) \setminus \{v\}$ that dominates u. Hence, the three properties hold.

For the other implication, let the three properties be valid for f. Let h be a minimal total Roman dominating function with $h \leq f$. This implies $V_2(h) \subseteq V_2(f)$. In the following we assume there is a $v \in V$ such that $h(v) < f(v)$ and go through all the cases.

Case 1: $f(v) = 2$. By Property 3, there is a $u \in (N(v) \cap V_0(f)) \setminus N(V_2(f) \setminus \{v\}) \subseteq (N(v) \cap V_0(h)) \setminus N(V_2(h))$. Since $v \notin V_2(h)$, u is not dominated. This would contradict that g is a Roman dominating function. Hence, we can assume $V_2(f) = V_2(h)$. This implies $V_1(h) \subseteq V_1(f)$.

Case 2: $h(v) = 0 < 1 = f(v)$. Property 2 implies that $N(v) \cap V_2(f) = N(v) \cap V_2(h) = \emptyset$ or $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ has an isolate. If the first part

[2] There is a different definition in [25]. In our opinion, the given definition fits better than the original, military motivation of distributing legions.

would hold, v would not be dominated by h which is a contradiction. Therefore, we can assume there is a $u \in N(v) \cap V_2(f)$ and $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ has isolates. As f is a total Roman dominating function, the isolated vertex x is in $N(v)$. If $x \in V_2(f) = V_2(h)$, x would be isolated in $G[V_1(h) \cup V_2(h)]$, since $V_1(h) \subseteq V_1(f)$ and $V_2(f) = V_2(h)$. This would contradict the totality of h. Hence, $x \in V_1(f)$. Since x is an isolate in $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$, x has no neighbor in $V_2(f)$. Thus, under condition $h(x) = 0$, h would not be a Roman dominating function. So, $h(x) = 1$ which implies that x is an isolate on the subgraph $G[V_1(h) \cup V_2(h)]$ of $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$. This would contradict the totality of h. Hence the equivalence holds. □

Lemma 8. *Let $G = (V, E)$ be a graph and $f \in \{0,1,2\}^V$. f is a minimal connected Roman dominating function if and only if the following conditions hold:*

1. *f is a connected Roman dominating function,*
2. *for all $v \in V_1(f)$, $N(v) \cap V_2(f) = \emptyset$ or $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ is not connected and*
3. *for all $v \in V_2(f)$, $P_{G',V_2(f)}(v) \nsubseteq \{v\}$.*

Proof. First let f is a minimal connected Roman dominating function. Hence, f is a connected Roman dominating function. Assume there is a $v \in V_1(f)$ where $N(v) \cap V_2(f) \neq \emptyset$ and $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ is connected. Define $g \coloneqq f - \chi_{\{v\}}$. Clearly, each $x \in V_0(f) = V_0(g) \setminus \{v\}$ is dominated by g as it is by f. Since $N(v) \cap V_2(f) \neq \emptyset$, v is also dominated and g a Roman dominating function. As $G[V_1(g) \cup V_2(g)] = G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ is connected, g is a connected Roman dominating function.

Now, assume there is a $v \in V_2(f)$ such that $P_{G',V_2(f)}(v) \subseteq \{v\}$. Define $g \coloneqq f - \chi_{\{v\}}$. If u is a neighbor of v, there exists a vertex $w \in N_G[u] \cap (V_2(f) \setminus \{v\}) = N_G[u] \cap V_2(g)$. For $u \in V_0(g)$ with $u \notin N(v)$, there is a vertex $w \in V_2(f) \setminus \{v\}$ that dominates u. Furthermore, $V_1(f) \cup V_2(f) = V_1(g) \cup V_2(g)$ is connected, which implies that g is a connected Roman dominating function. Hence, the three properties hold.

For the other implication, let the three properties be valid for f. Let h be a minimal connected Roman dominating function with $h \leq f$. This implies $V_2(h) \subseteq V_2(f)$. In the following we assume there is a $v \in V$ such that $h(v) < f(v)$ and go through all the cases.

Case 1: $f(v) = 2$**.** By Property 3, there is a $u \in (N(v) \cap V_0(f)) \setminus N(V_2(f) \setminus \{v\}) \subseteq (N(v) \cap V_0(h)) \setminus N(V_2(h))$. Since $v \notin V_2(h)$, u is not dominated. This would contradict that g is a Roman dominating function. Hence, we can assume $V_2(f) = V_2(h)$. This implies $V_1(h) \subseteq V_1(f)$.

Case 2: $h(v) = 0 < 1 = f(v)$**.** Property 2 implies that $N(v) \cap V_2(f) = N(v) \cap V_2(h) = \emptyset$ or $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ is not connected. If the first part would hold, v would not be dominated by h which is a contradiction. Therefore, we can assume there is a $u \in N(v) \cap V_2(f)$ and $G[(V_1(f) \cup V_2(f)) \setminus \{v\}]$ is not connected. $h \leq f$ implies $\overline{(V_1(f) \cup V_2(f))} = V_0(f) \subseteq V_0(h) = \overline{(V_1(h) \cup V_2(h))}$.

Thus, $V_1(h) \cup V_2(h) \subseteq V_1(f) \cup V_2(f)$. So, if $V_1(f) \cup V_2(f)$ is not connected then $V_1(h) \cup V_2(h)$ is neither. Hence the equivalence holds. $\square$

This result is important for the remaining section, especially for the next result:

Theorem 4. *$\mathcal{CRDF}$ and $\mathcal{TRDF}$ are nice Roman domination properties.*

Proof. Clearly, every connected Roman dominating function (resp. total Roman dominating function) is also a Roman dominating function. Let $G = (V, E)$ be a graph and f a connected Roman dominating function (resp. total Roman dominating function) on G. Since f is Roman dominating function, $V_0(f) \subseteq N(V_1(f) \cup V_2(f))$. Hence, for all $v \in V_0(f)$, $\{v\} \cup V_1(f) \cup V_2(f)$ is connected (resp. has no isolated vertex). This implies, $f + \chi_{\{v\}}$ is still a connected Roman dominating function (resp. total Roman dominating function) for all $v \in V_0(f)$. Therefore, we only need to consider the Constraint 3.

