EXTERIOR POWER OF STABLE VECTOR BUNDLE DESTABILIZED BY FROBENIUS PULL-BACK

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Abstract. In this paper, we prove that for any smooth projective curve C of genus $g \geq 2$ over an algebraically closed field of positive characteristic, there exists a stable vector bundle over C whose exterior power is not semi-stable.

1. Introduction

Let C be a smooth irreducible projective curve of genus $g > 0$ over an algebraically closed field k. As we know, when $\mathrm{char}(k) = 0$, the semi-stability of vector bundles over C is preserved under the operations of tensor product, symmetric product and exterior product operations (cf. [2, 3]). However, this is not the case in positive characteristic. In fact, in [1] the author proves that, for any prime p and integer $g > 1$, there exists a smooth curve C of genus g in characteristic p and a semi-stable bundle E of rank 2 over C such that the Frobenius pullback $E^{(p)} := F^*E$ and thus the symmetric power $S^p(E)$ are not semi-stable. The examples of non-ample semi-stable bundles constructed by J.-P. Serre (cf. [3, Sect. 3]) and generally by H. Tango (cf. [8, Sect. 5, Exam. 3]), as well as the push-forward F_*E of a vector bundle E, possess the required property. For the case of the exterior product, we found no relevant assertions; therefore, in this note, we present a similar conclusion regarding the exterior power of semi-stable bundles in positive characteristic.

First, we adopt a computational approach to obtain the following result.

Theorem 1.1. *(Theorem 3.3) Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$, and E be a vector bundle of rank r on C. Then $F^n_*E \wedge F^n_*E$ is not semi-stable whenever $r > 1$, $p > 3$ or $n > 1$.*

Therefore, as a corollary, we derive a similar result in case of exterior power of a stable vector bundle.

Theorem 1.2. *(Corollary 3.3) Let C be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$, then there exists a stable vector bundle E on C, whose double exterior power is not semi-stable.*

By definition, cohomological stability implies stability in the usual sense.In characteristic zero, any exterior power of a semi-stable bundle is also semi-stable, so cohomological semi-stability is equivalent to semi-stability.However, in positive characteristic, since exterior product no longer preserves semi-stability, cohomological semi-stability is not equivalent to slope semi-stability. So we intended to find counterexamples of vector bundles that are semi-stable but not cohomologically semi-stable and we only confirm that stability is not equivalent to cohomological stability in positive characteristic.

Proposition 1.3. *(Proposition 3.4) Let C be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$, then there exists a vector bundle E which is stable but not cohomologically stable.*

Based on the above observation, we propose the following conjecture.

Conjecture 1.4. *Let C be a smooth projective curve defined over an algebraically closed field k of characteristic $p > 0$, and E be a vector bundle over C. Then E is semi-stable if and only if E is cohomologically semi-stable.*

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2. Preliminary

In this section, we recall some definitions and facts about the stability of vector bundles on curves.

2.1. **Notation.** Let C be a smooth projective curve of genus $g > 1$ over an algebraically closed field k of positive characteristic and Ω_C be the sheaf of 1-forms on C. The absolute Frobenius morphism $F : C \to C$ is defined by the map $f \to f^p$ in $\mathcal{O}_C$, and the n-th Frobenius morphism F^n is n-times composition of F, where $n \geq 0$. If $n = 0$, we mean $F^0 = Id$. Let B_1 denote the exact 1-form which sits in the following exact sequence

$$0 \to \mathcal{O}_C \overset{F^{\#}}{\to} F_*\mathcal{O}_C \to B_1 \to 0.$$

and is semi-stable of degree $(p-1)(g-1)$.

2.2. **Stability.**

Definition 2.1. Let E be a vector bundle on a smooth projective curve C. Recall that the *slope* of E is the rational number

$$\mu(E) = \frac{\deg E}{\mathrm{rk}(E)},$$

and E is called *stable* (resp. *semi-stable*) if

$$\mu(F) < \mu(E) \text{ (resp. } \mu(F) \leq \mu(E))$$

for every sub-bundle $F \subset E$ with $\mathrm{rk}(F) < \mathrm{rk}(E)$.