Let $v \in V_2(f)$ and $f_v := f - \chi_{\{v\}}$. Clearly, $V_1(f_v) \cup V_2(f_v) = V_1(f) \cup V_2(f)$. Therefore, if f_v is no connected Roman dominating function (resp total Roman dominating function), there is a $u \in V_0(f_v)$ which is not dominated by $V_2(f_V) = V_2(f) \setminus \{v\}$. Hence, $u \in N(v)$ is a private neighbor of v. If $P_{G[V \setminus V_1(f)], V_2(f)}(v) \subseteq \{v\}$, then for each $w \in V_0(f) = V_0(f_v)$ there exists a $u \in (V_2(f) \setminus \{v\}) \cap N(w) = V_2(f_v) \cap N(w)$. Therefore, $V_0(f) = V_0(f_v)$ is dominated by $V_2(f) \setminus \{v\} = V_2(f_v)$. $\square$

5.1 Minimal connected/total Roman dominating function on Cobipartite Graphs

A graph $G = (V, E)$ is called *cobipartite* if there is a partition of V into two cliques C_1, C_2. It is known by [17] that enumerating minimal connected dominating sets is as hard as enumerating minimal hitting sets. The goal of this subsection is the following theorem:

Theorem 5. *For all cobipartite graphs $G = (V, E)$ and $A \subseteq V$, $C_{\mathcal{CRDF},G}[A]$ and $C_{\mathcal{TRDF},G}[A]$ can be enumerated with polynomial delay and $|C_{\mathcal{CRDF},G}[A]| \in \mathcal{O}(|V|^2)$ and $|C_{\mathcal{TRDF},G}[A]| \in \mathcal{O}(|V|^2)$. Furthermore, there is a polynomial-delay algorithm enumerating all minimal connected Roman dominating functions (or total Roman dominating functions) on cobipartite graphs of order n within time $\mathcal{O}^*(\sqrt[3]{3}^n)$.*

We will shortly describe the idea of the proof. Let $G = (V, E)$ be a cobipartite graph with the two cliques C_1, C_2, $A \subseteq V$, $f_A := 2 \cdot \chi_A + \chi_{V \setminus N[A]}$ and $g \in C_{\mathcal{XRDF},G}[A]$, where $\mathcal{X} \in \{\mathcal{T}, \mathcal{C}\}$. We can conclude $|(V_1(g) \cap C_i) \setminus V_1(f_A)| \leq 1$ for $i \in \{1, 2\}$. Therefore, $|C_{\mathcal{XRDF},G}[A]| \leq \mathcal{O}(|V|^2)$ and we can enumerate all elements in $C_{\mathcal{XRDF},G}[A]$ with polynomial delay.

Now we want to prove Theorem 5. For this we will first consider some lemmas.

Lemma 9. *Let $G = (V, E)$ be cobipartite graph where C_1, C_2 are two disjoint cliques with $V = C_1 \cup C_2$. For all minimal connected Roman dominating functions $f \in \{0, 1, 2\}^V$ on G, $|V_1(f) \cap N(V_2(f) \cap C_1)| \leq 1$ and $|V_1(f) \cap N(V_2(f) \cap C_2)| \leq 1$.*

Proof. Let $T \coloneqq V_2(f) \cap C_1$ and $v, u \in N(T) \cap V_1(f)$. Denote some neighbor of v and u in T by x_v and x_u. Since v, u are dominated by T, $(V_1(f) \cup V_2(f)) \setminus \{v\}$ and $(V_1(f) \cup V_2(f)) \setminus \{u\}$ are not connected. As C_1, C_2 are cliques, $C_1 \cap (V_1(f) \cup V_2(f))$ is connected. Hence, there is a $y \in (V_1(f) \cup V_2(f)) \cap C_2 \cap (N(u) \setminus N(T \cup \{v\}))$. Thus, for all $z \in N(v) \setminus N(\{u, x_u, x_v\}) \subseteq C_2$, x_v, x_u, u, y, z is a path from x_v to z. Therefore, $(V_1(f) \cup V_2(f)) \setminus \{v\}$ is connected and $f - \chi_{\{v\}}$ a connected Roman dominating function. $\square$

Lemma 10. *Let $G = (V, E)$ be a cobipartite graph of order n and $A \subseteq V$. Then $|C_{\mathcal{CRDF},G}[A]| \in \mathcal{O}(n^2)$ and $C_{\mathcal{CRDF},G}[A]$ can be enumerated in polynomial time.*

Proof. Let C_1, C_2 be disjoint cliques with $C_1 \cup C_2 = V$. By Lemma 3, for each $g \in C_{\mathcal{CRDF},G}[A]$, $f_A \coloneqq 2 \cdot \chi_A + \chi_{V \setminus N[A]} \leq g$. If $A = \emptyset$, $f_A = \chi_V$ which is a minimal connected Roman dominating function.

Let $A \cap C_1 \neq \emptyset$ and $A \cap C_2 = \emptyset$ (the other way around works analogously). If $V_1(f_A) = \emptyset$, then f_A is a connected Roman dominating function. Otherwise f_A is no connected Roman dominating function, as $V_0(f_A)$ separates $V_1(f_A)$ from $V_2(f_A) = A$. So we consider the case $V_1(f_A) \neq \emptyset$. By Lemma 9, for all minimal connected Roman dominating functions g on G with $V_2(g) = A$, $|V_1(g) \cap N(A \cap C_1)| \leq 1$. Thus, $C_{\mathcal{CRDF},G}[A] \subseteq \{f_A + \chi_{\{v\}} \mid v \in V_0(f)\}$. Since $|\{f_A + \chi_{\{v\}} \mid v \in V_0(f)\}| \leq n$ and checking if a function is a minimal connected Roman dominating function can be done in polynomial time, we can enumerate all elements in polynomial time.

The only case we still need to consider is $A \cap C_1 \neq \emptyset$ and $A \cap C_2 \neq \emptyset$. In this case A is a dominating set, which implies $V_1(f_A) = \emptyset$. Let g be a minimal connected Roman dominating function with $V_2(g) = A$. By Lemma 9,

$$\begin{aligned} |V_1(g)| = |(V_1(g) \cap C_1) \cup (V_1(g) \cap C_2)| &\leq |V_1(g) \cap C_1| + |V_1(g) \cap C_2| \\ &\leq |V_1(g) \cap N(A \cap C_2)| + |V_1(g) \cap N(A \cap C_2)| \leq 2 \end{aligned}$$

Thus, $C_{\mathcal{CRDF},G}[A] \subseteq \{f_A\} \cup \{f_A + \chi_{\{v\}} \mid v \in V_0(f_A)\} \cup \{f_A + \chi_{\{v,u\}} \mid v, u \in V_0(f)\}$ which implies $|C_{\mathcal{CRDF},G}[A]| \leq n^2 + n + 1$. Since checking if a function is a minimal connected Roman dominating function can be done in polynomial time, we can enumerate all elements in polynomial time. $\square$