Moreover, E is called *cohomologically stable* (resp. *cohomologically semi-stable*) if for every line bundle A of degree a and every integer $t < \mathrm{rk}(E)$,

$$H^0(\wedge^t E \otimes A^{-1}) = 0$$

whenever $a \geq t\mu(E)$ (resp. $a > t\mu(E)$).

Let E be a vector bundle on C. Then there exists a unique filtration, which is called the *Harder-Narasimhan filtration*,

$$0 = E_0 \subset E_1 \subset \cdots \subset E_s = E$$

such that E_i/E_{i-1} is semi-stable of slope μ_i and

$$\mu_1 > \mu_2 > \cdots > \mu_s.$$

Proposition 2.2. *Let E be a vector bundle on C. Then*

$$\mu(F^n_*(E)) = \frac{\mu(E)}{p^n} + (1 - \frac{1}{p^n})(g-1).$$

Proof. Note that $\chi(F^n_*E) = \chi(E)$, then by the Riemann-Roch theorem we have $\deg(F^n_*E) = \deg(E) + \mathrm{rk}(E)(p^n - 1)(g-1)$ and hence the result. □

2.3. **A canonical filtration.** Let E be a vector bundle on C. Then $V := F^*(F_*E)$ has a canonical connection

$$\nabla : V \to V \otimes \Omega_C$$

with zero p-curvature (cf. [5, Theorem 5.1]). So there is a canonical filtration

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_1 \subset V_0 = V \qquad \#$$

where $V_1 = \ker(\mathrm{V} = \mathrm{F^*F_*E} \twoheadrightarrow \mathrm{E})$ and

$$V_{l+1} = \ker(V_l \xrightarrow{\nabla} V \otimes \Omega_C \twoheadrightarrow V/V_l \otimes \Omega_C)$$

when $l > 0$. Then we have the following lemma by a local calculation (cf.[7, Lem 2.1]).

Lemma 2.3. *([4, Theorem 5.3]) Under the above assumption, we have*

1. *$V_0/V_1 \cong E, \nabla(V_{l+1}) \subset V_l \otimes \Omega_C$ for $l > 0$,*
2. *$V_l/V_{l+1} \xrightarrow{\nabla} (V_{l-1}/V_l) \otimes \Omega_C$ is an isomorphism for $1 \leq l \leq p-1$, and*
3. *if $g \geq 2$ and E is semi-stable, then the canonical filtration (#) is nothing but the Harder-Narasimhan filtration.*

This lemma can be used to prove the following result (see [6] for the case of line bundles).

Lemma 2.4. *([7, Theorem 2.2]) Let C be a smooth projective curve of genus $g \geq 1$. Then F_*E is semi-stable whenever E is semi-stable. If $g \geq 2$, then F_*E is stable whenever E is stable.*

2.4. **Combination number modulo p.**

Lemma 2.5. *Let $p > 0$ be a prime number. Then we have*

- $\binom{p-1}{h} \equiv (-1)^h \bmod(p)$, *and*
- $\binom{p-2}{h} \equiv (-1)^h(h+1) \bmod(p)$.

Proof. Note that

$$\binom{p-1}{h} = \frac{(p-1)!}{(p-1-h)!h!} = \binom{p-1}{h-1}\frac{p-h}{h} \equiv -\binom{p-1}{h-1} \bmod(p)$$

and

$$\binom{p-2}{h} = \frac{(p-2)!}{(p-2-h)!h!} = \binom{p-2}{h-1}\frac{p-h-1}{h} \equiv -\frac{h+1}{h}\binom{p-2}{h-1} \bmod(p).$$

Then we obtain the result by iteration. □

3. Instability of $F^n_*E \wedge F^n_*E$

In this section, we first make the following observation: $F^n_*(E) \wedge F^n_*(E)$ admits a canonical subbundle. We then utilize this observation to derive our main result by computing the slopes of the bundles involved in the subsequent assertions.

Lemma 3.1. *Let C be a smooth projective curve defined over an algebraically closed field k of characteristic $p > 0$, and E be a vector bundle of rank r on C. Then*

1. *when $r > 1$, there is a natural injection*
$$F^n_*(E \wedge E \otimes \Omega_C^{p^n-1}) \hookrightarrow F^n_*(E) \wedge F^n_*(E),$$
where $n \in \mathbb{Z}^+$;
2. *when $r = 1$ and $p > 2$, there is a natural injection*
$$F^n_*(E \otimes E \otimes \Omega_C^{p^n-2}) \hookrightarrow F^n_*(E) \wedge F^n_*(E),$$
where $n \in \mathbb{Z}^+$;

(3) *when $r=1$ and $p=2$, there is a natural injection*

$$F_*^{n-1}(B_1\otimes E\otimes \Omega_C^{p^{n-1}-1})\hookrightarrow F_*^n(E)\wedge F_*^n(E),$$

where $n\in\mathbb{Z}^+$.

Proof. By Lemma 2.3 there is a canonical filtration

$$0=V_p\subset\cdots\subset V_1\subset V_0=F^*F_*(E) \tag{$*$}$$

with $V_i/V_{i+1}\cong E\otimes\Omega_C^i$. After tensoring the above filtration with E and pushing it forward by F_*, by projection formula we obtain the following filtration of $F_*E\otimes F_*E$:

$$0=F_*(E\otimes V_p)\subset\cdots\subset F_*(E\otimes V_1)\subset F_*(E\otimes V_0)\cong F_*E\otimes F_*E$$

with $F_*(E\otimes V_i)/F_*(E\otimes V_{i+1})\cong F_*(E\otimes E\otimes\Omega_C^i)$.

Next, we give a local basis of the first non-zero sub-bundle $F_*(E\otimes V_{p-1})\cong F_*(E\otimes E\otimes\Omega_C^{p-1})$. Since it is a local problem, we first reduce to the case $E=\mathcal{O}_C$ and $C=\operatorname{Spec}\mathrm{k}[[\mathrm{t}]]$, then $F^*F_*\mathcal{O}_C=k[[t]]\otimes_{k[[s]]}k[[t]]$ where $s=t^p$. Set $I_0=k[[t]]\otimes_{k[[s]]}k[[t]]$ and $I_1=\ker(k[[t]]\otimes_{k[[s]]}k[[t]]\twoheadrightarrow k[[t]])$ which is also an ideal of the $k[[t]]$-algebra $I_0=k[[x]]\otimes_{k[[s]]}k[[x]]$. Set $\{\alpha:=1\otimes t-t\otimes 1\}$, and it is easy to see that $\{\alpha,\alpha^2,\ldots,\alpha^{p-1}\}$ is a basis of I_1 as $k[[t]]$-module ($\alpha^p=0$). Note that the filtration $(*)$ is defined by $I_{l+1}=\ker(I_l\xrightarrow{\nabla}I_0\otimes\Omega_C\twoheadrightarrow I_0/I_l\otimes\Omega_C)$ and $\nabla(\alpha^l)=-l\alpha^{l-1}\otimes\mathrm{d}t$. Thus, as a free $\mathcal{O}$-module, I_l has a basis $\{\alpha^l,\alpha^{l+1},\ldots,\alpha^{p-1}\}$.

Now, we return to the case when $r>1$ and suppose that locally $E=\mathcal{O}\cdot e_1\oplus\cdots\oplus\mathcal{O}\cdot e_r$. Then the local basis of $F_*(E\otimes E\otimes\Omega_C^{p-1})$ consists of:

$$\begin{aligned}\{(e_i\otimes e_j)\cdot t^k\alpha^{p-1} &:= \sum_{h=0}^{p-1}\binom{p-1}{h}(-1)^h t^{k+h}e_i\otimes t^{p-1-h}e_j\\ &= \sum_{h=0}^{p-1}t^{k+h}e_i\otimes t^{p-1-h}e_j,\ i,j=1,2,\ldots,r,k=0,1,\ldots,p-1\}\end{aligned}$$

since $\binom{p-1}{h}\equiv(-1)^h \bmod(p)$ by Lemma 2.5. Note that

$$\begin{aligned}t^k\alpha^{p-1} &= \sum_{h=0}^{p-1}t^{k+h}\otimes t^{p-1-h}=\sum_{i=k}^{p-1+k}t^i\otimes t^{p-1+k-i}=\sum_{i=k}^{p-1}t^i\otimes t^{p-1+k-i}+\sum_{i=p}^{p-1+k}t^i\otimes t^{p-1+k-i}\\ &= \sum_{i=k}^{p-1}t^i\otimes t^{p-1+k-i}+\sum_{j=0}^{k-1}t^{p+j}\otimes t^{k-1-j}=\sum_{i=k}^{p-1}t^i\otimes t^{p-1+k-i}+s\sum_{j=0}^{k-1}t^j\otimes t^{k-1-j},\end{aligned}$$