Lemma 4, Theorem 4 and Lemma 10 together with [2] imply:

Corollary 6. *There is a polynomial-delay algorithm enumerating all minimal connected Roman dominating functions on cobipartite of order n graphs within time $\mathcal{O}^*(\sqrt[3]{3}^n)$.*

With the same strategy, we can obtain similar results for total Roman dominating function on cobipartite graphs:

Lemma 11. *Let $G = (V, E)$ be a cobipartite graph of order n and $A \subseteq V$. Then $|C_{\mathcal{TRDF},G}[A]| \in \mathcal{O}(n^2)$ and $C_{\mathcal{TRDF},G}[A]$ can be enumerated in polynomial time.*

Proof. Let C_1, C_2 be disjoint cliques with $C_1 \cup C_2 = V$. By Lemma 3, for each $g \in C_{\mathcal{CRDF},G}[A]$, $f_A \coloneqq 2 \cdot \chi_A + \chi_{V \setminus N[A]} \leq g$.

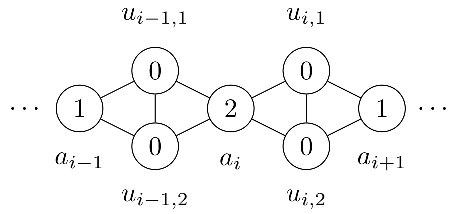

Figure 3: Construction of G_n for $i \in \{2i \mid i \in [\lfloor \frac{n}{2} \rfloor]\}$.

Claim. $|(V_1(g) \cap C_1) \setminus V_1(f_A)| \leq 1$

Proof. Assume there are two $v, u \in (V_1(g) \cap C_1) \setminus V_1(f_A)$ with $v \neq u$. Since $g \geq f_A$ and f_A is a Roman dominating function, there are $x_v, x_u \in A$ such that $\{v, x_v\}, \{u, x_u\} \in E$. Thus, $(V_1(g) \cup V_2(g)) \setminus \{u\}$ and $(V_1(g) \cup V_2(g)) \setminus \{v\}$ have isolates. We will now consider $(V_1(g) \cup V_2(g)) \setminus \{u\}$. Since $v \in C_1$ and $x_v \in N(v)$, there is no isolate in $C_1 \cap ((V_1(g) \cup V_2(g)) \setminus \{u\})$. Hence, the isolate has to be in C_2. Thus, $|C_2 \cap ((V_1(g) \cup V_2(g)) \setminus \{u\})| = 1$. Let y be this isolate. Therefore, $N(y) \cap (V_1(g) \cup V_2(g)) = \{u\}$ and $N(v) \cap C_2 = \emptyset$. This implies $(V_1(g) \cup V_2(g)) \setminus \{v\}$ has no isolates which is a contradiction to the minimality of g. $\Diamond$

Thus, $C_{\mathcal{TRDF},G}[A] \subseteq \{f_A\} \cup \{f_A + \chi_{\{v\}} \mid v \in V_0(f_A)\} \cup \{f_A + \chi_{\{v,u\}} \mid v, u \in V_0(f)\}$ which implies $|C_{\mathcal{TRDF},G}[A]| \leq n^2 + n + 1$. Since checking if a function is a minimal total Roman dominating function can be done in polynomial time, we can enumerate all elements in polynomial time. $\square$

Corollary 7. *There is a polynomial-delay algorithm enumerating all minimal total Roman dominating functions on cobipartite of order n graphs within time $\mathcal{O}^*(\sqrt[3]{3}^n)$.*

5.2 Interval graphs

In this subsection, we will consider connected Roman dominating functions on interval graphs. A graph $G = (V, E)$ is called an *interval graph* if for a each $v \in V$ there is an interval $I_v = [\ell_v, r_v]$ such that $\{u, v\} \in E$ if and only if $I_v \cap I_u \neq \emptyset$. We make use of *locally definable properties* from [5].
Before going into details, we show that we cannot apply Lemma 4 in this case:

Example 1. We consider the graph $G_n \coloneqq (V_n, E_n)$ with

$$V_n \coloneqq \{a_i \mid i \in [n]\} \cup \{u_{i,j} \mid i \in [n-1], j \in [2]\},$$
$$E_n \coloneqq \{\{a_i, u_{i,j}\} \mid i \in [n-1], j \in [2]\} \cup \{\{a_i, u_{i-1,j}\} \mid i \in [n] \setminus \{1\}, j \in [2]\}.$$

G_n is a unit interval graph with the representation $I_{a_i} = [i - 0.3, i + 0.3]$ and $I_{u_{k,j}} = [k + 0.2, k + 0.8]$ for $i \in [n], k \in [n-1], j \in [2]$. Furthermore, G_n is planar which can be observed in Figure 3. We define $A \coloneqq \{a_{2i} \mid i \in [\lfloor \frac{n}{2} \rfloor]\}$. Hence, $N(A) = \{u_{i,j} \mid i \in [n-1], j \in [2]\}$. For $f_A \coloneqq 2\chi_A + \chi_{V \setminus N[A]}$, $V_1(f_A) \cup V_2(f_A)$

is an independent set. To extend f_A to minimal connected Roman dominating function, we must add for all $i \in [n-1]$ exactly one of $u_{i,1}$ and $u_{i,2}$ to $V_1(f_A)$. Therefore, $|C_{G_n,\mathcal{CRDF}}[A]| \geq \sqrt[3]{2}^n$.

For our enumeration algorithm, we order the vertices by ℓ_v (for all $v, u \in V$, $v \leq u$ if and only if $\ell_v \leq \ell_u$). Let $A \subseteq V$. Define $f_A \coloneqq 2\chi_A + \chi_{V\setminus N[A]}$. We call a set $X \subseteq N(A) \setminus A$, *$f_A$-connected* if and only if $f_A + \chi_X$ is a minimal connected Roman dominating function.

In this section, we want to prove that f_A-connected is a locally definable property. This concept was introduced in [5].

Let $(\mathcal{G}, \leq)$ be a uniformly linearly ordered set of graphs. For a $G = (V, E) \in \mathcal{G}$, $X \subseteq V$ and $k \in [|X|]$,

$$\mathbb{W}^k(X) \coloneqq \{(x_1, \ldots, x_k) \in X^k | \forall i \in [k-1], x \in X \setminus \{x_i, x_{i+1}\} : \neg(x_i \leq x \leq x_{i+1})\}$$

A locally definable property on $(\mathcal{G}, \leq)$ $\mathcal{P}$ is a uniform vertex set property if there is a $k \in \mathbb{N} \setminus \{0\}$ and three uniform vertex constraints $\phi, \phi^{\min}, \phi^{\max}$ of respective arities $k, k-1, k-1$ such that for any $G = (V, E) \in \mathcal{G}$ and $X \subseteq V$ with $|X| \geq k$, X has property $\mathcal{P}$ with respect to G if and only if:

1. $\forall (x_1, \ldots, x_k) \in \mathbb{W}^k(X)$, $(x_1, \ldots, x_k) \in \phi$,
2. $\forall (x_1, \ldots, x_{k-1}) \in \mathbb{W}^{k-1}(X)$:
   - $(\forall x \in X : x_1 \leq x) \Rightarrow (x_1, \ldots, x_{k-1}) \in \phi^{\min}$,
   - $(\forall x \in X : x \leq x_{k-1}) \Rightarrow (x_1, \ldots, x_{k-1}) \in \phi^{\max}$,

Now we want to define $\phi, \phi^{\min}, \phi^{\max}$:

Let $s \coloneqq \arg\min\{l_v \mid v \in V_1(f_A) \cup V_2(f_A)\}$ and $t = \arg\max\{r_v \mid v \in V_1(f_A) \cup V_2(f_A)\}$. We define $\phi^{min}_{G,A}, \phi^{max}_{G,A} \subseteq (N(A) \setminus A)^3$ and $\phi_{G,A} \subseteq (N(A) \setminus A)^4$ as follows:

1. $(x, y, z) \in \phi^{min}_{G,A}$ if and only if for all $v \in A$, $P_{G[V\setminus(V_1(f_A)\cup\{x,y,z\})],A}(v) \nsubseteq \{v\}$ and s, z are in one connected component of $V_1(f_A) \cup A \cup \{s, x, y, z\}$ but not in one connected component of $V_1(f_A) \cup A \cup \{s, x, z\}$ and not in one connected component of $V_1(f_A) \cup A \cup \{s, y, z\}$.
2. $(x, y, z) \in \phi^{max}_{G,A}$ if and only if for all $v \in A$, $P_{G[V\setminus(V_1(f_A)\cup\{x,y,z\})],A}(v) \nsubseteq \{v\}$ and t, x are in one connected component of $V_1(f_A) \cup A \cup \{x, y, z, t\}$ but not in one connected component of $V_1(f_A) \cup A) \cup \{x, z, t\}$ and not in one connected component of $V_1(f_A) \cup V_2(f_A) \cup \{x, y, t\}$.
3. $(w, x, y, z) \in \phi_{G,A}$ if and only if for all $v \in A$, $P_{G[V\setminus(V_1(f_A)\cup\{w,x,y,z\})],A}(v) \nsubseteq \{v\}$ and w, z are in one connected component of $V_1(f_A) \cup A \cup \{w, x, y, z\}$ but not in one connected component of $V_1(f_A) \cup A \cup \{w, x, z\}$ and not in one connected component of $V_1(f_A) \cup A \cup \{w, y, z\}$.

Lemma 12. *Let $G = (V, E)$ be an interval graph, $k \in \mathbb{N} \setminus [3]$, $A \subseteq V$ and $X = \{x_1, \ldots, x_k\} \subseteq N(A) \setminus A$ with $x_i < x_j$ if and only if $i < j$ for $i, j \in [k]$. X is f_A-connected if and only if $(x_1, x_2, x_3) \in \phi^{min}_{G,A}$, $(x_{k-2}, x_{k-1}, x_k) \in \phi^{max}_{G,A}$ and for $i \in [k-3]$, $(x_i, x_{i+1}, x_{i+2}, x_{i+3}) \in \phi_{G,A}$.*

Proof. Define $g = f_A + \chi_X$. Let X be f_A-connected. Hence g is a minimal connected Roman dominating function. Hence, there is an $x \in P_{G[V\setminus V_1(g)],V_2(g)}(v) \setminus \{v\} \subseteq P_{G[V\setminus(V_1(f_A)\cup T)],V_2(g)}(v) \setminus \{v\}$ for each $T \subseteq X$ and $v \in A = V_2(f_A) = V_2(g)$. This leaves to prove the connectivity property of $\phi_{G,A}, \phi^{min}_{G,A}, \phi^{max}_{G,A}$. We will only consider $\phi_{G,A}$. The other ones are analogous. Let $i \in [k-3]$. Assume x_i and x_{i+3} are in the same connected component in $V_1(f_A) \cup V_2(f_A) \cup \{x_i, x_{i+2}, x_{i+3}\}$. Hence, there is a path $x_i = v_1, \ldots, v_\alpha = x_{i+3}$ in $V_1(f_A) \cup V_2(f_A) \cup \{x_i, x_{i+2}, x_{i+3}\}$. Obviously, the union of two intersecting intervals is also an interval. By an inductive argument, $[\ell_{x_i}, r_{x_{i+3}}] \subseteq \bigcup_{j=1}^{\alpha} I_{v_j}$. Because of minimality of g, for all $p, q \in [k]$ with $p \neq q$, $I_{x_p} \nsubseteq I_{x_q}$. Hence, $I_{x_{i+1}} \subseteq [\ell_{x_i}, r_{x_{i+3}}]$, which implies $N[x_{i+1}] \subseteq N[\{v_1, \ldots, v_\alpha\}]$. This contradicts the minimality of g. Thus, g fulfills the properties.