which implies that $t^k\alpha^{p-1},k=0,1,\ldots,p-1$ are all symmetric (if $m>n$, we mean $\Sigma_{i=m}^n a_i=0$). So the symmetric part of $F_*(E\otimes E\otimes\Omega_C^{p-1})$ is generated by

$$(e_i\otimes e_i)\cdot t^k\alpha^{p-1},i=1,2,\ldots,r,k=0,1,\ldots,p-1,\ \text{and}$$
$$(e_i\otimes e_j+e_j\otimes e_i)\cdot t^k\alpha^{p-1},i\neq j,k=0,1,\ldots,p-1,$$

which form a basis of the kernel of the natural map: $F_*(E\otimes E\otimes\Omega_C^{p-1})\twoheadrightarrow F_*(E\wedge E\otimes\Omega_C^{p-1})$ as $k[[s]]$-module. Therefore, we obtain an injection

$$F_*(E\wedge E\otimes\Omega_C^{p-1})\hookrightarrow F_*(E)\wedge F_*(E).$$

Moreover, by an n-times composition of the above map, we get the desired conclusion:

$$F_*^n(E\wedge E\otimes\Omega_C^{p^n-1})\hookrightarrow F_*^n(E)\wedge F_*^n(E).$$

When $r=1$, by the above argument, we see that the sections in the sub-bundle $F_*(E\otimes V_{p-1})\subset F_*(E\otimes V_{p-2})$ are all symmetric. So we consider the second non-zero sub-bundle $F_*(E\otimes V_{p-2})\subset F_*E\wedge F_*E$ and hence the local basis of the quotient $F_*(E\otimes V_{p-2})/F_*(E\otimes V_{p-1})\cong F_*(E\otimes E\otimes\Omega_C^{p-2})$.

Suppose that locally $E = \mathcal{O}\cdot e$. As the above argument, the local basis of $F_*(E\otimes E\otimes\Omega_C^{p-2})$ consists of:

$$\begin{aligned}\{(e\otimes e)\cdot t^k\alpha^{p-2} &:= \sum_{h=0}^{p-2}\binom{p-2}{h}(-1)^h t^{k+h}e\otimes t^{p-2-h}e \\ &= \sum_{h=0}^{p-2}(h+1)t^{k+h}e\otimes t^{p-2-h}e,\ k=0,1,\dots,p-1\}\end{aligned}$$

since $\binom{p-2}{h}\equiv(-1)^h(h+1)\ \mathrm{mod}(p)$ by Lemma 2.5. Note that

$$\begin{aligned}t^k\alpha^{p-2} &= \sum_{h=0}^{p-2}(h+1)t^{k+h}\otimes t^{p-2-h} = \sum_{i=k}^{p-2+k}(i-k+1)t^i\otimes t^{p-2+k-i} \\ &= \sum_{i=k}^{p-2}(i-k+1)t^i\otimes t^{p-2+k-i} + (p-k)t^{p-1}\otimes t^{k-1} + \sum_{i=p}^{p-2+k}(i-k+1)t^i\otimes t^{p-2+k-i} \\ &= \sum_{i=k}^{p-2}(i-k+1)t^i\otimes t^{p-2+k-i} + (p-k)t^{p-1}\otimes t^{k-1} + \sum_{j=0}^{k-2}(j+p-k+1)t^{j+p}\otimes t^{k-2-j} \\ &= \sum_{i=k}^{p-2}(i-k+1)t^i\otimes t^{p-2+k-i} + (p-k)t^{p-1}\otimes t^{k-1} + s\sum_{j=0}^{k-2}(j-k+1)t^j\otimes t^{k-2-j},\end{aligned}$$