Let X fulfill $(x_1, x_2, x_3) \in \phi^{min}_{G,A}$, $(x_{k-2}, x_{k-1}, x_k) \in \phi^{max}_{G,A}$ and for $i \in [k-3]$, $(x_i, x_{i+1}, x_{i+2}, x_{i+3}) \in \phi_{G,A}$. By an inductive argument on the connectivity property of $\phi^{min}_{G,A}, \phi^{max}_{G,A}$ and $\phi_{G,A}$, $s, t, x_1, \ldots, x_k$ are in one connected component of $V_1(g) \cup V_2(g)$. Therefore, there is a path between s, t in $V_1(g) \cup V_2(g)$ and $[\ell_s, r_t] \subseteq \bigcup_{v \in V_1(g)\cup V_2(g)} I_v$. Since s, t are minimal respectively maximal, $V_1(g) \cup V_2(g)$ is connected and g a connected Roman dominating function. No element in $V_1(g) \setminus X$ has a neighbor in A. The elements in X are dominated. Assume there is an $i \in [k-2] \setminus \{1\}$ such that $V_1(g) \cup V_2(g) \setminus \{x_i\}$ is connected. Since $(x_{i-1}, x_i, x_{i+1}, x_{i+2}) \in \phi_{G,A}$, $V_1(f_A) \cup V_2(f_A) \cup \{x_{i-1}, x_{i+1}, x_{i+2}\}$ is not connected. Thus, there is a path between x_{i-1} and x_{i+1} in $V_1(g) \cup V_2(g) \setminus \{x_i\}$, including at least one element in $X \setminus \{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$. Hence,

$$I_{x_i} \setminus \bigcup_{v \in V_1(f_A)\cup V_2(f_A)\cup\{x_{i-1},x_{i+1},x_{i+2}\}} I_v \subseteq \bigcup_{v \in X\setminus\{x_{i-1},x_i,x_{i+1},x_{i+2}\}} I_v.$$

This implies that there is a $j \in [i-2]$ such that $I_{x_{i-1}} \subseteq I_{x_j}$. Therefore, either $I_{x_{j+1}} \subseteq I_{x_j}$ or $I_{x_{i-1}} \subseteq I_{x_{j+1}}$. In the first case, $V_1(f_A) \cup V_2(f_A) \cup \{x_j, x_{j+2}, x_{j+3}\}$ is still connected, which contradicts $(x_j, x_{j+1}, x_{j+2}, x_{j+3}) \in \phi_{G,A}$. In the second case, either $j+1 = i$ or we continue the argument with x_{j+1}. If $j+1 = i$, $V_1(f_A) \cup V_2(f_A) \cup \{x_j, x_{j+2}, x_{j+3}\}$ is still connected, which contradicts $\phi_{G,A}$. For $i = 1$, the argument is the same with $x_0 \coloneqq s$. For $i \in \{k-1, k\}$, consider the intervals after multiplying with -1. This leaves to prove the private neighborhood condition for all $a \in A$. First, there is no $i \in [k]$ with $I_{x_i} \subseteq I_a$, as otherwise $V_1(f_A) \cup V_2(f_A) \cup \{x_{i-1}, x_{i+1}, x_{i+2}\}$ is still connected, which contradicts $\phi_{G,A}$. Therefore, either $\ell_a < \ell_{x_1}$, $r_{x_k} < r_a$, or there is an $i \in [k-1]$ such that $I_a \subseteq [\ell_{x_i}, r_{x_{i+1}}]$. We assume the last case. The other ones are analogous. Hence, there exists a $v \in P_{G[V\setminus(V_1(f)\cup\{x_{i-1},x_i,x_{i+1},x_{i+2}\})],V_2(f)}(a) \setminus \{a\}$. As in the connectivity property case, there are no elements $p, q \in [k]$ such that $I_{x_p} \subseteq I_{x_q}$. Therefore, $v \notin X$. Thus, for all $a \in A$, $P_{G',V_2(f)}(v) \nsubseteq \{v\}$. □

As membership in $\phi^{min}_{G,A}, \phi^{max}_{G,A}$ and $\phi_{G,A}$ can be decided in polynomial time, f_A-connectivity is a locally definable property on interval graphs. With [5, Theorem 3.2], we can deduce:

Corollary 8. *Let $G = (V, E)$ be an interval graph and $A \subseteq V$. The set of all f_A-connected sets can be enumerated with polynomial delay.*

Since there is a bijection between the f_A-connected sets on G and $C_{G,\mathcal{CRDF}}[A]$ which can be calculated in polynomial time, we obtain:

Theorem 6. *There is a polynomial-delay algorithm for enumerating all minimal connected Roman dominating functions on interval graphs.*

Remark 2. The approach described in [5] use of an existing polynomial delay algorithm for enumerating paths on directed graphs. Therefore, in this subsection we consider the combination of our (extension) approach together with this one. This is another strength of the nice Roman domination property approach. By considering the vertices of value 1 after fixing the vertices of value 2, we can combine different approaches. Even if we do not consider further such cases it would be interesting to see more such combinations (for example with the supergraph method [17,18]).

6 Downsides of the Approach

Is this section, we show that using this method naively is not helpful every time. To this end, we define the following problem for any nice Roman domination property $\mathcal{P}$:

Problem name: $\mathcal{P}$-Non-Emptiness **Given:** Graph $G = (V, E)$ and $A \subseteq V$. **Question:** Is $C_{\mathcal{P},G}[A] \neq \emptyset$?

Theorem 7. *$\mathcal{CRDF}$-Non-Emptiness is* NP*-complete even on bipartite 2-degenerate graphs of maximum degree 4. $|V_1(f)|$-Extension Connected Roman Domination is* para-NP*-hard even on bipartite graphs of maximum degree 4.*

Proof. For the hardness we use Monotone 3-Sat-(2,2). This is a variation of 3-Sat, where each variable appears twice positively and twice negatively. [11] provides an NP-completeness proof for this problem. So, let $X := \{x_1, \ldots, x_n\}$ be the set of variables and $C := \{c_1, \ldots, c_m\}$ the set of clauses. To simplify the notation, denote $X' = \{x_i, \overline{x_i} \mid i \in [n]\}$. We can assume that there is no pair $(i, j) \in [n] \times [m]$ such that $\{x_i, \overline{x}_i\} \subseteq c_j$. Otherwise, we can just delete c_j from C. For each $j \in [m]$, let $c_j = \{l_{j,1}, l_{j,2}, l_{j,3}\} \in C$ with $l_{j,1}, l_{j,2}, l_{j,3} \in X'$.

Define $G = (V, E)$ with

$$\begin{aligned}
V &:= \{v_i, \overline{v_i}, w_i \mid i \in [n]\} \cup \{p_j \mid j \in [m]\} \cup \{u_i, u_i' \mid i \in [n-1]\},\\
E &:= \{\{v_i, w_i\}, \{\overline{v_i}, w_i\} \mid i \in [n]\} \cup \{\{u_i, w_i\}, \{u_i, w_{i+1}\}, \{u_i, u_i'\} \mid i \in [n-1]\} \cup\\
&\quad \{\{v_i, p_j\} \mid j \in [m], i \in [n], \exists k \in [3] : x_i = l_{j,k}\} \cup\\
&\quad \{\{\overline{v_i}, p_j\} \mid j \in [m], i \in [n], \exists k \in [3] : \overline{x_i} = l_{j,k}\}.
\end{aligned}$$

This graph has the partition classes $\{v_i, \overline{v}_i \mid i \in [n]\} \cup \{u_i \mid i \in [n-1]\}$ and $\{w_i \mid i \in [n]\} \cup \{u_i' \mid i \in [n-1]\} \cup \{p_j \mid j \in [m]\}$. Thus, G is a bipartite graph. Now we will show that G is 2-degenerate. At first, for all $i \in [n-1]$, $\deg(u_i') = 1$. After deleting these vertices from G, u_i has degree 2 for all $i \in [n-1]$ (u_i has

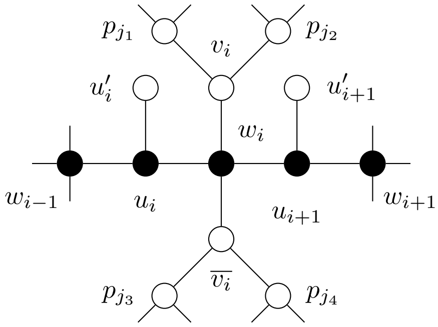

Figure 4: Construction of Theorem 7 where the black vertices are in A.

degree 3 in G). Let G' be the graph after deleting also these vertices. Now let $i \in [n]$. While w_i has degree at most 4 in G, the degree in G' is 2. After deleting $\{w_r \mid r \in [n]\}$, $v_i, \overline{v_i}$ have degree 2. In G, $v_i, \overline{v_i}$ have degree 3. Therefore, we only need consider p_j for $j \in [m]$. But $G[\{p_j \mid j \in [m]\}]$ has no edges. Hence, G is a 2-degenerate graph with maximum degree 4.

Let $A \coloneqq \{w_i \mid i \in [n]\} \cup \{u_i \mid i \in [n-1]\}$. The gadget is also depicted in Figure 4.

Claim. There is an assignment ϕ satisfying C if and only if $C_{\mathcal{CRDF},G}[A] \neq \emptyset$.

Proof. Let ϕ be a satisfying assignment. Let $I = [n]$. If there is an $i \in I$, such that $\phi - (1 - \phi(x_i))\chi_{\{x_i\}}$ is still satisfying C, delete i from I. By the fact that there is no pair $(i,j) \in [n] \times [m]$ with $\{x_i, \overline{x}_i\} \subseteq C_j$, ϕ restricted to I already satisfies C. Define $h \coloneqq 2 \cdot \chi_A + \chi_{\{v_i \mid i \in I, \phi(x_i)=1\} \cup \{\overline{v_i} \mid i \in I, \phi(x_i)=0\}} + \chi_{\{p_1,\ldots,p_m\}}$. Clearly, h is a Roman dominating function, as $V \setminus N(A) = \{p_1, \ldots, p_m\}$. As ϕ satisfies C, $N(p_j) \cap V_1(h) \neq \emptyset$ for all $j \in [m]$. Thus, for each $j \in [m]$ there is an $i \in I$ such that w_i is connected to p_j in $V_1(h) \cup V_2(h)$. For $k, \ell \in [n]$ with $k < \ell$, w_k and w_ℓ are connected by $w_k, u_k, w_{k+1}, \ldots, w_\ell$. Hence, $V_1(h) \cup V_2(h)$ is connected and h a connected Roman dominating function. Next, we want to prove that each element in $V_2(h) = \{w_1, u_1, w_2, \ldots, u_{n-1}, w_n\}$ has a private neighbor. Since ϕ is assignment, $\emptyset \neq \{v_i, \overline{v}_i\} \cap V_0(h) = N(w_i) \cap V_0(h)$ for all $i \in [n]$. Further, u'_i is the private neighbor of u_i for $i \in [n-1]$. This implies Property 3 of Lemma 8. For $j \in [m]$, $N(p_j) \cap V_2(h) = \emptyset$. This leaves only to show that $G[(V_1(h) \cup V_2(h)) \setminus \{t\}]$ is not connected for any $t \in \{v_i, \overline{v}_i \mid i \in I\} \cap V_1(h)$. Let $i \in I$ and without loss of generality $h(v_i) = 1$ (for $h(\overline{v}_i) = 1$ works analogously). By definition of I there is a $j \in [m]$ such that C_j is only satisfied by x_i. Hence, p_j has no neighbor in $V_1(h - \chi_{\{v_i\}}) \cup V_2(h - \chi_{\{v_i\}})$. Therefore, h is a minimal connected Roman dominating function.

Conversely, assume there is a minimal connected Roman dominating function $g \in \{0,1,2\}^V$ with $f \leq g$. Define $\phi \coloneqq \chi_{\{x_i \mid i \in [n], g(v_i) \neq 0\}}$. Let $i \in [n]$. Since $w_i \in V_2(f) \subseteq V_2(g)$, it needs a private neighbor and $\{v_i, \overline{v}_i\} \cap V_0(g) \neq \emptyset$. Let $j \in [m]$. Without loss of generality, we consider C_j includes only negative literals.

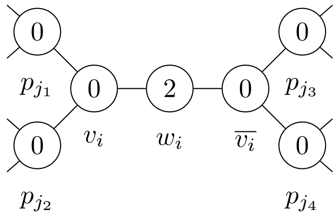

Figure 5: Construction for Theorem 8

Since g is a connected Roman dominating function, there is an $i \in [n]$ with $\overline{v}_i \in N(p_j) \cap (V_1(g) \cup V_2(g))$. Hence, $g(v_i) = 0$, $\phi(x_i) = 0$ and ϕ satisfies C_j. $\Diamond$

Since the graph has order $3n + m$ and $6n$ edges, G and f can be constructed in polynomial time. $\square$

There are simular results for total Roman dominating functions:

Theorem 8. *$\mathcal{TRDF}$-Non-Emptiness is NP-complete even on bipartite graphs of maximum degree 3.*

Proof. For the hardness we use Monotone 3-Sat-(2,2). It is known by [11], that this restricted Sat version is NP-complete. So, let $X := \{x_1, \ldots, x_n\}$ be the set of variables and $C := \{C_1, \ldots, C_m\}$ the set of clauses. To simplify the notation, denote $X' = \{x_i, \overline{x_i} \mid i \in [n]\}$. For each $j \in [m]$, let $C_j = l_{j,1} \vee l_{j,2} \vee l_{j,3} \in C$ with $l_{j,1}, l_{j,2}, l_{j,3} \in X'$.