which implies that $t^k\alpha^{p-2}$ is symmetric if and only if $k=0$ and $p=2$. (Here, if $m>n$ and $a<0$, we mean $\Sigma_{i=m}^n a_i = 0$ and $t^b\otimes t^a = 0$ respectively.) So when $p>2$ the symmetric part of $F_*(E\otimes V_{p-2})$ is exactly the sub-bundle $F_*(E\otimes E\otimes\Omega_C^{p-1})$. Therefore, we obtain an injection

$$F_*(E\otimes E\otimes\Omega_C^{p-2})\hookrightarrow F_*(E)\wedge F_*(E).$$

Moreover, by considering $E := F_*E$ which is of rank >1 in (1), we have

$$F_*^n(E\otimes E\otimes\Omega_C^{p^n-2})\hookrightarrow F_*^n(E)\wedge F_*^n(E).$$

When $p=2$, note that $F_*(E)\wedge F_*(E)\cong B_1\otimes E$ and by (1) again we have the required injection:

$$F_*^{n-1}(B_1\otimes E\otimes\Omega_C^{p^{n-1}-1})\hookrightarrow F_*^n(E)\wedge F_*^n(E)$$

□

Theorem 3.2. *Let C be a smooth projective curve of genus $g\geq 2$ defined over an algebraically closed field k of characteristic $p>0$, and E be a vector bundle of rank r on C then $F_*^nE\wedge F_*^nE$ is not semi-stable when $r>1$, $p>3$ or $n>1$.*

Proof. When $r>1$, by Lemma 3.1, we have the following injection

$$F_*^n(E\wedge E\otimes\Omega_C^{p^n-1})\hookrightarrow F_*^n(E)\wedge F_*^n(E).$$

After some computation of the slopes by using Proposition 2.2, we have:

$$\mu(F_*^nE\wedge F_*^nE) = 2\mu(F_*^nE) = \frac{2}{p^n}\mu(E) + \frac{2(p^n-1)(g-1)}{p^n}$$

and

$$\begin{aligned}\mu(F_*^n(E\wedge E\otimes\Omega_C^{p^n-1})) &= \frac{1}{p^n}\mu(E\wedge E\otimes\Omega_C^{p^n-1})+\frac{(p^n-1)(g-1)}{p^n}\\ &= \frac{1}{p^n}(2\mu(E)+\mu(\Omega_C^{p^n-1}))+\frac{(p^n-1)(g-1)}{p^n}\\ &= \frac{2}{p^n}\mu(E)+\frac{3(p^n-1)(g-1)}{p^n}\\ &> \mu(F_*^nE\wedge F_*^nE).\end{aligned}$$

So it is not semi-stable.

When $r=1$ and $p>2$, by Lemma 3.1, we have the following injection

$$F_*^n(E\otimes E\otimes\Omega_C^{p^n-2})\hookrightarrow F_*^n(E)\wedge F_*^n(E).$$

After some computation of the slopes by using Proposition 2.2, we have:

$$\mu(F_*^nE\wedge F_*^nE)=2\mu(F_*^nE)=\frac{2}{p^n}\mu(E)+\frac{2(p^n-1)(g-1)}{p^n}$$

and

$$\begin{aligned}\mu(F_*^n(E\otimes E\otimes\Omega_C^{p^n-2})) &= \frac{1}{p^n}\mu(E\otimes E\otimes\Omega_C^{p^n-2})+\frac{(p^n-1)(g-1)}{p^n}\\ &= \frac{1}{p^n}(2\mu(E)+\mu(\Omega_C^{p^n-2}))+\frac{(p^n-1)(g-1)}{p^n}\\ &= \frac{2}{p^n}\mu(E)+\frac{(3p^n-5)(g-1)}{p^n}\\ &> \mu(F_*^nE\wedge F_*^nE),\end{aligned}$$

when $p>3$ or $p=3$ and $n>1$. So $F_*^nE\wedge F_*^nE$ is not semi-stable when $p>3$ or $p=3$ and $n>1$.