Define $G = (V, E)$ with

$$\begin{aligned} V &:= \{v_i, \overline{v_i}, w_i \mid i \in [n]\} \cup \{p_j \mid j \in [m]\} \\ E &:= \{\{v_i, w_i\}, \{\overline{v_i}, w_i\} \mid i \in [n]\} \cup \\ &\quad \{\{v_i, p_j\} \mid j \in [m], k \in [3], i \in [n], x_i = l_{j,k}\} \cup \\ &\quad \{\{\overline{v_i}, p_j\} \mid j \in [m], k \in [3], i \in [n], \overline{x_i} = l_{j,k}\}. \end{aligned}$$

This graph has the partition classes $\{v_i, \overline{v}_i \mid i \in [n]\}$ and $\{w_i \mid i \in [n]\} \cup \{p_j \mid j \in [m]\}$. Because of the special Sat variant we consider, for all $i \in [n]$ and $j \in [m]$,

$$\deg(w_i) = 2 < 3 = \deg(v_i) = \deg(\overline{v}_i) = \deg(p_j).$$

Thus G is a bipartite graph with degree at most 3. Let $A := \{w_i \mid i \in [n]\}$ and $f := 2 \cdot \chi_A \in \{0, 1, 2\}^V$. The gadgets and the values of f are also depicted in Figure 5.

Claim. There is an assignment ϕ satisfying C if and only if $C_{\mathcal{TRDF},G}[A] \neq \emptyset$ if and only if there is a minimal total Roman dominating function h on G with $f \leq h$.

Proof. By Lemma 3, $C_{\mathcal{TRDF},G}[A] \neq \emptyset$ implies the existence of a minimal total Roman dominating function h on G with $f \leq h$.

Let ϕ be a satisfying assignment. Then define

$$h \coloneqq f + \chi_{\{v_i | i\in[n],\phi(x_i)=1\}\cup\{v_i | i\in[n],\phi(x_i)=0\}} + \chi_{\{p_j | j\in[m]\}} .$$

Clearly $V_0(h) \subseteq \{v_i, \overline{v}_i \mid i \in [n]\} \subseteq N(\{w_i \mid i \in [n]\}) \subseteq N(V_2(h))$. Therefore, h is a Roman dominating function. Since ϕ is an assignment, for each $i \in [n]$, there is $u_i \in N(w_i)\cap V_1(h)$. Further ϕ satisfies C. Hence, for all $j \in [m]$, $N(p_j)\cap V_1(h) \neq \emptyset$. Thus, h is a total Roman dominating function. Since $V_2(h)) = \{w_i \mid i \in [n]\}$ and ϕ is assignment, $\emptyset \neq \{v_i, \overline{v}_i\}\cap V_0(h) = N(w_i)\cap V_0(h)$. This implies Property 3 of Lemma 7. For $j \in [m]$, $N(p_j) \cap V_2(f) = \emptyset$. This leaves only to show that $G[(V_1(h) \cup V_2(h)) \setminus \{u\}]$ has an isolated vertex for $u \in \{v_i, \overline{v}_i \mid i \in [n]\} \cap V_1(h)$. Let $i \in [n]$ and without loss of generality $f(v_i) = 1$. As ϕ is an assignment, $f(\overline{v}_i) = 0$. Because $\overline{v}_i$ is besides v_i the only neighbor of w_i, w_i would be isolated of $G[(V_1(h)\cup V_2(h))\setminus\{v_i\}]$. By Lemma 7, h is a minimal total Roman dominating function. Furthermore, $f \leq h \in C_{\mathcal{TRDF},G}[A]$.

Conversely, assume there is a $g \in \{0,1,2\}^V$ be a minimal total Roman dominating function with $f \leq g$. Define $\phi \coloneqq \chi_{\{x_i | i\in[i],g(v_i)\neq 0\}}$. Let $i \in [n]$. Since $w_i \in V_2(f) \subseteq V_2(g)$, it needs a private neighbor and $\{v_i, \overline{v}_i\} \cap V_0(g) \neq \emptyset$. Further, $N(w_i) \cap (V_1(g) \cup V_2(g)) \neq \emptyset$, as otherwise g would not be a total Roman dominating function.

Let $j \in [m]$. Without loss of generality, we consider C_j includes only negative literals. Assume $g(p_j) = 0$. Then there exists an $i \in [n]$ with $\overline{v}_i \in N(p_j) \cap V_2(g)$. Hence, $g(v_i) = 0$ and $\phi(x_i) = 0$ and ϕ satisfies C_j. This leaves to consider $g(p_j) \neq 0$. As g is a total Roman dominating function, there is a $\overline{v}_i \in N(p_j) \cap (V_1(g) \cap V_2(g))$ and analogously to $g(p_j) = 0$, ϕ satisfies C_j. $\diamondsuit$

Since the graph has order $3n + m$ and $6n$ edges, G and f can be constructed in polynomial time. $\square$

We have even some hardness results for split graphs. For the proof we use the following lemma:

Lemma 13. *Let $G = (V, E)$ be a connected split graph without universal vertices. Then $f \in \{0,1,2\}^V$ is a total Roman dominating function on G if and only if f is a connected Roman dominating function.*

Proof. Let $G = (V, E)$ be split graph with the maximal clique C and independent set I as well as $f \in \{0,1,2\}^V$. Define $V' \coloneqq V_1(f) \cup V_2(f)$ and $G' \coloneqq G[V']$. Let f be a total Roman dominating function. Let $v, u \in V'$ with $v \neq u$. Let $v, u \in I$. Since f is a total Roman dominating function there exists an $x \in N(v) \cap V' \subseteq C \cap V'$. With the same argument, there is a $y \in N(u) \cap V' \subseteq C \cap V'$. If $x = y$, vxu forms a path on G'. Otherwise, $vxyu$ forms a path. Analogously, there is a path from v to u if $\{v, u\} \cap C \neq \emptyset$. Therefore, G' is connected.