When $r=1$ and $p=2$, by Lemma 3.1 we have the following injection

$$F_*^{n-1}(B_1\otimes E\otimes\Omega_C^{p^{n-1}-1})\hookrightarrow F_*^n(E)\wedge F_*^n(E).$$

After some computation of the slopes by using Proposition 2.2, we have:

$$\mu(F_*^nE\wedge F_*^nE)=2\mu(F_*^nE)=\frac{2}{p^n}\mu(E)+\frac{2(p^n-1)(g-1)}{p^n}$$

and

$$\begin{aligned}\mu(F_*^{n-1}(B_1\otimes E\otimes\Omega_C^{p^{n-1}-1})) &= \frac{1}{p^{n-1}}\mu(B_1\otimes E\otimes\Omega_C^{p^{n-1}-1})+\frac{(p^{n-1}-1)(g-1)}{p^{n-1}}\\ &= \frac{1}{p^{n-1}}(\mu(B_1)+\mu(E)+\mu(\Omega_C^{p^{n-1}-1}))+\frac{(p^{n-1}-1)(g-1)}{p^{n-1}}\\ &= \frac{2}{p^n}\mu(E)+\frac{(3p^n-4)(g-1)}{p^n}\\ &> \mu(F_*^nE\wedge F_*^nE),\end{aligned}$$

when $n>1$. So $F_*^nE\wedge F_*^nE$ is not semi-stable when $n>1$. ☐

Corollary 3.3. *Let C be a smooth projective curve of genus $g\geq 2$ defined over an algebraically closed field k of characteristic $p>0$, then there exists a stable vector bundle E on C, whose double exterior product is not semi-stable.*

Proof. Take a line bundle L on C, then $E:=F_*^nL$ is stable by Lemma 2.4. Thus by the above theorem we see that $E\wedge E$ is not semi-stable when $n>1$. ☐

Proposition 3.4. *Let C be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$, then there exists a vector bundle E which is stable but not cohomologically stable.*

Proof. Take $E = F_*^n L$ for some line bundle L of degree d and some integer $n > 1$, then E is stable by Lemma 2.4. By Definition 2.1, we must prove that there is an integer $t < rk(E)$ and a line bundle A of degree $a \geq t\mu(E)$ satisfying $H^0(\wedge^t E_L \otimes A^*) \neq 0$ When $p = 2$ by Lemma 3.1 we have

$$F_*^{n-1}(B_1 \otimes L \otimes \Omega_C^{p^{n-1}-1}) \hookrightarrow F_*^n L \wedge F_*^n L.$$

We take the line bundle L of degree d such that $d + (p^n - 1)(g-1)$ can be divided by p^{n-1} and then take a line bundle A of degree $\frac{d+(p^n-1)(g-1)}{p^{n-1}}$ such that $A^{p^{n-1}} \cong B_1 \otimes L \otimes \Omega_C^{p^{n-1}-1}$. Thus we have $\deg A \geq 2\mu(F_*^n L) = \frac{d+(p^n-1)(g-1)}{p^{n-1}}$, but $h^0(F_*^n L \wedge F_*^n L \otimes A^{-1}) \geq h^0(F_*^{n-1}(B_1 \otimes L \otimes \Omega_C^{p^{n-1}-1}) \otimes A^{-1}) = h^0(F_*^{n-1}(B_1 \otimes L \otimes \Omega_C^{p^{n-1}-1} \otimes A^{-p^{n-1}})) = h^0(\mathcal{O}_C) \neq 0$, which implies that $F_*^n L$ is not cohomologically stable.

When $p > 2$ by Lemma 3.1 we have

$$F_*^n(L \otimes L \otimes \Omega_C^{p^n-1}) \hookrightarrow F_*^n L \wedge F_*^n L.$$

We take the line bundle L of degree d such that $2d + 2(p^n - 1)(g-1)$ can be divided by p^n and then take a line bundle A of degree $\frac{2d+2(p^n-1)(g-1)}{p^n}$ such that $A^{p^n} \cong L \otimes L \otimes \Omega_C^{p^n-1}$. Thus we have $\deg A \geq 2\mu(F_*^n L) = \frac{2d+2(p^n-1)(g-1)}{p^n}$. But $h^0(F_*^n L \wedge F_*^n L \otimes A^{-1}) \geq h^0(F_*^n(L \otimes L \otimes \Omega_C^{p^n-1}) \otimes A^{-1}) = h^0(F_*^n(L \otimes L \otimes \Omega_C^{p^n-1} \otimes A^{-p^n})) = h^0(\mathcal{O}_C) \neq 0$, which implies that $F_*^n L$ is not cohomologically stable. $\square$

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