Now we consider f is a connected Roman dominating function on G. Since there is no universal vertex, there is no vertex $v \in V$ such that $\{v\}$ is a dominating set. As V' is a dominating set, $|V'| \geq 2$. Hence, each $v \in V'$ has a neighbor in V' and f is total. $\square$

Theorem 9. *Enum $C_{\mathcal{CRDF}}$ and Enum $C_{\mathcal{TRDF}}$ are at least as hard as Enum Tr even on split graphs.*

Proof. Let $H = (X, S)$ be the hypergraph with $X \coloneqq \{x_1, \ldots, x_n\}$ and $S \coloneqq \{s_1, \ldots, s_m\}$. We can assume that there is no $i \in [n]$ such that $x_i \in s_j$ for all $j \in [m]$. Otherwise, $Tr(H) = \{\{x_i\}\} \cup Tr((X \setminus \{x_i\}, S))$. Define $G = (V, E)$ with

$$V \coloneqq \{a, b\} \cup \{u_i \mid i \in [n]\} \cup \{w_j \mid j \in [m]\}$$
$$E \coloneqq \binom{\{a, b\} \cup \{u_i \mid i \in [n]\}}{2} \cup \{\{u_i, w_j\} \mid i \in [n], j \in [m], x_i \in s_j\}.$$

Claim. $F : C_{\mathcal{CRDF},G}[\{a\}] \to Tr(H)$, $g \mapsto \{x_i \mid i \in [n], g(u_i) = 1\}$ is a bijection.

Proof. Let $g \in C_{\mathcal{CRDF},G}[\{a\}]$. By Lemma 3, $2 \cdot \chi_{\{a\}} + \chi_{\{w_j \mid j \in [m]\}} \leq g$. Define $Z \coloneqq F(g)$. Since $V_1(g) \cup \{a\}$ needs to be connected and $\{w_1, \ldots, w_j\} \subseteq V_1(g)$, for all $j \in [m]$, $Z \cap s_j \neq \emptyset$. Hence Z is a hitting set. Assume Z is not minimal. Then there is an $i \in [n]$ such that for all $j \in [m]$, $s_j \cap (Z \setminus \{x_i\}) \neq \emptyset$. Hence, for $g_i \coloneqq g - \chi_{\{u_i\}}$, $V_1(g_i) \cup V_2(g_i)$ is connected. Since u_i is dominated by a, g_i is a connected Roman dominating function. This contradicts the minimality of g. Thus, Z is minimal.

Conversely, let $D \subseteq X$ be a minimal hitting set. Define $g \coloneqq 2 \cdot \chi_{\{a\}} + \chi_{\{u_i \mid x_i \in D\}} + \chi_{\{w_j \mid j \in [m]\}}$. Clearly, $D = F(g)$. Since D is a dominating set, for all $j \in [m]$ there is an $i \in [n]$ with $x_i \in s_j$ and $u_i \in N(w_j) \cap V_1(g)$. As $\{a, b\} \cup \{u_i \mid i \in [n]\}$ is a clique, $V_1(g) \cup V_2(g)$ is connected. Furthermore, $V_0(g) \subseteq \{a, b\} \cup \{u_i \mid i \in [n]\} \subseteq N[a]$. Hence, g is a connected Roman dominating function. b is a private neighbor of a. Assume there is an $i \in [n]$ such that $g_i \coloneqq g - \chi_{\{u_i\}}$ is a connected Roman dominating function. As D is minimal, there exists a $j \in [m]$ with $s_j \cap D = \{x_i\}$. Thus, w_j would be isolated in $G[V_1(g_i) \cup V_1(g_i)]$. Furthermore, $\emptyset \neq \{w_j \mid j \in [m]\} \cap N[a] = \{w_j \mid j \in [m]\} \cap N[V_2(g)]$. Therefore, g is minimal. This implies F is surjective.

Let $g, h \in C_{\mathcal{CRDF},G}[\{a\}]$ be minimal connected Roman dominating functions with $D \coloneqq F(g) = F(h)$. Therefore, $V_1(g) \cap \{u_i \mid i \in [n]\} = V_1(h) \cap \{u_i \mid i \in [n]\}$. Since $V_2(g) = \{a\} = V_2(h)$, $\{w_j \mid j \in [m]\} \subseteq V_1(g) \cap V_1(h)$. This implies g and h can only differ in b. If they differ in just one vertex, $h \leq g$ or $g \leq h$. This would contradict the minimality of g or h and implies $g = h$. Hence, F is injective and therefore bijective. $\Diamond$

Since F is bijection for which $F(g)$ can be calculated in polynomial time for all $g \in C_{\mathcal{CRDF},G}[\{a\}]$, we could enumerate all minimal hitting set of H in output polynomial time if we could enumerate all functions in $C_{\mathcal{CRDF},G}[\{a\}]$.

It should be mentioned that G has no universal vertex. Hence, each connected Roman dominating function is a total Roman dominating function and the other way around. $\square$

Corollary 9. *$|V_2(f)|$-Extension Total Roman Domination,$|V_2(f)|$-Extension Connected Roman Domination are para-NP-hard on split graphs.*

Remark 3. The hardness result in this section imply that this approach is not helpful in every case. For the proof of Theorem 1, we need a output-polynomial enumeration algorithm for $C_{\mathcal{P}}[A]$. This is not possible, if it is NP-complete to decide if there is element in $C_{\mathcal{P}}[A]$.

The algorithm of Theorem 1 does not consider $C_{\mathcal{P},G}[B]$ for $B \subseteq A$. The supergraph method, for example, uses already existing solutions to compute new once. This implies the question, whether we could use different approaches (besides extension) for results like Theorem 1.

By Theorem 9, we know, that even the idea of using $C_{\mathcal{CRDF},G}[B]$ for $C_{\mathcal{CRDF},G}[A]$, when $B \subseteq A$, is not helpful in every case. In this proof $A = \{a\}$. So, the only proper subset is $\emptyset$. By the definition, $C_{\mathcal{CRDF},G}[\emptyset] = \{\chi_V\}$. Hence, using proper subsets is not helpful in every case.

7 Conclusion

We introduced the notion of nice Roman domination property and used it to obtain some enumeration results for different Roman domination variations like maximal Roman dominating function, total Roman dominating function, connected Roman dominating function. Furthermore, we proved that our approach is not always helpful unless P = NP. In these cases, it does not prove that the enumeration is hard. Thus, it would be interesting to prove some results for enumerating minimal connected Roman dominating functions/total Roman dominating functions on general graphs. There could be even more graph classes with good enumeration results. Here, it could be interesting to consider to use different approaches/methods as mentioned in Remark 2 and Remark 3. If we have hardness results we could also use the approach to obtain FPT delay algorithms. Improving the input-sensitive bounds for enumerating minimal Roman dominating functions could strengthen results implied by Lemma 4.

Furthermore, not all Roman domination variations are nice Roman domination property. For example, unique response Roman domination (see [27]) or Italian domination (see [8,15]) fail to be. Finding more nice Roman domination properties, or even a generalization of nice Roman domination property which also handles some of these examples could be a future research direction. Another question is if there are results beyond enumeration which could use nice Roman domination property.

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